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# GAUSSIAN TYPE BOUNDS FOR THE NEUMANN GREEN FUNCTION OF A GENERAL PARABOLIC OPERATOR

MOURAD CHOULLI AND LAURENT KAYSER

ABSTRACT. Based on the fact that the Neumann Green function can be constructed as a perturbation of the fundamental solution by a single-layer potential, we establish a gaussian lower bound and gaussian type upper bound for the Neumann Green function for a general parabolic operator. We build our analysis on classical tools coming from the construction of a fundamental solution of a general parabolic operator by means of the so-called parametrix method. At the same time we provide a simple proof for the gaussian two-sided bounds for the fundamental solution. We also indicate how our method can be adapted to get a gaussian lower bound for the Neumann heat kernel of a compact Riemannian manifold with boundary having non negative Ricci curvature.

**Key words :** parabolic operator, fundamental solution, Neumann Green function, parametrix, heat kernel, compact Riemannian manifold with boundary, Ricci curvature.

**Mathematics subject classification 2010 :** 65M80

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## 1. INTRODUCTION

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^n$  with  $C^{1,1}$ -smooth boundary. Let  $t_0 < t_1$ , we set  $Q = \Omega \times (t_0, t_1)$  and we consider the second order differential operator

$$L = a_{ij}(x, t)\partial_{ij}^2 + b_k(x, t)\partial_k + c(x, t) - \partial_t.$$

Here and henceforth we use the usual Einstein's summation convention.

We make the following assumptions on the coefficients of  $L$ :

(i) the matrix  $(a_{ij}(x, t))$  is symmetric for any  $(x, t) \in \overline{Q}$ ,

(ii)  $a_{ij} \in W^{1,\infty}(Q)$ ,  $b_k, c \in C([t_0, t_1], C^1(\overline{\Omega}))$ ,

(iii)  $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ ,  $(x, t) \in \overline{Q}$ ,  $\xi \in \mathbb{R}^n$ ,

(iv)  $\|a_{ij}\|_{W^{1,\infty}(Q)} + \|b_k\|_{L^\infty(Q)} + \|c\|_{L^\infty(Q)} \leq A$ ,

where  $\lambda > 0$  and  $A > 0$  are two given constants.<sup>1</sup>

Since we will use the fundamental solution in the whole space, we begin by extending the coefficients of  $L$  in a neighborhood  $\tilde{\Omega}$  of  $\overline{\Omega}$  to coefficients having the same regularity. We observe that this is possible in view

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<sup>1</sup>These assumptions are surely not the best possible if one want to construct a fundamental solution or a Green function. But there are sufficient to carry out our analysis.

of the regularity of  $\Omega$ . For simplicity we keep the same symbols for the extended coefficients. We may also assume that the ellipticity condition holds for the extended coefficients with the same  $\lambda$ . Pick  $\psi \in C_0^\infty(\tilde{\Omega})$  satisfying  $0 \leq \psi \leq 1$  and  $\psi = 1$  in a neighborhood of  $\bar{\Omega}$ . We set

$$\tilde{a}_{ij} = a_{ij}\psi + \lambda\delta_{ij}(1 - \psi), \quad \tilde{b}_k = b_k\psi, \quad \tilde{c} = c\psi$$

and

$$\tilde{L} = \tilde{a}_{ij}(x, t)\partial_{ij}^2 + \tilde{b}_k(x, t)\partial_k + \tilde{c}(x, t) - \partial_t.$$

Clearly the coefficients of  $\tilde{L}$  satisfy the same assumptions as those of  $L$ . So in the sequel we will use the same symbol  $L$  for  $L$  or its extension  $\tilde{L}$ .

We recall that the function

$$\mathcal{G}(x, t) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \quad t > 0,$$

is usually called the gaussian Kernel. We set

$$\mathcal{G}_c(x, t) = c^{-1}\mathcal{G}(\sqrt{c}x, t), \quad c > 0.$$

It is important to observe that the map  $c \rightarrow \mathcal{G}_c$  is a non increasing.

We are interested in gaussian two-sided bounds for the Neumann Green function associated to the operator  $L$ . More specifically, denoting by  $G$  the Neumann Green function for  $L$ , we want to prove an estimate of the form

$$\mathcal{G}_C(x - \xi, t - \tau) \leq G(x, t; \xi, \tau) \leq \mathcal{G}_{\tilde{C}}(x - \xi, t - \tau), \quad (x, t; \xi, \tau) \in Q^2, \quad t > \tau,$$

where the constants  $C$  and  $\tilde{C}$  depend only on  $\Omega$ ,  $T = t_1 - t_0$  and  $A$ .

We succeed in proving that the above gaussian lower bound holds true. On the other hand, we are only able to prove an upper bound which is weaker than a gaussian upper bound. Namely we prove an upper bound of the form:

$$G(x, t, \xi, \tau) \leq (t - \tau)^{-1/2} \mathcal{G}_{\tilde{C}}(x - \xi, t - \tau), \quad (x, t; \xi, \tau) \in Q^2, \quad t > \tau,$$

for some constant  $\tilde{C}$  that can depend only on  $\Omega$ ,  $T$  and  $A$ .

To our knowledge the gaussian lower bound have never been established before for the Neumann Green function of a general parabolic operator. The upper bound we prove is weaker than that established by Daners [10]. However, our result is extended to Robin Green functions without the positivity assumption on the boundary zero order coefficient (compare with [10]). We also provide a simple proof of the upper gaussian type estimate. Choi and Kim [7] obtained a gaussian upper bound for a system of operators in divergence form under the assumption that the corresponding Neumann boundary value problem possesses a De Giorgi-Nash-Moser type estimate at the boundary. We will see in Section 4 how the gaussian upper bound for the Neumann Green function can be obtained in a straightforward way from the gaussian upper bound for the fundamental solution when  $\Omega$  is a half space.

The problem is quite different for a Dirichlet Green function since this later vanishes on the boundary. One can prove in an obvious manner, with the help of the maximum principle, that the Dirichlet Green function is non negative and dominated pointwise by the fundamental solution and so it has a gaussian upper bound. Aronson [2] (Theorem 8 in page 670)<sup>2</sup> established an interior gaussian lower bound for a Dirichlet Green function. Later, Cho [5], Cho, Kim and Park [6] extended this result to a global weighted gaussian lower bound involving the distance to the boundary.

For parabolic operators with time-independent coefficients, a fundamental solution or a Green function is reduced to a heat kernel. We mention that there is a tremendous literature dealing with gaussian bounds for heat kernels. We quote the classical books by Davies [11], Grigor'yan [16], Ouhabaz [27] Saloff-Coste [29] and Stroock [30], but of course there are many other references on the subject.

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<sup>2</sup>Let us observe that Theorem 8 in page 670 of [2] can be used to extend the results of Section 3 of [14] to a general parabolic operator. In other words, one can obtain a proof of a continuity theorem by Nash [25] and the Moser's Harnack inequality [24] for a general divergence form parabolic operator, since they rely on the two-sided gaussian bounds for the fundamental solution.

As we said in the summary, the main ingredient in our analysis relies on the classical construction of the fundamental solution by means of the so-called parametrix method. We revisit this construction in the next section and we derive from it the gaussian two-sided bounds for the fundamental solution. Special attention is paid to the dependence of the gaussian upper bound on the coefficients of orders one and zero. This dependence will be in the heart of the proof of the gaussian type upper bound for the Neumann Green function since we will conjugate  $L$  with  $e^\psi$ , for some function  $\psi$  appropriately chosen. This is a classical argument to derive a gaussian upper bound from a Nash upper bound. Note however that this argument is not enough to get a gaussian type upper bound for the Neumann Green function. We need to conjugate again with a special function together with a maximum principle argument in order to get a gaussian type upper bound. This is done in Section 4. Before, we prove in Section 3 a gaussian lower bound for the Neumann Green function. To do so, we construct the Neumann Green function as a perturbation of the fundamental solution by a single-layer potential. The gaussian lower bound is then derived from the smoothing effect of the single-layer potential. In the last section we adapt our method to prove a gaussian lower bound for the Neumann heat kernel of a compact Riemannian manifold with boundary having non negative Ricci curvature.

## 2. THE PARAMETRIX METHOD REVISITED

We are concerned in this section with gaussian two-sided bounds for the fundamental solution. For a systematic study of the fundamental solution we refer to the classical monographs by A. Friedman [15] and O. A. Ladyzhenskaja, V. A. Solonnikov and N. N. Ural'tzeva [19].

In the sequel  $P = \mathbb{R}^n \times (t_0, t_1)$ .

We recall that a fundamental solution of  $Lu = 0$  in  $P$  is a function  $E(x, t; \xi, \tau)$  which is  $C^{2,1}$  in  $P^2 \cap \{t > \tau\}$ , satisfies

$$LE(\cdot, \cdot; \xi, \tau) = 0 \text{ in } \mathbb{R}^n \times \{\tau < t \leq t_1\}, \text{ for any } (\xi, \tau) \in \mathbb{R}^n \times [t_0, t_1[$$

and, for any  $f \in C_0(\mathbb{R}^n)$ ,

$$\lim_{t \searrow \tau} \int_{\mathbb{R}^n} E(x, t; \xi, \tau) f(\xi) d\xi = f(x), \quad x \in \mathbb{R}^n.^3$$

The construction of a fundamental solution by means of the so-called parametrix method was initiated by E. E. Levi [21]. Let  $a = (a^{ij})$  be the inverse matrix of  $(a_{ij})$ ,  $|a|$  the determinant of  $a$  and

$$Z(x, t; \xi, \tau) = [4\pi(t - \tau)]^{-n/2} \sqrt{|a(\xi, \tau)|} e^{-\frac{a(\xi, \tau)(x - \xi) \cdot (x - \xi)}{4(t - \tau)}}, \quad (x, t; \xi, \tau) \in P^2 \cap \{t > \tau\}.$$

This function is called the parametrix. It satisfies

$$(2.1) \quad L_0 Z(\cdot, \cdot; \xi, \tau) = 0 \text{ in } \mathbb{R}^n \times \{\tau < t \leq t_1\} \text{ for any } (\xi, \tau) \in \mathbb{R}^n \times [t_0, t_1[,$$

where

$$L_0 = a_{ij}(\xi, \tau) \partial_{ij}^2 - \partial_t.$$

In the parametrix method we seek  $E$ , a fundamental solution of  $Lu = 0$  in  $P$ , of the form

$$(2.2) \quad E(x, t; \xi, \tau) = Z(x, t; \xi, \tau) + \int_{\tau}^t \int_{\mathbb{R}^n} Z(x, t; \eta, \sigma) \Phi(\eta, \sigma; \xi, \tau) d\eta d\sigma,$$

where  $\Phi$  is to be determined in order to satisfy  $LE(\cdot, \cdot; \xi, \tau) = 0$  for any  $(\xi, \tau) \in \mathbb{R}^n \times [t_0, t_1[$ .

Following Formulas (4.4) and (4.5) in page 14 of [15],  $\Phi$  is given by the series

$$\Phi = \sum_{\ell=1}^{\infty} \Phi_{\ell},$$

where  $\Phi_1(x, t; \xi, \tau) = LZ(x, t; \xi, \tau)$  and

$$\Phi_{\ell+1}(x, t; \xi, \tau) = \int_{\tau}^t \int_{\mathbb{R}^n} \Phi_1(x, t; \eta, \sigma) \Phi_{\ell}(\eta, \sigma; \xi, \tau) d\eta d\sigma, \quad \ell \geq 1.$$

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<sup>3</sup>One can take a larger class of functions. Namely a class of continuous functions satisfying a certain growth condition at infinity (e.g. formulas (6.1) and (6.2) in page 22 of [15]).

Here for simplicity we write  $LZ(x, t; \xi, \tau)$  instead of  $[LZ(\cdot, \cdot, \xi, \tau)](x, t)$ .

Let  $d_i$ ,  $1 \leq i \leq n$ , given by

$$d_i = d_i(x, t; \xi, \tau) = -\frac{a^{ij}(\xi, \tau)(x_j - \xi_j)}{2(t - \tau)}, \quad (x, t; \xi, \tau) \in P^2 \cap \{t > \tau\}.$$

Then

$$\partial_i Z = d_i Z, \quad \partial_{ij}^2 Z = \left[ -\frac{a^{ij}(\xi, \tau)}{2(t - \tau)} + d_j d_i \right] Z.$$

Therefore, taking into account (2.1), we get

$$LZ = LZ - L_0 Z = \left\{ (a_{ij}(x, t) - a_{ij}(\xi, \tau)) \left[ -\frac{a^{ij}(\xi, \tau)}{2(t - \tau)} + d_j d_i \right] + b_k d_k + c \right\} Z.$$

We write  $LZ = \Psi Z$ , where

$$\Psi = (a_{ij}(x, t) - a_{ij}(\xi, \tau)) \left[ -\frac{a^{ij}(\xi, \tau)}{2(t - \tau)} + d_j d_i \right] + b_k d_k + c.$$

Let

$$M = \max_{i,j} \|a_{ij}\|_{W^{1,\infty}(Q)}, \quad N = \max(\max_k \|b_k\|_{L^\infty(Q)}, \|c\|_{L^\infty(Q)}, 1).^4$$

Since

$$\begin{aligned} |d_i| &\leq \frac{|x - \xi|}{2\lambda(t - \tau)}, \\ |a_{ij}(x, t) - a_{ij}(\xi, \tau)| &\leq M(|x - \xi| + t - \tau), \end{aligned}$$

we have

$$(2.3) \quad |\Psi(x, t; \xi, \tau)| \leq N \frac{1}{\sqrt{t - \tau}} P \left( \frac{|x - \xi|}{\sqrt{t - \tau}} \right).$$

Here  $P$  is a polynomial function of degree less or equal to three whose coefficients depend only on  $M$ .

Unless otherwise stated, all the constants we use now do not depend on  $N$ .

In light of (2.3) we obtain

$$|LZ| \leq CN(t - \tau)^{-(n+1)/2} P(\rho) e^{-(\lambda/4)\rho^2} = CN(t - \tau)^{-(n+1)/2} \left[ P(\eta) e^{-(\lambda/8)\rho^2} \right] e^{-(\lambda/8)\rho^2},$$

with

$$\rho = \frac{|x - \xi|}{\sqrt{t - \tau}}.$$

But the function  $\rho \in (0, +\infty) \rightarrow P(\rho) e^{-(\lambda/8)\rho^2}$  is bounded. Consequently, where  $\lambda^* = \lambda/8$ ,

$$(2.4) \quad |\Phi_1(x, t; \xi, \tau)| = |LZ(x, t; \xi, \tau)| \leq N \tilde{C} (t - \tau)^{-(n+1)/2} e^{-\frac{\lambda^* |x - \xi|^2}{t - \tau}}.$$

The following lemma will be useful in the sequel. Its proof is given in page 15 of [15].

**Lemma 2.1.** *Let  $c > 0$  and  $-\infty < \gamma, \beta < n/2 + 1$ . Then*

$$\begin{aligned} \int_{\tau}^t \int_{\mathbb{R}^n} (t - \sigma)^{-\gamma} e^{-\frac{c|x-\eta|^2}{t-\sigma}} (\sigma - \tau)^{-\beta} e^{-\frac{c|\eta-\xi|^2}{\sigma-\tau}} d\eta d\sigma \\ = \left( \frac{4\pi}{c} \right)^{n/2} B(n/2 - \gamma + 1, n/2 - \beta + 1) (t - \tau)^{n/2+1-\gamma-\beta} e^{-\frac{c|x-\xi|^2}{t-\tau}}, \end{aligned}$$

where  $B$  is the usual beta function.

<sup>4</sup>We note that we can also take  $M$  equal to the maximum of Lipschitz constants of  $a_{ij}$ .

We want to show

$$(2.5) \quad |\Phi_\ell(x, t; \xi, \tau)| \leq (N\tilde{C})^\ell \widehat{C}^{\ell-1} (t-\tau)^{-(n+2-\ell)/2} \prod_{j=1}^{\ell-1} B(1/2, j/2) e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau}}, \quad \ell \geq 2.$$

Here  $\tilde{C}$  is the same constant as in (2.4) and  $\widehat{C} = \left(\frac{4\pi}{\lambda^*}\right)^{n/2}$ .

As

$$\Phi_2(x, t; \xi, \tau) = \int_\tau^t \int_{\mathbb{R}^n} \Phi_1(x, t; \eta, \sigma) \Phi_1(\eta, \sigma; \xi, \tau) d\eta d\sigma,$$

estimate (2.4) and Lemma 2.1 with  $\gamma = \beta = n/2 + 1$  show that (2.5) holds true with  $\ell = 2$ . The general case follows by an induction argument in  $\ell$ . Indeed, using

$$\Phi_{\ell+1}(x, t; \xi, \tau) = \int_\tau^t \int_{\mathbb{R}^n} \Phi_1(x, t; \eta, \sigma) \Phi_\ell(\eta, \sigma; \xi, \tau) d\eta d\sigma,$$

(2.4), (2.5) for  $\ell$  and Lemma 2.1 with  $\gamma = n/2 + 1$  and  $\beta = (n+2-\ell)/2$ , we obtain easily that (2.5) holds true with  $\ell+1$  in place of  $\ell$ .

If  $\Gamma$  is the usual gamma function, we recall that

$$B(1/2, j/2) = \frac{\Gamma(1/2)\Gamma(j/2)}{\Gamma((j+1)/2)}.$$

Therefore

$$(2.6) \quad \prod_{j=1}^{\ell-1} B(1/2, j/2) = \frac{\Gamma(1/2)^\ell}{\Gamma(\ell/2)} = \frac{\sqrt{\pi}^\ell}{\Gamma(\ell/2)}.$$

(2.4)-(2.6) entail that

$$(2.7) \quad |\Phi(x, t; \xi, \tau)| \leq \sum_{\ell \geq 1} |\Phi_\ell(x, t; \xi, \tau)| \leq N\tilde{C}(1+S)(t-\tau)^{-(n+1)/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau}},$$

with

$$S = \sum_{\ell \geq 1} \left[ CN(t-\tau)^{1/2} \right]^\ell / \Gamma((\ell+1)/2).$$

We have  $\Gamma((\ell+1)/2) = \Gamma(m+1/2) \geq \Gamma(m) = (m-1)!$  if  $\ell = 2m$  and  $\Gamma((\ell+1)/2) = \Gamma(m+1) = m!$  if  $\ell = 2m+1$ . Then

$$\begin{aligned} S &= \sum_{m \geq 1} \frac{1}{\Gamma(m+1/2)} \left[ CN(t-\tau)^{1/2} \right]^{2m} + \sum_{m \geq 0} \frac{1}{\Gamma(m+1)} \left[ CN(t-\tau)^{1/2} \right]^{2m+1} \\ &\leq \sum_{m \geq 1} \frac{1}{(m-1)!} \left[ CN(t-\tau)^{1/2} \right]^{2m} + \sum_{m \geq 0} \frac{1}{m!} \left[ CN(t-\tau)^{1/2} \right]^{2m+1}. \end{aligned}$$

Whence

$$1 + S \leq \tilde{C} e^{\tilde{C}N^2(t-\tau)}.$$

Plugging this estimate into (2.7), we obtain

$$(2.8) \quad |\Phi(x, t; \xi, \tau)| \leq \tilde{C}N(t-\tau)^{-(n+1)/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + \tilde{C}N^2(t-\tau)}.$$

With the help of Lemma 2.1, estimate (2.8) yields

$$(2.9) \quad \left| \int_\tau^t \int_{\mathbb{R}^n} Z(x, t; \eta, \sigma) \Phi(\eta, \sigma; \xi, \tau) d\eta d\sigma \right| \leq \tilde{C}N(t-\tau)^{-(n-1)/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + \tilde{C}N^2(t-\tau)}.$$

Noting that this inequality can be rewritten as

$$\left| \int_\tau^t \int_{\mathbb{R}^n} Z(x, t; \eta, \sigma) \Phi(\eta, \sigma; \xi, \tau) d\eta d\sigma \right| \leq \tilde{C} \left[ N(t-\tau)^{1/2} e^{-\tilde{C}N^2(t-\tau)} \right] (t-\tau)^{-n/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + 2\tilde{C}N^2(t-\tau)}$$

and using that  $\rho \rightarrow \rho e^{-\tilde{C}\rho^2}$  is bounded function on  $[0, +\infty)$ , we obtain

$$(2.10) \quad \left| \int_{\tau}^t \int_{\mathbb{R}^n} Z(x, t; \eta, \sigma) \Phi(\eta, \sigma; \xi, \tau) d\eta d\sigma \right| \leq \widehat{C}(t - \tau)^{-n/2} e^{-\frac{\lambda^* |x - \xi|^2}{t - \tau} + 2\tilde{C}N^2(t - \tau)}.$$

An immediate consequence of (2.10) is

$$(2.11) \quad |E(x, t; \xi, \tau)| \leq \tilde{C}(t - \tau)^{-n/2} e^{-\frac{\lambda^* |x - \xi|^2}{t - \tau} + \tilde{C}N^2(t - \tau)}.$$

This estimate is essential when establishing the gaussian type upper bound for the Neumann Green function.

In the rest of this section we forsake the explicit dependence on  $N$ . So the constants below may depend on  $\Omega$ ,  $\lambda$ ,  $A$ , and  $T$ .

From (2.9) we deduce in a straightforward manner that there exists  $\delta > 0$  such that

$$(2.12) \quad E(x, t; \xi, \tau) \geq \widehat{C}(t - \tau)^{-n/2}, \quad (x, t; \xi, \tau) \in P^2, \quad t > \tau, \quad \tilde{C}|x - \xi|^2 < t - \tau \leq \delta.$$

By Theorem 11 in page 44 of [15],  $E$  is positive. Moreover  $E$  satisfies the following identity, usually called the reproducing property,

$$(2.13) \quad E(x, t; \xi, \tau) = \int_{\mathbb{R}^n} E(x, t; \eta, \sigma) E(\eta, \sigma; \xi, \tau) d\eta, \quad x, \xi \in \mathbb{R}^n, \quad t_0 \leq \tau < \sigma < t \leq t_1.$$

We can now paraphrase the proof of Theorem 2.7 in page 334 of [14] to get a gaussian lower bound for  $E$  when  $0 < t - \tau \leq \delta$ . To pass from  $t - \tau \leq \delta$  to  $t - \tau \leq T$  one can use again a reproducing property argument. We detail the same argument in the proof of Theorem 3.1.

We sum up our analysis in the following theorem.

**Theorem 2.1.** *The fundamental solution  $E$  satisfies the gaussian two-sided bounds:*

$$(2.14) \quad \mathcal{G}_C(x - \xi, t - \tau) \leq E(x, t; \xi, \tau) \leq \mathcal{G}_{\tilde{C}}(x - \xi, t - \tau), \quad (x, t, \xi, \tau) \in P^2 \times \{t > \tau\}.$$

*Remark 2.1.* Let us assume that conditions (i)-(iv) above hold in the whole space  $\mathbb{R}^n \times \mathbb{R}$  instead of  $Q$  only. Taking into account the exponential term in  $N^2$  in (2.11), we prove, once again with the help of the reproducing property, the following global estimate in time:

$$(2.15) \quad e^{-\kappa N^2(t - \tau)} \mathcal{G}_C(x - \xi, t - \tau) \leq E(x, t; \xi, \tau) \leq e^{\kappa N^2(t - \tau)} \mathcal{G}_{\tilde{C}}(x - \xi, t - \tau),$$

for some constant  $\kappa > 0$ , where  $(x, t, \xi, \tau) \in (\mathbb{R}^n \times \mathbb{R})^2$ .

We point out that (2.15) does not give the two-sided gaussian bounds by Fabes and Stroock [14] for the divergence form operator  $\partial_i(a_{ij}(x, t)\partial_j \cdot) - \partial_t$  with  $(C^\infty)$  smooth coefficients. This is not surprising since the arguments we used for proving (2.15) are not well adapted to divergence form operator. We note however that the approach developed in [14] for establishing the gaussian two-sided bounds is more involved.

Gaussian two-sided bounds were obtained by S. D. Eidel'man and F. O. Porper [12] when the coefficients of  $L$  satisfy the uniform Dini condition with respect to  $x$ . The main tool in [12] is a parabolic Harnack inequality. We refer also to [1], [13], [18] and [26], where the reader can find various results on bounds for the fundamental solution.

We mentioned in the introduction that the Moser's Harnack inequality in [14] can be extended to a general divergence form parabolic operator. Let us show briefly how this Moser's Harnack inequality still holds for a general parabolic operator. First, we recall that the Dirichlet Green function was constructed in [19] as a perturbation of the fundamental solution by a double-layer potential (formula (16.7) in page 408). Therefore, in light of formula (16.10) in page 409 and estimate (16.14) in page 411 of [19], we can assert that Lemma 5.1 of [14] remains true for our  $L$ . Next, paraphrasing the proof of Lemma 5.2 and Theorem 5.4 in [14] (more detailed proofs are given in [30]), we can state the following Moser's Harnack inequality.

**Theorem 2.2.** *Let  $\eta, \mu, \varrho \in (0, 1)$ . Then there is  $M > 0$ , depending on  $n, \lambda, A, \eta, \mu$  and  $\varrho$  such that for all  $(x, s) \in \mathbb{R}^n \times \mathbb{R}$ , all  $R > 0$  and all non negative  $u \in C^{2,1}(\overline{B}(x, R) \times [s - R^2, s])$  satisfying  $Lu = 0$  one has*

$$u(y, t) \leq Mu(x, s) \quad \text{for all } (y, t) \in \overline{B}(x, \varrho R) \times [s - \eta R^2, s - \mu R^2].$$

## 3. GAUSSIAN LOWER BOUND FOR THE NEUMANN GREEN FUNCTION

The conormal vector at  $x \in \partial\Omega$  is denoted by  $\nu = (\nu_1(x), \dots, \nu_n(x))$ . That is  $\nu_i = a_{ij}\mathbf{n}_j$ , where  $\mathbf{n} = \mathbf{n}(x)$  is the unit outward normal vector at  $x$ .

For  $\tau \in [t_0, t_1[$ , we set  $Q_\tau = \Omega \times (\tau, t_1)$  and  $\Sigma_\tau = \partial\Omega \times (\tau, t_1)$ . We consider the Neumann initial-boundary value problem (abbreviated to IBVP) for the operator  $L$ :

$$(3.1) \quad \begin{cases} Lu = 0 & \text{in } Q_\tau, \\ u(\cdot, \tau) = \psi & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \Sigma_\tau. \end{cases}$$

From Theorem 2 in page 144 of [15] and its proof, for any  $\psi \in C_0^\infty(\Omega)$ , the IBVP (3.1) has a unique solution  $u \in C(\overline{Q_\tau}) \cap C^{2,1}(Q_\tau)$  given by

$$(3.2) \quad u(x, t) = \int_\tau^t \int_{\partial\Omega} E(x, t; \xi, \sigma) \varphi(\xi, \sigma) d\xi d\sigma + \int_\Omega E(x, t; \xi, \tau) \psi(\xi) d\xi.$$

Here

$$(3.3) \quad \varphi(x, t) = F_\tau(x, t) - 2 \sum_{\ell \geq 1} \int_\tau^t \int_{\partial\Omega} M_\ell(x, t; \xi, \sigma) F_\tau(\xi, \sigma) d\xi d\sigma,$$

with

$$F_\tau(x, t) = -2 \int_\Omega \partial_\nu E(x, t; \xi, \tau) \psi(\xi) d\xi,$$

$$M_1 = -2\partial_\nu E,$$

$$M_{\ell+1}(x, t; \xi, \tau) = \int_\tau^t \int_{\partial\Omega} M_1(x, t; \eta, \sigma) M_\ell(\eta, \sigma; \xi, \tau) d\eta d\sigma.$$

Let, where  $(x, t) \in \Sigma_\tau$  and  $\xi \in \Omega$ ,

$$\mathcal{N}(x, t; \xi, \tau) = -2\partial_\nu E(x, t; \xi, \tau) - 2 \sum_{\ell \geq 1} \int_\tau^t \int_{\partial\Omega} M_\ell(x, t; \eta, \sigma) \partial_\nu E(\eta, \sigma; \xi, \tau) d\eta d\sigma.$$

Assume for the moment (see the proof below) that

$$(3.4) \quad \varphi(x, t) = \int_\Omega \mathcal{N}(x, t; \xi, \tau) \psi(\xi) d\xi.$$

We set

$$(3.5) \quad G(x, t, \xi, \tau) = \int_\tau^t \int_{\partial\Omega} E(x, t; \eta, \sigma) \mathcal{N}(\eta, \sigma; \xi, \tau) d\eta d\sigma + E(x, t; \xi, \tau).$$

It follows from Fubini's theorem that

$$(3.6) \quad u(x, t) = \int_\Omega G(x, t; \xi, \tau) \psi(\xi) d\xi.$$

The function  $G$  is called the Neumann Green function for  $Lu = 0$  in  $Q$ .

We have, for any  $0 \leq \psi \in C_0^\infty(\Omega)$ ,  $u \geq 0$  according to the maximum principle (e.g. Theorem 2.9 and remarks following it in page 15 of [23]) ; whence  $G \geq 0$ . From the uniqueness of the solution of the IBVP (3.1) we have also

$$\int_\Omega G(x, t; \xi, \tau) \psi(\xi) d\xi = \int_\Omega G(x, t, \eta, \sigma) d\eta \int_\Omega G(\eta, \sigma, \xi, \tau) \psi(\xi) d\xi \quad \text{for any } \psi \in C_0^\infty(\Omega), \tau < \sigma < t.$$

Therefore

$$(3.7) \quad G(x, t; \xi, \tau) = \int_\Omega G(x, t, \eta, \sigma) G(\eta, \sigma, \xi, \tau) d\eta, \quad \tau < \sigma < t.$$

That is  $G$  has the reproducing property.



We note that when  $c = 0$ ,  $G$  satisfies in addition

$$\int_{\Omega} G(x, t; \xi, \tau) d\xi = 1.$$

The key point in the proof of the gaussian lower bound for  $G$  is the following lemma.

**Lemma 3.1.** *For  $1/2 < \mu \leq \frac{n}{2}$ , we have*

$$(3.8) \quad |\mathcal{N}(x, t; \xi, \tau)| \leq C(t - \tau)^{-\mu} |x - \xi|^{-n+2\mu}, \quad (x, t) \in \Sigma_{\tau}, \quad \xi \in \Omega, \quad x \neq \xi.$$

The lemma below appears in page 137 of [15] as Lemma 1. It is needed for proving Lemma 3.1.

**Lemma 3.2.** *Let  $0 < a, b < n - 1$ . Then*

$$(3.9) \quad \int_{\partial\Omega} |x - \eta|^{-a} |\eta - \xi|^{-b} d\eta \leq \widehat{C} |x - \xi|^{n-1-(a+b)}.$$

*Proof of Lemma 3.1.* Since  $\Omega$  is of class  $C^{1,1}$ , we obtain, by paraphrasing the proof of formula (2.12) in page 137 of [15],

$$|\partial_{\nu} E(x, t; \xi, \tau)| \leq C(t - \tau)^{-\mu} |x - \xi|^{-n+2\mu},$$

for any  $\mu > 0$ , and then

$$(3.10) \quad |M_1(x, t; \xi, \tau)| \leq C(t - \tau)^{-\mu} |x - \xi|^{-n+2\mu}.$$

We assume first that  $1/2 < \mu < 1$ . Since

$$|M_2(x, t; \xi, \tau)| \leq \int_{\tau}^t \int_{\partial\Omega} |M_1(x, t; \eta, \sigma)| |M_1(\eta, \sigma; \xi, \tau)| d\eta d\sigma,$$

(3.10) leads

$$(3.11) \quad |M_2(x, t; \xi, \tau)| \leq C^2 \int_{\tau}^t (t - \sigma)^{-\mu} (\tau - \sigma)^{-\mu} d\sigma \int_{\partial\Omega} |x - \eta|^{-n+2\mu} |\xi - \eta|^{-n+2\mu} d\eta.$$

Or from Lemma 3.2

$$(3.12) \quad \int_{\partial\Omega} |x - \eta|^{-n+2\mu} |\xi - \eta|^{-n+2\mu} d\eta \leq \widehat{C} |x - \xi|^{-n+4\mu-1}.$$

On the other hand

$$(3.13) \quad \int_{\tau}^t (t - \sigma)^{-\mu} (\tau - \sigma)^{-\mu} d\sigma = (t - \tau)^{-\mu+(1-\mu)} \int_0^1 s^{-\mu} (1-s)^{1-\mu} ds = (t - \tau)^{-\mu+(1-\mu)} B(1 - \mu, 1 - \mu).$$

We plug (3.12) and (3.13) into (3.11) ; we obtain

$$|M_2(x, t; \xi, \tau)| \leq C^2 \widehat{C} (t - \tau)^{-\mu+(1-\mu)} B(1 - \mu, 1 - \mu) |x - \xi|^{-n+2\mu+(2\mu-1)}.$$

Now an induction argument in  $\ell$  yields

$$|M_{\ell}(x, t; \xi, \tau)| \leq C^{\ell} \widehat{C}^{\ell-1} (t - \tau)^{-\mu+(\ell-1)(1-\mu)} \frac{\Gamma(1 - \mu)^{\ell}}{\Gamma(\ell(1 - \mu))} |x - \xi|^{\min(0, -n+2\mu+(\ell-1)(2\mu-1))}.$$

Let  $\tilde{\ell}$  be the smallest integer  $\ell$  such that  $-\mu + (\ell - 1)(1 - \mu) \geq 0$  and  $-n + 2\mu + (\ell - 1)(2\mu - 1) \geq 0$ . Since from the Stirling asymptotic formula

$$\Gamma(\ell(1 - \mu)) \sim (e^{-1}(\ell(1 - \mu) - 1))^{\ell(1-\mu)-1} \sqrt{2\pi(\ell(1 - \mu) - 1)} \quad \text{as } \ell \rightarrow +\infty,$$

it follows that the series

$$S = \sum_{\ell \geq \tilde{\ell}} C^{\ell} \widehat{C}^{\ell-1} T^{-\mu+(\ell-1)(1-\mu)} \frac{\Gamma(1 - \mu)^{\ell}}{\Gamma(\ell(1 - \mu))}$$

converges.

It is straightforward to check that

$$|\mathcal{N}(x, t; \xi, \tau)| \leq \sum_{\ell=1}^{\tilde{\ell}-1} |M_\ell(x, t; \xi, \tau)| + S.$$

Hence it is enough to prove that  $\tilde{\mathcal{N}}(x, t; \xi, \tau) = \sum_{\ell=1}^{\tilde{\ell}-1} |M_\ell(x, t; \xi, \tau)|$  satisfies (3.8). To this end, we observe that

$$\begin{aligned} |M_2(x, t; \xi, \tau)| &\leq \int_{\tau}^{(\tau+t)/2} \int_{\partial\Omega} |M_1(x, t; \eta, \sigma)| |M_1(\eta, \sigma; \xi, \tau)| d\eta d\sigma \\ &\quad + \int_{(\tau+t)/2}^t \int_{\partial\Omega} |M_1(x, t; \eta, \sigma)| |M_1(\eta, \sigma; \xi, \tau)| d\eta d\sigma. \end{aligned}$$

If  $1/2 < \mu \leq n/2$  and  $1/2 < \alpha < 1$ , we have

$$\begin{aligned} \int_{\tau}^{(\tau+t)/2} \int_{\partial\Omega} |M_1(x, t; \eta, \sigma)| |M_1(\eta, \sigma; \xi, \tau)| d\eta d\sigma &\leq C^2 \int_{\tau}^{(\tau+t)/2} (\sigma - \tau)^{-\alpha} (t - \sigma)^{-\mu} d\sigma \\ &\quad \int_{\partial\Omega} |x - \eta|^{-n+2\mu} |\xi - \eta|^{-n+2\alpha} d\eta \\ &\leq C^2 \left(\frac{t - \tau}{2}\right)^{-\mu} \int_{\tau}^{(\tau+t)/2} (\sigma - \tau)^{-\alpha} d\sigma |x - \xi|^{-n+2\mu+2\alpha-1} \\ &\leq C^2 \left(\frac{t - \tau}{2}\right)^{-\mu-\alpha+1} |x - \xi|^{-n+2\mu+2\alpha-1} \\ &\leq C'(t - \tau)^{-\mu} |x - \xi|^{-n+2\mu} \end{aligned}$$

Similarly,

$$\int_{(\tau+t)/2}^t \int_{\partial\Omega} |M_1(x, t; \eta, \sigma)| |M_1(\eta, \sigma; \xi, \tau)| d\eta d\sigma \leq C(t - \tau)^{-\mu} |x - \xi|^{-n+2\mu}$$

Therefore,  $M_2$  satisfies (3.8). Repeating the same argument we conclude that each  $M_\ell$  verifies (3.8) implying that  $\tilde{\mathcal{N}}$  satisfies also (3.8).  $\square$

*Proof of (3.4).* Let

$$\begin{aligned} \mathcal{N}_k(x, t; \xi, \tau) &= -2\partial_\nu E(x, t; \xi, \tau) - 2 \sum_{\ell \geq 1}^k \int_{\tau}^t \int_{\partial\Omega} M_\ell(x, t; \eta, \sigma) \partial_\nu E(\eta, \sigma; \xi, \tau) d\eta d\sigma, \\ \varphi_k(x, t) &= -2F_\tau(x, t) - 2 \sum_{\ell \geq 1}^k \int_{\tau}^t \int_{\partial\Omega} M_\ell(x, t; \xi, \sigma) F_\tau(\xi, \sigma) d\xi d\sigma. \end{aligned}$$

In light of Lemma 3.2 and with the help Lebesgue's dominated convergence theorem we can assert that

$$\int_{\Omega} \mathcal{N}_k(x, t; \xi, \tau) \psi(\xi) d\xi \longrightarrow \int_{\Omega} \mathcal{N}(x, t; \xi, \tau) \psi(\xi) d\xi \quad \text{as } k \longrightarrow +\infty.$$

Or according to Funini's theorem

$$\varphi_k(x, t) = \int_{\Omega} \mathcal{N}_k(x, t; \xi, \tau) \psi(\xi) d\xi.$$

But  $\varphi_k(x, t) \rightarrow \varphi(x, t)$  when  $k$  tends to infinity. Then the uniqueness of the limit yields

$$\varphi(x, t) = \int_{\Omega} \mathcal{N}(x, t; \xi, \tau) \psi(\xi) d\xi.$$

$\square$

We are now ready to prove

**Theorem 3.1.** *The Neumann Green function  $G$  satisfies the gaussian lower bound:*

$$(3.14) \quad \mathcal{G}_C(x - \xi, t - \tau) \leq G(x, t; \xi, \tau), \quad (x, t; \xi, \tau) \in Q^2 \cap \{t > \tau\}.$$

*Proof.* Let

$$G_0(x, t; \xi, \tau) = \int_{\tau}^t \int_{\partial\Omega} E(x, t; \eta, \sigma) N(\eta, \sigma; \xi, \tau) d\eta d\sigma.$$

From the gaussian upper bound for  $E$  we obtain in a straightforward way that, for any  $\mu > 0$ ,

$$|E(x, t; \xi, \tau)| \leq C(t - \tau)^{-\mu} |x - \xi|^{-n+2\mu}.$$

Let  $1/2 < \mu \leq n/2$  and  $1/2 < \alpha < 1$ . The same argument we used in the preceding proof for showing that  $M_2$  satisfies (3.8) enables us to prove

$$|G_0(x, t; \xi, \tau)| \leq C(t - \tau)^{-\mu-\alpha+1} |x - \xi|^{-n+2\mu+2\alpha-1}.$$

The peculiar choice of  $\mu = n/2$  gives

$$|G_0(x, t; \xi, \tau)| \leq C(t - \tau)^{-n/2-\alpha+1}.$$

On the other hand, we know from (2.12) that

$$E(x, t; \xi, \tau) \geq C(t - \tau)^{-n/2}, \quad (x, t; \xi, \tau) \in P^2, \quad t > \tau, \quad \widehat{C}|x - \xi|^2 < t - \tau.$$

Hence

$$\begin{aligned} G(x, t; \xi, \tau) &\geq E(x, t; \xi, \tau) - |G_0(x, t; \xi, \tau)| \\ &\geq C(t - \tau)^{-n/2} (1 - \widetilde{C}(t - \tau)^{1-\alpha}), \quad t > \tau, \quad \widehat{C}|x - \xi|^2 < t - \tau. \end{aligned}$$

Consequently, we find  $\delta > 0$  for which

$$G(x, t; \xi, \tau) \geq C(t - \tau)^{-n/2}, \quad (x, t; \xi, \tau) \in Q^2, \quad 0 < t - \tau \leq \delta, \quad \widetilde{C}|x - \xi|^2 < t - \tau.$$

Or equivalently

$$(3.15) \quad G(x, t; \xi, \tau) \geq C(t - \tau)^{-n/2}, \quad (x, t; \xi, \tau) \in Q^2, \quad 0 < t - \tau \leq \delta, \quad |x - \xi| < \widehat{C}(t - \tau)^{1/2}.$$

As  $\Omega$  is connected, we find a path  $\gamma : [0, 1] \rightarrow \Omega$  connecting  $x$  to  $\xi$  which is piecewise constant. Let  $k$  be a positive integer and set  $y_i = \gamma(i/k)$ ,  $0 \leq i \leq k$ . Then it is not difficult to show that there exists a constant  $c \geq 1$  not depending on  $k$  such that

$$(3.16) \quad |y_{i+1} - y_i| \leq \frac{c}{k} |x - \xi|, \quad 0 \leq i \leq k - 1.$$

When  $2c|x - \xi| \leq \widehat{C}(t - \tau)^{1/2}$  (implying  $|x - \xi| \leq \widehat{C}(t - \tau)^{1/2}$ ), (3.14) follows immediately from (3.15). Therefore we may assume that  $2c|x - \xi| > \widehat{C}(t - \tau)^{1/2}$ . We choose  $m \geq 2$  as the smallest integer satisfying

$$2c \frac{|x - y|}{m^{1/2}} \leq \widehat{C}(t - \tau)^{1/2}.$$

Set  $x_i = \gamma(i/m)$ ,  $0 \leq i \leq m$ , and

$$r = \frac{1}{4} \widehat{C} \left( \frac{t - \tau}{m} \right)^{1/2}.$$

In light of the reproducing property and the positivity of  $G$  we obtain

$$\begin{aligned} G(x, t; \xi, \tau) &= \int_{\Omega} \dots \int_{\Omega} G \left( x, t; \xi_1, \frac{(m-1)t + \tau}{m} \right) \dots G \left( \xi_{m-1}, \frac{t + (m-1)\tau}{m}; \xi, \tau \right) d\xi_1 \dots d\xi_{m-1} \\ &\geq \int_{B(x_1, r) \cap \Omega} \dots \int_{B(x_{m-1}, r) \cap \Omega} G \left( x, t; \xi_1, \frac{(m-1)t + \tau}{m} \right) \dots \\ &\quad \dots G \left( \xi_{m-1}, \frac{t + (m-1)\tau}{m}; \xi, \tau \right) d\xi_1 \dots d\xi_{m-1}. \end{aligned} \tag{3.17}$$

Because  $\Omega$  is  $C^{1,1}$ -smooth, it has the uniform interior cone property and in particular it satisfies the following: there exist two positive constants  $d$  and  $r_0$  such that, for any  $z \in \overline{\Omega}$  and  $0 < \rho \leq r_0$ ,

$$(3.18) \quad d\rho^n \leq |B(z, \rho) \cap \Omega|.^5$$

In the sequel shortening if necessary the constant  $\widehat{C}$  we assume that  $r \leq r_0$ .

Let  $\xi_0 = x$  and  $\xi_m = \xi$ ; we have

$$|\xi_{i+1} - \xi_i| \leq |x_{i+1} - x_i| + 2r \leq c \frac{|x - \xi|}{m} + 2r \leq c \frac{|x - \xi|}{m^{1/2}} + 2r \leq 4r, \quad 0 \leq i \leq m-1.$$

Whence

$$|\xi_{i+1} - \xi_i| \leq \widehat{C} \left( \frac{t - \tau}{m} \right)^{1/2}, \quad 0 \leq i \leq m-1.$$

It follows from (3.15) and (3.18) that

$$\begin{aligned} G(x, t; \xi, \tau) &\geq \int_{B(x_1, r) \cap \Omega} \cdots \int_{B(x_{m-1}, r) \cap \Omega} C^m \left( \frac{t - \tau}{m} \right)^{-nm/2} d\xi_1 \cdots d\xi_{m-1} \\ &\geq (dr^n)^{m-1} C^m \left( \frac{t - \tau}{m} \right)^{-nm/2} \\ &\geq d^{m-1} \left[ \frac{\widehat{C}^2}{16} \left( \frac{t - \tau}{m} \right) \right]^{n(m-1)/2} C^m \left( \frac{t - \tau}{m} \right)^{-nm/2} \\ &\geq \widetilde{C} C^m (t - \tau)^{-n/2}. \end{aligned}$$

Hence

$$(3.19) \quad G(x, t; \xi, \tau) \geq \widetilde{C} e^{-Cm} (t - \tau)^{-n/2}.$$

Or from the definition of  $m$  we have

$$(3.20) \quad m - 1 \leq \left( \frac{2c}{\widehat{C}} \right)^2 \frac{|x - y|^2}{t - \tau}.$$

Finally, a combination of (3.19) and (3.20) leads to (3.14) when  $t - \tau \leq \delta$ .

We complete the proof by showing that we can remove the assumption  $t - \tau \leq \delta$  in (3.15). Let then  $0 < t - \tau \leq T$ , such that  $t - \tau > \delta$ , and let  $m \geq 2$  be the smallest integer such that  $\delta^{-1}(t - \tau) \leq m$ . We set

$$(3.21) \quad r = r_0 T^{-1/2} m^{-1/2} (t - \tau)^{1/2}$$

and we denote by  $p$  the smallest integer satisfying

$$\frac{2cD}{rm} \leq p, \quad \text{with } D = \text{diam}(\Omega).$$

If we choose  $k = pm$  in (3.16), we obtain

$$|y_{i+1} - y_i| \leq \frac{c|x - \xi|}{pm} \leq \frac{cD}{pm} \leq r/2.$$

Let us denote by (3.18\*) the inequality (3.18) in which we take  $r/2$  in place of  $r$ , with  $r$  given as in (3.21), and  $m$  in changed to  $pm$ .

Taking into account that

$$p < 1 + 2cDr_0^{-1} T^{1/2} \delta^{-1/2} = p^*,$$

we get

$$(3.22) \quad \frac{pm|\xi_{i+1} - \xi_i|^2}{t - \tau} \leq \frac{pmr^2}{t - \tau} < p^* r_0^2 T^{-1}.$$

---

<sup>5</sup>Choi and Kim [7] has observed that this condition is necessary for domains having a De Giorgi-Nash-Moser type estimate at the boundary.

As  $(pm)^{-1}(t - \tau) \leq \delta$ , (3.14) holds true. Therefore in light of (3.22), we obtain from (3.18\*)

$$\begin{aligned} G(x, t; \xi, \tau) &\geq \left( d \left[ 2^{-1} r_0 T^{-1/2} m^{-1/2} (t - \tau)^{1/2} \right]^n \right)^{pm-1} C^{pm} [(pm)^{-1}(t - \tau)]^{-pnm/2} \\ &\geq \tilde{C} \widehat{C}^{pm} (t - \tau)^{-n/2} \\ &\geq \tilde{C} e^{-pm |\ln \widehat{C}|} (t - \tau)^{-n/2} \\ &\geq \tilde{C} e^{-p^* m^* |\ln \widehat{C}|} (t - \tau)^{-n/2}, \quad \text{with } m^* = \delta^{-1} T + 1. \end{aligned}$$

This estimate completes the proof.  $\square$

Theorem 3.1 can be easily extended to a Robin Green function. Indeed, if we replace the Neumann boundary condition by the following Robin boundary condition

$$\partial_\nu u + q(x, t)u = 0 \quad \text{in } \Sigma_\tau,$$

where  $q \in C(\Sigma_\tau)$ , then  $\mathcal{N}$  have to be changed to

$$\begin{aligned} \mathcal{N}_q(x, t; \xi, \tau) &= -2[\partial_\nu E(x, t; \xi, \tau) + q(x, t)E(x, t; \xi, \tau)] \\ &\quad - 2 \sum_{\ell \geq 1} \int_\tau^t \int_{\partial\Omega} M_\ell(x, t; \eta, \sigma) [\partial_\nu E(\eta, \sigma; \xi, \tau) + q(\eta, \sigma)E(\eta, \sigma; \xi, \tau)] d\eta d\sigma. \end{aligned}$$

Here

$$\begin{aligned} M_1(x, t; \xi, \tau) &= -2[\partial_\nu E(x, t; \xi, \tau) + q(x, t)E(x, t; \xi, \tau)] \\ M_{\ell+1}(x, t; \xi, \tau) &= \int_\tau^t \int_{\partial\Omega} M_1(x, t; \eta, \sigma) M_\ell(\eta, \sigma; \xi, \tau) d\eta d\sigma, \quad \ell \geq 1. \end{aligned}$$

Apart the positivity of the Green function which can be obtained by an adaptation of Proposition 3.2 in [8], one can see that, without any difficulty, the rest of the previous analysis holds true when  $\mathcal{N}$  is changed to  $\mathcal{N}_q$ .

We already mentioned that, when  $L$  has time-independent coefficients, the Neumann Green function is nothing else but the Neumann heat kernel. Let us then assume that the coefficients of the operator  $L$  are time-independent. In other words  $L$  is of the form

$$(3.23) \quad L = a_{ij}(x)\partial_{ij}^2 + b_k(x)\partial_k + c(x) - \partial_t$$

and the following assumptions are satisfied

- (i') the matrix  $(a_{ij}(x))$  is symmetric for any  $x \in \overline{\Omega}$ ,
- (ii')  $a_{ij} \in W^{1,\infty}(\Omega)$ ,  $b_k, c \in C^1(\overline{\Omega})$ ,
- (iii')  $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ ,  $(x, t) \in \overline{\Omega}$ ,  $\xi \in \mathbb{R}^n$ ,
- (iv')  $\|a_{ij}\|_{W^{1,\infty}(\Omega)} + \|b_k\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \leq A$ ,

where  $\lambda > 0$  and  $A > 0$  are two given constants.<sup>6</sup>

We consider the unbounded operator

$$A_L = a_{ij}(x)\partial_{ij}^2 + b_k(x)\partial_k + c(x)$$

with domain

$$D(A_L) = \{u \in L^2(\Omega); A_L u \in L^2(\Omega) \text{ and } \partial_\nu u = 0 \text{ on } \partial\Omega\}.$$
<sup>7</sup>

It is a classical result that  $A_L$  generates a continuous semigroup  $e^{tA_L}$  on  $L^2(\Omega)$  (e.g. [27]).

Let  $\psi \in C_0^\infty(\Omega)$ . Since  $u(t) = e^{tA_L}\psi$  is the solution of the IBVP (3.1) with  $\tau = 0$  and  $t_1 = T$ ,

$$e^{tA_L}\psi = \int_\Omega G(x, t; \xi, 0)\psi(\xi)d\xi, \quad 0 < t \leq T.$$

<sup>6</sup>Here again, the assumptions on the coefficients are not necessary the best possible.

<sup>7</sup> $\partial_\nu u = 0$  is to be understood in some appropriate weak sense.

We rewrite this equality as follows:

$$e^{tA_L}\psi = \int_{\Omega} K(x, \xi, t)\psi(\xi)d\xi, \quad 0 < t \leq T.$$

The function

$$K(x, \xi, t) = G(x, t; \xi, 0)$$

is usually called the heat kernel of the semigroup  $e^{tA_L}$ .

A straightforward consequence of Theorems 3.1 is

**Corollary 3.1.** *The Neumann heat kernel  $K$  satisfies the gaussian lower bound:*

$$(3.24) \quad \mathcal{G}_C(x - \xi, t) \leq K(x, \xi, t), \quad (x, \xi) \in \Omega^2, \quad 0 < t \leq T.$$

The gaussian lower bound for the Neumann heat kernel was proved in [9] when  $L$  is the laplacien. The key point is the Hölder continuity of  $x \rightarrow K(x, \xi, t)$  which relies on the fact that  $\mu - A_L$  is an isomorphism from  $H^s(\Omega)$  into  $H^{s-2}(\Omega)$ , for some large  $\mu$  and  $s$  of order  $n/2 + 1$ . We note that a quick examination of the proof in [9] shows that this result can be extended to a divergence form operator with  $C^\infty$ -smooth coefficients.

We end this section by showing that we can obtain a strong maximum from Theorem 3.1. Let  $\psi \in C(\overline{\Omega})$ ,  $f \in C(\overline{Q_\tau})$ ,  $g \in C(\overline{\Sigma_\tau})$  and consider the IBVP

$$(3.25) \quad \begin{cases} Lu = f & \text{in } Q_\tau, \\ u(\cdot, \tau) = \psi & \text{in } \Omega, \\ \partial_\nu u = g & \text{on } \Sigma_\tau. \end{cases}$$

**Corollary 3.2.** *We assume that  $\psi \geq 0$ ,  $f \geq 0$ ,  $g \geq 0$  and at least one of the functions  $\psi$ ,  $f$  and  $g$  is non identically equal to zero. If  $u \in C(\overline{Q_\tau}) \cap C^{2,1}(Q_\tau)$  is the solution of the IBVP (3.25), then*

$$u > 0 \text{ in } \Omega \times ]\tau, t_1].$$

*Proof.* Immediate from Theorem 3.1 since (e.g. formula (3.5) in page 144 of [15])

$$u(x, t) = \int_{\Omega} G(x, t; \xi, \tau)\psi(\xi)d\xi + \int_{\tau}^t \int_{\Omega} G(x, t; \xi, s)f(\xi, s)d\xi ds + \int_{\tau}^t \int_{\partial\Omega} G(x, t; \xi, s)g(\xi, s)dS_\xi ds.$$

□

#### 4. GAUSSIAN TYPE UPPER BOUND FOR THE NEUMANN GREEN FUNCTION

For sake of simplicity, even if it is not necessary, we assume that  $\Omega$  is of class  $C^{2,\alpha}$ , for some  $0 < \alpha < 1$ . We recall that the constant  $N$  defined in Section 2 is given as follows

$$N = \max(\max_k \|b_k\|_{L^\infty(Q)}, \|c\|_{L^\infty(Q)}, 1).$$

We first observe that the Green function  $G$  satisfies a Nash type upper bound. Indeed, an adaptation of the proof of Lemma (3.1) yields

$$|\mathcal{N}(x, t; \xi, \tau)| \leq (t - \tau)^{n/2} e^{CN^2(t-\tau)}, \quad (x, t; \xi, \tau) \in Q^2, \quad t > \tau.$$

Therefore, we can slightly modify the beginning of the proof of Theorem (3.1) to deduce that

$$|G_0(x, t; \xi, \tau)| \leq (t - \tau)^{n/2} e^{CN^2(t-\tau)}, \quad (x, t; \xi, \tau) \in Q^2, \quad t > \tau.$$

In light of (2.11) and the last estimate, we can state

**Lemma 4.1.** *The Green function  $G$  satisfies the Nash type upper bound:*

$$(4.1) \quad G(x, t; \xi, \tau) \leq C(t - \tau)^{-n/2} e^{CN^2(t-\tau)}, \quad (x, t; \xi, \tau) \in Q^2, \quad t > \tau.$$

To the operator  $L$  we can associate the evolution system  $U(t, \tau)$ ,  $t_0 \leq \tau < t \leq t_1$ , given by

$$U(t, \tau)\psi(x) = \int_{\Omega} G(x, t; \xi, \tau)\psi(\xi)d\xi, \quad \psi \in C_0^\infty(\Omega).$$

In light of (4.1), we can assert that  $U(t, \tau)$  is extended to a bounded operator from  $L^1(\Omega)$  into  $L^\infty(\Omega)$  and

$$\|U(t, \tau)\|_{1, \infty} \leq C(t - \tau)^{-n/2} e^{CN^2(t - \tau)},^8$$

where  $\|\cdot\|_{1, \infty}$  is the natural norm on  $\mathcal{L}(L^1(\Omega), L^\infty(\Omega))$ .

Let  $\zeta \in \mathbb{R}^n$  and set  $|\nu|_\infty = \|\nu\|_\infty$ . As  $\Omega$  is of class  $C^{2, \alpha}$ , we obtain from a variant of Lemma 3.1 in [8]<sup>9</sup> that there exists  $\theta_\zeta \in C^{2, \alpha}(\overline{\Omega})$  having the properties

$$(4.2) \quad \theta_\zeta \geq 1, \quad \theta_\zeta = 1 \text{ on } \partial\Omega, \quad -\partial_\nu \theta_\zeta \geq |\nu|_\infty |\zeta| \text{ on } \partial\Omega.$$

Moreover examining the proof of Lemma 3.1 in [8] we obtain in a straightforward manner that

$$(4.3) \quad |\partial_i \theta_\zeta|, |\partial_{ij} \theta_\zeta| \leq \varrho |\zeta|,$$

where  $\varrho$  is a constant independent on  $\zeta$ .

We now conjugate  $L$  by  $e^{\zeta \cdot x + K|\zeta|^2 t} \theta_\zeta^{-1}$ , where  $K$  is a constant to be specified later. To this end we consider

$$L^\zeta = e^{-\zeta \cdot x - K|\zeta|^2 t} \theta_\zeta(x) L e^{\zeta \cdot x + K|\zeta|^2 t} \theta_\zeta^{-1}(x).$$

An elementary calculation gives  $L^\zeta = \widehat{L}^\zeta + c^\zeta$  with

$$\widehat{L}^\zeta = a_{ij} \partial_{ij}^2 + b_p \partial_p + 2a_{kl} (\zeta_l - \theta_\zeta^{-1} \partial_l \theta_\zeta) \partial_k + b_m (\zeta_m - \theta_\zeta^{-1} \partial_m \theta_\zeta) + c - \theta_\zeta^{-1} a_{ij} \partial_{ij}^2 \theta_\zeta - \partial_t$$

and

$$c^\zeta = a_{pq} \zeta_p \zeta_q + 2\theta_\zeta^{-1} a_{ij} \zeta_j \partial_i \theta_\zeta + 2a_{ij} \theta_\zeta^{-2} \partial_i \theta_\zeta \partial_j \theta_\zeta - K|\zeta|^2.$$

In light of (4.2) and (4.3) we can choose  $K$ , independent on  $\zeta$ , such that  $c^\zeta \leq 0$  for any  $\zeta \in \mathbb{R}^n$ .

Let  $0 \leq \psi \in C_0^\infty(\Omega)$  and  $u$  the corresponding solution of the IBVP (3.1). Then  $v = e^{-\zeta \cdot x - K|\zeta|^2 t} \theta_\zeta u$  solves the following IBVP

$$(4.4) \quad \begin{cases} L^\zeta v = 0 & \text{in } Q_\tau, \\ v(\cdot, \tau) = e^{-\zeta \cdot x - K|\zeta|^2 \tau} \theta_\zeta \psi & \text{in } \Omega, \\ \partial_\nu v + (\zeta \cdot \nu - \partial_\nu \theta_\zeta) v = 0 & \text{on } \Sigma_\tau. \end{cases}$$

Next, let  $\widehat{v}$  be the solution of the IBVP

$$(4.5) \quad \begin{cases} \widehat{L}^\zeta \widehat{v} = 0 & \text{in } Q_\tau, \\ \widehat{v}(\cdot, \tau) = e^{-\zeta \cdot x - K|\zeta|^2 \tau} \theta_\zeta \psi & \text{in } \Omega, \\ \partial_\nu \widehat{v} = 0 & \text{on } \Sigma_\tau. \end{cases}$$

Since  $v \geq 0$ ,

$$\begin{cases} \widehat{L}^\zeta (v - \widehat{v}) = -c^\zeta v \geq 0 & \text{in } Q_\tau, \\ (v - \widehat{v})(\cdot, \tau) = 0 & \text{in } \Omega, \\ \partial_\nu (v - \widehat{v}) = -(\zeta \cdot \nu - \partial_\nu \theta_\zeta) v \leq 0 & \text{on } \Sigma_\tau. \end{cases}$$

The last inequality is obtained from the third inequality in (4.2). We can then apply Theorem 2.9 [23] to deduce that  $v \leq \widehat{v}$ .

Let  $\widehat{E}^\zeta$  and  $\widehat{G}^\zeta$  be respectively the fundamental solution and the Neumann Green function corresponding to  $\widehat{L}^\zeta$ . It follows from estimate (2.11) that

$$\widehat{E}^\zeta(x, t; \xi, \tau) \leq \widetilde{C}(t - \tau)^{-n/2} e^{-\frac{\lambda^* |x - \xi|^2}{t - \tau} + \widehat{C}\Psi(|\zeta|)(t - \tau)}.$$

<sup>8</sup>We mention that this type of estimate is essential in the approach of Daners [10].

<sup>9</sup>In this lemma the result is given in terms of the outward normal derivative. With the help of Theorem 7 in page 65 of [28] one can see that the normal derivative can be replaced by any derivative in the a direction  $\nu$  with  $\nu \cdot \mathbf{n} > 0$ .

where  $\Psi(\rho) = 1 + \rho + \rho^2$ ,  $\rho > 0$ . Here and henceforth all generic constants we use may depend on  $\|b_k\|_\infty$  and  $\|c\|_\infty$ .

By Lemma 4.1  $\widehat{G}^\zeta$  satisfies the Nash type upper bound

$$(4.6) \quad \widehat{G}^\zeta(x, t; \xi, \tau) \leq C(t - \tau)^{-n/2} e^{\widehat{C}\Psi(|\zeta|)(t - \tau)}.$$

Therefore, for any  $0 \leq \psi \in C_0^\infty(\Omega)$ ,

$$e^{-\zeta \cdot x - K|\zeta|^2 t} \theta_\zeta(x) u(x, t) = v(x, t) \leq \int_\Omega \widehat{G}^\zeta(x, t; \xi, \tau) e^{-\zeta \cdot \xi - K|\zeta|^2 \tau} \theta_\zeta(\xi) \psi(\xi) d\xi.$$

Or

$$u(x, t) = \int_\Omega G(x, t; \xi, \tau) \psi(\xi) d\xi.$$

Whence

$$G(x, t, \xi, \tau) \leq e^{\zeta \cdot x + K|\zeta|^2 t} \widehat{G}^\zeta(x, t; \xi, \tau) e^{-\zeta \cdot \xi - K|\zeta|^2 \tau} \theta_\zeta(x)^{-1} \theta_\zeta(\xi).$$

This and (4.6) imply

$$(4.7) \quad G(x, t, \xi, \tau) \leq C(t - \tau)^{-n/2} e^{\zeta \cdot (x - \xi) + \widehat{C}\Psi(|\zeta|)(t - \tau)} \theta_\zeta(x)^{-1} \theta_\zeta(\xi)$$

and then

$$G(x, t, \xi, \tau) \leq C|\zeta|(t - \tau)^{-n/2} e^{\zeta \cdot (x - \xi) + \widehat{C}\Psi(|\zeta|)(t - \tau)}.$$

Taking  $\zeta = (\xi - x)/[2\widehat{C}(t - \tau)]$  in the last estimate we get the following result.

**Theorem 4.1.** *The Neumann Green function satisfies the upper bound:*

$$G(x, t, \xi, \tau) \leq (t - \tau)^{-1/2} \mathcal{G}_C(x - \xi, t - \tau), \quad (x, t; \xi, \tau) \in Q^2, \quad t > \tau.$$

*Remark 4.1.* 1) Returning to the proof of Lemma 3.1 in [8], we prove that for any compact subset  $K$  of  $\Omega$  there is a constant  $C_K$  such that

$$|\theta_\zeta(x) \theta_\zeta^{-1}(\xi)| \leq C_K, \quad x \in K, \quad \xi \in \overline{\Omega} \text{ and } \zeta \in \mathbb{R}^n.$$

We derive from (4.7) the following gaussian upper bound

$$G(x, t, \xi, \tau) \leq C_K (t - \tau)^{-n/2} e^{-\frac{C|x - \xi|^2}{t - \tau}}, \quad (x, t; \xi, \tau) \in Q^2 \cap \{x \in K, t > \tau\}.$$

2) The construction of the Neumann Green function is quite simple when  $\Omega$  is a half space, say

$$\Omega = \mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n > 0\}.$$

Let us assume for simplicity that the original coefficients of  $L$  satisfy assumptions (i)-(iv) in the whole space  $\mathbb{R}^n$  and set

$$G(x, t; \xi, \tau) = \frac{1}{2} [E(x, t; \xi, \tau) + E(x^s, t; \xi, \tau)], \quad x, \xi \in \mathbb{R}_+^n, \quad \tau, t \in (t_0, t_1), \quad \tau < t,$$

where  $x^s = (x', -x_n)$  if  $x = (x', x_n)$ .

It is not hard to check that  $G$  is the Neumann Green function for  $L$  in  $\mathbb{R}_+^n \times (t_0, t_1)$ . In addition, observing that  $|x - \xi| \leq |x^s - \xi|$  for any  $x, \xi \in \mathbb{R}_+^n$ , we see the gaussian upper bound for  $E$  yields the gaussian upper bound for  $G$ .

2) As for the gaussian lower bound, Theorem 4.1 is extended to the case of Robin boundary condition without the positivity assumption on the zero order boundary coefficient.

For  $1 < p, q \leq \infty$ , let  $\|\cdot\|_{p,q}$  denotes the norm in  $L^q(t_0, t_1; L^p(\Omega))$ . We have similarly to Corollary 7.1 in page 668 of [2] the following result which is a direct consequence of the upper bound in Theorem 4.1.



**Corollary 4.1.** *Let  $1 < p, q \leq \infty$  be such that*

$$\frac{n}{p} + \frac{2}{q} < 1.$$

*Then*

$$\|G(x, t; \cdot, \cdot)\|_{p', q'}, \quad \|G(\cdot, \cdot; \xi, \tau)\|_{p', q'} \leq C,$$

*where  $p', q'$  are the respective Hölder conjugate exponents of  $p$  and  $q$ .*

Let  $f \in C_0^\infty(Q_\tau)$  and  $p, q$  be as in the previous lemma. We recall that the solution of the following BVP

$$\begin{cases} Lu = f & \text{in } Q_\tau, \\ u(\cdot, \tau) = 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \Sigma_\tau \end{cases}$$

can be represented by the formula

$$u(x, t) = \int_\tau^t \int_\Omega G(x, t; \xi, s) f(\xi, s) d\xi ds.$$

In light of this formula we have the following immediate consequence of Corollary 4.1:

$$\|u\|_{\infty, \infty} \leq C \|f\|_{p, q}.$$

## 5. GAUSSIAN LOWER BOUND FOR THE NEUMANN HEAT KERNEL OF A RIEMANNIAN MANIFOLD

Let  $M = (M, g)$  be a  $n$ -dimensional compact and connected Riemannian manifold without boundary and  $\Omega$  a regular domain in  $M$  with boundary  $\Sigma$  carrying the outward normal vector field  $\nu$ . If  $d$  is the Riemannian distance function, we set

$$\mathcal{E}(x, y, t) = (4\pi t)^{-n/2} e^{-d^2(x, y)/4t}.$$

In general  $\mathcal{E}$  is not a heat kernel on  $M$ . However the parametrix method by S. Minakshisundaram and Å.Pleijel shows that any compact Riemannian manifold has an almost eucliden heat kernel (e.g. [3, 5]). In particular  $p$ , the heat kernel of  $M$ , satisfies the pointwise gaussian upper bound

$$p(x, y, t) \leq C \mathcal{E}(x, y, t), \quad (x, y, t) \in M \times M \times (0, +\infty),$$

where  $C$  is a constant.

Let  $\Delta = \Delta_g$  be the Laplace-Beltrami operator associated to the metric  $g$ . Following the idea in section 2, we construct the heat kernel for the Neumann problem

$$(5.1) \quad \begin{cases} (\partial_t - \Delta)u = 0 & \text{in } M \times (0, +\infty), \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \Sigma \times (0, +\infty), \end{cases}$$

as a perturbation of the heat kernel  $p$  by a single-layer potential. As a first step, we seek the solution the following IBVP

$$(5.2) \quad \begin{cases} (\partial_t - \Delta)u = 0 & \text{in } M \times (0, +\infty), \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \Sigma \times (0, +\infty), \\ u(\cdot, 0) = \psi \in C_0^\infty(\Omega) \end{cases}$$

of the form

$$u(x, t) = \int_0^t \int_\Sigma p(x, y, s) \varphi(y, t - s) dA(y) ds + \int_\Omega p(x, y, t) \psi(y) dV(y), \quad (x, t) \in M \times (0, +\infty).$$

Since  $u$  is  $C^1$  with respect to  $x$  up to the boundary, we obtain from the jump relation in Theorem 2 in page 161 of [5] that  $\varphi$  must be the solution of the following integral equation

$$\varphi(x, t) = -2 \int_0^t \int_\Sigma \frac{\partial p}{\partial \nu_x}(x, y, s) \varphi(y, t - s) dA(y) ds - 2 \int_\Omega \frac{\partial p}{\partial \nu_x}(x, y, t) \psi(y) dV(y), \quad (x, t) \in \Sigma \times (0, +\infty).$$

As usual this integral equation is solved by successive approximation method. We get

$$\varphi(x, t) = \varphi_0(x, t) + \int_0^t \int_{\Sigma} r(x, y, s) \varphi_0(y, t - s) dA(y) ds,$$

Here

$$\begin{aligned} \varphi_0(x, t) &= -2 \int_M \frac{\partial p}{\partial \nu_x}(x, y, t) \psi(y) dV(y) \\ r(x, y, t) &= \sum_{j \geq 1} r^j(x, y, t) \end{aligned}$$

with

$$\begin{aligned} r^1(x, y, t) &= -2 \frac{\partial p}{\partial \nu_x}(x, y, t) \\ r^{j+1}(x, y, t) &= -2 \int_0^t \int_{\Sigma} r^1(x, z, t - s) r^j(z, y, s) dA(z) ds \quad j \geq 1. \end{aligned}$$

In other words

$$r(x, y, t) = -2 \frac{\partial p}{\partial \nu_x}(x, y, t) - 2 \sum_{j \geq 1} \int_0^t \int_{\Sigma} r^j(x, z, s) \frac{\partial p}{\partial \nu_x}(z, y, t - s) dA(z) ds$$

With the help of inequality (5.3) below, we prove similarly to (3.4)

$$\varphi(x, t) = \int_{\Omega} r(x, y, t) \psi(y) dV(y).$$

Then

$$u(x, t) = \int_{\Omega} q(x, y, t) \psi(y) dV(y),$$

where

$$q(x, y, t) = p(x, y, t) + \int_0^t \int_{\Sigma} p(x, z, s) r(z, y, t - s) dA(z) ds.$$

We call this function the Neumann heat kernel for the problem (5.1). We leave to the reader to verify that  $q$  satisfies the following reproducing property

$$q(x, y, t) = \int_{\Omega} q(x, z, t - s) q(z, y, s) dV(z), \quad 0 < s < t,$$

and

$$\int_{\Omega} q(x, y, t) dV(y) = 1.$$

The following two inequalities will be useful in the sequel. They are taken from [5].

For any  $\mu > 0$  and  $T > 0$ , there is a constant  $C_0$  such that

$$(5.3) \quad \left| \frac{\partial p}{\partial \nu_x}(x, y, t) \right| \leq C_0 t^{-\mu} d^{-n+2\mu}(x, y),$$

for any  $x, y \in \Sigma$  and  $t \in (0, T]$ .

There exist a constant  $C_1$  such that for any  $\alpha, \beta \in (0, n - 1)$ ,

$$(5.4) \quad \int_{\Sigma} d^{-\alpha}(x, z) d^{-\beta}(z, y) dV(z) \leq C_1 d^{m-1-(\alpha+\beta)}(x, y),$$

for all  $x, y \in \Sigma$ ,  $x \neq y$ .

In light of (5.3) and (5.4), reasoning as in the proof of Lemma 3.1, we obtain

**Lemma 5.1.** *Let  $1/2 < \mu \leq n/2$ . Then*

$$|r(x, y, t)| \leq Ct^{-\mu}d^{-n+2\mu}(x, y),$$

for any  $x, y \in \Sigma$ ,  $x \neq y$ ,  $t \in (0, T]$ .

Let

$$q_0(x, y, t) = \int_0^t \int_{\Sigma} p(x, z, s)r(z, y, t-s)dA(z)ds.$$

Let  $1/2 < \alpha < 1$ . Proceeding as in the Section 3, we show that the estimate in Lemma 5.1 yields

$$(5.5) \quad |q_0(x, y, t)| \leq Ct^{-n/2+1-\alpha}.$$

We recall (e.g. Theorem 5.6.1 in page 173 in [11]) that when  $M$  has Ricci curvature  $\text{Ric} \geq 0$  then  $p$  is bounded from below by  $\mathcal{E}$ . That is

$$(5.6) \quad p(x, y, t) \geq \mathcal{E}(x, y, t), \quad x, y \in M, t > 0.$$

From this, we deduce that,

$$(5.7) \quad p(x, y, t) \geq Ct^{-n/2}, \quad \text{if } d(x, y) \leq \sqrt{t}.$$

Now a combination of (5.5) and (5.7) leads

$$p(x, y, t) \geq p(x, y, t) - |q_0(x, y, t)| \geq Ct^{-n/2}(1 - ct^{1-\alpha}), \quad \text{if } d(x, y) \leq \sqrt{t}.$$

In consequence, there is  $\delta > 0$  such that

$$(5.8) \quad p(x, y, t) \geq Ct^{-n/2}, \quad \text{if } 0 < t \leq \delta \text{ and } d(x, y) \leq \sqrt{t}.$$

By Lemma 6.2 in page 94 of [21], the Riemannian distance induce the same topology as the given manifold topology. Then  $(M, d)$  as a compact connected metric space satisfies the chain condition: there is a constant  $k$  such that for  $x, y \in M$  and  $m \in \mathbb{N}^*$ , there exists a sequence  $(x_i)_{0 \leq i \leq m}$  satisfying  $x_0 = x$ ,  $x_m = y$  and

$$(5.9) \quad d(x_i, y_{i+1}) \leq \frac{kd(x, y)}{m}, \quad i = 0, \dots, m-1.$$

Since  $\min_{x \in M} |B_d(x, 1)| > 0$ , with  $|B_d(x, 1)| = \text{Vol}_g(B_d(x, 1))$ , by a result due to Gromov (e.g. Theorem 1.5 in page of [17]), we proceed in a similar manner to the three lines in the beginning of the proof of Lemma 3.16 of [17] to deduce that

$$c^{-1}r^n \leq |B_d(x, r)| \leq cr^n,$$

for some constant  $c = c(n) \geq 1$ . Hence

$$(5.10) \quad \tilde{c}^{-1}r^n \leq |B_d(x, r) \cap \Omega| \leq \tilde{c}r^n, \quad 0 < r \leq r_0.$$

Here  $\tilde{c} = \tilde{c}(n, \Omega)$  and  $r_0 = r_0(n, \Omega)$  are two constants. We note that the proof of (5.10) follows from the same inequalities in the flat case.

In light of (5.9) and (5.10), we can paraphrase the proof of Theorem 3.1 to get the following result

**Theorem 5.1.** *We assume that  $M$  has Ricci curvature  $\text{Ric} \geq 0$ . Then  $q$  satisfies a gaussian lower bound*

$$q(x, y, t) \geq \lambda \mathcal{E}(x, y, \lambda t), \quad x, y \in M \text{ and } 0 < t \leq T.$$

Here  $\lambda$  is a constant.

This result completes the one already proved by Li and Yau [22] under the additional condition that  $\Sigma$  is convex

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