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GAUSSIAN TYPE BOUNDS FOR THE NEUMANN-GREEN FUNCTION OF A GENERAL PARABOLIC OPERATOR

MOURAD CHOULLI AND LAURENT KAYSER

ABSTRACT. Based on the fact that the Neumann-Green function can be constructed as a perturbation of the fundamental solution by a single-layer potential, we establish a gaussian lower bound and a gaussian type upper bound for the Neumann-Green function for a general parabolic operator. We build our analysis on classical tools coming from the construction of a fundamental solution of a general parabolic operator by means of the so-called parametrix method. At the same time we provide a simple proof for the gaussian two-sided bounds for the fundamental solution.

Key words : parabolic operator, fundamental solution, Neumann-Green function, parametrix, heat kernel.

Mathematics subject classification 2010 : 65M80

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1. INTRODUCTION

Let Ω be a bounded domain of \mathbb{R}^n with $C^{1,\alpha}$ -smooth boundary, where $0 < \alpha < 1$. Let $t_0 < t_1$, set $Q = \Omega \times (t_0, t_1)$ and consider the second order differential operator

$$L = a_{ij}(x, t)\partial_{ij}^2 + b_k(x, t)\partial_k + c(x, t) - \partial_t.$$

Here and henceforth we use the usual Einstein's summation convention.

We make the following assumptions on the coefficients of L :

(i) the matrix $(a_{ij}(x, t))$ is symmetric for any $(x, t) \in \overline{Q}$,

(ii) $a_{ij} \in W^{1,\infty}(Q)$, $b_k, c \in C([t_0, t_1], C^\alpha(\overline{\Omega}))$,

(iii) $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $(x, t) \in \overline{Q}$, $\xi \in \mathbb{R}^n$,

(iv) $\|a_{ij}\|_{W^{1,\infty}(Q)} + \|b_k\|_{L^\infty(Q)} + \|c\|_{L^\infty(Q)} \leq A$,

where $\lambda > 0$ and $A > 0$ are two given constants.

Since we will use the fundamental solution in the whole space, we begin by extending the coefficients of L in a neighborhood $\tilde{\Omega}$ of $\overline{\Omega}$ to coefficients having the same regularity. We observe that this is possible in view of the regularity of Ω . For simplicity we keep the same symbols for the extended coefficients. We may also assume that the ellipticity condition holds for the extended coefficients with the same λ . Pick $\psi \in C_0^\infty(\tilde{\Omega})$ satisfying $0 \leq \psi \leq 1$ and $\psi = 1$ in a neighborhood of $\overline{\Omega}$. We set

$$\tilde{a}_{ij} = a_{ij}\psi + \lambda\delta_{ij}(1 - \psi), \quad \tilde{b}_k = b_k\psi, \quad \tilde{c} = c\psi$$

and

$$\tilde{L} = \tilde{a}_{ij}(x, t)\partial_{ij}^2 + \tilde{b}_k(x, t)\partial_k + \tilde{c}(x, t) - \partial_t.$$

Clearly the coefficients of \tilde{L} satisfy the same assumptions as those of L . So in the sequel we will use the same symbol L for L or its extension \tilde{L} .

We recall that the function

$$\mathcal{G}(x, t) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \quad t > 0,$$

is usually called the Gaussian Kernel. We set

$$\mathcal{G}_c(x, t) = c^{-1}\mathcal{G}(\sqrt{c}x, t), \quad c > 0.$$

It is important to observe that the map $c \rightarrow \mathcal{G}_c$ is a non increasing.

We are interested in gaussian two-sided bounds for the Neumann-Green function associated to the operator L . More specifically, denoting by G the Neumann-Green function for L , we want to prove an estimate of the form

$$\mathcal{G}_C(x - \xi, t - \tau) \leq G(x, t; \xi, \tau) \leq \mathcal{G}_{\tilde{C}}(x - \xi, t - \tau), \quad (x, t; \xi, \tau) \in Q^2, \quad t > \tau,$$

where the constants C and \tilde{C} depend only on Ω , $T = t_1 - t_0$ and A .

We succeed in proving that the above gaussian lower bound holds true. But we are only able to prove an upper bound which is weaker than a gaussian upper bound. Namely we prove an upper bound of the form:

$$G(x, t; \xi, \tau) \leq (t - \tau)^{-1/2} \mathcal{G}_{\tilde{C}}(x - \xi, t - \tau), \quad (x, t; \xi, \tau) \in Q^2, \quad t > \tau,$$

for some constant C that can depend only on Ω , T and A .

To our knowledge the gaussian lower bound have never been established before for the Neumann-Green function of a general parabolic operator. Also, it is an open problem to know whether the gaussian upper bound is true for a general parabolic operator. The problem is completely different for operators in divergence form possessing a variational formulation. The main feature is that the divergence form operators having a variational formulation generate evolution systems. Using this fact, Daners [Dan] proved an upper bound for Lipschitz domains with minimal assumptions on coefficients. Choi and Kim [CK] obtained an upper gaussian bound for a system of operators in divergence form under the assumption that the corresponding Neumann boundary value problem possesses a De Giorgi-Nash-Moser type estimate at the boundary. We will see in Section 4 how the gaussian upper bound for the Neumann-Green function can be obtained in a straightforward way from the gaussian upper bound for the fundamental solution when Ω is a half space.

The situation is quite different for the Dirichlet-Green function since this later vanishes on the boundary. One can prove in an obvious manner, with the help of the maximum principle, that the Dirichlet-Green function is non negative and dominated pointwise by the fundamental solution and so it has a gaussian upper bound. Aronson [Ar2] (Theorem 8 in page 670)¹ established an interior gaussian lower bound for the Dirichlet-Green function. Later Cho [Ch], Cho, Kim and Park [CKP] extended this result to a global weighted gaussian lower bound involving the distance to the boundary.

For parabolic operators with time-independent coefficients, generating a continuous semigroup, a fundamental solution or a Green function is reduced to a heat kernel. We mention that there is a wide literature dealing with gaussian bounds for heat kernels. We quote the classical books by Davies [Dav], Grigor'yan [Gr], Ouhabaz [Ou] Saloff-Coste [Sa] and Stroock [St], but of course there are many other references on the subject.

As we said in the summary, the main ingredient in our analysis relies on the classical construction of the fundamental solution by the so-called parametrix method. We revisit this construction in the next section and we derive from it the gaussian two-sided bounds for the fundamental solution. Special attention is paid

¹Let us observe that Theorem 8 in page 670 of [Ar2] can be used to extend the results of Section 3 of [FS] to a general parabolic operator. In other words, one can obtain a proof of a continuity theorem by Nash [Na] and the Moser's Harnack inequality [Mo] for a general divergence form parabolic operator, since they rely on the two-sided gaussian bounds for the fundamental solution.

to the dependence of the gaussian upper bound on the coefficients of orders one and zero. This dependence will be in the heart of the proof of the gaussian type upper bound for the Neumann-Green function since we will conjugate L with e^ψ , for some function ψ appropriately chosen. This is a classical argument to derive a gaussian upper bound from a Nash upper bound. Note however that this argument is not enough to get a gaussian upper bound for the Neumann-Green function. We need in our case to conjugate again with a special function together with a maximum principle argument in order to get a gaussian type upper bound. This is done in Section 4. Before, we prove in Section 3 a gaussian lower bound for the Neumann-Green function. To do so, we construct the Neumann-Green function as a perturbation of the fundamental solution by a single-layer potential. The gaussian lower bound is derived from an estimate for the “kernel” of the single-layer potential which is the key point in the proof.

2. THE PARAMETRIX METHOD REVISITED

We are concerned in this section with gaussian two-sided bounds for the fundamental solution. For a systematic study of the fundamental solution we refer to the classical monographs by A. Friedman [Fr] and O. A. Ladyzhenskaja, V. A. Solonnikov and N. N. Ural'tzeva [LSU].

In the sequel $P = \mathbb{R}^n \times (t_0, t_1)$.

We recall that a fundamental solution of $Lu = 0$ in P is a function $E(x, t; \xi, \tau)$ which is $C^{2,1}$ in $P^2 \cap \{t > \tau\}$, satisfies

$$LE(\cdot, \cdot; \xi, \tau) = 0 \text{ in } \mathbb{R}^n \times \{\tau < t \leq t_1\}, \text{ for any } (\xi, \tau) \in \mathbb{R}^n \times [t_0, t_1[$$

and, for any $f \in C_0(\mathbb{R}^n)$,

$$\lim_{t \searrow \tau} \int_{\mathbb{R}^n} E(x, t; \xi, \tau) f(\xi) d\xi = f(x), \quad x \in \mathbb{R}^n.^2$$

The construction of a fundamental solution by the so-called parametrix method was initiated by E. E. Levi [Le]. Let $a = (a^{ij})$ be the inverse matrix of (a_{ij}) , $|a|$ the determinant of a and

$$Z(x, t; \xi, \tau) = [4\pi(t - \tau)]^{-n/2} \sqrt{|a(\xi, \tau)|} e^{-\frac{a(\xi, \tau)(x - \xi) \cdot (x - \xi)}{4(t - \tau)}}, \quad (x, t; \xi, \tau) \in P^2 \cap \{t > \tau\}.$$

This function is called the parametrix. It satisfies

$$L_0 Z(\cdot, \cdot; \xi, \tau) = 0 \text{ in } \mathbb{R}^n \times \{\tau < t \leq t_1\} \text{ for any } (\xi, \tau) \in \mathbb{R}^n \times [t_0, t_1[,$$

where

$$(2.1) \quad L_0 = a_{ij}(\xi, \tau) \partial_{ij}^2 - \partial_t.$$

In the parametrix method we seek E , a fundamental solution of $Lu = 0$ in P , of the form

$$(2.2) \quad E(x, t; \xi, \tau) = Z(x, t; \xi, \tau) + \int_{\tau}^t \int_{\mathbb{R}^n} Z(x, t; \eta, \sigma) \Phi(\eta, \sigma; \xi, \tau) d\eta d\sigma,$$

where Φ is to be determined in order to satisfy $LE(\cdot, \cdot; \xi, \tau) = 0$ for any $(\xi, \tau) \in \mathbb{R}^n \times [t_0, t_1[$.

Following Formulas (4.4) and (4.5) in page 14 of [Fr], Φ is given by the series

$$\Phi = \sum_{\ell=1}^{\infty} \Phi_{\ell},$$

where $\Phi_1(x, t; \xi, \tau) = LZ(x, t; \xi, \tau)$ and

$$\Phi_{\ell+1}(x, t; \xi, \tau) = \int_{\tau}^t \int_{\mathbb{R}^n} \Phi_1(x, t; \eta, \sigma) \Phi_{\ell}(\eta, \sigma; \xi, \tau) d\eta d\sigma, \quad \ell \geq 1.$$

Here for simplicity we write $LZ(x, t; \xi, \tau)$ instead of $[LZ(\cdot, \cdot; \xi, \tau)](x, t)$.

²One can take a larger class of functions. Namely a class of continuous functions satisfying a certain growth condition at infinity (e.g. formulas (6.1) and (6.2) in page 22 of [Fr]).

Let d_i , $1 \leq i \leq n$, given by

$$d_i = d_i(x, t; \xi, \tau) = -\frac{a^{ij}(\xi, \tau)(x_j - \xi_j)}{2(t - \tau)}, \quad (x, t; \xi, \tau) \in P^2 \cap \{t > \tau\}.$$

Then

$$\partial_i Z = d_i Z, \quad \partial_{ij}^2 Z = \left[-\frac{a^{ij}(\xi, \tau)}{2(t - \tau)} + d_j d_i \right] Z.$$

Therefore, taking into account (2.1), we get

$$LZ = LZ - L_0 Z = \left\{ (a_{ij}(x, t) - a_{ij}(\xi, \tau)) \left[-\frac{a^{ij}(\xi, \tau)}{2(t - \tau)} + d_j d_i \right] + b_k d_k + c \right\} Z.$$

We write $LZ = \Psi Z$, where

$$\Psi = (a_{ij}(x, t) - a_{ij}(\xi, \tau)) \left[-\frac{a^{ij}(\xi, \tau)}{2(t - \tau)} + d_j d_i \right] + b_k d_k + c.$$

Let

$$M = \max_{i,j} \|a_{ij}\|_{W^{1,\infty}(Q)}, \quad N = \max(\max_k \|b_k\|_{L^\infty(Q)}, \|c\|_{L^\infty(Q)}, 1).^3$$

Since

$$\begin{aligned} |d_i| &\leq \frac{|x - \xi|}{2\lambda(t - \tau)}, \\ |a_{ij}(x, t) - a_{ij}(\xi, \tau)| &\leq M(|x - \xi| + t - \tau), \end{aligned}$$

we have

$$(2.3) \quad |\Psi(x, t; \xi, \tau)| \leq N \frac{1}{\sqrt{t - \tau}} P \left(\frac{|x - \xi|}{\sqrt{t - \tau}} \right).$$

Here P is a polynomial function of degree less or equal to three whose coefficients depend only on M .

Unless otherwise stated, all the constants we use now do not depend on N .

In light of (2.3) we obtain

$$|LZ| \leq CN(t - \tau)^{-(n+1)/2} P(\rho) e^{-(\lambda/4)\rho^2} = CN(t - \tau)^{-(n+1)/2} \left[P(\eta) e^{-(\lambda/8)\rho^2} \right] e^{-(\lambda/8)\rho^2},$$

with

$$\rho = \frac{|x - \xi|}{\sqrt{t - \tau}}.$$

But the function $\rho \in (0, +\infty) \rightarrow P(\rho) e^{-(\lambda/8)\rho^2}$ is bounded. Consequently, where $\lambda^* = \lambda/8$,

$$(2.4) \quad |\Phi_1(x, t; \xi, \tau)| = |LZ(x, t; \xi, \tau)| \leq N \tilde{C} (t - \tau)^{-(n+1)/2} e^{-\frac{\lambda^* |x - \xi|^2}{t - \tau}}.$$

The following lemma will be useful in the sequel. Its proof is given in page 15 of [Fr].

Lemma 2.1. *Let $c > 0$ and $-\infty < \gamma, \beta < n/2 + 1$. Then*

$$\begin{aligned} \int_{\tau}^t \int_{\mathbb{R}^n} (t - \sigma)^{-\gamma} e^{-\frac{c|x-\eta|^2}{t-\sigma}} (\sigma - \tau)^{-\beta} e^{-\frac{c|\eta-\xi|^2}{\sigma-\tau}} d\eta d\sigma \\ = \left(\frac{4\pi}{c} \right)^{n/2} B(n/2 - \gamma + 1, n/2 - \beta + 1) (t - \tau)^{n/2+1-\gamma-\beta} e^{-\frac{c|x-\xi|^2}{t-\tau}}, \end{aligned}$$

where B is the usual beta function.

³We note that we can also take M equal to the maximum of Lipschitz constants of a_{ij} .

We want to show

$$(2.5) \quad |\Phi_\ell(x, t; \xi, \tau)| \leq (N\tilde{C})^\ell \widehat{C}^{\ell-1} (t-\tau)^{-(n+2-\ell)/2} \prod_{j=1}^{\ell-1} B(1/2, j/2) e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau}}, \quad \ell \geq 2.$$

Here \tilde{C} is the same constant as in (2.4) and $\widehat{C} = \left(\frac{4\pi}{\lambda^*}\right)^{n/2}$.

As

$$\Phi_2(x, t; \xi, \tau) = \int_\tau^t \int_{\mathbb{R}^n} \Phi_1(x, t; \eta, \sigma) \Phi_1(\eta, \sigma; \xi, \tau) d\eta d\sigma,$$

estimate (2.4) and Lemma 2.1 with $\gamma = \beta = n/2 + 1$ show that (2.5) holds true with $\ell = 2$. The general case follows by an induction argument in ℓ . Indeed, using

$$\Phi_{\ell+1}(x, t; \xi, \tau) = \int_\tau^t \int_{\mathbb{R}^n} \Phi_1(x, t; \eta, \sigma) \Phi_\ell(\eta, \sigma; \xi, \tau) d\eta d\sigma,$$

(2.4), (2.5) for ℓ and Lemma 2.1 with $\gamma = n/2 + 1$ and $\beta = (n + 2 - \ell)/2$, we obtain easily that (2.5) holds true with $\ell + 1$ in place of ℓ .

If Γ is the usual gamma function, we recall that

$$B(1/2, j/2) = \frac{\Gamma(1/2)\Gamma(j/2)}{\Gamma((j+1)/2)}.$$

Therefore

$$(2.6) \quad \prod_{j=1}^{\ell-1} B(1/2, j/2) = \frac{\Gamma(1/2)^\ell}{\Gamma(\ell/2)} = \frac{\sqrt{\pi}^\ell}{\Gamma(\ell/2)}.$$

(2.4)-(2.6) entail that

$$(2.7) \quad |\Phi(x, t; \xi, \tau)| \leq \sum_{\ell \geq 1} |\Phi_\ell(x, t; \xi, \tau)| \leq N\tilde{C}(1+S)(t-\tau)^{-(n+1)/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau}},$$

with

$$S = \sum_{\ell \geq 1} \left[CN(t-\tau)^{1/2} \right]^\ell / \Gamma((\ell+1)/2).$$

We have $\Gamma((\ell+1)/2) = \Gamma(m+1/2) \geq \Gamma(m) = (m-1)!$ if $\ell = 2m$ and $\Gamma((\ell+1)/2) = \Gamma(m+1) = m!$ if $\ell = 2m+1$. Then

$$\begin{aligned} S &= \sum_{m \geq 1} \left[CN(t-\tau)^{1/2} \right]^{2m} / \Gamma(m+1/2) + \sum_{m \geq 0} \left[CN(t-\tau)^{1/2} \right]^{2m+1} / \Gamma(m+1) \\ &\leq \sum_{m \geq 1} \left[CN(t-\tau)^{1/2} \right]^{2m} / (m-1)! + \sum_{m \geq 0} \left[CN(t-\tau)^{1/2} \right]^{2m+1} / m!. \end{aligned}$$

Whence

$$1 + S \leq \tilde{C} e^{\tilde{C}N^2(t-\tau)}.$$

Plugging this estimate into (2.7), we obtain

$$(2.8) \quad |\Phi(x, t; \xi, \tau)| \leq \tilde{C}N(t-\tau)^{-(n+1)/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + \tilde{C}N^2(t-\tau)}.$$

With the help of Lemma 2.1, estimate (2.8) yields

$$(2.9) \quad \left| \int_\tau^t \int_{\mathbb{R}^n} Z(x, t; \eta, \sigma) \Phi(\eta, \sigma; \xi, \tau) d\eta d\sigma \right| \leq \tilde{C}N(t-\tau)^{-(n-1)/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + \tilde{C}N^2(t-\tau)}.$$

Noting that this inequality can be rewritten as

$$\left| \int_\tau^t \int_{\mathbb{R}^n} Z(x, t; \eta, \sigma) \Phi(\eta, \sigma; \xi, \tau) d\eta d\sigma \right| \leq \tilde{C} \left[N(t-\tau)^{1/2} e^{-\tilde{C}N^2(t-\tau)} \right] (t-\tau)^{-n/2} e^{-\frac{\lambda^* |x-\xi|^2}{t-\tau} + 2\tilde{C}N^2(t-\tau)}$$

and using that $\rho \rightarrow \rho e^{-\tilde{C}\rho^2}$ is bounded function on $[0, +\infty)$, we obtain

$$(2.10) \quad \left| \int_{\tau}^t \int_{\mathbb{R}^n} Z(x, t; \eta, \sigma) \Phi(\eta, \sigma; \xi, \tau) d\eta d\sigma \right| \leq \widehat{C}(t - \tau)^{-n/2} e^{-\frac{\lambda^* |x - \xi|^2}{t - \tau}} + 2\tilde{C}N^2(t - \tau).$$

An immediate consequence of (2.10) is

$$(2.11) \quad |E(x, t; \xi, \tau)| \leq \tilde{C}(t - \tau)^{-n/2} e^{-\frac{\lambda^* |x - \xi|^2}{t - \tau}} + \tilde{C}N^2(t - \tau).$$

This estimate is essential when establishing the gaussian type upper bound for the Neumann-Green function.

In the rest of this section we forsake the explicit dependence on N . So the constants below may depend on Ω , λ , A , and T .

From (2.9) we deduce in a straightforward manner that there exists $\delta > 0$ such that

$$(2.12) \quad E(x, t; \xi, \tau) \geq \widehat{C}(t - \tau)^{-n/2}, \quad (x, t; \xi, \tau) \in P^2, \quad t > \tau, \quad \tilde{C}|x - \xi|^2 < t - \tau \leq \delta.$$

By Theorem 11 in page 44 of [Fr], E is positive. Moreover E satisfies the following identity, usually called the reproducing property,

$$(2.13) \quad E(x, t; \xi, \tau) = \int_{\mathbb{R}^n} E(x, t; \eta, \sigma) E(\eta, \sigma; \xi, \tau) d\eta, \quad x, \xi \in \mathbb{R}^n, \quad t_0 \leq \tau < \sigma < t \leq t_1.$$

We can now paraphrase the proof of Theorem 2.7 in page 334 of [FS] to get the gaussian lower bound for E when $0 < t - \tau \leq \delta$. To pass from $t - \tau \leq \delta$ to $t - \tau \leq T$ one can use again a reproducing property argument. We detail the same argument in the proof of Theorem 3.1.

We sum up our analysis in the following theorem.

Theorem 2.1. *The fundamental solution E satisfies the gaussian two-sided bounds:*

$$(2.14) \quad \mathcal{G}_C(x - \xi, t - \tau) \leq E(x, t; \xi, \tau) \leq \mathcal{G}_{\tilde{C}}(x - \xi, t - \tau), \quad (x, t, \xi, \tau) \in P^2 \times \{t > \tau\}.$$

Remark 2.1. Let us assume that conditions (i)-(iv) above hold in the whole space $\mathbb{R}^n \times \mathbb{R}$ instead of Q only. Taking into account the exponential term in N^2 in (2.11), we prove, once again with the help of the reproducing property, the following global estimate in time:

$$(2.15) \quad e^{-\kappa N^2(t - \tau)} \mathcal{G}_C(x - \xi, t - \tau) \leq E(x, t; \xi, \tau) \leq e^{\kappa N^2(t - \tau)} \mathcal{G}_{\tilde{C}}(x - \xi, t - \tau),$$

for some constant $\kappa > 0$, where $(x, t, \xi, \tau) \in (\mathbb{R}^n \times \mathbb{R})^2$.

We point out that (2.15) does not give the two-sided gaussian bounds by Fabes and Stroock [FS] for the divergence form operator $\partial_i(a_{ij}(x, t)\partial_j \cdot) - \partial_t$ with (C^∞) smooth coefficients. This is not surprising since the arguments we used for proving (2.15) are not well adapted to divergence form operator. We note however that the approach developed in [FS] for establishing the gaussian two-sided bounds is more involved.

Gaussian two-sided bounds were obtained by S. D. Eidel'man and F. O. Porper [EP] when the coefficients of L satisfy the uniform Dini condition with respect to x . The main tool in [EP] is a parabolic Harnack inequality. We refer also to [Ar1], [Fa], [It] and [NS], where the reader can find various results on bounds for the fundamental solution.

We mentioned in the introduction that the Moser's Harnack inequality in [FS] can be extended to a general divergence form parabolic operator. Let us show briefly how this Moser's Harnack inequality still holds for a general parabolic operator. First, we recall that Dirichlet-Green function was constructed in [LSU] as a perturbation of the fundamental solution by a double-layer potential (formula (16.7) in page 408). Therefore, in light of formula (16.10) in page 409 and estimate (16.14) page 411 of [LSU], we can assert that Lemma 5.1 of [FS] remains true for our L . Next, paraphrasing the proof of Lemma 5.2 and Theorem 5.4 in [FS] (more detailed proofs are given in [St]), we can state the following Moser's Harnack inequality.

Theorem 2.2. *Let $\eta, \mu, \varrho \in (0, 1)$. Then there is an M , depending on n, λ, A, η, μ and ϱ such that for all $(x, s) \in \mathbb{R}^n \times \mathbb{R}$, all $R > 0$ and all non negative $u \in C^{2,1}(\overline{B}(x, R) \times [s - R^2, s])$ satisfying $Lu = 0$ one has*

$$u(y, t) \leq Mu(x, s) \quad \text{for all } (y, t) \in \overline{B}(x, \varrho R) \times [s - \eta R^2, s - \mu R^2].$$

3. GAUSSIAN LOWER BOUND FOR THE NEUMANN-GREEN FUNCTION

The unit outward normal vector at $x \in \partial\Omega$ is denoted by $\nu = \nu(x)$.

For $\tau \in [t_0, t_1]$, set $Q_\tau = \Omega \times (\tau, t_1)$ and $\Sigma_\tau = \partial\Omega \times (\tau, t_1)$. We consider the Neumann initial-boundary value problem (abbreviated to IBVP) for the operator L :

$$(3.1) \quad \begin{cases} Lu = 0 & \text{in } Q_\tau, \\ u(\cdot, \tau) = \psi & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \Sigma_\tau. \end{cases}$$

From Theorem 2 in page 144 of [Fr] and its proof, for any $\psi \in C_0^\infty(\Omega)$, the IBVP (3.1) has a unique solution $u \in C(\overline{Q_\tau}) \cap C^{2,1}(Q_\tau)$ given by

$$(3.2) \quad u(x, t) = \int_\tau^t \int_{\partial\Omega} E(x, t; \xi, \sigma) \varphi(\xi, \sigma) d\xi d\sigma + \int_\Omega E(x, t; \xi, \tau) \psi(\xi) d\xi.$$

Here

$$(3.3) \quad \varphi(x, t) = 2F_\tau(x, t) + 2 \sum_{\ell \geq 1} \int_\tau^t \int_{\partial\Omega} M_\ell(x, t; \xi, \sigma) F_\tau(\xi, \sigma) d\xi d\sigma,$$

with

$$\begin{aligned} F_\tau(x, t) &= \int_\Omega \partial_\nu E(x, t; \xi, \tau) \psi(\xi) d\xi, \\ M_1 &= 2\partial_\nu E, \\ M_{\ell+1}(x, t; \xi, \tau) &= \int_\tau^t \int_{\partial\Omega} M_1(x, t; \eta, \sigma) M_\ell(\eta, \sigma; \xi, \tau) d\eta d\sigma. \end{aligned}$$

Let, where $(x, t) \in \Sigma_\tau$ and $\xi \in \Omega$,

$$N(x, t; \xi, \tau) = 2\partial_\nu E(x, t; \xi, \tau) + 2 \sum_{\ell \geq 1} \int_\tau^t \int_{\partial\Omega} M_\ell(x, t; \eta, \sigma) \partial_\nu E(\eta, \sigma; \xi, \tau) d\eta d\sigma.$$

Assume for the moment (see the proof below) that

$$(3.4) \quad \varphi(x, t) = \int_\Omega N(x, t; \xi, \tau) \psi(\xi) d\xi.$$

We set

$$(3.5) \quad G(x, t, \xi, \tau) = \int_\tau^t \int_{\partial\Omega} E(x, t; \eta, \sigma) N(\eta, \sigma; \xi, \tau) d\eta d\sigma + E(x, t; \xi, \tau).$$

It follows from Fubini's theorem that

$$(3.6) \quad u(x, t) = \int_\Omega G(x, t; \xi, \tau) \psi(\xi) d\xi.$$

The function G is called the Neumann-Green function for $Lu = 0$ in Q .

We have, for any $0 \leq \psi \in C_0^\infty(\Omega)$, $u \geq 0$ according to the maximum principle (e.g. Theorem 2.9 and remarks following it in page 15 of [Li]) ; whence $G \geq 0$. From the uniqueness of the solution of the IBVP (3.1) we have also

$$\int_\Omega G(x, t; \xi, \tau) \psi(\xi) d\xi = \int_\Omega G(x, t, \eta, \sigma) d\eta \int_\Omega G(\eta, \sigma, \xi, \tau) \psi(\xi) d\xi \quad \text{for any } \psi \in C_0^\infty(\Omega), \tau < \sigma < t.$$

Therefore

$$(3.7) \quad G(x, t; \xi, \tau) = \int_\Omega G(x, t, \eta, \sigma) G(\eta, \sigma, \xi, \tau) d\eta, \quad \tau < \sigma < t.$$

That is G has the reproducing property.

We note that when $c = 0$, G satisfies in addition

$$\int_{\Omega} G(x, t; \xi, \tau) d\xi = 1.$$

We shall need the following key lemma for establishing the gaussian lower bound for G .

Lemma 3.1. *For $1 - \alpha/2 < \mu < 1$, we have*

$$(3.8) \quad |N(x, t; \xi, \tau)| \leq C(t - \tau)^{-\mu} |x - \xi|^{-n+1}, \quad (x, t) \in \Sigma_{\tau}, \quad \xi \in \Omega, \quad x \neq \xi.$$

The lemma below appears in page 137 of [Fr] as Lemma 1. It is needed for proving Lemma 3.1.

Lemma 3.2. *Let $0 \leq a, b < n - 1$. Then*

$$(3.9) \quad \int_{\partial\Omega} |x - \eta|^{-a} |\eta - \xi|^{-b} d\eta \leq \widehat{C} |x - \xi|^{n-1-(a+b)}.$$

Proof of Lemma 3.1. Let $1 - \alpha/2 < \mu < 1$ be given. From formula (2.12) in page 137 of [Fr], we have

$$|\partial_{\nu} E(x, t; \xi, \tau)| \leq C(t - \tau)^{-\mu} |x - \xi|^{-n+1+(2\mu+\alpha-2)}$$

and then

$$(3.10) \quad |M_1(x, t; \xi, \tau)| \leq C(t - \tau)^{-\mu} |x - \xi|^{-n+1+(2\mu+\alpha-2)}.$$

Since

$$|M_2(x, t; \xi, \tau)| \leq \int_{\tau}^t \int_{\partial\Omega} |M_1(x, t; \eta, \sigma)| |M_1(\eta, \sigma; \xi, \tau)| d\eta d\sigma,$$

(3.10) leads

$$(3.11) \quad |M_2(x, t; \xi, \tau)| \leq C^2 \int_{\tau}^t (t - \sigma)^{-\mu} (\tau - \sigma)^{-\mu} d\sigma \int_{\partial\Omega} |x - \eta|^{-n+1+(2\mu+\alpha-2)} |\xi - \eta|^{-n+1+(2\mu+\alpha-2)} d\eta.$$

Or from Lemma 3.2

$$(3.12) \quad \int_{\partial\Omega} |x - \eta|^{-n+1+(2\mu+\alpha-2)} |\xi - \eta|^{-n+1+(2\mu+\alpha-2)} d\eta \leq \widehat{C} |x - \xi|^{-n+1+2(2\mu+\alpha-2)}.$$

On the other hand

$$(3.13) \quad \int_{\tau}^t (t - \sigma)^{-\mu} (\tau - \sigma)^{-\mu} d\sigma = (t - \tau)^{-\mu+(1-\mu)} \int_0^1 s^{-\mu} (1 - s)^{1-\mu} ds = (t - \tau)^{-\mu+(1-\mu)} B(1 - \mu, 1 - \mu).$$

We plug (3.12) and (3.13) into (3.11) ; we obtain

$$|M_2(x, t; \xi, \tau)| \leq C^2 \widehat{C} (t - \tau)^{-\mu+(1-\mu)} B(1 - \mu, 1 - \mu) |x - \xi|^{-n+1+2(2\mu+\alpha-2)}.$$

Now an induction argument in ℓ yields

$$|M_{\ell}(x, t; \xi, \tau)| \leq C^{\ell} \widehat{C}^{\ell-1} (t - \tau)^{-\mu+(\ell-1)(1-\mu)} \frac{\Gamma(1 - \mu)^{\ell}}{\Gamma(\ell(1 - \mu))} |x - \xi|^{-n+1+\ell(2\mu+\alpha-2)}.$$

It follows from $\Gamma(\ell(1 - \mu)) \geq [\ell(1 - \mu)]!$ (here $[\cdot]$ is the entire part) that the series

$$S(x, t; \xi, \tau) = \sum_{\ell \geq 1} C^{\ell} \widehat{C}^{\ell-1} (t - \tau)^{(\ell-1)(1-\mu)} \frac{\Gamma(1 - \mu)^{\ell}}{\Gamma(\ell(1 - \mu))} |x - \xi|^{\ell(2\mu+\alpha-2)}$$

converges uniformly in all its arguments. We complete the proof by noting that

$$N(x, t; \xi, \tau) = (t - \tau)^{-\mu} |x - \xi|^{-n+1} S(x, t; \xi, \tau).$$

□

Proof of (3.4). Let

$$N_k(x, t; \xi, \tau) = 2\partial_\nu E(x, t; \xi, \tau) + 2 \sum_{\ell \geq 1}^k \int_\tau^t \int_{\partial\Omega} M_\ell(x, t; \eta, \sigma) \partial_\nu E(\eta, \sigma; \xi, \tau) d\eta d\sigma,$$

$$\varphi_k(x, t) = 2F_\tau(x, t) + 2 \sum_{\ell \geq 1}^k \int_\tau^t \int_{\partial\Omega} M_\ell(x, t; \xi, \sigma) F_\tau(\xi, \sigma) d\xi d\sigma.$$

In light of Lemma 3.2 and with the help Lebesgue's dominated convergence theorem we can assert that

$$\int_\Omega N_k(x, t; \xi, \tau) \psi(\xi) d\xi \longrightarrow \int_\Omega N(x, t; \xi, \tau) \psi(\xi) d\xi \text{ as } k \longrightarrow +\infty.$$

Or according to Funini's theorem

$$\varphi_k(x, t) = \int_\Omega N_k(x, t; \xi, \tau) \psi(\xi) d\xi.$$

But $\varphi_k(x, t) \rightarrow \varphi(x, t)$ when k tends to infinity. Then the uniqueness of the limit yields

$$\varphi(x, t) = \int_\Omega N(x, t; \xi, \tau) \psi(\xi) d\xi.$$

□

We are now ready to prove

Theorem 3.1. *The Neumann-Green function G satisfies the gaussian lower bound:*

$$(3.14) \quad \mathcal{G}_C(x - \xi, t - \tau) \leq G(x, t; \xi, \tau), \quad (x, t; \xi, \tau) \in Q^2 \cap \{t > \tau\}.$$

Proof. Let

$$G_0(x, t; \xi, \tau) = \int_\tau^t \int_{\partial\Omega} E(x, t; \eta, \sigma) N(\eta, \sigma; \xi, \tau) d\xi d\sigma.$$

Let $0 < \beta < 1$ be fixed. From the gaussian upper bound for E we obtain in a straightforward way that

$$|E(x, t; \xi, \tau)| \leq C(t - \tau)^{-\beta} |x - \xi|^{-n+2\beta}.$$

This estimate together with (3.8) imply

$$|G_0(x, t; \xi, \tau)| \leq \tilde{C} \int_\tau^t (t - \sigma)^{-\mu} (\sigma - \tau)^{-\beta} d\sigma \int_{\partial\Omega} |x - \eta|^{-n+1} |\eta - \xi|^{-n+2\beta} d\eta$$

$$\leq \tilde{C}(t - \tau)^{1-\mu-\beta} |x - \xi|^{-n+2\beta},$$

where we used Lemma 3.2. That is

$$(3.15) \quad |G_0(x, t; \xi, \tau)| \leq \tilde{C}(t - \tau)^{-n/2+1-\mu} [(t - \tau)^{-1} |x - \xi|^2]^{-n/2+\beta}.$$

But we already know from (2.12) that

$$E(x, t; \xi, \tau) \geq C(t - \tau)^{-n/2}, \quad (x, t; \xi, \tau) \in P^2, \quad t > \tau, \quad \widehat{C}|x - \xi|^2 < t - \tau.$$

Hence

$$G(x, t; \xi, \tau) \geq E(x, t; \xi, \tau) - |G_0(x, t; \xi, \tau)|$$

$$\geq C(t - \tau)^{-n/2} (1 - \tilde{C}(t - \tau)^{1-\mu}), \quad t > \tau, \quad \widehat{C}|x - \xi|^2 < t - \tau.$$

Consequently, we find $\delta > 0$ for which

$$G(x, t; \xi, \tau) \geq C(t - \tau)^{-n/2}, \quad (x, t; \xi, \tau) \in Q^2, \quad 0 < t - \tau \leq \delta, \quad \tilde{C}|x - \xi|^2 < t - \tau.$$

Or equivalently

$$(3.16) \quad G(x, t; \xi, \tau) \geq C(t - \tau)^{-n/2}, \quad (x, t; \xi, \tau) \in Q^2, \quad 0 < t - \tau \leq \delta, \quad |x - \xi| < \widehat{C}(t - \tau)^{1/2}.$$

As Ω is connected, we find a path $\gamma : [0, 1] \rightarrow \Omega$ connecting x to ξ which is piecewise constant. Let k be a positive integer and set $y_i = \gamma(i/k)$, $0 \leq i \leq k$. Then it is not difficult to show that there exists a constant $c \geq 1$ not depending on k such that

$$(3.17) \quad |y_{i+1} - y_i| \leq \frac{c}{k} |x - \xi|, \quad 0 \leq i \leq k-1.$$

When $2c|x - \xi| \leq \widehat{C}(t - \tau)^{1/2}$ (implying $|x - \xi| \leq \widehat{C}(t - \tau)^{1/2}$), (3.14) follows immediately from (3.16). Therefore we may assume that $2c|x - \xi| > \widehat{C}(t - \tau)^{1/2}$. We choose $m \geq 2$ as the smallest integer satisfying

$$2c \frac{|x - \xi|}{m^{1/2}} \leq \widehat{C}(t - \tau)^{1/2}.$$

Set $x_i = \gamma(i/m)$, $0 \leq i \leq m$, and

$$r = \frac{1}{4} \widehat{C} \left(\frac{t - \tau}{m} \right)^{1/2}.$$

In light of the reproducing property and the positivity of G we obtain

$$(3.18) \quad \begin{aligned} G(x, t; \xi, \tau) &= \int_{\Omega} \dots \int_{\Omega} G \left(x, t; \xi_1, \frac{(m-1)t + \tau}{m} \right) \dots G \left(\xi_{m-1}, \frac{t + (m-1)\tau}{m}; \xi, \tau \right) d\xi_1 \dots d\xi_{m-1} \\ &\geq \int_{B(x_1, r) \cap \Omega} \dots \int_{B(x_{m-1}, r) \cap \Omega} G \left(x, t; \xi_1, \frac{(m-1)t + \tau}{m} \right) \dots \\ &\quad \dots G \left(\xi_{m-1}, \frac{t + (m-1)\tau}{m}; \xi, \tau \right) d\xi_1 \dots d\xi_{m-1}. \end{aligned}$$

As Ω is $C^{1,\alpha}$ -smooth, it has the uniform interior cone property and in particular it satisfies the following: there exist two positive constants d and r_0 such that, for any $z \in \overline{\Omega}$ and $0 < \rho \leq r_0$,

$$(3.19) \quad d\rho^n \leq |B(z, \rho) \cap \Omega|.^4$$

In the sequel shortening if necessary the constant \widehat{C} we assume that $r \leq r_0$.

Let $\xi_0 = x$ and $\xi_m = \xi$; we have

$$|\xi_{i+1} - \xi_i| \leq |x_{i+1} - x_i| + 2r \leq c \frac{|x - \xi|}{m} + 2r \leq c \frac{|x - \xi|}{m^{1/2}} + 2r \leq 4r, \quad 0 \leq i \leq m-1.$$

Whence

$$|\xi_{i+1} - \xi_i| \leq \widehat{C} \left(\frac{t - \tau}{m} \right)^{1/2}, \quad 0 \leq i \leq m-1.$$

It follows from (3.16) and (3.19) that

$$\begin{aligned} G(x, t; \xi, \tau) &\geq \int_{B(x_1, r) \cap \Omega} \dots \int_{B(x_{m-1}, r) \cap \Omega} C^m \left(\frac{t - \tau}{m} \right)^{-nm/2} d\xi_1 \dots d\xi_{m-1} \\ &\geq (dr^n)^{m-1} C^m \left(\frac{t - \tau}{m} \right)^{-nm/2} \\ &\geq d^{m-1} \left[\frac{\widehat{C}^2}{16} \left(\frac{t - \tau}{m} \right) \right]^{n(m-1)/2} C^m \left(\frac{t - \tau}{m} \right)^{-nm/2} \\ &\geq \widetilde{C} C^m (t - \tau)^{-1/2}. \end{aligned}$$

Hence

$$(3.20) \quad G(x, t; \xi, \tau) \geq \widetilde{C} e^{-C^m (t - \tau)^{-1/2}}.$$

⁴Choi and Kim [CK] has observed that this condition is necessary for domains having a De Giorgi-Nash-Moser type estimate at the boundary.

Or from the definition of m we have

$$(3.21) \quad m - 1 \leq \left(\frac{2c}{\widehat{C}} \right)^2 \frac{|x - y|^2}{t - \tau}.$$

Finally, a combination of (3.20) and (3.21) leads to (3.14) when $t - \tau \leq \delta$.

We complete the proof by showing that we can remove the assumption $t - \tau \leq \delta$ in (3.16). Let then $0 < t - \tau \leq T$, such that $t - \tau > \delta$, and let $m \geq 2$ be the smallest integer such that $\delta^{-1}(t - \tau) \leq m$. We set

$$(3.22) \quad r = r_0 T^{-1/2} m^{-1/2} (t - \tau)^{1/2}$$

and we denote by p the smallest integer satisfying

$$\frac{2cD}{rm} \leq p, \quad \text{with } D = \text{diam}(\Omega).$$

If we choose $k = pm$ in (3.17), we obtain

$$|y_{i+1} - y_i| \leq \frac{c|x - \xi|}{pm} \leq \frac{cD}{pm} \leq r/2.$$

Let us denote by (3.18*) the inequality (3.18) in which we take $r/2$ in place of r , with r given as in (3.22), and m in changed to pm .

Taking into account that

$$p < 1 + 2cDr_0^{-1}T^{1/2}\delta^{-1/2} = p^*,$$

we get

$$(3.23) \quad \frac{pm|\xi_{i+1} - \xi_i|^2}{t - \tau} \leq \frac{pmr^2}{t - \tau} < p^*r_0^2T^{-1}.$$

As $(pm)^{-1}(t - \tau) \leq \delta$, (3.14) holds true. Therefore in light of (3.23), we obtain from (3.18*)

$$\begin{aligned} G(x, t; \xi, \tau) &\geq \left(d \left[2^{-1}r_0T^{-1/2}m^{-1/2}(t - \tau)^{1/2} \right]^n \right)^{pm-1} C^{pm} [(pm)^{-1}(t - \tau)]^{-npm/2} \\ &\geq \widetilde{C}\widehat{C}^{pm}(t - \tau)^{-n/2} \\ &\geq \widetilde{C}e^{-pm|\ln \widehat{C}|}(t - \tau)^{-n/2} \\ &\geq \widetilde{C}e^{-p^*m^*|\ln \widehat{C}|}(t - \tau)^{-n/2}, \quad \text{with } m^* = \delta^{-1}T + 1 \end{aligned}$$

This last estimate finishes the proof. \square

We already mentioned that, when L has time-independent coefficients and generates a continuous semi-group, the Neumann-Green function is nothing else but the Neumann-heat kernel. Let us then assume that the coefficients of the operator L are time-independent. In other words L is of the form

$$(3.24) \quad L = a_{ij}(x)\partial_{ij}^2 + b_k(x)\partial_k + c(x) - \partial_t$$

and the following assumptions are satisfied

(i') the matrix $(a_{ij}(x))$ is symmetric for any $x \in \overline{\Omega}$,

(ii') $a_{ij} \in W^{1,\infty}(\Omega)$, $b_k, c \in C^\alpha(\overline{\Omega})$,

(iii') $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $(x, t) \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$,

(iv') $\|a_{ij}\|_{W^{1,\infty}(\Omega)} + \|b_k\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \leq A$,

where $\lambda > 0$ and $A > 0$ are two given constants.

We consider the unbounded operator

$$A_L = a_{ij}(x)\partial_{ij}^2 + b_k(x)\partial_k + c(x)$$

with domain

$$D(A_L) = \{u \in L^2(\Omega); A_L u \in L^2(\Omega) \text{ and } \partial_\nu u = 0 \text{ on } \partial\Omega\}^5$$

We make the assumption that A_L generates a continuous semigroup e^{tA_L} on $L^2(\Omega)$.⁶

Let $\psi \in C_0^\infty(\Omega)$. Since $u(t) = e^{tA_L}\psi$ is the solution of the IBVP (3.1) with $\tau = 0$ and $t_1 = T$,

$$e^{tA_L}\psi = \int_{\Omega} G(x, t; \xi, 0)\psi(\xi)d\xi, \quad 0 < t \leq T.$$

We rewrite this equality as follows:

$$e^{tA_L}\psi = \int_{\Omega} K(x, \xi, t)\psi(\xi)d\xi, \quad 0 < t \leq T.$$

The function

$$K(x, \xi, t) = G(x, t; \xi, 0)$$

is usually called the heat kernel of the semigroup e^{tA_L} .

A straightforward consequence of Theorems 3.1 is

Corollary 3.1. *The Neumann heat kernel K satisfies the gaussian lower bound:*

$$(3.25) \quad \mathcal{G}_C(x - \xi, t) \leq K(x, \xi, t), \quad (x, \xi) \in \Omega^2, \quad 0 < t \leq T.$$

The gaussian lower bound for the Neumann heat kernel was proved in [COY] when L is the laplacien. The key point is the Hölder continuity of $x \rightarrow K(x, \xi, t)$ which relies on the fact that $\mu - A_L$ is an isomorphism from $H^s(\Omega)$ into $H^{s-2}(\Omega)$, for some large μ and s of order $n/2 + 1$. We note that a quick examination of the proof in [COY] shows that this result can be extended to a divergence form operators with C^∞ -smooth coefficients and having a variational formulation (or equivalently they possess an associated quadratic form).

We end this section by showing that we can obtain a strong maximum from Theorem 3.1. Let $\psi \in C(\overline{\Omega})$, $f \in C(\overline{Q_\tau})$, $g \in C(\overline{\Sigma_\tau})$ and consider the IBVP

$$(3.26) \quad \begin{cases} Lu = f & \text{in } Q_\tau, \\ u(\cdot, \tau) = \psi & \text{in } \Omega, \\ \partial_\nu u = g & \text{on } \Sigma_\tau. \end{cases}$$

Corollary 3.2. *Assume that $\psi \geq 0$, $f \geq 0$, $g \geq 0$ and at least one of the functions ψ , f and g is non identically equal to zero. If $u \in C(\overline{Q_\tau}) \cap C^{2,1}(Q_\tau)$ is the solution of the IBVP (3.26), then*

$$u > 0 \text{ in } \Omega \times]\tau, t_1].$$

Proof. Immediate from Theorem 3.1 since

$$u(x, t) = \int_{\Omega} G(x, t; \xi, \tau)\psi(\xi)d\xi + \int_{\tau}^t \int_{\Omega} G(x, t; \xi, s)f(\xi, s)d\xi ds + \int_{\tau}^t \int_{\partial\Omega} G(x, t; \xi, s)g(\xi, s)dS_\xi ds.$$

□

4. GAUSSIAN TYPE UPPER BOUND FOR THE NEUMANN-GREEN FUNCTION

In all of this section, Ω is assumed to be of class $C^{2,\alpha}$. We recall that N defined in Section 2 is given as follows

$$N = \max(\max_k \|b_k\|_{L^\infty(Q)}, \|c\|_{L^\infty(Q)}, 1).$$

We first prove that the Green function G satisfies a Nash type upper bound.

Lemma 4.1. *The Green function G satisfies the Nash type upper bound:*

$$(4.1) \quad G(x, t; \xi, \tau) \leq C(t - \tau)^{-n/2} e^{CN^2(t-\tau)}, \quad (x, t; \xi, \tau) \in Q^2, \quad t > \tau.$$

⁵ $\partial_\nu u = 0$ is to be understood in some appropriate weak sense.

⁶This is the case for instance if $a_{ij}(x) = \delta_{ij}$ (e.g. [Ou]).

Proof. We proceed as in the beginning of the proof of Theorem 3.1. We get by taking into account the dependence on N that, for any $0 < \beta < 1$,

$$(4.2) \quad G(x, t; \xi, \tau) \leq C(t - \tau)^{-\beta} |x - \xi|^{-n+2\beta} e^{CN^2(t-\tau)}.$$

Or by (3.7) (reproducing property)

$$(4.3) \quad G(x, t; y, \tau) = \int_{\Omega} G(x, t, \xi, (t - \tau)/2) G(\xi, (t - \tau)/2, y, \tau) d\xi.$$

In light of the inequality in Lemma 2 in page 14 of [Fr]⁷, it follows from (4.2) and (4.3)

$$G(x, t; \xi, \tau) \leq C(t - \tau)^{-2\beta} |x - \xi|^{-n+2(2\beta)} e^{CN^2(t-\tau)}.$$

We find by repeating this argument k -times

$$(4.4) \quad G(x, t; \xi, \tau) \leq C(t - \tau)^{-k\beta} |x - \xi|^{-n+k(2\beta)} e^{CN^2(t-\tau)}.$$

We let k be the smallest integer satisfying $n < 2k$ and we take $\beta = n/(2k)$ in (4.4) in order to get (4.1). \square

Let $\zeta \in \mathbb{R}^n$ and set $|\nu|_{\infty} = \|\nu\|_{\infty}$. As Ω is of class $C^{2,\alpha}$, we obtain from Lemma 3.1 in [Cho] that there exists $\theta_{\zeta} \in C^{2,\alpha}(\bar{\Omega})$ having the properties

$$(4.5) \quad \theta_{\zeta} \geq 1, \quad \theta_{\zeta} = 1 \text{ on } \partial\Omega, \quad -\partial_{\nu}\theta_{\zeta} \geq |\nu|_{\infty}|\zeta| \text{ on } \partial\Omega.$$

Moreover examining the proof of Lemma 3.1 in [Cho] we obtain in a straightforward manner that

$$(4.6) \quad |\partial_i\theta_{\zeta}|, |\partial_{ij}\theta_{\zeta}| \leq \varrho|\zeta|,$$

where ϱ is a constant independent on ζ .

We now conjugate L by $e^{\zeta \cdot x + K|\zeta|^2 t} \theta_{\zeta}^{-1}$, where K is a constant to be specified later. To this end we consider

$$L^{\zeta} = e^{-\zeta \cdot x - K|\zeta|^2 t} \theta_{\zeta} L e^{\zeta \cdot x + K|\zeta|^2 t} \theta_{\zeta}^{-1}.$$

An elementary calculation gives $L^{\zeta} = \widehat{L}^{\zeta} + c^{\zeta}$ with

$$\widehat{L}^{\zeta} = a_{ij}\partial_{ij}^2 + b_p\partial_p + 2a_{kl}(\zeta_l - \theta_{\zeta}^{-1}\partial_l\theta_{\zeta})\partial_k + b_m(\zeta_m - \theta_{\zeta}^{-1}\partial_m\theta_{\zeta}) + c - \theta_{\zeta}^{-1}a_{ij}\partial_{ij}^2\theta_{\zeta} - \partial_t$$

and

$$c^{\zeta} = a_{pq}\zeta_p\zeta_q + 2\theta_{\zeta}^{-1}a_{ij}\zeta_j\partial_i\theta_{\zeta} + 2a_{ij}\theta_{\zeta}^{-2}\partial_i\theta_{\zeta}\partial_j\theta_{\zeta} - K|\zeta|^2.$$

In light of (4.5) and (4.6) we can choose K , independent on ζ , such that $c^{\zeta} \leq 0$ for any $\zeta \in \mathbb{R}^n$.

Let $0 \leq \psi \in C_0^{\infty}(\Omega)$ and u the corresponding solution of the IBVP (3.1). Then $v = e^{-\zeta \cdot x - K|\zeta|^2 t} \theta_{\zeta} u$ solves the following IBVP

$$(4.7) \quad \begin{cases} L^{\zeta}v = 0 & \text{in } Q_{\tau}, \\ v(\cdot, \tau) = e^{-\zeta \cdot x - K|\zeta|^2 \tau} \theta_{\zeta} \psi & \text{in } \Omega, \\ \partial_{\nu}v + (\zeta \cdot \nu - \partial_{\nu}\theta_{\zeta})v = 0 & \text{on } \Sigma_{\tau}. \end{cases}$$

Next, let \widehat{v} be the solution of the IBVP

$$(4.8) \quad \begin{cases} \widehat{L}^{\zeta}\widehat{v} = 0 & \text{in } Q_{\tau}, \\ \widehat{v}(\cdot, \tau) = e^{-\zeta \cdot x - K|\zeta|^2 \tau} \theta_{\zeta} \psi & \text{in } \Omega, \\ \partial_{\nu}\widehat{v} = 0 & \text{on } \Sigma_{\tau}. \end{cases}$$

Since $v \geq 0$,

$$\begin{cases} \widehat{L}^{\zeta}(v - \widehat{v}) = -c^{\zeta}v \geq 0 & \text{in } Q_{\tau}, \\ (v - \widehat{v})(\cdot, \tau) = 0 & \text{in } \Omega, \\ \partial_{\nu}(v - \widehat{v}) = -(\zeta \cdot \nu - \partial_{\nu}\theta_{\zeta})v \leq 0 & \text{on } \Sigma_{\tau}. \end{cases}$$

⁷Let $0 < \alpha, \beta < n$. Then there exists a positive constant C such that for any $(x, \xi) \in \Omega^2$, $x \neq \xi$, we have

$$\int_{\Omega} |x - \eta|^{-\alpha} |\eta - \xi|^{-\beta} d\eta \leq C|x - \xi|^{n-\alpha-\beta} \text{ if } \alpha + \beta > n \text{ and } \int_{\Omega} |x - \eta|^{-\alpha} |\eta - \xi|^{-\beta} d\eta \leq C \text{ if } \alpha + \beta \leq n.$$

The last inequality is obtained from the third inequality in (4.5). We invoke Theorem 2.9 [Li] to deduce that $v \leq \widehat{v}$.

Let \widehat{E}^ζ and \widehat{G}^ζ be respectively the fundamental solution and the Neumann-Green function corresponding to \widehat{L}^ζ . It follows from estimate (2.11) that

$$\widehat{E}^\zeta(x, t; \xi, \tau) \leq \widetilde{C}(t - \tau)^{-n/2} e^{-\frac{\lambda^* |x - \xi|^2}{t - \tau} + \widehat{C}\Psi(|\zeta|)(t - \tau)},$$

where $\Psi(\rho) = 1 + \rho + \rho^2$, $\rho > 0$. Here and henceforth all generic constants we use may depend on $\|b_k\|_\infty$ and $\|c\|_\infty$.

By Lemma 4.1 \widehat{G}^ζ satisfies the Nash type upper bound

$$(4.9) \quad \widehat{G}^\zeta(x, t; \xi, \tau) \leq C(t - \tau)^{-n/2} e^{\widehat{C}\Psi(|\zeta|)(t - \tau)}.$$

Therefore, for any $0 \leq \psi \in C_0^\infty(\Omega)$,

$$e^{-\zeta \cdot x - K|\zeta|^2 t} \theta_\zeta(x) u(x, t) = v(x, t) \leq \int_\Omega \widehat{G}^\zeta(x, t; \xi, \tau) e^{-\zeta \cdot \xi - K|\zeta|^2 \tau} \theta_\zeta(\xi) \psi(\xi) d\xi.$$

Or

$$u(x, t) = \int_\Omega G(x, t; \xi, \tau) \psi(\xi) d\xi.$$

Whence

$$G(x, t, \xi, \tau) \leq e^{\zeta \cdot x + K|\zeta|^2 t} \widehat{G}^\zeta(x, t; \xi, \tau) e^{-\zeta \cdot \xi - K|\zeta|^2 \tau} \theta_\zeta(x)^{-1} \theta_\zeta(\xi).$$

This and (4.9) imply

$$(4.10) \quad G(x, t, \xi, \tau) \leq C(t - \tau)^{-n/2} e^{\zeta \cdot (x - \xi) + \widehat{C}\Psi(|\zeta|)(t - \tau)} \theta_\zeta(x)^{-1} \theta_\zeta(\xi)$$

and then

$$G(x, t, \xi, \tau) \leq C|\zeta|(t - \tau)^{-n/2} e^{\zeta \cdot (x - \xi) + \widehat{C}\Psi(|\zeta|)(t - \tau)}.$$

Taking $\zeta = (\xi - x)/[2\widehat{C}(t - \tau)]$ in the last estimate we get the following result.

Theorem 4.1. *The Neumann-Green function satisfies the upper bound:*

$$G(x, t, \xi, \tau) \leq (t - \tau)^{-1/2} \mathcal{G}_C(x - \xi, t - \tau), \quad (x, t; \xi, \tau) \in Q^2, \quad t > \tau.$$

Remark 4.1. 1) Returning to the proof of Lemma 3.1 in [Cho], we prove that for any compact subset K of Ω there is a constant C_K such that

$$|\theta_\zeta(x) \theta_\zeta^{-1}(\xi)| \leq C_K, \quad x \in K, \quad \xi \in \overline{\Omega} \text{ and } \zeta \in \mathbb{R}^n.$$

We derive from (4.10) the following gaussian upper bound

$$G(x, t, \xi, \tau) \leq C_K (t - \tau)^{-n/2} e^{-\frac{C|x - \xi|^2}{t - \tau}}, \quad (x, t; \xi, \tau) \in Q^2 \cap \{x \in K, t > \tau\}.$$

1) The construction of the Neumann-Green function is quite simple when Ω is a half space, say

$$\Omega = \mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n > 0\}.$$

Assume for simplicity that the original coefficients of L satisfy assumptions (i)-(iv) in the whole space \mathbb{R}^n and set

$$G(x, t; \xi, \tau) = \frac{1}{2} [E(x, t; \xi, \tau) + E(x^s, t; \xi, \tau)], \quad x, \xi \in \mathbb{R}_+^n, \quad \tau, t \in (t_0, t_1), \quad \tau < t,$$

where $x^s = (x', -x_n)$ if $x = (x', x_n)$.

It is not hard to check that G is the Neumann-Green function for L in $\mathbb{R}_+^n \times (t_0, t_1)$. In addition, observing that $|x - \xi| \leq |x^s - \xi|$ for any $x, \xi \in \mathbb{R}_+^n$, we see the gaussian upper bound for E yields the gaussian upper bound for G .

2) We used above a parabolic maximum principle from [Li]. In fact, this theorem is not only valid for the Neumann boundary condition but for any non tangential operator $B = \beta_j \partial_j + \gamma$. On the other hand the construction of the Neumann-Green function from the tools in [Fr] can be extended to an operator with

Robin boundary condition. Therefore, we believe that Theorem 4.1 holds true if the Neumann boundary condition is changed to a Robin boundary condition.

For $1 < p, q \leq \infty$, let $\|\cdot\|_{p,q}$ denotes the norm in $L^q(t_0, t_1; L^p(\Omega))$. We have similarly to Corollary 7.1 in page 668 of [Ar2] the following result which is a direct consequence of the upper bound in Theorem 4.1.

Corollary 4.1. *Let $1 < p, q \leq \infty$ be such that*

$$\frac{n}{p} + \frac{2}{q} < 1.$$

Then

$$\|G(x, t; \cdot, \cdot)\|_{p',q'}, \quad \|G(\cdot, \cdot; \xi, \tau)\|_{p',q'} \leq C,$$

where p', q' are the respective Hölder conjugate exponents of p and q .

Let $f \in C_0^\infty(Q_\tau)$ and p, q be as in the previous lemma and recall that the solution of the following BVP

$$\begin{cases} Lu = f & \text{in } Q_\tau, \\ u(\cdot, \tau) = 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \Sigma_\tau \end{cases}$$

can be represented by the formula

$$u(x, t) = \int_\tau^t \int_\Omega G(x, t; \xi, s) f(\xi, s) d\xi ds.$$

In light of this formula we have the following immediate consequence of Corollary 4.1:

$$\|u\|_{\infty, \infty} \leq C \|f\|_{p, q}.$$

REFERENCES

- [Ar1] D. G. ARONSON, *Bounds for the fundamental solution of a parabolic equation*, Bulletin of American Math. Soc. 73 (1967), 890-896.
- [Ar2] D. G. ARONSON, *Non-negative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa 22 (3) (1968), 607-694.
- [Ch] S. CHO, *Two-sided global estimates of Green's function of parabolic equations*, Potential Anal. 25 (2006), 387-398.
- [CKP] S. CHO, S. KIM AND H. PARK, *Two-sided estimates on Dirichlet heat kernels for time-dependent parabolic operators with singular drifts*, J. Diffent. Equat. 252 (2012), 1101-1145.
- [CK] J. CHOI AND S. KIM, *Greens function for second order parabolic systems with Neumann boundary condition*, J. Diffent. Equat. 252 (2013), 2834-2860.
- [Cho] M. CHOULLI, *On the determination of an unknown boundary function in a parabolic equation*, Inverse Problems 15 (1999) 659-667.
- [COY] M. CHOULLI, E. M. OUHABAZ AND M. YAMAMOTO, *Stable determination of a semilinear term in a parabolic equation* Commun. Pure Appl. Anal. 5 (3) (2006) 447-462.
- [Dan] D. DANERS, *Heat Kernel estimates for operators with boundary conditions*, Math Nachr. 217 (2000), 13-41.
- [Dav] E. B. DAVIES, *Heat kernels and spectral theory*, Cambridge Tracts in Math. 92, Cambridge University Press, London 1989.
- [EP] S. D. EIDEL'MAN AND F. O. PORPER, *Two-sided estimates for fundamental solutions of second-order parabolic equations, and some applications* Uspekhi. Mat. Nauk 39 (3) (1984), 107-156 ; Russian Math. Surveys 39 (3) (1984), 119-178.
- [Fa] E. FABES, *Gaussian upper bounds on fundamental solutions of parabolic equations ; the method of Nash*, Dirichlet forms (Varenna, 1992), 120, Lecture Notes in Math., 1563, Springer, Berlin, 1993.
- [FS] E. FABES AND D. W. STROOCK, *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*, Arch. Rat. Mech. Anal. 96 (1986), 327-338.
- [Fr] A. FRIEDMAN, *Partial differential equations of parabolic type*, Englewood Cliffs NJ, Prentice-Hall, 1964.
- [Gr] A. GRIGOR'YAN, *Heat kernel and analysis on manifolds*, AMS/IP Studies in Advanced Mathematics, 47. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009.
- [It] S. ITÔ, *Diffusion equations*, Transaction of Mathematical Monographs 114, Providence, RI, 1991.
- [LSU] O. A. LADYZHENSKAJA, V. A. SOLONNIKOV AND N. N. URAL'TZEVA, *Linear and quasilinear equations of parabolic type*, Nauka, Moscow, 1967 in Russian; English translation: American Math. Soc., Providence, RI, 1968.
- [Le] E. E. LEVI, *Sulle equazioni lineari totalmente alle derivate parziali*, Rend. Circ. Mat. Palermo, 24 (1907), 275-317.
- [Li] G. LIEBERMAN, *Second order parabolic differential equations*, World Scientific Publishing, Singapore, 1996.

- [Mo] J. MOSER, *Harnack inequality for parabolic differential equations*, Commun. Pure Appl. Math. 17 (1964), 101-134 ; Correction to "*Harnack inequality for parabolic differential equations*" Commun. Pure Appl. Math. 20 (1967) 231-236.
- [Na] J. NASH, *Continuity of solutions of parabolic and elliptic equations*, American J. Math. 80 (1958), 931-954.
- [NS] J. R. NORRIS AND D. W. STROOCK, *Estimates on the fundamental solution to heat flows with uniformly elliptic coefficients*, Proc. London Math. Soc. 62 (3) (1991), 373-402.
- [Ou] E. M. OUHABAZ, *Analysis of heat equations on domains*, London Math. Soc. Monographs, vol. 31, Princeton University Press 2004.
- [Sa] L. SALOFF-COSTE, *Aspects of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series, 289. Cambridge University Press, Cambridge, 2002.
- [St] D. W. STROOCK, *Partial differential equations for probabilists*, Cambridge Studies in Advanced Mathematics, 112. Cambridge University Press, Cambridge, 2008.

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