Liouville Brownian motion

Christophe Garban, Rémi Rhodes, Vincent Vargas

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Abstract

We construct a stochastic process, called the **Liouville Brownian motion** which we conjecture to be the scaling limit of random walks on large planar maps which are embedded in the euclidean plane or in the sphere in a conformal manner. Our construction works for all universality classes of planar maps satisfying $\gamma < \gamma_c = 2$. In particular, this includes the interesting case of $\gamma = \sqrt{8/3}$ which corresponds to the conjectured scaling limit of large uniform planar p-angulations (with fixed $p \geqslant 3$).

We start by constructing our process from some fixed point $x \in \mathbb{R}^2$ (or $x \in \mathbb{S}^2$). This amounts to changing the speed of a standard two-dimensional brownian motion B_t depending on the local behaviour of the Liouville measure "$M_\gamma(dz) = e^{\gamma X} dz$" (where X is a Gaussien Free Field, say on $\mathbb{S}^2$). A significant part of the paper focuses on extending this construction simultaneously to all points $x \in \mathbb{R}^2$ or $\mathbb{S}^2$ in such a way that one obtains a semi-group $P_{\mathbf{t}}$ (the Liouville semi-group). We prove that the associated Markov process is a Feller diffusion for all $\gamma < \gamma_c = 2$ and that for $\gamma < \sqrt{2}$, the Liouville measure M_γ is invariant under $P_{\mathbf{t}}$ (which in some sense shows that it is the right quantum gravity diffusion to consider).

This Liouville Brownian motion enables us to give sense to part of the celebrated Feynman path integrals which are at the root of Liouville quantum gravity, the Liouville Brownian ones. Finally we believe that this work sheds some new light on the difficult problem of constructing a quantum metric out of the exponential of a Gaussian Free Field (see conjecture 2).

Key words or phrases: Random measures, Liouville quantum gravity, Liouville Brownian motion, multiplicative chaos.
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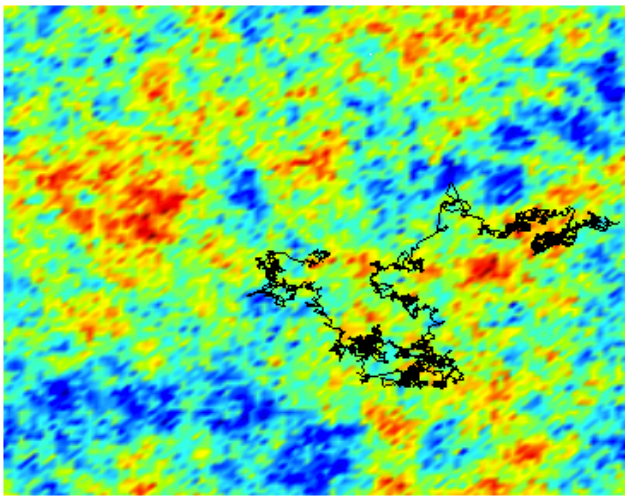

Figure 1: Simulation of a massive LBM on the unit torus. The background stands for the height landscape of the GFF on the torus: red for high values and blue for small values. The evolution of the LBM is plotted in black

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1 Introduction

Let $\mathcal{T}_n$ be the set of (conformally equivalent classes of) all triangulations of the two-dimensional sphere $\mathbb{S}^2$ with n faces with no loops or multiple edges. Choose uniformly at random a triangulation $T = T_n$ of $\mathcal{T}_n$ and consider a simple random walk on T. We want to identify the limit in law as $n \to \infty$ of this random walk. As stated, the problem seems to be ill-posed because there is some flexibility in the choice of the embedding of the triangulation into the sphere: we may apply a Möbius transformation of the sphere and obtain a conformally equivalent triangulation. As suggested in [11], one possible way to remove this flexibility is as follows. Consider a circle packing $P = (P_c)_{c \in C}$ in the sphere $\mathbb{S}^2$ such that the contact graph of P is T. This packing is unique up to Möbius transformations. In order to fix the Möbius transformation without spoiling the symmetries of the problem, one may consider among all circle packings P on the sphere $\mathbb{S}^2$ which realize the triangulation T, the packing P whose "barycenter" is the centre of the sphere. We refer to [11] for more detail and for a justification through the Poincaré-Beardon and the Douady-Earle theorems that one can indeed find such a packing, which is unique up to rotations. This embedding will be denoted by P_T and constitutes a canonical discrete conformal structure for the triangulation T.

Now if one chooses uniformly at random a triangulation $T = T_n$ of $\mathcal{T}_n$ and if μ_T denotes the random probability measure on $\mathbb{S}^2$ which assigns a measure $1/n$ to

each face of P_T, it is conjectured in [11] that μ_T converges in law towards a limiting probability measure given in terms of the Liouville measure

$$\text{“}M_{\sqrt{8/3}}(dz) = e^{\sqrt{8/3}X(z)}\, dz\text{”}\,,$$

where X is a Gaussian Free Field on the sphere $\mathbb{S}^2$ with vanishing mean (see subsection 3.1.1). Since there are several natural ways how to renormalize M_γ in order to obtain a probability measure on $\mathbb{S}^2$, a precise conjecture for the scaling limit of μ_T is still missing. Yet, in any case the limiting measure is believed to be absolutely continuous with respect to $M_{\sqrt{8/3}}$. See [11, 26, 31, 60] for more on this topic.

Conjecture 1. *Choose uniformly at random a triangulation T of $\mathcal{T}_n$ and consider a simple random walk on T. As n goes to ∞, the law of the simple random walk converges towards the law of the main object of this paper: the* **Liouville Brownian motion** *on the sphere for $\gamma^2 = \frac{8}{3}$.*

As such the purpose of this paper is to construct what can be thought of as the natural scaling limit of random walks on random triangulations (or similar models: quadrangulations, p-angulations) or, more generally, random walks on random triangulations weighted by the partition functions of suitable statistical physics models: Ising, $O(n)$, Potts models,...in which case, the γ in conjecture 1 must be adapted to the central charge of the model (see [28, 31, 32] for instance).

Nevertheless, the interest of the Liouville Brownian motion (LBM for short) that we are going to explain goes beyond the possibility of providing a natural candidate for scaling limits of random walks on planar maps. Indeed, recall that the ultimate mathematical problem in (critical) $2d$-Liouville quantum gravity is to construct a random metric on a two dimensional Riemannian manifold D, say a domain of $\mathbb{R}^2$ (or the sphere) equipped the Euclidean metric dz^2, which takes on the form

$$e^{\gamma X(z)} dz^2 \tag{1.1}$$

where X is a Gaussian Free Field (GFF) on the manifold D and $\gamma \in [0, 2)$ is a coupling constant that can be expressed in terms of the central charge of the underlying model (see [43, 15] for further details and also [18, 19, 32, 52] for insights in Liouville quantum gravity). The simplicity of such an expression hides many highly non trivial mathematical difficulties. Indeed, the correlation function of a GFF presents a short scale logarithmically divergent behaviour that makes relation (1.1) non rigorous. One has to apply a cutoff procedure to smooth down the singularity of the GFF and the method to do this at a "metric level" remains unclear. However, many geometric quantities are related to this metric and for some of them, the cutoff procedure may be applied properly without having a direct access to the metric. For instance, Duplantier and Sheffield [26] focused on the volume form M associated to the metric, sometimes called the Liouville measure, and defined it rigorously. Their method falls

under the scope of the theory of Gaussian multiplicative chaos developed by Kahane [36], which allows us to give a rigorous meaning to the expression

$$M(A) = \int_A e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X(z)^2]}\, dz, \tag{1.2}$$

where dz stands for the volume form (Lebesgue measure) on D (to be exhaustive, one should integrate against $h(z)\, dz$ where h is a deterministic function involving the conformal radius at z but this term does not play an important role for our concerns). This strategy made possible an interpretation in terms of measures of the Knizhnik-Polyakov-Zamolodchikov formula (KPZ for short, see [43]) relating the fractal dimensions of sets as seen by the Lebesgue measure or the Liouville measure. The KPZ formula is proved in [26] when considering the fractal notion of expected box counting dimension whereas the fractal notion of almost sure Hausdorff dimension is considered in [56]. This measure approach made also possible in [8] a mathematical understanding of *duality* in Liouville quantum gravity as well as a rigorous proof of the dual KPZ formula (see [3, 4, 13, 21, 22, 27, 28, 35, 40, 41, 42, 44] for an account of physics literature).

Another powerful tool in describing a Riemannian geometry is the Brownian motion. With it are attached several analytic objects serving to describe the geometry: Laplace-Beltrami operator, heat kernel, Dirichlet forms... Therefore, a relevant way to have further insights into Liouville quantum gravity geometry is to define the *Liouville Brownian motion*. It can be constructed on any $2d$ background Riemannian manifold equipped with a Gaussian Free Field. However, in this paper and for pedagogical purposes, we will mostly describe the following situations: the whole plane $\mathbb{R}^2$, the sphere $\mathbb{S}^2$ or the torus $\mathbb{T}^2$, or planar bounded domains. Working on the whole plane $\mathbb{R}^2$ is convenient to avoid unnecessary complications related to stochastic calculus on manifolds. In that case, it is necessary to introduce a long scale infrared regulator in order to have a well defined field X. So we will consider a Massive Gaussian Free Field X on $\mathbb{R}^2$ (MFF for short). Then, we will describe the situation when the manifold is the sphere $\mathbb{S}^2$, the torus $\mathbb{T}^2$ or a planar bounded domain equipped with a GFF X (here compactness plays the role of infrared regulator). More generally, it is also clear that our methodology may apply to n-dimensional Riemannian manifold without boundary and yields the same results, whatever the structure of the manifold: we just work in dimension 2 because of motivations related to $2d$ Liouville quantum gravity.

Important enough, we feel that the construction of the Liouville semigroup might lead to the construction of the Liouville metric. Let us formulate here a conjecture at a rough level. The reader is referred to Section 4 for more details. In this paper, we will construct the Liouville semigroup $(P^X_{\mathbf{t}})_{\mathbf{t}\geqslant 0}$ of the Liouville Brownian motion and prove that it is symmetric with respect to the Liouville measure M. Following the standard steps of the theory of symmetric Dirichlet forms (see [30] for instance),

we can associate to this semi-group a symmetric Dirichlet form Σ by:

$$\Sigma(f,f) = \lim_{t\to 0} \int\int \big(f(x)-f(y)\big)^2 P^X(t,x,dy)M(dx) \tag{1.3}$$

with domain $\mathcal{D}$, which is defined as the set of functions $f \in L^2(\mathbb{R}^2, M)$ for which the above limit exists. The geometric aspect of Dirichlet forms allows to interpret the theory of Dirichlet forms as an extension of Riemannian geometry applicable to non differential structures and to describe stochastic processes in terms of intrinsically defined geometric quantities. Let us then consider the *length of the gradient* Γ associated to this Dirichlet form (see [61, 62, 63]).

Conjecture 2. *For $\gamma^2 < 4$, almost surely in the (massive) Gaussian free field X, the Liouville Dirichlet form $(\mathcal{D}, \Sigma)$ defined in subsection 4.3 is strongly local and regular. Furthermore, its associated intrinsic metric*

$$d_X(x,y) = \sup\{f(x)-f(y); f \in \mathcal{D}_{loc} \cap C_0(D), \Gamma(f,f) \leqslant M\}$$

is finite and its associated topology is Euclidean.

If true, this conjecture ensures that (D, d_X) is a length space (see [61, Theorem 5.2]). Observe that, in the context of pure gravity ($c = 0, \gamma = \sqrt{8/3}$), this question is (conjectured to be) related to the topology of large planar maps [46, 47, 50]. The fact that the associated topology should be Euclidean is related in the case $\gamma = \sqrt{8/3}$ to the fact that the scaling limit of large planar maps a.s. have the topology of a $2d$-sphere (see [45, 48, 49]).

Let us now comment on our results and explain the thread of our approach in the situation of the whole plane $\mathbb{R}^2$ equipped with a MFF X. As previously explained, giving sense to (1.1) requires first to apply an ultraviolet cutoff to the MFF (this procedure is described in subsection 2.1) in order to smooth down the singularity of the covariance kernel. Let us denote by X_n the field resulting from a cutoff procedure at level n: we do not need to make this statement more precise now, let us just say that the field X has been smoothed up to some extent that is encoded by n: the larger n is, the closer to X the field X_n is. We can then consider a Riemannian metric tensor on $\mathbb{R}^2$:

$$g_n(z) = e^{\gamma X_n(z) - \frac{\gamma^2}{2}\mathbb{E}[X_n(z)^2]} dz^2, \tag{1.4}$$

which appears as a n-regularized form of (1.1). The renormalization term $e^{\frac{\gamma^2}{2}\mathbb{E}[X_n(z)^2]}$ appears for future renormalization purposes, which have the same origins as in (1.2), but does not play a role now in the geometry associated with this metric. The couple $(\mathbb{R}^2, g_n)$ is a Riemannian manifold. We further stress that the Riemannian volume of this manifold is nothing but the n-regularized version of (1.2):

$$M_n(A) = \int_A e^{\gamma X_n(z) - \frac{\gamma^2}{2}\mathbb{E}[X_n(z)^2]}\, dz,$$

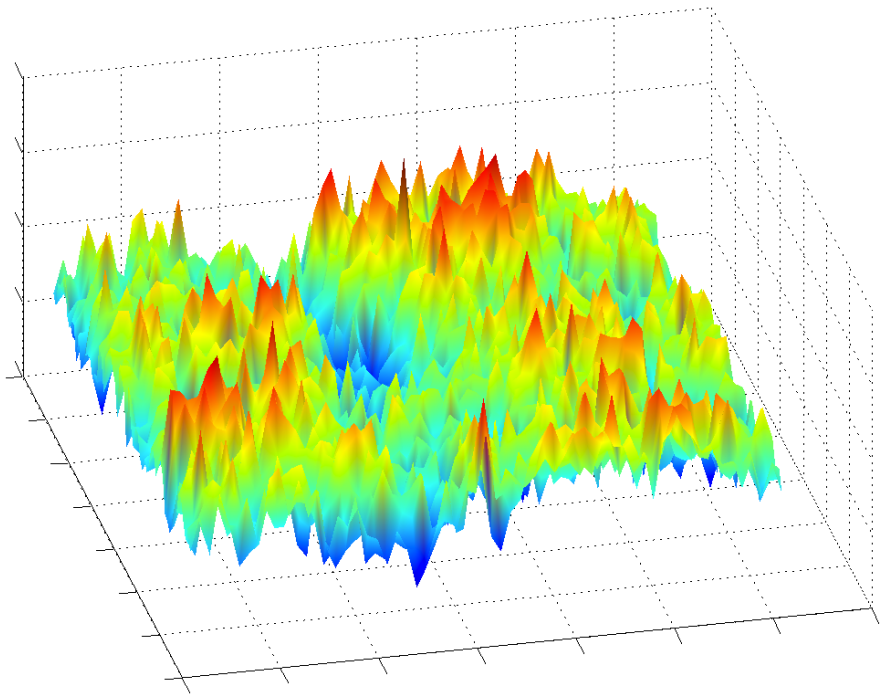

Figure 2: Simulation of a massive GFF on the unit torus.

which converges as $n \to \infty$ (meaning when the cutoff is removed) towards the Liouville measure M. One can associate to the Riemannian manifold $(\mathbb{R}^2, g_n)$ a Brownian motion $\mathcal{B}^n$ in a standard way: consider a standard 2-dimensional Brownian motion $\bar{B}$ and define the n-regularized Liouville Brownian motion:

$$\begin{cases} \mathcal{B}^{n,x}_{\mathbf{t}=0} = x \\ d\mathcal{B}^{n,x}_{\mathbf{t}} = e^{-\frac{\gamma}{2}X_n(\mathcal{B}^{n,x}_{\mathbf{t}})+\frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}^{n,x}_{\mathbf{t}})^2]}\, d\bar{B}_{\mathbf{t}}. \end{cases} \tag{1.5}$$

It is the solution of a SDE on $\mathbb{R}^2$. At first sight, it is not obvious to understand in which way the above n-regularized LBM will converge as $n \to \infty$. To get a better idea, let us use the Dambis-Schwarz theorem and rewrite (1.5) as

$$\mathcal{B}^{n,x}_{\mathbf{t}} \overset{law}{=} x + B_{\langle \mathcal{B}^{n,x} \rangle_{\mathbf{t}}}\,, \tag{1.6}$$

where $(B_r)_{r \geqslant 0}$ is another two-dimensional Brownian motion and the quadratic variation $\langle \mathcal{B}^{n,x} \rangle$ of $\mathcal{B}^{n,x}$ is given by:

$$\langle \mathcal{B}^{n,x} \rangle_{\mathbf{t}} := \inf\{ s \geqslant 0 \,:\, \int_0^s e^{\gamma X_n(x+B_u)-\frac{\gamma^2}{2}\mathbb{E}\left[X_n(x+B_u)^2\right]}\, du \geqslant \mathbf{t} \}\,. \tag{1.7}$$

Therefore, the n-regularized LBM appears as a time change of a standard Brownian motion and studying its convergence thus boils down to proving the convergence of its quadratic variations, which are entirely encoded by the mapping:

$$F_n(x,t) = \int_0^t e^{\gamma X_n(x+B_u)-\frac{\gamma^2}{2}\mathbb{E}\left[X_n(x+B_u)^2\right]}\, du. \tag{1.8}$$

More precisely, the quadratic variation of $\mathcal{B}^{n,x}$ is nothing but the inverse mapping of $F_n(x,\cdot)$. Notice also that $F_n(x,\cdot)$ can be seen as a Gaussian multiplicative chaos along the Brownian paths of B. Gaussian multiplicative chaos theory thus enters the game in order to establish the convergence of F_n. This can be done for all values of $\gamma \in [0,2)$. Nevertheless, much more work is needed to deduce the convergence of the n-regularized LBM: we have to prove not only that $F_n(x,\cdot)$ converges towards a continuous, strictly increasing mapping but also to prove that this convergence holds, almost surely in X, for **all** the starting points $x \in \mathbb{R}^2$ in order to obtain a properly defined limiting Markov process. So we claim:

Theorem 1.1. *Assume $\gamma^2 < 4$ and fix $x \in \mathbb{R}^2$. Almost surely in X and in B, the n-regularized Brownian motion $(\mathcal{B}^{n,x})_n$ defined by Definition 1.6 converges in the space $C(\mathbb{R}_+,\mathbb{R}^2)$ equipped with the supremum norm on compact sets towards a continuous random process $\mathcal{B}^x$, which we call (massive)* **Liouville Brownian motion** *starting from x, characterized by:*

$$\mathcal{B}^x_{\mathbf{t}} = x + B_{\langle \mathcal{B}^x \rangle_{\mathbf{t}}}$$

where $\langle \mathcal{B}^x \rangle$ is defined by

$$F(x, \langle \mathcal{B}^x \rangle_{\mathbf{t}}) = \mathbf{t}.$$

As a consequence, almost surely in X, the n-regularized Liouville Brownian motion defined in Definition 1.5 converges in law under $\mathbb{P}$ in $C(\mathbb{R}_+,\mathbb{R}^2)$ towards $\mathcal{B}^x$.

This result will allow us to prove that, almost surely in X, we can define the law of the Liouville Brownian motion for all possible starting point $y \in \mathbb{R}^2$:

Theorem 1.2. *Assume $\gamma^2 < 4$. Almost surely in X, for all $y \in \mathbb{R}^2$, the family $(F^n(y,\cdot))_n$ converges in law under $\mathbb{P}^B$ in $C(\mathbb{R}_+)$ equipped with the sup-norm topology towards a continuous increasing mapping $F(y,\cdot)$. Let us define the process $\mathbf{t} \mapsto \langle \mathcal{B}^y \rangle_{\mathbf{t}}$ by:*

$$\forall \mathbf{t} \geqslant 0, \quad F(y, \langle \mathcal{B}^y \rangle_{\mathbf{t}}) = \mathbf{t}.$$

The law of the Liouville Brownian motion $\mathcal{B}^y$ starting from y is then given by

$$\mathcal{B}^y_{\mathbf{t}} = y + B_{\langle \mathcal{B}^y \rangle_{\mathbf{t}}}.$$

Almost surely in X, for all $y \in \mathbb{R}^2$, the process $\mathcal{B}^y$ is the limit in law in $C(\mathbb{R}_+)$ of the family $(\mathcal{B}^{n,y})_n$. Furthermore, almost surely in X and under $\mathbb{P}^B$, the law of the mapping $y \mapsto \mathcal{B}^y$ is continuous in $C(\mathbb{R}_+)$.

Therefore, the Liouville Brownian motion can be thought of as the solution of the following formal SDE

$$\begin{cases} \mathcal{B}^x_{\mathbf{t}=0} = x \\ d\mathcal{B}^x_{\mathbf{t}} = e^{-\frac{\gamma}{2}X(\mathcal{B}^x_{\mathbf{t}}) + \frac{\gamma^2}{4}\mathbb{E}[X(\mathcal{B}^x_{\mathbf{t}})^2]}\, d\bar{B}_{\mathbf{t}}. \end{cases} \tag{1.9}$$

We will prove the following intriguing result. Once the environment X is fixed, the n-regularized Liouville Brownian motion $\mathcal{B}^{n,x}$ appearing in (1.5) is of course a

measurable function of the Euclidean Brownian motion $\bar{B}$. We will prove not only that the couple $(\mathcal{B}^{n,x}, \bar{B})$ converges in law towards the couple $(\mathcal{B}^x, \bar{B})$, but also that $\mathcal{B}^x$ is independent of $\bar{B}$. In a way, this can be interpreted as a creation of randomness by strongly pinching the Brownian curve $\bar{B}$ in order to create a new randomness $\mathcal{B}^x$ independent of $\bar{B}$.

It is natural to expect that the Liouville Brownian motion is Markovian. Actually, we can prove much more:

Theorem 1.3. *The Liouville Brownian motion is a Feller Markov process with continuous sample paths.*

As a Markov process, it is natural to wonder whether the Liouville Brownian motion possesses an invariant measure. We can carry out the argument for $\gamma^2 < 2$:

Theorem 1.4. *For $\gamma^2 < 2$, the Liouville Brownian motion is reversible with respect to the Liouville measure. Therefore the Liouville measure is invariant.*

This property hints that the Liouville Brownian motion is the right diffusion to consider if one wants to study the geometry of quantum gravity through the eyes of random walks and diffusions. This Markov process has a generator, which we call *Liouville Laplacian*, which formally reads

$$\Delta_X = e^{-\gamma X(x) + \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}\Delta.$$

and can be thought of as the Laplace-Beltrami operator of $2d$ Liouville quantum gravity.

As the LBM is Markovian, we can consider the Liouville semi-group $(P_t^X)_t$, i.e. the transition probabilities of the LBM. From Theorem 1.2, it is obvious to check that the Liouville semi-group is the limit as $n \to \infty$ of the semi-group of the n-regularized LBM.

In forthcoming works, we will address the question of establishing the main properties of the Liouville semi-group. In particular, we aim at establishing that the Liouville semi-group is strong Feller in order to prove the following theorem, which entails the existence of the *Liouville heat kernel*:

Conjecture 3 (Liouville heat kernel)**.** *The Liouville semi-group $(P_t^X)_t$ is absolutely continuous with respect to the Liouville measure and can therefore be written as*

$$P_{\mathbf{t}}^X f(x) = \int_{\mathbb{R}^2} f(y) p^X(x, y, \mathbf{t})\, M(dy)$$

for continuous functions f vanishing at infinity. The family $(p^X(\cdot, \cdot, \mathbf{t}))_{\mathbf{t}}$ will be called Liouville heat kernel.

We will try to extend these properties until the threshold $\gamma^2 < 4$.

We point out that, in the physics literature, we have tracked down the notions of heat kernel or Laplace-Beltrami operator of $2d$-Liouville quantum gravity at least in

[4, 16] (quoting all physics references is beyond the scope of this paper and certainly beyond our skills too) and this paper is mostly inspired by [16]. It may be worth pointing out that our methods allow us to give sense to Feynman path integrals of the type

$$\int_{C([0,T];\mathbb{R}^2)} f(\sigma) \exp\Big(-\frac{1}{2}\int_0^T e^{\gamma X(\sigma(s))-\frac{\gamma^2}{2}\mathbb{E}[X^2]}|\sigma'(s)|^2\,ds\Big)\mathcal{D}\sigma,$$

or

$$\int_{C([0,T];\mathbb{R}^2)} f(\sigma) \exp\Big(-\frac{1}{2}\int_0^T |\sigma'(s)|^2 + \mu e^{\gamma X(\sigma(s))-\frac{\gamma^2}{2}\mathbb{E}[X^2]}\,ds\Big)\mathcal{D}\sigma$$

appearing throughout the physics literature.

Finally, we stress that we are convinced that our approach opens many doors on this topic and, actually, raises many more questions than we can possibly address, at least in this paper. So, a whole section 4 is devoted to describing several related questions, with various ambition level. In particular, in conjecture 2 and along subsection 4.4, we suggest a construction of the Liouville metric via the Dirichlet form associated to the Liouville Brownian motion.

Index of notations

- Liouville Brownian motion: $(\mathcal{B}_{\mathbf{t}})_{\mathbf{t}\geqslant 0}$
- Classical time: t v.s. Quantum time: $\mathbf{t}$

  We will thus distinguish the (quantum) time $\mathbf{t}$ along $\mathcal{B}_{\mathbf{t}}$ and the (classical) time along B_t.
- (massive) Gaussian Free Field: X.
- Liouville measure: $M = M_\gamma$.
- Space of continuous functions with compact support in a domain D: $C_c(D)$,
- Space of continuous functions vanishing at infinity on $\mathbb{R}^2$: $C_0(\mathbb{R}^2)$,
- Space of continuous bounded functions on D: $C_b(D)$,
- Space of continuous functions on $\mathbb{R}_+$: $C(\mathbb{R}_+)$ equipped with the sup-norm topology over compact sets.
- Laplace-Beltrami operator on a manifold: Δ

In what follows, we will consider Brownian motions B or $\bar{B}$ on $\mathbb{R}^2$ or the sphere $\mathbb{S}^2$ independent of the underlying Free Field. We will denote by $\mathbb{E}^Y$ or $\mathbb{P}^Y$ expectations and probability with respect to a field Y. For instance, $\mathbb{E}^X$ or $\mathbb{P}^X$ (resp. $\mathbb{E}^B$ or $\mathbb{P}^B$) stand for expectation and probability with respect to the (M)GFF (resp. the Brownian motion B).

2 Liouville Brownian motion on the plane

In this section, we set out to construct the (massive) Liouville Brownian motion on the whole plane. As explained in introduction, we need to introduce an infrared regulator to get a well defined Free Field on the whole plane: put in other words, the massless GFF cannot be defined on the whole plane so that we have to apply a large scale cutoff. There are many ways of applying a large scale cutoff. Regarding physics literature, a natural way to do this is to consider a whole plane Massive Gaussian Free Field (MFF for short). So we first remind the reader of the construction of the MFF.

2.1 Massive Gaussian Free Field on the plane

We consider a whole plane Massive Gaussian Free Field (MFF) (see [33, 59] for an overview of the construction of the MFF and applications). It is a standard Gaussian in the Hilbert space defined as the closure of Schwartz functions over $\mathbb{R}^2$ with respect to the inner product

$$(f,g)_m = m^2(f,g)_2 - (f, \triangle g)_2,$$

where $(\cdot,\cdot)_2$ is the standard inner product on $L^2(\mathbb{R}^2)$. The real $m > 0$ is called the mass. Its action on $L^2(\mathbb{R}^2)$ can be seen as a Gaussian distribution with covariance kernel given by the Green function G_m of the operator $m^2 - \triangle$, i.e.:

$$(m^2 - \triangle)G_m(x,\cdot) = 2\pi\delta_x.$$

The main differences with the GFF are that the MGFF can perfectly be defined on the whole plane since the massive term makes coercive the associated Dirichlet form $(\cdot,\cdot)_h$ on the plane: the mass term acts as a long-scale cutoff (or infrared cutoff/regulator). Furthermore, the MGFF does not possess conformal invariance properties.

It is a standard fact that the massive Green function can be written as an integral of the transition densities of the Brownian motion weighted by the exponential of the mass:

$$\forall x, y \in \mathbb{R}^2, \quad G_m(x,y) = \int_0^\infty e^{-\frac{m^2}{2}u - \frac{|x-y|^2}{2u}} \frac{du}{2u}. \tag{2.1}$$

Clearly, it is a kernel of σ-positive type in the sense of Kahane [36] since we integrate a continuous function of positive type with respect to a positive measure. It is furthermore a star-scale invariant kernel [2]: it can be rewritten as

$$G_m(x,y) = \int_1^{+\infty} \frac{k_m(u(x-y))}{u}\, du. \tag{2.2}$$

for some continuous covariance kernel k_m:

$$k_m(z) = \frac{1}{2}\int_0^\infty e^{-\frac{m^2}{2v}|z|^2 - \frac{v}{2}}\, dv.$$

One can check that

$$G_m(x,y) = \ln_+ \frac{1}{|x-y|} + g_m(x,y)$$

for some continuous and bounded function g_m, which decays exponentially fast to 0 when $|x-y| \to \infty$.

Let us consider an unbounded increasing sequence $(c_n)_{n \geqslant 1}$ such that $c_1 = 1$. For each $n \geqslant 1$, we consider a centered smooth Gaussian process Y_n with covariance kernel given by

$$\mathbb{E}[Y_n(x)Y_n(y)] = \int_{c_n}^{c_{n+1}} \frac{k_m(u(x-y))}{u}\, du.$$

The MGFF is the Gaussian distribution defined by

$$X(x) = \sum_{n \geqslant 1} Y_n(x)$$

where the processes $(Y_n)_n$ are assumed to be independent. We define the n-regularized field by

$$X_n(x) = \sum_{k=1}^{n} Y_k(x). \tag{2.3}$$

Actually, based on Kahane's theory of multiplicative chaos [36], the choice of the decomposition 2.3 will not play a part in the forthcoming results, excepted that it is important that the covariance kernel of X_n be smooth in order to associate to this field a Riemannian geometry.

2.2 n-regularized Riemannian geometry

We can consider a Riemannian metric tensor on $\mathbb{R}^2$:

$$g_n(x)(dx_1, dx_2) = e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}(dx_1^2 + dx_2^2).$$

The factor $\gamma \geqslant 0$ is a parameter. The renormalization term $e^{\frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}$ appears for future renormalization purposes but does not play a role now in the geometry associated with this metric. Since the Riemannian manifold $(\mathbb{R}^2, g_n)$ is connected, it carries the structure of distance. More precisely, we denote by $DC(\mathbb{R}^2)$ the family of all parameterized differentiable curves $\sigma : [a,b] \to \mathbb{R}^2$. Given $\sigma \in DC(\mathbb{R}^2)$, the length of σ is defined by

$$L_n(\sigma) = \int_a^b \sqrt{g_n(\sigma_t)(\sigma_t', \sigma_t')}\, dt.$$

This definition is independent of the parameterization. In particular, the curve can be parameterized by arclength, that is $g_n(\sigma_t)(\sigma_t', \sigma_t') = 1$ for all $t \in [a,b]$. The distance $d_n : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty[$ is defined by

$$d_n(x,y) = \inf\left\{L_n(\sigma); \sigma \in DC(\mathbb{R}^2), \sigma : x \to y\right\}. \tag{2.4}$$

The topology induced by d_n on $\mathbb{R}^2$ coincides with the Euclidean topology. In particular, $(\mathbb{R}^2, d_n)$ is complete, which implies that the Riemannian manifold $(\mathbb{R}^2, g_n)$ is geodesically complete (by the Hopf-Rinow theorem).

The Riemannian volume on the manifold $(\mathbb{R}^2, g_n)$ is given by:

$$M_n(dx) = e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\, dx$$

and will be called n-regularized Liouville measure. Classical theory of Gaussian multiplicative chaos ensures that, almost surely in X, the family $(M_n)_n$ weakly converges towards a limiting Radon measure M, which is called the Liouville measure. The limiting measure is non trivial if and only if $\gamma \in [0, 2)$. Concerning the theory of Gaussian multiplicative chaos, the reader is referred to Kahane's original paper [36] (or [2] as the MGFF is star scale invariant). A few results of the theory are also gathered in Appendix A. We will denote by ξ_M the power law spectrum of M (see [2, 56] for instance):

$$\forall p \geqslant 0, \quad \xi_M(p) = (2 + \frac{\gamma^2}{2})p - \frac{\gamma^2}{2}p^2.$$

2.3 Definition of the n-regularized Brownian motion

One can associate to the Riemannian manifold $(\mathbb{R}^2, g_n)$ a Brownian motion $\mathcal{B}^n$. It is an intrinsic stochastic tool describing the geometry of the manifold:

Definition 2.1 (n-regularized Liouville Brownian motion)**.** *For any $n \geqslant 1$ fixed, we define the following diffusion on $\mathbb{R}^2$. For any $x \in \mathbb{R}^2$,*

$$\begin{cases} \mathcal{B}^{n,x}_{\mathbf{t}=0} = x \\ d\mathcal{B}^{n,x}_{\mathbf{t}} = e^{-\frac{\gamma}{2}X_n(\mathcal{B}^{n,x}_{\mathbf{t}}) + \frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}^{n,x}_{\mathbf{t}})^2]}\, d\bar{B}_{\mathbf{t}} \end{cases} \tag{2.5}$$

where $\bar{B}_t$ is a standard two-dimensional Brownian motion. Equivalently,

$$\mathcal{B}^{n,x}_{\mathbf{t}} = x + \int_0^{\mathbf{t}} e^{-\frac{\gamma}{2}X_n(\mathcal{B}^{n,x}_{\mathbf{u}}) + \frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}^{n,x}_{\mathbf{u}})^2]}\, d\bar{B}_{\mathbf{u}}\,. \tag{2.6}$$

By using the Dambis-Schwarz Theorem, one can define the n-regularized Liouville Brownian motion as follows.

Definition 2.2. *For any $n \geqslant 1$ fixed and $x \in \mathbb{R}^2$,*

$$\mathcal{B}^{n,x}_{\mathbf{t}} = x + B_{\langle \mathcal{B}^{n,x} \rangle_{\mathbf{t}}}\,, \tag{2.7}$$

where $(B_r)_{r \geqslant 0}$ is a two-dimensional Brownian motion independent of the Massive Free Field X and where the **quadratic variation** *$\langle \mathcal{B}^{n,x} \rangle$ of $\mathcal{B}^{n,x}$ is defined as follows:*

$$\langle \mathcal{B}^{n,x} \rangle_{\mathbf{t}} := \inf\{s \geqslant 0 \,:\, \int_0^s e^{\gamma X_n(x+B_u) - \frac{\gamma^2}{2}\mathbb{E}\left[X_n(x+B_u)^2\right]}\, du \geqslant \mathbf{t}\}\,. \tag{2.8}$$

Remark 2.3. *Note that in the two above equivalent definitions of the n-regularized LBM, the "driving" Brownian motions $\bar{B}_t$ and B_t are both independent of the Gaussian Free field X. Nevertheless, it is not correct that $(X, \bar{B}_t, B_t)$ are mutually independent. There is some dependency between (B_t) and $(\bar{B}_t)$ which depends on the field X and the value of $n \geqslant 1$. We shall see later that as the regularization $n \to \infty$, these two Brownian motions are asymptotically independent.*

Observe that (2.8) amounts to saying that the increasing process $\langle \mathcal{B}^{n,x} \rangle : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the differential equation:

$$\langle \mathcal{B}^{n,x} \rangle_{\mathbf{t}} = \int_0^{\mathbf{t}} e^{-\gamma X_n(x + B_{\langle \mathcal{B}^{n,x} \rangle_u}) + \frac{\gamma^2}{2} \mathbb{E}[X_n(x + B_{\langle \mathcal{B}^{n,x} \rangle_u})^2]} \, du. \tag{2.9}$$

We stress that $(\mathbb{R}^2, g_n)$ is a stochastically complete manifold. Mathematically, this means that

$$\forall \mathbf{t} \geqslant 0, \quad \mathbb{P}^B[\mathcal{B}^{n,x}_{\mathbf{t}} \in \mathbb{R}^2] = 1,$$

or equivalently that the Brownian motion $\mathcal{B}^n$ runs for all time (no killing effect).

Several standard facts can be deduced from the smoothness of X_n:

Proposition 2.4. *Let $n \geqslant 1$ be fixed. Clearly, since $x \in \mathbb{R}^2 \mapsto X_n(x)$ is a.s. (in X) a smooth function, the above n-regularized Liouville Brownian motion $\mathcal{B}^n$ a.s. induces a Feller diffusion on $\mathbb{R}^2$. Let us denote by $(P^n_{\mathbf{t}})_{\mathbf{t} \geqslant 0}$ its semi-group. Also, $\mathcal{B}^n$ is reversible with respect to the Riemannian volume M_n, which is therefore invariant for $\mathcal{B}^n$.*

Definition 2.5 (transition kernel)**.** *One has the existence of transition kernels $p^n(x, y, \mathbf{t})$ so that for any $f \in C_c(\mathbb{R}^2)$ and any $x \in \mathbb{R}^2$,*

$$P^n_{\mathbf{t}} f(x) = \int_{y \in \mathbb{R}^2} f(y) p^n(x, y, \mathbf{t}) dM_n(y) \, , \tag{2.10}$$

The transition kernel p^n is the minimal fundamental solution of

$$\partial_t p^n(t, x, \cdot) = \frac{1}{2} e^{-X_n(x) + \frac{\gamma^2}{2} \mathbb{E}[X_n(x)^2]} \triangle p^n(t, x \cdot), \quad \lim_{t \to 0} p^n(t, x, y) M_n(dy) = \delta_x(dy).$$

It is known that the heat kernel $p^n :]0, +\infty[\times \mathbb{R}^2 \times \mathbb{R}_2 \to]0, +\infty[$ is a positive C^∞-function. This follows from Hörmander's criterion for hypoellipticity [34]. Positivity is established in [1].

2.4 Convergence of the quadratic variations when starting from a given fixed point

In order to study the behavior of the quadratic variation $\langle \mathcal{B}^n \rangle_t$, it will be useful to define the following quantities:

Definition 2.6. *Let F^n be the following random function on $\mathbb{R}^2 \times \mathbb{R}_+$:*

$$F^n(x,s) = \int_0^s e^{\gamma X_n(x+B_u) - \frac{\gamma^2}{2}\mathbb{E}\left[X_n(x+B_u)^2\right]}\, du. \tag{2.11}$$

The interest of such quantity lies in the relation defining $\langle \mathcal{B}^{n,x} \rangle$, or equivalently by solving equation (2.9).

Lemma 2.7. *Fix $x \in \mathbb{R}^2$. The process $\langle \mathcal{B}^{n,x} \rangle$ is entirely characterized by:*

$$\int_0^{\langle \mathcal{B}^{n,x} \rangle_{\mathbf{t}}} e^{\gamma X_n(x+B_u) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x+B_u)^2]}\, du = \mathbf{t}.$$

Proof . By differentiating with respect to $\mathbf{t}$ the equation (2.9), we get:

$$\langle \mathcal{B}^{n,x} \rangle'_{\mathbf{t}} = e^{-\gamma X_n(B_{\langle \mathcal{B}^{n,x} \rangle_t}) + \frac{\gamma^2}{2}\mathbb{E}[X_n(B_{\langle \mathcal{B}^{n,x} \rangle_t})^2]}.$$

It is plain to deduce that:

$$\int_0^{\langle \mathcal{B}^{n,x} \rangle_{\mathbf{t}}} e^{\gamma X_n(x+B_u) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x+B_u)^2]}\, du = \mathbf{t}.$$

□

We first focus on the convergence of the quadratic variations when we consider only one arbitrary starting point.

Theorem 2.8. *Assume $\gamma^2 < 4$ and fix $x \in \mathbb{R}^2$. Almost surely in X and in B, the mapping*

$$F^n(x,\cdot) : \mathbf{t} \mapsto \int_0^{\mathbf{t}} e^{\gamma X_n(x+B_r) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x+B_r)^2]}\, dr$$

converges in the space $C(\mathbb{R}_+)$ towards a continuous increasing mapping $F(x,\cdot)$ on $[0,+\infty[$. Furthermore, a.s. in X and in B,

$$\lim_{t\to\infty} F(x,t) = +\infty. \tag{2.12}$$

Proof of Theorem 2.8. It is enough to prove the convergence on each interval $[0,T]$ for $T > 0$. The quantity $F^n(x,t)$ converges almost surely as $n \to \infty$ because it is a nonnegative martingale with respect to the parameter n. Let us prove that it is uniformly integrable.

Let us denote by ν_t the occupation measure of the Brownian motion B between 0 and t. Recall that for each bounded continuous function $f : \mathbb{R}^2 \to \mathbb{R}$

$$\int_{\mathbb{R}^2} f(x)\, d\nu_t(dx) = \int_0^t f(B_r)\, dr.$$

From Theorem A.3, we just have to prove that almost surely in B the measure ν_t is in the class R_α for $\alpha < 2$. This is a standard simple result, which we recall here for the sake of completeness. Further details on the topic may be found in [17] for instance. To be in the class R_α, it is enough to prove that for any $\alpha < 2$, the integral

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha}\,\nu_t(dx)\,\nu_t(dy)$$

is almost surely finite. It is enough to prove

$$\mathbb{E}^B\Big[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha}\,\nu_t(dx)\,\nu_t(dy)\Big] < +\infty.$$

So, let us compute this expectation:

$$\begin{aligned}\mathbb{E}^B\Big[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha}\,\nu_t(dx)\,\nu_t(dy)\Big] &= \mathbb{E}^B\Big[\int_0^t\int_0^t \frac{1}{|B_r-B_s|^\alpha}\,dr\,ds\Big]\\ &= \mathbb{E}^B\Big[\frac{1}{|B_1|^\alpha}\Big]\int_0^t\int_0^t \frac{1}{|r-s|^{\alpha/2}}\,dr\,ds\\ &< +\infty.\end{aligned}$$

This shows that, almost surely with respect to B, the measure ν_B is in the class R_α for $\alpha < 2$. Theorem A.3 implies that, for each measurable set A with finite ν_t-measure, the quantity

$$\nu_t^n(x,A) = \int_A e^{\gamma X_n(x+z)-\frac{\gamma^2}{2}\mathbb{E}[X_n(x+z)^2]}\,\nu_t(dz)$$

is a uniformly integrable martingale with respect to the parameter n. Therefore it converges almost surely towards a non trivial limit.

Since $\mathbb{R}^2$ has a finite ν_t-measure, we deduce that

$$F_n(x,t) = \nu_t^n(x,\mathbb{R}^2) \to \widetilde{\nu}_t(x,\mathbb{R}^2) \quad \text{as } n\to\infty.$$

In particular, we deduce that for a countable family of couples (s,t) with $s<t$ such that the intervals $[s,t]$ generate the Borel topology on $\mathbb{R}_+$, the family $(F^n(x,t) - F^n(x,s))_n$ converges almost surely towards a non trivial limit. Therefore, almost surely in X, the family $(F^n(x,dr))_n$ of random measures on $\mathbb{R}_+$ weakly converges towards a limiting random measure $F(x,dr)$.

Let us prove that the measure $F(x,dr)$ has no atom and full support. Let us consider an interval I. Obviously, almost surely in B, the event $\{F(x,I)>0\}$ is an event in the asymptotic sigma algebra generated by the random processes $(Y_n)_n$ (involved in the construction of X). Therefore a.s. in B, it has $\mathbb{P}^X$-probability 0 or 1. Since we have $\mathbb{P}^B$ a.s.

$$\mathbb{E}^X[F(x,I)] = \widetilde{\nu}_t(x,\mathbb{R}^2) = t,$$

it has $\mathbb{P}^X$-probability 1. Then we can consider a countable family $(I_n)_n$ of intervals generating the Borel sigma algebra on $\mathbb{R}_+$. Almost surely in X and B, we have $F(x, I_n) > 0$ for all n. This shows that $F(x, dr)$ has full support, which equivalently means that the random mapping $t \mapsto F(x, t)$ is increasing, a.s. in X and B.

Let us prove that the measure $F(x, dr)$ has no atom, which equivalently means that the random mapping $t \mapsto F(x, t)$ is continuous, a.s. in X and B. It suffices to prove that it has no atom on each interval $[0, T]$ for each $T > 0$. Without loss of generality, we may assume that $T = 1$. Also, it suffices to prove that $F(x, dr)$ has no atom of size $\delta > 0$ for all $\delta > 0$. Observe that

$$\{F(x, \cdot) \text{ has an atom of size } \delta\} \subset \bigcap_{n \geqslant 1} \bigcup_{k=0}^{n-1} \big\{F(x, [\frac{k}{n}, \frac{k+1}{n}]) \geqslant \delta\big\}.$$

Using the Markov inequality, it is enough to show:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \mathbb{E}^B \mathbb{E}^X \Big(F(x, [\frac{k}{n}, \frac{k+1}{n}])^q \Big) = 0$$

for some $q > 0$. Let us admit for a while that Proposition 2.11 below is true. Then, it suffices to choose any $q \in]1, 4/\gamma^2[$ because $\xi(q) > 1$ for such a q.

Now that we have proved that the mapping $F(x, \dot{})$ is continuous and increasing, the Dini theorem ensures that $(F^n(x, \cdot))_n$ converges in $C(\mathbb{R}_+)$ equipped with the sup-norm topology over compact sets towards $F(x, \cdot)$.

Finally, we prove that

$$\lim_{t\to\infty} F(x, t) = +\infty.$$

We give two proofs of this fact. A first simple proof in the case $0 \leqslant \gamma^2 < 2$ and a more elaborate proof to treat the general case $0 \leqslant \gamma^2 < 4$. Furthermore, we stress that the $0 \leqslant \gamma^2 < 2$ proof yields a stronger result on the asymptotic behaviour of F so that, in a way, the two proofs do not overlap.

Proof 1. (simple case) We will use the weak law of large numbers (WLLN) for covariance stationary sequences of random variables. Let us set

$$W_n = F(x,]n, n+1]).$$

We have $\mathbb{E}^X \mathbb{E}^B[W_n] = 1$, and

$$\begin{aligned} \mathbb{E}^X \mathbb{E}^B[W_0 W_k] &= \mathbb{E}^B \int_0^1 \int_k^{k+1} e^{\gamma^2 G_m(B_r, B_u)}\, dr\, du \\ &= \int_0^1 \int_k^{k+1} \mathbb{E}^B \big[e^{\gamma^2 G_m(B_r, B_u)}\big]\, dr\, du. \end{aligned}$$

To apply the WLLN theorem, we must check that

$$\frac{1}{n} \sum_{k=0}^{n} \mathbb{E}^X \mathbb{E}^B[W_0 W_k] \to 0, \quad \text{as } n \to \infty. \tag{2.13}$$

We have (for some constant C independent of n, which may vary along lines):

$$\begin{aligned}\sum_{k=0}^{n}\mathbb{E}^X\mathbb{E}^B[W_0W_k] &\leqslant C\int_0^1\int_0^{n+1}\mathbb{E}^B\Big[\frac{1}{|B_r-B_u|^{\gamma^2}}\Big]\,drdu\\ &= C\int_0^1\int_0^{n+1}\mathbb{E}^B\Big[\frac{1}{|B_1|^{\gamma^2}}\Big]\frac{1}{|r-u|^{\frac{\gamma^2}{2}}}\,drdu\\ &= C\sum_{k=1}^{n}\frac{1}{k^{\gamma^2/2}}.\end{aligned}$$

It is plain to deduce that criterion (2.13) holds. We deduce that in $\mathbb{P}^X\otimes\mathbb{P}^B$-probability

$$\lim_{n\to\infty}\frac{1}{n}F(x,n)=1.$$

Since $F(x,\cdot)$ is increasing, it is plain to deduce that its limit as $t\to\infty$ is ∞.

Proof 2. We consider the following sequence of stopping times associated to the Brownian motion:

$$T_n=\inf\{t\geqslant 0,\ |B_t|=n\},\quad \bar{T}_n=\inf\{t\geqslant T_n,\ |B_t-B_{T_n}|=\frac{1}{4},\ \}$$

We consider a subsequence $(n_j)_{j\geqslant 1}$ such that the following property holds for all $l\leqslant k$:

$$\sum_{l\leqslant j,j'\leqslant k}\alpha_{j,j'}\leqslant k-l$$

where $\alpha_{j,j'}=\sup_{|x|\leqslant n_j,|y|\geqslant n_{j'}}G_m(x,y)$

We get:

$$\begin{aligned}&\mathbb{P}^B\mathbb{P}^X(\cap_{l\leqslant j\leqslant k}(F(x,]T_{n_j},\bar{T}_{n_j}])\leqslant c))\\ &\leqslant c^{k-l}\mathbb{E}^X\mathbb{E}^B\big[\Pi_{l\leqslant j\leqslant k}\frac{1}{F(x,]T_{n_j},\bar{T}_{n_j}])}\big]\\ &= c^{k-l}\mathbb{E}^X\mathbb{E}^B\Big[\frac{1}{\int_{]T_{n_l},\bar{T}_{n_l}]\times\cdots\times]T_{n_k},\bar{T}_{n_k}]}e^{\gamma(X(B_{s_l})+\cdots+X(B_{s_k}))-\frac{\gamma^2}{2}(\mathbb{E}[X(B_{s_l})^2]+\cdots+\mathbb{E}[X(B_{s_k})^2])}}\Big]\\ &\leqslant c^{k-l}\mathbb{E}^X\mathbb{E}^B\Big[\frac{1}{\int_{]T_{n_l},\bar{T}_{n_l}]\times\cdots\times]T_{n_k},\bar{T}_{n_k}]}e^{\gamma(X(B_{s_l})+\cdots+X(B_{s_k}))-\frac{\gamma^2}{2}\mathbb{E}[(X(B_{s_l})+\cdots+X(B_{s_k}))^2)]}}\Big]\end{aligned}$$

Now, if we introduce $\bar{X}$ the free field with a cutoff (say $\mathbb{E}[\bar{X}_x\bar{X}_y]=\ln_+\frac{1}{|y-x|}$),

then by Kahane's inequality, we get that (let Y be a standard Gaussian variable):

$$\begin{aligned}
&\mathbb{P}^B\mathbb{P}^X\big(\cap_{l\leqslant j\leqslant k}\{F(x,]T_{n_j},\bar{T}_{n_j}])\leqslant c\}\big)\\
&\leqslant c^{k-l}\mathbb{E}^X\mathbb{E}^B\Big[\frac{1}{\int_{]T_{n_l},\bar{T}_{n_l}]\times\cdots\times]T_{n_k},\bar{T}_{n_k}]}e^{\gamma(X(B_{s_l})+\cdots+X(B_{s_k}))-\frac{\gamma^2}{2}\mathbb{E}[(X(B_{s_l})+\cdots+X(B_{s_k}))^2]}}\Big]\\
&\leqslant c^{k-l}\mathbb{E}^{\bar{X}}\mathbb{E}^B\Big[\frac{1}{\int_{]T_{n_l},\bar{T}_{n_l}]\times\cdots\times]T_{n_k},\bar{T}_{n_k}]}e^{\gamma(\bar{X}(B_{s_l})+\cdots+\bar{X}(B_{s_k}))-\frac{\gamma^2}{2}\mathbb{E}[(\bar{X}(B_{s_l})+\cdots+\bar{X}(B_{s_k}))^2]}}\Big]\mathbb{E}[\frac{1}{e^{\gamma\sqrt{k-l}Y-\frac{\gamma^2}{2}(k-l)}}]\\
&\leqslant (ce^{\gamma^2})^{k-l}\mathbb{E}^{\bar{X}}\mathbb{E}^B\Big[\frac{1}{\int_{]T_{n_1},\bar{T}_{n_1}]}e^{\gamma\bar{X}(B_{s_1})-\frac{\gamma^2}{2}\mathbb{E}[(\bar{X}(B_{s_1})^2]}}\Big]^{k-l}.
\end{aligned}$$

Observe that the latter expectation is finite (see Lemma 2.12 below). One then chooses c such that

$$ce^{\gamma^2}\mathbb{E}^{\bar{X}}\mathbb{E}^B[\frac{1}{\int_{]T_{n_1},\bar{T}_{n_1}]}e^{\gamma\bar{X}(B_{s_1})-\frac{\gamma^2}{2}\mathbb{E}^{\bar{X}}[(\bar{X}(B_{s_1})^2]}}]<1.$$

By letting k go to infinity, we conclude that:

$$\mathbb{P}^B\mathbb{P}^X(\cap_{l\leqslant j\leqslant\infty}(F(x,]T_{n_j},\bar{T}_{n_j}])\leqslant c))=0.$$

Hence, we get that:

$$\mathbb{P}^B\mathbb{P}^X(\cap_l\cup_{l\leqslant j\leqslant\infty}(F(x,]T_{n_j},\bar{T}_{n_j}])>c))=1.$$

Since

$$\lim_{t\to\infty}F(x,t)\geqslant c\sum_{j\geqslant 1}1_{\{F(x,]T_{n_j},\bar{T}_{n_j}])>c\}},$$

the proof is complete. □

Remark 2.9. *Let us emphasize that Kahane's theory of Gaussian multiplicative chaos ensures that the law of the limiting mapping $F(x,\cdot)$ does not depend on the chosen regularization $(X_n)_n$ of X.*

Corollary 2.10. *Assume $\gamma^2<4$ and fix $x\in\mathbb{R}^2$. Almost surely in X and in B, the family $(B,\langle\mathcal{B}^{n,x}\rangle)_n$ converges in the space $C(\mathbb{R}_+,\mathbb{R}^2)\times C(\mathbb{R}_+,\mathbb{R}_+)$ equipped with the supremum norm on compact sets towards the couple $(B,\langle\mathcal{B}^x\rangle)$, characterized by:*

$$\forall\mathbf{t}\geqslant 0,\quad F(x,\langle\mathcal{B}^x\rangle_{\mathbf{t}})=\mathbf{t}.$$

As such, the mapping $\mathbf{t}\mapsto\langle\mathcal{B}^x\rangle_{\mathbf{t}}$ is continuous and increasing.

Proof of Corollary 2.10. Almost surely in X and B, the family $(F^n(x,\cdot))_n$ converges in $C(\mathbb{R}_+,\mathbb{R}_+)$ towards $F(x,\cdot)$. Since

$$F^n(x,\langle\mathcal{B}^{n,x}\rangle_t)=t,$$

and

$$\lim_{t\to\infty} F(x,t) = +\infty,$$

it is plain to deduce that the family $(\langle \mathcal{B}^{n,x}\rangle)_n$ converges in $C(\mathbb{R}_+,\mathbb{R}_+)$ towards a continuous increasing process characterized by:

$$F(x,\langle \mathcal{B}^x\rangle_{\mathbf{t}}) = \mathbf{t}. \quad \square$$

2.5 Study of the moments

Proposition 2.11. *If $\gamma^2 < 4$ and $x \in \mathbb{R}^2$, the mapping $F(x,\cdot)$ possesses moments of order $0 \leqslant q < \min(2, 4/\gamma^2)$.*

Furthermore, if F admits moments of order $q \geqslant 0$ then, for all $s \in [0,1]$ and $t \in [0,T]$:

$$\mathbb{E}^X\mathbb{E}^B[F(x,[t,t+s])^q] \leqslant C_q s^{\xi(q)},$$

where $C_q > 0$ (independent of x, T) and

$$\xi(q) = \big(1+\frac{\gamma^2}{4}\big)q - \frac{\gamma^2}{4}q^2.$$

Proof. For pedagogical purposes, we give here a short argument to prove the finiteness of the moments when $\gamma^2 < 2$. There is here no exception to the rule in multiplicative chaos that studying finiteness of the moments beyond the L^2 threshold is much more involved. So, the whole Appendix B is devoted to investigating the general case (in particular $2 \leqslant \gamma^2 < 4$).

Given a point $x \in \mathbb{R}^2$, we have:

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{E}^X\mathbb{E}^B[F^n(x,t)^2] \\
&\quad = \limsup_{n\to\infty} \mathbb{E}^X\mathbb{E}^B\Big[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\gamma X_n(x+u)+\gamma X_n(x+v)-\frac{\gamma^2}{2}\mathbb{E}[X_n(x+u)^2]-\frac{\gamma^2}{2}\mathbb{E}[X_n(x+v)^2]}\Big]\nu_t(du)\nu_t(dv) \\
&\quad = \mathbb{E}^B\Big[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\gamma^2 G_m(u-v)}\nu_t(du)\nu_t(dv)\Big] \\
&\quad = \mathbb{E}^B\Big[\int_0^t\int_0^t e^{\gamma^2 G_m(B_r-B_s)}dsdr\Big] \\
&\quad < +\infty
\end{aligned}$$

when $\gamma^2 < 2$.

Now we assume that $\gamma^2 < 4$ and that F possesses moments of order q. We prove the estimate concerning the power law spectrum. We first prove it when $t = 1$ and starting from $x \in \mathbb{R}^2$ and then we deduce the uniform estimate in t when starting

from x.

$$\begin{aligned} F(x,s) &= \int_0^s e^{\gamma X(x+B_r)-\frac{\gamma^2}{2}\mathbb{E}[X(x+B_r)^2]}\,dr \\ &= s\int_0^1 e^{\gamma X(x+B_{us})-\frac{\gamma^2}{2}\mathbb{E}[X(x+B_{us})^2]}\,du \\ &\overset{law}{=} s\int_0^1 e^{\gamma X(\sqrt{s}B_u)-\frac{\gamma^2}{2}\mathbb{E}[X(\sqrt{s}B_u)^2]}\,du. \end{aligned}$$

By taking the q-th power and expectation and by using star scale invariance of X, in particular

$$G_m(\sqrt{s}u,\sqrt{s}v) \leqslant \ln\frac{1}{\sqrt{s}} + G_m(u,v),$$

and Kahane's convexity inequalities (see Lemma A.4), we get

$$\begin{aligned} \mathbb{E}^X\mathbb{E}^B[F(x,s)^q] &\leqslant s^q\mathbb{E}^B\mathbb{E}^X\left[\left(e^{\gamma\Omega_{\sqrt{s}}-\frac{\gamma^2}{2}\mathbb{E}[\Omega^2_{\sqrt{s}}]}\int_0^1 e^{\gamma X(B_u)-\frac{\gamma^2}{2}\mathbb{E}[X(B_u)^2]}\,du\right)^q\right] \\ &= s^q\mathbb{E}^B\mathbb{E}^X\left[e^{q\gamma\Omega_{\sqrt{s}}-q\frac{\gamma^2}{2}\mathbb{E}[\Omega^2_{\sqrt{s}}]}\right]\mathbb{E}^B\mathbb{E}^X\left[\left(\int_0^1 e^{\gamma X(B_u)-\frac{\gamma^2}{2}\mathbb{E}[X(B_u)^2]}\,du\right)^q\right] \\ &= C_q s^{\xi(q)} \end{aligned}$$

where $\Omega_{\sqrt{s}}$ is a Gaussian random variable with mean 0 and variance $\ln\frac{1}{\sqrt{s}}$ and independent of $\int_0^t e^{\gamma X(B_u)-\frac{\gamma^2}{2}\mathbb{E}[X(B_u)^2]}\,du$, and $C_q = \mathbb{E}^B\mathbb{E}^X\left[\left(\int_0^1 e^{\gamma X(B_u)-\frac{\gamma^2}{2}\mathbb{E}[X(B_u)^2]}\,du\right)^q\right]$ is independent of s,x.

Now we treat the general case.

$$\begin{aligned} F(x,[t,t+s]) &= \int_t^{t+s} e^{\gamma X(x+B_r)-\frac{\gamma^2}{2}\mathbb{E}[X(x+B_r)^2]}\,dr \\ &= \int_0^s e^{\gamma X(x+B_t+\bar{B}_r)-\frac{\gamma^2}{2}\mathbb{E}[X(x+B_t+\bar{B}_r)^2]}\,dr \end{aligned}$$

where $\bar{B}_r = B_{r+t} - B_t$ for $r \geqslant 0$ is a Brownian motion starting from 0 and independent of $(x+B_u)_{u\leqslant t}$. We deduce:

$$\begin{aligned} &\mathbb{E}^X\mathbb{E}^B[F(x,[t,t+s])^q] \\ &\quad= \mathbb{E}^B\mathbb{E}^X\left[\left(\int_0^s e^{\gamma X(x+B_t+\bar{B}_r)-\frac{\gamma^2}{2}\mathbb{E}[X(x+B_t+\bar{B}_r)^2]}\,dr\right)^q\right] \\ &\quad= \mathbb{E}^B\mathbb{E}^X\left[\mathbb{E}^B\mathbb{E}^X\left[\left(\int_0^s e^{\gamma X(z+\bar{B}_r)-\frac{\gamma^2}{2}\mathbb{E}[X(z+\bar{B}_r)^2]}\,dr\right)^q | x+B_t=z\right]\right] \\ &\quad\leqslant C_q s^{\xi(q)}. \end{aligned}$$

The proof is complete. □

Now we investigate finiteness of moments of negative order:

Lemma 2.12. *Let us denote by T_r the first exit time of the Brownian motion B out of the disk of radius $r>0$ centered at x. For all $q>0$, there exists some constant $C>0$ (depending on q,r) such that:*

$$\sup_{n\geqslant 0}\mathbb{E}^X\mathbb{E}^B\Big[\Big(\frac{1}{\int_0^{T_r}e^{\gamma X_n(x+B_s)-\frac{\gamma^2}{2}\mathbb{E}\left[X_n(x+B_s)^2\right]}\,ds}\Big)^q\Big]\leqslant C\Big(\frac{1}{r}\Big)^{2q+\frac{q(1+q)}{2}\gamma^2}. \tag{2.14}$$

Proof. We assume $r<1$. Without loss of generality, we can take $x=0$ by stationarity of the field X. Furthermore, from Kahane's convexity inequalities, it suffices to prove the result for one log-correlated Gaussian field. Let us choose the exact scale invariant field $\bar{X}$ with covariance kernel given by:

$$\mathbb{E}[\bar{X}(x)\bar{X}(y)]=\ln_+\frac{1}{|x-y|}.$$

Let us also consider a white noise decomposition $(\bar{X}_\epsilon)_{\epsilon\in]0,1]}$ of $\bar{X}$ as constructed in [57]. In particular, the process $\epsilon\to\bar{X}_\epsilon$ has independent increments and $\bar{X}_{\epsilon,\epsilon'}:=\bar{X}_\epsilon-\bar{X}_{\epsilon'}$ has a correlation cutoff of length ϵ' (i.e. if the Euclidean distance between two sets A,B is greater than ϵ' then $(\bar{X}_{\epsilon,\epsilon'}(x))_{x\in A}$ and $(\bar{X}_{\epsilon,\epsilon'}(x))_{x\in B}$ are independent). The correlation structure of $(\bar{X}_\epsilon)_{\epsilon\in]0,1]}$ is given for $\epsilon\in]0,1]$ by:

$$\mathbb{E}[\bar{X}_\epsilon(x)\bar{X}_{\epsilon'}(y)]=\begin{cases}0 & \text{if } |x-y|>1\\ \ln\frac{1}{|x-y|} & \text{if } \epsilon\leqslant|x-y|\leqslant 1\\ \ln\frac{1}{\epsilon}+2(1-\frac{|x-y|^{1/2}}{\epsilon^{1/2}}) & \text{if } |y-x|\leqslant\epsilon\end{cases}.$$

Therefore, we have to prove

$$\sup_{\epsilon\in]0,1]}\mathbb{E}^{\bar{X}}\mathbb{E}^B\Big[\Big(\frac{1}{\int_0^{T_r}e^{\gamma \bar{X}_\epsilon(x+B_s)-\frac{\gamma^2}{2}\mathbb{E}\left[\bar{X}_\epsilon(x+B_s)^2\right]}\,ds}\Big)^q\Big]\leqslant C\Big(\frac{1}{r}\Big)^{2q+\frac{q(1+q)}{2}\gamma^2}. \tag{2.15}$$

Recall that the supremum is reached for $\epsilon\to 0$ by the martingale property. Now, if $\widetilde{T}_{\frac{1}{4}}$ is the first time $B_{t+T_{3/4}}-B_{T_{3/4}}$ hits the disk of radius $\frac{1}{4}$, we get:

$$\begin{aligned}&\int_0^{T_1}e^{\gamma\bar{X}_\varepsilon(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{\varepsilon}}\,ds\\ &\geqslant\int_0^{T_{\frac{1}{4}}}e^{\gamma\bar{X}_\varepsilon(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{\varepsilon}}\,ds+\int_0^{\widetilde{T}_{\frac{1}{4}}}e^{\gamma\bar{X}_\varepsilon(B_{s+T_{3/4}})-\frac{\gamma^2}{2}\ln\frac{1}{\varepsilon}}\,ds\\ &\geqslant e^{\gamma\inf_{|x|\leqslant 1}\bar{X}_{1/4}(x)-\frac{\gamma^2}{2}\ln 4}\left(\int_0^{T_{\frac{1}{4}}}e^{\gamma\bar{X}_{\varepsilon,1/4}(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\,ds+\int_0^{\widetilde{T}_{\frac{1}{4}}}e^{\gamma\bar{X}_{\varepsilon,1/4}(B_{s+T_{3/4}})-\frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\,ds\right)\end{aligned}$$

The main observation is that, under the annealed measure $\mathbb{E}^{\bar{X}}\mathbb{E}^{B}$, the above two integrals are independent random variables. Indeed, we get for two functionals F, G:

$$\begin{aligned}
&\mathbb{E}^{\bar{X}}\mathbb{E}^{B}\Big[F\left(\int_0^{T_{\frac{1}{4}}} e^{\gamma\bar{X}_{\varepsilon,1/4}(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\,ds\right)G\left(\int_0^{\widetilde{T}_{\frac{1}{4}}} e^{\gamma\bar{X}_{\varepsilon,1/4}(B_{s+T_{3/4}})-\frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\,ds\right)\Big]\\
&=\mathbb{E}^{B}\Big[\mathbb{E}^{\bar{X}}[F\left(\int_0^{T_{\frac{1}{4}}} e^{\gamma\bar{X}_{\varepsilon,1/4}(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\,ds\right)]\mathbb{E}^{\bar{X}}[G\left(\int_0^{\widetilde{T}_{\frac{1}{4}}} e^{\gamma\bar{X}_{\varepsilon,1/4}(B_{s+T_{3/4}})-\frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\,ds\right)]\Big]\\
&=\mathbb{E}^{B}\Big[\mathbb{E}^{\bar{X}}[F\left(\int_0^{T_{\frac{1}{4}}} e^{\gamma\bar{X}_{\varepsilon,1/4}(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\,ds\right)]\mathbb{E}^{\bar{X}}[G\left(\int_0^{\widetilde{T}_{\frac{1}{4}}} e^{\gamma\bar{X}_{\varepsilon,1/4}(B_{s+T_{3/4}}-B_{T_{3/4}})-\frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\,ds\right)]\Big]\\
&=\mathbb{E}^{\bar{X}}\mathbb{E}^{B}\Big[F\left(\int_0^{T_{\frac{1}{4}}} e^{\gamma\bar{X}_{\varepsilon,1/4}(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\,ds\right)\Big]\mathbb{E}^{\bar{X}}\mathbb{E}^{B}\Big[G\left(\int_0^{\widetilde{T}_{\frac{1}{4}}} e^{\gamma\bar{X}_{\varepsilon,1/4}(B_{s+T_{3/4}}-B_{T_{3/4}})-\frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\,ds\right)\Big]\\
&=\mathbb{E}^{\bar{X}}\mathbb{E}^{B}\Big[F\left(\int_0^{T_{\frac{1}{4}}} e^{\gamma\bar{X}_{\varepsilon,1/4}(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\,ds\right)\Big]\mathbb{E}^{\bar{X}}\mathbb{E}^{B}\Big[G\left(\int_0^{\widetilde{T}_{\frac{1}{4}}} e^{\gamma\bar{X}_{\varepsilon,1/4}(B_{s+T_{3/4}})-\frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\,ds\right)\Big],
\end{aligned}$$

where we have used the fact that $\bar{X}_{\varepsilon,1/4}$ has a correlation cutoff of length $1/4$ for the first equality and the fact that the field $\bar{X}_{\varepsilon,1/4}$ is stationary for the second equality.

For all $r\in]0,1]$, we have:

$$\begin{aligned}
\int_0^{T_r} e^{\gamma\bar{X}_{r\varepsilon}(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{r\varepsilon}}\,ds &= r^2\int_0^{\frac{T_r}{r^2}} e^{\gamma\bar{X}_{r\varepsilon}(r\frac{B_{r^2s'}}{r})-\frac{\gamma^2}{2}\ln\frac{1}{r\varepsilon}}\,ds'\\
&= r^2\int_0^{\widetilde{T}_1} e^{\gamma\bar{X}_{r\varepsilon}(r\widetilde{B}_{s'})-\frac{\gamma^2}{2}\ln\frac{1}{r\varepsilon}}\,ds'
\end{aligned}$$

where $\widetilde{B}_{s'}=\frac{B_{r^2s'}}{r}$ is a Brownian motion and $\widetilde{T}_1=\frac{T_r}{r^2}$ is the first time it hits the disk of radius 1. Therefore, we get the following scaling relation in distribution for all $r\in]0,1]$ under the annealed measure:

$$\int_0^{T_r} e^{\gamma\bar{X}_{r\varepsilon}(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{r\varepsilon}}\,ds \overset{(law)}{=} r^2 e^{\gamma\Omega_r-\frac{\gamma^2}{2}\ln\frac{1}{r}}\int_0^{T_1} e^{\gamma\bar{X}_{\varepsilon}(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{\varepsilon}}\,ds \tag{2.16}$$

where Ω_r is independent from B, $(\bar{X}_\epsilon)_\epsilon$ and with distribution $N(0,\ln\frac{1}{r})$. From this scaling relation and the above considerations, we deduce that we can find some variable N with negative moments and such that we have the following stochastic domination:

$$Y\geqslant N(Y_1+Y_2)$$

where (Y_1,Y_2) are i.i.d. of distribution Y and independent from N where Y is distributed like $\lim_{\varepsilon\to 0}\int_0^{T_1} e^{\gamma\bar{X}_{\varepsilon}(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{\varepsilon}}\,ds$. Then we get from adapting [51] that:

$$\sup_{\varepsilon>0}\mathbb{E}^{\bar{X}}\mathbb{E}^{B}\Big[\Big(\frac{1}{\int_0^{T_1} e^{\gamma\bar{X}_{\varepsilon}(B_s)-\frac{\gamma^2}{2}\ln\frac{1}{\varepsilon}}\,ds}\Big)^q\Big]<\infty$$

One then deduces inequality (2.15) from (2.16).

2.6 Convergence of the quadratic variations for all points

Now we want to define almost surely in X the change of times $F(y,\cdot)$ for all starting points $y \in \mathbb{R}^2$. The task is not obvious because most of the desired properties of $F(y,\cdot)$ can be established "almost surely" for a given fixed point. Therefore, apart from the obvious situation when one considers a countable quantity of points $y \in \mathbb{R}^2$, the properties of $F(y,\cdot)$ cannot be assumed to hold simultaneously for an uncountable quantity of points $y \in \mathbb{R}^2$. We briefly draw below the picture of our strategy:

1. First we prove that almost surely in X, under $\mathbb{P}^B$ the sequence $(F^n(y,\cdot))$ is tight in $C(\mathbb{R}_+)$ simultaneously for all possible starting points $y \in \mathbb{R}^2$,
2. From Theorem 2.8, we further know that it converges for a countably dense sequence of points of $\mathbb{R}^2$.
3. We prove some some uniform continuity estimates with respect to y and we deduce that the limit $F(y,\cdot)$ is continuous with respect to y (in some sense that we will make precise later).
4. Finally, we deduce that its inverse $\langle \mathcal{B}^y \rangle$ is also continuous w.r.t. y.

Now we come down into details. In what follows, we will assume that, almost surely in X, under $\mathbb{P}^B$ the sequence $(F^n(y,\cdot))$ converges in $C(\mathbb{R}_+)$ for all possible rational starting points $y \in \mathbb{Q}^2$.

In what follows, if B is a Brownian motion on $\mathbb{R}^2$, we will denote by B^1 and B^2 its components. We will further use throughout this subsection, the following coupling lemma, the proof of which is standard and left to the reader.

Lemma 2.13. *Fix $y_0 \in \mathbb{R}^2$ and let us start a Brownian motion B^{y_0} from y_0. Let us consider another independent Brownian motion B starting from 0 and denote by B^y, for a rational $y \in \mathbb{R}^2$, the Brownian motion $B^y = y + B$. Let us denote by τ_1^y the first time at which the first components of B^{y_0} and B^y coincide:*

$$\tau_1^y = \inf\{u > 0; B_u^{1,y_0} = B_u^{1,y}\}$$

and by τ_2^y the first time at which the second components coincide after τ_1^y:

$$\tau_2^y = \inf\{u > \tau_1; B_u^{2,y_0} = B_u^{2,y}\}$$

The random process $\overline{B}^{y,y_0}$ defined by

$$\overline{B}_t^{y,y_0} = \begin{cases} (B_t^{1,y_0}, B_t^{2,y_0}) & \text{if } \ t \leqslant \tau_1 \\ (B_t^{1,y}, B_t^{2,y_0}) & \text{if } \ \tau_1 < t \leqslant \tau_2 \\ (B_t^{1,y}, B_t^{2,y}) & \text{if } \ \tau_2 < t. \end{cases}$$

is a new Brownian motion on $\mathbb{R}^2$ *starting from* y_0*, and coincides with* B^y *for all times* $t > \tau_2^y$*. Furthermore, as* $y \to y_0$*, we have for all* $\eta > 0$*:*

$$\mathbb{P}(\tau_2^y > \eta) \to 0.$$

Now we claim:

Proposition 2.14. *Almost surely in* X*, for all* $y_0 \in \mathbb{R}^2$*, under* $\mathbb{P}^B$*, the family* $(F^n(y_0, \cdot))_n$ *is tight in* $C(\mathbb{R}_+)$*.*

Proof. Let us admit for a while the three following lemmas:

Lemma 2.15. *Almost surely in* X*, for all* $y_0 \in \mathbb{R}^2$*, for each fixed* $T > 0$*:*

$$\limsup_{R\to\infty} \limsup_{n\to\infty} \mathbb{P}^B\Big(F^n(y_0, T) \geqslant R\Big) = 0. \tag{2.17}$$

Lemma 2.16. *Almost surely in* X*, for all* $y_0 \in \mathbb{R}^2$*, for each fixed* $0 < T' < T$*:*

$$\forall \eta > 0, \quad \limsup_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}^B\Big(\sup_{\substack{T' \leqslant s,t \leqslant T \\ |t-s| \leqslant \delta}} |F^n(y_0, t) - F^n(y_0, s)| \geqslant \eta\Big) = 0. \tag{2.18}$$

Lemma 2.17. *Almost surely in* X*, for all* $y_0 \in \mathbb{R}^2$*,*

$$\lim_{t\to 0} \int_{\mathbb{R}^2} \int_0^t e^{-\frac{|u-y_0|^2}{2s}} \frac{ds}{2\pi s} M(du) = 0. \tag{2.19}$$

By combining Lemma 2.15 and Lemma 2.16, we deduce that, almost surely in X, for all $y_0 \in \mathbb{R}^2$, the sequence $(F^n(y_0, \cdot))_n$ is tight in $C(]0, +\infty[)$ equipped with the sup norm topology over compact sets. To prove tightness in $C(\mathbb{R}_+)$, it remains to prove that every possible limit $F(y_0, \cdot)$ of a converging subsequence satisfies

$$\lim_{t\to 0} F(y_0, t) = 0.$$

Because F is nondecreasing and nonnegative, it suffices to prove that $F(y_0, t)$ converges in probability towards 0 as $t \to 0$. Observe that

$$\begin{aligned}
\mathbb{P}^B\big(F(y_0, t) \geqslant \eta\big) &= \lim_{n\to\infty} \mathbb{P}^B\big(F^n(y_0, t) \geqslant \eta\big) \\
&\leqslant \lim_{n\to\infty} \frac{1}{\eta} \mathbb{E}[F^n(y_0, t)] \\
&= \lim_{n\to\infty} \frac{1}{\eta} \int_{\mathbb{R}^2} \int_0^t e^{-\frac{|u-y_0|^2}{2s}} \frac{ds}{2\pi s} M_n(du) \\
&= \frac{1}{\eta} \int_{\mathbb{R}^2} \int_0^t e^{-\frac{|u-y_0|^2}{2s}} \frac{ds}{2\pi s} M(du).
\end{aligned}$$

We complete the proof of Proposition 2.14 with the help of Lemma 2.17. □

*Proof of Lemma 2.16.*We use the coupling procedure of Lemma 2.13, consider the law of $F^n(y_0,\cdot)$ constructed with the Brownian motion $\overline{B}^{y,y_0}$ and find $y\in\mathbb{Q}^2$ such that $\mathbb{P}^B(\tau_2^y\geqslant T')\leqslant\epsilon$. We have:

$$\begin{aligned}
&\mathbb{P}^{\overline{B}^{y,y_0}}\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F^n(y_0,t)-F^n(y_0,s)|\geqslant\eta\Big)\\
&\quad\leqslant\mathbb{P}^{\overline{B}^{y,y_0}}\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F^n(y_0,t)-F^n(y_0,s)|\geqslant\eta;\tau_2^y\geqslant T'\Big)\\
&\qquad+\mathbb{P}^{\overline{B}^{y,y_0}}\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F^n(y_0,t)-F^n(y_0,s)|\geqslant\eta;\tau_2^y< T'\Big)\\
&\quad\leqslant\epsilon+\mathbb{P}^{B}\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F^n(y,t)-F^n(y,s)|\geqslant\eta\Big).
\end{aligned}$$

Since $y\in\mathbb{Q}^2$, we have

$$\limsup_{\delta\to0}\limsup_{n\to\infty}\mathbb{P}^{B}\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F^n(y,t)-F^n(y,s)|\geqslant\eta\Big)=0.$$

The proof is complete. □

Proof of Lemma 2.15. It suffices to prove that almost surely in X, for all y in a ball, say $y\in B(0,R)$ with $R>0$, we have

$$\sup_n\mathbb{E}^B[F^n(y,T)]<+\infty.$$

Observe that

$$\mathbb{E}^B[F^n(y,T)]=\int_{\mathbb{R}^2}\int_0^T e^{-\frac{|u-y|^2}{2s}}\frac{ds}{2\pi s}M_n(du).$$

To this purpose, it suffices to prove that

$$\sup_{y\in B(0,R)}\int_{\mathbb{R}^2}\int_0^T e^{-\frac{|u-y|^2}{2s}}\frac{ds}{2\pi s}M(du)<+\infty.$$

Let us set

$$f_y(u)=\int_0^T e^{-\frac{|u-y|^2}{2s}}\frac{ds}{2\pi s}.$$

We define two other functions where we separate the singularity of f_y:

$$g_y(u)=\mathbb{1}_{\{|u-y|\geqslant1\}}\int_0^T e^{-\frac{|u-y|^2}{2s}}\frac{ds}{2\pi s}\quad\text{and}\quad h_y(u)=\mathbb{1}_{\{|u-y|<1\}}\int_0^T e^{-\frac{|u-y|^2}{2s}}\frac{ds}{2\pi s}.$$

Finally we set, for some $T>0$,

$$g(u)=\sup_{y\in B(0,T)}g_y(u).$$

It is plain to check that $g \in L^1(\mathbb{R}^2)$. We deduce

$$\mathbb{E}^X[\int_{\mathbb{R}^2} g(u)\, M(du)] = \int_{\mathbb{R}^2} g(u)\, du < +\infty.$$

Therefore, almost surely in X, for all $y \in B(0,T)$, g_y is M-integrable over $\mathbb{R}^2$. It remains to prove that, for all $y \in B(0,T)$, h_y is M-integrable. To this purpose, we first observe that the divergence of h_y at y is logarithmic. Indeed, we have

$$\int_0^T e^{-\frac{|u-y|^2}{2s}} \frac{ds}{2\pi s} = \int_0^{\frac{T}{|u-y|^2}} e^{-\frac{1}{2s}} \frac{ds}{2\pi s}$$

and the mapping $x \mapsto \int_0^x e^{-\frac{1}{2s}} \frac{ds}{2\pi s}$ behaves as $\ln x$ for large x. To overcome this logarithmic divergence and prove the integrability oh h_y, it is therefore enough to establish that, for some $\epsilon > 0$,

$$\sup_{y \in B(0,T)} \sup_r \frac{M(B(y,r))}{r^\epsilon} < +\infty\,.$$

which is the content of Proposition 2.18 below and is valid provided that $\gamma^2 < 4$. □

Proof of Lemma 2.17. For $t > 0$ and $y \in \mathbb{R}^2$, we define

$$f_{y,t}(u) = \int_0^t e^{-\frac{|u-y|^2}{2s}} \frac{ds}{2\pi s}.$$

Fix $T > 0$. We will use the dominated convergence theorem. From Lemma 2.15, we know that, almost surely in X, for all $y \in \mathbb{R}^2$:

$$\int_{\mathbb{R}^2} f_{y,T}(u) M(du) < +\infty.$$

Furthermore, we have

$$\forall t < T, \forall u, y \in \mathbb{R}^2, \quad f_{y,t}(u) \leqslant f_{y,T}(u).$$

For all $u \neq y$, $f_{y,t}(u)$ converges towards 0 as $t \to 0$. Since M does not possess atom, we can thus claim that M-almost surely, the family of functions $(f_{y,t})_t$ converges towards 0 as $t \to 0$. The dominated convergence theorem thus ensures that:

$$\lim_{t \to 0} \int_{\mathbb{R}^2} \int_0^t e^{-\frac{|u-y|^2}{2s}} \frac{ds}{2\pi s} M(du) = 0.$$

The proof is over. □

We prove here an estimate on the modulus of continuity of the measure M:

Proposition 2.18. *Let $\epsilon > 0$ and $T > 0$. We set $\alpha = 2(1 - \frac{\gamma}{2})^2 > 0$. Almost surely, there exists a random constant $C > 0$ such that:*

$$\sup_{x \in [-T,T]^2} M(B(x,r)) \leqslant C r^{\alpha - \epsilon}, \quad \forall r > 0$$

Proof. Recall that:

$$\xi_M(p) = (2 + \frac{\gamma^2}{2})p - \frac{\gamma^2}{2}p^2.$$

We take $T = \frac{1}{2}$ for simplicity. Now, we partition $[-\frac{1}{2}, \frac{1}{2}]^2$ into 2^{2n} dyadic squares $(I_n^j)_{1 \leqslant j \leqslant 2^{2n}}$. If p belongs to $]0, \frac{4}{\gamma^2}[$, we get:

$$\begin{aligned} P(\sup_{1 \leqslant j \leqslant 2^{2n}} M(I_n^j) \geqslant \frac{1}{2^{(\alpha-\epsilon)n}}) &\leqslant 2^{p(\alpha-\epsilon)n}\mathbb{E}[\sum_{1 \leqslant j \leqslant 2^{2n}} M(I_n^j)^p] \\ &\leqslant C_p 2^{p(\alpha-\epsilon)n} 2^{(2-\xi_M(p))n} \\ &\leqslant \frac{C_p}{2^{(\xi_M(p)-2-(\alpha-\epsilon)p)n}} \end{aligned}$$

By tacking $p = \frac{2}{\gamma}$ in the above inequalities (i.e. $\xi_M(p) - 2 - (\alpha - \epsilon)p > 0$), we get that:

$$\sup_{1 \leqslant j \leqslant 2^{2n}} M(I_n^j) \leqslant \frac{C}{2^{(\alpha-\epsilon)n}}, \quad \forall n \geqslant 1.$$

Let $r > 0$. There exists n such that $\frac{1}{2^{n+1}} < r \leqslant \frac{1}{2^n}$. We conclude by the fact that the ball $B(x, r)$ is contained in at most 4 dyadic squares in $(I_{n-1}^j)_{1 \leqslant j \leqslant 2^{2(n-1)}}$. □

Now, we investigate continuity of the mapping $F(y, \cdot)$ with respect to the parameter y. We claim:

Lemma 2.19. *For all $0 < s < t$ and $\eta > 0$, almost surely in X, for all $x, y \in B(0, R)$, we have*

$$\lim_{|y-x| \to 0} \limsup_{n,n' \to 0} \mathbb{P}^B\Big(\Big|F^n(y, [s, t]) - F^{n'}(x, [s, t])\Big| \geqslant \eta\Big) = 0.$$

Proof. Let us fix $\epsilon > 0$. We can use the coupling procedure of Lemma 2.13 with three independent Brownian motions, one starting from x, one starting from y and one, call it B, starting from a rational point $z \in \mathbb{Q}^2$. We can then couple first the two Brownian motions starting from x and z to get a Brownian motion $B^{x,z}$ and then couple the two Brownian motions starting from y and z to get a Brownian motion $B^{y,z}$. Both of them coincide with the Brownian motion starting from z after some random time τ, which satisfies $\mathbb{P}(\tau > \eta) \to 0$ (for $\eta > 0$) when $\max(|x - z|, |y - z|) \to 0$. We can then consider the families $(F^n(x, \cdot))_n$ and $(F^n(y, \cdot))_n$, both of them respectively constructed with the Brownian motions $B^{x,z}$ and $B^{y,z}$. The family $(F^n(z, \cdot))_n$ is constructed with the Brownian motion B.

As $|x-y| \to 0$, we can choose z such that $\mathbb{P}(\tau \geqslant s) \leqslant \epsilon$. We have for $\eta > 0$:

$$\begin{aligned}
\mathbb{P}^{B^{x,z},B^{y,z}}\Big(\Big|F^n(y,[s,t]) - F^{n'}(x,[s,t])\Big| \geqslant \eta\Big) \\
&\leqslant \mathbb{P}^{B^{x,z},B^{y,z}}\Big(\Big|F^n(y,[s,t]) - F^{n'}(x,[s,t])\Big| \geqslant \eta, \tau \geqslant s\Big) \\
&\quad + \mathbb{P}^{B^{x,z},B^{y,z}}\Big(\Big|F^n(y,[s,t]) - F^{n'}(x,[s,t])\Big| \geqslant \eta, \tau < s\Big) \\
&\leqslant \epsilon + \mathbb{P}^{B}\Big(\Big|F^n(z,[s,t]) - F^{n'}(z,[s,t])\Big| \geqslant \eta\Big).
\end{aligned}$$

Since $z \in \mathbb{Q}^2$, we have

$$\limsup_{n,n' \to \infty} \mathbb{P}^{B}\Big(\Big|F^n(z,[s,t]) - F^{n'}(z,[s,t])\Big| \geqslant \eta\Big) = 0.$$

The proof is complete. □

Proposition 2.20. *Almost surely in X, for all $y \in \mathbb{R}^2$, the family $(F^n(y,\cdot))_n$ converges in law under $\mathbb{P}^B$ in $C(\mathbb{R}_+)$ equipped with the sup-norm topology towards a limiting function $F(y,\cdot)$, which is continuous, increasing and satisfies:*

$$\forall x \in \mathbb{R}^2, \quad \lim_{t \to \infty} F(x,t) = +\infty. \tag{2.20}$$

Furthermore the mapping

$$y \mapsto F(y,\cdot)$$

is continuous in law in $C(\mathbb{R}_+)$.

Proof of Proposition 2.20. By applying Theorem 2.8 on all the rational points of $\mathbb{R}^2$ together with Proposition 2.14, we prove that, almost surely in X, for all $x \in \mathbb{R}^2$, the family $(F^n(x,\cdot))_n$ is tight in law under $\mathbb{P}^B$ in $C(\mathbb{R}_+)$ equipped with the sup-norm topology. Furthermore, we may assume that convergence in law holds for all the rational points of $\mathbb{R}^2$. With the help of Lemma 2.19, we prove that, for each given point $x \in \mathbb{R}^2$, there is a unique possible limiting law, which is characterized by the laws of $(F(x,\cdot)_{x \in \mathbb{Q}^2}$. Therefore, for each $x \in \mathbb{R}^2$, the family $(F^n(x,\cdot))_n$ converges in law under $\mathbb{P}^B$ in $C(\mathbb{R}_+)$ equipped with the sup-norm topology towards a limiting functions $F(x,\cdot)$, which is continuous and nondecreasing in its input t. Let us denote by $\mathbb{P}^x$ the law of $F(x,\cdot)$ in $C(\mathbb{R}_+)$.

We deduce from Lemma 2.19 that, when x converges to x_0, the random measure $F(x,dr)$ converges in law under $\mathbb{P}^x$ in the sense of weak convergence of measures towards the law of the random measure $F(x_0,dr)$ under $\mathbb{P}^{x_0}$. From this and Lemma 2.21 below, we conclude that the law of $F(x,\cdot)$ under $\mathbb{P}^x$ converges in $C(\mathbb{R}_+)$ towards the law of $F(x_0,\cdot)$ under $\mathbb{P}^{x_0}$ when $x \to x_0$.

All what remains to prove is that for all y, the mapping $t \mapsto F(y,[0,t])$ is increasing under $\mathbb{P}^y$. From Theorem 2.8, we may assume that, almost surely in X, the mapping $t \mapsto F(y,[0,t])$ is a.s. increasing for all the rational points $y \in \mathbb{R}^2$ under

$\mathbb{P}^y$. So, let us fix $y_0 \in \mathbb{R}^2$ and let us prove that the mapping $t \mapsto F(y_0, [0,t])$ is a.s. increasing in $\mathbb{P}^{y_0}$. It is enough to prove that $\mathbb{P}^{y_0}(F(y_0,[s,t]) > 0)$ for a countable family of intervals $[s,t]$ generating the Borel topology on $\mathbb{R}_+$. Let us consider such an interval $[s,t]$ with $s > 0$. We will use a coupling argument.

In what follows, if B is a Brownian motion on $\mathbb{R}^2$, we will denote by B^1 and B^2 its components. Let us start a Brownian motion B^{y_0} from y_0. Let us consider another independent Brownian motion B starting from 0 and denote by B^y, for a rational $y \in \mathbb{R}^2$, the Brownian motion $B^y = y + B$. Let us denote by τ_1^y the first time at which the first components of B^{y_0} and B^y coincide:

$$\tau_1^y = \inf\{u > 0; B_u^{1,y_0} = B_u^{1,y}\}$$

and by τ_2^y the first time at which the second components coincide after τ_1^y:

$$\tau_2^y = \inf\{u > \tau_1; B_u^{2,y_0} = B_u^{2,y}\}$$

We can consider a new Brownian motion $\overline{B}^{y,y_0}$ by

$$\overline{B}_t^{y,y_0} = \begin{cases} (B_t^{1,y_0}, B_t^{2,y_0}) & \text{if} \quad t \leqslant \tau_1 \\ (B_t^{1,y}, B_t^{2,y_0}) & \text{if} \quad \tau_1 < t \leqslant \tau_2 \\ (B_t^{1,y}, B_t^{2,y}) & \text{if} \quad \tau_2 < t. \end{cases} .$$

Of course, the law $\mathbb{P}^{y_0}$ does not depend on the Brownian motion that we choose to define F^n. So we are free to choose the Brownian motion $\overline{B}^{y,y_0}$. Then we have

$$\begin{aligned} \mathbb{P}^{y_0}(F(y_0,[s,t]) > 0) &\geqslant \mathbb{P}^{y_0}(F(y_0,[s,t]) > 0, \tau_2^y < s) \\ &= \mathbb{P}^y(F(y,[s,t]) > 0, \tau_2^y < s) \\ &\geqslant \mathbb{P}^y(F(y,[s,t]) > 0) - \mathbb{P}(\tau_2^y \geqslant s) \\ &\geqslant 1 - \mathbb{P}(\tau_2^y \geqslant s). \end{aligned}$$

Clearly, $\mathbb{P}(\tau_2^y \geqslant s) \to 0$ as $y \to y_0$ in such a way that $\mathbb{P}^{y_0}(F(y_0,[s,t]) > 0) = 1$.

This coupling argument also applies to deduce from Theorem 2.8 that, a.s. in X, for all $y \in \mathbb{R}^2$, $\mathbb{P}^y$ a.s.:

$$\lim_{t \to \infty} F(y,t) = +\infty.$$

□

Lemma 2.21. *Let us denote by M_T the set of finite Radon measures on $[0,T]$ equipped with the topology of weak convergence of measures. Let $(\mu_n)_n$ be a sequence of random elements in M_T converging in law towards μ. Assume that μ and each μ_n is diffuse. Then the random mapping $t \mapsto \mu_n([0,t])$ converges in law in $C([0,T])$ as $n \to \infty$ towards the random mapping $t \mapsto \mu([0,t])$.*

Proof.

It is well-known that the topology of weak-convergence for measures on M_T is metrizable, the so-called Prohorov's metric being one of the possible choices. Recall

that the Prohorov metric on the space M_T is defined as follows (see for example [53]). For any $\mu, \nu \in M_T$, let

$$d_{M_T}(\mu,\nu) := \inf \left\{ \varepsilon > 0\,,\ \text{s.t } \forall \text{ closed set } A \subset [0,T]\,, \begin{array}{c} \mu(A^\varepsilon) \leq \nu(A) + \varepsilon \\ \text{and} \\ \nu(A^\varepsilon) \leq \mu(A) + \varepsilon \end{array} \right\}, \tag{2.21}$$

where A^ε-denotes the ε-neighborhood of A.

It is well-known (see [53]) that the metric space (M_T, d_{M_T}) is a complete separable metric space. In particular, one can apply Skorohod representation theorem. With a slight abuse of notation, we may thus couple μ_n and μ on the same probability space so that a.s. $d_{M_T}(\mu_n, \mu) \to 0$.

Let us define,

$$F_n(t) = \mu_n([0,t]), \quad F(t) = \mu([0,t]).$$

Since $[0,T]$ is compact and since μ is assumed to be diffuse, we have that F is a.s. uniformly continuous. Let δ_F be its modulus of continuity. For any fixed $\alpha > 0$, let $\varepsilon > 0$ be such that $\delta_F(2\varepsilon) < \alpha$ and N large enough so that for each $n \geqslant N$, $d_{M_T}(\mu_n, \mu) < \varepsilon$. As such we have for all $n \geqslant N$ and for all $t \in [0,T]$,

$$F(t-2\varepsilon) - 2\varepsilon \leqslant F_n(t) \leqslant F(t+2\varepsilon) + 2\varepsilon\,,$$

which shows that $\|F - F_n\|_\infty < \alpha$, for all $n \geqslant N$. Since α was arbitrary we thus showed that a.s. $\|F_n - F\|_\infty \to 0$. □

2.7 Defining the Liouville Brownian motion

By fixing a given point in $x \in \mathbb{R}^2$, we are now able to define the LBM starting from x almost surely w.r.t. the field X and the Brownian motion B:

Theorem 2.22. *Assume $\gamma^2 < 4$ and fix $x \in \mathbb{R}^2$. Almost surely in X and in B, the n-regularized Brownian motion $(\mathcal{B}^{n,x})_n$ defined by Definition 2.2 converges in the space $C(\mathbb{R}_+, \mathbb{R}^2)$ equipped with the supremum norm on compact sets towards a continuous random process $\mathcal{B}^x$, which we call (massive)* **Liouville Brownian motion** *starting from x, characterized by:*

$$\mathcal{B}^x_{\mathbf{t}} = x + B_{\langle \mathcal{B}^x \rangle_{\mathbf{t}}}$$

where $\langle \mathcal{B}^x \rangle$ is defined by

$$F(x, \langle \mathcal{B}^x \rangle_{\mathbf{t}}) = \mathbf{t}.$$

Furthermore, $\mathcal{B}^x$ is a local martingale.

As a consequence, almost surely in X, the n-regularized Liouville Brownian motion defined in Definition 2.1 converges in law under $\mathbb{P}$ in $C(\mathbb{R}_+, \mathbb{R}^2)$ towards $\mathcal{B}^x$.

Proof of Theorem 2.22. It is just a consequence of corollary 2.10. □

This result will allow us to prove that, almost surely in X, we can define the law of the Liouville Brownian motion for all possible starting point $y \in \mathbb{R}^2$:

Theorem 2.23. *Assume $\gamma^2 < 4$. Almost surely in X and in B, for all $y \in \mathbb{R}^2$, the n-regularized Brownian motion $(\mathcal{B}^{n,y})_n$ defined by Definition 2.2 converges in the space $C(\mathbb{R}_+)$ towards a continuous random process $\mathcal{B}^y$, characterized by:*

$$\mathcal{B}^y_{\mathbf{t}} = y + B_{\langle \mathcal{B}^y \rangle_{\mathbf{t}}}$$

where $\langle \mathcal{B}^y \rangle$ is defined by

$$F(y, \langle \mathcal{B}^y \rangle_{\mathbf{t}}) = \mathbf{t}.$$

As a consequence, almost surely in X, for all $y \in \mathbb{R}^2$, the n-regularized Liouville Brownian motion $(\mathcal{B}^{n,y})_n$ converges in law in $C(\mathbb{R}_+, \mathbb{R}^2)$ towards $\mathcal{B}^y$. Furthermore, almost surely in X and under $\mathbb{P}^B$, the law of the mapping $y \mapsto \mathcal{B}^y$ is continuous in law in $C(\mathbb{R}_+)$.

Proof of Theorem 2.23. All the statements of Theorem 2.23 are a direct consequence of Proposition 2.20 excepted the continuity in law in $C(\mathbb{R}_+)$ of the quadratic variations. Let us prove this statement. Fix $n \in \mathbb{N}$ and $0 \leqslant t_1 < \cdots < t_n$ and consider the mapping

$$\varphi \in C(\mathbb{R}_+) \mapsto (\varphi^{-1}(t_1), \ldots, \varphi^{-1}(t_n))$$

where

$$\varphi^{-1}(t) = \inf\{s \geqslant 0; \varphi(s) > t\} \quad \text{with } \inf \emptyset = +\infty.$$

It is well defined for all functions $\varphi \in C(\mathbb{R}_+)$. Observe that it is continuous at all those functions that are continuous increasing and going to ∞ as $t \to \infty$. Since $\mathbb{P}^x$ gives full measures to those functions, we deduce that, almost surely in X, $(F^{-1}(y, t_1), \ldots, F^{-1}(y, t_n))$ converges in law under $\mathbb{P}^y$ towards $(F^{-1}(x, t_1), \ldots, F^{-1}(x, t_n))$ as $y \to x$.

We can apply this result for all the dyadic points $t \in \mathbb{R}_+$. We deduce that, almost surely in X, for all $x \in \mathbb{R}^2$, the random measure $F^{-1}(y, dr)$ converges in law under $\mathbb{P}^y$ in the space of Radon measures on $\mathbb{R}_+$ towards $F^{-1}(x, dr)$ under $\mathbb{P}^x$ as $y \to x$. By using Lemma 2.21 once again, we deduce that $F^{-1}(y, \cdot)$ converges in law under $\mathbb{P}^y$ in $C(\mathbb{R}_+)$ towards $F^{-1}(x, \cdot)$ under $\mathbb{P}^x$ as $y \to x$. □

Corollary 2.24. *The Liouville Brownian motion is a Feller process.*

Proof of corollary 2.24. Let us consider a continuous bounded function f. Fix $x \in \mathbb{R}^2$. Since the mapping $y \mapsto \mathcal{B}^y$ is continuous in law in $C(\mathbb{R}_+)$, we deduce that $\mathcal{B}^y_t$ converges in law towards $\mathcal{B}^x_t$ as $y \to x$. Therefore the mapping $y \mapsto \mathbb{E}^y[f(\mathcal{B}^y_t)]$ is continuous at x. □

It is then a routine trick to deduce

Corollary 2.25. *The Liouville Brownian motion is a strong Markov process.*

Corollary 2.26. *Almost surely in X, the Liouville Brownian motion is recurrent, i.e. for all $x \in \mathbb{R}^2$ and for all $z \in \mathbb{R}^2$:*

$$\mathbb{P}^x\big[\liminf_{t \to \infty} |\mathcal{B}_t - z| = 0\big] = 1.$$

Furthermore, almost surely in X, for all $x \in \mathbb{R}^2$,

$$\mathbb{P}^x\big[\limsup_{t\to\infty} |\mathcal{B}_t| = \infty\big] = 1.$$

Remark 2.27. *The whole convergence of the quadratic variations in $C(\mathbb{R}_+)$ allows to deal with quite general a family of functionals of this process. As a straightforward consequence, we will see below that the semi-group of the n-regularized LBM converges towards the semi-group of the LBM. For instance, we can also deal with exit times of smooth enough domains (say satisfying a Zaremba's cone condition for instance), giving a probabilistic interpretation to PDEs involving the Liouville Laplacian (see below) with boundary conditions of Dirichlet type. For instance, one can give rigorous a meaning to equations of the type:*

$$\triangle u = \mu\, e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]} \quad \text{for } x \in D, \quad u(x) = f(x) \quad \text{for } x \in \partial D.$$

We let the reader think of all the other possible uses of this convergence.

2.8 Reversibility of the Liouville Brownian motion under the Liouville measure

Theorem 2.28. *If $\gamma^2 < 2$, almost surely in X, the massive Liouville Brownian motion is reversible w.r.t. the Liouville measure, which is therefore invariant too.*

So our Liouville Brownian motion a.s. preserves the so-called **Liouville measure**. It would be interesting to check that the semigroup converges towards the Liouville measure and to investigate at which rate this convergence occurs.

Proof of Theorem 2.28 for $\gamma^2 < 2$. For all $n \geqslant 1$ and every functions $f, g \in C_c(\mathbb{R}^2)$, we have:

$$\int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})]g(x)M_n(dx) = \int_{\mathbb{R}^2} \mathbb{E}^B[g(\mathcal{B}^{n,x}_{\mathbf{t}})]f(x)M_n(dx). \tag{2.22}$$

Let us prove that we can pass to the limit as $n \to \infty$ in each side of the above relation. For instance, we treat the left-hand side. Recall that, almost surely in X, the measures $(M_n)_n$ weakly converge towards a Radon measure M.

Let us consider a dense countable family $(f_k)_k$ in $C_c(\mathbb{R}^2)$ for the uniform topology. From Lemma 2.29 below, we may assume that, almost surely in X, for all $k, p \in \mathbb{N}$

$$\begin{aligned}\liminf_n \Big| \int_{\mathbb{R}^2} \mathbb{E}^B[f_k(\mathcal{B}^{n,x}_{\mathbf{t}})]f_p(x)M(dx) - \int_{\mathbb{R}^2} \mathbb{E}^B[f_k(\mathcal{B}^{n,x}_{\mathbf{t}})]f_p(x)M_n(dx)\Big| \\ + \Big| \int_{\mathbb{R}^2} \mathbb{E}^B[f_p(\mathcal{B}^{n,x}_{\mathbf{t}})]f_k(x)M(dx) - \int_{\mathbb{R}^2} \mathbb{E}^B[f_p(\mathcal{B}^{n,x}_{\mathbf{t}})]f_k(x)M_n(dx)\Big| = 0.\end{aligned} \tag{2.23}$$

It is plain to deduce that, almost surely in X, for every functions $f, g \in C_c(\mathbb{R}^2)$

$$\liminf_n \Big| \int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})]g(x)M(dx) - \int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})]g(x)M_n(dx)\Big| + \Big| \int_{\mathbb{R}^2} \mathbb{E}^B[g(\mathcal{B}^{n,x}_t)]f(x)M(dx) - \int_{\mathbb{R}^2} \mathbb{E}^B[g(\mathcal{B}^{n,x}_{\mathbf{t}})]f(x)M_n(dx)\Big| = 0. \quad (2.24)$$

From the dominated convergence theorem and Theorem 2.23, we have:

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})]g(x)M(dx) = \int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{x}_{\mathbf{t}})]g(x)M(dx). \quad (2.25)$$

By gathering (2.24) and (2.25) we deduce that for every functions $f, g \in C_c(\mathbb{R}^2)$ and (up to extracting the same subsequence):

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})]g(x)M_n(dx) = \int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{x}_{\mathbf{t}})]g(x)M(dx) \quad (2.26)$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \mathbb{E}^B[g(\mathcal{B}^{n,x}_{\mathbf{t}})]f(x)M_n(dx) = \int_{\mathbb{R}^2} \mathbb{E}^B[g(\mathcal{B}^{x}_{\mathbf{t}})]f(x)M(dx). \quad (2.27)$$

From (2.22)+(2.26)+(2.27), we deduce

$$\int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{x}_{\mathbf{t}})]g(x)M(dx) = \int_{\mathbb{R}^2} \mathbb{E}^B[g(\mathcal{B}^{x}_{\mathbf{t}})]f(x)M(dx)$$

holds for every functions $f, g \in C_c(\mathbb{R}^2)$ and $\mathbf{t} \geqslant 0$. □

Lemma 2.29. *If $\gamma^2 < 2$, for any functions $f, g \in C_c(\mathbb{R}^2)$, we have*

$$\mathbb{E}^X\Big[\Big| \int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})]g(x)M(dx) - \int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})]g(x)M_n(dx)\Big|\Big] \to 0 \quad \text{as } n \to \infty.$$

Proof. Let us denote by $\mathcal{F}_n$ the sigma algebra generated by the random processes $\{Y_k; k \leqslant n\}$. Let us also denote by

$$G^n_m(x,y) = \int_{c_{n+1}}^{\infty} \frac{k_m(u(x-y))}{u}\, du.$$

Choose $R > 0$ such that $\mathrm{Supp}(g) \subset B(0,R)$. Since we assumed $\gamma^2 < 2$, we may study the following second moment (in order then to apply Cauchy-Schwarz):

$$\begin{aligned}
&\mathbb{E}^X\Big[\Big(\int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})]g(x)M(dx) - \int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})]g(x)M_n(dx)\Big)^2\Big] \\
&= \mathbb{E}^X\Big[\mathbb{E}^X\Big[\Big(\int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})]g(x)M(dx) - \int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})]g(x)M_n(dx)\Big)^2 |\mathcal{F}_n\Big]\Big] \\
&= \mathbb{E}^X\Big[\int_{\mathbb{R}^2\times\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})]\mathbb{E}^B[f(\mathcal{B}^{n,y}_{\mathbf{t}})]g(x)g(y)e^{\gamma X_n(x)+\gamma X_n(y)-\gamma^2\mathbb{E}[X_n^2]}(e^{G^n_m(x,y)}-1)\,dxdy\Big] \\
&\leqslant \|f\|^2_\infty \|g\|^2_\infty \int_{B(0,R)\times B(0,R)} (e^{G^n_m(x,y)}-1)\,dxdy
\end{aligned}$$

It is plain to see that the last quantity goes to 0 as $n \to \infty$. Hence the lemma in the case $\gamma^2 < 2$. □

2.9 Asymptotic independence of the Liouville Brownian motion and the Euclidean Brownian motion

In this subsection, we make rigorous the statement in Remark 2.3. Recall that the Euclidean Brownian motion $\bar{B}$ is the one involved in Definition 2.1.

Theorem 2.30. *If $\gamma^2 < 4$, almost surely in X, the couple of processes $(\bar{B}, \mathcal{B}^n)_n$ converges in law towards a couple $(\bar{B}, \mathcal{B})$. The Euclidean Brownian motion $\bar{B}$ and the Liouville Brownian motion $\mathcal{B}$ (or equivalently B) are independent.*

Surprisingly, the above theorem shows that some extra-randomness is created by taking the limit $n \to \infty$. Indeed, the n-regularized Liouville Brownian motion is a measurable function of the Euclidean Brownian motion. Yet, Liouville/Euclidean Brownian motions are independent at the limit.

As a consequence the Liouville Brownian motion, as defined in (2.5) cannot converge in a stronger sense than in law. This justifies our approach of studying the convergence via the Dambis-Schwarz representation theorem.

Proof of Theorem 2.30. Before beginning the proof, let us first clarify a few points. The n-regularization of the Liouville Brownian motion (2.5) involves the Euclidean Brownian motion $\bar{B}$. An equivalent definition of this n-regularization is given in Definition 2.2 by means of another Brownian motion B, constructed via the Dambis-Schwarz theorem. As such, it implicitly depends on n as well as $\bar{B}$. It is therefore relevant to write explicitly this dependence in this proof. So we will write B^n instead of B. It turns out that, as $n \to \infty$, the proofs of Theorems 2.22 and 2.23 show that the Liouville Brownian motion is a measurable function of the Brownian motion B. The frame of our proof will thus be the following. First, we write explicitly the dependence structure between $\bar{B}$ and B^n. Second, we prove that, at the limit $n \to \infty$, they are independent.

So, as announced, we begin with writing explicitly the dependence between $\bar{B}$ and B^n. The Dambis-Schwarz theorem tells us that

$$x + B_t^{n,x} = \mathcal{B}^n_{\tau_t^{n,x}}$$

where

$$\tau_t^{n,x} = \inf\{s \geqslant 0; \langle \mathcal{B}^{n,x} \rangle_s > t\}.$$

From (2.9) and Lemma 2.7, we deduce:

$$\tau_t^{n,x} = F^n(x,t).$$

Therefore

$$B_t^{n,x} = \int_0^{F^n(x,t)} e^{-\frac{\gamma}{2} X_n(\mathcal{B}^{n,x}_{\mathbf{u}}) + \frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}^{n,x}_{\mathbf{u}})^2]}\, d\bar{B}_{\mathbf{u}} = \mathcal{B}^{n,x}_{F^n(t,x)}.$$

Now we prove asymptotic independence of $\mathcal{B}$ and $\bar{B}$. Let us compute their predictable bracket:

$$\begin{aligned}\langle \mathcal{B}^{n,x}, \bar{B}\rangle_t &= \int_0^t e^{-\frac{\gamma}{2}X_n(\mathcal{B}_r^{n,x})+\frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}_r^{n,x})^2]}\,dr \\ &= \int_0^t e^{-\frac{\gamma}{2}X_n(x+B_{\langle \mathcal{B}^{n,x}\rangle_r})+\frac{\gamma^2}{4}\mathbb{E}[X_n(x+B_{\langle \mathcal{B}^{n,x}\rangle_r})^2]}\,dr.\end{aligned}$$

By making the change of variables

$$u = \langle \mathcal{B}^{n,x}\rangle_r, \quad e^{\gamma X_n(x+B_u)-\frac{\gamma^2}{2}\mathbb{E}[X_n(x+B_u)^2]}\,du = dr,$$

we get:

$$\begin{aligned}\langle \mathcal{B}^{n,x}, \bar{B}\rangle_t &= \int_0^{\langle \mathcal{B}^{n,x}\rangle_t} e^{\frac{\gamma}{2}X_n(x+B_u)-\frac{\gamma^2}{4}\mathbb{E}[X_n(x+B_u)^2]}\,dr \\ &= e^{-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]}\int_0^{\langle \mathcal{B}^{n,x}\rangle_t} e^{\frac{\gamma}{2}X_n(x+B_u)-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]}\,dr.\end{aligned}$$

Let us prove that this latter quantity converges in probability towards 0 when t is fixed. Theorem 2.8 implies that, almsot surely in X, the mapping

$$t \mapsto \int_0^t e^{\frac{\gamma}{2}X_n(x+B_u)-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]}\,dr$$

converges in law in $C(\mathbb{R}_+)$. Therefore

$$e^{-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]}\int_0^{\langle \mathcal{B}^{n,x}\rangle_t} e^{\frac{\gamma}{2}X_n(x+B_u)-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]}\,du$$

converges in law in $C(\mathbb{R}_+)$ towards 0 since $e^{-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]} \to 0$ as $n \to \infty$ (this quantity is independent of x, u and $\mathrm{Var}(X_n) \to \infty$ as $n \to \infty$). Furthermore, for t fixed,

$$\mathbb{P}^{\bar{B}}(\langle \mathcal{B}^{n,x}\rangle_t > R) \to 0, \quad \text{uniformly w.r.t } n.$$

It is plain to deduce that, almost surely in X, under $\mathbb{P}^{\bar{B}}$ the sequence $(\langle \mathcal{B}^{n,x}, \bar{B}\rangle_t)_n$ converges in law towards 0 s $n \to \infty$ and therefore in $\mathbb{P}^{\bar{B}}$-probability. Since the mapping $t \mapsto \langle \mathcal{B}^{n,x}, \bar{B}\rangle_t$ is nondecreasing, we deduce that in $\mathbb{P}^{\bar{B}}$-probability, the process $t \mapsto \langle \mathcal{B}^{n,x}, \bar{B}\rangle_t$ converges towards 0 in $C(\mathbb{R}_+)$. Knight's theorem implies that B and $\bar{B}$ are independent. As a measurable function of B, the Liouville Brownian motion is independent of $\bar{B}$. □

2.10 Liouville heat kernel and Liouville Laplacian

The LBM is a time-homogeneous strong Markov Feller diffusion on $\mathbb{R}^2$, which is invariant under the Liouville measure for $\gamma^2 < 2$. Therefore one can associate a Feller

semi-group $(P_{\mathbf{t}}^X)_{\mathbf{t}}$ acting on $C_0(\mathbb{R}^2)$. Furthermore and almost surely in X, Theorem 2.28 entails that it extends to a strongly continuous semigroup on $L^p(\mathbb{R}^2, M)$ for all $1 \leqslant p < +\infty$. The **Liouville Laplacian** Δ_X is defined as the generator of the Liouville Brownian motion times the usual extra factor $\sqrt{2}$. The Liouville Laplacian corresponds to an operator which can formally be written as

$$\Delta_X = e^{-\gamma X(x) + \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}\Delta.$$

Finally we conclude this section with a remark about the **fractional Liouville Laplacian**. Indeed, one may also wishes to define rigorously the fractional Liouville Laplacian for $0 < \alpha < 1$. The underlying Markov process can be obtained by subordinating the Liouville Brownian motion with an α-stable Lévy subordinator. The fractional Liouville Laplacian is then nothing but the generator of this Markov process.

Let us try to give an intuition of this operator. The Euclidean fractional Laplacian in dimension 2 can be formally written as

$$(-\triangle)^\alpha f(x) = P.V.\ c_\alpha \int_{\mathbb{R}^2} \frac{f(x+z) - f(x)}{|z|^{2+2\alpha}} dz.$$

On a Riemannian manifold the above expression remains true provided that we replace the "$x+z$" quantity by the exponential map of the manifold (see for instance [5] and references therein). In Liouville quantum gravity, this expression should remain valid, provided that we can give sense to the exponential map.

Works in progress

In future works, we will investigate the main properties of the Liouville semi-group. The main question is to prove that, almost surely in X, the Liouville semi-group is strong Feller. As a consequence, it is absolutely continuous w.r.t. the Liouville measure. It thus possesses a density, call it p_t^X such that:

$$\forall f \in C_c^0(\mathbb{R}^2), \quad P_{\mathbf{t}}^X f(x) = \int_{\mathbb{R}^2} f(y) p^X(x, y, \mathbf{t})\, M(dy).$$

The family $(p^X(\cdot, \cdot, \mathbf{t}))_{\mathbf{t}}$ will be called **Liouville heat kernel**. We will try to prove these properties until the threshold $\gamma^2 < 4$.

2.11 Remarks about associated Feynman path integrals

Feynman path integrals have been introduced in order to produce probability measures on curves the energy of which are expressed in terms of Lagrangians instead of Hamiltonians. Remind that the standard Wiener measure gives a rigorous interpretation of the heuristic path integral on $\mathbb{R}^2$

$$\frac{1}{Z_0} \int_{C([0,T];\mathbb{R}^2)} f(\sigma) \exp\Big(-\frac{1}{2} \int_0^T |\sigma'(s)|^2\, ds \Big) \mathcal{D}\sigma \tag{2.28}$$

where Z_0 appears as a normalization constant and $f : C([0,T],\mathbb{R}^2) \to \mathbb{R}$ is a bounded continuous function. It turns out that the construction of the Liouville Brownian motion allows us to make sense of several Feynman path integrals appearing in the Liouville quantum gravity literature. We discuss below these integrals.

The "Wiener measure" associated to the Liouville Brownian motion has the following path integral interpretation:

$$\mathbb{E}^B\big[f\big((\mathcal{B}_t)_{0\leqslant t\leqslant T}\big)\big] = \frac{1}{Z_1}\int_{C([0,T];\mathbb{R}^2)} f(\sigma)\exp\Big(-\frac{1}{2}\int_0^T e^{\gamma X(\sigma(s))-\frac{\gamma^2}{2}\mathbb{E}[X^2]}|\sigma'(s)|^2\,ds\Big)\mathcal{D}\sigma,$$

where Z_1 is a normalization constant, valid for all bounded continuous function $f : C([0,T],\mathbb{R}^2) \to \mathbb{R}$. This claim can be further illustrated by large deviation arguments (see Section 4).

We can also give a rigorous meaning to the heuristic path integral on $\mathbb{R}^2$:

$$\begin{aligned}\frac{1}{Z_0}\int_{C([0,T];\mathbb{R}^2)} f(\sigma)\exp\Big(-\frac{1}{2}\int_0^T |\sigma'(s)|^2 + \mu e^{\gamma X(\sigma(s))-\frac{\gamma^2}{2}\mathbb{E}[X^2]}\,ds\Big)\mathcal{D}\sigma \\ = \mathbb{E}^B\Big[f\big((B_t)_{0\leqslant t\leqslant T}\big)e^{-\mu\int_0^T e^{\gamma X(B_r)-\frac{\gamma^2}{2}\mathbb{E}[X^2]}\,dr}\Big] \qquad (2.29)\end{aligned}$$

Of course, Z_0 is the renormalization constant of the Wiener measure so that, as written in (2.29), the path integral is not normalized. We can renormalize it by replacing Z_0 by

$$Z_\mu = \mathbb{E}^B\Big[e^{-\mu\int_0^T e^{\gamma X(B_r)-\frac{\gamma^2}{2}\mathbb{E}[X^2]}\,dr}\Big].$$

Further comments can be made if we further assume that f is nonnegative.

Proposition 2.31. *Assume that f is nonnegative. Then the Feynman path integral*

$$\frac{1}{Z_0}\int_{C([0,T];\mathbb{R}^2)} f(\sigma)\exp\Big(-\frac{1}{2}\int_0^T |\sigma'(s)|^2 + \mu e^{\gamma X(\sigma(s))-\frac{\gamma^2}{2}\mathbb{E}[X^2]}\,ds\Big)\mathcal{D}\sigma$$

is a continuous non-increasing function of μ. Expectation of this path integral with respect to the field X is a non-decreasing function of γ.

Proof. The claimed properties with respect to the parameter μ just results from standard theorems of parameterized integrals. Let us prove that taking MGFF expectation $\mathbb{E}_X$ of this path integral with respect to γ yields a non-decreasing function. This is just a consequence of the fact that the mapping $x \mapsto e^{-\mu x}$ is convex and Kahane's convexity inequalities. □

3 Liouville Brownian motion defined on other geometries: torus, sphere and planar domains

So far, we constructed in detail the Liouville Brownian motion for the (Massive) Free Field on $\mathbb{R}^2$. In this section, we wish to briefly discuss how one can extend this construction to the following cases:

1. Liouville Brownian motion on the sphere $\mathbb{S}^2$ equipped with a standard Gaussian Free Field (GFF) with vanishing average.

2. Liouville Brownian motion on the torus $\mathbb{T}^2$ equipped with a GFF with vanishing average.

3. Liouville Brownian motion on a domain D (i.e. a simply connected domain $D \subsetneq \mathbb{C}$), equipped with a GFF with Dirichlet boundary conditions.

Will will not detail the proofs since the whole machinery works the same as in the plane, especially in the first two cases where the field is stationary as the MFF was in $\mathbb{R}^2$. In the third case, we will briefly explain two ways to build a LBM on the domain D: either by adapting the machinery to a non-stationary GFF or by relying on an appropriate coupling argument which avoids any additional technicality. As such, we will essentially focus on formulating precisely the respective frameworks.

3.1 Liouville Brownian motion on the sphere

3.1.1 Gaussian Free Field on the sphere

We consider a Gaussian Free Field (GFF for short) on the sphere $\mathbb{S}^2$ with vanishing average. It is a standard Gaussian in the Hilbert space defined as the closure of Schwartz functions with vanishing integral over $\mathbb{S}^2$ with respect to the inner product

$$(f, g)_h = -(f, \triangle g)_2,$$

where $(\cdot, \cdot)_2$ is the standard inner product on $L^2(\mathbb{S}^2)$. Its action on $L^2(\mathbb{S}^2)$ can be seen as a Gaussian distribution with covariance kernel given by the Green function G of the operator $-\triangle$ with vanishing mean (times the normalization factor 2π).

Let us consider the sequence $(\lambda_n)_{n \geqslant 1}$ of (positive) eigenvalues of the operator $-\triangle$ and let $(e_n)_n$ denote an orthonormal basis of $L^2(\mathbb{S}^2)$ made up of associated eigenfunctions (precisely the spherical harmonics excepted the constant one). The GFF on $\mathbb{S}^2$ is the Gaussian distribution defined by

$$X(x) = \sqrt{2\pi} \sum_{k \geqslant 1} \lambda_k^{-1/2} e_k(x) \alpha_k$$

where $(\alpha_k)_k$ is a sequence of i.i.d. standard Gaussian random variables. In that case, we define the n-regularized smooth Gaussian field

$$X_n(x) = \sqrt{2\pi} \sum_{k=1}^{n} \lambda_k^{-1/2} e_k(x) \alpha_k. \tag{3.1}$$

We stress that the correlations of the GFF on the sphere behaves at short scales like the logarithm of the Riemannian distance

$$\mathbb{E}[X(x)X(y)] = \ln_+ \frac{1}{d(x, y)} + g(x, y)$$

where g is some bounded continuous function on the sphere and d is the distance induced by the Riemannian metric of the sphere. Therefore, Kahane's theory [36] applies for the GFF on the sphere.

3.1.2 Brownian motion on the sphere

Consider the unit sphere $\mathbb{S}^2$ as a submanifold of $\mathbb{R}^3$. Using classical terminology, let us denote by $\mathbf{T}\mathbb{S}^2 = \bigcup_{x\in\mathbb{S}^2} \mathbf{T}_x\mathbb{S}^2$ the tangent bundle of the sphere. The Laplace-Beltrami operator on the sphere, here denoted by $\triangle$ can be written in the form of a sum of squares:

$$\triangle = \sum_{i=1}^{3} P_i^2$$

where P_i is the projection of the i-th coordinate unit vector e_i on the tangent space $\mathbf{T}_x\mathbb{S}^2$. Each P_i is a vector field on $\mathbb{S}^2$. The projection to the tangent sphere at x is given by

$$P(x)\xi = \xi - \langle \xi, x\rangle x, \quad x \in \mathbb{S}^2, \ \ \xi \in \mathbb{R}^3,$$

in such a way that the matrix $P = \{P_1, P_2, P_3\}$ can be explicitly written as:

$$P(x)_{ij} = \delta_{ij} - x_i x_j.$$

Consider the following Stratonovich stochastic differential on $\mathbb{S}^2$ driven by a 3-dimensional euclidian Brownian motion W:

$$dB_t = \sum_{i=1}^{3} P_i(B_t) \circ dW_t^i, \quad X_0 \in \mathbb{S}^2.$$

This is a stochastic differential equation on $\mathbb{S}^2$ because P_i are vector fields on $\mathbb{S}^2$. Extending P_i arbitrarily to the whole ambient space, we can solve this equation as if it is an equation on $\mathbb{R}^3$. It can be checked that if the initial condition lies on the manifold $\mathbb{S}^2$, then the solution B lies on $\mathbb{S}^2$ for all times. Furthermore, it is a diffusion process generated by $\frac{1}{2}\triangle$.

Therefore, Brownian motion on $\mathbb{S}^2$ is the solution of the stochastic differential equation

$$B_t^i = B_0^i + \int_0^t (\delta_{ij} - B_s^i B_s^j) \circ dW_s^j, \quad B_0 \in \mathbb{S}^2.$$

This is Stroock's representation of spherical Brownian motion.

3.1.3 Construction of the Liouville Brownian motion on the sphere

The construction of the n-regularized Brownian motion on the sphere is quite similar to the standard spherical Brownian motion. It is the solution of the following stochastic differential equation:

$$\mathcal{B}_t^{n,x,i} = \mathcal{B}_0^{n,x,i} + \int_0^t (\delta_{ij} - \mathcal{B}_s^{n,x,i}\mathcal{B}_s^{n,x,j}) e^{-\frac{\gamma}{2}X_n(\mathcal{B}_s^{n,x}) + \frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}_s^{n,x})]} \circ dW_s^j, \quad \mathcal{B}_0^{n,x} \in \mathbb{S}^2.$$

All the results stated in section 2 apply since Kahane's theory remains valid on C^1-manifolds (see [36]). Intuitively, this is just because such manifolds are locally isometric to open domains of the Euclidean space. In particular, curvature does not play a fundamental part.

3.2 Liouville Brownian motion on the torus $\mathbb{T}^2$

The standard GFF on $\mathbb{T}^2$ with vanishing average is defined exactly in the same fashion as on the sphere. Namely, let $(e_n)_n$ be an orthonormal basis of $L^2(\mathbb{T}^2)$ made up of eigenfunctions of $-\triangle$ with eigenvalues $(\lambda_n)_n$. (One could be more explicit here but this will not be needed). The GFF on $\mathbb{T}^2$ with vanishing average is defined as well by

$$X(x) = \sqrt{2\pi} \sum_{k \geqslant 1} \lambda_k^{-1/2} e_k(x) \alpha_k \, ,$$

where $(\alpha_k)_k$ is a sequence of i.i.d. standard Gaussian random variables. Exactly as in the case of the sphere, one can define a Liouville Brownian motion $(\mathcal{B}_{\mathbf{t}})_{\mathbf{t} \geqslant 0}$ on $\mathbb{T}^2$ (furthermore, there no curvature effect here).

Remark 3.1. *There is at least one point in our proofs that must be changed in order to apply to the torus or the sphere, or any bounded manifold without boundary: the fact that* $\lim_{t\to\infty} F(x,t) = +\infty$. *Indeed, our proof uses the "infinite volume" of the plane. In the case of the torus or sphere, the strategy is much simpler because of compactness arguments: the standard Brownian motion on* $\mathbb{S}^2$ *to* $\mathbb{T}^2$ *possesses an invariant probability measure, call it* μ. *Apply the ergodic theorem to prove that* $\mathbb{P}^X \otimes \mathbb{P}^\mu$ *almost surely:*

$$\lim_{t\to\infty} \frac{F(x,t)}{t} = G,$$

for some random variable G, *which is shift-invariant. Since the Brownian on the sphere is ergodic,* G *is measurable with respect to the sigma algebra* $\sigma\{X_x; x \in \mathbb{T}^2 \text{ or } \mathbb{S}^2\}$. *It is not clear that* G *is constant. Yet, the set* $\{G > 0\}$ *is measurable with respect to the asymptotic sigma-algebra of the* $(X_{n+1} - X_n)_n$. *Therefore the set* $\{G > 0\}$ *has probability* 0 *or* 1. *Since* G *has expectation* 1, *this set has probability* 1. *Therefore,* $\mathbb{P}^X$ *almost surely, the change of times* $F(x,t)$ *goes to* ∞ *as* $t \to \infty$ *for* μ *almost every* x. *Then use the coupling trick to deduce that the property holds for all starting points.*

3.3 Liouville Brownian motion on a bounded planar domain $D \subsetneq \mathbb{C}$

Let $D \subsetneq \mathbb{C}$ be a bounded simply connected domain of the plane. Let X be the GFF on D with Dirichlet boundary conditions and let $M = M_\gamma(X)$ denote the Liouville measure for $\gamma \in [0,2)$ on the domain D. We wish to briefly discuss how to construct a Liouville Brownian motion for $\gamma \in [0,4)$ on the domain D. We highlight two approaches.

3.3.1 Comparison with the massive Liouville Brownian motion on $\mathbb{R}^2$ through Kahane's convexity inequalities

In order to extend the approach we developed in $\mathbb{R}^2$ for the (massive)-free field to the case of our bounded domain D, one has to deal with two differences. First the field X in D with Dirichlet boundary conditions is no longer stationary in $x \in D$. Second, the quadratic variation $\langle \mathcal{B} \rangle_{\mathbf{t}}$ will no longer tend to infinity due to the fact that the LBM will eventually leave the domain D. To compare the situation in D with the situation in $\mathbb{R}^2$, endowed with a massive free field X_m, it is fruitful to rely on Kahane's convexity inequalities given in Lemma A.4. But to rely on these inequalities, one needs to show that for any $x, y \in D$, $\mathrm{Cov}\big[X(x), X(y)\big] \leqslant \mathrm{Cov}\big[X_m(x), X_m(y)\big]$. Recall that in a domain D (see for example [59]), one has

$$\mathrm{Cov}\big[X(x), X(y)\big] = G_D(x,y) = \log \frac{1}{|x-y|} + H^x(y)\,,$$

where G_D is the Green function of D and where $y \mapsto H^x(y)$ is the harmonic extension of the function $y \mapsto \log|x-y|$ on ∂D. In particular, one always have

$$\mathrm{Cov}\big[X(x), X(y)\big] \leqslant \frac{1}{|x-y|} + \log \mathrm{diam}(D)\,.$$

Now recall from subsection 2 that

$$\mathrm{Cov}\big[X_m(x), X_m(y)\big] = \log_+ \frac{1}{|x-y|} + g_m(x,y)\,,$$

where g_m is a continuous bounded function $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. As in the Proof 2. that $F(x,t) \to \infty$ in subsection 2.4, let Y be an independent standard Gaussian variable of variance $\mathbb{E}\big[Y^2\big] = \log_+ \mathrm{diam}(D) + \|g_m\|_\infty$ in such a way that on has

$$\mathrm{Cov}\big[X\big] \leqslant \mathrm{Cov}\big[X_m + Y\big]\,.$$

The contribution of the independent variable Y will factorize in each required computation leading to an additional constant $c = c(Y)$. As such, by Lemma A.4, one can control all expectations of the form

$$\mathbb{E}\big[G(\int_D e^{\gamma X_r - \frac{\gamma^2}{2}\mathbb{E}\left[X_r^2\right]}\sigma(dr))\big]\,,$$

for any convex function $G : \mathbb{R}_+ \to \mathbb{R}$ with polynomial growth and for any $\gamma < 4$ (since we have estimates on such functionals for X_m only when $\gamma < 4$). By applying this Kahane convexity inequality to the occupation time measure σ of a Brownian motion $(B_t)_t$ stopped on ∂D, all the steps required to prove the analog of Theorem 1.2 can now be made rigorous. The proofs which analyse the behaviour of the quadratic variation of $\mathcal{B}_{\mathbf{t}}$ would require more effort in this case (in particular if $\gamma \in [2,4)$ due to the negative moments in Lemma 2.12). Yet on the intuitive level, since the field X has Dirichlet boundary conditions, the Liouville Brownian motion should not be slowed down near ∂D (in general, the LBM is slowed down in areas where X is large). This means that one should have $\langle \mathcal{B}_{\mathbf{t}\wedge\tau_D} \rangle_\infty < \infty$ a.s.

3.3.2 Using a coupling argument

Let us briefly sketch as second reason why it is not hard to extend the above Liouville Brownian motions to the case of a bounded domain D.

Even though we did not find a proper reference, the following Lemma in the spirit of [20, section 4.5] and [58, section 3.1] should hold.

Lemma 3.2. *Let $D' \subset D$ be two domains with $\bar{D}' \subset D$. Let $T > 0$ such that $D \subset [-T, T]^2$. Furthermore, let X_D be the GFF in D with Dirichlet boundary conditions and let X_T denote the GFF with vanishing mean in the $2d$-Torus with fundamental domain $[-T, T]^2$. Then the law of $(X_D)_{|D'}$ restricted to the domain D' is absolutely continuous with respect to the law of $(X_T)_{|D'}$.*

This lemma easily enables us to extend the above construction of a LBM on the $2T$-Torus to a LBM defined on D. Indeed, for each $\varepsilon > 0$, one applies the above lemma with $D' := D^{(\varepsilon)} := \{x \in D, \mathrm{dist}(x, \partial D) > \varepsilon\}$.

Let us end this section with the following remark.

Remark 3.3. *As is well known from Levy's theorem, standard two-dimensional Brownian motion is a conformally invariant object (modulo an explicit change of time-parametrization). Since the Liouville measures are conformally co-variant (see [26]), (which follows from the conformal invariance property of the GFF, see [59]), it is not hard to obtain that the Liouville Brownian motion is a conformally invariant object as well, up to a different time-parametrization which may be written explicitly.*

4 Conjectures and open problems

4.1 About the construction for all possible values of γ^2

Question 1. Prove that for $\gamma^2 \geqslant 4$, the changes of times $(F^n)_n$ converge to 0.

Question 2. For $\gamma^2 = 4$, construct the critical Liouville Brownian motion in the spirit of [24, 25]. In particular, construct the derivative change of times

$$F(x,t) = \int_0^t \big(2\mathbb{E}[X(B_r)^2] - X(B_r)\big) e^{2X(B_r) - 2\mathbb{E}[X(B_r)^2]}\, dr.$$

Question 3. About the maximum of the GFF. Determine the asymptotic behaviour of the maximum of the GFF along the Brownian curve. More precisely, in section 2.1, choose the sequence $(c_n)_n$ in order to make sure that $\mathrm{Var}(X_n) = \ln n$. Is it true that, for $t > 0$ the family

$$\max_{0 \leqslant s \leqslant t} X_n(B_s) - 2\ln n + \frac{3}{4}\ln\ln n$$

converges in law as $n \to \infty$? Can one express the limiting law as a shift of a Gumbel law by the (log) derivative time change of Question 2? Formulate a similar result

for the maximum of a discrete GFF along the path of a simple random walk on the vertices of a regular lattice of $\mathbb{R}^2$.

Question 4. For $\gamma^2 > 4$, construct the dual Liouville Brownian motion in the spirit of [8].

4.2 Structure of the Liouville Brownian motion

Question 5. Prove that the Liouville semi-group is absolutely continuous with respect to the Liouville measure and define in this way the Liouville heat kernel $p^X(t,x,y)$. Prove that $p^X(t,x,y)$ is continuous with respect to x,y and that it is positive. It is not sure that we can get further "Euclidean" regularity. C^∞-smoothness is more likely to make sense for the Liouville metric (see subsection 4.4).

Question 6. Give a proper construction of the Liouville Green function, sometimes called the (massless for GFF/massive for MFF) scalar propagator.

Question 7. Prove a martingale representation theorem for the Liouville Brownian motion.

Question 8. Prove the convergence of the semigroup of the Liouville Brownian motion towards the Liouville measure as $\mathbf{t} \to \infty$. For instance, if one can provide a positive answer to question 5 in the case of the sphere, one can use the Doeblin lemma to prove exponential convergence of the semi-group to the Liouville measure. Can we characterize this exponential decay?

Question 9. What become all the functional inequalities (Poincaré, log-Sobolev,...) in the world of Liouville quantum gravity?

4.3 Related Dirichlet forms

In this subsection, we explain our heuristics in the case of the massive Liouville Brownian motion on the whole plane $\mathbb{R}^2$. Of course, a straightforward adaptation of the arguments for the Liouville Brownian motion on the sphere is possible.

We consider the Hilbert space $L^2(\mathbb{R}^2, M)$. Observe that the measure M is almost surely a Radon measure over $\mathbb{R}^2$, with support $\mathbb{R}^2$. Furthermore, a.s. in X, $L^2(\mathbb{R}^2, M)$ contains the space $C_c(\mathbb{R}^2)$.

Since the Markovian semi-group $(P^X_{\mathbf{t}})_{\mathbf{t}\geqslant 0}$ is almost surely in X reversible under the measure M, it extends to a strongly continuous self-adjoint semi-group on $L^2(\mathbb{R}^2, M)$. Following the standard steps of the theory of symmetric Dirichlet forms (see [30] for instance), we can associate to this semi-group a resolvent family, a non-positive definite, self-adjoint operator on $L^2(\mathbb{R}^2, M)$ (which is nothing but the Liouville Laplacian) and a symmetric Dirichlet form Σ by:

$$\Sigma(f,f) = \lim_{t\to 0} \int\int \big(f(x)-f(y)\big)^2 P^X(t,x,dy)M(dx). \tag{4.1}$$

The domain $\mathcal{D}$ of Σ is precisely those functions f for which the above limit is finite.

Question 10. Prove that, almost surely w.r.t. X, the Dirichlet form $(\mathcal{D}, \Sigma)$ is strongly local and regular.

Question 11. Can one give a tractable description of the domain (or a suitable core) of the Dirichlet form $(\mathcal{D}, \Sigma)$?

4.4 Towards the metric

The geometric aspect of Dirichlet forms allows to interpret the theory of Dirichlet forms as an extension of Riemannian geometry applicable to non differential structures and to describe stochastic processes in terms of intrinsically defined geometric quantities. This may be a way of going rigorously until the metric via the metric angle of Dirichlet forms. Let us explain. Denote by $\mathcal{M}$ the collection of all signed Radon measures on $\mathbb{R}^2$. For each $f, g \in \mathcal{D}$,

$$\Sigma(f,g) = \int_{\mathbb{R}^2} d\Gamma(f,g),$$

where Γ is $\mathcal{M}$-valued nonnegative definite and symmetric bilinear form defined by the formula

$$\int_{\mathbb{R}^2} \phi \, d\Gamma(f,g) = \frac{1}{2}\Big[\Sigma(f,\phi g) + \Sigma(f\phi, g) - \Sigma(fg,\phi)\Big]$$

for all $f, g \in \mathcal{D} \cap L^\infty(\mathbb{R}^2, M)$ and $\phi \in \mathcal{D} \cap C_c(\mathbb{R}^2)$. Equivalently, one can express $\Gamma(f,f)$ as:

$$\int_{\mathbb{R}^2} \phi \, d\Gamma(f,f) = \lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^2} \mathbb{E}^x[|f(\mathcal{B}_t^x) - f(\mathcal{B}_0^x)|^2]\phi(x)\, M(dx).$$

Let us denote by $\partial_M \Gamma(f,g)$ the Radon-Nykodim derivative of $\Gamma(f,g)$ with respect to $M \in \mathcal{M}$. The quantity

$$\sqrt{\partial_M \Gamma(f,f)}$$

is then called the length of the gradient. Let us consider the space $\mathcal{D}_{loc}$ defined as the collection of all $f \in L^2_{loc}(\mathbb{R}^2, M)$ satisfying that for each relatively compact set $K \subset \mathbb{R}^2$, there exists a function $g \in \mathcal{D}$ such that $f = g$ M-almost everywhere on K. Observe that both Σ and Γ can be defined for $f, g \in \mathcal{D}_{loc}$. Since $\mathcal{B}$ is a right process, it induces intrinsically a distance on $\mathbb{R}^2$, which is defined by (see [61, 62, 63]):

$$d_X(x,y) = \sup\{f(x) - f(y); f \in \mathcal{D}_{loc} \cap C^0(\mathbb{R}^2), \Gamma(f,f) \leqslant M\}$$

where $\Gamma(f,f) \leqslant M$ means that $\Gamma(f,f)$ is absolutely continuous with respect to M and $\partial_M \Gamma(f,f) \leqslant 1$ M-almost everywhere. This metric is a *metric in the wide sense*, meaning that it may possibly take values in $[0, +\infty]$.

What needs to be investigated is the following question:

Question 12. Prove that, almost surely in X, the topology induced by d_X is the Euclidean topology. If true, the metric d_X is finite (i.e. does not take ∞ as possible value) and $(\mathbb{R}^2, d_X)$ is a length space (see [61, Theorem 5.2]). Observe that, in the context of pure gravity ($c = 0, \gamma = \sqrt{8/3}$), this question is (conjectured to be) related to the topology of large planar maps [46, 47, 50].

4.5 Fractal geometry of Liouville quantum gravity

Question 13. Spectral dimension of quantum gravity. Prove rigorously that the spectral dimension of two-dimensional quantum gravity is 2 (see [4]). Tackling this problem requires the construction of the Liouville heat kernel as explained in subsection 2.10. The proof consists in a careful analysis of the quadratic variations of the Liouville Brownian motion.

Question 14. Brownian formulation of the KPZ formula. We want here to address the issue of formulating a Brownian version of the KPZ formula. Remind that the KPZ formula is a relation between the fractal dimensions of sets as seen by the Euclidean metric or the quantum geometry. In [26, 56], the KPZ formula is proved by treating the field $e^{\gamma X}$ as a random measure, but not as a random metric. Quantum exponents are then computed via standard ball covering arguments, the quantum mass of a ball being given by the mass assigned by the Liouville measure to this ball. Hence the underlying metric stays in a way "classical" or Euclidean. Another derivation of the KPZ formula is suggested in [16]: express the KPZ formula in terms of time spend by the Liouville Brownian motion to cover Euclidean balls. We want to describe below a possible way of making this rigorous, which differs from the approach developed in [16].

The quantum mass that we assign to a ball of Euclidean radius R will be the time spent by the Liouville Brownian motion to leave this ball. If one wants to figure out what this quantum time looks like, then one has to introduce the stopping time

$$\tau^R_{\mathrm{LQG}} = \inf\{u \geqslant 0; \mathcal{B}^x_u \notin B(x, R)\}.$$

This quantity stands for the time spent by the LBM to leave the Euclidean ball of radius R. It can be expressed in terms of the time spent by the Euclidean Brownian motion to leave this ball

$$\tau^R_{\mathrm{E}} = \inf\{u \geqslant 0; x + B_u \notin B(x, R)\},$$

by the relation

$$F(x, \tau^R_{\mathrm{E}}) = \tau^R_{\mathrm{LQG}}.$$

Since for small R we have $\tau^R_{\mathrm{E}} \simeq R^2$, one could thus assign to the Euclidean ball of radius R centered at x the quantum mass

$$\mathbb{E}^B[F(x, R^2)].$$

The question is then to formulate a suitable definition of the Euclidean fractal dimension of a set K, call it $\dim_E(K)$, as well as the quantum fractal dimension of K, call it $\dim_{\mathrm{LQG}}(K)$ and to prove that (if possible almost surely in X)

$$\dim_E(K) = \big(1 + \frac{\gamma^2}{4}\big)\dim_{\mathrm{LQG}}(K) - \frac{\gamma^2}{4}\dim_{\mathrm{LQG}}(K)^2.$$

Observe that the power law spectrum of F computed in Lemma 2.11 is a strong hint that the above conjecture should be true.

Question 15. Metric formulation of KPZ formula. If one can construct the metric d_X (as suggested in subsection 4.4), prove the KPZ formula with respect to the metric.

Question 16. Hausdorff dimension of quantum gravity. Compute the Hausdorff dimension of the sphere equipped with the Liouville metric d_X. The reader is referred to [14, 39, 23] for further references about geometry of $2d$ quantum gravity. We feel that it is unlikely to be given by 4 for all values of the central charge c between -2 and 1, as claimed in [23].

Question 17. Let us consider a Liouville Brownian bridge from x to y with lifetime t, call it $\mathcal{B}^{x,y,\mathbf{t}}$. It can be obtained by conditioning the Liouville Brownian motion starting from x to reach y at time $\mathbf{t}$. Prove that the family of curves $\{\mathcal{B}^{x,y,\mathbf{t}}_{st}; 0 \leqslant s \leqslant 1\}$ is sequentially compact as $\mathbf{t} \to 0$ and that any possible limit is a minimizing geodesic between x and y.

4.6 Connection with Brownian motion on planar maps

Question 18. Prove the convergence of Brownian motion on planar maps towards the Liouville Brownian motion as explained in conjecture 1.

4.7 Further questions in related stochastic calculus

Question 19. Can one construct a theory of SDE with respect to the Liouville Brownian motion? Can one give some kind of Liouville version of the Schauder theory expressed in terms of the metric d_X?

Question 20. If one provides a satisfactory answer to question 7, this opens the doors of a theory of Backward Stochastic Differential Equations (BSDE) with respect to the Liouville Brownian motion. In particular, one should be able to give a probabilistic representation of nonlinear problems of the type:

$$\partial_t u = \triangle_X u + f(x, u, \nabla_L u)$$

with suitable initial condition $u(0, \cdot)$.

4.8 Related models

Question 21. Construct the analog of the Liouville Brownian motion in the context of Mandelbrot's multiplicative cascades (see [7, 37, 38] or [9, 12] in the KPZ context) or even Branching Random Walks. Beyond the fact that it is a toy model, it is a powerful laboratory to understand the continuous case. Though independence structure is reinforced in this model, it has been for long illustrated that the main qualitative phenomena observed in the context of cascades remain true in the continuous case. Furthermore, Kahane's convexity inequalities provide a (one-sided) powerful bridge between cascades and multiplicative chaos (see [36, 24] for instance).

Question 22. Construct the analog of the Liouville Brownian motion with respect to log-correlated infinitely divisible fields instead of the GFF or MFF. The reader is referred to [10, 29, 55, 54] for insights of the topic. We also would like to ask the following question: what is the planar maps counterpart of these (non-lognormal) other geometries?

A Background about Gaussian multiplicative chaos theory

Here we recall a few material about Gaussian multiplicative chaos theory that can be found in [36]. First we remind of the following regularity notion for a measure

Definition A.1. *A measure σ on (a bounded domain of) $\mathbb{R}^d$ is said to be in the class R_α for $\alpha > 0$ if for all $\epsilon > 0$ there is $\delta > 0$ and a compact set $A \subset D$ such that $\sigma(D \setminus A) \leqslant \epsilon$ and:*

$$\forall O \text{ open set}, \quad \sigma(O \cap A) \leqslant C\mathrm{diam}(O)^{\alpha+\delta}, \tag{A.1}$$

where $\mathrm{diam}(O)$ *stands for the Euclidean diameter of* O.

Now we consider a Radon measure σ on (a bounded domain of) $\mathbb{R}^d$ and the associated chaos:

$$M_\sigma(dx) = \lim_{n\to\infty} M_\sigma^n(dx),$$

where

$$M_\sigma^n(dx) = \int_{\cdot} e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\,\sigma(dx).$$

Recall (see [36]):

Theorem A.2 (Law of the chaos)**.** *The law of the measure M_σ does not depend on the choice of the decomposition of the covariance kernel of the Gaussian field X into a sum of nonnegative continuous kernels of positive type.*

Further reinforcements of the above theorem are established in [57]. In particular, we can deal with kernels that may take negative values.

Theorem A.3 (Non-degeneracy)**.** *Assume that the measure σ is in the class R_α for some $\alpha > 0$. If $\gamma^2 < 2\alpha$ then, for all Borelian set A with finite σ-measure, the sequence $(M^n_\sigma(A))_n$ is uniformly integrable. Furthermore, the chaos M is non degenerate and belongs to $R_{\alpha-\frac{\gamma^2}{2}}$.*

Lemma A.4. *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be some convex function such that*

$$\forall x \in \mathbb{R}_+, \quad |F(x)| \leqslant M(1 + |x|^\beta),$$

for some positive constants M, β, and σ be a Radon measure on the Borelian subsets of a locally compact separable metric space (D, d). Given two $(X_r)_{r\in D}, (Y_r)_{r\in D}$ be two continuous centered Gaussian processes with continuous covariance kernels k_X and k_Y such that

$$\forall u, v \in D, \quad k_X(u, v) \leqslant k_Y(u, v).$$

Then

$$\mathbb{E}\Big[F\Big(\int_D e^{X_r - \frac{1}{2}\mathbb{E}[X_r^2]}\,\sigma(dr)\Big)\Big] \leqslant \mathbb{E}\Big[F\Big(\int_D e^{Y_r - \frac{1}{2}\mathbb{E}[Y_r^2]}\,\sigma(dr)\Big)\Big].$$

Star scale invariance

The need of characterizing Gaussian multiplicative chaos may be achieved via a functional equation, called lognormal $\star$-scale invariance [2]:

Definition A.5. Log-normal $\star$-scale invariance. *A random Radon measure M is lognormal $\star$-scale invariant if for all $0 < \varepsilon \leqslant 1$, M obeys the cascading rule*

$$\big(M(A)\big)_{A\in\mathcal{B}(\mathbb{R}^d)} \overset{law}{=} \Big(\int_A e^{\omega_\varepsilon(x)} M^\varepsilon(dx)\Big)_{A\in\mathcal{B}(\mathbb{R}^d)} \tag{A.2}$$

where ω_ε is a stationary stochastically continuous Gaussian process and M^ε is a random measure independent from ω_ε satisfying the scaling relation

$$\big(M^\varepsilon(A)\big)_{A\in\mathcal{B}(\mathbb{R}^d)} \overset{law}{=} \big(M(\frac{A}{\varepsilon})\big)_{A\in\mathcal{B}(\mathbb{R}^d)}. \tag{A.3}$$

□

Notice that the process ω_ε is unknown. Roughly speaking, we look for random measures that scale with an independent lognormal factor on the whole space. This property is shared by many examples of Gaussian multiplicative chaos as we will see below, but not all. And for those Gaussian multiplicative chaos that do not share this property, they are very close to satisfying it. If the reader is familiar with branching random walks (BRW), here is an explanation that may help intuition. If we consider a BRW the reproduction law of which does not change with time (i.e. is the same at each generation), the law of the branching random walk will be characterized by a discrete version of the above $\star$-scale invariance (in the lognormal case of course). If

the reproduction law evolves in time, then we have to change things a bit to adapt to this time evolution. The same argument holds for the log-normal $\star$-scale invariance: it characterizes these Gaussian multiplicative chaos that do not vary along scales.

It is proved in [2] that a Gaussian multiplicative chaos

$$M(A) = \int_A e^{X_x - \frac{1}{2}\mathbb{E}[X_x^2]}\, dx$$

with respect to a stationary Gaussian field X with covariance kernel of the type

$$K(x,y) = \int_1^{+\infty} \frac{k((x-y)u)}{u}\, du, \tag{A.4}$$

for some continuous covariance function k such that $k(0) \leqslant \frac{2d}{1+\delta}$, is star scale invariant. The converse is also studied in [2] and it is proved that lognormal star scale invariant measures are essentially of this form provided that the measure M possesses a moment of order $1+\delta$ for some $\delta > 0$. When a kernel K takes on the form A.4, it will be said star scale invariant. Observe that such kernels satisfy the scaling relation for all $\epsilon \in]0,1]$:

$$K(x,y) = K(\frac{x}{\epsilon}, \frac{y}{\epsilon}) + k_\epsilon(x,y), \quad \text{with } k_\epsilon(x,y) = \int_1^{\frac{1}{\epsilon}} \frac{k((x-y)u)}{u}\, du.$$

In particular, we have:

$$K(x,y) = K(\frac{x}{\epsilon}, \frac{y}{\epsilon}) + k(0)\ln\frac{1}{\epsilon}. \tag{A.5}$$

This relation turns out to be very useful when combined with Kahane's convexity inequalities (Lemma A.4).

B Finiteness of the moments

In this section, our only goal is to prove that

$$\mathbb{E}^X\mathbb{E}^B[F(x,t)^p] < +\infty$$

for $p \in [0, 4/\gamma^2[\cap[0,2]$. Of course, it suffices to compute the moments of order 2 when $\gamma^2 < 2$: this situation is trivial. So, due to their technicality, the following computations only make sense for $2 \leqslant \gamma^2 < 4$.

When p is less than 1, finiteness of the moments directly result from the uniform integrability of the sequence $(F^n(x,t))_n$. So it remains to investigate the case when $1 < p < 4/\gamma^2$ (and therefore $p < 2$). As $(F^n(x,t))_n$ is a martingale, it suffices to prove $\mathbb{E}^X\mathbb{E}^B[F(x,t)^p] < +\infty$. Furthermore, by stationarity of the field X, we may

assume that $x = 0$. By using the concavity of the mapping $x \mapsto x^{p/2}$ and the Jensen inequality, we get:

$$\begin{aligned}\mathbb{E}^X\mathbb{E}^B\Big[\Big(F(0,t)\Big)^p\Big] &\leqslant \mathbb{E}^X\Big[\mathbb{E}^B\Big[F(0,t)^2\Big]^{p/2}\Big] \\ &= \mathbb{E}^X\Big[\Big(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_0^t\int_0^t e^{-\frac{|x|^2}{2s}-\frac{|y-x|^2}{2|r-s|}}\,\frac{drds}{4\pi^2 s|r-s|}M(dx)M(dy)\Big)^{p/2}\Big] \\ &\leqslant \mathbb{E}^X\Big[\Big(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x,y)M(dx)M(dy)\Big)^{p/2}\Big],\end{aligned}$$

where we have set

$$f(x,y) = \int_0^t\int_s^t e^{-\frac{|x|^2}{2s}-\frac{|y-x|^2}{2|r-s|}}\,\frac{drds}{4\pi^2 s|r-s|}.$$

So one just has to prove that the above expectation is finite. This is what we are going to prove.

In what follows, ξ_M stands for the structure exponent of the measure M. Recall that, in dimension d, it reads

$$\xi_M(p) = (d + \frac{\gamma^2}{2})p - \frac{\gamma^2}{2}p^2.$$

Of course, we can take here $d = 2$. But is is worth recalling this fact since it will happen that some arguments below will be carried out in dimension 1. So the reader will take care of replacing d by 1 when reading a proof in dimension 1. The main idea of our proof is the following. First, we observe that the function f possesses singularities. They are logarithmic (see below) when x or $|x-y|$ is close to 0. So we have to provide estimates on M that ensure that this logarithmic divergence can be overcome by the measure M. We will also have to treat the behaviour near infinity. We will split the space $\mathbb{R}^2$ into 3 domains $\{|x| \leqslant 1, |x-y| \leqslant 1\}$, $\{|x| \geqslant 1, |x-y| \leqslant 1\}$ and $\{|x-y| \geqslant 1\}$.

Domain $\{|x| \leqslant 1, |x-y| \leqslant 1\}$

We claim:

Lemma B.1. *For any $\gamma^2 < 4$ and $p \in]1, \frac{4}{\gamma^2}[$, there exists $\delta > 0$ such that*

$$\mathbb{E}\Big[\Big(\int_{\max(|x|,|y|)\leqslant 1}\frac{1}{|x|^\delta|x-y|^\delta}M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

To prove this lemma, we first prove

Lemma B.2. *For any $\gamma^2 < 4$ and $p \in]1, \frac{4}{\gamma^2}[$, there exists $\delta > 0$ such that*

$$\mathbb{E}\Big[\Big(\int_{\max(|x|,|y|)\leqslant 1}\frac{1}{|x|^\delta}M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

Furthermore, there exists a constant $C > 0$ such that for all n:

$$\mathbb{E}\Big[\Big(\int_{\max(|x|,|y|)\leqslant 2^{-n}} \frac{1}{|x|^{\delta}} M(dx)M(dy)\Big)^{p/2}\Big] \leqslant C 2^{-n(\xi_M(p)-\frac{\delta p}{2})}.$$

Proof. We carry out the proof in dimension 1 since, apart from notational issues, the dimension 2 does not raise any further difficulty. In that case, we have to prove

$$\mathbb{E}\Big[\Big(\int_{(x,y)\in[0,1]^2} \frac{1}{|x|^{\delta}} M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

Furthermore, from Kahane's convexity inequalities, it suffices to prove the above lemma for any log-correlated Gaussian field. We choose the perfect kernel of [6] (the reader may consult [55] to adapt the proof to higher dimension). We will only use scaling properties of this kernel. We further stress that a direct argument may be carried out with the help of Kahane's convexity inequalities (direct means without using the perfect kernel). We also remind the reader that the above integral is finite of $\delta = 0$ (see [36]). Therefore, by using sub-additivity of the mapping $x \mapsto x^{p/2}$, we have:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{(x,y)\in[0,1]^2} \frac{1}{|x|^{\delta}} M(dx)M(dy)\Big)^{p/2}\Big] \\
&\qquad = \mathbb{E}\Big[\Big(\sum_{n=0}^{\infty} \int_{2^{-n-1}\leqslant x\leqslant 2^{-n}} \frac{1}{|x|^{\delta}} M(dx)M(dy)\Big)^{p/2}\Big] \\
&\qquad \leqslant \sum_{n=0}^{\infty} \mathbb{E}\Big[\Big(\int_{2^{-n-1}\leqslant x\leqslant 2^{-n}} \frac{1}{|x|^{\delta}} M(dx)M(dy)\Big)^{p/2}\Big] \\
&\qquad \leqslant \sum_{n=0}^{\infty} 2^{\delta(n+1)} \mathbb{E}\Big[\Big(M([2^{-n-1},2^{-n}])M([0,1])\Big)^{p/2}\Big].
\end{aligned}$$

Now we use the standard inequality $ab \leqslant \epsilon a^2 + \frac{b^2}{\epsilon}$ for any $\epsilon > 0$ and sub-additivity of the mapping $x \mapsto x^{p/2}$ to get:

$$(ab)^{p/2} \leqslant \epsilon^{p/2} a^p + \epsilon^{-p/2} b^p.$$

Therefore, with $a = M([2^{-n-1},2^{-n}])$, $b = M([0,1])$ and $\epsilon = 2^{(n+1)\xi_M(p)/p}$, we obtain:

$$\begin{aligned}
\mathbb{E}\Big[\Big(M([2^{-n-1},2^{-n}])M([0,1])\Big)^{p/2}\Big] &\leqslant (2^{(n+1)\xi_M(p)/p})^{p/2} \mathbb{E}\Big[M([2^{-n-1},2^{-n}])^p\Big] \\
&\quad + (2^{(n+1)\xi_M(p)/p})^{-p/2} \mathbb{E}\Big[M([0,1])^p\Big].
\end{aligned}$$

By using now the exact scale invariance of the measure M, we get

$$\mathbb{E}\Big[M([2^{-n-1},2^{-n}])^p\Big] = 2^{-(n+1)\xi_M(p)} \mathbb{E}\Big[M([0,1])^p\Big]$$

and plugging this relation into the above expression yields:

$$\mathbb{E}\Big[\Big(M([2^{-n-1},2^{-n}])M([0,1])\Big)^{p/2}\Big] \leqslant 2\times 2^{-(n+1)\xi_M(p)/2}\mathbb{E}\Big[M([0,1])^p\Big].$$

To sum up, we have:

$$\mathbb{E}\Big[\Big(\int_{(x,y)\in[0,1]^2}\frac{1}{|x|^\delta}M(dx)M(dy)\Big)^{p/2}\Big] \leqslant \sum_{n=0}^{\infty}2^{-(n+1)(\xi_M(p)/2-\delta)}\times 2\mathbb{E}\Big[M([0,1])^p\Big].$$

So, δ can clearly be chosen small enough to make the above series convergent.

The second statement results from the finiteness of the expectation that we have just proved and a straightforward scaling argument. □

Lemma B.3. *For any $\gamma^2<4$ and $p\in]1,\frac{4}{\gamma^2}[$, there exist $\delta>0$ and a constant $C>0$ (only depending on $\mathbb{E}\big[M\big([0,1]\big)^p\big]$) such that for all n:*

$$\mathbb{E}\Big[\Big(\int_{\substack{\max(|x|,|y|)\leqslant 1\\ 2^{-n-1}\leqslant|x-y|\leqslant 2^{-n}}}\frac{1}{|x|^\delta}M(dx)M(dy)\Big)^{p/2}\Big]\leqslant\frac{C}{1-\delta}2^{-n(\xi_M(p)-2-\frac{\delta p}{2})}.$$

Proof. Once again, we carry out the proof in dimension 1 since, apart from notational issues, the dimension 2 does not raise any further difficulty. In that case, we have to prove

$$\mathbb{E}\Big[\Big(\int_{\substack{(x,y)\in[0,1]^2\\ 2^{-n-1}\leqslant|x-y|\leqslant 2^{-n}}}\frac{1}{|x|^\delta}M(dx)M(dy)\Big)^{p/2}\Big]\leqslant C2^{-n(\xi_M(p)-1)}.$$

Furthermore, from Kahane's convexity inequalities, it suffices to prove the above lemma for any log-correlated Gaussian field. Once again, we choose the perfect kernel of [6] and we will only use scaling properties of this kernel. A direct argument may be again carried out with the help of Kahane's convexity inequalities. We will use the following elementary geometric argument: for any $n\geqslant 1$, the set of points 2^{-n}-close to the diagonal

$$\{(x,y)\in[0,1]^2;|x-y|\leqslant 2^{-n}\}$$

is entirely recovered by the union for $k=0,\dots,2^n-2$ of the (overlapping) squares $[\frac{k}{2^n},\frac{k+2}{2^n}]^2$. Therefore, by using sub-additivity of the mapping $x\mapsto x^{p/2}$, we have:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\substack{x,y\in[0,1]\\ 2^{-n-1}\leqslant|x-y|\leqslant 2^{-n}}}\frac{1}{|x|^\delta}M(dx)M(dy)\Big)^{p/2}\Big]\\
&\quad\leqslant\mathbb{E}\Big[\Big(\sum_{k=0,\dots,2^n-2}\int_{x,y\in[\frac{k}{2^n},\frac{k+2}{2^n}]^2}\frac{1}{|x|^\delta}M(dx)M(dy)\Big)^{p/2}\Big]\\
&\quad\leqslant\sum_{k=0,\dots,2^n-2}\mathbb{E}\Big[\Big(\int_{x,y\in[\frac{k}{2^n},\frac{k+2}{2^n}]^2}\frac{1}{|x|^\delta}M(dx)M(dy)\Big)^{p/2}\Big]\\
&\quad=\mathbb{E}\Big[\Big(\int_{x,y\in[0,2^{-n+1}]^2}\frac{1}{|x|^\delta}M(dx)M(dy)\Big)^{p/2}\Big]+\sum_{k=1,\dots,2^n-2}\frac{2^{n\delta}}{k^\delta}\mathbb{E}\Big[M\Big([\frac{k}{2^n},\frac{k+2}{2^n}]\Big)^p\Big]
\end{aligned}$$

By stationarity and scale invariance, we get:

$$\begin{aligned}\sum_{k=1,\dots,2^n-2} \frac{2^{n\delta}}{k^\delta}\mathbb{E}\Big[M\Big([\frac{k}{2^n},\frac{k+2}{2^n}]\Big)^p\Big] &\leqslant 2^{n\delta}\sum_{k=1,\dots,2^n-2}\frac{1}{k^\delta}\mathbb{E}\Big[M\Big([0,2^{-n+1}]\Big)^p\Big]\\ &\leqslant 2^{n\delta}2^{-(n-1)\xi_M(p)}\mathbb{E}\Big[M\Big([0,1]\Big)^p\Big]\sum_{k=1,\dots,2^n-2}\frac{1}{k^\delta}\\ &\leqslant \frac{C}{1-\delta}2^{-n(\xi_M(p)-1)}\end{aligned}$$

where C only depends on $\mathbb{E}\big[M\big([0,1]\big)^p\big]$. We conclude with Lemma B.2. □

Proof of Lemma B.1. Choose another δ' such that $0<\delta'+\delta<\frac{2(\xi_M(p)-2)}{p}$. By using Lemma B.3, we have:

$$\begin{aligned}&\mathbb{E}\Big[\Big(\int_{\max(|x|,|y|)\leqslant 1}\frac{1}{|x|^\delta|x-y|^{\delta'}}M(dx)M(dy)\Big)^{p/2}\Big]\\ &\quad=\mathbb{E}\Big[\Big(\sum_{n=0}^{+\infty}\int_{\substack{\max(|x|,|y|)\leqslant 1\\ 2^{-n-1}\leqslant|x-y|\leqslant 2^{-n}}}\frac{1}{|x|^\delta|x-y|^{\delta'}}M(dx)M(dy)\Big)^{p/2}\Big]\\ &\quad\leqslant\sum_{n=0}^{+\infty}\mathbb{E}\Big[\Big(\int_{\substack{\max(|x|,|y|)\leqslant 1\\ 2^{-n-1}\leqslant|x-y|\leqslant 2^{-n}}}\frac{1}{|x|^\delta|x-y|^{\delta'}}M(dx)M(dy)\Big)^{p/2}\Big]\\ &\quad\leqslant\sum_{n=0}^{+\infty}2^{(n+1)\frac{\delta' p}{2}}\mathbb{E}\Big[\Big(\int_{\substack{\max(|x|,|y|)\leqslant 1\\ 2^{-n-1}\leqslant|x-y|\leqslant 2^{-n}}}\frac{1}{|x|^\delta}M(dx)M(dy)\Big)^{p/2}\Big]\\ &\quad\leqslant\sum_{n=0}^{+\infty}2^{(n+1)\frac{\delta' p}{2}}C2^{-n(\xi_M(p)-2-\delta p/2)}.\end{aligned}$$

Since the latter series converges, the proof is complete. □

Now we prove that the function f satisfies:

$$\mathbb{E}^X\Big[\Big(\int_{|x|\leqslant 1,|x-y|\leqslant 1}f(x,y)M(dx)M(dy)\Big)^{p/2}\Big]<+\infty.$$

To this purpose, it is enough to prove that the divergence of f when approaching the diagonal is logarithmic:

Lemma B.4. *We claim:*

$$f(x,y)\leqslant D(1+\ln_+\frac{1}{|x-y|})(1+\ln_+\frac{1}{|x|}),$$

for some constant D that only depends on t.

Proof. Recall that:

$$f(x,y) = \int_0^t \int_s^t e^{-\frac{|x|^2}{2s} - \frac{|y-x|^2}{2|r-s|}} \frac{drds}{4\pi^2 s|r-s|}.$$

By making successive changes of variables, we obtain:

$$\begin{aligned} f(x,y) &= \int_0^t \int_0^{\frac{t-s}{|x-y|^2}} e^{-\frac{|x|^2}{2s} - \frac{1}{2r}} \frac{drds}{4\pi^2 sr} \\ &= \int_0^{\frac{t}{|x|^2}} \int_0^{\frac{t-s|x|^2}{|x-y|^2}} e^{-\frac{1}{2s} - \frac{1}{2r}} \frac{drds}{4\pi^2 sr} \\ &\leqslant g(\frac{t}{|x|^2}) g(\frac{t}{|x-y|^2}), \end{aligned}$$

where we have set

$$g(t) = \int_0^t e^{-\frac{1}{2s}} \frac{ds}{2\pi s}.$$

It is obvious to check that, for some constant $D > 0$, we have

$$g(t) \leqslant D(1 + \ln_+ t).$$

The proof is complete. □

By gathering Lemma B.4 and Lemma B.1, we deduce

Corollary B.5.

$$\mathbb{E}^X \Big[\Big(\int_{|x| \leqslant 1, |x-y| \leqslant 1} f(x,y) M(dx) M(dy) \Big)^{p/2} \Big] < +\infty.$$

Domain $\{|x-y| \geqslant 1\}$

Let us now investigate the situation when $|x-y| \geqslant 1$. This is the easy part because, in that case, the measures $M(dx)$ and $M(dy)$ are "almost" independent. Therefore, we can proceed more directly in the computations. We claim:

Lemma B.6. *The following expectation is finite:*

$$\mathbb{E}^X \Big[\Big(\int_{|x-y| \geqslant 1} f(x,y) M(dx) M(dy) \Big)^{p/2} \Big] < +\infty.$$

Proof. We use the Jensen inequality with the concave function $x \mapsto x^{p/2}$ to get:

$$\begin{aligned} &\mathbb{E}^X \Big[\Big(\int_{|x-y| \geqslant 1} f(x,y) M(dx) M(dy) \Big)^{p/2} \Big] \\ &\quad \leqslant \Big(\mathbb{E}^X \Big[\int_{|x-y| \geqslant 1} f(x,y) M(dx) M(dy) \Big] \Big)^{p/2} \\ &\quad \leqslant \Big(\int_{|x-y| \geqslant 1} f(x,y) e^{\gamma^2 G_m(x,y)} \, dxdy \Big)^{p/2}. \end{aligned}$$

Since $|x-y| \geqslant 1$, we have $G_m(x,y) \leqslant C$ for some fixed positive constant C. We deduce:

$$\begin{aligned}\mathbb{E}^X\Big[\Big(\int_{|x-y|\geqslant 1} f(x,y)M(dx)M(dy)\Big)^{p/2}\Big] &\leqslant e^{Cp/2}\Big(\int_{\mathbb{R}^2\times\mathbb{R}^2} f(x,y)\,dxdy\Big)^{p/2}\\ &= e^{Cp/2}.\end{aligned}$$

We complete the proof of the lemma. □

Domain $\{|x| \geqslant 1, |x-y| \leqslant 1\}$

The final part of the proof consists in checking the following lemma:

Lemma B.7. *The following expectation is finite:*

$$\mathbb{E}^X\Big[\Big(\int_{|x|\geqslant 1, |x-y|\leqslant 1} f(x,y)M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

To prove this lemma, the first step consists in identifying the behaviour of f on the domain $\{|x| \geqslant 1, |x-y| \leqslant 1\}$. Following the lines of Lemma B.4, the reader may check the following lemma:

Lemma B.8. *There exists a constant $D > 0$ such that*

$$f(x,y) \leqslant D(1 + \ln_+ \frac{1}{|x-y|}) \exp\big(-\frac{|x|^2}{2t}\big),$$

for all $|x| \geqslant 1$ and $|x-y| \leqslant 1$.

Therefore, the proof of Lemma B.7 just boils down to proving:

Lemma B.9. *Fix $t > 0$. For any $\gamma^2 < 4$ and $p \in]1, \frac{4}{\gamma^2}[$, there exist $\delta > 0$ and a constant $C > 0$ (only depending on $\mathbb{E}\big[M\big([0,1]\big)^p\big]$) such that:*

$$\mathbb{E}\Big[\Big(\int_{\substack{|x|\geqslant 1\\ |x-y|\leqslant 1}} \frac{\exp\big(-\frac{|x|^2}{2t}\big)}{|x-y|^\delta} M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

Proof. We go on carrying out the proof in dimension 1 to avoid notational issues. Once again, we first need to estimate the above expectation on stripes of the type $\{|x| \geqslant 1, 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}\}$. So we claim:

Lemma B.10. *Fix $t > 0$. For any $\gamma^2 < 4$ and $p \in]1, \frac{4}{\gamma^2}[$, there exists a constant $C > 0$ (only depending on $\mathbb{E}\big[M\big([0,1]\big)^p\big]$) such that for all n:*

$$\mathbb{E}\Big[\Big(\int_{\substack{|x|\geqslant 1\\ 2^{-n-1}\leqslant |x-y|\leqslant 2^{-n}}} \exp\big(-\frac{|x|^2}{2t}\big) M(dx)M(dy)\Big)^{p/2}\Big] \leqslant C 2^{-n(\xi_M(p)-2)}.$$

Let us admit for a while the above lemma to finish the proof of Lemma B.9. Choose δ such that $0 < \delta < \frac{2(\xi_M(p)-2)}{p}$. By using Lemma B.10, we have:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\substack{|x|\geqslant 1\\ |x-y|\leqslant 1}} \frac{\exp\big(-\frac{|x|^2}{2t}\big)}{|x-y|^\delta} M(dx)M(dy)\Big)^{p/2}\Big] \\
&\quad = \mathbb{E}\Big[\Big(\sum_{n=0}^{+\infty} \int_{\substack{|x|\geqslant 1\\ 2^{-n-1}\leqslant |x-y|\leqslant 2^{-n}}} \frac{\exp\big(-\frac{|x|^2}{2t}\big)}{|x-y|^\delta} M(dx)M(dy)\Big)^{p/2}\Big] \\
&\quad \leqslant \sum_{n=0}^{+\infty} \mathbb{E}\Big[\Big(\int_{\substack{|x|\geqslant 1\\ 2^{-n-1}\leqslant |x-y|\leqslant 2^{-n}}} \frac{\exp\big(-\frac{|x|^2}{2t}\big)}{|x-y|^\delta} M(dx)M(dy)\Big)^{p/2}\Big] \\
&\quad \leqslant \sum_{n=0}^{+\infty} 2^{(n+1)\frac{\delta p}{2}} \mathbb{E}\Big[\Big(\int_{\substack{|x|\geqslant 1\\ 2^{-n-1}\leqslant |x-y|\leqslant 2^{-n}}} \exp\big(-\frac{|x|^2}{2t}\big) M(dx)M(dy)\Big)^{p/2}\Big] \\
&\quad \leqslant \sum_{n=0}^{+\infty} 2^{(n+1)\frac{\delta p}{2}} C 2^{-n(\xi_M(p)-2)}.
\end{aligned}$$

Since the latter series converges, the proof is complete. □

Proof of Lemma B.10. Once again the choice of the log-correlated Gaussian field is left open thanks to Kahane's convexity inequalities and we choose the perfect kernel of [6]. It is also plain to check that the expectation is finite thanks to the exponential term. We will prove the result when integrating only over the domain $\{x \geqslant 1, 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}\}$. It will then be obvious to complete the proof (for instance by using invariance of M in law under reflection). As previously, the reader may check that the stripe $\{x \geqslant 1, 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}\}$ may be covered by the squares $[\frac{k}{2^n}, \frac{k+2}{2^n}]^2$ for k running over the set $K_n = \mathbb{Z} \cap [2^n, +\infty[$. Therefore, by using sub-additivity of the mapping $x \mapsto x^{p/2}$, we have:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\substack{x\geqslant 1\\ 2^{-n-1}\leqslant |x-y|\leqslant 2^{-n}}} \exp\big(-\frac{|x|^2}{2t}\big) M(dx)M(dy)\Big)^{p/2}\Big] \\
&\quad \leqslant \mathbb{E}\Big[\Big(\sum_{k\in K_n} \int_{[\frac{k}{2^n}, \frac{k+2}{2^n}]^2} \exp\big(-\frac{|x|^2}{2t}\big) M(dx)M(dy)\Big)^{p/2}\Big] \\
&\quad \leqslant \sum_{k\in K_n} \mathbb{E}\Big[\Big(\int_{[\frac{k}{2^n}, \frac{k+2}{2^n}]^2} \exp\big(-\frac{k^2}{t2^{2n+1}}\big) M(dx)M(dy)\Big)^{p/2}\Big] \\
&\quad = \sum_{k\in K_n} \exp\big(-\frac{k^2 p}{t2^{2n+2}}\big) \mathbb{E}\Big[M\Big([\frac{k}{2^n}, \frac{k+2}{2^n}]\Big)^p\Big]
\end{aligned}$$

By stationarity and scale invariance, we get:

$$\begin{aligned}\sum_{k\in K_n}\exp\big(-\frac{k^2p}{t2^{2n+2}}\big)\mathbb{E}\Big[M\Big([\frac{k}{2^n},\frac{k+2}{2^n}]\Big)^p\Big]&\\ =\sum_{k\in K_n}\exp\big(-\frac{k^2p}{t2^{2n+2}}\big)\mathbb{E}\Big[M\Big([0,2^{n-1}]\Big)^p\Big]&\\ =2^{-(n-1)\xi_M(p)}\sum_{k\in K_n}\exp\big(-\frac{k^2p}{t2^{2n+2}}\big)\mathbb{E}\Big[M\Big([0,1]\Big)^p\Big]&\\ \leqslant C2^{-n(\xi_M(p)-1)}.&\end{aligned}$$

where C only depends on $\mathbb{E}\big[M\big([0,1]\big)^p\big]$. The last line uses the standard trick of convergence of Riemann sums. □

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