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Closed prime ideals for discontinuous algebra seminorms on $\mathcal{C}(K)$ (preliminary version)

J. Esterle

1 Introduction

Let K be a compact space, let $\mathcal{C}(K)$ (resp. $\mathcal{C}_{\mathbb{R}}(K)$) denote the algebra of continuous complex valued (resp. real valued) functions on K . An algebra seminorm $\|\cdot\|$ on $\mathcal{C}(K)$ is a seminorm satisfying $\|fg\| \leq \|f\|\|g\|$ for every $f, g \in \mathcal{C}(K)$, and such a seminorm is said to be continuous if there exists $k > 0$ such that $\|f\| \leq k\|f\|_K$ for every $f \in \mathcal{C}(K)$, where $\|f\|_K := \max_{t \in K} |f(t)|$ denotes the usual norm on $\mathcal{C}(K)$. A classical result of Kaplansky [19] shows that if $\|\cdot\|$ is any algebra norm on $\mathcal{C}(K)$ we have

$$\|f\| \geq \|f\|_K \quad \text{for every } f \in \mathcal{C}(K).$$

The existence of a discontinuous algebra seminorm on $\mathcal{C}(K)$, which is equivalent to the existence of a discontinuous algebra norm on $\mathcal{C}(K)$ and to the existence of a discontinuous homomorphism from $\mathcal{C}(K)$ into a Banach algebra, is the well-known Kaplansky's problem, which turns out to be undecidable in ZFC. H.G. Dales and the author [4], [11], [6] proved independently that if $2^{\aleph_0} = \aleph_1$, which means that the continuum hypothesis (CH) is assumed, then discontinuous algebra seminorms exist on $\mathcal{C}(K)$ for every compact space K . Those commutative Banach algebras A for which a discontinuous homomorphism $\phi : \mathcal{C}(K) \rightarrow A$ does exist under CH were characterized in [12], see also [5]. On the other direction Solovay and Woodin constructed models of set theory including the axiom of choice and Martin's axiom in which all algebra seminorms on $\mathcal{C}(K)$ are continuous, see [7] for details. Notice that models of set theory in which $2^{\aleph_0} = \aleph_2$ and in which all algebra seminorms on $\mathcal{C}(K)$ are continuous for every compact space K were constructed independently by Frankiewicz-Zbierski and Woodin [15] [29].

The structure of closed ideals of $\mathcal{C}(K)$ for discontinuous algebra seminorms was investigated by A.M. Sinclair [27] and later, independently, by the author [9], who showed that the closure of an ideal is the intersection of all closed prime ideals which contain it. Also a chain of nonmaximal closed prime ideals is well-ordered with respect to inclusion, see [9]. If K is an F -space, which means that f and $|f|$ generate the same ideal of $\mathcal{C}(K)$ for every $f \in \mathcal{C}(K)$, then the family $\text{Prim}(q)$ of all nonmaximal prime ideals which are closed with respect to an algebra seminorm q on $\mathcal{C}(K)$ is a finite union of well-ordered chains of nonmaximal prime ideals.

Pham [22], [23], [24] showed that the situation is much more complicated in the general case. To describe his deep contributions we will need the following notions ;

Definition 1.1 A family $(F_\lambda)_{\lambda \in \Lambda}$ of subsets of a set E is said to be

- **pseudo-finite** if the set $\{\lambda \in \Lambda \mid a \notin F_\lambda\}$ is finite for every $a \in \cup_{\lambda \in \Lambda} F_\lambda$.
- **intersection redundant** if $\cap_{\mu \neq \lambda} F_\mu \subseteq F_\lambda$ for some $\lambda \in \Lambda$.
- **intersection non-redundant** if no subfamily of $(F_\lambda)_{\lambda \in \Lambda}$ is intersection redundant.

Definition 1.2 Let q be an algebra seminorm on $\mathcal{C}(K)$. The **continuity ideal** $\mathcal{I}(q)$ of q is the set of all $f \in \mathcal{C}(K)$ such that there exists $k_f > 0$ satisfying $q(fg) \leq k_f\|g\|_K$ for every $g \in \mathcal{C}(K)$.

It follows from the definition of the continuity ideal that it contains every ideal I such that the restriction of q to I is continuous, and it is known that the restriction of q to its continuity ideal is in fact continuous. Also the continuity ideal is the intersection of all minimal elements of $\text{Prim}(q)$. Pham proved in [22] that if there exists a discontinuous algebra seminorm q on $\mathcal{C}(K)$ such that the continuity ideal of q is not the intersection of a finite family of prime ideals then $\mathcal{C}(K)$ possesses an infinite intersection non-redundant pseudo-finite family of nonmaximal prime ideals. Conversely if I is the intersection of an infinite intersection non-redundant pseudo-finite family of nonmaximal prime ideals, and if $|\mathcal{C}(K)/I| = 2^{\aleph_0}$, then assuming (CH) there exists an algebra seminorm q on $\mathcal{C}(K)$ (or, equivalently, an algebra norm q on $\mathcal{C}(K)$, see [9]) such that the continuity ideal of q equals I .

Now if K is compact define ∂K to be the set of all non-isolated points of K , so that $\partial K \neq \emptyset$ unless K is finite. Starting with an infinite compact space K , we define a non-increasing sequence $(\partial_n K)_{n \in \mathbb{Z}^+}$ of compact subsets of K as follows :

- (i) $\partial_0 K = K$
- (ii) for each $n \in \mathbb{Z}^+$, $\partial_{n+1} K = \partial \partial_n K$.

Define $\partial_\infty(K) = \bigcap_{n \in \mathbb{Z}^+} \partial_n(K)$. Then either $\partial_\infty K \neq \emptyset$, or $\partial_m K = \emptyset$ for some $m \in \mathbb{Z}^+$. In the former case, we say that K has infinite limit level ; in the latter, we say that K has finite limit level. Assume that K is a compact metric space. Pham proved in [22] that if $t \in K$, then the maximal ideal $\mathcal{M}_t := \{f \in \mathcal{C}(K) \mid f(t) = 0\}$ contains an infinite pseudo-finite family of nonmaximal prime ideals if and only if $t \in \partial_\infty K$. In particular if K is countable compact metric space of infinite level then assuming CH there exists a discontinuous algebra seminorm q on $\mathcal{C}(K)$ such that the continuity ideal $\mathcal{I}(q)$ of q is not a finite intersection of prime ideals. The situation is even more complicated on $\mathcal{C}([0, 1])$: it follows from [23] that the maximal ideal \mathcal{M}_t contains for every $t \in [0, 1]$ an intersection non-redundant pseudo-finite family $(I_\lambda)_{\lambda \in \Lambda}$ such that $|\Lambda| = 2^{\aleph_0}$, and so assuming (CH) there exists a discontinuous algebra seminorm on $\mathcal{C}([0, 1])$ such that the continuity ideal of q is not the intersection of any countable family of nonmaximal prime ideals.

In [24], Pham discusses the structure of continuity ideals in the general case. He shows that the continuity ideal of a discontinuous algebra seminorm on $\mathcal{C}(K)$ is always the intersection of an intersection non-redundant family \mathcal{J} of nonmaximal prime ideals such that every infinite sequence of elements of \mathcal{J} contains an infinite pseudo-finite subsequence. He obtains a partial converse, assuming (CH) : if \mathcal{J} is an intersection non-redundant family \mathcal{J} of nonmaximal prime ideals such that every infinite sequence of elements of \mathcal{J} contains an infinite pseudo-finite subsequence, and if the two following conditions are satisfied

1. $|\mathcal{C}(K)/\bigcap_{I \in \mathcal{J}} I| = 2^{\aleph_0}$
2. $\bigcap_{I \in \mathcal{J}} I$ is the intersection of a countable family of non-maximal prime ideals

then there exists a discontinuous algebra seminorm q on $\mathcal{C}(K)$ such that the continuity ideal of q equals $\bigcap_{I \in \mathcal{J}} I$.

A discussion at the end of [24] suggests that condition 2 above is not necessary. We were not able to remove this condition in the present article, in which we present a slightly different approach to these questions.

In section 2, we discuss the set $\text{Prim}(q)$ of all nonmaximal prime ideals which are closed with respect to some discontinuous algebra seminorm q on $\mathcal{C}(K)$, and the related set $\mathcal{U}(\text{Prim}(q))$ of all unions of elements of $\text{Prim}(q)$. Using results of [12], we show that *every chain of elements of $\mathcal{U}(\text{Prim}(q))$ is well-ordered with respect to inclusion*.

Recall that an ideal I in a commutative ring A is said to be *semiprime* if I contains every $a \in A$ such that $a^n \in I$ for some positive integer n , which is equivalent to the fact that I equals the intersection of the prime ideals of A which contain it. We will say that a semiprime ideal I of a

commutative algebra A is *pure* if for every maximal ideal \mathcal{M} of A containing I there exists a prime ideal J of A such that $I \subset J \subsetneq \mathcal{M}$.

Our discussion of the continuity ideal $\mathcal{I}(q)$ involves the following notion.

Definition 1.3 *Let A be a commutative ring. A Badé-Curtis ideal of A is an ideal I of A such that, for every sequence $(f_n)_{n \geq 1}$ of elements of A such that $f_n f_m = 0$ for $n \neq m$, there exists $p \geq 1$ such that $f_n \in I$ for every $n \geq p$.*

A standard consequence of the Badé-Curtis main boundedness theorem [2], theorem 2.1, and of Sinclair's stability lemma, [5], 5.2.7 or [28], 1.6 is that *the continuity ideal of a discontinuous algebra seminorm on $\mathcal{C}(K)$ is a semiprime pure Badé-Curtis ideal of $\mathcal{C}(K)$* (a proof of this result using a general property of linear seminorms on $\mathbb{R}^{\mathbb{N}}$ instead of using the stability lemma can be found in [12]). In fact we will show in section 2 that if I is an ideal of $\mathcal{C}(K)$, then the following conditions imply each other :

1. I is a (pure) semiprime Badé-Curtis ideal
2. There exists a family \mathcal{I} of (nonmaximal) prime ideals of $\mathcal{C}(K)$ such that $\cap\{J : J \in \mathcal{I}\} = I$ and such that every chain of unions of elements of \mathcal{I} is well-ordered with respect to inclusion.
3. There exists an intersection non-redundant family \mathcal{J} of (nonmaximal) prime ideals of $\mathcal{C}(K)$ such that $\cap\{J : J \in \mathcal{J}\} = I$ and such that every chain of unions of elements of \mathcal{J} is well-ordered with respect to inclusion.

Now let E be a set, and let $\mathcal{F} = (F_\lambda)_{\lambda \in \Lambda}$ be a family of subsets of E . We observe in section 2 that the following conditions imply each other :

1. Every chain of unions of subfamilies of the family \mathcal{F} is well-ordered with respect to inclusion.
2. Every infinite sequence of elements of the family \mathcal{F} contains an infinite pseudo-finite subsequence.

In particular, the class of semiprime Badé-Curtis ideals of $\mathcal{C}(K)$ is the same as the class of abstract continuity ideals of $\mathcal{C}(K)$ introduced by Pham in [24]. Also the fact that if $\mathcal{I} := (I_\lambda)_{\lambda \in \Lambda}$ is a family of prime ideals of $\mathcal{C}(K)$ satisfying condition 1 above then the union of any subfamily of \mathcal{I} is a finite union of prime ideals, observed by the author in some unpublished notes, follows from lemma 5.7 of [24].

In section 3 we discuss a general version of a lifting result which shows that if I and J a prime ideal of $\mathcal{C}(K)$, and if $I \subset J$, then there exists a subalgebra $A_{I,J}$ of the quotient algebra $\mathcal{C}(K)/J$ such that $\mathcal{C}(K)/J = A_{I,J} \oplus \pi_J(I)$, where $\pi_J : \mathcal{C}(K) \rightarrow \mathcal{C}(K)/J$ denotes the canonical surjection. It follows immediately from this result that if a nonmaximal prime ideal J is the kernel of some discontinuous algebra seminorm q on $\mathcal{C}(K)$, and if I is a nonmaximal prime ideal containing J , then there exists another discontinuous seminorm \tilde{q} on $\mathcal{C}(K)$ such that $\text{Ker}(\tilde{q}) = I$. Our general version of the lifting theorem provides, given a nonmaximal prime ideal I of $\mathcal{C}(K)$, a subalgebra B_I of the quotient algebra $\mathcal{C}(K)/\mathcal{J}(I)$ such that $\mathcal{C}(K)/\mathcal{J}(I) = B_I \oplus \pi(I)$, where $\mathcal{J}(I)$ denotes the intersection of all prime ideals of $\mathcal{C}(K)$ contained in I and where $\pi : \mathcal{C}(K) \rightarrow \mathcal{C}(K)/\mathcal{J}(I)$ denotes the canonical surjection.

If it were possible to construct the algebras A_I so that $\pi_{\mathcal{J}(I')}^{-1}(A'_I) \subset \pi_{\mathcal{J}(I)}^{-1}(A_I)$ for every pair (I, I') of nonmaximal prime ideals of $\mathcal{C}(K)$ such that $I \subset I'$, it would be possible to remove the condition that I is a countable intersection of prime ideals in theorem 6.7 (ii) of [24], which would give a complete characterization of the ideals I of $\mathcal{C}(K)$ satisfying $|\mathcal{C}(K)/I| = 2^{\aleph_0}$ which are the continuity ideal of some discontinuous algebra seminorm on $\mathcal{C}(K)$ (assuming (CH)).

We were not able to do this, but we conjecture that there exists (assuming CH) a discontinuous norm on $\mathcal{C}(K)$ satisfying $\text{Prim}(q) = \mathcal{U}$ for every family \mathcal{U} of nonmaximal prime ideals of $\mathcal{C}(K)$ satisfying the three following conditions

- every sequence of elements of \mathcal{U} contains a pseudo-finite subsequence,
- $\mathcal{U} \cup \mathcal{M}_{\mathcal{U}}$ is "almost stable under unions", where $\mathcal{M}_{\mathcal{U}}$ denotes the set of maximal ideals of $\mathcal{C}(K)$ containing some element of \mathcal{U} ,
- $|\mathcal{C}(K)/I| = 2^{\aleph_0}$ for every $I \in \mathcal{U}$.

This conjecture is proved in [14] in the case where \mathcal{U} is a chain (in this situation the first condition just means that \mathcal{U} is well ordered).

A major success in the works related to Kaplansky's problem in the seventies was the fact that, if the continuum hypothesis is assumed, every complex non-unital commutative algebra U of cardinality 2^{\aleph_0} possesses an algebra norm. There were two ways to reach this result. The first one, followed by the author, consisted in showing that a 'big' algebra of power series, denoted by \mathbb{C}_{ω_1} , does have an algebra norm, and then use the "fundamental theorem on extension of places" to embed U in the maximal ideal of a complex valuation algebra which, assuming the continuum hypothesis, can be embedded into the normable algebra \mathbb{C}_{ω_1} . The second one, followed by Dales and Woodin, consisted in proving that there always exists an ultrafilter \mathcal{U} on \mathbb{N} such that U embeds into the quotient algebra c_0/\mathcal{U} , and then use Dales' construction [4] to show that, if CH is assumed, the quotient algebra c_0/\mathcal{U} is normable. Pham uses in theorem 6.5 of [22] and proposition 6.2 of [24] arguments based on Woodin's embeddings of nonunital integral domains of cardinality 2^{\aleph_0} , based on the existence of "almost disjoint" infinite families of integers and on Hilbert's Nullstellensatz, to obtain the major results of [22] and [24].

In section 4 we propose an alternative to this approach. We show that, if I and J are prime ideals of $\mathcal{C}(K)$, with $I \subset J$, and if B is a subalgebra of the quotient algebra $\mathcal{C}(K)/I$ such that $\mathcal{C}(K)/I = B \oplus \pi_I(J)$, where $\pi_I : \mathcal{C}(K) \rightarrow \mathcal{C}(K)/I$ denotes the canonical surjection, then every one-to-one homomorphism $\phi : B \rightarrow \mathbb{C}_{\omega_1}$ into a "small" subalgebra of \mathbb{C}_{ω_1} can be extended to a one to one homomorphism $\psi : \mathcal{C}(K)/I \rightarrow \mathbb{C}_{\omega_1}$. Using this result it is for example easy to see that if $(I_\lambda)_{\lambda \in \Lambda}$ is a pseudo-finite family of ideals of $\mathcal{C}(K)$, and if $|\mathcal{C}(K)/I_\lambda| = 2^{\aleph_0}$ for $\lambda \in \Lambda$, then the quotient algebra $\mathcal{C}(K)/\bigcap_{\lambda \in \Lambda} I_\lambda$ has an algebra norm if CH is assumed, a result slightly stronger than corollary 7.3 of [22]. The proof of our "extension theorem" is based on the "fundamental theorem on extension of places" and on Kaplansky's embedding theorem of valued fields into maximal valued fields [18].

A survey of the current state-of-the-art concerning Kaplansky's problem and Michael's problem on continuity of characters on Fréchet algebras is proposed by the author in [13], where some of the results of the present paper are announced without proof.

2 Nonmaximal prime ideals closed for a discontinuous algebra norm on $\mathcal{C}(K)$

In what follows K denotes an infinite compact space. Let $\text{Prim}(q)$ be the set of nonmaximal prime ideals of $\mathcal{C}(K)$ which are closed with respect to a discontinuous algebra seminorm on $\mathcal{C}(K)$, and let $\mathcal{I}(q)$ be the continuity ideal of q , see definition 1.2. As indicated in the introduction, it follows from automatic continuity theory that we have the following properties.

Theorem 2.1 (i) $\mathcal{I}(q) = \bigcap \{I : I \in \text{Prim}(q)\}$.

(ii) $\mathcal{I}(q)$ is the largest ideal I of $\mathcal{C}(K)$ such that the restriction of q to I is continuous.

(iii) $\mathcal{I}(q)$ is a pure semi-prime Badé-Curtis ideal of $\mathcal{C}(K)$.

If I is an ideal of $\mathcal{C}(K)$, set $Z(I) := \{t \in K : I \subset \mathcal{M}_t\}$, where $\mathcal{M}_t = \{f \in \mathcal{C}(K) : f(t) = 0\}$. The fact that $Z(\mathcal{I}(q))$ is finite was proved by Badé and Curtis in their seminal paper [2]. In fact if $Z(I)$ is infinite for some ideal I of $\mathcal{C}(K)$ then it is possible to construct by induction a sequence $(t_n)_{n \geq 1}$

of distinct elements of $Z(I)$ and a sequence U_n of disjoint open subsets of K such that $t_n \in U_n$ for $n \geq 1$. There exists for every $n \geq 1$ a function $f_n \in \mathcal{C}(K)$, with $\text{Supp}(f_n) \subset U_n$, such that $f_n(t_n) = 1$. Hence $f_n f_m = 0$ for $n \neq m$ and $f_n \notin I$ for every $n \geq 1$, which shows that I is not a Badé-Curtis ideal (this argument applies to Badé-Curtis ideals of semi-simple commutative regular Banach algebras).

Let Ω_q be the Stone-Čech compactification of $K \setminus Z(\mathcal{I}(q))$, identify the closure $\bar{\mathcal{I}}(q) := \{f \in \mathcal{C}(K) : f(t) = 0 \ \forall t \in Z(\mathcal{I}(q))\}$ of $\mathcal{I}(q)$ in $\mathcal{C}(K)$ to the set of functions $f \in \mathcal{C}(\Omega_q)$ vanishing on $\Omega_q \setminus (K \setminus Z(\mathcal{I}(q)))$, and denote by $\tilde{\mathcal{I}}(q)$ the set of all functions $f \in \mathcal{C}(\Omega_q)$ such that $fg \in \mathcal{I}(q)$ for every $g \in \bar{\mathcal{I}}(q)$. Using similar arguments, it is shown in [9] that the set $Z(\tilde{\mathcal{I}}(q))$ is a finite subset of $\Omega_q \setminus (K \setminus Z(\mathcal{I}(q)))$.

Recall that a linear subspace L of $\mathcal{C}(K)$ is said to be *absolutely convex* if $g \in L$ for every $g \in \mathcal{C}(K)$ satisfying $|g| \leq |f|$ for some $f \in L$. We will use the following lemma.

Lemma 2.2 *Let \mathcal{L} be a family of absolutely convex linear subspaces of $\mathcal{C}(K)$, and let $\mathcal{U}(\mathcal{L})$ be the family of all sets of the form $S_{\mathcal{N}} := \cup\{L : L \in \mathcal{N}\}$ where $\mathcal{N} \subset \mathcal{L}$. If $\mathcal{U}(\mathcal{L})$ contains a chain which is not well-ordered with respect to inclusion, then at least one of the two following conditions holds*

- (i) *there exists a sequence $(L_n)_{n \geq 1}$ of elements of \mathcal{L} such that $L_{n+1} \subsetneq L_n$ for every $n \geq 1$,*
- (ii) *there exists a sequence $(L_n)_{n \geq 1}$ of elements of \mathcal{L} such that $L_n \not\subseteq \cup_{m \neq n} L_m$ for every $n \geq 1$.*

Proof : There exists a sequence $(\mathcal{L}_n)_{n \geq 1}$ of subsets of \mathcal{L} such that $\cup\{L : L \in \mathcal{L}_{n+1}\} \subsetneq \cup\{L : L \in \mathcal{L}_n\}$, and we can find a sequence $(L_n)_{n \geq 1}$ of elements of \mathcal{L} such that $L_n \in \mathcal{L}_n$ and $L_n \not\subseteq \cup_{m \geq n+1} L_m$ for $n \geq 1$. Let S be the set of all integers $n \geq 1$ such that L_n does not contain L_m for any $m \neq n$.

If S is empty or finite there exists $p \geq 1$ such that $n \notin S$ for $n \geq p$. Let $n \geq p$, and let $m \neq n$ such that $L_m \subset L_n$. Since L_m is not contained in $\cup_{k > m} L_k$, we have $m > n$, and we can construct by induction a strictly increasing sequence $(n_k)_{k \geq 1}$ of positive integers such that $L_{n_{k+1}}$ is strictly contained in L_{n_k} for $k \geq 1$.

Now if S is infinite, let $(n_k)_{k \geq 1}$ be a strictly increasing sequence of positive integers such that $n_k \in S$ for every $k \geq 1$. Set $\tilde{L}_k = L_{n_k}$, and let $k \geq 2$. Then \tilde{L}_k is not contained in $\cup_{l > k} \tilde{L}_l$, and so there exists $g_k \in \tilde{L}_k$ such that $g_k \notin \cup_{l > k} \tilde{L}_l$. Now let $j \in \{0, \dots, k-1\}$. Since $n_j \in S$, \tilde{L}_j does not contain \tilde{L}_k and so there exists a function $g_{k,j} \in \tilde{L}_k \setminus \tilde{L}_j$. Using absolute convexity, we see that $f_k := |g_k| + \sum_{j=1}^{k-1} |g_{k,j}| \in \tilde{L}_k$, but that $f_k \notin \tilde{L}_l$ for $l \neq k$, which completes the proof of the lemma. \square

The link with Pham's discussion of continuity ideals is given by the following observation.

Lemma 2.3 *Let E be a set, let \mathcal{F} be a family of subsets of E , and let $\mathcal{U}(\mathcal{F})$ be the family of all sets of the form $S_{\mathcal{G}} := \cup\{F : F \in \mathcal{G}\}$, where $\mathcal{G} \subset \mathcal{F}$. Then the following conditions imply each other :*

- (i) *Every chain of elements of $\mathcal{U}(\mathcal{F})$ is well-ordered with respect to inclusion.*
- (ii) *Every sequence of elements of \mathcal{F} possesses a pseudo-finite subsequence.*
- (iii) *Every sequence of elements of $\mathcal{U}(\mathcal{F})$ possesses a pseudo-finite subsequence.*

Proof : Assume that a sequence $(F_n)_{n \geq 1}$ of elements of \mathcal{F} does not possess any pseudo-finite subsequence. Then we could construct by induction a sequence $(A_n)_{n \geq 1}$ of infinite subsets of \mathbb{Z}^+ and a sequence $(u_n)_{n \geq 1}$ of elements of E such that $A_{n+1} \subset A_n$, $u_n \in \cup_{n \in A_n} F_n$ and $u_n \notin \cup_{n \in A_{n+1}} F_n$, which contradicts the fact that every chain of elements of $\mathcal{U}(\mathcal{F})$ is well-ordered with respect to inclusion. Hence (i) implies (ii).

Now assume that some chain of elements of $\mathcal{U}(\mathcal{F})$ is not well-ordered with respect to inclusion. Then there exists a sequence \mathcal{G}_n of subsets of \mathcal{F} such that $\cup\{F : F \in \mathcal{G}_{n+1}\} \subsetneq \cup\{F : F \in \mathcal{G}_n\}$ for $n \geq 1$. We can then construct a sequence $(F_n)_{n \geq 1}$ of elements of \mathcal{F} such that $F_n \in \mathcal{G}_n$ and $F_n \not\subseteq \cup\{F : F \in \mathcal{G}_{n+1}\}$. Now let $(n_p)_{p \geq 1}$ be a strictly increasing sequence of positive integers. We

have $F_{n_p} \not\subseteq \cup_{q \geq p+1} F_{n_q}$ for $p \geq 1$. In particular there exists $a \in F_{n_1}$ such that $a \notin F_{n_q}$ for $q \geq 2$, and so the sequence $(F_{n_p})_{p \geq 1}$ is not pseudo-finite. Hence (ii) implies (i).

Clearly, (iii) implies (ii). Now assume that (i) holds, and let $(U_n)_{n \geq 1}$ be a sequence of elements of $\mathcal{U}(\mathcal{F})$. There exists an infinite subset W_0 of \mathbb{N} such that $\cup_{n \in W} U_n = \cup_{n \in W_0} U_n$ for every infinite subset W of W_0 , which means that the family $\{U_n\}_{n \in W_0}$ is pseudo-finite. Hence (i) implies (iii). \square

It follows from [9] that every chain of nonmaximal prime ideals of $\mathcal{C}(K)$ which are closed with respect to a discontinuous algebra norm on $\mathcal{C}(K)$ is well-ordered with respect to inclusion. The following theorem gives an improvement of this result.

Theorem 2.4 *Let q be an discontinuous algebra seminorm on $\mathcal{C}(K)$, let $\text{Prim}(q)$ of all nonmaximal prime ideals of $\mathcal{C}(K)$ which are closed with respect to q , and let $\mathcal{U}(\text{Prim}(q))$ be the family of all sets of the form $S_N := \cup\{I : I \in N\}$, where $N \subset \text{Prim}(q)$. The set $\text{Prim}(q)$ satisfies the following two equivalent conditions*

- (i) *Every chain of elements of $\mathcal{U}(\text{Prim}(q))$ is well-ordered with respect to inclusion.*
- (ii) *Every sequence $(I_n)_{n \geq 1}$ of elements of $\text{Prim}(q)$ possesses a pseudo-finite subsequence.*

Proof : Assume that (i) is not satisfied. Since any chain of elements of $\text{Prim}(q)$ is well-ordered with respect to inclusion, it would follow from lemma 2.2 that there exists a sequence $(I_n)_{n \geq 1}$ of elements of $\text{Prim}(q)$ such that $I_n \not\subseteq \cup_{m \neq n} I_m$. It would then follow from lemma 3.9 of [9] that there exists a sequence $(f_n)_{n \geq 1}$ of elements of $\mathcal{C}(K)$ such that $f_n \notin \cap_{m \geq 1} I_m$ for $n \geq 1$ whereas $f_m f_n = 0$ for $m \neq n$, which contradicts the fact that the continuity ideal $\mathcal{I}(q) = \cap\{I : I \in \text{Prim}(q)\}$ of q is a Badé-Curtis ideal of $\mathcal{C}(K)$.

The fact that conditions (i) and (ii) are equivalent is given by lemma 2.3. \square

The following result is essentially a reformulation of lemma 5.7 of [24], where Pham shows that every "compact" family of prime ideals of $\mathcal{C}(K)$ admits a finite number of "roofs", in a slightly more general context.

Proposition 2.5 *Let \mathcal{F} be a family of prime ideals of $\mathcal{C}(K)$ such that every sequence of elements of \mathcal{F} has a pseudofinite subsequence, and let $\mathcal{V}(\mathcal{F})$ be the set of ideals of $\mathcal{C}(K)$ which belong to $\mathcal{U}(\mathcal{F})$.*

- (i) *Let $S \in \mathcal{U}(\mathcal{F})$, and let \mathcal{G}_S be the set of all $J \in \mathcal{V}(\mathcal{F})$ contained in S . Then the set Δ_S of maximal elements of \mathcal{G}_S is finite, and $S = \cup\{J : J \in \Delta_S\}$.*
- (ii) *If $S \in \mathcal{V}(\mathcal{F})$, and if $(A_n)_{n \geq 1}$ is a family of disjoint maximal chains of elements of $\mathcal{G}_S \setminus \{S\}$, then $S = \cup_{n \geq 1} J_n$ for every sequence J_n of ideals of $\mathcal{C}(K)$ such that $J_n \in A_n$ for $n \geq 1$.*

Proof : (i) It follows from Zorn's lemma that every element of \mathcal{G}_S belongs to a maximal chain of elements of \mathcal{G}_S . Since the union of a chain of elements of \mathcal{G}_S is a prime ideal contained in S , we see that every maximal chain of elements of \mathcal{G}_S has a largest element which is a maximal element of \mathcal{G}_S . This shows that $S = \cup\{J : J \in \Delta_S\}$.

Assume that Δ_S is infinite, and let I_n be a sequence of distinct elements of Δ_S . It follows from lemma 2.3 that the sequence $(I_n)_{n \geq 1}$ admits a pseudo-finite subsequence $(I_{n_p})_{p \geq 1}$. Set $I := \cup_{p \geq 1} I_{n_p}$. Then $I \in \mathcal{G}_S$, which contradicts the maximality of the ideals I_{n_p} . Hence Δ_S is finite.

(ii) Since every sequence of elements of $\mathcal{V}(\mathcal{F})$ possesses a pseudofinite subsequence, we can assume without loss of generality that the sequence $(J_n)_{n \geq 1}$ is pseudofinite. Let $J = \cup_{n \geq 1} J_n$, and let $L \in A_n$. If $L \subset J_n$, then $L \subset J$, and if $J_n \subset L$ then either $L \subset J$ or $J \subset L$. So $A_n \cup J$ is a chain of elements of \mathcal{G}_S for $n \geq 1$. But if $J \neq S$ we would have $J \in \cap_{n \geq 1} A_n$, which is impossible. Hence $J = S$. \square

We obtain the following intriguing property, which had been already noticed by the author in some unpublished notes before he heard of the paper [24].

Corollary 2.6 *Let q be a discontinuous algebra seminorm on $\mathcal{C}(K)$, and let $S \in \mathcal{U}(\text{Prim}(q))$. Then there exists $n \geq 1$ and $J_1, \dots, J_n \in \mathcal{U}(\text{Prim}(q))$ satisfying the following properties*

- (i) J_1, \dots, J_n are prime ideals of $\mathcal{C}(K)$, and $J_i \not\subseteq J_j$ if $i \neq j$.
- (ii) $S = \cup_{1 \leq j \leq n} J_j$.

Notice that if $(I_\lambda)_{\lambda \in \Lambda}$ is an uncountable pseudo-finite family of elements of $\text{Prim}(q)$ (such families are constructed in [23] under (CH) for some compact sets) then $\cup_{\lambda \in \Lambda} I_\lambda \in \text{Prim}(q)$. To see this assume that $f \in \mathcal{C}(K)$ satisfies $\lim_{n \rightarrow \infty} q(f - f_n) = 0$, with $f_n \in \cup_{\lambda \in \Lambda} I_\lambda$ for $n \geq 1$. Then the set $\cup_{n \geq 1} \{\lambda \in \Lambda : f_n \notin I_\lambda\}$ is at most countable, and so there exists $\mu \in \Lambda$ such that $f_n \in I_\mu$ for every $n \geq 1$, so that $f \in I_\mu \subset \cup_{\lambda \in \Lambda} I_\lambda$. Similarly if ω is a limit ordinal which does not admit any cofinal sequence, and if $(I_\zeta)_{\zeta < \omega}$ is a well-ordered chain of elements of $\text{Prim}(q)$ satisfying $I_\zeta \subsetneq I_{\zeta'}$ for $\zeta < \zeta' < \omega$, then $\cup_{\zeta < \omega} I_\zeta \in \text{Prim}(q)$.

If I is an ideal of a commutative algebra A , we set $I : a = \{b \in A : ab \in I\}$ for $a \in A$. We now introduce a notion used by Pham in [24].

Definition 2.7 *Let A be a commutative algebra. An ideal I of A is an abstract continuity ideal if, for each sequence $(a_n)_{n \geq 1}$ in A , there exists $n_0 \geq 1$ such that*

$$I : a_1 \dots a_n = I : a_1 \dots a_{n+1} \quad (n \geq n_0)$$

It is possible to deduce the following observation from lemmas 5.6 and 5.7 of [24]. We give a proof for the sake of completeness.

Proposition 2.8 *Let $(P_\alpha)_{\alpha \in \Lambda}$ be a family of prime ideals of a commutative ring A such that $P_\alpha \not\subseteq P_\beta$ for $\alpha \neq \beta$. If the family $(P_\alpha)_{\alpha \in \Lambda}$ is intersection redundant, then there exists a sequence $(\alpha_n)_{n \geq 1}$ such that $\cup_{p \geq n+1} P_{\alpha_p} \subsetneq \cup_{p \geq n} P_{\alpha_p}$ for $n \geq 1$.*

Proof : Let α be such that $\cap_{\beta \neq \alpha} P_\beta \subset P_\alpha$, let $\alpha_1 \neq \alpha$, and let $f_1 \in P_{\alpha_1} \setminus P_\alpha \cap P_{\alpha_1}$. Since $\cap_{\beta \neq \alpha} P_\beta \subset P_\alpha$, there exists $\alpha_2 \in \Lambda \setminus \{\alpha, \alpha_1\}$ such that $f_1 \notin P_{\alpha_2}$.

Starting with f_1, α_1 and α_2 , we now construct two sequences $(f_n)_{n \geq 1}$ and $(\alpha_n)_{n \geq 1}$ such that $f_n \in P_{\alpha_n} \setminus P_{\alpha_n} \cap P_\alpha$ and $f_1 \dots f_n \notin P_{\alpha_{n+1}}$ for $n \geq 1$. Assume that f_1, \dots, f_n and $\alpha_1, \dots, \alpha_{n+1}$ have been constructed for some $n \geq 1$. Take $f_{n+1} \in P_{\alpha_{n+1}} \setminus P_{\alpha_{n+1}} \cap P_\alpha$. Then $f_1 \dots f_{n+1} \in \cap_{1 \leq p \leq n+1} P_{\alpha_p}$, and since $f_1 \dots f_{n+1} \notin P_\alpha$ there exists $\alpha_{n+2} \in \Lambda \setminus \{\alpha, \alpha_1, \dots, \alpha_{n+1}\}$ such that $f_1 \dots f_{n+1} \notin P_{\alpha_{n+2}}$. So we can construct the sequences $(f_n)_{n \geq 1}$ and $(\alpha_n)_{n \geq 1}$ by induction.

We have $f_n \in P_{\alpha_n} \subset \cup_{p \geq n} P_{\alpha_p}$. But if $p \geq n+1$ we have $f_1 \dots f_{p-1} \notin P_{\alpha_p}$. A fortiori $f_n \notin P_{\alpha_p}$ and $f_n \notin \cup_{p \geq n+1} P_{\alpha_p}$, which concludes the proof of the proposition. \square

Corollary 2.9 *Let \mathcal{L} be a family of prime ideals in a commutative ring A satisfying one of the equivalent conditions of lemma 2.3. If $J \not\subseteq L$ for $I \in \mathcal{L}, J \in \mathcal{L}, J \neq L$, then the family \mathcal{L} is intersection non-redundant.*

Proof : Since every subfamily of \mathcal{L} also satisfies the equivalent conditions of lemma 2.3, the corollary follows immediately from proposition 2.8. \square

We will say that an abstract continuity ideal is *pure* if every for every maximal ideal \mathcal{M} of A containing I there exists a prime ideal J of A such that $I \subseteq J \subsetneq \mathcal{M}$. Now if \mathcal{L} is a family of ideals of $\mathcal{C}(K)$, we will denote again by $\mathcal{U}(\mathcal{L})$ the family of sets of the form $S_{\mathcal{N}} := \cup\{I : I \in \mathcal{N}\}$, where $\mathcal{N} \subset \mathcal{L}$.

Using corollary 4.12 of [24], we obtain the following result.

Theorem 2.10 : Let I be an ideal of $\mathcal{C}(K)$, let $\mathcal{P}(I)$ be the set of prime ideals of $\mathcal{C}(K)$ containing I and let $\mathcal{P}_0(I)$ be the set of minimal elements of $\mathcal{P}(I)$. The following conditions imply each other.

- (i) I is a pure semiprime Badé-Curtis ideal.
- (ii) I is a pure abstract continuity ideal.
- (iii) There exists a family \mathcal{L} of nonmaximal prime ideals of $\mathcal{C}(K)$ such that every chain of elements of $\mathcal{U}(\mathcal{L})$ is well-ordered with respect to inclusion which satisfies $I = \cap\{P : P \in \mathcal{L}\}$.
- (iv) There exists a family \mathcal{L} of nonmaximal prime ideals of $\mathcal{C}(K)$ such that every sequence of elements of \mathcal{L} contains a pseudo-finite subsequence which satisfies $I = \cap\{P : P \in \mathcal{L}\}$.
- (v) I is semiprime, every element of $\mathcal{P}_0(I)$ is nonmaximal, and every chain of elements of $\mathcal{U}(\mathcal{P}_0(I))$ is well-ordered with respect to inclusion.
- (vi) I is semiprime, every element of $\mathcal{P}_0(I)$ is nonmaximal, and every sequence of elements of $\mathcal{P}_0(I)$ possesses a pseudo-finite subsequence.

If these conditions are satisfied then $\mathcal{P}_0(I)$ is intersection non-redundant, and every $S \in \mathcal{U}(\mathcal{P}_0(I))$ can be written under the form $S = \cup_{1 \leq j \leq n} J_j$, where $J_i \in \mathcal{U}(\mathcal{P}_0(I))$ is a prime ideal of $\mathcal{C}(K)$ for $1 \leq j \leq n$, and where $J_i \not\subseteq J_j$ for $i \neq j$. Also if \mathcal{L} is a family of prime ideals such that $P \not\subseteq Q$ and $Q \not\subseteq L$ for $P \in \mathcal{L}, Q \in \mathcal{L}, P \neq Q$, and if $I = \cap\{P : P \in \mathcal{L}\}$, then $\mathcal{L} = \mathcal{P}_0(I)$.

Proof : It follows from lemma 2.3 that (iii) and (iv) are equivalent and that (v) and (vi) are equivalent, and it follows from corollary 4.13 of [24] that (ii), (iv) and (vi) are equivalent. Since every subfamily of a family of ideals satisfying one of the equivalent conditions of lemma 2.3 also satisfies these conditions, (iii) implies (v) and (v) obviously implies (iii). Now assume that I is semiprime, that no element of $\mathcal{P}_0(I)$ is maximal and that some chain of elements of $\mathcal{U}(\mathcal{P}_0(I))$ is not well-ordered with respect to inclusion. Since every chain of elements of $\mathcal{P}_0(I)$ is a singleton, it follows from lemma 2.3 that there exists a sequence $(P_n)_{n \geq 1}$ of elements of $\mathcal{P}_0(I)$ such that $P_n \not\subseteq \cup_{m \neq n} P_m$ for $n \geq 1$. It follows then from lemma 3.9 of [9] that there exists a sequence $(f_n)_{n \geq 1}$ of elements of $\mathcal{C}(K)$ such that $f_n \notin \cap_{m \geq 1} P_m$ for $n \geq 1$ whereas $f_m f_n = 0$ for $n \neq m$. Hence I is not a Badé-Curtis ideal, which shows that (i) implies (v).

Now assume that (v) holds. Since I is semiprime, we have $I = \cup\{P : P \in \mathcal{P}_0(I)\}$ and $\mathcal{P}_0(I)$ does not contain any maximal ideal of $\mathcal{C}(K)$. Let $(f_n)_{n \geq 1}$ be a sequence of elements of $\mathcal{C}(K)$ such that $f_m f_n = 0$ for $n \neq m$. Assume that there exists a strictly increasing sequence $(n_m)_{m \geq 1}$ of positive integers such that $f_{n_m} \notin I$ for every $m \geq 1$.

Set $g_p := \sum_{m=p+1}^{+\infty} \frac{f_{n_m}}{2^m \|f_{n_m}\|}$ for $p \geq 1$. We have $g_p f_{n_m} = \frac{f_{n_m}^2}{2^m \|f_{n_m}\|} \notin P_{\lambda_m}$ for $m \geq p+1$, and so $g_p \notin \cup_{m \geq p+1} P_{\lambda_m}$. On the other hand $g_p f_{n_p} = 0$, and so $g_p \in P_{\lambda_p}$. Hence $\cup_{m \geq p+1} P_{\lambda_m} \subsetneq \cup_{m \geq p} P_{\lambda_m}$ for $p \geq 1$, which contradicts the fact that every chain of elements of $\mathcal{U}(\mathcal{P}_0(I))$ is well-ordered with respect to inclusion. So I is a Badé-Curtis ideal, and (v) implies (i).

Also it follows from corollary 2.9 that condition (vi) implies that $\mathcal{P}_0(I)$ is intersection non-redundant, and it follows from lemma 5.7 of [24] that every $S \in \mathcal{U}(\mathcal{P}_0(I))$ can be written under the form $S = \cup_{1 \leq j \leq n} J_j$, where $J_j \in \mathcal{U}(\mathcal{P}_0(I))$ is a prime ideal of $\mathcal{C}(K)$ for $1 \leq j \leq n$, and where $J_i \not\subseteq J_j$ if $i \neq j$.

Now assume that \mathcal{L} is a family of prime ideals such that $P \not\subseteq Q$ and $Q \not\subseteq L$ for $P \in \mathcal{L}, Q \in \mathcal{L}, P \neq Q$, and such that $I = \cap\{P : P \in \mathcal{L}\}$. If some chain of elements of $\mathcal{U}(\mathcal{L})$ were not well-ordered with respect to inclusion, it would follow from lemma 2.3 that there would exist a sequence $(P_n)_{n \geq 1}$ of elements of \mathcal{L} such that $P_n \not\subseteq \cup_{m \neq n} P_m$ for $n \geq 1$, and it would follow again from lemma 3.9 of [9] that there would exist a sequence $(f_n)_{n \geq 1}$ of elements of $\mathcal{C}(K)$ such that $f_n \notin \cap_{m \geq 1} P_m$ for $n \geq 1$ whereas $f_m f_n = 0$ for $n \neq m$, which contradicts the fact that I is a Badé-Curtis ideal of $\mathcal{C}(K)$. It follows again from corollary 2.9 that the family \mathcal{L} is intersection non-redundant. In particular for every $P \in \mathcal{L}$ there exists $f \in \mathcal{C}(K)$ such that $f \notin P$ but $f \in Q$ for every $Q \in \mathcal{L} \setminus P$. Let J be a prime

ideal of $\mathcal{C}(K)$ such that $I \subset J \subset P$. If $g \in P$ we have $fg \in I \subset J$. Hence $g \in J$ since $f \notin J$, and $P \in \mathcal{P}_0(I)$. So $\mathcal{L} \subset \mathcal{P}_0(I)$. But it follows from the fact that $\mathcal{P}_0(I)$ is intersection non-redundant that the intersection of elements of any proper subfamily of $\mathcal{P}_0(I)$ strictly contains I , and so $\mathcal{L} = \mathcal{P}_0(I)$. \square

3 Lifting results and problems

If A is a commutative ring we will denote by $A[x_1, \dots, x_m]$ (resp. by $A[x]$ if $m = 1$) the ring of all polynomials in m indeterminates x_1, \dots, x_m with coefficients in A . Also if B is a commutative ring containing A , and if $c := [c_1, \dots, c_m] \in B^m$, we will denote by $A[c_1, \dots, c_m] := \{p(c_1, \dots, c_m) \mid p \in A[x_1, \dots, x_m]\}$ the subring of B generated by $A \cup \{c_1, \dots, c_m\}$.

For $f \in \mathcal{C}(K)$ we set $Z(f) := \{t \in K \mid |f(t)| = 0\}$. For $t \in K$, $\mathcal{M}_t := \{f \in \mathcal{C}(K) \mid f(t) = 0\}$ denotes the maximal ideal associated to t , and \mathcal{O}_t denotes the space of all functions $f \in \mathcal{C}(K)$ for which $Z(f)$ is a neighborhood of t , so that \mathcal{O}_t is dense in \mathcal{M}_t . We will also use the following notations, some of which involving the quotient order on $\mathcal{C}_{\mathbb{R}}(K)/P$ when P is a prime ideal of $\mathcal{C}_{\mathbb{R}}(K)$, with respect to which $\mathcal{C}_{\mathbb{R}}(K)/P$ is totally ordered, see [5], theorem 4.8.13 (Theorem 4.8.13 of [5] also shows that the field of fractions of $\mathcal{C}_{\mathbb{R}}(K)/P$ is also a totally ordered field with respect to the order associated to the quotient order on $\mathcal{C}_{\mathbb{R}}(K)/P$).

Definition 3.1 Let K be an infinite compact space, and let I be a prime ideal of $\mathcal{C}(K)$. Denote by $\mathcal{J}(I)$ the intersection of all prime ideals of $\mathcal{C}(K)$ contained in I , set $A_I := \mathcal{C}(K)/I$, $I_{\mathbb{R}} := \mathcal{C}_{\mathbb{R}}(K) \cap I$, $A_{I, \mathbb{R}} := \mathcal{C}_{\mathbb{R}}(K)/I_{\mathbb{R}}$, and denote by $\pi_I : \mathcal{C}(K) \rightarrow A_I$ the canonical surjection. For $f \in \mathcal{C}(K)$, set $|\pi_I(f)| := \pi_I(|f|)$, denote by K_I (resp. $K_I(\mathbb{R})$) the field of fractions of A_I (resp. $A_{I, \mathbb{R}}$) and set $K_I^+ := \cup_{n \geq 1} \{u \in A_I \mid |u| \leq n.1\}$ (resp $K_{I, \mathbb{R}}^+ := \cup_{n \geq 1} \{u \in A_{I, \mathbb{R}} \mid |u| \leq n.1\}$).

A subalgebra A of A_I is said to be algebraically closed if A contains every $u \in A_I$ satisfying $p(u) = 0$ for some nonzero $p \in A[x]$.

Notice that since every prime ideal I of $\mathcal{C}(K)$ is absolutely convex, the inequality $||f| - |g|| \leq |f - g|$ shows that the map $u \rightarrow |u|$ is well-defined on A_I .

It is a standard fact that K_I is algebraically closed, or equivalently, that $K_{I, \mathbb{R}}$ is "real-closed", see [8], p.38. More precisely if $a_0, \dots, a_{n-1} \in A_I$, then there exists $u_1, \dots, u_n \in A_I$ such that $a_0 + a_1x + \dots + a_{n-1}x^{n-1} + x^n = (x - u_1) \dots (x - u_n)$, (this can be proved using exercise 13A of [16] to adapt to the complex case the argument used in the proof of theorem 3.4 of [16] to prove an analogous result in the quotient algebra $\mathcal{C}_{\mathbb{R}}(K)/I_{\mathbb{R}}$ when n is odd). Now assume that A is an algebraically closed subalgebra of A_I , let $K \subset K_I$ be the field of fractions of A and let $u \in K_I \setminus \{0\}$ be algebraic over K . There exists $a_0, \dots, a_n \in A$, with $a_0 \neq 0, a_n \neq 0$, satisfying $a_0 + a_1u + \dots + a_nu^n = 0$. Set $v = a_0/u$. Since $v^n + a_1v^{n-1} + \dots + a_0^{n-2}a_{n-1}v + a_0^{n-1}a_n = 0$, we have $v \in A_I$, and so $v \in A$ and $u = a_0/v \in K$. This tautological observation shows that the field of fractions of an algebraically closed subalgebra of A_I is algebraically closed. Notice also that if H is a proper nonzero ideal of A_I then the quotient field of the unital algebra $H \oplus \mathbb{C}.1$ is K_I , which is algebraically closed, but $H \oplus \mathbb{C}.1$ is obviously not an algebraically closed subalgebra of A_I in the sense of definition 3.1.

Let I be a prime ideal of $\mathcal{C}(K)$. It is well known that if J is a prime ideal contained in I , and if $f \notin I, g \in I$, then $f + J$ is a divisor of $g + J$ in $\mathcal{C}(K)/J$. In fact the following more precise result is true (a less general result in the same direction is given by lemma 5.8 of [24]).

Lemma 3.2 Let I be a prime ideal of $\mathcal{C}(K)$. If $f \notin I$, then $f + \mathcal{J}(I)$ is a divisor of $g + \mathcal{J}(I)$ in $\mathcal{C}(K)/\mathcal{J}(I)$ for every $g \in I$.

Proof : Assume that $f \notin I$ and that $g \in I$. Since I is absolutely convex, we have $|f| \notin I$ and $|g| \in I$. Hence $|f|^2 \notin I$ and $[|f|^2 - |g|]^+ - [|f|^2 - |g|]^- = |f|^2 - |g| \notin I$.

We have $0 \leq [|f|^2 - |g|]^- = [|g| - |f|^2]^+ \leq |g|$, and so $[|f|^2 - |g|]^- \in I$, and $[|f|^2 - |g|]^+ \notin I$. Since $[|f|^2 - |g|]^+ \cdot [|f|^2 - |g|]^- = 0$, we see that $[|f|^2 - |g|]^- \in \mathcal{J}(I)$.

Set $h = |f|^2 + [|f|^2 - |g|]^-$, so that $0 \leq |g| \leq |h|$, and set $u(t) = 0$ if $h(t) = 0$, $u(t) = \frac{g(t)\overline{f}(t)}{h(t)}$ otherwise. Then $u \in \mathcal{C}(K)$.

Now set $v = \sqrt{[|f|^2 - |g|]^-}$, and set $w(t) = 0$ if $h(t) = 0$, $w(t) = \frac{g(t)v(t)}{h(t)}$ otherwise, so that $w \in \mathcal{C}(K)$. Since $\mathcal{J}(I)$ is semiprime, it contains v and $g - fu = vw \in \mathcal{J}(I)$. \square

Corollary 3.3 *Let I, J be prime ideals of $\mathcal{C}(K)$, with $I \subseteq J$, and let $\pi_{I,J} : f + I \rightarrow f + J$ be the canonical surjection of A_I onto A_J . If A is an algebraically closed subalgebra of A_I , then $\pi_{I,J}(A)$ is an algebraically closed subalgebra of A_J .*

Proof : Set $B = \pi_{I,J}(A)$. Let $u \in A_J \setminus \{0\}$, and assume that $b_0 = b_1 u$ with $b_0, b_1 \in B$, $b_0 \neq 0$. Let $a_0 \in \pi_{I,J}^{-1}(b_0)$, let $a_1 \in \pi_{I,J}^{-1}(b_1)$, let $f_0 \in \pi_I^{-1}(a_0)$, let $f_1 \in \pi_I^{-1}(a_1)$, and let $g \in \pi_J^{-1}(u)$. Then $f_0 - f_1 g \in J$, $f_1 \notin J$, and it follows from the lemma that there exists $h \in \mathcal{C}(K)$ such that $f_0 - f_1 g - f_1 h \in \mathcal{J}(J) \subset I$. Then $h \in J$, since J is prime, and $\pi_I(g + h) \in A$, since $a_0 - a_1 \pi_I(g + h) = 0$. Hence $u = \pi_{I,J}(\pi_I(g)) = \pi_{I,J}(\pi_I(g + h)) \in B$.

Now let again $u \in A_J \setminus \{0\}$, and assume that $b_0 + b_1 u + \dots + b_n u^n = 0$, with $b_0, \dots, b_n \in B$, $b_0 \neq 0, b_n \neq 0$. Let $a_i \in \pi_{I,J}^{-1}(b_i)$. There exists $v_1, \dots, v_n \in A_I$ such that $x^n + a_1 x + \dots + a_0^{n-1} a_n = (x - v_1) \dots (x - v_n)$, and $v_i \in A$ for $1 \leq i \leq n$ since A is an algebraically closed subalgebra of A_I . Set $w = b_0/u$. Then $0 = w^n + b_1 w^{n-1} + \dots + b_0^{n-1} b_n = (w - w_1) \dots (w - w_n)$, where $w_i = \pi_{I,J}(v_i) \in B$. Hence $w_i u = b_0$ for some $i \leq n$, and $u \in B$. \square

A folklore result, which has been known to the author since the seventies, shows that if I is a prime ideal of $\mathcal{C}(K)$, and if J is a prime ideal of $\mathcal{C}(K)$ contained in I , then there exists a subalgebra $A_{I,J}$ of the quotient algebra $\mathcal{C}(K)/J$ such that $\mathcal{C}(K)/J = A_{I,J} \oplus \pi(I)$, where $\pi : \mathcal{C}(K) \rightarrow \mathcal{C}(K)/J$ denotes the canonical surjection. The following lemma gives a more general property, which is also more general than proposition 5.9 of [24]. Notice that the proof uses an argument similar to an argument used by G.R. Allan in his seminal paper [1] to perform algebraic extensions of embeddings of subalgebras of $\mathbb{C}[[X]]$, the algebra of all formal power series, into some Banach algebras.

Lemma 3.4 *Let I be a prime ideal of $\mathcal{C}(K)$, let J be a prime ideal of $\mathcal{C}(K)$ contained in I , let B_0 be a unital subalgebra of $\mathcal{C}(K)$ such that $B_0 \cap I \subset \mathcal{J}(I)$, and let B_1 be a subalgebra of $\mathcal{C}(K)$ containing $B_0 + \mathcal{J}(J)$ such that $\pi_J(B_1)$ is an algebraically closed subalgebra of A_J . Then there exists a subalgebra B of B_1 containing $B_0 + \mathcal{J}(I)$ which possesses the following properties*

- (1) $B \cap I = \mathcal{J}(I)$
- (2) $B_1 = B + I \cap B_1$.

Proof : Let \mathcal{F} be the family of all subalgebras B of B_1 containing $B_0 + \mathcal{J}(I)$ such that $B \cap I = \mathcal{J}(I)$, so that $\tilde{\pi}_I(B) \cap \tilde{\pi}_I(I) = \{0\}$ for every $B \in \mathcal{F}$. Since the union of any family of elements of \mathcal{F} linearly ordered with respect to inclusion belongs to \mathcal{F} , it follows from Zorn's lemma that \mathcal{F} possesses a maximal element. Set $A_0 := \pi_I(B_0)$ and $A_1 = \pi_I(B_1)$.

We claim that if $u \in \mathcal{C}(K) \setminus I$, and if $p(u) \in \mathcal{J}(I) \subset J \cap B_1$ for some $p \in B_1[x]$ such that $p'(u) \notin I$, then $u \in I$.

To see this notice that since $\pi_J(B_1)$ is an algebraically closed subalgebra of A_J , there exists $v \in B_1$ and $r \in J$ such that $u = v - r$. We have

$$p(v) - p(u) = r \left(p'(u) + \sum_{j=2}^n \frac{p^{(j)}(u)}{j} r^{j-1} \right),$$

and so $p(u) - p(v) \in B_1 \cap J \subset \mathcal{J}(I)$.

It follows from the lemma that there exists $s \in J$ and $\rho \in \mathcal{J}(I)$ such that $r = p'(u)s + \rho$. We obtain

$$p'(u)^2 s \left(1 + \sum_{j=2}^n \frac{p^{(j)}(u)}{j} p'(u)^{n-2} s^{n-1} \right) \in \mathcal{J}(J).$$

Since the quotient algebra $\mathcal{C}(K)/\mathcal{J}(J)$ is a local ring, $1 + \sum_{j=2}^n \frac{p^{(j)}(u)}{j} p'(u)^{n-2} s^{n-1} + \mathcal{J}(J)$ is invertible in $\mathcal{C}(K)/\mathcal{J}(J)$. Since $p'(u)^2 + \mathcal{J}(J)$ is not a divisor of 0 in $\mathcal{C}(K)/\mathcal{J}(J)$, we have $s \in \mathcal{J}(J)$. Hence $r \in \mathcal{J}(J) \subset B_1$, and $u \in B_1$, which proves our claim.

Now let B be an element of \mathcal{F} such that $B + I \cap B_1 \subsetneq B_1$. For $p = a_0 + \dots + a_n x^n \in B[x]$, set $\tilde{p} = \pi_I(a_0) + \dots + \pi_I(a_n)x^n \in \pi_I(B)[x]$.

1) Assume that there exists $a \in B \setminus I \cap B$ and $b \in B$ such that $au - b \in I$ for some $u \in B_1$ such that $u \notin B + I \cap B_1$. It follows from the lemma that there exists $r \in \mathcal{C}(K)$ such that $a - bu - br \in \mathcal{J}(I)$. Set $v = u + r$. It follows from the claim that $v \in B_1$, and $r \in I$, since I is prime, which shows that $v \notin B + I \cap B_1$. Set $\tilde{B} = B(v)$. Let $y = a_0 + a_1 v + \dots + a_n v^n \in \tilde{B} \cup I$. Set $z = a_0 a^n + a_1 a^{n-1} b + \dots + a_n b^n$. Then $a^n y - z \in \mathcal{J}(I)$, and $z \in I \cap B \subset \text{in } \mathcal{J}(I)$. Hence $a^n y \in \mathcal{J}(I)$, and so $y \in \mathcal{J}(I)$ since $a^n \notin I$. This shows that $\tilde{B} \in \mathcal{F}$, and B is not maximal.

2) Assume that $b - au \notin I$ for $a, b \in B \setminus I \cap B$, $u \in B_1 \setminus B + I \cap B_1$, but that there exists a polynomial $p \in B[x]$, of degree $n \geq 2$, with $\tilde{p} \neq 0$, such that $p(u) \in I$ for some $u \in B_1 \setminus B + I \cap B_1$. with $\alpha_0 \in B \setminus I \cap B$, $\alpha_n \in B \setminus I \cap B$. We can assume that the degree of p is minimal among the degree of polynomials having this property, so that $d^o(\tilde{p}) = n$, and $p'(u) \notin I$. Hence $p'(u)^2 \notin I$, and there exists $\rho \in \mathcal{C}(K)$ such that $p(u) - \rho p'(u)^2 \in \mathcal{J}(I)$. Then $\rho \in I$, since I is prime. We have, by Taylor's formula, for $h \in \mathcal{C}(K)$,

$$p(u + p'(u)h) = p(u) + p'(u)^2 h + \sum_{k=2}^n \frac{p'(u)^k p^{(k)}(u)}{k!} h^k = z + p'(u)^2 \left[\rho + h + \sum_{k=2}^n \frac{p'(u)^{k-2} p^{(k)}(u)}{k!} h^k \right],$$

where $z \in \mathcal{J}(I)$.

Set $\phi(h) = h + \sum_{k=2}^n \frac{p'(u)^{k-2} p^{(k)}(u)}{k!} h^k$ for $h \in \mathcal{M}_t$, where t denotes the unique element of $Z(I)$. Since $D(\phi)(0)$ is the identity map, it follows from the inverse function theorem there exists $\epsilon > 0$ such that the equation $\phi(h) = g$ has a solution in \mathcal{M}_t for every $g \in \mathcal{M}_t$ such that $\|g\| < \epsilon$. Since $\mathcal{J}(I)$ is dense in \mathcal{M}_t , this implies that there exists $h \in \mathcal{M}_t$ such that $\rho + h + \sum_{k=2}^n \frac{p'(u)^{k-2} p^{(k)}(u)}{k!} h^k \in \mathcal{J}(I)$. Also $1 + \sum_{k=2}^n \frac{q'(u)^{k-2} q^{(k)}(u)}{k!} \pi(y)^{k-1} + \mathcal{J}(I)$ is invertible in $\mathcal{C}(K)/\mathcal{J}(I)$, and $h \in I$. Set $v = u + q'(u)h$. Then $v \notin B + I$, $q(v) \in \mathcal{J}(I)$, and it follows from the claim that $v \in B_1$.

Now let $q \in B[x]$ such that $q(v) \in I$. There exists $\gamma \in B \setminus I \cap B$, $\delta \in \mathcal{J}(I)$ and $q_1, q_2 \in B[x]$, with $d^o(q_2) < n$, such that $\gamma \tilde{q} = \tilde{p} \tilde{q}_1 - \tilde{q}_2$. Hence $q_2(v) = \gamma q(v) - p(v)q_1(v) \in I$, and so $\tilde{q}_2 = 0$ since $d^o(q_2) < n$. So $q(v) \in \mathcal{J}(I)$, since $\gamma + \mathcal{J}(I)$ is not a divisor of 0 in $\mathcal{C}(K)/\mathcal{J}(I)$. Hence $B(v) \in \mathcal{F}$, so that B is not a maximal element of \mathcal{F} .

3) If neither (1) nor (2) are not satisfied, and if $u \in B_1 \setminus (B + I \cap B_1)$, then $p(u) \notin I$ for every $p \in B[x]$ such that $\tilde{\pi}p \neq 0$. In this situation $B(h) \in \mathcal{F}$ and B is obviously not maximal.

Hence $B_1 = B + I \cap B_1$ for every maximal element B of \mathcal{F} . \square

We introduce the following natural notion.

Definition 3.5 Let I be a prime ideal of $\mathcal{C}(K)$. A subalgebra B of $\mathcal{C}(K)$ is said to be a lifting of the quotient algebra A_I if the following two conditions are satisfied

- (1) $B \cap I = \mathcal{J}(I)$
- (2) $\mathcal{C}(K) = B + I$.

If $t \in K$, then the algebra $\mathbb{C}.1 + \mathcal{O}_t$ of functions $f \in \mathcal{C}(K)$ which are constant on some neighbourhood of t is a lifting of the quotient algebra $A_{\mathcal{M}_t} \simeq \mathbb{C}$. Also if I is a minimal prime ideal of $\mathcal{C}(K)$ then $\mathcal{C}(K)$ is obviously a lifting of A_I .

We deduce from the lemma the following corollary.

Corollary 3.6 *Let I, J, L be prime ideals of $\mathcal{C}(K)$ such that $J \subsetneq I \subsetneq L$, let B_J be a lifting of A_J and let B_L be a lifting of A_L such that $B_L \subset B_J$. Then there exists a lifting B_I of A_I such that $B_L \subset B_I \subset B_K$.*

This corollary suggests the following problem.

Problem 3.7 *Let I be a minimal nonmaximal prime ideal of $\mathcal{C}(K)$ and let \mathcal{F} be the set of prime ideals of $\mathcal{C}(K)$ containing I . Does there exist a family $\{B_J\}_{J \in \mathcal{F}}$ of subalgebras of $\mathcal{C}(K)$ which possesses the following properties.*

- (i) B_J is a lifting of the quotient algebra A_J for every $J \in \mathcal{F}$.
- (ii) If $J \in \mathcal{F}, L \in \mathcal{F}$, and if $J \subset L$, then $B_L \subset B_J$.

A positive answer to this problem would give a direct proof the fact that every chain \mathcal{F} of nonmaximal closed prime ideals of $\mathcal{C}(K)$ such that $|\mathcal{C}(K)/I| = 2^{\aleph_0}$, where I denotes the smallest element of \mathcal{F} , is contained in $\text{Prim}(q)$ for some discontinuous algebra norm on $\mathcal{C}(K)$ if CH is assumed [14].

If A is a family of linear subspaces on a complex linear space E we will denote again by $\mathcal{U}(A)$ the set of all unions of subfamilies of A , and by $\mathcal{V}(A)$ the set of all linear subspaces of E which belong to $\mathcal{U}(A)$. We will say that A is stable under unions if $\mathcal{V}(A) = A$. We will need the following notion.

Definition 3.8 *Let E be a linear space and let \mathcal{F} be a family of linear subspaces of E . We will say that the family \mathcal{F} is almost stable under unions if there exists a sequence $(\mathcal{F}_n)_{n \geq 1}$ of subfamilies of \mathcal{F} such that $\cup_{n \geq 1} \mathcal{V}(\mathcal{F}_n) \subset \mathcal{F}$.*

A countable family of linear subspaces of E is indeed almost stable under unions. Also, since $\mathcal{V}(\mathcal{V}(A)) = \mathcal{V}(A)$ for every family A of linear subspaces of E , we see that if \mathcal{F} is almost stable under unions there exists a sequence $(\mathcal{F}_n)_{n \geq 1}$ of subfamilies of \mathcal{F} stable under unions such that $\mathcal{F} = \cup_{n \geq 1} \mathcal{F}_n$.

Notice that if \mathcal{F} is almost stable under unions, and if \mathcal{G} is a chain of elements of \mathcal{F} which does not admit any countable cofinal subset, then $\mathcal{G} \cup \mathcal{F}_n$ is cofinal in \mathcal{G} for some $n \geq 1$, and so $\cup\{L : L \in \mathcal{G}\} \in \mathcal{F}$. Similarly if $(I_\lambda)_{\lambda \in \Lambda}$ is an uncountable pseudo-finite family of elements of \mathcal{F} , then $\cup_{\lambda \in \Lambda} I_\lambda \in \mathcal{F}$, since every sequence of elements of $\cup_{\lambda \in \Lambda} I_\lambda$ is contained in I_λ for some $\lambda \in \Lambda$.

Let \mathcal{F} be a well-ordered chain of prime nonmaximal ideals of $\mathcal{C}(K)$ satisfying $|\mathcal{C}(K)/I_{\mathcal{F}}| = 2^{\aleph_0}$, where $I_{\mathcal{F}}$ denotes the smallest element of \mathcal{F} and let \mathcal{M} be the maximal ideal of $\mathcal{C}(K)$ containing the elements of \mathcal{F} . The author proved in [14], theorem 2.8 that if $\mathcal{F} \cup \mathcal{M}$ is almost stable under unions, then there exists a discontinuous algebra norm on $\mathcal{C}(K)$ such that $\mathcal{F} = \text{Prim}(q)$.

If \mathcal{U} is a family of nonmaximal prime ideals of $\mathcal{C}(K)$, denote by $\mathcal{M}_{\mathcal{U}}$ the set of maximal ideals of $\mathcal{C}(K)$ which contain some element of \mathcal{U} . Theorem 2.8 of [14] suggests the following conjecture

Conjecture 3.9 (CH) *Let \mathcal{U} be a family of nonmaximal prime ideals of $\mathcal{C}(K)$ satisfying the three following properties*

- 1) *every sequence of elements of \mathcal{U} has a pseudo-finite subsequence,*

- 2) $\mathcal{U} \cup \mathcal{M}_{\mathcal{U}}$ is almost stable under unions,
 3) $|\mathcal{C}(K)/I| = 2^{\aleph_0}$ for every minimal element I of \mathcal{U} .
 Then there exists an algebra norm on $\mathcal{C}(K)$ such that $\mathcal{U} = \text{Prim}(q)$.

A weaker form of the conjecture is obtained by replacing condition (2) by the condition
 2') $\mathcal{U} \cup \mathcal{M}_{\mathcal{U}}$ is stable under unions.

This weak form of the conjecture would imply that every ideal I such that $|\mathcal{C}(K)/I| = 2^{\aleph_0}$ satisfying the equivalent conditions of theorem 2.10 is the continuity ideal of some discontinuous homomorphism from $\mathcal{C}(K)$ if CH is assumed. In other terms this would show that the continuity ideals of discontinuous homomorphisms are exactly the pure semiprime Badé-Curtis ideals under CH (and allow to remove the countability condition in assertion (ii) of theorem 6.7 of [24]). This result would present some heuristic interest since the fact that the continuity ideal of a pure semiprime Badé-Curtis ideals is a consequence of the two main tools of automatic continuity theory, the "main boundedness theorem" of Badé-Curtis [2] and the "stability lemma" of [28] (see also [5], corollary 5.2.7). These two tools would thus provide all the information concerning partial continuity of homomorphisms from $\mathcal{C}(K)$ if CH is assumed.

4 An extension theorem

We present here an alternative approach to Pham's method based on almost disjoint infinite families of integers and embeddings of quotient algebras $A_I = \mathcal{C}(K)/I$ into algebras of the form $(\mathbb{C}^\kappa/\mathcal{U})^o$, where κ is a cardinal, \mathcal{U} an ultrafilter on κ and $(\mathbb{C}^\kappa/\mathcal{U})^o$ is the algebra of infinitesimal elements of the field $\mathbb{C}^\kappa/\mathcal{U}$. This method is used by Pham to obtain theorem 6.5 of [22] and proposition 6.2 of [24], which are both extensions of Woodin's embedding theorem, which shows that every complex algebra A such that $|A| = 2^{\aleph_0}$ can be embedded in c_0/\mathcal{U} for some free ultrafilter \mathcal{U} on \mathbb{N} , see theorem 5.25 of [24] (this result does not depend on the continuum hypothesis). Theorem 6.5 of [22] and proposition 6.2 of [24] are the key steps for the proofs of theorem 9.5 of [22] and theorem 6.7 of [24], which show that continuity ideals are not in general finite intersections of prime ideal.

To formulate our result we need to introduce some objects used in the author's construction of discontinuous homomorphisms from $\mathcal{C}(K)$.

Let ω_1 be the smallest uncountable ordinal. We denote by $S_{\omega_1} \subset \{0, 1\}^{\omega_1}$ the set of all transfinite dyadic sequences $x = (x_\zeta)_{\zeta < \omega_1}$ for which there exists $\eta(x) < \omega_1$ such that $x_{\eta(x)} = 1$ and such that $x_\zeta = 0$ for every $\zeta > \eta(x)$.

Equipped with the lexicographic order, S_{ω_1} is a linearly ordered set, and a classical result of Sierpiński [26], see also [5] shows that every linearly ordered set of cardinal $\leq \aleph_1$ is order-isomorphic to a subset of S_{ω_1} .

Denote by $G_{\omega_1} \subset S_{\omega_1}^{\mathbb{R}}$ the set of all real-valued functions ϕ on S_{ω_1} such that $\text{Supp}(\phi) := \{s \in S_{\omega_1} \mid \phi(s) \neq 0\}$ is well-ordered and at most countable. For $\phi \in G_{\omega_1} \setminus \{0\}$, denote by $\rho(\phi)$ the smallest element of $\text{Supp}(\phi)$. By definition, a nonzero element $\phi \in G_{\omega_1}$ is said to be strictly positive if $\phi(\rho(\phi)) > 0$. Equipped with the linear structure inherited from the linear structure of $S_{\omega_1}^{\mathbb{R}}$, G_{ω_1} is a linearly ordered real vector space, which contains a copy of every linearly ordered group of cardinal \aleph_1 .

Now let G be a linearly ordered group, and let k be a field. We will denote by $\mathcal{F}(G, k)$ the set of all functions $f : G \rightarrow k$ such that $\text{Supp}(f) := \{\tau \in G \mid f(\tau) \neq 0\}$ is well-ordered, and we set

$$\mathcal{F}_{(1)}(G, k) := \{f \in \mathcal{F}(G, k) \mid |\text{supp}(f)| \leq \aleph_0\}.$$

Now let $f, g \in \mathcal{F}(G, k)$, and let $\tau \in G$. If $\tau \notin \text{Supp}(f) + \text{Supp}(g) := \{\alpha + \beta \mid \alpha \in \text{Supp}(f), \beta \in \text{Supp}(g)\}$, set $(fg)(\tau) = 0$. Otherwise set

$$(fg)(\tau) = \sum_{\substack{\alpha \in \text{Supp}(f), \beta \in \text{Supp}(g) \\ \alpha + \beta = \tau}} f(\alpha)g(\beta).$$

Then fg is well-defined, since the set $\{(\alpha, \beta) \in \text{Supp}(f) \times \text{Supp}(g) \mid \alpha + \beta = \tau\}$ is finite for every $\tau \in \text{Supp}(f) + \text{Supp}(g)$, and $fg \in \mathcal{F}(G, k)$. In fact Hahn observed in 1907 in [17] that $\mathcal{F}(G, k)$ is a field. Set $v(f) = \inf(\text{Supp}(f))$ for $f \in \mathcal{F}(G, k) \setminus \{0\}$. Then v is a valuation on the field $\mathcal{F}(G, k)$, and the valued field $\mathcal{F}(G, k)$ is maximal : if U is a field containing $\mathcal{F}(G, k)$, and if w is a valuation on U with values in G such that $w(f) = v(f)$ for every $f \in \mathcal{F}(G, k) \setminus \{0\}$, then $U = \mathcal{F}(G, k)$.

Mac Lane showed in [20] that $\mathcal{F}(G, k)$ is algebraically closed if k is algebraically closed and if the equation $nt = \tau$ has a solution in G for every $\tau \in G$ and every integer $n \geq 2$. In particular, the fields $\mathcal{F}(G_{\omega_1}, \mathbb{C})$ and $\mathcal{F}_{(1)}(G_{\omega_1}, \mathbb{C})$ are algebraically closed.

We set, with the convention $v(0) = +\infty > \tau$ for every $\tau \in \mathbf{G}$,

$$\mathbb{C}_{\omega_1} := \{f \in \mathcal{F}_{(1)}(G_{\omega_1}, \mathbb{C}) \mid v(f) \geq 0\}, \mathcal{M}_{\omega_1} := \{f \in \mathcal{F}_{(1)}(G_{\omega_1}, \mathbb{C}) \mid v(f) > 0\},$$

so that the radical complex algebra \mathcal{M}_{ω_1} is the unique maximal ideal of the commutative unital complex algebra \mathbb{C}_{ω_1} . The algebra \mathbb{C}_{ω_1} is universal : if the continuum hypothesis is assumed then \mathbb{C}_{ω_1} contains a copy of every commutative unital complex algebra of cardinality 2^{\aleph_0} which is an integral domain and possesses a character.

If I is a prime ideal of $\mathcal{C}(K)$ we denote again by A_I the quotient algebra $\mathcal{C}(K)/I$ and we denote by $\pi_I : \mathcal{C}(K) \rightarrow \mathcal{C}(K)/I$ the canonical surjection.

Let \mathbb{K} be a field. A \mathbb{K} -algebra is an algebra over \mathbb{K} , and a \mathbb{K} -character over a \mathbb{K} -algebra A is an algebra homomorphism from A into \mathbb{K} . A \mathbb{K} -field is a field which is also a \mathbb{K} -algebra.

We will need the following classical result called the fundamental theorem for the extension of places.

Theorem 4.1 *Let \mathbb{K} be an algebraically closed field, and let A be a commutative unital \mathbb{K} -algebra which possesses a \mathbb{K} -character ϕ_A . Assume that A is contained in a \mathbb{K} -field L . Then there exists a \mathbb{K} -algebra V of quotient field L containing A which possesses the following properties :*

- (i) V is a valuation ring.
- (ii) There exists a \mathbb{K} -character ϕ_V on V such that $\phi_V|_A = \phi_A$.

This result is given in theorem 2.19 of [8] in the case where $\mathbb{K} = \mathbb{C}$. The proof given in [8] extends without modification to algebras over any algebraically closed \mathbb{K} . \square

Notice that If V is a \mathbb{K} -valuation algebra which possesses a \mathbb{K} -character ϕ_V , then $\text{Ker}(\phi_V)$ is the unique maximal ideal M_V of the valuation ring V , and so M_V is a \mathbb{K} -subalgebra of V , and $V/M_V \approx \mathbb{K}$. Let L be the field of fractions of V . Set

$$\Gamma_V = (L \setminus \{0\})/Inv(V),$$

and denote by $v : L \setminus \{0\} \rightarrow \Gamma_V$ be the canonical surjection. Set $v(a) \leq v(b)$ for $a, b \in L \setminus \{0\}$ if $b \in aV$. We define an additive law on Γ_V by using the formula

$$v(a) + v(b) = v(ab) \quad (a \in L \setminus \{0\}, b \in L \setminus \{0\}).$$

Standard routine verifications show that Γ_V is a linearly ordered group, which is divisible if L is algebraically closed, and that $v : L \setminus \{0\} \rightarrow \Gamma_V$ is a valuation on L . If we set $v(0) = +\infty$, we have

$$V = \{a \in L : v(a) \geq 0\}, \text{Ker}(\phi_V) = M_V = \{a \in L : v(a) > 0\}.$$

Now assume that L is algebraically closed. If L_0 is a subfield of L which is a \mathbb{K} algebra, we will say that a \mathbb{K} -homomorphism $\psi_0 : L_0 \rightarrow \mathcal{F}(G, \mathbb{K})$ is valuation preserving if $v(\psi_0(a)) = v(a)$ for every $a \in L_0 \setminus \{0\}$, where we also denote by v the standard valuation on $\mathcal{F}(G, \mathbb{K})$. Kaplansky's embedding theorem [18] shows that if we set $G_0 = v(L_0 \setminus \{0\})$ then for every valuation preserving homomorphism $\psi_0 : L_0 \rightarrow \mathcal{F}(G_0, \mathbb{K})$ there exists a valuation-preserving homomorphism $\psi : L \rightarrow \mathcal{F}(G, \mathbb{K})$ such that $\psi|_{L_0} = \psi_0$. We now deduce from Kaplansky's embedding theorem the following lemma.

Lemma 4.2 (CH) *Let \mathbb{K} be an algebraically closed field, and let A be a commutative unital \mathbb{K} -algebra which possesses a \mathbb{K} -character ϕ_A . If $|A| = 2^{\aleph_0}$, then there exists a linearly ordered group G , with $|G| = 2^{\aleph_0}$, and a one-to-one unital \mathbb{K} -algebra homomorphism $\psi : A \rightarrow \mathcal{F}_{(1)}(G^+, \mathbb{K})$ such that $v(\phi(a)) > 0$ for every $a \in \text{Ker}(\phi_A)$.*

Proof : Recall that a linearly ordered set S is said to be an α_1 -set if every nonempty subset of S has a countable coinital and cofinal subset G , see definition 1.7 of [8]. Let L be the algebraic closure of the quotient field of A , which is a \mathbb{K} -algebra. Let V be a \mathbb{K} -algebra of quotient field L satisfying conditions (i) and (ii) of theorem 4.1. Set $G = \Gamma_V$, and let $v : L \setminus \{0\} \rightarrow G$ be the valuation on L associated to V . Let $(a_\zeta)_{\zeta < \omega}$ be a transcendence basis of L over \mathbb{K} , where $\omega \leq \omega_1$ is an ordinal.

Set $L_0 = \mathbb{K}$, and for $0 < \sigma \leq \omega$ denote by L_σ the algebraic closure of the field of fractions of $L_\zeta(a_\sigma)$ if $\sigma = \zeta + 1$ is a successor ordinal, and set $L_\sigma = \cup_{\zeta < \sigma} L_\zeta$ if σ is a limit ordinal. Set $G_\sigma = v(L_\sigma \setminus \{0\})$ for $\sigma < \omega$.

Let $\sigma < \omega$ is a successor ordinal. Denote by U the field of fractions of $L_\sigma(a_{\sigma+1})$. Since L_σ is algebraically closed, a standard argument given for example in the proof of proposition 2.27 of [8] shows that there exists $\tau \in G$ such that $v(U \setminus \{0\}) = G_\zeta + \mathbb{Z} \cdot \tau$. Another standard argument given in the proof of proposition 2.6 of [8] shows that $G_\sigma = \cup_{n \geq 1} \frac{1}{n} v(U \setminus \{0\})$. So $G_{\sigma+1}$ is an α_1 -group if G_σ is an α_1 -group. Since any countable union of a nondecreasing family of α_1 -groups is an α_1 -group, we see by transfinite induction that G_σ is an α_1 group for every $\sigma < \omega$. Set $\phi_0(k \cdot 1) = k$ for $k \in \mathbb{K}$. Applying Kaplansky's embedding theorem, we can define by transfinite induction for $\sigma < \omega$ a one-to-one homomorphism $\phi_\sigma : L_\sigma \rightarrow \mathcal{F}(G_\sigma, \mathbb{K}) = \mathcal{F}_{(1)}(G_\sigma, \mathbb{K})$ satisfying $v(\phi_\sigma(a)) = v(a)$ for every $a \in L_\sigma \setminus \{0\}$ such that $\phi_{\sigma|_\zeta} = \phi_\zeta$ for $\zeta < \sigma$, and it follows from the definition of ϕ_0 that ϕ_σ is a \mathbb{K} -algebra homomorphism. Then the homomorphism $\phi : L \rightarrow \mathcal{F}_{(1)}(G, \mathbb{K}) = \cup_{\sigma < \omega} \mathcal{F}_{(1)}(G_\sigma, \mathbb{K})$ defined by the condition $\phi|_{L_\sigma} = \phi_\sigma$ for $\sigma < \omega$ satisfies $v(\phi(a)) = v(a)$ for every $a \in L \setminus \{0\}$. Since $A \subset V$, we have $\phi(a) \in \mathcal{F}_{(1)}(G, \mathbb{K})$ for every $a \in A$, and $v(\phi(a)) = v(a) > 0$ for every $a \in \text{Ker}(\phi_A)$. \square

Theorem 4.3 (CH) *Let I be a nonmaximal prime ideal of $\mathcal{C}(K)$ such that $|A_I| = 2^{\aleph_0}$, and let J be a nonmaximal prime ideal of $\mathcal{C}(K)$ containing I . Let B be a subalgebra of A_I such that $A_I = B \oplus \pi_I(J)$. Let $\phi : B \rightarrow \mathbb{C}_{\omega_1}$ be a one-to-one homomorphism. If there exists a noncofinal subgroup H_0 of G_{ω_1} such that $\text{Supp}(\phi(b)) \subset H_0$ for every $b \in B$, then there exists a one-to-one homomorphism $\psi : A_I \rightarrow \mathbb{C}_{\omega_1}$ such that $\psi|_B = \phi$.*

Proof : Let \mathbb{K} be the field of fractions of B , and let \mathbb{U} be the field of fractions of A_I . Then \mathbb{K} and \mathbb{U} are algebraically closed. Set $D = \{u = ab^{-1} : a \in A_I, b \in B \setminus \{0\}\}$. Since $a \in bA_I$ for every $a \in \pi_I(I)$ and every $b \in B \setminus \{0\}$, and since $A_I = B \oplus \pi_I(J)$, we have $D = \mathbb{K} + \pi_I(J)$, $\pi_I(J)$ is an ideal of D , and D is a \mathbb{K} -algebra. Since $1 \notin \pi_I(J)$, we have $\mathbb{K} \cap \pi_I(J) = \{0\}$, and so $D = \mathbb{K} \oplus \pi_I(J)$. Hence there exists a \mathbb{K} -character ϕ_D on D such that $\pi_I(J) = \text{Ker}(\phi_D)$. Hence every $x \in D$ admits a unique decomposition of the form $x = \alpha(x) + y(x)$, where $\alpha(x) \in \mathbb{K}$ and $y(x) \in \pi_I(J)$, and the map $\phi_D : x \rightarrow \alpha(x)$ is a \mathbb{K} -character on D such that $\text{Ker}(\phi_D) = \pi_I(J)$.

It follows from the lemma that there exists a linearly ordered group G satisfying $|G| = 2^{\aleph_0}$ and a one-to-one \mathbb{K} -algebra homomorphism $\delta : D \rightarrow \mathcal{F}_{(1)}(G^+, \mathbb{K})$ such that $v(\theta(a)) > 0$ for every $a \in \text{Ker}(\phi_D) = \pi_I(J)$.

Now assume that $\phi : B \rightarrow \mathbb{C}_{\omega_1}$ is a one-to-one algebra homomorphism and that H_0 is a noncofinal subgroup of G containing $\text{Supp}(b)$ for every $b \in B$, and let $\tau \in G_{\omega_1}$ such that $\tau > s$ for every $s \in H_0$.

Denote by $r \in S_{\omega_1}$ the smallest element of $\text{supp}(\tau)$, and set $S = \{s \in S_{\omega_1} : s < r\}$. Since S_{ω_1} is a η_1 -set, S is also a η_1 -set, and so the group $H := \{\sigma \in G : \text{supp}(\sigma) \subset S\}$ is a η_1 -group, see theorem 1.19 in [8] (and it follows from corollary 1.49 of [8] that H is in fact isomorphic to G_{ω_1} as an ordered group). Since CH is assumed, It follows from theorem 1.46 of [8] that there exist a one-to-one order preserving homomorphism $\rho : G \rightarrow H$. We now define an application $\theta : \mathcal{F}_{(1)}(G, \mathbb{K}) \rightarrow \mathcal{F}_{(1)}(G_{\omega_1}, \mathbb{C})$ by the formula

$$\begin{cases} \theta(u)(\tau + \rho(s)) = \tilde{\phi}(u(s))(\tau) & \text{for } \tau \in H_0, s \in G \\ \theta(u)(\sigma) = 0 & \text{for } \sigma \notin H_0 + \rho(G) \end{cases}$$

We have $|\sigma| < |t|$ for every $\sigma \in H_0$ and every $t \in H_1$. Equip the cartesian product $H_0 \times G$ with lexicographic order, i.e. $(s_1, t_1) < (s_2, t_2)$ if $t_1 < t_2$ or if $t_1 = t_2$ and $s_1 < s_2$. Then the map $(\tau, s) \rightarrow \tau + s$ is an order-preserving group homomorphism from $H_0 \times G$ into G_{ω_1} , and it follows from this observation that $\text{Supp}(\theta(u))$ is well-ordered for every $u \in \mathcal{F}_{(1)}(G, \mathbb{K})$. Since a countable union of countable sets is countable, we see that $\theta(u) \in \mathcal{F}_{(1)}(G_{\omega_1}, \mathbb{C})$ for every $u \in \mathcal{F}_{(1)}(G, \mathbb{K})$, and θ is clearly \mathbb{K} -linear and one-to-one.

We have, for $\tau \in H_0$, $s \in G$, $u, v \in \mathcal{F}_{(1)}(G, \mathbb{K})$,

$$\begin{aligned} \theta(uv)(\tau + \rho(s)) &= \tilde{\phi}(uv(s))(\tau) = \tilde{\phi}\left(\sum_{s_1+s_2=s} u(s_1)v(s_2)\right)(\tau) \\ &= \sum_{s_1+s_2=s} [\tilde{\phi}(u(s_1))\tilde{\phi}(v(s_2))](\tau) = \sum_{\tau_1+\tau_2=\tau, s_1+s_2=s} [\tilde{\phi}(u)(s_1)](\tau_1)[\tilde{\phi}(v)(s_2)](\tau_2) \\ &= \sum_{(\tau_1+\rho(s_1))+(\tau_2+\rho(s_2))=\tau+\rho(s)} \theta(u)(\tau_1 + \rho(s_1))\theta(v)(\tau_2 + \rho(s_2)) \\ &= [\theta(u)\theta(v)](\tau + \rho(s)). \end{aligned}$$

So $\theta(uv) = \theta(u)\theta(v)$ and θ is a \mathbb{K} -algebra homomorphism.

Denote by ψ the restriction of $\theta \circ \delta$ to A_I , and let $a \in A_I$. If $b \in B$ we have $\delta(b) = bX^0$, donc $(\theta \circ \delta)(b) = \tilde{\phi}(b) = \phi(b) \in \mathbb{C}_{\omega_1}$. Now if $c \in \pi_I(J)$ we have $s > 0$ for every $s \in \text{Supp}(\delta(c))$, and so every $t \in \text{Supp}(\psi(c))$ has the form $t = \tau + \rho(s)$ where $\tau \in H_0$ and where $\rho(s)$ is a strictly positive element of H_1 . This shows that $t > 0$ for every $t \in \text{Supp}(\phi(c))$. In particular $v(\phi(c)) > 0$, and $\psi(c) \in \mathbb{C}_{\omega_1}$. Since $A_I = B \oplus \pi_J(I)$, we have $\phi(A_I) \subset \mathbb{C}_{\omega_1}$, which concludes the proof of the theorem. \square

We obtain the following result, which is slightly stronger than corollary 7.3 of [24].

Corollary 4.4 (CH) *Let $(I_\lambda)_{\lambda \in \Lambda}$ be a pseudo-finite family of prime ideals of $\mathcal{C}(K)$. If $|\mathcal{C}(K)/I_\lambda| = 2^{\aleph_0}$ for $\lambda \in \Lambda$, then the quotient algebra $\mathcal{C}(K)/\bigcap_{\lambda \in \Lambda} I_\lambda$ possesses an algebra norm.*

Proof. Set $J = \bigcup_{\lambda \in \Lambda} I_\lambda$. Then J is a prime ideal of $\mathcal{C}(K)$. Denote by $\mathcal{J}(J)$ the intersection of all prime ideals of $\mathcal{C}(K)$ contained in J , and let U be a subalgebra of $\mathcal{C}(K)$ containing $\mathcal{J}(J)$ such that $U \cap J \subset \mathcal{J}(J)$ and such that $U + J = \mathcal{C}(K)$. Denote by $\pi : \mathcal{C}(K) \rightarrow \mathcal{C}(K)/\mathcal{J}(J)$ and for $\lambda \in \Lambda$ denote by $\pi_\lambda : \mathcal{C}(K)/I_\lambda$ the canonical surjections. Choose a strictly positive $\tau \in G_{\omega_1}$ and set $G := \{s \in G_{\omega_1} : n|s| < \tau \ \forall n \geq 1\}$. Then G is an η_1 -subgroup of G_{ω_1} which is not cofinal in G_{ω_1} . If $\pi(u)\pi(v) = 0$, with $u, v \in U$, and if $u \notin I_\lambda$ for some $\lambda \in \Lambda$, then $v \in I_\lambda \cap U \subset \mathcal{J}(J)$, and so $\pi(v) = 0$. So the algebra $\pi(U)$ is an integral domain, and it follows again from theorem 1.46 of [8] that there exists a one-to-one algebra homomorphism $\phi : \pi(U) \rightarrow \mathcal{F}_{(1)}(G, \mathbb{C}) \subset \mathbb{C}_{\omega_1}$.

Since $U \cap I_\lambda \subset \mathcal{J}(J)$, the map $\theta_\lambda : f + \mathcal{J}(J) \rightarrow f + I_\lambda$ defines an isomorphism from $\pi(U)$ onto $\pi_\lambda(U)$ for $\lambda \in \Lambda$, and we have $A_{I_\lambda} = \mathcal{C}(K)/I_\lambda = \pi_\lambda(U) \oplus \pi_\lambda(J)$. Now set $\phi_\lambda := \phi \circ \theta_\lambda^{-1}$. The map $\phi_\lambda : \pi_\lambda(U) \rightarrow \mathbb{C}_{\omega_1}$ satisfies the conditions of theorem 4.3, and so there exists a one-to-one homomorphism $\psi_\lambda : A_{I_\lambda} \rightarrow \mathbb{C}_{\omega_1}$ such that $\psi_\lambda(a) = \phi_\lambda(a)$ for every $a \in \pi_\lambda(U)$.

If $u \in U$, we have $(\psi_\lambda \circ \pi_\lambda)(u) = (\phi_\lambda \circ \pi_\lambda)(u) = (\phi \circ \theta_\lambda^{-1} \circ \pi_\lambda)(u) = (\phi \circ \pi)(u)$, which does not depend on λ . Now if $g \in J$ then the set $\{\lambda \in \Lambda : \pi_\lambda(g) \neq 0\}$ is finite. Since $\mathcal{C}(K) = U + J$, we see that the set $\{\psi_\lambda(f)\}_{\lambda \in \Lambda}$ is finite for every $f \in \mathcal{C}(K)$.

Let p be an algebra norm on \mathbb{C}_{ω_1} , and set, for $f \in \mathcal{C}(K)$,

$$\|f\| = \max_{\lambda \in \Lambda} p(\psi_\lambda(f)).$$

We obtain an algebra seminorm on $\mathcal{C}(K)$, and the kernel of this seminorm is $\bigcap_{\lambda \in \Lambda} I_\lambda$, which gives an algebra norm on the quotient algebra $\mathcal{C}(K) / \bigcap_{\lambda \in \Lambda} I_\lambda$. \square

We thus see that it is possible to use the extension theorem instead of theorem 6.5 of [22] to complete the proof of the main results of [22]. It is also possible to use the extension theorem instead of proposition 6.2 of [24] to complete the proof of the main results of [24]. We leave the details to the reader.

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