Elastic waves interacting with a thin, pre-stessed, fiber-reinforced surface film

David Steigmann

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D.J. Steigmann
Department of Mechanical Engineering
University of California
Berkeley, CA. 94720
USA

Abstract: Elastic surface waves propagating at the interface between an isotropic substrate and a thin, transversely isotropic film are analyzed. The transverse isotropy is conferred by fibers lying parallel to the interface. A rigorous leading-order model of the thin-film/substrate interface is derived from the equations of three-dimensional elasticity for prestressed, transversely isotropic films having non-uniform properties. This is used to study Love waves.

1. Introduction

In this work we derive a rigorous leading-order-in-thickness model for the dynamics of a thin, fiber-reinforced elastic film bonded to an isotropic elastic substrate. The fibers are assumed to lie in the plane of the film and equations of motion are derived for the displacement of the interface. Related work based on a variational argument is given in [1], where references to the pertinent literature may be found. Among these works we direct the reader's particular attention to [2,3,4]. Related developments are reported in [5,6].

2. Basic equations

Standard notation is used throughout. Thus, we use bold face for vectors and tensors and indices to denote their components. Latin indices take values in $\{1, 2, 3\}$; Greek in $\{1, 2\}$. The latter are associated with surface coordinates and associated vector and tensor components. A dot between bold symbols is used to denote the standard inner product. Thus, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are second-order tensors, then $\mathbf{A}_1 \cdot \mathbf{A}_2 = tr(\mathbf{A}_1\mathbf{A}_2^t)$, where $tr(\cdot)$ is the trace and the superscript ^{t} is used to denote the transpose. The notation $\otimes$ identifies the standard tensor product of vectors. If $\mathbf{C}$ is a fourth-order tensor, then $\mathbf{C}[\mathbf{A}]$ is the second-order tensor with orthogonal components $C_{ijkl}A_{kl}$. Finally, we use symbols such as Div and D to denote the three-dimensional divergence and gradient operators, while div and ∇ are reserved for their two-dimensional counterparts. Thus, for example, $Div\mathbf{A} = A_{ij,j}\mathbf{e}_i$ and $div\mathbf{A} = A_{i\alpha,\alpha}\mathbf{e}_i$, where $\{\mathbf{e}_i\}$ is an orthonormal basis and subscripts preceded by commas are used to denote partial derivatives with respect to Cartesian coordinates.

The three-dimensional equation of motion without body force is

$$Div\mathbf{P} = \rho\ddot{\mathbf{u}}, \tag{1}$$

where

$$\mathbf{P} = \mathbf{S} + \mathbf{HS} + \mathbf{C}[\mathbf{H}] \tag{2}$$

is the linear approximation to the Piola stress, $\mathbf{S}$ is the (symmetric) residual stress, $\mathbf{H} = D\mathbf{u}$ is the gradient of the displacement field $\mathbf{u}(\mathbf{x},t)$, $\ddot{\mathbf{u}}$ is the acceleration, and $\mathbf{C}$ is the fourth-order tensor of elastic moduli. The moduli possess the usual minor and major symmetries, the latter ensuring that

$$\mathbf{P} = U_{\mathbf{H}}, \tag{3}$$

where

$$U(\mathbf{H};\mathbf{x}) = \mathbf{S}\cdot\mathbf{H} + \tfrac{1}{2}(\mathbf{HS}\cdot\mathbf{H} + \mathbf{H}\cdot\mathbf{C}[\mathbf{H}]) \tag{4}$$

is the quadratic-order approximation to the strain energy per unit volume of the material region R in which explicit dependence on $\mathbf{x} \in R$ is present if the material is non-uniform. Any such dependence occurs through the residual stress and the moduli. Here we take these to be uniform and thus restrict attention to unform materials.

We suppose that traction data

$$\mathbf{t} = \mathbf{Pn} \tag{5}$$

are assigned on a part of the boundary ∂R with exterior unit normal $\mathbf{n}$.

We impose the strong-ellipticity condition

$$(\mathbf{w}\cdot\mathbf{Sw})\mathbf{v}\cdot\mathbf{v} + \mathbf{v}\otimes\mathbf{w}\cdot\mathbf{C}[\mathbf{v}\otimes\mathbf{w}] > 0 \quad \text{for all} \quad \mathbf{v}\otimes\mathbf{w} \neq \mathbf{0}. \tag{6}$$

This is necessary for the undeformed body to be a minimizer of the total strain energy. It is also necessary for minimizers of the potential energy in standard mixed traction/displacement boundary-value problems [7].

3. Motion of the film/substrate interface

The undeformed film/substrate interface is a plane denoted by Ω. Let $\mathbf{k}$ be the unit vector that orients the interface, directed away from the substrate. The reference placement of the film is described by

$$\mathbf{x} = \mathbf{r} + \varsigma\mathbf{k}, \tag{7}$$

where $\mathbf{r} \in \Omega$, $\mathbf{k}$ is the fixed orientation of the film, and $\varsigma \in [0, h]$ in the film, where h is the film thickness. The origin of the position $\mathbf{r}$ is assumed to lie on Ω. In the present work the film-substrate combination is a half space that supports a propagating surface wave whose wavelength l (the reciprocal of the wavenumber k) furnishes the only length scale to which h can be compared. Henceforth we regard h as being small in the sense that $h/l \ll 1$. In the present section it simplifies matters to adopt l as the unit of length (i.e.; $l = 1$, $h \ll 1$).

Let $\mathbf{u}(\mathbf{r},\varsigma, t)$ be the function obtained by substituting (7) into $\mathbf{u}(\mathbf{x},t)$, and let $\nabla(\cdot)$ and $(\cdot)'$, respectively, stand for the (two-dimensional) gradient with respect to $\mathbf{r}$ at fixed ς and the derivative $\partial(\cdot)/\partial\varsigma$ at fixed $\mathbf{r}$. Further, let

$$\mathbf{1} = \mathbf{I} - \mathbf{k}\otimes\mathbf{k}, \tag{8}$$

where $\mathbf{I}$ is the identity for three-space; this is the projection onto the translation (vector) space Ω' of Ω. In [1] it is used to derive

$$\mathbf{H1} = \nabla\mathbf{u}, \quad \mathbf{Hk} = \mathbf{u}', \tag{9}$$

and the consequent orthogonal decomposition

$$\mathbf{H} = \nabla\mathbf{u} + \mathbf{u}' \otimes \mathbf{k}. \tag{10}$$

Using (8) with $\mathbf{P} = \mathbf{PI}$, we also obtain

$$\mathbf{P} = \mathbf{P1} + \mathbf{Pk} \otimes \mathbf{k}, \tag{11}$$

and write (1) in the form

$$div(\mathbf{P1}) + \mathbf{P}'\mathbf{k} = \rho\ddot{\mathbf{u}}, \tag{12}$$

where div is the two-dimensional divergence on Ω. This holds at all points of the thin film, and in the limit $\varsigma \searrow 0^+$ in particular, yielding the interfacial equation of motion

$$div(\mathbf{P}_0\mathbf{1}) + \mathbf{P}_0'\mathbf{k} = \rho_0\ddot{\mathbf{u}}_0, \tag{13}$$

where, here and henceforth, the susbscript $(\cdot)_0$ is used to denote the values of functions on the interface Ω defined by $\varsigma = 0$.

If $\mathbf{t}^+$ is the traction exerted by the environment on the film, then $\mathbf{t}^+ = \mathbf{P}^+\mathbf{k}$ where $\mathbf{P}^+$ is the stress at $\varsigma = h$. A Taylor expansion furnishes

$$\mathbf{t}^+ = \mathbf{P}_0\mathbf{k} + h\mathbf{P}_0'\mathbf{k} + o(h). \tag{14}$$

Accordingly, (13) and (14) combine to yield

$$div(\mathbf{P}_0\mathbf{1}) + h^{-1}(\mathbf{t}^+ - \mathbf{P}_0\mathbf{k}) + h^{-1}o(h) = \rho_0\ddot{\mathbf{u}}_0. \tag{15}$$

For this to furnish a well-defined problem in the limit of small thickness it is necessary that

$$\mathbf{t}^+ - \mathbf{P}_0\mathbf{k} = O(h) \quad \text{as} \quad h \to 0. \tag{16}$$

This is the net traction acting on the lateral surfaces of the film.

Let $\boldsymbol{\sigma}$ be the stress in the substrate, assumed to occupy the half-space defined by $\varsigma < 0$. Let $\boldsymbol{\sigma}_0$ be the limit of $\boldsymbol{\sigma}$ as $\varsigma \nearrow 0^-$. Then the traction exerted *on* the substrate *by* the film at the film-substrate interface Ω is $\boldsymbol{\sigma}_0\mathbf{k}$, whereas that exerted *by* the substrate *on* the film is $-\mathbf{P}_0\mathbf{k}$. An elementary pillbox argument yields the exact result

$$\boldsymbol{\sigma}_0\mathbf{k} = \mathbf{P}_0\mathbf{k}, \tag{17}$$

which is independent of h. Combining this with (15) and passing to the limit in (15) and (16) yields the leading-order model

$$div(h\mathbf{P}_0\mathbf{1}) + \mathbf{t}^+ - \boldsymbol{\sigma}_0\mathbf{k} = h\rho_0\ddot{\mathbf{u}}_0 \quad \text{and} \quad \mathbf{t}^+ - \mathbf{P}_0\mathbf{k} = \mathbf{0}. \tag{18}$$

These equations, together with interfacial continuity of displacements, couple the responses of the film and substrate. In particular, if $\mathbf{w}(\mathbf{x},t)$ is the displacement field in the substrate, then

$$\mathbf{w}_0(\mathbf{r},t) = \mathbf{u}_0, \tag{19}$$

where $\mathbf{w}_0 = \mathbf{w}_{|\Omega}$. We assume the substrate to be free of residual stress, so that

$$\boldsymbol{\sigma} = \mathbf{E}[D\mathbf{w}], \tag{20}$$

where $\mathbf{E}$ is the associated tensor of elastic moduli, possessing the usual minor and major symmetries. Thus, in contrast to the film stress $\mathbf{P}$, $\boldsymbol{\sigma}$ is symmetric. This satisfies the equation of motion

$$Div\boldsymbol{\sigma} = \rho_s \ddot{\mathbf{w}}, \tag{21}$$

where ρ_s is the mass density of the substrate.

The foregoing model agrees precisely with the leading-order system derived elsewhere [1] via a variational argument.

We demonstrate that $(18)_2$ can be solved uniquely for the derivative $\mathbf{u}_0'$. To this end we fix $\mathbf{u}_0$ and define

$$W(\mathbf{a}) = U(\nabla\mathbf{u}_0 + \mathbf{a}\otimes\mathbf{k}) - \mathbf{a}\cdot\mathbf{t}^+. \tag{22}$$

Let ϵ be a parameter and consider the one-parameter family $\mathbf{a}(\epsilon)$. Let $g(\epsilon) = W(\mathbf{a}(\epsilon))$ and let $()^{\cdot} = d()/d\epsilon$. Then (3) yields

$$\dot{g} = W_{\mathbf{a}}\cdot\dot{\mathbf{a}} \quad \text{and} \quad \ddot{g} = W_{\mathbf{a}}\cdot\ddot{\mathbf{a}} + \dot{\mathbf{a}}\cdot(W_{\mathbf{aa}})\dot{\mathbf{a}}, \tag{23}$$

with

$$W_{\mathbf{a}} = \mathbf{P}_0\mathbf{k} - \mathbf{t}^+ \quad \text{and} \quad (W_{\mathbf{aa}})\dot{\mathbf{a}} = (\mathbf{P}_0\mathbf{k})^{\cdot}, \tag{24}$$

wherein $\mathbf{P}_0$ is given by (2) and (10), with $\mathbf{a}(\epsilon)$ substituted in place of $\mathbf{u}_0'$. From $(18)_2$ we find that W is stationary at $\mathbf{a} = \mathbf{u}_0'$, while (2) furnishes

$$\dot{\mathbf{a}}\cdot(\mathbf{P}_0\mathbf{k})^{\cdot} = (\mathbf{k}\cdot\mathbf{S}_0\mathbf{k})\dot{\mathbf{a}}\cdot\dot{\mathbf{a}} + \dot{\mathbf{a}}\otimes\mathbf{k}\cdot\mathbf{C}_0[\dot{\mathbf{a}}\otimes\mathbf{k}]. \tag{25}$$

This is strictly positive by virtue of (6), implying that $W_{\mathbf{aa}}$ is positive definite. In particular, $\ddot{g} > 0$ on straight-line paths defined by $\mathbf{a}(\epsilon) = (1-\epsilon)\mathbf{a}_1 + \epsilon\mathbf{a}_2$ with $\mathbf{a}_1, \mathbf{a}_2$ fixed and $\epsilon \in [0,1]$. These paths belong to the domain of $W(\cdot)$, the convex set generated by the linear space of 3-vectors. Integrating with respect to ϵ yields $\dot{g}(\epsilon) > \dot{g}(0)$ and $g(1) - g(0) > \dot{g}(0)$, proving that $W(\mathbf{a})$ is strictly convex; i.e.,

$$W(\mathbf{a}_2) - W(\mathbf{a}_1) > W_{\mathbf{a}}(\mathbf{a}_1)\cdot(\mathbf{a}_2 - \mathbf{a}_1), \tag{26}$$

for all unequal pairs $\mathbf{a}_1, \mathbf{a}_2$. Because strictly convex functions have unique stationary points, it follows that

$$\mathbf{u}_0' = \bar{\mathbf{a}}(\nabla\mathbf{u}_0; \mathbf{t}^+) \tag{27}$$

where $\bar{\mathbf{a}}$ is the value of $\mathbf{a}$ at which $W_{\mathbf{a}}$ vanishes.

In the present work we impose the condition $\mathbf{t}^+ = \mathbf{0}$ for all deformations. Equations (2) and $(18)_2$, specialized to the case of zero displacement, then require that

$$\mathbf{S}_0\mathbf{k} = \mathbf{0}, \tag{28}$$

which in turn may be used to simplify (28) for arbitrary displacements. This yields

$$\mathbf{u}_0' = -\mathbf{A}_0^{-1}(\mathbf{C}_0[\nabla\mathbf{u}_0])\,\mathbf{k}, \tag{29}$$

where $\mathbf{A}_0$ is the value on Ω of the acoustic tensor of the film material, defined, for arbitrary $\mathbf{v}$, by

$$\mathbf{A}_0\mathbf{v} = (\mathbf{C}_0[\mathbf{v}\otimes\mathbf{k}])\,\mathbf{k}. \tag{30}$$

That this is positive definite and hence invertible follows from (6), with (29). Accordingly, $(18)_1$ furnishes a system for the interfacial displacement field $\mathbf{u}_0(\mathbf{r},t)$.

Following Spencer [8] we model the film as a transversely isotropic solid. The axes of transverse isotropy are coincident with the direction fields of the (straight) fiber trajectories. We drop the subscript $_0$ from all notation denoting interfacial values of variables. The components of $\mathbf{C}$ relative to the basis $\{\mathbf{e}_i\}$ are [1]

$$\begin{aligned} C_{ijkl} \;=\;& \lambda\delta_{ij}\delta_{kl} + \mu_T(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \alpha(\delta_{ij}m_km_l + m_im_j\delta_{kl}) \\ &+ (\mu_L - \mu_T)(m_im_k\delta_{jl} + m_im_l\delta_{jk} + m_jm_k\delta_{il} + m_jm_l\delta_{ik}) + \beta m_im_jm_km_l, \end{aligned} \tag{31}$$

where δ_{ij} is the Kronecker delta; $\alpha, \beta, \lambda, \mu_T$ and μ_L are material constants; and the unit vector $\mathbf{m}$, with components m_i, is the fiber axis, assumed here to be uniform and lying in the plane of the film. Spencer [8] shows that μ_T is the shear modulus for shearing in planes transverse to $\mathbf{m}$, whereas μ_L is the shear modulus for shearing parallel to $\mathbf{m}$. The remaining material constants in (32) may be interpreted in terms of extensional moduli and Poisson ratios [8].

The general form of the residual stress may be derived by enumerating the strain invariants for transverse isotropy that are linear in the (infinitesimal) strain. These are [8] $\mathbf{I}\cdot\mathbf{H}$ and $\mathbf{m}\otimes\mathbf{m}\cdot\mathbf{H}$. Comparison with the linear term in (4) then furnishes

$$\mathbf{S} = S_T(\mathbf{I} - \mathbf{m}\otimes\mathbf{m}) + S_L\mathbf{m}\otimes\mathbf{m}, \tag{32}$$

where S_T is the constant residual stress in the isotropic plane and S_L is the constant residual uniaxial stress along $\mathbf{m}$.

In this work we consider the film to be a prismatic body formed by the parallel translation of a midplane Ω in the direction of its unit normal $\mathbf{k}(=\mathbf{e}_3)$. The fibers are assumed to lie parallel to Ω, so that $m_3 = 0$ and $\mathbf{m} = m_\alpha\mathbf{e}_\alpha$. Equation (29) then yields

$$\mathbf{S}_0 = S_0\mathbf{m}\otimes\mathbf{m}, \tag{33}$$

where S_0 is the residual interfacial stress.

The acoustic tensor defined by (31) is

$$\mathbf{A} = \lambda\mathbf{k}\otimes\mathbf{k} + \mu_T(\mathbf{I} + \mathbf{k}\otimes\mathbf{k}) + (\mu_L - \mu_T)\mathbf{m}\otimes\mathbf{m}, \tag{34}$$

with eigenvalues μ_L, μ_T and $\lambda + 2\mu_T$. Inequality (6), with (29), requires that these be strictly positive. Equation (30) then yields

$$\mathbf{u}_0' = -(\lambda + 2\mu_T)^{-1}[\lambda(div\mathbf{v}) + \alpha\mathbf{m}\cdot(\nabla\mathbf{v})\mathbf{m}]\mathbf{k} - \nabla w, \tag{35}$$

where $div\mathbf{v} = v_{\alpha,\alpha}$, $\mathbf{m}\cdot(\nabla\mathbf{v})\mathbf{m} = v_{\alpha,\beta}m_\alpha m_\beta$ and

$$\mathbf{v} = \mathbf{1}\mathbf{u}_0 \quad \text{and} \quad w = \mathbf{k}\cdot\mathbf{u}_0 \tag{36}$$

are the in-plane and transverse interfacial displacements, respectively; i.e.,

$$\mathbf{u}_0 = \mathbf{v} + w\mathbf{k}. \tag{37}$$

4. Example: Love waves

We are concerned with the acoustic interaction of the film and substrate, the former coinciding with the r_1, r_2- plane and the latter with the half-space defined by $\varsigma < 0$. Accordingly, we study harmonic surface waves whose amplitudes decay with depth in the substrate. For the sake of simplicity we confine attention to Love waves. For waves propagating along the r_1- direction, these have the form

$$u_i = \delta_{i2} F(r_1, \varsigma, t); \quad F(r_1, \varsigma, t) = f(r_1, t)\exp(\eta k \varsigma), \tag{38}$$

where η and k are positive constants, and

$$f(r_1, t) = A\exp[ik(r_1 - ct)] \tag{39}$$

in which c is the wavespeed and A is a constant. The induced displacement of the film/substrate interface is

$$w = u_3 = 0, \quad v_\alpha = u_\alpha = \delta_{\alpha 2} f(r_1, t) \tag{40}$$

For uniform isotropic materials, the film-substrate interaction term is [1]

$$\boldsymbol{\sigma}\mathbf{k} = \mu_s \eta k f(r_1, t)\mathbf{e}_2. \tag{41}$$

In the substrate, eq. (22) is satisfied provided that [2, 3]

$$\eta = \sqrt{1 - s^2}, \tag{42}$$

where

$$s = c/c_s < 1 \quad \text{and} \quad c_s = \sqrt{\mu_s/\rho_s} \tag{43}$$

is the transverse wavespeed in the substrate.

According to $(40)_2$, we have $div\mathbf{v} = 0$ identically at the interface. Using (36) and eqs. (76) and (82) of [9] we find, after much manipulation, omitted here for the sake of brevity, that $(18)_1$ becomes

$$\begin{aligned} & hS(\nabla\nabla\mathbf{v})\mathbf{m}\otimes\mathbf{m} + h\mu_T\Delta\mathbf{v} + h\gamma\nabla[\mathbf{m}\cdot(\nabla\mathbf{v})\mathbf{m}] + \\ & h\delta\{\mathbf{m}\cdot[(\boldsymbol{\nabla\nabla}\mathbf{v})\mathbf{m}\otimes\mathbf{m}]\} + \\ & h\,(\mu_L - \mu_T)[(\mathbf{m}\cdot\Delta\mathbf{v})\mathbf{m} \;+\; (\nabla\nabla\mathbf{v})\mathbf{m}\otimes\mathbf{m}] - \boldsymbol{\sigma}\mathbf{k} \\ = \; & h\rho\mathbf{v}_{tt}, \end{aligned} \tag{44}$$

where

$$\gamma = \tfrac{-\lambda\alpha}{\lambda + 2\mu_T} + \alpha + \mu_L - \mu_T, \qquad \delta = \beta - \tfrac{\alpha^2}{\lambda + 2\mu_T}, \tag{45}$$

and where $(\nabla\nabla\mathbf{v})\mathbf{m}\otimes\mathbf{m} = v_{\gamma,\alpha\beta}m_\alpha m_\beta \mathbf{e}_\gamma$ and $\Delta\mathbf{v} = v_{\gamma,\alpha\alpha}\mathbf{e}_\gamma$. Substituting (38) reduces this to

$$hf''\{Sm_1^2\mathbf{e}_2 + \gamma m_1 m_2 \mathbf{e}_1 + \delta m_1^2 m_2 \mathbf{m} + (\mu_L - \mu_T)(m_2\mathbf{m} + m_1^2\mathbf{e}_2) + \mu_T\mathbf{e}_2\} - \mu_s\eta k f\mathbf{e}_2 = h\rho\ddot{f}\mathbf{e}_2, \tag{46}$$

wherein the primes and dots refer to derivatives of f with respect to r_1 and t, respectively. This is equivalent to its projections onto $\mathbf{e}_1$ and $\mathbf{e}_2$, given respectively by

$$hf''m_1m_2(\gamma+\delta m_1^2+\mu_L-\mu_T)=0 \tag{47}$$

and

$$hf''(Sm_1^2+\delta m_1^2m_2^2+\mu_L)-\mu_s\eta kf=h\rho\ddot{f}. \tag{48}$$

We note that in the course of obtaining these equations no use was made of the special form (39) of the function f. In particular, (47) and (48) are satisfied by taking f to be a function linear in r_1, with time-dependent coefficients. The associated interfacial strain vanishes. The solution with the interfacial displacement bounded everywhere is a harmonic oscillation with frequency $\omega=\sqrt{\mu_s\eta k/h\rho}$. The interface effectively vibrates as a rigid body, and the absence of a length scale associated with this vibration means that ηk remains indeterminate. For wave-like solutions with f given by (39), eq. (47) requires that the parenthesis vanishes or that m_1m_2 vanishes. For typical data on carbon-fiber/epoxy-resin composites [8], we find that the first alternative has no solution for real-valued m_1. The remaining alternative yields the two possibilities $\mathbf{m}=\pm\mathbf{e}_1$ and $\mathbf{m}=\pm\mathbf{e}_2$, corresponding to waves propagating parallel and transverse to the fibers, respectively. In the first case; i.e., for waves traveling along the fibers, (48) reduces to

$$hk(S+\mu_L)+\mu_s\eta=h\rho kc^2, \tag{49}$$

yielding

$$\eta=hk(rs^2-\tfrac{S+\mu_L}{\mu_s}). \tag{50}$$

where $r=\rho/\rho_s$, and

$$s=c/c_s<1, \quad \text{where} \quad c_s=\sqrt{\mu_s/\rho_s} \tag{51}$$

is the transverse shear wave speed in the substrate. Eliminating η between (42) and (50) yields the dispersion relation

$$\sqrt{1-s^2}=\epsilon(rs^2-\tfrac{S+\mu_L}{\mu_s}), \tag{52}$$

where $\epsilon=hk$. The theory used here to obtain this relation purports to be valid only if $\epsilon\ll 1$, so that $1-s^2=O(\epsilon^2)$. Accordingly, we assume that $s^2=1-\epsilon^2C^2+o(\epsilon^2)$, solve for C and obtain

$$s\sim 1-\tfrac{1}{2}\epsilon^2(r-\tfrac{S+\mu_L}{\mu_s})^2, \tag{53}$$

to leading order. The case of waves traveling in the direction transverse to the fibers leads to the same results, but with S set to zero. We observe that the relevant film stiffness in both cases is the longitudinal modulus μ_L. This is to be expected because the transverse modulus μ_T pertains to shearing in the isotropic plane orthogonal to $\mathbf{m}$, whereas deformations of the type considered induce shearing in a plane containing $\mathbf{m}$. It is easy to show, from (31), that such shear deformations generate a stress equal to $2\mu_L$ times the strain, provided that $\mathbf{m}$ is oriented along either of the axes of shear; i.e., along $\pm\mathbf{e}_1$ or $\pm\mathbf{e}_2$, as in the foregoing solutions.

References

1. D.J. Steigmann and R.W. Ogden, 2007, Surface waves supported by thin-film/substrate interactions. IMA J. Appl. Math. 72, 730-47.

2. J.D. Achenbach, 1973, *Wave Propagation in Elastic Solids,* North-Holland, Amsterdam.

3. I. Murdoch, 1976, The propagation of surface waves in bodies with material boundaries. J. Mech. Phys. Solids 24, 137-46.

4. A.V. Pichugin and G.A. Rogerson, 2002, An asymptotic membrane-like theory for long-wave motion in a pre-stressed elastic plate. Proc. R. Soc. Lond. A458, 1447-68.

5. A.L. Shuvalov and A.G. Every, 2008, On the long-wave onset of dispersion of the surface-wave velocity in coated solids. Wave Motion 45, 857-63.

6. T.C.T. Ting, 2009, Steady waves in an anisotropic elastic layer attached to a half-space or between two half-spaces: a generalization of Love waves and Stoneley waves. Math. Mech. Solids 14, 52-71.

7. R.J. Knops and L.E. Payne, 1971, *Uniqueness Theorems in Linear Elasticity,* Springer Tracts in Natural Philosophy, Vol. 19.

8. A.J.M. Spencer, 1984, Constitutive theory for strongly anisotropic solids, in: Continuum Theory of the Mechanics of Fibre-Reinforced Composites (A.J.M. Spencer, Ed.). CISM Courses and Lectures No. 282., pp. 1-32. Springer, Wien.

9. D.J. Steigmann, 2009, Linear theory for the bending and extension of a thin, residually stressed, fiber-reinforced lamina. Int. J. Engng. Sci. 47, 1367-78.