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Wall-crossing formulas in Hamiltonian geometry

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Summary. In this article, we study the local invariants associated to the Hamiltonian action of a compact torus. Our main results are wall-crossing formulas between invariants attached to adjacent connected components of regular values of the moment map.

Key words: moment map, equivariant cohomology, geometric quantization, transversally elliptic.

1.1 Introduction

Let (M, Ω) be a compact symplectic manifold with the Hamiltonian action of a compact torus T , and moment map $\Phi : M \rightarrow \mathfrak{t}^*$. Let us assume that the action is *effective*. We are interested here in two global invariants:

1. the Duistermaat-Heckman measure $\text{DH}(M)$ which is the pushforward by Φ of the Liouville volume form,
2. the Riemann-Roch characters $RR(M, L^{\otimes k})$, $k \geq 1$, which are virtual representations of T . Here the data (M, Ω, Φ) is prequantized by a Kostant-Souriau line bundle L .

Let $\Lambda^* \subset \mathfrak{t}^*$ be the weight lattice of T . For every couple $(\mu, k) \in \Lambda^* \times \mathbb{Z}^{>0}$, we denote by $m(\mu, k) \in \mathbb{Z}$ the multiplicity of the weight μ in $RR(M, L^{\otimes k})$.

One striking property of the moment map is that its image $\Phi(M)$ is a convex polytope in \mathfrak{t}^* . In fact, as noted for example in [17] or [20], each component of the set of regular values of Φ is either an open convex polytope contained in $\Phi(M)$, or the open subset $\mathfrak{c}_{\text{ext}} = \mathfrak{t}^* \setminus \Phi(M)$.

Let us fix a connected component \mathfrak{c} of regular values of Φ . A celebrated Theorem of Duistermaat and Heckman [15] tells us that the measure $\text{DH}(M)$ is equal to a *polynomial* $\text{DH}_{\mathfrak{c}}$ times a Lebesgue measure on the open subset \mathfrak{c} . Note that $\text{DH}_{\mathfrak{c}_{\text{ext}}}$ is the zero polynomial.

The “quantization commutes with reduction” Theorem [28, 29] shows that there exists a *periodic polynomial* $m_c : \Lambda^* \times \mathbb{Z} \rightarrow \mathbb{Z}$ which coincides with the multiplicity map $m : \Lambda^* \times \mathbb{Z}^{>0} \rightarrow \mathbb{Z}$ on the cone of $\mathfrak{t}^* \times \mathbb{R}$ generated by $\mathfrak{c} \times \{1\}$. The periodic polynomial m_c is defined by a Kawasaki-Riemann-Roch formula on a symplectic quotient $\mathcal{M}_a = \Phi^{-1}(a)/T$ where $a \in \mathfrak{c}$. As a corollary, we get that DH_c is the semi-classical limit of m_c : one has

$$\lim_{k \rightarrow \infty} \frac{m_c(k\mu, k)}{k^d} = \frac{1}{(2\pi)^d} \text{DH}_c(\mu) \quad (1.1)$$

for every $\mu \in \Lambda^*$. Here $d = \frac{1}{2} \dim \mathcal{M}_a$.

We have seen that the *global* invariants $\text{DH}(M)$, $RR(M, L^{\otimes k})$, $k \geq 1$ gives rises to a family of *local* invariants DH_c , m_c , where c runs over the connected component of regular values of Φ .

This paper is concerned by the differences $\text{DH}_{c_+} - \text{DH}_{c_-}$ and $m_{c_+} - m_{c_-}$ when c_{\pm} are two *adjacent* connected components of regular values of Φ . Let $\Delta \subset \mathfrak{t}^*$ be the hyperplane that separates c_{\pm} . Some continuity properties are known:

1. the polynomial $\text{DH}_{c_+} - \text{DH}_{c_-}$ is divisible by a certain power of the equation the hyperplane Δ (see [17] and [12]),
2. the periodic polynomial $m_{c_+} - m_{c_-}$ vanishes on

$$\{(\mu, k) \in \Lambda^* \times \mathbb{Z} \mid \mu \in k\Delta\}. \quad (1.2)$$

See [29].

In this paper, we compute explicitly the difference $\text{DH}_{c_+} - \text{DH}_{c_-}$, and we show that $m_{c_+} - m_{c_-}$ vanishes also on some translates of (1.2).

Let us introduce some notations. We denote by $T_{\Delta} \subset T$ the subtorus of dimension 1 that has for Lie algebra the one dimensional subspace \mathfrak{t}_{Δ} which is orthogonal to the direction of Δ . Let $\beta \in \mathfrak{t}_{\Delta}$ be the primitive element of the lattice $\ker(\exp : \mathfrak{t} \rightarrow T)$ which is pointing out of c_- .

We make the choice of a decomposition $T = T/T_{\Delta} \times T_{\Delta}$, where T/T_{Δ} denotes a subtorus de T . At the level of Lie algebras, we have then $\mathfrak{t} = (\mathfrak{t}/\mathfrak{t}_{\Delta}) \oplus \mathfrak{t}_{\Delta}$ and $\mathfrak{t}^* = (\mathfrak{t}/\mathfrak{t}_{\Delta})^* \oplus \mathfrak{t}_{\Delta}^*$: hence $\xi + (\mathfrak{t}/\mathfrak{t}_{\Delta})^* = \Delta$ for any $\xi \in \Delta$. We denote $\mathcal{S}(\mathfrak{t})$ the algebra of polynomials on the vector space \mathfrak{t}^* . We will consider the polynomial $\text{DH}_{c_+} - \text{DH}_{c_-} \in \mathcal{S}(\mathfrak{t})$ relatively to the decomposition

$$\mathcal{S}(\mathfrak{t}) = \bigoplus_{j \in \mathbb{N}} \mathcal{S}(\mathfrak{t}/\mathfrak{t}_{\Delta}) \beta^j.$$

Let us choose $\xi \in \Delta$ in the relative interior of $\bar{c}_+ \cap \bar{c}_-$ in Δ . We consider the family \mathcal{F} of connected components $Z \subset M^{T_{\Delta}}$ such that $\xi \in \Phi(Z) \subset \Delta$. It is easy to see that \mathcal{F} does not depend of the choice of ξ : we have $\bar{c}_+ \cap \bar{c}_- \subset \Phi(Z)$ for all $Z \in \mathcal{F}$. For each $Z \in \mathcal{F}$, we denote

$$\Phi_Z : Z \rightarrow (\mathfrak{t}/\mathfrak{t}_\Delta)^*$$

the restriction of the map $\Phi - \xi$ to the symplectic sub-manifold Z . The map Φ_Z is a moment map relative to the Hamiltonian action of T/T_Δ on Z . Let $\text{DH}(Z)$ be Duistermaat-Heckman measure on $(\mathfrak{t}/\mathfrak{t}_\Delta)^*$ associated to the moment map Φ_Z . Since 0 is a regular value of Φ_Z , we may consider the Duistermaat-Heckman polynomial

$$\text{DH}_0(Z) \in \mathcal{S}(\mathfrak{t}/\mathfrak{t}_\Delta)$$

such that $\text{DH}(Z)(a') = \text{DH}_0(Z)(a')da'$ for a' in a neighborhood of 0 in $(\mathfrak{t}/\mathfrak{t}_\Delta)^*$.

For $Z \in \mathcal{F}$, we consider the symplectic reduction

$$\mathcal{Z}_\xi = \Phi_Z^{-1}(0)/(T/T_\Delta),$$

and the normal bundle N_Z of Z in M . Let $2d_Z$ be the dimension of \mathcal{Z}_ξ and $2r_Z$ be the (real) rank of N_Z . We prove in Section 1.2 the following.

Theorem A. *We have*

$$(\text{DH}_{\mathfrak{c}_+} - \text{DH}_{\mathfrak{c}_-})(a) = \sum_{Z \in \mathcal{F}} \mathbf{D}_Z(a - \xi), \quad a \in \mathfrak{t}^*$$

where each polynomial $\mathbf{D}_Z \in \mathcal{S}(\mathfrak{t})$ admits the following decomposition

$$\mathbf{D}_Z = \frac{\beta^{r_Z-1}}{\det_Z^{1/2}(\frac{-\mathcal{L}_\beta}{2\pi})} \left(\frac{\text{DH}_0(Z)}{(r_Z - 1)!} + \sum_{k=1}^{d_Z} \beta^k \mathbf{Q}_{Z,k} \right).$$

Each polynomial $\mathbf{Q}_{Z,k}$ belongs to $\mathcal{S}(\mathfrak{t}/\mathfrak{t}_\Delta)$ and is of degree less than $d_Z - k$. The term $\det_Z^{1/2}(\frac{-\mathcal{L}_\beta}{2\pi}) \in \mathbb{Z}$ is the Pfaffian of the infinitesimal action of $\frac{-\beta}{2\pi}$ on the fibers of the normal bundle N_Z .

Theorem A generalizes previous results of Guillemin-Lerman-Sternberg [17] and Brion-Procesi [12]. In Section 1.2.4 we give the precise definition of the polynomials $\mathbf{Q}_{Z,k}$.

Suppose now that M is prequantized by a Kostant-Souriau line bundle L . The hyperplane Δ is defined by the equation

$$\frac{\langle a, \beta \rangle}{2\pi} - r_\Delta = 0, \quad a \in \mathfrak{t}^*, \tag{1.3}$$

for some $r_\Delta \in \mathbb{Z}$. The bundle N_Z decomposes as the sum of two polarized sub-bundles $N_Z^{\pm, \beta}$. Let $s_Z^\pm \in \mathbb{N}$ be the absolute value of the trace of $\frac{1}{2\pi}\mathcal{L}_\beta$ on $N_Z^{\pm, \beta}$. Note that the integer $s_Z^+ + s_Z^-$ is larger than half of the codimension of Z in M .

We prove in Section 1.3.5 the following

Theorem B *Let $s^\pm := \inf_{Z \in \mathcal{F}} s_Z^\pm$. We have $m_{\mathfrak{c}_+}(\mu, k) = m_{\mathfrak{c}_-}(\mu, k)$ when*

$$-s^- < \frac{\langle \mu, \beta \rangle}{2\pi} - kr_\Delta < s^+. \quad (1.4)$$

Note that the symplectic orbifolds $\mathcal{Z}_\xi, Z \in \mathcal{F}$ are the connected component of the symplectic reduction

$$\mathcal{M}_\xi^\Delta := (\Phi^{-1}(\xi) \cap M^{T_\Delta}) / (T/T_\Delta).$$

We have the following refinement au Theorem **B**.

Theorem C *If \mathcal{M}_ξ^Δ is connected, the inequalities (1.4) are optimal, i.e. there exists (μ, k) such that $\frac{\langle \mu, \beta \rangle}{2\pi} - kr_\Delta = \pm s^\pm$ and $m_{\mathfrak{c}_+}(\mu, k) \neq m_{\mathfrak{c}_-}(\mu, k)$.*

In Section 1.4 we apply Theorem **B** to the particular cases where M is a integral coadjoint orbit of a compact Lie group G . In Section 1.4.4, we study more precisely the case $G = \mathrm{SU}(n)$: here our result precises some of the results of Billey-Guillemin-Rassart [10].

In Section 1.5, we obtain a strong version of Theorem **B** in the case of an action of a torus T on a complex vector spaces \mathbb{C}^d . The quantization of this action is in some sense the vector space $\mathrm{Pol}(\mathbb{C}^d)$ of complex polynomials on \mathbb{C}^d . The T -multiplicities of $\mathrm{Pol}(\mathbb{C}^d)$ are given by a partition function $N_R : \Lambda^* \rightarrow \mathbb{N}$. It was observed in [13, 35] that there exists a finite decomposition of the vector space \mathfrak{t}^* in *conic chambers* such that N_R is periodic polynomial on each piece.

Let \mathfrak{c}_\pm be two adjacents chambers, and let $P_{\mathfrak{c}_\pm}$ be the corresponding periodic polynomials computing N_R on each chambers. The main result of Section 1.5 is the formula (1.109) which depicts the periodic polynomial $P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-}$ as a convolution of distributions. Recently¹, Boyal-Vergne [11] and De Concini-Procesi-Vergne [14] proposed differents proofs of this formula.

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Notations

Throughout the paper T will denote a compact, connected abelian Lie group, and \mathfrak{t} its Lie algebra. The integral lattice $\Lambda \subset \mathfrak{t}$ is defined as the kernel of $\exp : \mathfrak{t} \rightarrow T$, and the real weight lattice $\Lambda^* \subset \mathfrak{t}^*$ is defined by : $\Lambda^* := \mathrm{hom}(\Lambda, 2\pi\mathbb{Z})$. Every $\mu \in \Lambda^*$ defines a 1-dimensional T -representation, denoted by \mathbb{C}_μ , where $t = \exp X$ acts by $t^\mu := e^{i\langle \mu, X \rangle}$. We denote by $R(T)$ the ring of characters of finite-dimensional T -representations. We denote by $R^{-\infty}(T)$ the set of generalized characters of T . An element $\chi \in R^{-\infty}(T)$ is of

¹ Our present paper is a revised version of the preprint math.SG/0411306

the form $\chi = \sum_{\mu \in \Lambda^*} a_\mu \mathbb{C}_\mu$, where $\mu \mapsto a_\mu$, $\Lambda^* \rightarrow \mathbb{Z}$ has at most polynomial growth.

The symplectic manifolds are oriented by their Liouville volume forms. If (Z, o_Z) is an oriented submanifold of an oriented manifold (M, o_M) , we take on the fibers of the normal bundle N of Z in M , the orientation o_N satisfying $o_M = o_Z \cdot o_N$.

1.2 Duistermaat-Heckman measures

Let (M, Ω) be a symplectic manifold of dimension $2n$ equipped with an Hamiltonian action of a torus T , with Lie algebra \mathfrak{t} . The moment map $\Phi : M \rightarrow \mathfrak{t}^*$ satisfies the relations $\Omega(X_M, -) + d\langle \Phi, X \rangle = 0$, $X \in \mathfrak{t}$. We assume in this section that Φ is *proper*, and that the generic stabiliser Γ_M of T on M is *finite*.

The Duistermaat-Heckman measure $\text{DH}(M)$ is defined as the pushforward by Φ of the Liouville volume form $\frac{\Omega^n}{n!}$ on M . For every $f \in \mathcal{C}^\infty(\mathfrak{t}^*)$ with compact support one has $\int_{\mathfrak{t}^*} \text{DH}(M)(a) f(a) = \int_M f(\Phi) \frac{\Omega^n}{n!}$. In other terms

$$\text{DH}(M)(a) = \int_M \delta(a - \Phi) \frac{\Omega^n}{n!}, \quad a \in \mathfrak{t}^*.$$

We can define $\text{DH}(M)$ in terms of equivariant forms as follows. Let $\mathcal{A}(M)$ be the space of differential forms on M with complex coefficients. We denote by $\mathcal{A}_{temp}^{-\infty}(\mathfrak{t}, M)$ the space of tempered generalized functions over \mathfrak{t} with values in $\mathcal{A}(M)$, and by $\mathcal{M}_{temp}^{-\infty}(\mathfrak{t}^*, M)$ the space of tempered distributions over \mathfrak{t}^* with values in $\mathcal{A}(M)$. Let $\mathcal{F} : \mathcal{A}_{temp}^{-\infty}(\mathfrak{t}, M) \rightarrow \mathcal{M}_{temp}^{-\infty}(\mathfrak{t}^*, M)$ be the Fourier transform normalized by the condition that $\mathcal{F}(X \mapsto e^{i\langle \xi, X \rangle})$ is equal to the Dirac distribution $a \mapsto \delta(a - \xi)$.

Let $\Omega_{\mathfrak{t}}(X) = \Omega - \langle \Phi, X \rangle$ be the equivariant symplectic form. We have then $\mathcal{F}(e^{-i\Omega_{\mathfrak{t}}}) = e^{-i\Omega} \delta(a - \Phi)$ and so

$$\text{DH}(M) = (i)^n \int_M \mathcal{F}(e^{-i\Omega_{\mathfrak{t}}}). \tag{1.5}$$

1.2.1 Equivariant cohomology and localization

We first recall the Cartan model of equivariant cohomology with polynomial coefficients and the extension to generalized coefficients defined by Kumar and Vergne [26]. We give after a brief account to the method of localization developed in [30, 31],

Let M be a manifold provided with an action of a compact connected Lie group K with Lie algebra \mathfrak{k} . Let $d : \mathcal{A}(M) \rightarrow \mathcal{A}(M)$ be the exterior differentiation. Let $\mathcal{A}_c(M)$ be the sub-algebra of compactly supported differential forms. If V is a vector field on M we denote by $c(V) : \mathcal{A}(M) \rightarrow \mathcal{A}(M)$ the

contraction by V . The action of K on M gives a morphism $X \rightarrow X_M$ from \mathfrak{k} to the Lie algebra of vector fields on M .

We consider the space of K -equivariant maps $\mathfrak{k} \rightarrow \mathcal{A}(M)$, $X \mapsto \eta(X)$, equipped with the derivation $(D\eta)(X) := (d - c(X_M))(\eta(X))$, $X \in \mathfrak{k}$. Since $D^2 = 0$, one can define the cohomology space $\ker D/\text{Im}D$. The Cartan model [7, 21] considers *polynomial* maps and the associated cohomology is denoted $\mathcal{H}_K^*(M)$. Kumar and Vergne [26] studied the cohomology spaces $\mathcal{H}_K^{\pm\infty}(M)$ obtained by taking $\mathcal{C}^{\pm\infty}$ maps. Recall the construction $\mathcal{H}_K^{-\infty}(M)$.

The space $\mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}(M))$ of generalized functions on \mathfrak{k} with values in the space $\mathcal{A}(M)$ is, by definition, the space $\text{Hom}(m_c(\mathfrak{k}), \mathcal{A}(M))$ of continuous \mathbb{C} -linear maps from the space $m_c(\mathfrak{k})$ of smooth compactly supported densities on \mathfrak{k} to the space $\mathcal{A}(M)$, both endowed with the \mathcal{C}^∞ -topologies. We define $\mathcal{A}_K^{-\infty}(M) := \mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}(M))^K$ as the space of K -equivariant $\mathcal{C}^{-\infty}$ -maps from \mathfrak{k} to $\mathcal{A}(M)$. The differential D defined on $\mathcal{C}^\infty(\mathfrak{k}, \mathcal{A}(M))$ admits a natural extension to $\mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}(M))$ and $D^2 = 0$ on $\mathcal{A}_K^{-\infty}(M)$ [26]. The cohomology associated to $(\mathcal{A}_K^{-\infty}(M), D)$ is called the K -equivariant cohomology with generalized coefficients and is denoted by $\mathcal{H}_K^{-\infty}(M)$. The subspace $\mathcal{A}_{K,c}^{-\infty}(M) := \mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}_c(M))^K$ is stable under the differential D , and we denote by $\mathcal{H}_{K,c}^{-\infty}(M)$ the associated cohomology. When M is oriented, the integration over M gives rise to a map $\int_M : \mathcal{H}_{K,c}^{-\infty}(M) \rightarrow \mathcal{C}^{-\infty}(\mathfrak{k})^K$.

Localization procedure. Let λ be a K -invariant 1-form on M and let

$$\Phi_\lambda : M \rightarrow \mathfrak{k}^* \quad (1.6)$$

be the K -equivariant map defined by $\langle \Phi_\lambda(m), X \rangle = \lambda(X_M)_m$: then $D\lambda(X) = d\lambda - \langle \Phi_\lambda, X \rangle$. The localization procedure developed in [30, 31] is based on the existence of an inverse $[D\lambda]^{-1}$ of the K -equivariant form $D\lambda$. It is an equivariantly closed element of $\mathcal{A}_K^{-\infty}(M - \Phi_\lambda^{-1}(0))$ defined by the integral

$$[D\lambda]^{-1}(X) = i \int_0^\infty e^{-itD\lambda(X)} dt. \quad (1.7)$$

An open subset $\mathcal{U} \subset M$ is called *adapted to λ* if \mathcal{U} is K -invariant and if $(\partial\mathcal{U}) \cap \Phi_\lambda^{-1}(0) = \emptyset$. In [31], we associate to an open subset \mathcal{U} adapted to λ , the following equivariantly closed form with generalized coefficients

$$P_\lambda^\mathcal{U} = \chi^\mathcal{U} + d\chi^\mathcal{U}[D\lambda]^{-1}\lambda. \quad (1.8)$$

Here $\chi^\mathcal{U} \in \mathcal{C}^\infty(M)$ is a K -invariant function supported in \mathcal{U} which is equal to 1 in a neighborhood of $\mathcal{U} \cap \Phi_\lambda^{-1}(0)$. The cohomology class defined by $P_\lambda^\mathcal{U}$ in $\mathcal{H}_K^{-\infty}(M)$ does not depend of $\chi^\mathcal{U}$. In particular $P_\lambda^\mathcal{U} = 0$ in $\mathcal{H}_K^{-\infty}(M)$ if $\mathcal{U} \cap \Phi_\lambda^{-1}(0) = \emptyset$. If $\mathcal{U} \cap \Phi_\lambda^{-1}(0)$ is compact, we take $\chi^\mathcal{U}$ with compact support, then $P_\lambda^\mathcal{U}$ defines a cohomology class in $\mathcal{H}_{K,c}^{-\infty}(M)$.

1.2.2 Localization of $\text{DH}(M)$

We come back to the situation of a Hamiltonian action of a torus T on a symplectic manifold (M, ω) . We need two auxiliary data : a T -invariant Riemannian metric on M denoted $(\cdot, \cdot)_M$, and a scalar product (\cdot, \cdot) on \mathfrak{t}^* which induces an identification $\mathfrak{t}^* \simeq \mathfrak{t}$.

Let \mathcal{H} be the Hamiltonian vector field of the function $\frac{-1}{2}\|\Phi\|^2 : M \rightarrow \mathbb{R}$: for $m \in M$ we have $\mathcal{H}_m = (\Phi(m))_M|_m$. Then for every $\xi \in \mathfrak{t}^*$, the Hamiltonian vector field of $\frac{-1}{2}\|\Phi - \xi\|^2$ is $\mathcal{H} - \xi_M$, and we consider the following T -invariant 1-form

$$\lambda_\xi = (\mathcal{H} - \xi_M, \cdot)_M \tag{1.9}$$

with corresponding map $\Phi_{\lambda_\xi} : M \rightarrow \mathfrak{t}^*$ (see (1.6)). Here $\Phi_{\lambda_\xi}^{-1}(0)$ coincides with the subset $\text{Cr}(\|\Phi - \xi\|^2) \subset M$ of critical points of the function $\|\Phi - \xi\|^2$, and $m \in \text{Cr}(\|\Phi - \xi\|^2)$ if and only if $(\Phi(m) - \xi)_M$ vanishes at m [30, 31].

Definition 1.2.1 *Let $P_\xi \in \mathcal{H}_{T,c}^{-\infty}(M)$ be the cohomology class defined by $P_{\lambda_\xi}^{\mathcal{U}}$, where \mathcal{U} is a T -invariant relatively compact neighborhood of $\Phi^{-1}(\xi)$ such that $\overline{\mathcal{U}} \cap \text{Cr}(\|\Phi - \xi\|^2) = \Phi^{-1}(\xi)$.*

The cohomology class P_ξ will be used to localized the Duistermaat-Heckman measure. For every $\xi \in \mathfrak{t}^*$, we define the distribution $\text{DH}_\xi(M)$ by

$$\text{DH}_\xi(M) = (i)^n \mathcal{F} \left(\int_M P_\xi e^{-i\Omega_t} \right). \tag{1.10}$$

Here we can put the Fourier transform outside the integral because P_ξ is compactly supported on M . For any $\xi \in \mathfrak{t}^*$, let $r_\xi > 0$ be the smallest non-zero critical value of the function $\|\Phi - \xi\|^2$. As a particular case of Proposition 3.8 in [31], we have

Proposition 1.2.2 *Let ξ be any point in \mathfrak{t}^* . The following equality of distributions on \mathfrak{t}^**

$$\text{DH}(M) = \text{DH}_\xi(M)$$

holds in the open ball $B(\xi, r_\xi) \subset \mathfrak{t}^$.*

We will now use the last Proposition, first to recover the classical result of Duistermaat and Heckman [15] concerning the polynomial behaviour of $\text{DH}(M)$ on the open subset of regular values of Φ . After we determine the difference taken by $\text{DH}(M)$ between two adjacent regions of regular values.

1.2.3 Polynomial behaviour

We recall now the computation of the cohomology class P_ξ when ξ is a regular value of Φ , that is given in [30][Section 6] for the torus case (see [31] [Section 3.1] for the case of Hamiltonian action of a compact Lie group). First recall the following basic result which shows that $\xi \mapsto \text{DH}_\xi(M)$ is locally constant on the open subset of regular values of Φ .

Lemma 1.2.3 ([33]) *If ξ and ξ' belong to the same connected component of regular values of Φ , we have $P_\xi = P_{\xi'}$ in $\mathcal{H}_{T,c}^{-\infty}(M)$.*

If we combine Lemma 1.2.3 with Proposition 1.2.2, we see that for any connected component \mathfrak{c} of regular values of Φ , we have

$$\mathrm{DH}(M)(a) = \mathrm{DH}_\xi(M)(a), \quad a \in \mathfrak{c},$$

for any $\xi \in \mathfrak{c}$. We have to compute $\mathrm{DH}_\xi(M)$ when ξ a regular value of Φ .

We consider the T -principal bundle $\Phi^{-1}(\xi) \rightarrow \mathcal{M}_\xi := \Phi^{-1}(\xi)/T$ with curvature form $\omega_\xi \in \mathcal{H}^2(\mathcal{M}_\xi) \otimes \mathfrak{t}$. The orbifold \mathcal{M}_ξ carries a canonical symplectic 2-form Ω_ξ . We denote

$$\mathbf{Kir}_\xi : \mathcal{H}_T^\infty(M) \rightarrow \mathcal{H}^*(\mathcal{M}_\xi)$$

the Kirwan morphism. For any $\psi \in \mathcal{C}^\infty(\mathfrak{t})$ and $\eta \in \mathcal{H}_T^\infty(M)$ we have $\mathbf{Kir}_\xi(\eta\psi) = \mathbf{Kir}_\xi(\eta)\psi(\omega_\xi)$, where the characteristic class $\psi(\omega_\xi)$ is the value of the differential operator $e^{\omega_\xi(\frac{\partial}{\partial X}|_0)}$ against ψ . After [31][Prop. 3.11], we know that the integral

$$\int_{\mathfrak{t}} \int_M P_\xi(X)\eta(X)\psi(X)dX$$

is equal to

$$\frac{(-2i\pi)^{\dim T} \mathrm{vol}(T, dX)}{|I_M|} \int_{\mathcal{M}_\xi} \mathbf{Kir}_\xi(\eta)\psi(\omega_\xi) \quad (1.11)$$

for every equivariant class $\eta \in \mathcal{H}_T^\infty(M)$. Here $\mathrm{vol}(T, dX)$ is the volume of T for the Haar measure compatible with dX , and $|I_M|$ is the cardinal of I_M (Note that the generic stabilizer of T on $\Phi^{-1}(\xi)$ is I_M). In other words, for every $\eta \in \mathcal{H}_T^\infty(M)$ we have the following equality of generalized functions on \mathfrak{t}^* supported at 0

$$\int_M P_\xi(X)\eta(X) = \frac{(-2i\pi)^{\dim T}}{|I_M|} \int_{\mathcal{M}_\xi} \mathbf{Kir}_\xi(\eta)e^{\omega_\xi(\frac{\partial}{\partial X}|_0)} \mathrm{vol}(T, -). \quad (1.12)$$

For $\eta = e^{-i\Omega_\xi}$ we have $\mathbf{Kir}_\xi(\eta) = e^{-i(\Omega_\xi - \langle \xi, \omega_\xi \rangle)}$, and a small computation shows that

$$\mathcal{F}\left(e^{\omega_\xi(\frac{\partial}{\partial X}|_0)} \mathrm{vol}(T, -)\right)(a) = e^{-i\langle a, \omega_\xi \rangle} \frac{da}{(2\pi)^{\dim T}}, \quad a \in \mathfrak{t}^*. \quad (1.13)$$

where da is the Lebesgue measure on \mathfrak{t}^* normalized by the condition: $\mathrm{vol}(T, dX) = 1$ for the Lebesgue measure dX on \mathfrak{t} which is dual to da .

Finally (1.10), (1.12) and (1.13) give

$$\begin{aligned} \mathrm{DH}_\xi(M)(a) &= \frac{(i)^d}{|I_M|} \int_{\mathcal{M}_\xi} e^{-i(\Omega_\xi + \langle a - \xi, \omega_\xi \rangle)} da \\ &= \frac{1}{|I_M|} \int_{\mathcal{M}_\xi} \frac{(\Omega_\xi + \langle a - \xi, \omega_\xi \rangle)^d}{d!} da, \end{aligned} \quad (1.14)$$

where $2d = \dim \mathcal{M}_\xi$.

Definition 1.2.4 For any connected component \mathfrak{c} of regular values of Φ we denote $\text{DH}_{\mathfrak{c}}$ the polynomial function $a \mapsto \frac{1}{|\Gamma_M|} \int_{\mathcal{M}_{\xi}} \frac{(\Omega_{\xi} + \langle a - \xi, \omega_{\xi} \rangle)^d}{d!}$, where ξ is any point of \mathfrak{c} .

With the help of Proposition 1.2.2 we recover the classical result of Duistermaat and Heckman [15] that says that the measure $\text{DH}(M)$ is locally polynomial² on the open subset of regular values of Φ , and its value at a regular element ξ is equal to the symplectic volume of the reduce space \mathcal{M}_{ξ} (times $|\Gamma_M|^{-1}$). More precisely we have shown that for a connected component \mathfrak{c} of regular values of Φ we have

$$\text{DH}(M)(a) = \text{DH}_{\mathfrak{c}}(a)da, \quad a \in \mathfrak{c}. \quad (1.15)$$

1.2.4 Wall-crossing formulas

Consider now two connected regions \mathfrak{c}_{\pm} of regular values of Φ separated by an hyperplane $\Delta \subset \mathfrak{t}^*$. In this section we compute the polynomial $\text{DH}_{\mathfrak{c}_+} - \text{DH}_{\mathfrak{c}_-}$. It generalizes previous results of Guillemin-Lerman-Sternberg [17] and Brion-Procesi [12].

Let ξ_+, ξ_- be respectively two elements of \mathfrak{c}_+ and \mathfrak{c}_- . We know from (1.2.2), (1.14) and Definition (1.2.4) that

$$(\text{DH}_{\mathfrak{c}_+} - \text{DH}_{\mathfrak{c}_-})(a)da = (i)^n \mathcal{F} \left(\int_M (\text{P}_{\xi_+} - \text{P}_{\xi_-}) e^{-i\Omega_{\mathfrak{t}}} \right) (a), \quad a \in \mathfrak{t}^*. \quad (1.16)$$

We recall now the computation of the cohomology class $\text{P}_{\xi_+} - \text{P}_{\xi_-} \in \mathcal{H}_{T,c}^{-\infty}(M)$ done in [33]. We use the notation defined in the introduction.

Definition 1.2.5 We denote M^{Δ} the union of the connected component Z of the fixed point set $M^{T_{\Delta}}$ for which we have $\Phi(Z) \subset \Delta$. Let M_o^{Δ} be the T -invariant open subset of M^{Δ} where T/T_{Δ} acts locally freely.

For a connected component $Z \subset M^{\Delta}$, one has either $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-} \subset \Phi(Z)$ or $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-} \cap \Phi(Z) = \emptyset$. It is due to the fact that for any ξ in relative interior of $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-}$ in Δ , and any $m \in \Phi^{-1}(\xi)$ the stabilizer $\mathfrak{t}_m \subset \mathfrak{t}$ is either equal to \mathfrak{t}_{Δ} or reduced to $\{0\}$.

The symplectic manifold M^{Δ} carries a Hamiltonian action of T/T_{Δ} with moment map $\Phi|_{M^{\Delta}} : M^{\Delta} \rightarrow \Delta$ equal to the restriction of Φ on M^{Δ} .

Let ξ be a point in the relative interior of $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-}$ in Δ . From the previous discussion, we know that ξ is a regular value of $\Phi|_{M^{\Delta}}$, i.e. $\Phi^{-1}(\xi) \cap M^{T_{\Delta}}$ is a submanifold of M_o^{Δ} . Following Definition 1.2.1 we associate to ξ the cohomology class

$$\text{P}_{\xi}^{\Delta} \in \mathcal{H}_{T/T_{\Delta},c}^{-\infty}(M_o^{\Delta}).$$

² It is a polynomial times a Lebesgue measure on \mathfrak{t}^* .

Let $\mathcal{H}^*(M_o^\Delta)^{bas}$ be the sub-algebra of $\mathcal{H}^*(M_o^\Delta)$ formed by the T -basic elements. Since the T_Δ -action on M_o^Δ is trivial we have a canonical product operation

$$\mathcal{H}_{T/T_\Delta, c}^{-\infty}(M_o^\Delta) \times \mathcal{C}^{-\infty}(\mathfrak{t}_\Delta, \mathcal{H}^*(M_o^\Delta)^{bas}) \xrightarrow{\wedge} \mathcal{H}_{T, c}^{-\infty}(M_o^\Delta). \quad (1.17)$$

Proposition 1.2.6 ([33]) *There exists a generalized function supported at 0, $\delta^\Delta \in \mathcal{C}^{-\infty}(\mathfrak{t}_\Delta, \mathcal{H}^*(M_o^\Delta)^{bas})$, such that*

$$P_{\xi^+} - P_{\xi^-} = (i_\Delta)_* (P_\xi^\Delta \wedge \delta^\Delta) \quad \text{in } \mathcal{H}_{T, c}^{-\infty}(M).$$

Here $(i_\Delta)_* : \mathcal{H}_{T, c}^{-\infty}(M_o^\Delta) \rightarrow \mathcal{H}_{T, c}^{-\infty}(M)$ is the direct image map relative to the inclusion $i_\Delta : M_o^\Delta \hookrightarrow M$.

We will now give the precise definition of δ^Δ . The decomposition $T = T_\Delta \times T/T_\Delta$ and the trivial action of T_Δ on M_o^Δ determine a canonical isomorphism

$$j_\Delta : \mathcal{H}_T^*(M_o^\Delta) \xrightarrow{\sim} \mathcal{S}(\mathfrak{t}_\Delta^*) \otimes \mathcal{H}_{T/T_\Delta}^*(M_o^\Delta),$$

where $\mathcal{S}(\mathfrak{t}_\Delta^*)$ is the algebra of complex polynomial functions on \mathfrak{t}_Δ . Since the T/T_Δ -action on M_o^Δ is locally free, we have the Chern-Weil isomorphism

$$\mathbf{cv}_\Delta : \mathcal{H}_{T/T_\Delta}^*(M_o^\Delta) \xrightarrow{\sim} \mathcal{H}^*(M_o^\Delta)^{bas}.$$

Let N_Δ be the T -equivariant normal bundle of M^Δ in M , and let

$$\text{Eul}(N_\Delta) \in \mathcal{H}_T^*(M^\Delta)$$

be the T -equivariant Euler class of N_Δ . Now we consider the restriction of $\text{Eul}(N_\Delta)$ on the open subset $M_o^\Delta \subset M^\Delta$, that we look through the isomorphism $\mathbf{cv}_\Delta \circ j_\Delta$ as an element of $\mathcal{S}(\mathfrak{t}_\Delta^*) \otimes \mathcal{H}^*(M_o^\Delta)^{bas}$ (for simplicity we keep the same notations $\text{Eul}(N_\Delta)$ for this element). Following [30], we define inverses $\text{Eul}_{\pm\beta}^{-1}(N_\Delta) \in \mathcal{C}^{-\infty}(\mathfrak{t}_\Delta, \mathcal{H}^*(M_o^\Delta)^{bas})$ by

$$\text{Eul}_{\pm\beta}^{-1}(N_\Delta)(X) = \lim_{s \rightarrow +\infty} \frac{1}{\text{Eul}(N_\Delta)(X \pm is\beta)}. \quad (1.18)$$

Here $\beta \in \mathfrak{t}_\Delta$ is chosen so that $\langle \xi^+ - \xi^-, \beta \rangle > 0$.

Definition 1.2.7 *The generalized function $\delta^\Delta \in \mathcal{C}^{-\infty}(\mathfrak{t}_\Delta, \mathcal{H}^*(M_o^\Delta)^{bas})$ is defined by*

$$\delta^\Delta := \text{Eul}_\beta^{-1}(N_\Delta) - \text{Eul}_{-\beta}^{-1}(N_\Delta). \quad (1.19)$$

Since the polynomial $\text{Eul}(N_\Delta)$ is invertible in a smooth manner on $\mathfrak{t}_\Delta \setminus \{0\}$ the generalized function δ^Δ is supported at 0.

Let ξ be a point in the relative interior of $\bar{\mathfrak{c}}_+ \cap \bar{\mathfrak{c}}_-$ in Δ . We consider the symplectic reduction

$$\mathcal{M}_\xi^\Delta := \left(M^\Delta \cap \Phi^{-1}(\xi) \right) / (T/T_\Delta).$$

If we restrict δ^Δ to the submanifold $M^\Delta \cap \Phi^{-1}(\xi)$ we get the generalized function

$$\delta_\xi^\Delta \in \mathcal{C}^{-\infty}(\mathfrak{t}_\Delta, \mathcal{H}^*(\mathcal{M}_\xi^\Delta)).$$

Now we are able to compute the right hand side of (1.16). Let $\omega_\xi^\Delta \in \mathcal{H}^2(\mathcal{M}_\xi^\Delta) \otimes \mathfrak{t}/\mathfrak{t}_\Delta$ be the curvature of the T/T_Δ -principal bundle $M^\Delta \cap \Phi^{-1}(\xi) \rightarrow \mathcal{M}_\xi^\Delta$. Let $|S_\xi^\Delta|$ be locally constant function on $M^\Delta \cap \Phi^{-1}(\xi)$ which is equal to the cardinal of the generic stabilizer of T/T_Δ . From (1.12) and Proposition 1.2.6 we have

$$\begin{aligned} & \int_M (\mathbb{P}_{\xi_+} - \mathbb{P}_{\xi_-})(X) e^{-i\Omega_t(X)} \\ &= \int_{M_\phi^\Delta} \mathbb{P}_\xi^\Delta(X') \delta^\Delta(X'') e^{-i\Omega_t(X'+X'')} \\ &= \frac{(-2i\pi)^{\dim T-1}}{|S_\xi^\Delta|} \int_{\mathcal{M}_\xi^\Delta} e^{\omega_\xi^\Delta(\frac{\partial}{\partial X'}|_0)} \text{vol}(T/T_\Delta, -) \mathbf{Kir}_\xi^\Delta(e^{-i\Omega_t})(X'') \delta_\xi^\Delta(\mathbb{K}^{\mathbb{Z}}) \end{aligned}$$

In the last equation the notations are the following :

1. $X = X' + X''$ with $X' \in \mathfrak{t}/\mathfrak{t}_\Delta$ and $X'' \in \mathfrak{t}_\Delta$,
2. the Kirwan map $\mathbf{Kir}_\xi^\Delta : \mathcal{H}_T^\infty(M) \rightarrow \mathcal{C}^\infty(\mathfrak{t}_\Delta, \mathcal{H}^*(\mathcal{M}_\xi^\Delta))$ is the composition of the restriction $\mathcal{H}_T^\infty(M) \rightarrow \mathcal{H}_T^\infty(M^\Delta \cap \Phi^{-1}(\xi))$ with the Chern-Weil isomorphism $\mathcal{H}_T^\infty(M^\Delta \cap \Phi^{-1}(\xi)) \xrightarrow{\sim} \mathcal{C}^\infty(\mathfrak{t}_\Delta, \mathcal{H}^*(\mathcal{M}_\xi^\Delta))$.

A direct computation gives that $\mathbf{Kir}_\xi^\Delta(\Omega_t)(X'') = \Omega_\xi^\Delta - \langle \xi, \omega_\xi^\Delta + X'' \rangle$ where Ω_ξ^Δ is the induced symplectic form on the reduced space \mathcal{M}_ξ^Δ . If we take the Fourier transform in (1.20) we get

$$\begin{aligned} & (\text{DH}_{\mathfrak{c}_+} - \text{DH}_{\mathfrak{c}_-})(a) da \\ &= \frac{(i)^{n+1-\dim T}}{|S_\xi^\Delta|} \left(\int_{\mathcal{M}_\xi^\Delta} e^{-i(\Omega_\xi^\Delta + \langle a', \omega_\xi^\Delta \rangle)} da' \mathcal{F}_{\mathfrak{t}_\Delta}(\delta_\xi^\Delta)(a'') \right) (a - \xi), \\ &= \sum_{Z \in \mathcal{F}} \frac{(i)^{n+1-\dim T}}{|S_\xi^Z|} \left(\int_{Z_\xi} e^{-i(\Omega_\xi^Z + \langle a', \omega_\xi^Z \rangle)} da' \mathcal{F}_{\mathfrak{t}_\Delta}(\delta_\xi^Z)(a'') \right) (a - \xi) \end{aligned} \quad (1.21)$$

where $a = a' + a''$ with $a' \in (\mathfrak{t}/\mathfrak{t}_\Delta)^*$ and $a'' \in (\mathfrak{t}_\Delta)^*$. In (1.21), we write $\int_{\mathcal{M}_\xi^\Delta} = \sum_{Z \in \mathcal{F}} \int_{Z_\xi}$ where the sum is taken over the set \mathcal{F} of connected components Z of M^Δ that intersects $\Phi^{-1}(\xi)$: we take then

$$Z_\xi = \left(Z \cap \Phi^{-1}(\xi) \right) / (T/T_\Delta).$$

The 2-forms $\Omega_\xi^\Delta, \omega_\xi^\Delta$, the generic stabiliser S_ξ^Δ , the vector bundle N_Δ , the generalized function δ_ξ^Δ restrict to each component Z : we denote them respectively $\Omega_\xi^Z, \omega_\xi^Z, S_\xi^Z, N_Z, \delta_\xi^Z$.

We recall now the computation of the Fourier transform of the inverses $\text{Eul}_{\pm\beta}^{-1}(N_Z) := \text{Eul}_{\pm\beta}^{-1}(N_\Delta)|_Z$ that is given in [30][Proposition 4.8.]. We consider a T -invariant scalar product on the fibers of the bundle N_Δ . Let $R \in \mathcal{A}^2(M_o^\Delta, \text{so}(N_\Delta))^{bas}$ be the curvature of a T -invariant and T/T_Δ -horizontal Euclidean connexion on N_Δ : we denote by $R^Z \in \mathcal{A}^2(Z, \text{so}(N_Z))^{bas}$ the restriction of R to a component $Z \in \mathcal{F}$. The curvature commutes with the infinitesimal action \mathcal{L}_X of $X \in \mathfrak{t}_\Delta$, and with the complex structure $J_\beta = \mathcal{L}_\beta(-\mathcal{L}_\beta^2)^{1/2}$ on N_Δ defined by $\beta \in \mathfrak{t}_\Delta$.

We denote by S^\bullet the symmetric algebra of the complex vector bundle (N_Δ, J_β) . We keep the same notation for the restriction of S^\bullet on the submanifolds Z , $\Phi^{-1}(\xi) \cap M^\Delta$, and for the induced orbifold vector bundle on the reduced spaces \mathcal{Z}_ξ and \mathcal{M}_ξ^Δ . For each $k \in \mathbb{N}$, we denote by Tr_{S^k} the trace operator defined on the complex endomorphisms of S^k . For a complex endomorphism A of N_Δ , we denote by $A^{\otimes k}$ the induced endomorphism on S^k . For any $X \in \mathfrak{t}_\Delta$, the complex endomorphism $\mathcal{L}_X^{-1}R^Z$ is symmetric. Hence the trace $\text{Tr}_{S^k}((\mathcal{L}_X^{-1}R^Z)^{\otimes k})$ is a basic *real* differential form of degree $2k$ on Z which does not depend of the choice of complex structures (J_β or $J_{-\beta}$).

Let $\beta^* \in \mathfrak{t}_\Delta^*$ the dual of $\beta \in \mathfrak{t}_\Delta$.

Proposition 1.2.8 ([30]) *For a smooth function f on \mathfrak{t}_Δ^* with compact support we have $\int_{\mathfrak{t}_\Delta^*} \mathcal{F}_{\mathfrak{t}_\Delta}(\text{Eul}_\beta^{-1}(N_Z))(a'')f(a'') = \int_0^\infty P_Z(t)f(t\beta^*)dt$ where P_Z is the polynomial on \mathbb{R} defined by:*

$$P_Z(t) = \frac{(2\pi i)^{r_Z}}{\det_Z^{1/2}(\mathcal{L}_\beta)} \left(\frac{t^{r_Z-1}}{(r_Z-1)!} + \sum_{k=1}^{\dim(Z)/2} (i)^k \text{Tr}_{S^k}((\mathcal{L}_\beta^{-1}R^Z)^{\otimes k}) \frac{t^{r_Z-1+k}}{(r_Z-1+k)!} \right). \quad (1.22)$$

Here $\det_Z^{1/2}(\mathcal{L}_\beta)$ is the Pfaffian of \mathcal{L}_β on N_Z , and $r_Z = \text{rk}_\mathbb{C}(N_Z)$.

One checks then that

$$\begin{aligned} \int_{\mathfrak{t}_\Delta^*} \mathcal{F}_{\mathfrak{t}_\Delta}(\text{Eul}_\beta^{-1}(N_Z))(a'')f(a'') &= \int_0^\infty -P_Z(-t)f(-t\beta^*)dt \\ &= - \int_{-\infty}^0 P_Z(t)f(t\beta^*)dt. \end{aligned}$$

Hence the distribution $\mathcal{F}_{\mathfrak{t}_\Delta}(\delta^Z)$ is equal to $P_Z(\beta)d\beta$. From now one we fix β as the primitive element of $\mathfrak{t}_\Delta \cap \Lambda$ which point out \mathfrak{c}_- . Then $d\beta$ and $d\beta^*$ are then (dual) Lebesgue measure on \mathfrak{t}^* and \mathfrak{t} : we have $\text{vol}(T_\Delta, d\beta^*) = 1$.

Let R_ξ^Z be the restriction of the curvature R^Z to the submanifold $Z \cap \Phi^{-1}(\xi)$. Since R^Z is T/T_Δ -basic, $\text{Tr}_{S^k}((\mathcal{L}_\beta^{-1}R_\xi^Z)^{\otimes k})$ can be seen as a real differential form of degree $2k$ on the orbifold $\mathcal{Z}_\xi = (Z \cap \Phi^{-1}(\xi))/(T/T_\Delta)$.

Each connected component Z of M^Δ is a T/T_Δ Hamiltonian manifold: we take for moment map $\Phi_Z : Z \rightarrow (\mathfrak{t}/\mathfrak{t}_\Delta)^*$ the restriction of $\Phi - \xi$ to Z . Hence 0 is a regular value of Φ_Z . Let $\text{DH}_0(Z)$ be the polynomial function on

$(\mathfrak{t}/\mathfrak{t}_\Delta)^* = \{a \in \mathfrak{t}^* \mid \langle \beta, a \rangle = 0\}$ such that $\text{DH}(Z)(a') = \text{DH}_0(Z)(a')da'$ near 0. Finally (1.21) together with the proposition 1.2.8 give the following

Theorem 1.2.9 *We have $(\text{DH}_{\mathfrak{c}_+} - \text{DH}_{\mathfrak{c}_-})(a) = \sum_{Z \in \mathcal{F}} \mathbf{D}_Z(a - \xi)$, $a \in \mathfrak{t}^*$ where each polynomial $\mathbf{D}_Z \in \mathcal{S}(\mathfrak{t})$ admits the following decomposition*

$$\mathbf{D}_Z = \frac{\beta^{r_Z-1}}{\det_Z^{1/2}(\frac{-\mathcal{L}_\beta}{2\pi})} \left(\frac{\text{DH}_0(Z)}{(r_Z - 1)!} + \sum_{k=1}^{d_Z} \beta^k \mathbf{Q}_{Z,k} \right). \quad (1.23)$$

The polynomials $\mathbf{Q}_{Z,k} \in \mathcal{S}(\mathfrak{t}/\mathfrak{t}_\Delta)$ are defined by

$$\mathbf{Q}_{Z,k}(a') = \frac{(-1)^k}{(r_Z - 1 + k)! |S_\xi^Z|} \int_{Z_\xi} \frac{(\Omega_\xi^Z + \langle a', \omega_\xi^Z \rangle)^{d_Z - k}}{(d_Z - k)!} \text{Tr}_{S^k}((\mathcal{L}_\beta^{-1} R_\xi^Z)^{\otimes k}). \quad (1.24)$$

Here $2d_Z = \dim Z_\xi$ and $2r_Z = \dim M - \dim Z$.

Remark 1.2.10 • *The polynomial $\text{DH}_{\mathfrak{c}_+} - \text{DH}_{\mathfrak{c}_-}$ is divisible by the factor $a \mapsto \langle a - \xi, \beta \rangle^{r-1}$ with $r = \inf_{Z \in \mathcal{F}} r_Z$. If $\Delta \cap \Phi(M)$ is not a facet of the polytope $\Phi(M)$ we have $r_Z \geq 2$ for all connected component $Z \in \mathcal{F}$, hence $r - 1 \geq 1$.*

• *Suppose now that \mathfrak{c}_- is a connected component of regular values of Φ bordering a facet $\Phi(M) \cap \Delta$ of the polytope $\Phi(M)$. Here $Z = \Phi^{-1}(\Delta)$ is a connected component of the fixed point set M^{T_Δ} . In this situation we have $\text{DH}_{\mathfrak{c}_-} = -\mathbf{D}_Z$ where the polynomial \mathbf{D}_Z is defined by (1.23).*

1.3 Quantum version of Duistermaat-Heckman measures

We suppose here that the Hamiltonian T -manifold (M, ω, Φ) is prequantized by a T -equivariant Hermitian line bundle L over M , which is equipped with an Hermitian connection ∇ satisfying the Kostant formula

$$\mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi, X \rangle, \quad X \in \mathfrak{t}. \quad (1.25)$$

The former equation implies that the first Chern class of L is equal to $\frac{\Omega}{2\pi}$. In this section we suppose that M is compact and we still assume that the generic stabiliser Γ_M of T on M is finite. The quantization of (M, Ω) is defined by the Riemann-Roch character $RR(M, L) \in R(T)$ which is compute with a T -equivariant almost complex structure on M compatible with Ω [32]. For $k \geq 1$, we consider the tensor product $L^{\otimes k}$. Its Riemann-Roch character $RR(M, L^{\otimes k})$ decomposes as

$$RR(M, L^{\otimes k}) = \sum_{\mu \in \Lambda^*} m(\mu, k) \mathbb{C}_\mu. \quad (1.26)$$

Let us recall the well-known properties of the map $m : \Lambda^* \times \mathbb{Z}^{>0} \rightarrow \mathbb{Z}$. When $\frac{\mu}{k}$ is a regular value of Φ , the "Quantization commutes with Reduction Theorem" [28, 29] tell us that

$$m(\mu, k) = RR(\mathcal{M}_{\frac{\mu}{k}}, \mathcal{L}^{\mu, k}) \quad (1.27)$$

where $\mathcal{L}^{\mu, k} = (L^{\otimes k}|_{\Phi^{-1}(\frac{\mu}{k})} \otimes \mathbb{C}_{-\mu})/T$ is an orbifold line bundle over the symplectic orbifold $\mathcal{M}_{\frac{\mu}{k}} = \Phi^{-1}(\frac{\mu}{k})/T$. In particular if $\frac{\mu}{k}$ does not belong to $\Phi(M)$ we have $m(\mu, k) = 0$. When $\frac{\mu}{k} \in \Phi(M)$ is not necessarily a regular value of Φ , one proceed by shift desingularization. If $\xi \in \Phi(M)$ is a regular value of Φ close enough to $\frac{\mu}{k}$ then (1.27) becomes

$$m(\mu, k) = RR(\mathcal{M}_{\xi}, \mathcal{L}_{\xi}^{\mu, k}) \quad (1.28)$$

where $\mathcal{L}_{\xi}^{\mu, k} = (L^{\otimes k}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu})/T$ (for a proof see [29, 32]).

Definition 1.3.1 A function $f : \Xi \rightarrow \mathbb{Z}$ defined over a lattice $\Xi \simeq \mathbb{Z}^r$ is called periodic polynomial if

$$f(x) = \sum_{i=1}^p e^{i \frac{\langle \alpha_i, x \rangle}{N}} P_i(x), \quad x \in \Xi,$$

where $\alpha_1, \dots, \alpha_p \in \Xi^*$, $N \geq 1$, and the functions P_1, \dots, P_p are polynomials with complex coefficients.

Remark 1.3.2 Let \mathcal{C} a cone with non-empty interior in the real vector space $\Xi \otimes_{\mathbb{Z}} \mathbb{R}$. Any periodic-polynomial function $f : \Xi \rightarrow \mathbb{Z}$ is completely determined by its restriction on $\mathcal{C} \cap \Xi$.

Let $\mathfrak{c} \subset \mathfrak{t}^*$ be a connected component of regular values of Φ . In [29] Meinrenken and Sjamaar proved that there exists a periodic polynomial function $m_{\mathfrak{c}} : \Lambda^* \times \mathbb{Z} \rightarrow \mathbb{Z}$ such that $m_{\mathfrak{c}}(\mu, k) = m(\mu, k)$ for every (μ, k) in the cone

$$\text{Cone}(\mathfrak{c}) = \{(\xi, s) \in \mathfrak{t}^* \times \mathbb{R}^{>0} \mid \xi \in s \cdot \mathfrak{c}\}. \quad (1.29)$$

Consider now two adjacent connected regions \mathfrak{c}_{\pm} of regular values of Φ separated by an hyperplane $\Delta \subset \mathfrak{t}^*$. When Δ does not contain a facet of the polytope $\Phi(M)$, Meinrenken and Sjamaar proved also that

$$m_{\mathfrak{c}_+}(\mu, k) = m_{\mathfrak{c}_-}(\mu, k) = m(\mu, k) \quad (1.30)$$

for every $(\mu, k) \in \overline{\text{Cone}(\mathfrak{c}_+)} \cap \overline{\text{Cone}(\mathfrak{c}_-)} = \text{Cone}(\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-}) \subset \text{Cone}(\Delta)$.

The main objective of this section is to prove that (1.30) extends to a "strip" containing $\text{Cone}(\Delta)$.

Let $\beta \in \Lambda$ be the primitive orthogonal vector to the hyperplane $\Delta \subset \mathfrak{t}^*$ which is pointing out of \mathfrak{c}_- . Then $\Delta = \{\xi \in \mathfrak{t}^* \mid \frac{\langle \xi, \beta \rangle}{2\pi} = r_{\Delta}\}$ for some $r_{\Delta} \in \mathbb{Z}$, $\text{Cone}(\Delta) = \{(\xi, s) \in \mathfrak{t}^* \times \mathbb{R}^{\geq 0} \mid \frac{\langle \xi, \beta \rangle}{2\pi} - sr_{\Delta} = 0\}$ and $\mathfrak{c}_- \subset \{\xi \in \mathfrak{t}^* \mid \frac{\langle \xi, \beta \rangle}{2\pi} < r_{\Delta}\}$.

Let T_Δ be the subtorus of T generated by β . Let N_Δ be the normal vector bundle of M^{T_Δ} in M . The almost complex structure on M induces a complex structure J on the fibers of N_Δ . We have a decomposition $N_\Delta = \sum_s N_\Delta^s$ where $N_\Delta^s = \{v \in N_\Delta \mid \mathcal{L}_\beta v = s Jv\}$. We write $N_\Delta = N_\Delta^{+, \beta} \oplus N_\Delta^{-, \beta}$ where

$$N_\Delta^{\pm, \beta} = \sum_{\pm s > 0} N_\Delta^s. \tag{1.31}$$

Definition 1.3.3 For every connected component $Z \subset M^{T_\Delta}$ we define $s_Z^\pm \in \mathbb{N}$ respectively as the absolute value of the trace of $\frac{1}{2\pi} \mathcal{L}_\beta$ on $N_\Delta^{\pm, \beta}|_Z$.

Note that $s_Z^+ + s_Z^-$ is larger than half of the codimension of Z in M . We prove in Section 1.3.5 the following

Theorem 1.3.4 We have $m_{\mathfrak{c}_+}(\mu, k) = m_{\mathfrak{c}_-}(\mu, k)$ for all $(\mu, k) \in \Lambda^* \times \mathbb{Z}$ such that

$$-s^- < \frac{\langle \mu, \beta \rangle}{2\pi} - k r_\Delta < s^+ . \tag{1.32}$$

The number $s^-, s^+ \in \mathbb{N}$ are defined as follows. We take $s^\pm = \inf_Z s_Z^\pm$ where the minimum is taken over the connected components Z of M^{T_Δ} for which $\bar{\mathfrak{c}}_+ \cap \bar{\mathfrak{c}}_- \subset \Phi(Z)$.

Similar results were obtained by Billey-Guillemin-Rassart [10] in the case where M is a coadjoint orbit of $SU(n)$, and by Szenes-Vergne [36] in the case where M is a complex vector space. See Sections 1.4.4 and 1.5 where we study these two particular cases in details. In Proposition 1.3.25, we give also a criterium which says when the inequalities in (1.32) are optimal. This criterium is fulfilled when there is *only one* component Z of M^{T_Δ} such that $\bar{\mathfrak{c}}_+ \cap \bar{\mathfrak{c}}_- \subset \Phi(Z)$. Then (1.32) is *optimal* and $s^+ + s^-$ is larger than half of the codimension of Z in M .

The following easy Lemma (see Lemma 7.3. of [32]) gives some basic informations about the integer s_Z^\pm .

Lemma 1.3.5 Let (M, Ω, Φ) be a compact Hamiltonian T -manifold equipped with a T -invariant almost complex structure compatible with Ω . Consider a non-zero vector $\gamma \in \mathfrak{t}$ and let Z be a connected component of the fixed point set M^γ . Let N be the normal vector of Z in M and let $N^{-, \gamma}$ be the negative polarized normal bundle (see (1.31)). Then $N^{-, \gamma} = 0$ if and only if the function $\langle \Phi, \gamma \rangle : M \rightarrow \mathbb{R}$ takes its maximal value on Z .

This Lemma insures that $s^\pm \geq 1$ in Theorem 1.3.4 when $\Delta \cap \Phi(M)$ is not a facet of the polytope $\Phi(M)$.

Consider the situation where $\Delta \cap \Phi(M)$ is a facet of the polytope $\Phi(M)$ so that $\mathfrak{c}_+ \cap \Phi(M) = \emptyset$: hence $m_{\mathfrak{c}_+} = 0$. If we apply Lemma 1.3.5 with $\gamma = \beta$, one gets $N^{-, \beta} = 0$ and so $s^- = 0$. In this situation we get

Corollary 1.3.6 *Let \mathfrak{c}_- be a connected component of regular values of Φ bordering a facet $\Phi(M) \cap \Delta$ of the polytope $\Phi(M)$. Let $\beta \in \Lambda$ be the primitive orthogonal vector to the hyperplane $\Delta \subset \mathfrak{t}^*$ which is pointing out of \mathfrak{c}_- . Here $Z = \Phi^{-1}(\Delta)$ is a connected component of the fixed point set M^{T^Δ} . We have $m_{\mathfrak{c}_-}(\mu, k) = 0$ for all $(\mu, k) \in \Lambda^* \times \mathbb{Z}$ such that*

$$0 < \frac{\langle \mu, \beta \rangle}{2\pi} - kr_\Delta < s_Z^+. \quad (1.33)$$

Here $s_Z^+ \in \mathbb{N}$ is larger than half of the codimension of Z in M , and the inequalities (1.33) are optimal.

The rest of this section is dedicated to the proof of Theorem 1.3.4. We start by reviewing some of the results of [32].

1.3.1 Elliptic and transversally elliptic symbols

We work in the setting of a compact manifold M equipped with a smooth action of a torus T .

Let $p : \mathbf{T}M \rightarrow M$ be the projection, and let $(\cdot, \cdot)_M$ be a T -invariant Riemannian metric. If E^0, E^1 are T -equivariant vector bundles over M , a T -equivariant morphism $\sigma \in \Gamma(\mathbf{T}M, \text{hom}(p^*E^0, p^*E^1))$ is called a *symbol*. The subset of all $(m, v) \in \mathbf{T}M$ where $\sigma(m, v) : E_m^0 \rightarrow E_m^1$ is not invertible is called the *characteristic set* of σ , and is denoted by $\text{Char}(\sigma)$.

Let $\mathbf{T}_T M$ be the following subset of $\mathbf{T}M$:

$$\mathbf{T}_T M = \{(m, v) \in \mathbf{T}M, (v, X_M(m))_M = 0 \text{ for all } X \in \mathfrak{k}\}.$$

A symbol σ is *elliptic* if σ is invertible outside a compact subset of $\mathbf{T}M$ ($\text{Char}(\sigma)$ is compact), and is *transversally elliptic* if the restriction of σ to $\mathbf{T}_T M$ is invertible outside a compact subset of $\mathbf{T}_T M$ ($\text{Char}(\sigma) \cap \mathbf{T}_T M$ is compact). An elliptic symbol σ defines an element in the equivariant K -theory of $\mathbf{T}M$ with compact support, which is denoted by $\mathbf{K}_T(\mathbf{T}M)$, and the index of σ is a virtual finite dimensional representation of T [3, 4, 5, 6].

A *transversally elliptic* symbol σ defines an element of $\mathbf{K}_T(\mathbf{T}_T M)$, and the index of σ is defined as a trace class virtual representation of T (see [1] for the analytic index and [8, 9] for the cohomological one). Remark that any elliptic symbol of $\mathbf{T}M$ is transversally elliptic, hence we have a restriction map $\mathbf{K}_T(\mathbf{T}M) \rightarrow \mathbf{K}_T(\mathbf{T}_T M)$, and a commutative diagram

$$\begin{array}{ccc} \mathbf{K}_T(\mathbf{T}M) & \longrightarrow & \mathbf{K}_T(\mathbf{T}_T M) \\ \text{Index}_M^T \downarrow & & \downarrow \text{Index}_M^T \\ R(T) & \longrightarrow & R^{-\infty}(T) . \end{array} \quad (1.34)$$

Using the *excision property*, one can easily show that the index map $\text{Index}_U^T : \mathbf{K}_T(\mathbf{T}_T U) \rightarrow R^{-\infty}(T)$ is still defined when U is a T -invariant relatively compact open subset of a T -manifold (see [32][section 3.1]).

1.3.2 Localization of the Riemann-Roch character

We suppose now that the compact T -manifold M is equipped with a T -invariant almost complex structure J . Let us recall the definitions of the Thom symbol $\text{Thom}(M, J)$ and of the Riemann-Roch character [32].

Consider a T -invariant Riemannian metric q on M such that J is orthogonal relatively to q , and let h be the Hermitian structure on $\mathbf{T}M$ defined by : $h(v, w) = q(v, w) - \iota q(Jv, w)$ for $v, w \in \mathbf{T}M$. The symbol

$$\text{Thom}(M, J) \in \Gamma(M, \text{hom}(p^*(\wedge_{\mathbb{C}}^{\text{even}} \mathbf{T}M), p^*(\wedge_{\mathbb{C}}^{\text{odd}} \mathbf{T}M)))$$

at $(m, v) \in \mathbf{T}M$ is equal to the Clifford map

$$\text{Cl}_m(v) : \wedge_{\mathbb{C}}^{\text{even}} \mathbf{T}_m M \longrightarrow \wedge_{\mathbb{C}}^{\text{odd}} \mathbf{T}_m M, \quad (1.35)$$

where $\text{Cl}_m(v).w = v \wedge w - c_h(v).w$ for $w \in \wedge_{\mathbb{C}}^{\bullet} \mathbf{T}_m M$. Here $c_h(v) : \wedge_{\mathbb{C}}^{\bullet} \mathbf{T}_m M \rightarrow \wedge^{\bullet-1} \mathbf{T}_m M$ denotes the contraction map relative to h . Since the map $\text{Cl}_m(v)$ is invertible for all $v \neq 0$, the symbol $\text{Thom}(M, J)$ is elliptic.

The Riemann-Roch character $RR(M, -) : \mathbf{K}_T(M) \rightarrow R(T)$ is defined by the following relation

$$RR(M, E) = \text{Index}_M^T(\text{Thom}(M, J) \otimes p^*E) . \quad (1.36)$$

The important point is that for any T -vector bundle E , $\text{Thom}(M, J) \otimes p^*E$ corresponds to the *principal symbol* of the twisted Spin^c Dirac operator \mathcal{D}_E [16], hence $RR(M, E) \in R(T)$ is also defined as the (analytical) index of the elliptic operator \mathcal{D}_E .

Consider now the case of a *compact* Hamiltonian T -manifold (M, ω, Φ) . Here J is a T -invariant almost complex structure compatible with $\Omega : (v, w) \mapsto \Omega(v, Jw)$ defines a Riemannian metric on M . Like in Section 1.2.2, we make the choice of a scalar product (\cdot, \cdot) on \mathfrak{t}^* (which induces an identification $\mathfrak{t}^* \simeq \mathfrak{t}$) and we consider for any $\xi \in \mathfrak{t}^*$ the function $\frac{1}{2} \|\Phi - \xi\|^2 : M \rightarrow \mathbb{R}$ and its Hamiltonian vector field $\mathcal{H} - \xi_M$.

Definition 1.3.7 For any $\xi \in \mathfrak{t}^*$ and any T -invariant open subset $\mathcal{U} \subset M$ we define the symbol $\text{Thom}_{\xi}(\mathcal{U})$ by the relation

$$\text{Thom}_{\xi}(\mathcal{U})(m, v) := \text{Thom}(M, J)(m, v - (\mathcal{H} - \xi_M)(m)) \quad (m, v) \in \mathbf{T}\mathcal{U}$$

The characteristic set of $\text{Thom}_{\xi}(\mathcal{U})$ corresponds to $\{(m, v) \in \mathbf{T}\mathcal{U}, v = (\mathcal{H} - \xi_M)(m)\}$, the graph of the vector field $\mathcal{H} - \xi_M$ over \mathcal{U} . Since $\mathcal{H} - \xi_M$ belongs to the set of tangent vectors to the T -orbits, we have

$$\begin{aligned} \text{Char}(\text{Thom}_{\xi}(\mathcal{U})) \cap \mathbf{T}_T \mathcal{U} &= \{(m, 0) \in \mathbf{T}\mathcal{U} \mid (\mathcal{H} - \xi_M)(m) = 0\} \\ &\cong \{m \in \mathcal{U}, d \|\Phi - \xi\|_m^2 = 0\} . \end{aligned}$$

Therefore the symbol $\text{Thom}_{\xi}(\mathcal{U})$ is transversally elliptic if and only if

$$\text{Cr}(\|\Phi - \xi\|^2) \cup \partial \mathcal{U} = \emptyset. \quad (1.37)$$

Definition 1.3.8 When (1.37) holds we say that the couple (\mathcal{U}, ξ) is good.

Definition 1.3.9 Let (\mathcal{U}, ξ) be a good couple. For any T -vector bundle $E \rightarrow M$, the tensor product $\text{Thom}_\xi(\mathcal{U}) \otimes p^*E$ belongs to $\mathbf{K}_T(\mathbf{T}_T\mathcal{U})$ and we denote by

$$RR_{\mathcal{U}}^\xi(M, E) \in R^{-\infty}(T)$$

its index.

Proposition 1.3.10 Let (\mathcal{U}, ξ) be a good couple.

• If \mathcal{U} possess two T -invariant open subsets $\mathcal{U}^1, \mathcal{U}^2$ such that $\overline{\mathcal{U}^1} \cap \overline{\mathcal{U}^2} \cap \text{Cr}(\|\Phi - \xi\|^2) = \emptyset$ and $(\mathcal{U}^1 \cup \mathcal{U}^2) \cap \text{Cr}(\|\Phi - \xi\|^2) = \mathcal{U} \cap \text{Cr}(\|\Phi - \xi\|^2)$, then the couples (\mathcal{U}^1, ξ) and (\mathcal{U}^2, ξ) are good and

$$RR_{\mathcal{U}}^\xi(M, -) = RR_{\mathcal{U}^1}^\xi(M, -) + RR_{\mathcal{U}^2}^\xi(M, -).$$

In particular $RR_{\mathcal{U}}^\xi(M, -) = RR_{\mathcal{U}^1}^\xi(M, -)$ if \mathcal{U}^1 is an open subset of \mathcal{U} such that $\mathcal{U}^1 \cap \text{Cr}(\|\Phi - \xi\|^2) = \mathcal{U} \cap \text{Cr}(\|\Phi - \xi\|^2)$.

• If $\xi' \in \mathfrak{t}^*$ is close enough to ξ , then (\mathcal{U}, ξ') is good and

$$RR_{\mathcal{U}}^\xi(M, -) = RR_{\mathcal{U}}^{\xi'}(M, -).$$

Proof. The first point is a direct consequence of the excision property (see Proposition 4.1. in [32]).

Let us prove the second point. Consider now the scalar product

$$\phi(s) := (\mathcal{H} - \xi_M^s, \mathcal{H} - \xi_M)_M$$

where $\xi^s = s\xi' + (1-s)\xi$, $s \in [0, 1]$: $\phi(s)$ is a smooth function on M . We have $\phi(s) = \|\mathcal{H} - \xi_M\|^2 + s((\xi - \xi')_M, \mathcal{H} - \xi_M)$ and then the following inequality holds on M

$$\phi(s) \geq \|\mathcal{H} - \xi_M\|^2 \left(\|\mathcal{H} - \xi_M\| - s\|\xi_M - \xi'_M\| \right). \quad (1.38)$$

Since $\partial\mathcal{U}$ is compact we have the following inequalities on it: $\|\mathcal{H} - \xi_M\| \geq c_1 > 0$ and $\|X_M\| \leq c_2\|X\|$ for any $a \in \mathfrak{t}$. So (1.38) implies the following inequality on $\partial\mathcal{U}$:

$$\phi(s) \geq c_1(c_1 - s\|\xi - \xi'\|) \quad \text{for } s \in [0, 1].$$

So if ξ' is close enough to ξ , we have $\|\mathcal{H} - \xi_M^s\| \geq c_3 > 0$ on $\partial\mathcal{U}$ for any $s \in [0, 1]$. We have first prove that the couple (\mathcal{U}, ξ^s) is good for any $s \in [0, 1]$. We see then that the family of transversally elliptic symbols $\text{Thom}_{\xi^s}(\mathcal{U})$, $s \in [0, 1]$ defines an homotopy between $\text{Thom}_\xi(\mathcal{U})$ and $\text{Thom}_{\xi'}(\mathcal{U})$. Hence $\text{Thom}_\xi(\mathcal{U}) = \text{Thom}_{\xi'}(\mathcal{U})$ in $\mathbf{K}_T(\mathbf{T}_T\mathcal{U})$. \square

The first point of Proposition 1.3.10 shows that $RR_{\mathcal{U}}^\xi(M, -)$ depends closely of the intersection $\mathcal{U} \cap \text{Cr}(\|\Phi - \xi\|^2)$. In particular $RR_{\mathcal{U}}^\xi(M, -) = 0$ when $\mathcal{U} \cap \text{Cr}(\|\Phi - \xi\|^2) = \emptyset$. Recall that

$$\text{Cr}(\|\Phi - \xi\|^2) = \bigcup_{\gamma \in \mathcal{B}_\xi} M^\gamma \cap \Phi^{-1}(\gamma + \xi) \quad (1.39)$$

where $\mathcal{B}_\xi \subset \mathfrak{t}^*$ is a finite set [24].

Definition 1.3.11 For any $\xi \in \mathfrak{t}^*$ and $\gamma \in \mathcal{B}_\xi$, we denote simply by

$$RR_\gamma^\xi(M, -) : \mathbf{K}_T(M) \rightarrow R^{-\infty}(T)$$

the map $RR_{\mathcal{U}}^\xi(M, -)$, where \mathcal{U} is a T -invariant open neighborhood of $M^\gamma \cap \Phi^{-1}(\gamma + \xi)$ such that $\text{Cr}(\|\Phi - \xi\|^2) \cap \overline{\mathcal{U}} = M^\gamma \cap \Phi^{-1}(\gamma + \xi)$.

Proposition 1.3.10 insures that the maps $RR_\gamma^\xi(M, -)$ are well defined, and for any good couple (\mathcal{U}, ξ) we have

$$RR_{\mathcal{U}}^\xi(M, -) = \sum_{\gamma \in \mathcal{B}_\xi \cap \Phi(\mathcal{U})} RR_\gamma^\xi(M, -). \quad (1.40)$$

If one takes $\mathcal{U} = M$, we have $RR_{\mathcal{U}}^\xi(M, -) = RR(M, -) = \sum_{\gamma \in \mathcal{B}_\xi} RR_\gamma^\xi(M, -)$ (see [32][Section 4]).

1.3.3 Periodic polynomial behaviour of the multiplicities

We suppose here that the Hamiltonian T -manifold (M, Ω, Φ) is prequantized by a T -complex line bundle L satisfying (1.25) for a suitable invariant connection. In this section we will characterize the periodic polynomial behaviour of the multiplicities $m(\mu, k)$ with the help of the localized Riemann-Roch character $RR_0^\xi(M, -)$.

Let us introduce some vocabulary. We say that two generalized characters $\chi^\pm = \sum_{\mu \in \Lambda^*} a_\mu^\pm \mathbb{C}_\mu$ coincide on a region $D \subset \mathfrak{t}^*$, if $a_\mu^+ = a_\mu^-$ for every $\mu \in D \cap \Lambda^*$. A generalized character $\chi = \sum_{\mu} a_\mu \mathbb{C}_\mu$ is supported on a region $D \subset \mathfrak{t}^*$ if $a_\mu = 0$ for $\mu \notin D$. A weight $\mu \in \Lambda^*$ occurs in $\chi = \sum_{\mu} a_\mu \mathbb{C}_\mu$ if $a_\mu \neq 0$.

For $\xi \in \mathfrak{t}^*$, we define $r_\xi > 0$ as the smallest non-zero critical value of the function $\|\Phi - \xi\|$, and we denote by $B(\xi, r_\xi)$ the open ball of center ξ and radius r_ξ .

Theorem 1.3.12 ([32]) For any $\xi \in \mathfrak{t}^*$, the generalized character $RR_0^\xi(M, L^{\otimes k})$ coincides with $RR(M, L^{\otimes k})$ on the open ball $k \cdot B(\xi, r_\xi)$.

The arguments of [32] for the proof of this Theorem will be needed another time, so we recall them. Let $\xi \in \mathfrak{t}^*$. We start with the decomposition

$$RR(M, L^{\otimes k}) = \sum_{\gamma \in \mathcal{B}_\xi} RR_\gamma^\xi(M, L^{\otimes k}). \quad (1.41)$$

We recall now, for a non-zero $\gamma \in \mathcal{B}_\xi$, the localization of the map RR_γ^ξ on the fixed point set M^γ [32].

Let N be the normal bundle of M^γ in M . The almost complex structure on M induces an almost complex structure on M^γ and a complex structure on the bundles N and $N_{\mathbb{C}} := N \otimes \mathbb{C}$. Following (1.31) we define the γ -polarized complex vector bundles $N^{+\gamma}$ and $(N_{\mathbb{C}})^{+\gamma}$.

The manifold M^γ is a symplectic submanifold of M equipped with an induced Hamiltonian action of T : its moment map is the restriction of Φ on M^γ . Following Definition 1.3.11, we have on M^γ a localized Riemann-Roch character $RR_\gamma^\xi(M^\gamma, -)$. On M^γ , the Hamiltonian vector fields of the functions $\|\Phi - \xi\|^2$ and $\|\Phi - (\xi + \gamma)\|^2$ coincide, hence

$$RR_\gamma^\xi(M^\gamma, -) = RR_0^{\xi+\gamma}(M^\gamma, -). \quad (1.42)$$

We prove in [32][Theorem 5.8.] that

$$RR_\gamma^\xi(M, E) = \sum_{k \in \mathbb{N}} (-1)^l RR_\gamma^\xi(M^\gamma, E|_{M^\gamma} \otimes \det(N^{+\gamma}) \otimes S^k(N_{\mathbb{C}}^{+\gamma})) \quad (1.43)$$

for every T -vector bundle E . Here l is the locally constant function on M^γ equal to the complex rank of $N^{+\gamma}$.

Proposition 1.3.13 ([32], Section 5) *Let \overline{N} be the T -vector bundle N with the opposite complex structure on the fibers. The sum $(-1)^l \sum_{k \in \mathbb{N}} \det(N^{+\gamma}) \otimes S^k(N_{\mathbb{C}}^{+\gamma})$ is an inverse of $\wedge_{\mathbb{C}}^\bullet \overline{N}$ that we denote $[\wedge_{\mathbb{C}}^\bullet \overline{N}]_\gamma^{-1}$.*

If we use the notations of Proposition 1.3.13 and (1.42), the localization (1.43) can be rewritten as

$$RR_\gamma^\xi(M, E) = RR_0^{\xi+\gamma} \left(M^\gamma, E|_{M^\gamma} \otimes [\wedge_{\mathbb{C}}^\bullet \overline{N}]_\gamma^{-1} \right). \quad (1.44)$$

Let $i : T_\gamma \hookrightarrow T$ be the inclusion of the subtorus generated by γ . Let F be a T -vector bundle on M^γ .

Lemma 1.3.14 ([32], Lemma 9.4.) *A weight $\mu \in \Lambda^*$ occurs in $RR_\gamma^\xi(M^\gamma, F)$ only if $i^*(\mu)$ occurs as a weight for the T_γ -action on the fibers of $F \otimes [\wedge_{\mathbb{C}}^\bullet \overline{N}]_\gamma^{-1}$.*

Since the T_γ weights on the bundles $N_{\mathbb{C}}^{+\gamma}$ and $N^{+\gamma}$ are polarized by γ , the localization (1.43) gives the following

Corollary 1.3.15 *For a non-zero $\gamma \in \mathcal{B}_\xi$, the generalized character $RR_\gamma^\xi(M, L^{\otimes k})$ is supported on the half space $\{a \in \mathfrak{t}^* \mid (\gamma, a - k(\xi + \gamma)) \geq 0\}$.*

Since the condition $(\gamma, a - k(\xi + \gamma)) \geq 0$ implies that $\|a - k\xi\| \geq k \|\gamma\| \geq kr_\xi$, the last proposition shows that every weights of the open ball $k \cdot B(\xi, r_\xi)$ does not occurs in $RR_\gamma^\xi(M, L^{\otimes k})$. This last remark together with (1.41) prove Theorem 1.3.12.

For the localized Riemann-Roch character $RR_0^\xi(M, -)$ we have the following Lemma which is very similar to Lemma 1.2.3.

Lemma 1.3.16 *Let $\mathfrak{c} \subset \mathfrak{t}^*$ be a connected component of regular values of Φ . For every $\xi, \xi' \in \mathfrak{c}$, we have $RR_0^\xi(M, -) = RR_0^{\xi'}(M, -)$.*

Proof. We have to show that the map $\xi \mapsto RR_0^\xi(M, -)$ is locally constant on \mathfrak{c} . Let $\xi \in \mathfrak{c}$ and take an open neighborhood \mathcal{U} of $\Phi^{-1}(\xi)$ small enough such that the stabilizer $T_m = \{t \in T \mid t \cdot m = m\}$ is finite for every $m \in \bar{\mathcal{U}}$. We see then that $\bar{\mathcal{U}} \cap \text{Cr}(\|\Phi - \xi'\|^2) = \Phi^{-1}(\xi')$ and $\partial\mathcal{U} \cap \text{Cr}(\|\Phi - \xi'\|^2) = \emptyset$ if ξ' is close enough to ξ : hence $RR_0^{\xi'}(M, -) = RR_{\mathcal{U}}^{\xi'}(M, -)$ for ξ' close enough to ξ . The second point of Proposition 1.3.10 finishes the proof. \square

When ξ is a regular value of Φ , the localized Riemann-Roch character $RR_0^\xi(M, -)$ as been computed in [32] as follows. Let $RR(\mathcal{M}_\xi, -)$ be the Riemann-Roch map defined on the orbifold $\mathcal{M}_\xi = \Phi^{-1}(\xi)/T$ by means of an almost complex structure compatible with the induced symplectic structure. For every T -vector bundle $E \rightarrow M$ we define the following family of orbifold vector bundles over \mathcal{M}_ξ :

$$\mathcal{E}_\xi^\mu := \left(E|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu} \right) / T, \quad \mu \in \Lambda^*. \quad (1.45)$$

For every T -vector bundle E on M , we proved in [32][Section 6.2.] the following equality in $R^{-\infty}(T)$

$$RR_0^\xi(M, E) = \sum_{\mu \in \Lambda^*} RR(\mathcal{M}_\xi, \mathcal{E}_\xi^\mu) \mathbb{C}_\mu. \quad (1.46)$$

This decomposition was first obtained by Vergne [37] when T is the circle group and when M is Spin. The number $RR(\mathcal{M}_\xi, \mathcal{E}_\xi^\mu) \in \mathbb{Z}$ is then equal to the T -invariant part of the index $RR_0^\xi(M, E) \otimes \mathbb{C}_{-\mu}$.

Remark 1.3.17 *Let $t \rightarrow t^\lambda$ be a character of T . Suppose that a subgroup $H \subset T$ acts trivially on M and with the character $t \in H \rightarrow t^\lambda$ on the fibers of the T -vector bundle E . Then H acts with the character $t \in H \rightarrow t^{\lambda-\mu}$ on $RR_0^\xi(M, E) \otimes \mathbb{C}_{-\mu}$, and then $RR(\mathcal{M}_\xi, \mathcal{E}_\xi^\mu) \neq 0$ only if $t^{\lambda-\mu} = 1$ for every $t \in H$. So the sum in (1.46) can be restricted to $\lambda + \Lambda_H^*$, where Λ_H^* is the sub-lattice of Λ^* formed by the element $\alpha \in \Lambda^*$ satisfying $t^\alpha = 1, \forall t \in H$.*

This remark applies also on the usual character $RR(M, E) = \sum_{\mu \in \Lambda^} m_\mu \mathbb{C}_\mu$. The multiplicity $m_\mu \in \mathbb{Z}$ is equal to the (virtual) dimension of the T -invariant part of $RR(M, E) \otimes \mathbb{C}_{-\mu}$. With the same hypothesis than above we see that $m_\mu \neq 0$ only if $\mu \in \lambda + \Lambda_H^*$.*

Let Γ_M be the generic stabilizer for the action of T on M . Consider a weight α_o such that Γ_M acts on the fibers of L with the character $t \mapsto t^{\alpha_o}$. We define the sublattice $\Xi(M, L) \subset \Lambda^* \times \mathbb{Z}$ by

$$\Xi(M, L) := \{(\mu, k) \in \Lambda^* \times \mathbb{Z} \mid k\alpha_o - \mu \in \Lambda_{\Gamma_M}^*\}. \quad (1.47)$$

We know then that $m(\mu, k) = 0$ if $(\mu, k) \notin \Xi(M, L)$.

Proposition 1.3.18 *Let \mathfrak{c} be a connected component of regular values of Φ and let $\text{Cone}(\mathfrak{c})$ be the corresponding cone in $\mathfrak{t}^* \times \mathbb{R}^{>0}$ (see (1.29)). Let $\xi \in \mathfrak{c}$. For any $(\mu, k) \in \text{Cone}(\mathfrak{c}) \cap \Xi(M, L)$ we have*

$$m(\mu, k) = RR(\mathcal{M}_\xi, \mathcal{L}_\xi^{\mu, k}) \quad (1.48)$$

where

$$\mathcal{L}_\xi^{\mu, k} = (L^{\otimes k}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu})/T. \quad (1.49)$$

Proof. Let $(\mu, k) \in \text{Cone}(\mathfrak{c})$ and let $\xi' = \frac{\mu}{k} \in \mathfrak{c}$. We know from Theorem 1.3.12 that the generalized character $RR_0^{\xi'}(M, L^{\otimes k})$ coincides with $RR(M, L^{\otimes k})$ on the open ball $k \cdot B(\xi', r_{\xi'}) = B(\mu, kr_{\xi'})$. So $m(\mu, k)$ is equal to the μ -multiplicity in $RR_0^{\xi'}(M, L^{\otimes k})$. Take now any $\xi \in \mathfrak{c}$. We know after Lemma 1.3.16 that $RR_0^\xi(M, -) = RR_0^{\xi'}(M, -)$ and (1.46) shows that the μ -multiplicity in $RR_0^\xi(M, L^{\otimes k})$ is equal to $RR(\mathcal{M}_\xi, \mathcal{L}_\xi^{\mu, k})$. \square

Definition 1.3.19 *Take $\xi \in \mathfrak{c}$. The map $m_\mathfrak{c} : \Lambda^* \times \mathbb{Z} \rightarrow \mathbb{Z}$ is defined by the equation*

$$m_\mathfrak{c}(\mu, k) = RR(\mathcal{M}_\xi, \mathcal{L}_\xi^{\mu, k}), \quad (1.50)$$

where $\mathcal{L}_\xi^{\mu, k}$ is the orbifold line bundle defined by (1.49).

In other words, the map $m_\mathfrak{c}$ is defined by the following equality in $R^{-\infty}(T)$

$$\sum_{\mu \in \Lambda^*} m_\mathfrak{c}(\mu, k) \mathbb{C}_\mu = RR_0^\xi(M, L^{\otimes k}).$$

for all $k \in \mathbb{Z}$. After remark 1.3.17, we know that $m_\mathfrak{c}$ is supported on the sub-lattice $\Xi(M, L)$ defined in (1.47).

We will now exploit the Riemann-Roch for orbifold due to Kawasaki [23] to show that the map $m_\mathfrak{c}$ is a periodic polynomial.

1.3.4 Riemann-Roch-Kawasaki theorem

First we recall how is defined the Riemann-Roch character $RR(\mathcal{M}_\xi, \mathcal{E}_\xi)$ when ξ is a regular value of Φ , and $\mathcal{E}_\xi = E|_{\Phi^{-1}(\xi)}/T$ is the reduction of a complex T -vector bundle E over M . The number $RR(\mathcal{M}_\xi, \mathcal{E}_\xi) \in \mathbb{Z}$ is defined as the T -invariant part of the index of a transversally elliptic operator D_E on $\Phi^{-1}(\xi)$. Since the index of D_E depends only of the class of its symbol $\sigma(D_E)$ in $K_T(\mathbf{T}_T\Phi^{-1}(\xi))$, it is enough to define the transversally elliptic symbol $\sigma(D_E)$. Since the action of T on $\Phi^{-1}(\xi)$ is locally free, $V := \mathbf{T}_T\Phi^{-1}(\xi)$ is a vector bundle. It carries a canonical symplectic structure on the fibers and we choose any compatible complex structure making V into a Hermitian vector bundle. At $(m, v) \in \mathbf{T}\Phi^{-1}(\xi)$, the map $\sigma(D_E)(m, v)$ is the Clifford action

$$\text{Cl}_m(v_1) \otimes \text{Id}_{E_m} : (\wedge_{\mathbb{C}}^{\text{even}} V_m) \otimes E_m \longrightarrow (\wedge_{\mathbb{C}}^{\text{odd}} V_m) \otimes E_m.$$

where $v_1 \in V_m$ is the V -component of the vector $v \in \mathbf{T}_m \Phi^{-1}(\xi)$. We explain now the formula of Kawasaki for $RR(\mathcal{M}_\xi, \mathcal{E}_\xi)$ when $\xi \in \Phi(M)$ is a regular value of Φ [23].

Let \mathbb{F} be the collection of the finite subgroup of T which are stabilizer of points in M . Consider the orbit type stratification of $\Phi^{-1}(\xi)$ and denote by \mathbb{S}_ξ the set of its orbit type strata. Each stratum S is a connected component of the smooth submanifold

$$\Phi^{-1}(\xi)_{H_S} := \{m \in \Phi^{-1}(\xi) \mid \text{Stab}_T(m) = H_S\}. \quad (1.51)$$

for a unique $H_S \in \mathbb{F}$. The orbifold \mathcal{M}_ξ decomposes as a disjoint union $\cup_{S \in \mathbb{S}_\xi} S/T$ of smooth components, and each quotient \bar{S}/T is a suborbifold of \mathcal{M}_ξ . The generic stabilizer Γ_M of T on M is also the generic stabilizer of T on the fiber³ $\Phi^{-1}(\xi)$, and is associated to an open and dense stratum S_{max} .

Suppose that $E \rightarrow M$ is an Hermitian T -vector bundle. On each suborbifold \bar{S}/T , we get the orbifold complex vector bundle

$$\mathcal{E}_S := E_{|\bar{S}}/T. \quad (1.52)$$

We define twisted characteristic classes $\text{Ch}^-(\mathcal{E}_S)$ and $D^-(\mathcal{E}_S)$ by

$$\text{Ch}^\gamma(\mathcal{E}_S) := \mathbf{Tr} \left(\gamma^{\mathcal{E}_S} \cdot e^{\frac{i}{2\pi} R(\mathcal{E}_S)} \right), \quad \gamma \in H_S, \quad (1.53)$$

and

$$D^\gamma(\mathcal{E}_S) := \det \left(1 - (\gamma^{\mathcal{E}_S})^{-1} \cdot e^{-\frac{i}{2\pi} R(\mathcal{E}_S)} \right), \quad \gamma \in H_S. \quad (1.54)$$

Here $R(\mathcal{E}_S) \in \mathcal{A}^2(\bar{S}/T, \text{End}(\mathcal{E}_S))$ is the curvature of an horizontal Hermitian connection on $E_{|\bar{S}}$, and $\gamma \mapsto \gamma^{\mathcal{E}_S}$ is the linear action of H_S on the fibers of $E_{|\bar{S}}$.

Let N_S be the normal bundle of \bar{S} in $\Phi^{-1}(\xi)$. The symplectic structure on M induces a symplectic form Ω_S on each suborbifold \bar{S}/T , and a symplectic structure on the fibers of the bundle N_S . Choose a compatible almost complex structure on \bar{S}/T , and a compatible complex structure on the fibers of N_S making the tangent bundle of \bar{S}/T and $\mathcal{N}_S := N_S/T$ into Hermitian vector bundle. Consider a Hermitian connexion on $\mathbf{T}(\bar{S}/T)$, with curvature $R(\bar{S}/T)$, and let

$$\text{Todd}(\bar{S}/T) = \det \left(\frac{(i/2\pi)R(\bar{S}/T)}{1 - e^{-(i/2\pi)R(\bar{S}/T)}} \right) \quad (1.55)$$

be the corresponding Todd forms. Like in (1.54), we associate to the complex orbifold vector bundle \mathcal{N}_S , the twisted form $D^-(\mathcal{N}_S)$ which is a map from H_S to $\mathcal{A}^{even}(\bar{S}/T)$. The 0-degree part of $D^\gamma(\mathcal{N}_S)$ is equal to $\det(1 - (\gamma^{\mathcal{N}_S})^{-1})$, hence $D^\gamma(\mathcal{N}_S)$ is invertible in $\mathcal{A}^{even}(\bar{S}/T)$ when γ belongs to

³ Since a neighborhood of $\Phi^{-1}(\xi)$ in M is T -equivariantly diffeomorphic to $\Phi^{-1}(\xi) \times \mathfrak{t}^*$.

$$H_S^o = \{\gamma \in H_S \mid \det(1 - (\gamma^{\mathcal{N}_S})^{-1}) \neq 0\}. \quad (1.56)$$

Note that H_S^o corresponds to the set of $\gamma \in H_S$ for which \bar{S} is a connected component of $(\Phi^{-1}(\xi))^\gamma$.

Theorem 1.3.20 (Kawasaki) *The number $RR(\mathcal{M}_\xi, \mathcal{E}_\xi) \in \mathbb{Z}$ is given by the formula*

$$RR(\mathcal{M}_\xi, \mathcal{E}_\xi) = \sum_{S \in \mathbb{S}_\xi} \frac{1}{|H_S|} \sum_{\gamma \in H_S^o} \int_{\bar{S}/T} \frac{\text{Todd}(\bar{S}/T) \text{Ch}^\gamma(\mathcal{E}_S)}{D^\gamma(\mathcal{N}_S)}. \quad (1.57)$$

We exploit now Theorem 1.3.20 to show that the map $m_c : \Lambda^* \times \mathbb{Z} \rightarrow \mathbb{Z}$ which is defined by (1.50) is periodic polynomial. We need the classical computation of the first Chern class of the line bundle

$$\mathcal{L}_S^{\mu, k} = (L^{\otimes k} \otimes \mathbb{C}_{-\mu})|_{\bar{S}/T}. \quad (1.58)$$

The curvature form $\omega_\xi \in H^2(\mathcal{M}_\xi) \otimes \mathfrak{t}$ of the principal T -bundle $\Phi^{-1}(\xi) \rightarrow \mathcal{M}_\xi$ restricts to a curvature form $\omega_S \in H^2(\bar{S}/T) \otimes \mathfrak{t}$ on each strata.

Lemma 1.3.21 *The first Chern class of the line bundle $\mathcal{L}_S^{\mu, k}$ is given by*

$$c_1(\mathcal{L}_S^{\mu, k}) = \frac{1}{2\pi} \left(k\Omega_S - \langle k\xi - \mu, \omega_S \rangle \right).$$

For a strata S , we consider $\alpha_S \in \Lambda^*$ such that $\gamma \in H_S \mapsto \gamma^{\alpha_S}$ corresponds to the action of H_S on the fibers of $L|_{\bar{S}}$. Finally we have the decomposition

$$m_c(\mu, k) = \sum_{S \in \mathbb{S}_\xi} P_S(\mu, k), \quad (1.59)$$

where

$$P_S(\mu, k) = \frac{1}{|H_S|} \sum_{\gamma \in H_S^o} \gamma^{k\alpha_S - \mu} \int_{\bar{S}/T} \frac{\text{Todd}(\bar{S}/T)}{D^\gamma(\mathcal{N}_S)} e^{\frac{1}{2\pi} \left(k\Omega_S - \langle k\xi - \mu, \omega_S \rangle \right)}. \quad (1.60)$$

When S is the principal open dense stratum S_{max} the map P_S is

$$P_{max}(\mu, k) = \frac{\sum_{\gamma \in \Gamma_M} \gamma^{k\alpha_o - \mu}}{|\Gamma_M|} \int_{\mathcal{M}_\xi} \text{Todd}(\mathcal{M}_\xi) e^{\frac{1}{2\pi} (k\Omega_\xi - \langle k\xi - \mu, \omega_\xi \rangle)}. \quad (1.61)$$

The term $\frac{\sum_{\gamma \in \Gamma_M} \gamma^{k\alpha_o - \mu}}{|\Gamma_M|}$ is equal to 1 when (μ, k) belongs to the lattice $\Xi(M, L)$ (see (1.47)), and is equal to 0 in the other cases. From (1.60) we see that P_S is a periodic polynomial of degree less than $\frac{\dim(\bar{S}/T)}{2}$, and for $S = S_{max}$ we have on $\Xi(M, L)$

$$P_{max}(\mu, k) = \frac{1}{(2\pi)^d} \int_{\mathcal{M}_\xi} \frac{(k\Omega_\xi - \langle k\xi - \mu, \omega_\xi \rangle)^d}{d!} + O(d-1) \quad (1.62)$$

where $d = \frac{\dim \mathcal{M}_\xi}{2}$ and $O(d-1)$ denotes a polynomial of degree less than $d-1$. If we use the polynomial DH_c defined in Section 1.2 we can conclude our computations with the following

Proposition 1.3.22 *The map m_c is a periodic polynomial of degree $d = \frac{\dim \mathcal{M}_\xi}{2}$ supported on $\Xi(M, L)$. For $(\mu, k) \in \Xi(M, L)$ we have*

$$m_c(\mu, k) = |\Gamma_M| \frac{k^d}{(2\pi)^d} \text{DH}_c\left(\frac{\mu}{k}\right) + O(d-1),$$

where $O(d-1)$ means a periodic polynomial of degree less than $d-1$.

1.3.5 Wall-crossing formulas for the m_c

Let \mathfrak{c}_+ and \mathfrak{c}_- be two adjacent connected component of regular values of Φ separated by an hyperplane Δ . The aim of this section is to compute the periodic polynomial $m_{\mathfrak{c}_+} - m_{\mathfrak{c}_-}$.

We consider two points $\xi_\pm \in \mathfrak{c}_\pm$ such that $\xi = \frac{1}{2}(\xi_+ + \xi_-)$ belongs to the relative interior of $\bar{\mathfrak{c}}_+ \cap \bar{\mathfrak{c}}_-$ in Δ . We suppose furthermore that $\xi^+ - \xi^-$ is orthogonal to Δ . Using the identification $\mathfrak{t}^* \simeq \mathfrak{t}$ given by the scalar product the vector $\gamma = \frac{1}{2}(\xi_+ - \xi_-)$, seen as a vector of \mathfrak{t}_Δ , belongs⁴ to $\mathbb{R}^{>0}\beta$. We noticed in Section 1.2.4 that for all $m \in \Phi^{-1}(\xi)$ the stabilizer \mathfrak{t}_m is either equal to \mathfrak{t}_Δ or to $\{0\}$. Then there exists an open T -invariant neighborhood \mathcal{U} of $\Phi^{-1}(\xi)$ in M such that for all $m \in \bar{\mathcal{U}}$ either $\mathfrak{t}_m := \{0\}$, or $\mathfrak{t}_m = \mathfrak{t}_\Delta$ and $\Phi(m) \in \Delta$.

One see easily that the couple (\mathcal{U}, ξ) is good and the second point of Proposition 1.3.10 tells us that

$$RR_{\mathcal{U}}^{\xi}(M, -) = RR_{\mathcal{U}}^{\xi^-}(M, -) = RR_{\mathcal{U}}^{\xi^+}(M, -) \quad (1.63)$$

when ξ_\pm are close enough to ξ . Since $\mathcal{U} \cap \text{Cr}(\|\Phi - \xi\|^2) = \Phi^{-1}(\xi)$ we have $RR_{\mathcal{U}}^{\xi}(M, -) = RR_0^{\xi}(M, -)$. If ξ_\pm are close enough to ξ we have

$$\mathcal{U} \cap \text{Cr}(\|\Phi - \xi_\pm\|^2) = \Phi^{-1}(\xi_\pm) \bigcup M^\gamma \cap \Phi^{-1}(\xi). \quad (1.64)$$

The former decomposition is due to (1.39) and to the fact that the stabiliser of \mathfrak{t} on \mathcal{U} are either equal to \mathfrak{t}_Δ or to $\{0\}$. Notice that $\xi_- + \gamma = \xi_+ + \gamma = \xi$. The decomposition (1.64) gives

$$RR_{\mathcal{U}}^{\xi_\pm}(M, -) = RR_0^{\xi_\pm}(M, -) + RR_{\mp\gamma}^{\xi_\pm}(M, -), \quad (1.65)$$

where $RR_{\gamma}^{\xi^-}(M, -)$ (resp. $RR_{-\gamma}^{\xi^+}(M, -)$) is the Riemann-Roch character localized on $M^\gamma \cap \Phi^{-1}(\xi)$ by the vector field $\mathcal{H} - (\xi_-)_M$ (resp. $\mathcal{H} - (\xi_+)_M$). Now (1.63) and (1.65) prove the following

⁴ β is the primitive vector of $\mathfrak{t}_\Delta \cap \Lambda$ pointing out of \mathfrak{c}_-

Proposition 1.3.23 *If ξ_{\pm} are close enough to Δ , we have*

$$RR_0^{\xi_+}(M, -) - RR_0^{\xi_-}(M, -) = RR_{\gamma}^{\xi_-}(M, -) - RR_{-\gamma}^{\xi_+}(M, -).$$

We know from Proposition 1.3.18 that $m_{c_{\pm}}(\mu, k)$ is equal to the μ -multiplicity of T in $RR_0^{\xi_{\pm}}(M, L^{\otimes k})$. Hence $m_{c_+}(\mu, k) - m_{c_-}(\mu, k)$ is equal to the μ -multiplicity of T in $RR_{\gamma}^{\xi_-}(M, L^{\otimes k}) - RR_{-\gamma}^{\xi_+}(M, L^{\otimes k})$.

Let N_{Δ} be the normal bundle of $M^{T_{\Delta}}$ in M , and let $[\wedge_{\mathbb{C}}^{\bullet} \overline{N_{\Delta}}]_{\pm\beta}^{-1}$ be the polarized inverses of $\wedge_{\mathbb{C}}^{\bullet} \overline{N_{\Delta}}$ (see Proposition 1.3.13). Since $\xi = \xi_+ - \gamma = \xi_- + \gamma$ and $\gamma \in \mathbb{R}^{>0}\beta$, the localization (1.44) gives

$$\begin{aligned} RR_{\gamma}^{\xi_-}(M, L^{\otimes k}) &= \sum_{Z \in \mathcal{F}} RR_0^{\xi} \left(Z, L^{\otimes k}|_Z \otimes [\wedge_{\mathbb{C}}^{\bullet} \overline{N_Z}]_{\beta}^{-1} \right), \\ RR_{-\gamma}^{\xi_+}(M, L^{\otimes k}) &= \sum_{Z \in \mathcal{F}} RR_0^{\xi} \left(Z, L^{\otimes k}|_Z \otimes [\wedge_{\mathbb{C}}^{\bullet} \overline{N_Z}]_{-\beta}^{-1} \right). \end{aligned}$$

Finally $m_{c_+}(\mu, k) - m_{c_-}(\mu, k) = \sum_{Z \in \mathcal{F}} \mathcal{A}_Z(\mu, k)$ where $\mathcal{A}_Z(\mu, k)$ is equal to the μ -multiplicity of T in

$$RR_0^{\xi} \left(Z, L^{\otimes k}|_Z \otimes [\wedge_{\mathbb{C}}^{\bullet} \overline{N_Z}]_{-\beta}^{-1} \right) - RR_0^{\xi} \left(Z, L^{\otimes k}|_Z \otimes [\wedge_{\mathbb{C}}^{\bullet} \overline{N_Z}]_{\beta}^{-1} \right) \quad (1.66)$$

Let $\beta' \in \mathfrak{t}_{\Delta}^* \cap \Lambda^*$ which is defined by the relation $\langle \beta', \beta \rangle = 2\pi$, so that $\Lambda_{\mathfrak{t}_{\Delta}}^* = \mathbb{Z}\beta'$. Concerning the T_{Δ} -weights we have

1. The T_{Δ} -weight on $L^{\otimes k}|_Z$ is equal to $kr_{\Delta}\beta'$.
2. The T_{Δ} -weight on $\det(N_Z^{+, \pm\beta})$ is $\pm s_Z^{\pm}\beta'$ where $s_Z^{\pm} \in \mathbb{N}$ is the absolute value of the trace of $\frac{1}{2\pi}\mathcal{L}_{\beta}$ on $N_Z^{+, \pm\gamma}$.
3. The T_{Δ} -weights on $S^{>0}(N_Z^{+, \beta})$ (resp. $S^{>0}(N_Z^{+, -\beta})$) are of the form $p\beta'$ with $p > 0$ (resp. $p < 0$).

Now Lemma 1.3.14 shows that if a weight μ occurs in $RR_0^{\xi} \left(Z, L^{\otimes k}|_Z \otimes [\wedge_{\mathbb{C}}^{\bullet} \overline{N_Z}]_{\beta}^{-1} \right)$ we have $i^*(\mu) = (kr_{\Delta} + s_Z^+ + p)\beta'$ with $p \geq 0$, and then

$$\frac{\langle \mu, \beta \rangle}{2\pi} - kr_{\Delta} \geq s_Z^+.$$

Similarly, if a weight μ occurs in $RR_0^{\xi} \left(Z, L^{\otimes k}|_Z \otimes [\wedge_{\mathbb{C}}^{\bullet} \overline{N_Z}]_{-\beta}^{-1} \right)$ we have

$$\frac{\langle \mu, \beta \rangle}{2\pi} - kr_{\Delta} \leq -s_Z^-.$$

Finally, $\mathcal{A}_Z(\mu, k) = 0$ when $-s_Z^- < \frac{\langle \mu, \beta \rangle}{2\pi} - kr_{\Delta} < s_Z^+$. We have proved the following

Theorem 1.3.24 *Let $s^\pm = \inf_Z s_Z^\pm$ where the infimum is taken over the connected components Z of M^{T_Δ} for which $\bar{c}_+ \cap \bar{c}_- \subset \Phi(Z)$. For every $(\mu, k) \in \Lambda^* \times \mathbb{Z}$, we have $m_{c_+}(\mu, k) = m_{c_-}(\mu, k)$ if*

$$-s^- < \frac{\langle \mu, \beta \rangle}{2\pi} - kr_\Delta < s^+. \quad (1.67)$$

Sometimes the inequalities (1.67) are optimal.

Proposition 1.3.25 • *Consider the connected components $Z \in \mathcal{F}$ for which s_Z^+ is minimal. Among them consider the subset \mathcal{F}_+ where $\dim(Z)$ is maximal. If the integers $\text{rk}_{\mathbb{C}}(N_Z^{+, \beta})$, $Z \in \mathcal{F}_+$ have the same parity, then the condition “ $\frac{\langle \mu, \beta \rangle}{2\pi} - kr_\Delta < s^+$ ” is optimal in (1.67).*

• *In the same way, consider the connected components $Z \in \mathcal{F}$ for which s_Z^- is minimal. Among them consider the subset \mathcal{F}_- where $\dim(Z)$ is maximal. If the integers $\text{rk}_{\mathbb{C}}(N_Z^{+, \beta})$, $Z \in \mathcal{F}_+$ have the same parity, then the condition “ $-s^- < \frac{\langle \mu, \beta \rangle}{2\pi} - kr_\Delta$ ” is optimal in (1.67).*

Remark 1.3.26 *The last Proposition applies when there is a unique connected component Z of M^{T_Δ} for which $\bar{c}_+ \cap \bar{c}_- \subset \Phi(Z)$.*

Proof. We consider only the first point since the other point works similarly. We restrict our attention to the couples (μ, k) such that $\frac{\langle \mu, \beta \rangle}{2\pi} - kr_\Delta = s^+$. They are of the form

$$\mu = (kr_\Delta + s^+)\beta' + \mu_2 \quad (1.68)$$

with $\mu_2 \in \Lambda_{t/t_\Delta}^*$. Let us denote $D(\mu_2, k)$ the restriction of $m_{c_+}(\mu, k) - m_{c_-}(\mu, k)$ to the set of couples (μ, k) parametrized by (1.68). We want to prove that $D(\mu_2, k)$ is not identically equal to zero.

From the previous discussion one knows that

$$D(\mu_2, k) = \sum_{Z, s_Z^+ = s^+} (-1)^{\text{rk}_{\mathbb{C}}(N_Z^{+, \beta})} D_Z(\mu_2, k), \quad (1.69)$$

where $D_Z(\mu_2, k)$ is the μ -multiplicity of T in

$$RR_0^\xi \left(Z, L^{\otimes k}|_Z \otimes \det(N_Z^{+, \beta}) \right).$$

Let us make few remarks concerning the maps $RR_0^\xi(Z, -) : \mathbf{K}_T(Z) \rightarrow R^{-\infty}(T)$. Since T_Δ acts trivially on Z , the decomposition $T = T/T_\Delta \times T_\Delta$ induces a canonical isomorphism $\mathbf{K}_T(Z) \simeq \mathbf{K}_{T/T_\Delta}(Z) \otimes R(T_\Delta)$: i.e. every T -equivariant vector bundle $E \rightarrow Z$ decomposes as

$$E = \sum_{\mu_1 \in \mathbb{Z}\beta'} E^{\mu_1} \otimes \mathbb{C}_{\mu_1}. \quad (1.70)$$

Here each E^{μ_1} is a T/T_Δ -equivariant vector bundle on Z and \mathbb{C}_{μ_1} denotes the one dimensional T_Δ -representation associated to $\mu_1 \in \Lambda_{t_\Delta}^*$.

For every T -equivariant vector bundle $E \rightarrow Z$, the character $RR_0^\xi(Z, E)$ is equal to the T -equivariant index of the T -transversally elliptic symbol $\text{Thom}_\xi(\mathcal{V}) \otimes p^*(E)$, where \mathcal{V} is a small neighborhood of $\Phi^{-1}(\xi) \cap Z$ in Z (see Definition 1.3.9). Since the T_Δ action is trivial on Z the symbol $\text{Thom}_\xi(\mathcal{V})$ is also T/T_Δ transversally elliptic and the action of T_Δ is trivial on it. We have then

$$RR_0^\xi(Z, E) = \sum_{\mu_1 \in \mathbb{Z}\beta'} RR_0^\xi(Z, E^{\mu_1}) \otimes \mathbb{C}_{\mu_1}. \quad (1.71)$$

where the character $RR_0^\xi(Z, E^{\mu_1}) \in R^{-\infty}(T/T_\Delta)$ is computed by Theorem 1.46 applied to the Hamiltonian T/T_Δ -manifold Z . For every T -vector bundle $E \rightarrow Z$ we define the family $\mathcal{E}_\xi^{\mu_1, \mu_2}, \mu_1 \in \mathbb{Z}\beta', \mu_2 \in \Lambda_{\mathfrak{t}/\mathfrak{t}_\Delta}^*$ of orbifold vector bundles over the reduced space $\mathcal{Z}_\xi = Z \cap \Phi^{-1}(\xi)/(T/T_\Delta)$ by

$$\mathcal{E}_\xi^{\mu_1, \mu_2} := (E^{\mu_1} \otimes \mathbb{C}_{-\mu_2})|_{\Phi^{-1}(\xi) \cap Z / (T/T_\Delta)}. \quad (1.72)$$

Finally (1.46) and (1.71) give the following

$$\begin{aligned} RR_0^\xi(Z, E) &= \sum_{\mu_1 \in \mathbb{Z}\beta'} \sum_{\mu_2 \in \Lambda_{\mathfrak{t}/\mathfrak{t}_\Delta}^*} RR(\mathcal{Z}_\xi, \mathcal{E}_\xi^{\mu_1, \mu_2}) \otimes \underbrace{\mathbb{C}_{\mu_1}}_{\in R(T_\Delta)} \otimes \underbrace{\mathbb{C}_{\mu_2}}_{\in R(T/T_\Delta)} \\ &= \sum_{\mu \in \Lambda^*} RR(\mathcal{Z}_\xi, \mathcal{E}_\xi^{\mu_1, \mu_2}) \mathbb{C}_\mu. \end{aligned} \quad (1.73)$$

In (1.73) we write $\mu \in \Lambda^*$ as a sum of $\mu_1 \in \mathbb{Z}\beta'$ with $\mu_2 \in \Lambda_{\mathfrak{t}/\mathfrak{t}_\Delta}^*$ so that $\mathbb{C}_\mu \in R(T)$ is equal to the tensor product $\mathbb{C}_{\mu_1} \otimes \mathbb{C}_{\mu_2}$.

When the vector bundle $E \rightarrow Z$ is the line bundle $\mathbb{L} := L^{\otimes k}|_Z \otimes \det(N_Z^{+, \beta})$ we have $\mathbb{L} = \mathbb{L}^{j\beta'} \otimes \mathbb{C}_{j\beta'}$ for $j = kr_\Delta + s^+$. Finally, we have

$$D_Z(\mu_2, k) = RR(\mathcal{Z}_\xi, \mathbb{L}_\xi^{(kr_\Delta + s^+)\beta', \mu_2}).$$

Now we use the results of Section 1.3.4 to study the map

$$D_Z : \Lambda_{\mathfrak{t}/\mathfrak{t}_\Delta}^* \times \mathbb{Z} \longrightarrow \mathbb{Z}. \quad (1.74)$$

Let $\Gamma_Z \subset T/T_\Delta$ be the generic stabiliser of T/T_Δ on a component Z . Let $\alpha_Z, \delta_Z \in \Lambda_{\mathfrak{t}/\mathfrak{t}_\Delta}^*$ such that the action of Γ_Z on the fibers of $L|_Z$ and $\det(N_Z^{+, \beta})$ are respectively $t \rightarrow t^{\alpha_Z}$ and $t \rightarrow t^{\delta_Z}$. After Remark 1.3.17 we know that the map (1.74) is supported on the subset

$$\Xi_Z := \{(\mu_2, k) \in \Lambda_{\mathfrak{t}/\mathfrak{t}_\Delta}^* \times \mathbb{Z} \mid t^{k\alpha_Z + \delta_Z + \mu_2} = 1, \forall t \in \Gamma_Z\}. \quad (1.75)$$

The only difference with the computations done in Section 1.3.4 is the line bundle $\det(N_Z^{+, \beta})$. But this do not change the global behaviour of the map (1.74) on Ξ_Z : it is a periodic polynomial map of degree $d_Z = \dim(\mathcal{Z}_\xi)/2$ and we have

$$D_Z(\mu_2, k) = \frac{1}{(2\pi)^{d_Z}} \int_{Z_\xi} \frac{(k\Omega_{Z_\xi} - \langle k\xi - \mu_2, \omega_{Z_\xi} \rangle)^{d_Z}}{d_Z!} + O(d_Z - 1) \quad (1.76)$$

for all $(\mu_2, k) \in \Xi_Z$.

Suppose now that all the signs $(-1)^{\text{rk}_{\mathbb{C}}(N_Z^{\pm, \beta})}$ coincide when $Z \in \mathcal{F}_+$. From (1.69), we get that $D(\mu_2, k)$ does not vanish for large values of (μ_2, k) . \square

1.4 Multiplicities of group representations

Let K be a semi-simple compact Lie group with Lie algebra \mathfrak{k} , and let T be a maximal torus in K with Lie algebra \mathfrak{t} . In this section we denote $(-, -)$ the scalar product on \mathfrak{k} induced by the Killing form, and we keep the same notation for the induced scalar products on \mathfrak{k}^* and on \mathfrak{t} .

Let $\Lambda^* \subset \mathfrak{k}^*$ be the weight lattice, and let $\mathfrak{R} \subset \Lambda^*$ be the set of roots for the action of T on $\mathfrak{k} \otimes \mathbb{C}$: we denote $\Lambda_{\mathfrak{R}}^*$ the sub-lattice of Λ^* generated by \mathfrak{R} . We choose a system of positive roots $\mathfrak{R}^+ \subset \mathfrak{R}$, and we denote \mathfrak{t}_+^* the corresponding Weyl chamber.

The irreducible representations of K are parametrized by the set $\Lambda_+^* = \Lambda^* \cap \mathfrak{t}_+^*$. For $\lambda \in \Lambda_+^*$ we denote by V_λ the irreducible representation of K with highest weight λ . Here we are interested in the T -multiplicities in $V_\lambda|_T$. Let $m : \Lambda^* \times \Lambda_+^* \rightarrow \mathbb{N}$ be the map defined by

$$V_\lambda|_T = \sum_{\mu \in \Lambda^*} m(\mu, \lambda) \mathbb{C}_\mu \quad (1.77)$$

for every $\lambda \in \Lambda_+^*$.

Definition 1.4.1 *For every $\lambda \in \Lambda_+^*$, we denote $m^\lambda : \Lambda^* \times \mathbb{Z}^{>0} \rightarrow \mathbb{N}$ the map defined by $m^\lambda(\mu, k) = m(\mu, k\lambda)$. So $m^\lambda(\mu, k)$ is equal to the multiplicity of \mathbb{C}_μ in $V_{k\lambda}|_T$.*

1.4.1 Borel-Weil Theorem

First we recall the realization of the K -representation V_λ given by the Borel-Weil Theorem. The coadjoint orbit $K \cdot \lambda$ is equipped with the Kirillov-Kostant-Souriau symplectic form Ω which is defined by:

$$\Omega(X_M, Y_M)_m = \langle m, [X, Y] \rangle, \quad \text{for } m \in K \cdot \lambda \quad \text{and } X, Y \in \mathfrak{k}. \quad (1.78)$$

The action of K on $K \cdot \lambda$ is Hamiltonian with moment map $K \cdot \lambda \hookrightarrow \mathfrak{k}^*$ equal to the inclusion. The action of T on $K \cdot \lambda$ is also Hamiltonian with moment map $\Phi : K \cdot \lambda \rightarrow \mathfrak{t}^*$ equal to the composition of the inclusion $K \cdot \lambda \hookrightarrow \mathfrak{k}^*$ with the projection map $\mathfrak{k}^* \rightarrow \mathfrak{t}^*$.

There exists a unique K -invariant complex structure on $K \cdot \lambda$ compatible with the symplectic form. In this situation the Kostant-Souriau prequantum line bundle over $K \cdot \lambda$ is

$$\mathbb{C}_{[\lambda]} = K \times_{K_\lambda} \mathbb{C}_\lambda.$$

Here we use the canonical identification $K/K_\lambda \simeq K \cdot \lambda$, $[k] \mapsto k \cdot \lambda$, where K_λ is the stabilizer of λ in K . The line bundle $\mathbb{C}_{[\lambda]}$ over the complex manifold $K \cdot \lambda$ carries a canonical holomorphic structure. If one work with the symplectic form $k\Omega$, for an integer $k \geq 1$, the corresponding Kostant-Souriau prequantum line bundle is $\mathbb{C}_{[\lambda]}^{\otimes k} = K \times_{K_\lambda} \mathbb{C}_{k\lambda} = \mathbb{C}_{[k\lambda]}$.

Let $\mathcal{H}^q(K \cdot \lambda, \mathbb{C}_{[\lambda]}^{\otimes k})$ be q th cohomology group of the sheaf of holomorphic section of $\mathbb{C}_{[\lambda]}^{\otimes k}$ over $K \cdot \lambda$. The Borel-Weil Theorem tells us that

$$\mathcal{H}^0(K \cdot \lambda, \mathbb{C}_{[\lambda]}^{\otimes k}) = V_{k\lambda} \quad (1.79)$$

and

$$\mathcal{H}^q(K \cdot \lambda, \mathbb{C}_{[\lambda]}^{\otimes k}) = 0 \quad \text{for } q \geq 1. \quad (1.80)$$

If $RR^K(K \cdot \lambda, -) : \mathbf{K}_K(K \cdot \lambda) \rightarrow R(K)$ is the K -equivariant Riemann-Roch character defined by the compatible complex structure, (1.79) and (1.80) give

$$RR^K(K \cdot \lambda, \mathbb{C}_{[\lambda]}^{\otimes k}) = V_{k\lambda} \quad \text{in } R(K). \quad (1.81)$$

Now if we denote by $RR(K \cdot \lambda, -) : \mathbf{K}_T(K \cdot \lambda) \rightarrow R(T)$ the T -equivariant Riemann-Roch character, we have $V_{k\lambda}|_T = RR(K \cdot \lambda, \mathbb{C}_{[\lambda]}^{\otimes k})$. The multiplicity function $m^\lambda : A_+^* \times \mathbb{N}^* \rightarrow \mathbb{N}$ is characterized by the relation

$$RR(K \cdot \lambda, \mathbb{C}_{[\lambda]}^{\otimes k}) = \sum_{\mu \in A^*} m^\lambda(\mu, k) \mathbb{C}_\mu, \quad \text{in } R(T), \quad (1.82)$$

for $k \geq 1$.

The sub-lattice $A_{\mathfrak{R}}^*$ of A^* generated by the roots is characterized by the (finite) center $Z(K)$ of K as follows. For $\alpha \in A^*$ we have

$$\lambda \in A_{\mathfrak{R}}^* \iff t^\lambda = 1, \quad \forall t \in Z(K), \quad (1.83)$$

and for $t \in T$ we have $t \in Z(K) \iff t^\lambda = 1, \quad \forall \lambda \in A_{\mathfrak{R}}^*$. The finite abelian group $A^*/A_{\mathfrak{R}}^*$ is then naturally identified with the dual of $Z(K)$. We have the following well-known fact.

Lemma 1.4.2 *The multiplicity map m^λ is supported on the sub-lattice $\Xi_\lambda = \{(\mu, k) \in A^* \times \mathbb{Z} \mid \mu - k\lambda \in A_{\mathfrak{R}}^*\}$.*

Proof. The center $Z(K)$ of K acts trivially on $K \cdot \lambda$ and with the character $t \in Z(K) \mapsto t^{k\lambda}$ on the fibers of the line bundle $\mathbb{C}_{[\lambda]}^{\otimes k}$. Since $m^\lambda(\mu, k)$ is equal to the dimension of the T -invariant subspace of $RR(K \cdot \lambda, \mathbb{C}_{[\lambda]}^{\otimes k}) \otimes \mathbb{C}_{-\mu}$, we have following Lemma 1.3.17 that $m^\lambda(\mu, k) \neq 0$ only if $t^{\mu - k\lambda} = 1, \quad \forall t \in Z(K)$. We conclude then with (1.83). \square

In this section we study the periodic polynomials

$$m_c^\lambda : \Lambda^* \times \mathbb{Z} \longrightarrow \mathbb{Z}. \tag{1.84}$$

defined for every connected component $\mathfrak{c} \subset \mathfrak{t}^*$ of regular values of the moment map $\Phi : K \cdot \lambda \rightarrow \mathfrak{t}^*$. We know that m_c^λ is also supported on the sub-lattice Ξ_λ (see Section 1.3.3).

In order to apply Theorem 1.3.24 to the periodic polynomials m_c^λ , we have to compute the critical values of the moment map $\Phi : K \cdot \lambda \rightarrow \mathfrak{t}^*$.

1.4.2 Critical points of $\Phi : K \cdot \lambda \rightarrow \mathfrak{t}^*$

Let $\{\alpha_1, \dots, \alpha_{\dim T}\}$ be the simple roots of the set \mathfrak{R}_+ of positive weights. The fundamental weights $\varpi_k, 1 \leq k \leq \dim T$ are defined by the conditions

$$2 \frac{(\varpi_i, \alpha_j)}{|\alpha_j|^2} = \delta_{i,j} \quad \text{for all } 1 \leq i, j \leq \dim T. \tag{1.85}$$

Recall that the fundamental weights generate the lattice Λ_{alg}^* of algebraic integral element of \mathfrak{t}^* . We have $\Lambda^* \subset \Lambda_{alg}^*$ and equality holds only if K is simply-connected.

Let W be the Weyl group of (K, T) . We will look at

$$\mathcal{G} = \{\sigma \cdot \varpi_i \mid \sigma \in W, 1 \leq i \leq \dim T\}. \tag{1.86}$$

as a subset of \mathfrak{t} modulo the identification $\mathfrak{t} \simeq \mathfrak{t}^*$ given by the scalar product. The singular points of Φ have the following nice description. This result first appeared in Heckman's Thesis [22].

Proposition 1.4.3 ([22, 17]) *The critical points of $\Phi : K \cdot \lambda \rightarrow \mathfrak{t}^*$ is the union of the fixed points set $(K \cdot \lambda)^\beta, \beta \in \mathcal{G}$. For each $\beta \in \mathcal{G}$ we have*

$$(K \cdot \lambda)^\beta = \bigcup_{\sigma \in W} K^\beta \cdot \sigma \lambda,$$

where K^β is the stabilizer subgroup of β in K .

The fixed points of the action of T on $K \cdot \lambda$ characterize the image of Φ completely: $\Phi(K \cdot \lambda)$ is the convex polytope

$$\text{conv}(W \cdot \lambda) := \text{convex hull of } W \cdot \lambda. \tag{1.87}$$

This result was first proved by Kostant [25]. This is particular case of the convexity theorem of Atiyah, Guillemin and Sternberg [2, 18]. From Proposition 1.4.3, we know that the singular values of $\Phi : K \cdot \lambda \rightarrow \mathfrak{t}^*$ are the convex polytopes

$$\text{conv}(W^\beta \cdot \sigma \lambda), \quad \beta \in \mathcal{F}, \sigma \in W/W^\beta, \tag{1.88}$$

where W^β is the stabilizer⁵ of β in W , i.e. W^β is the Weyl group of (K^β, T) . Each convex polytope $\text{conv}(W^\beta \cdot \sigma \lambda)$ lies in the hyperplane

⁵ When $\beta = \varpi_i$, we denote W^i the stabilizer of ϖ_i in W .

$$\Delta_{\beta,\sigma} = \{\xi \in \mathfrak{t}^* \mid (\xi - \sigma\lambda, \beta) = 0\}. \quad (1.89)$$

We will use the following

Lemma 1.4.4 • $K^\beta \cdot \sigma\lambda = K^\beta \cdot \sigma'\lambda$ if and only if $\sigma\lambda \in W^\beta \sigma'\lambda$,
 • $\text{conv}(W^\beta \cdot \sigma\lambda) \cap \text{conv}(W^\beta \cdot \sigma'\lambda) \neq \emptyset$ if and only if $\Delta_{\beta,\sigma} = \Delta_{\beta,\sigma'}$.

Proof. The first point follows from the fact that the intersection of a coadjoint orbit $K^\beta \cdot \mu$, $\mu \in \mathfrak{t}^*$ with \mathfrak{t}^* is equal to $W^\beta \cdot \mu$.

It is sufficient to prove the second point for $\beta = \varpi_i$. The half-line $\mathbb{R}^{>0}\varpi_i$ is an edge of the Weyl chamber. It is well known that the following vector subspaces coincides:

- the line $\mathbb{R}\varpi_i$,
- the vector sub-space of K^{ϖ_i} -invariant element of \mathfrak{t}^* ,
- the vector sub-space of W^i -invariant element of \mathfrak{t}^* .

Each convex polytope $\text{conv}(W^i \cdot \sigma\lambda)$ contains the W^i -invariant element

$$\frac{1}{|W^i|} \sum_{\tau \in W^i} \tau \cdot \sigma\lambda$$

which is equal to the intersection of the hyperplane $\Delta_{\beta,\sigma}$ with the line $\mathbb{R}\varpi_i$. Hence, if $\Delta_{\beta,\sigma} = \Delta_{\beta,\sigma'}$, the intersection $\text{conv}(W^\beta \cdot \sigma\lambda) \cap \text{conv}(W^\beta \cdot \sigma'\lambda)$ contains a W^i -invariant element, and then is not empty.

Definition 1.4.5 An element $\lambda \in \Lambda_+^*$ is generic if for every fundamental root ϖ_i and any $\sigma, \sigma' \in W$, we have

$$\Delta_{\beta,\sigma} \neq \Delta_{\beta,\sigma'} \quad (1.90)$$

each times the submanifolds $K^\beta \cdot \sigma\lambda$ and $K^\beta \cdot \sigma'\lambda$ are not equal.

This condition of genericity imposes that $(\sigma\lambda, \varpi_i) \neq (\sigma'\lambda, \varpi_i)$ when $\sigma\lambda \notin W^i \sigma'\lambda$.

Example 1.4.6 Consider the case of $\text{SU}(4)$. Take the coadjoint orbit through $\lambda = (2, 1, -1, -2)$, and σ, σ' such that $\sigma\lambda = (2, -2, 1, -1)$ and $\sigma'\lambda = (1, -1, 2, -2)$. Take the fundamental weight $\varpi_2 = \frac{1}{2}(1, 1, -1, -1)$. In this case λ is not “generic” since $\sigma\lambda \notin W^i \sigma'\lambda$ but $(\sigma\lambda, \varpi_2) = 0 = (\sigma'\lambda, \varpi_2)$.

1.4.3 Main theorems

Let \mathfrak{c}_+ and \mathfrak{c}_- be two adjacent connected components of regular values of $\Phi : K \cdot \lambda \rightarrow \mathfrak{t}^*$. The intersection $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-}$ is contained in an hyperplane orthogonal to $\beta \in \mathcal{F}$.

Definition 1.4.7 Let $\mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ be the set of all $[\sigma] \in W/W^\beta$ such that the convex polytope $\text{conv}(W^\beta \cdot \sigma\lambda)$ contains $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-}$.

The following

$$\bigcup_{[\sigma] \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)} K^\beta \cdot \sigma\lambda$$

is the union of the connected components of $(K \cdot \lambda)^\beta$ that intersect $\Phi^{-1}(\xi)$ when $\xi \in \overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-}$.

Remark 1.4.8 When λ is a regular element of \mathfrak{t}^* , all polytopes $\text{conv}(W^\beta \cdot \sigma\lambda)$ are of codimension 1. When λ is “generic” (see Def. 1.4.5), the set $\mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ is reduced to one element.

The multiplicity function $m^\lambda : \Lambda^* \times \mathbb{Z}^{>0} \rightarrow \mathbb{N}$ is invariant under the action of the Weyl group: $m^\lambda(\sigma\mu, k) = m^\lambda(\mu, k)$ for every $\sigma \in W$. The set of connected component of regular values of Φ is also invariant under the action of W .

So, for the rest of this section we restrict our attention to case where \mathfrak{c}_+ and \mathfrak{c}_- are separated by an hyperplane orthogonal to a fundamental weight $\beta = \varpi_i$: the vector ϖ_i is pointing out of \mathfrak{c}_- . We denote K^i the stabilizer of ϖ_i in K .

Consider $[\sigma] \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ and let $K^i \cdot \sigma\lambda$ be the corresponding connected component of $(K \cdot \lambda)^\beta$. The tangent space of $K \cdot \lambda$ at $\sigma\lambda$ is the following $K^{\sigma\lambda}$ -module

$$\mathbf{T}_{\sigma\lambda}(K \cdot \lambda) = \sum_{(\alpha, \sigma\lambda) > 0} \mathfrak{k}_\alpha \quad (1.91)$$

where $\mathfrak{k}_\alpha \subset \mathfrak{k} \otimes \mathbb{C}$ is the one-dimensional complex subspace associated to the weight $\alpha \in \mathfrak{R}$. In the same way, the tangent space of $K^i \cdot \sigma\lambda$ at $\sigma\lambda$ is the $K^i \cap K^{\sigma\lambda}$ -module defined by

$$\mathbf{T}_{\sigma\lambda}(K^i \cdot \sigma\lambda) = \sum_{\substack{(\alpha, \sigma\lambda) > 0 \\ (\alpha, \varpi_i) = 0}} \mathfrak{k}_\alpha \quad (1.92)$$

Finally the normal bundle of $K^i \cdot \sigma\lambda \simeq K^i / (K^i \cap K^{\sigma\lambda})$ in $K \cdot \lambda$ is $\mathcal{N}_{\sigma,i} = K^i \times_{K^i \cap K^{\sigma\lambda}} N_{\sigma,i}$ where

$$N_{\sigma,i} = \sum_{\substack{(\alpha, \sigma\lambda) > 0 \\ (\alpha, \varpi_i) \neq 0}} \mathfrak{k}_\alpha \quad (1.93)$$

For an element $\mu \in \mathfrak{t}^*$, we have $\mu = \sum_{i=1}^{\dim T} [\mu]_k \alpha_k$ where

$$[\mu]_k = 2 \frac{(\varpi_k, \mu)}{|\alpha_k|^2} \in \mathbb{R}.$$

Note that $[\mu]_k \in \mathbb{Z}$ when μ belongs to the lattice $\Lambda_{\mathfrak{R}}^*$.

Definition 1.4.9 For $[\sigma] \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ we define the positive integers

$$s_{\sigma,i}^{\pm} = \pm \sum_{\substack{(\alpha, \sigma\lambda) > 0 \\ \pm(\alpha, \varpi_i) > 0}} [\alpha]_i.$$

Note that $s_{\sigma,i}^+ + s_{\sigma,i}^-$ is larger than half of the codimension of $K^i \cdot \sigma\lambda$ in $K \cdot \lambda$.

Theorem 1.4.10 Let \mathfrak{c}_+ and \mathfrak{c}_- be two adjacent connected component of regular values of $\Phi : K \cdot \lambda \rightarrow \mathfrak{t}^*$ separated by an hyperplane orthogonal to a fundamental weight ϖ_i : we denote r_i the commum value $[\xi]_i$ for all ξ in this hyperplane. Let $m_{\mathfrak{c}_{\pm}}^{\lambda} : \Lambda^* \times \mathbb{Z} \rightarrow \mathbb{Z}$ be the corresponding periodic polynomials which are supported on the sub-lattice $\Xi_{\lambda} := \{(\mu, k) \mid \mu \in k\lambda + \Lambda_{\mathfrak{R}}^*\}$.

For all $(\mu, k) \in \Xi_{\lambda}$, we have $m_{\mathfrak{c}_+}^{\lambda}(\mu, k) = m_{\mathfrak{c}_-}^{\lambda}(\mu, k)$ when the integer $[\mu]_i - kr_i$ satisfies

$$-s_i^- < [\mu]_i - kr_i < s_i^+. \quad (1.94)$$

Here the positive integer s_i^{\pm} are defined by

$$s_i^{\pm} = \inf_{[\sigma] \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)} s_{\sigma,i}^{\pm}. \quad (1.95)$$

When $\mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ is reduced to one element σ , for example if λ is “generic”, the integer $s_i^+ + s_i^-$ is larger than half of the codimension of $K^i \cdot \sigma\lambda$ in $K \cdot \lambda$.

Another way to express the result of Theorem 1.4.10 is to introduce like in [36] the convex polytope

$$\square(\mathfrak{c}_+, \mathfrak{c}_-) = \bigcap_{\sigma \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)} \left(\sum_{(\alpha, \sigma\lambda) > 0} [0, 1[\alpha] \right). \quad (1.96)$$

Let Δ be the hyperplane which separates \mathfrak{c}_+ and \mathfrak{c}_- . Equation (1.94) is equivalent to saying that

$$m_{\mathfrak{c}_+}^{\lambda}(\mu, k) = m_{\mathfrak{c}_-}^{\lambda}(\mu, k) \quad \text{if} \quad \mu \in k\Delta + \square(\mathfrak{c}_+, \mathfrak{c}_-). \quad (1.97)$$

Corollary 1.4.11 Let \mathfrak{c} be a connected component of regular values of Φ which is bording a facet of the polytope $\Phi(K \cdot \lambda)$ orthogonal to the fundamental weight ϖ_i : the facet is $\text{conv}(W^i \cdot \sigma\lambda)$ for a unique $\sigma \in W/W^i$. We suppose that ϖ_i is pointing out of \mathfrak{c} . We denote r_i the commum value $[\xi]_i$ for all ξ in the facet. For all $(\mu, k) \in \Xi_{\lambda}$, we have $m_{\mathfrak{c}}^{\lambda}(\mu, k) = 0$ when the integer $[\mu]_i - kr_i$ satisfies

$$-s_{\sigma,i}^- < [\mu]_i - kr_i < s_{\sigma,i}^+. \quad (1.98)$$

Proof. Theorem 1.4.10 is a direct consequence of Theorem 1.3.24. The main difference between them is the decomposition of the lattice supporting the periodic polynomials. In the former we use the decomposition $\Lambda^* = \Lambda_{t_\Delta}^* \oplus \Lambda_{t/t_\Delta}^*$ associated to the choice of a subtorus T/T_Δ . Here we use the decomposition $\Lambda_{\mathfrak{R}}^* = \mathbb{Z}\alpha_i \oplus \sum_{k \neq i} \mathbb{Z}\alpha_k$. Note first that for $(\mu, k) \in \Xi_\lambda$, we have $\mu - \sigma\lambda \in \Lambda_{\mathfrak{R}}^*$ and then $[\mu - \sigma\lambda]_i = [\mu]_i - kr_i$ is an integer.

We start like after Proposition 1.3.23: $m_{c_+}^\lambda(\mu, k) - m_{c_-}^\lambda(\mu, k)$ is equal to the μ -multiplicity in $\sum_{\sigma \in \mathcal{A}(c_+, c_-)} A_\sigma^- - A_\sigma^+$ where

$$A_\sigma^\pm = RR_0^\xi \left(K^i \cdot \sigma\lambda, \mathbb{C}_{[\lambda]}^{\otimes k} \otimes [\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}_{\sigma,i}}]_{\mp \varpi_i}^{-1} \right) \quad (1.99)$$

Here ξ belongs to the relative interior of $\overline{c_+} \cap \overline{c_-}$, the line bundle $\mathbb{C}_{[\lambda]}^{\otimes k}$ is equal to $K^i \times_{K^i \cap K^{\sigma\lambda}} \mathbb{C}_{k\sigma\lambda}$ and $[\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}_{\sigma,i}}]_{\pm \varpi_i}^{-1}$ corresponds to $(-1)^{\text{rk}_{\mathbb{C}}(N_{\sigma,i}^\pm)}$ times

$$K^i \times_{K^i \cap K^{\sigma\lambda}} (\det(N_{\sigma,i}^\pm) \otimes S^\bullet((N_{\sigma,i} \otimes \mathbb{C})^\pm)),$$

with

$$N_{\sigma,i}^\pm = \sum_{\substack{(\alpha, \sigma\lambda) > 0 \\ \pm(\alpha, \varpi_i) > 0}} \mathfrak{k}_\alpha,$$

and

$$(N_{\sigma,i} \otimes \mathbb{C})^\pm = \sum_{\substack{(\alpha, \sigma\lambda) \neq 0 \\ \pm(\alpha, \varpi_i) > 0}} \mathfrak{k}_\alpha.$$

Now we can apply Remark 1.3.17 with the subgroup $H \subset T$ equal to the center $Z(K^i)$ of K^i : an element $\gamma \in \Lambda^*$ belong to $\sum_{k \neq i} \mathbb{Z}\alpha_k$ if and only if $t^\gamma = 1$ for all $t \in Z(K^i)$.

The group $Z(K^i)$ acts trivially on the manifolds $K^i \cdot \sigma\lambda$, and with the characters associated to the weights

$$k\sigma\lambda + \sum_{\substack{(\alpha, \sigma\lambda) > 0 \\ (\alpha, \varpi_i) > 0}} \alpha + \delta \quad \text{with} \quad (\delta, \varpi_i) \geq 0$$

on the bundle $\mathbb{C}_{[\lambda]}^{\otimes k} \otimes [\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}_{\sigma,i}}]_{\varpi_i}^{-1}$, and with the characters associated to the weights

$$k\sigma\lambda + \sum_{\substack{(\alpha, \sigma\lambda) > 0 \\ (\alpha, \varpi_i) < 0}} \alpha + \delta \quad \text{with} \quad (\delta, \varpi_i) \leq 0$$

on the bundle $\mathbb{C}_{[\lambda]}^{\otimes k} \otimes [\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}_{\sigma,i}}]_{-\varpi_i}^{-1}$. Now the μ -multiplicity in A_σ^\pm is not equal to 0 only if

$$k\sigma\lambda + \sum_{\substack{(\alpha, \sigma\lambda) > 0 \\ \pm(\alpha, \varpi_i) > 0}} \alpha + \delta - \mu \in \sum_{k \neq i} \mathbb{Z}\alpha_k \quad \text{with} \quad \pm(\delta, \varpi_i) \geq 0. \quad (1.100)$$

Condition (1.100) implies that $[\mu]_i \geq k[\sigma\lambda]_i + s_{\sigma,i}^+$ or $[\mu]_i \leq k[\sigma\lambda]_i - s_{\sigma,i}^-$. Finally we have prove that $m_{\mathfrak{c}_+}^\lambda(\mu, k) = m_{\mathfrak{c}_-}^\lambda(\mu, k)$ if

$$-s_{\sigma,i}^- < [\mu]_i - k[\sigma\lambda]_i < s_{\sigma,i}^+$$

for all $\sigma \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$. \square

1.4.4 The case of $SU(n)$

Let T be the maximal torus of $SU(n)$ consisting of the diagonal matrices. The dual \mathfrak{t}^* can be identified with the subspace $x_1 + \cdots + x_n = 0$ of \mathbb{R}^n . The roots are $\mathfrak{R} = \{e_i - e_j \mid 1 \leq i \neq j \leq n\}$ and we will choose the positives ones to be $\mathfrak{R}^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\}$. The simple roots are then $\alpha_i = e_i - e_{i+1}$, for $1 \leq i \leq n-1$, and for these simple roots, the fundamental weights are

$$\varpi_k = \frac{1}{n} \left(\underbrace{n-k, n-k, \dots, n-k}_{k \text{ times}}, \underbrace{-k, -k, \dots, -k}_{n-k \text{ times}} \right), \quad 1 \leq k \leq n-1. \quad (1.101)$$

Consider now the coadjoint orbit O_λ for $\lambda \in \mathfrak{t}^*$. Let $\Phi : O_\lambda \rightarrow \mathfrak{t}^*$ the moment map associated to the Hamiltonian action of T on O_λ . The center of $SU(n)$, that we denote \mathbb{Z}_n corresponds to the set of matrices zI with $z^n = 1$. Recall the following well-known fact.

Lemma 1.4.12 *Let ξ be a regular value of $\Phi : O_\lambda \rightarrow \mathfrak{t}^*$. Then for every $m \in \Phi^{-1}(\xi)$ the stabilizer subgroup $T_m := \{t \in T \mid t \cdot m = m\}$ is equal to \mathbb{Z}_n .*

Proof. Since ξ is a regular value, we know that T_m is finite for every $m \in \Phi^{-1}(\xi)$. The dual of the Lie algebra $\mathfrak{su}(n)$ decomposes as $\mathfrak{su}(n)^* = \mathfrak{t}^* \oplus \sum_{\alpha \in \mathfrak{R}^+} \mathfrak{su}(n)_\alpha^*$ where $\mathfrak{su}(n)_\alpha^* \simeq \mathbb{C}_{-\alpha}$ as T -module. For $m \in \Phi^{-1}(\xi)$, we have $m = m_0 + \sum_{\alpha \in \mathfrak{R}^+} m_\alpha$ with $m_\alpha \in \mathfrak{su}(n)_\alpha^*$, and then $T_m = \bigcap_{m_\alpha \neq 0} \ker(t \mapsto t^\alpha)$. So the lattice Λ_m^* generated by the set $\{\alpha \in \mathfrak{R}^+ \mid m_\alpha \neq 0\}$ is a subgroup of $\Lambda_{\mathfrak{R}^+}^*$ with $\Lambda_{\mathfrak{R}^+}^*/\Lambda_m^*$ finite. We have to show that $\Lambda_m^* = \Lambda_{\mathfrak{R}^+}^*$. For this purpose we introduce the following equivalence relation on $\{1, \dots, n\}$:

$$i \sim j \iff e_i - e_j \in \Lambda_m^*.$$

Suppose that $\{1, \dots, n\}/\sim$ is not reduced to a point: let C_1 and C_2 be two distinct equivalent classes and let $\beta = (\beta_1, \dots, \beta_n)$ be the element of \mathfrak{t}^* defined by: $\beta_i = \frac{1}{|C_1|}$ if $i \in C_1$, $\beta_i = \frac{-1}{|C_2|}$ if $i \in C_2$, and $\beta_i = 0$ in the other cases. We see then that $(\beta, \alpha) = 0$ for all $\alpha \in \Lambda_m^*$: it is in contradiction with the fact that $\Lambda_{\mathfrak{R}^+}^*/\Lambda_m^*$ is finite. We have proved that $e_i - e_j \in \Lambda_m^*$ for all $i, j \in \{1, \dots, n\}$. \square

We are in the particularly nice situation where the symplectic reduction $(O_\lambda)_\xi = \Phi^{-1}(\xi)/T$ is a smooth manifold for any regular value ξ .

Suppose now that λ is a positive weight, and let \mathfrak{c} a connected component of regular values of $\Phi : O_\lambda \rightarrow \mathfrak{t}^*$. We know that $m_\mathfrak{c}^\lambda : \Lambda^* \times \mathbb{Z} \rightarrow \mathbb{Z}$ is supported on the sub-lattice $\Xi_\lambda := \{(\mu, k) \mid \mu \in k\lambda + \Lambda_{\mathfrak{R}^+}^*\}$.

Corollary 1.4.13 *The map $m_c^\lambda : \Xi_\lambda \rightarrow \mathbb{Z}$ is a polynomial of degree $\frac{(n-1)(n-2)}{2} - d_\lambda$, where d_λ is the number of positive roots orthogonal to λ .*

Proof. Take $\xi \in \mathfrak{c}$. Following Proposition 1.3.18, the periodic-polynomial m_c^λ is defined by $m_c^\lambda(\mu, k) = RR((O_\lambda)_\xi, \mathcal{L}_{\xi, \mu}^k)$ for all $(\mu, k) \in \Xi_\lambda$. Here $(O_\lambda)_\xi = \Phi^{-1}(\xi)/T$ is a smooth manifold, and the line bundle $\mathcal{L}_{\xi, \mu}^k = (L^{\otimes k}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu})/T$ is also smooth since the center \mathbb{Z}_n acts trivially on $L^{\otimes k}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu}$. Now the Atiyah-Singer integral formula for the Riemann-Roch number $RR((O_\lambda)_\xi, \mathcal{L}_{\xi, \mu}^k)$ shows that m_c^λ is a polynomial of degree $\frac{\dim(O_\lambda)_\xi}{2} = \frac{\dim O_\lambda}{2} - (n-1) = \frac{(n-1)(n-2)}{2} - d_\lambda$. \square

Now we rewrite Theorem 1.4.10 for the group $SU(n)$. Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$ be a positive weight and let \mathfrak{c}_+ and \mathfrak{c}_- be two adjacent connected components of regular values of $\Phi : O_\lambda \rightarrow \mathfrak{t}^*$ separated by an hyperplane orthogonal to a fundamental weight ϖ_i : the vector ϖ_i is pointing out of \mathfrak{c}_- , and let $(\varpi_i, \xi) - r_i = 0$ be the equation of this hyperplane. We consider the linear map

$$Q(\xi, t) := (\varpi_i, \xi) - tr_i.$$

The hyperplane $\{Q = 0\} \subset \mathfrak{t}^* \times \mathbb{R}$ separates $\text{Cone}(\mathfrak{c}_+)$ and $\text{Cone}(\mathfrak{c}_-)$.

The conditions $(e_k - e_l, \sigma\lambda) > 0$ and $(e_k - e_l, \varpi_i) > 0$ are respectively equivalent to $\lambda_{\sigma(k)} > \lambda_{\sigma(l)}$ and $k \leq i < l$. For $SU(n)$, the number $[\alpha]_i$ is equal to 0, 1 or -1 for any roots α and any $i = 1, \dots, n-1$. Hence for every $\sigma \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$, the integers $s_{\sigma, i}^-, s_{\sigma, i}^+ \geq 0$ introduced in Definition (1.4.9) are equal to

$$s_{\sigma, i}^+ = \text{rk}_{\mathbb{C}}(N_{\sigma, i}^+) = \#\{k \leq i < l \text{ such that } \lambda_{\sigma(k)} > \lambda_{\sigma(l)}\}, \quad (1.102)$$

$$s_{\sigma, i}^- = \text{rk}_{\mathbb{C}}(N_{\sigma, i}^-) = \#\{k \leq i < l \text{ such that } \lambda_{\sigma(k)} < \lambda_{\sigma(l)}\}, \quad (1.103)$$

and the sum $s_{\sigma, i}^+ + s_{\sigma, i}^-$ is equal to half of the codimension of $K^i \cdot \sigma\lambda$ in $K \cdot \lambda$, that is $s_{\sigma, i}^+ + s_{\sigma, i}^- = i(n-i) - \dim(K^{\sigma\lambda}/K^i \cap K^{\sigma\lambda})/2$.

Now we precise the results of [10].

Theorem 1.4.14 • *The polynomial $m_{\mathfrak{c}_-}^\lambda - m_{\mathfrak{c}_+}^\lambda : \Xi_\lambda \rightarrow \mathbb{Z}$ is divisible by the linear factors*

$$(Q - s_i^- + 1), (Q - s_i^- + 2), \dots, Q, \dots, (Q + s_i^+ - 2), (Q + s_i^+ - 1),$$

where $s_i^\pm = \inf_{[\sigma] \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)} s_{\sigma, i}^\pm$.

• *The linear factors $(Q - s_i^-)$ and $(Q - s_i^+)$ do not divide $m_{\mathfrak{c}_-}^\lambda - m_{\mathfrak{c}_+}^\lambda$.*

Proof. The first part is the translation of Theorem 1.4.10. We have just to prove that the linear factors $(Q - s_i^-)$ and $(Q - s_i^+)$ do not divide $m_{\mathfrak{c}_-}^\lambda - m_{\mathfrak{c}_+}^\lambda$. This point is a direct application of Proposition 1.3.25. The only fact we use here is that $\text{rk}_{\mathbb{C}}(N_{\sigma, i}^\pm) = s_{\sigma, i}^\pm$. So the number $\text{rk}_{\mathbb{C}}(N_{\sigma, i}^\pm)$ is constant for all $\sigma \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ for which $s_{\sigma, i}^\pm = s_i^\pm$. \square

We rewrite now Theorem 1.4.14 in the particular case where $\mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ contains just one element: it happens when λ is a “generic” positive weight (see Definition 1.4.5), or when \mathfrak{c}_+ does not intersect $\Phi(O_\lambda)$. Here a positive weight $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$ is “generic” if for every couple of permutations σ, σ' and any $k = 1, \dots, n-1$, we have

$$\sum_{i=1}^k \lambda_{\sigma(i)} \neq \sum_{i=1}^k \lambda_{\sigma'(i)}$$

when $(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}) \notin \mathfrak{S}_k \times \mathfrak{S}_{n-k}(\lambda_{\sigma'(1)}, \dots, \lambda_{\sigma'(n)})$.

Corollary 1.4.15 *Let λ be a regular weight. Let \mathfrak{c}_+ and \mathfrak{c}_- be two adjacent connected components of regular values of $\Phi : O_\lambda \rightarrow \mathfrak{t}^*$ and suppose that $\mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ contains just one element σ . Then the polynomial $m_{\mathfrak{c}_-}^\lambda - m_{\mathfrak{c}_+}^\lambda : \Xi_\lambda \rightarrow \mathbb{Z}$ is divisible by the $i(n-i)$ linear factors*

$$(Q - s_i^- + 1), (Q - s_i^- + 2), \dots, q, \dots, (Q + s_i^+ - 2), (Q + s_i^+ - 1),$$

where $s_i^\pm = s_{\sigma, i}^\pm$ are defined by (1.102) and (1.103). Moreover the linear factors $(Q - s_i^-)$ and $(Q - s_i^+)$ do not divide $m_{\mathfrak{c}_-}^\lambda - m_{\mathfrak{c}_+}^\lambda$.

1.5 Vector partition functions

Let T be a torus with Lie algebra \mathfrak{t} and let $\Lambda^* \subset \mathfrak{t}^*$ be the weight lattice. Let $R = \{\alpha_1, \dots, \alpha_d\}$ be a subset of not necessarily distinct elements of Λ^* which are in an open halfspace of \mathfrak{t}^* . We associate with the collection R a function

$$N_R : \Lambda^* \longrightarrow \mathbb{N}$$

called the *vector partition function* associated to R . By definition, for a weight μ , the value $N_R(\mu)$ is the number of solutions of the equation

$$\sum_{j=1}^d k_j \alpha_j = \mu, \quad k_j \in \mathbb{Z}^{\geq 0}, \quad j = 1, \dots, d. \quad (1.104)$$

Let $C(R) \subset \mathfrak{t}^*$ be the closed convex cone generated by the elements of R , and denote by $\Lambda_R^* \subset \Lambda^*$ the sublattice generated by R . Obviously, $N_R(\mu)$ vanishes if μ does not belong to $C(R) \cap \Lambda_R^*$.

Suppose now that R generates the vector space \mathfrak{t}^* . Following [36], we will call a vector *singular* with respect to R if it is in a cone $C(\nu)$ generated by a subset $\nu \subset R$ of cardinality strictly less than $\dim T$. The connected components of $\mathfrak{t}^* \setminus \{\text{singular vectors}\}$ are called *conic chambers*. The periodic polynomial behavior of N_R on closures of conic chambers of the cone $C(R)$ is proved in [35]. We have the following refinement due to Szenes and Vergne [36]. Let us introduce the convex polytope

$$\square(R) = \sum_{j=1}^d [0, 1] \alpha_j. \quad (1.105)$$

We remark that $\mathfrak{c} - \square(R)$ is a neighborhood of $\bar{\mathfrak{c}}$ for any conic chamber \mathfrak{c} of the cone $C(R)$. We have the following qualitative result.

Theorem 1.5.1 ([36]) *Let \mathfrak{c} be a conic chamber of the cone $C(R)$. There exists a periodic polynomial $P_{\mathfrak{c}}$ on Λ^* such that for each $\mu \in \mathfrak{c} - \square(R)$, we have*

$$N_R(\mu) = P_{\mathfrak{c}}(\mu).$$

In Section 1.5.4 we will give another proof of Theorem 1.5.1.

Let $\mathfrak{c}_{\pm} \subset \mathfrak{t}^*$ be two adjacent conic chambers separated by the hyperplane $\Delta = \{\xi \in \mathfrak{t}^* \mid \langle \xi, \beta \rangle = 0\}$. Here $\beta \in \mathfrak{t}$ is chosen so that $\mathfrak{c}_{\pm} \subset \{\xi \in \mathfrak{t}^* \mid \pm \langle \xi, \beta \rangle > 0\}$. The aim of this Section is to give a wall-crossing formula for the periodic polynomial $P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-}$.

Note that the vector space Δ is generated by $R \cap \Delta$. We polarize the elements of R that are outside Δ . We define

$$R' = \{\epsilon_j \alpha_j \mid \langle \alpha_j, \beta \rangle \neq 0 \text{ and } \epsilon_j = \text{sign } \langle \alpha_j, \beta \rangle\}, \quad (1.106)$$

$$\delta^{\pm} = \sum_{\pm \langle \alpha_j, \beta \rangle > 0} \alpha_j, \quad (1.107)$$

and

$$r^{\pm} = \#\{j \mid \pm \langle \alpha_j, \beta \rangle > 0\}. \quad (1.108)$$

We now look at the vector space Δ equipped with the subset $R \cap \Delta \subset \Lambda^* \cap \Delta$ which lie entirely in an open halfspace: let $N_{R \cap \Delta} : \Lambda^* \cap \Delta \rightarrow \mathbb{N}$ be the corresponding vector partition function. It is easy to see that $\bar{\mathfrak{c}}_+ \cap \bar{\mathfrak{c}}_-$ is contained in the closure of a conic chamber $\mathfrak{c}' \subset \Delta$ relative to $R \cap \Delta$. Following Proposition 1.5.1 there exists a periodic polynomial $P_{\mathfrak{c}'}$ on $\Lambda^* \cap \Delta$ such that for each $\mu \in \bar{\mathfrak{c}}' \cap \Lambda^*$, we have

$$N_{R \cap \Delta}(\gamma) = P_{\mathfrak{c}'}(\gamma).$$

Let $N_{R'} : \Lambda^* \rightarrow \mathbb{N}$ be the vector partition function associated to the polarized set of weight R' (see (1.106)). The main result of this Section is the following

Theorem 1.5.2 *The periodic polynomial $P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-} : \Lambda^* \rightarrow \mathbb{Z}$ satisfies*

$$P_{\mathfrak{c}_+}(\mu) - P_{\mathfrak{c}_-}(\mu) = \sum_{\gamma \in \Lambda^* \cap \Delta} D(\mu - \gamma) P_{\mathfrak{c}'}(\gamma), \quad \mu \in \Lambda^*, \quad (1.109)$$

where $D : \Lambda^* \rightarrow \mathbb{Z}$ is defined by

$$D(\mu) = (-1)^{r^-} N_{R'}(\mu + \delta^-) - (-1)^{r^+} N_{R'}(-\mu - \delta^+).$$

The proof of Theorem 1.5.2 will be given in Section 1.5.5.

Corollary 1.5.3 $P_{\mathfrak{c}_+}(\mu) = P_{\mathfrak{c}_-}(\mu)$ for all the weights $\mu \in \Lambda^*$ satisfying the condition

$$-\langle \delta^+, \beta \rangle < \langle \mu, \beta \rangle < -\langle \delta^-, \beta \rangle.$$

The former inequalities are optimal since

$$(P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-})(-\delta^- + \gamma) = (-1)^{r^-} P_{\mathfrak{c}'}(\gamma)$$

and

$$(P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-})(-\delta^+ + \gamma) = (-1)^{1+r^+} P_{\mathfrak{c}'}(\gamma)$$

for all $\gamma \in \Lambda^* \cap \Delta$.

Proof. In (1.109), the term $D(\mu - \gamma)P_{\mathfrak{c}'}(\gamma)$ does not vanish only if $\mu - \gamma \in -\delta^- + C(R')$ or $-(\mu - \gamma) \in \delta^+ + C(R')$ for some $\gamma \in C(R \cap \Delta)$. These two conditions impose respectively that $\langle \mu, \beta \rangle \geq -\langle \delta^-, \beta \rangle$ and $\langle \mu, \beta \rangle \leq -\langle \delta^+, \beta \rangle$. If one take $\mu = -\delta^- + \gamma$ with $\gamma \in \Lambda^* \cap \Delta$, (1.109) becomes $(P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-})(-\delta^- + \gamma) = \sum_{\gamma' \in \Lambda^* \cap \Delta} D(-\delta^- + \gamma - \gamma')P_{\mathfrak{c}'}(\gamma')$ with

$$D(-\delta^- + \gamma - \gamma') = (-1)^{r^-} N_{R'}(\gamma - \gamma') - (-1)^{r^+} N_{R'}(\delta^- - \delta^+ - \gamma + \gamma').$$

Since the cone $C(R')$ intersects Δ only at $\{0\}$, $N_{R'}(\gamma - \gamma') = 0$ if $\gamma \neq \gamma'$. Since $\langle \delta^- - \delta^+, \beta \rangle < 0$ we always have $N_{R'}(\delta^- - \delta^+ - \gamma + \gamma') = 0$. We get finally that $(P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-})(-\delta^- + \gamma) = (-1)^{r^-} P_{\mathfrak{c}'}(\gamma)$. One can show in the same way that $(P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-})(-\delta^+ + \gamma) = -(-1)^{r^+} P_{\mathfrak{c}'}(\gamma)$. \square

1.5.1 Quantization of \mathbb{C}^d

We consider the complex vector space \mathbb{C}^d equipped with the canonical symplectic form $\Omega = \frac{i}{2} \sum_{j=1}^d dz_j \wedge d\bar{z}_j$. The standard complex structure J on \mathbb{C}^d is compatible with Ω . Let T be a torus, let $\alpha_j \in \mathfrak{t}^*$, $j = 1, \dots, d$ be weights of T , and let T acts on \mathbb{C}^d as

$$t \cdot (z_1, \dots, z_d) = (t^{-\alpha_1} z_1, \dots, t^{-\alpha_d} z_d). \quad (1.110)$$

The action of T preserve the symplectic form Ω and the moment map associated with this action is

$$\Phi(z) = \frac{1}{2} \sum_{j=1}^d |z_j|^2 \alpha_j. \quad (1.111)$$

The pre-quantization data $(L, \langle, \rangle, \nabla)$ on the Hamiltonian T -manifold $(\mathbb{C}^d, \Omega, \Phi)$ is a trivial line bundle L with a trivial action of T equipped with the Hermitian structure $\langle s, s' \rangle_z = e^{-\frac{|z|^2}{2}} \overline{ss'}$ and the Hermitian connexion $\nabla = d - \theta$ where $\theta = \frac{1}{2} \sum_{j=1}^d \bar{z}_j dz_j$.

The quantization of the Hamiltonian T -manifold (\mathbb{C}^d, Ω) , that we denote $\mathcal{Q}^T(\mathbb{C}^d)$, is the *Bargman space* of entire holomorphic functions on \mathbb{C}^d which are \mathcal{L}^2 integrable with respect to the Gaussian measure $e^{-\frac{|z|^2}{2}} \Omega^d$.

We suppose now that the set of weights $R = \{\alpha_1, \dots, \alpha_d\}$ is *polarized* by $\eta \in \mathfrak{t}$, which means that $\langle \alpha_j, \eta \rangle > 0$ for all j . The T -representation $\mathcal{Q}^T(\mathbb{C}^d)$ is then admissible and we have the following equality in $R^{-\infty}(T)$:

$$\mathcal{Q}^T(\mathbb{C}^d) = \sum_{\mu \in \Lambda^*} N_R(\mu) \mathbb{C}_\mu, \tag{1.112}$$

where $N_R : \Lambda^* \rightarrow \mathbb{N}$ is the vector partition function associated to R . In other words, the generalized character of $\mathcal{Q}^T(\mathbb{C}^d)$ coincides with the generalized character of the symmetric algebra $S^\bullet(\overline{\mathbb{C}^d})$, where $\overline{\mathbb{C}^d}$ means \mathbb{C}^d with the opposite complex structure.

For the remaining part of Section 1.5, we assume that the set of weights $R = \{\alpha_1, \dots, \alpha_d\}$ is *polarized*, and *generates* the vector space \mathfrak{t}^* . The first assumption is equivalent to the fact that the moment map $\Phi : \mathbb{C}^d \rightarrow \mathfrak{t}^*$ is *proper*, and the second assumption is equivalent to the fact that the generic stabiliser of T on \mathbb{C}^d is *finite*. Notice that the vectors of \mathfrak{t}^* which are *singular* with respect to R correspond to the *singular values* of Φ .

In the next section we will show that $\mathcal{Q}^T(\mathbb{C}^d)$, viewed as an element of $R^{-\infty}(T)$, can be realized as the index of transversally elliptic symbols on \mathbb{C}^d . After we will apply the techniques developed in Section 1.3. The main difference here is that we work with the *non-compact* manifold \mathbb{C}^d .

1.5.2 Transversally elliptic symbols on \mathbb{C}^d

Let $p : \mathbf{T}\mathbb{C}^d \rightarrow \mathbb{C}^d$ be the canonical projection. We consider the Thom symbol

$$\text{Thom}(\mathbb{C}^d) \in \Gamma(\mathbf{T}\mathbb{C}^d, \text{hom}(p^*(\wedge_{\mathbb{C}}^{\text{even}} \mathbf{T}\mathbb{C}^d), p^*(\wedge_{\mathbb{C}}^{\text{odd}} \mathbf{T}\mathbb{C}^d)))$$

associated to the standard Hermitian structure on \mathbb{C}^d . Obviously the symbol $\text{Thom}(\mathbb{C}^d)$ is not elliptic since its characteristic set is equal to the zero section in $\mathbf{T}\mathbb{C}^d$ (hence is not compact).

Now we deform the symbol $\text{Thom}(\mathbb{C}^d)$ in order to obtain transversally elliptic symbols. Since \mathbb{C}^d can be realized as an open subset of a compact T -manifold we have a well defined index map

$$\text{Index}_{\mathbb{C}^d}^T : \mathbf{K}_T(\mathbf{T}_T \mathbb{C}^d) \longrightarrow R^{-\infty}(T).$$

Definition 1.5.4 For any $\eta \in \mathfrak{t}$, we define the symbol $\text{Thom}^\eta(\mathbb{C}^d)$ by

$$\text{Thom}^\eta(\mathbb{C}^d)(z, v) = \text{Thom}(\mathbb{C}^d)(z, v - \eta_{\mathbb{C}^d}(z)), \quad (z, v) \in \mathbf{T}\mathbb{C}^d,$$

where $\eta_{\mathbb{C}^d}$ is the vector field on \mathbb{C}^d generated by η .

The symbols $\text{Thom}^\eta(\mathbb{C}^d)$ were studied in [32]. It is easy to see that $\text{Thom}^\eta(\mathbb{C}^d)$ is transversally elliptic if and only if the vector subspace $(\mathbb{C}^d)^\eta$ is reduced to $\{0\}$, i.e. if $\langle \alpha_j, \eta \rangle \neq 0$ for all $j = 1, \dots, d$. We prove in Proposition 5.4. of [32] that

$$\text{Index}_{\mathbb{C}^d}^T(\text{Thom}^\eta(\mathbb{C}^d)) = S^\bullet(\overline{\mathbb{C}^d}) \quad \text{in } R^{-\infty}(T), \quad (1.113)$$

when $\langle \alpha_j, \eta \rangle > 0$ for all $j = 1, \dots, d$.

In order to compute the multiplicities $N_R(\mu)$ of $\mathcal{Q}^T(\mathbb{C}^d)$ we introduce the following transversally elliptic symbols. Take a scalar product $b(\cdot, \cdot)$ on \mathfrak{t}^* , and denote by $\xi \mapsto \xi^b, \mathfrak{t}^* \simeq \mathfrak{t}$ the induced isomorphism. For each $\xi \in \mathfrak{t}^*$, the Hamiltonian vector field of the function $\frac{1}{2}\|\Phi - \xi\|_b^2$ is the vector field

$$z \mapsto \left((\Phi(z) - \xi)^b \right)_{\mathbb{C}^d}(z).$$

that we denote $\mathcal{H}^b - \xi_{\mathbb{C}^d}^b$.

Definition 1.5.5 For any $\xi \in \mathfrak{t}^*$, and any scalar product $b(\cdot, \cdot)$ on \mathfrak{t}^* , we define the symbol $\text{Thom}_{\xi, b}(\mathbb{C}^d)$ by

$$\text{Thom}_{\xi, b}(\mathbb{C}^d)(z, v) = \text{Thom}(\mathbb{C}^d)(z, v - (\mathcal{H}^b - \xi_{\mathbb{C}^d}^b)(z)), \quad (z, v) \in \mathbf{T}\mathbb{C}^d.$$

Let $\text{Char}(\text{Thom}_{\xi, b}(\mathbb{C}^d)) \subset \mathbf{T}\mathbb{C}^d$ be the characteristic set of $\text{Thom}_{\xi, b}(\mathbb{C}^d)$. We know that $\text{Char}(\text{Thom}_{\xi, b}(\mathbb{C}^d)) \cap \mathbf{T}_T\mathbb{C}^d$ is equal to the critical set $\text{Cr}(\|\Phi - \xi\|_b^2)$ of the function $\|\Phi - \xi\|_b^2 : \mathbb{C}^d \rightarrow \mathbb{R}$ (see Section 1.3.2). A straightforward computation gives that $z \in \text{Cr}(\|\Phi - \xi\|_b^2)$ if and only if

$$b(\Phi(z) - \xi, \alpha_j) z_j = 0 \quad \text{for all } j = 1, \dots, d. \quad (1.114)$$

The former relations implies in particular that $b(\Phi(z) - \xi, \Phi(z)) = \frac{1}{2} \sum_j b(\Phi(z) - \xi, \alpha_j) |z_j|^2 = 0$. Hence $\|\Phi(z)\|_b^2 = b(\Phi(z), \xi)$ which implies

$$\|\Phi(z)\|_b \leq \|\xi\|_b. \quad (1.115)$$

Take now $\eta \in \mathfrak{t}$ such that $\langle \alpha_j, \eta \rangle > 0$ for all j , and let $\eta_b \in \mathfrak{t}^*$ such that $(\eta_b)^b = \eta$. We have then

$$C_\eta \|z\|^2 \leq \langle \Phi(z), \eta \rangle = b(\Phi(z), \eta_b) \leq \|\Phi(z)\|_b \|\eta_b\|_b \quad (1.116)$$

where $C_\eta = \frac{1}{2} \inf_j \langle \alpha_j, \eta \rangle$, and $z \mapsto \|z\|^2$ is the usual hermitian form on \mathbb{C}^d . With (1.114) and (1.116) we get the following

Lemma 1.5.6 The critical set $\text{Cr}(\|\Phi - \xi\|_b^2) \subset \mathbb{C}^d$ is contained in the ball of radius

$$\frac{\|\xi\|_b \|\eta_b\|_b}{C_\eta},$$

where $\eta \in \mathfrak{t}$ is such that $C_\eta = \frac{1}{2} \inf_j \langle \alpha_j, \eta \rangle > 0$.

We have then proved that the symbols $\text{Thom}_{\xi,b}(\mathbb{C}^d)$ are transversally elliptic.

Proposition 1.5.7 *The class of the transversally elliptic symbol $\text{Thom}_{\xi,b}(\mathbb{C}^d)$ in $\mathbf{K}_T(\mathbf{T}_T\mathbb{C}^d)$ does not depend of the data ξ, b , and is equal to the class defined by $\text{Thom}^\eta(\mathbb{C}^d)$ where $\eta \in \mathfrak{t}$ is chosen so that $\langle \alpha_j, \eta \rangle > 0$ for all j .*

Proof. After Lemma 1.5.6, we know that for any scalar product $b(\cdot, \cdot)$ on \mathfrak{t}^* , the characteristic set of $\text{Thom}_{0,b}(\mathbb{C}^d)$ intersects $\mathbf{T}_T\mathbb{C}^d$ at $\{0\}$. If b_0 and b_1 are two scalar products on \mathfrak{t}^* we consider the family $b_t = tb_1 + (1-t)b_0$, $0 \leq t \leq 1$, of scalar products on \mathfrak{t}^* . Hence $\text{Thom}_{0,b_t}(\mathbb{C}^d)$, $t \in [0, 1]$, defines an homotopy of transversally elliptic symbols. We have proved that $\text{Thom}_{0,b_0}(\mathbb{C}^d) = \text{Thom}_{0,b_1}(\mathbb{C}^d)$ in $\mathbf{K}_T(\mathbf{T}_T\mathbb{C}^d)$ for any $\xi \in \mathfrak{t}^*$.

Fix now the scalar product b and an element $\xi \in \mathfrak{t}^*$. For any $t \in [0, 1]$ the characteristic set of $\text{Thom}_{t\xi,b}(\mathbb{C}^d)$ intersects $\mathbf{T}_T\mathbb{C}^d$ in the ball of radius

$$\frac{\|\xi\|_b \|\eta_b\|_b}{C_\eta}.$$

Hence $\text{Thom}_{t\xi,b}(\mathbb{C}^d)$, $t \in [0, 1]$, defines an homotopy of transversally elliptic symbols: $\text{Thom}_{\xi,b}(\mathbb{C}^d) = \text{Thom}_{0,b}(\mathbb{C}^d)$ in $\mathbf{K}_T(\mathbf{T}_T\mathbb{C}^d)$. We have proved that the class of the transversally elliptic symbol $\text{Thom}_{\xi,b}(\mathbb{C}^d)$ in $\mathbf{K}_T(\mathbf{T}_T\mathbb{C}^d)$ does not depend of the data ξ, b .

Since the weights α_j lie entirely in an open halfspace of \mathfrak{t}^* , there exists a scalar product $b_+(\cdot, \cdot)$ on \mathfrak{t}^* for which we have

$$b_+(\alpha_i, \alpha_j) > 0$$

for all $i, j = 1, \dots, d$. Let \mathcal{H}^{b_+} be the Hamiltonian vector field of the function $\frac{1}{2}\|\Phi\|_{b_+}^2$, and let $\eta_{\mathbb{C}^d}$ be the vector field on \mathbb{C}^d generated by $\eta \in \mathfrak{t}$ such that $\langle \alpha_j, \eta \rangle > 0$ for all j . A straightforward computation gives that

$$(\mathcal{H}^{b_+}(z), \eta_{\mathbb{C}^d}(z)) > 0 \tag{1.117}$$

for all non zero $z \in \mathbb{C}^d$. Consider now the following family of symbols on \mathbb{C}^d

$$\sigma_t(z, v) = \text{Thom}(\mathbb{C}^d)(z, v - (t\mathcal{H}^{b_+} + (1-t)\eta_{\mathbb{C}^d})(z)), \quad (z, v) \in \mathbf{T}\mathbb{C}^d.$$

so that $\sigma_0 = \text{Thom}^\eta(\mathbb{C}^d)$ and $\sigma_1 = \text{Thom}_{0,b_+}(\mathbb{C}^d)$. The inequality (1.117) shows that $\text{Char}(\sigma_t) \cap \mathbf{T}_T\mathbb{C}^d = \{0\}$ for all $t \in [0, 1]$. Hence σ_t , $t \in [0, 1]$, defines an homotopy of transversally elliptic symbols: $\text{Thom}^\eta(\mathbb{C}^d) = \text{Thom}_{0,b_+}(\mathbb{C}^d)$ in $\mathbf{K}_T(\mathbf{T}_T\mathbb{C}^d)$. \square

For the remaining part of this paper we fix a scalar product on \mathfrak{t}^* , and we consider the family of transversally elliptic symbols $\text{Thom}_\xi(\mathbb{C}^d)$, $\xi \in \mathfrak{t}^*$ (to simplify, we do not mention the scalar product in the notation). Proposition 1.5.7 and (1.113) imply the following

Proposition 1.5.8 *For every $\xi \in \mathfrak{t}^*$, $\mathcal{Q}^T(\mathbb{C}^d)$ is equal to the generalized character $\text{Index}_{\mathbb{C}^d}^T(\text{Thom}_\xi(\mathbb{C}^d))$.*

Now we apply the techniques developed in Section 1.3 in order to compute the multiplicities of $\text{Index}_{\mathbb{C}^d}^T(\text{Thom}_\xi(\mathbb{C}^d))$.

1.5.3 Localization in a non-compact setting

Like in Section 1.3.2 we start with the

Definition 1.5.9 *For any $\xi \in \mathfrak{t}^*$ and any T -invariant relatively compact open subset $\mathcal{U} \subset \mathbb{C}^d$ we define the symbol $\text{Thom}_\xi(\mathcal{U})$ by the relation*

$$\text{Thom}_\xi(\mathcal{U})(z, v) := \text{Thom}(\mathbb{C}^d)(z, v - (\mathcal{H} - \xi_{\mathbb{C}^d})(z)) \quad (z, v) \in \mathbf{T}\mathcal{U}.$$

The symbol $\text{Thom}_\xi(\mathcal{U})$ is transversally elliptic when $\text{Cr}(\|\Phi - \xi\|^2) \cap \partial\mathcal{U} = \emptyset$ (the couple (\mathcal{U}, ξ) is called *good*) and we denote by

$$RR_{\mathcal{U}}^\xi(\mathbb{C}^d) \in R^{-\infty}(T)$$

its index. Proposition 1.3.10 is still valid here. In particular, for a good couple (\mathcal{U}, ξ) , we have $RR_{\mathcal{U}}^{\xi'}(\mathbb{C}^d) = RR_{\mathcal{U}}^\xi(\mathbb{C}^d)$ if ξ' is close enough to ξ . Consider now the decomposition

$$\text{Cr}(\|\Phi - \xi\|^2) = \bigcup_{\gamma \in \mathcal{B}_\xi} (\mathbb{C}^d)^\gamma \cap \Phi^{-1}(\gamma + \xi).$$

Here $\mathcal{B}_\xi \subset \mathfrak{t}^*$ is finite set since \mathbb{C}^d has a finite number of stabilizer. Since $0 \in (\mathbb{C}^d)^\gamma$ and $z \mapsto \langle \Phi(z), \gamma \rangle$ is constant on $(\mathbb{C}^d)^\gamma$, we have

$$(\gamma + \xi, \gamma) = 0 \tag{1.118}$$

for all $\gamma \in \mathcal{B}_\xi$.

Definition 1.5.10 *For any $\xi \in \mathfrak{t}^*$ and $\gamma \in \mathcal{B}_\xi$, we denote simply by*

$$RR_\gamma^\xi(\mathbb{C}^d) \in R^{-\infty}(T)$$

the generalized character $RR_\gamma^\xi(\mathbb{C}^d)$, where \mathcal{U} is a T -invariant relatively compact open neighborhood of $(\mathbb{C}^d)^\gamma \cap \Phi^{-1}(\gamma + \xi)$ such that $\text{Cr}(\|\Phi - \xi\|^2) \cap \bar{\mathcal{U}} = (\mathbb{C}^d)^\gamma \cap \Phi^{-1}(\gamma + \xi)$.

Since $RR_{\mathbb{C}^d}^\xi(\mathbb{C}^d)$ is equal to $\mathcal{Q}^T(\mathbb{C}^d)$ (see Proposition 1.5.8), part a) of Proposition 1.3.10 insures that we have the decomposition

$$\mathcal{Q}^T(\mathbb{C}^d) = \sum_{\gamma \in \mathcal{B}_\xi} RR_\gamma^\xi(\mathbb{C}^d).$$

Let $\mathfrak{c} \subset \mathfrak{t}^*$ be a conic chamber of the cone $C(R)$, and take ξ in \mathfrak{c} . Then ξ is a regular value of the moment map $\Phi : \mathbb{C}^d \rightarrow \mathfrak{t}^*$ defined in (1.111). Let Ω_ξ be the symplectic structure on the orbifold $(\mathbb{C}^d)_\xi = \Phi^{-1}(\xi)/T$ that is induced

from Ω . The orbifold $(\mathbb{C}^d)_\xi$ is also equipped with a *complex structure* J_ξ that is induced from the standard complex structure on \mathbb{C}^d , in such a way that the orbifold $((\mathbb{C}^d)_\xi, \Omega_\xi, J_\xi)$ is a Kähler orbifold. If ξ belongs to the lattice Λ^* , the reduced space $(\mathbb{C}^d)_\xi$ is the Kähler toric variety corresponding to the polytope $\{s \in (\mathbb{R}^{\geq 0})^d \mid \sum s_j \alpha_j = \xi\}$ of \mathbb{R}^d . For every $\mu \in \Lambda$ we consider the holomorphic orbifold line bundle

$$\mathcal{L}_{\xi, \mu} = (\Phi^{-1}(\xi) \times \mathbb{C}_{-\mu})/T$$

on $(\mathbb{C}^d)_\xi$.

Definition 1.5.11 *The periodic polynomial $P_{\mathfrak{c}} : \Lambda^* \rightarrow \mathbb{Z}$ associated to the conic chamber \mathfrak{c} is given by*

$$P_{\mathfrak{c}}(\mu) = RR((\mathbb{C}^d)_\xi, \mathcal{L}_{\xi, \mu}), \quad (1.119)$$

where the right hand side is the Riemann-Roch number associated to the holomorphic orbifold line bundle $\mathcal{L}_{\xi, \mu}$.

Another way to define the periodic polynomial $P_{\mathfrak{c}}$ is to consider the generalized character $RR_0^\xi(\mathbb{C}^d)$ for $\xi \in \mathfrak{c}$: here $\gamma = 0$ parametrizes the component $\Phi^{-1}(\xi)$ of $\text{Cr}(\|\Phi - \xi\|^2)$. Following (1.46) we have

$$RR_0^\xi(\mathbb{C}^d) = \sum_{\mu \in \Lambda^*} P_{\mathfrak{c}}(\mu) \mathbb{C}_\mu \quad \text{in } R^{-\infty}(T). \quad (1.120)$$

After Lemma 1.3.16, we know that $RR_0^{\xi'}(\mathbb{C}^d) = RR_0^\xi(\mathbb{C}^d)$ when ξ, ξ' are two elements of \mathfrak{c} : hence the polynomial $P_{\mathfrak{c}}$ does not depend of the choice of ξ in \mathfrak{c} .

1.5.4 Proof of Theorem 1.5.1

Consider a weight $\mu \in (\mathfrak{c} - \square(R)) \cap \Lambda^*$ of the form $\mu = \xi' - \sum_j t_j \alpha_j$ with $\xi' \in \mathfrak{c}$ and $t_j \in [0, 1]$. We start with the decomposition

$$\mathcal{Q}^T(\mathbb{C}^d) = \sum_{\gamma \in \mathcal{B}_{\xi'}} RR_\gamma^{\xi'}(\mathbb{C}^d).$$

Since $N_R(\mu)$ and $P_{\mathfrak{c}}(\mu)$ are respectively the multiplicity of \mathbb{C}_μ in $\mathcal{Q}^T(\mathbb{C}^d)$ and in $RR_0^{\xi'}(\mathbb{C}^d)$, the proof will be complete if we show that the multiplicity of \mathbb{C}_μ in $RR_\gamma^{\xi'}(\mathbb{C}^d)$ is equal to zero when $\gamma \neq 0$.

Consider a *non-zero* element γ in $\mathcal{B}_{\xi'}$. For the character $RR_\gamma^{\xi'}(\mathbb{C}^d)$ the localization (1.44) gives

$$RR_\gamma^{\xi'}(\mathbb{C}^d) = RR_0^{\xi' + \gamma}((\mathbb{C}^d)^\gamma) \otimes [\wedge_{\mathbb{C}}^{\bullet} \overline{N}]_\gamma^{-1}, \quad (1.121)$$

where $N = \sum_{(\alpha_j, \gamma) \neq 0} \mathbb{C}_{-\alpha_j}$ corresponds to the normal bundle of $(\mathbb{C}^d)^\gamma$ in \mathbb{C}^d . The inverse $[\wedge_{\mathbb{C}}^\bullet \bar{N}]_\gamma^{-1}$ is equal to $(-1)^l \mathbb{C}_{\delta(\gamma)} \otimes S^\bullet(N_{\mathbb{C}}^{+, \gamma})$ where

$$\delta(\gamma) = - \sum_{(\alpha_j, \gamma) < 0} \alpha_j.$$

Since γ acts trivially on $(\mathbb{C}^d)^\gamma$ all the weights $\mu' \in \Lambda^*$ that appear in $RR_0^{\xi' + \gamma}((\mathbb{C}^d)^\gamma)$ satisfy $(\mu', \gamma) = 0$. Since the weights of $N_{\mathbb{C}}^{+, \gamma}$ are polarized by γ , we see from (1.121) that all the weights $\mu' \in \Lambda^*$ that appear in $RR_\gamma^{\xi'}(\mathbb{C}^d)$ must satisfy

$$(\mu', \gamma) \geq (\delta(\gamma), \gamma). \quad (1.122)$$

Consider now the weight $\mu = \xi' - \sum_j t_j \alpha_j$. Since $\xi' \in \mathfrak{c}$, the equality (1.118) implies $(\xi', \gamma) < 0$ and then

$$(\mu, \gamma) = \underbrace{(\xi', \gamma)}_{< 0} + \underbrace{\sum_{(\alpha_j, \gamma) > 0} -t_j(\alpha_j, \gamma)}_{\leq 0} - \sum_{(\alpha_j, \gamma) < 0} t_j(\alpha_j, \gamma) < - \sum_{(\alpha_j, \gamma) < 0} (\alpha_j, \gamma).$$

So we have proved that $(\mu, \gamma) < (\delta(\gamma), \gamma)$, hence the multiplicity of \mathbb{C}_μ in $RR_\gamma^{\xi'}(\mathbb{C}^d)$ is equal to zero. \square

1.5.5 Proof of Theorem 1.5.2

Let \mathfrak{c}_\pm be two adjacent conic chambers separated by the hyperplane $\Delta = \{\xi \in \mathfrak{t}^* \mid \langle \xi, \beta \rangle = 0\}$. Here β is pointing out \mathfrak{c}_- .

We consider two points $\xi_\pm \in \mathfrak{c}_\pm$ such that $\xi = \frac{1}{2}(\xi^+ + \xi^-) \in \Delta$ belongs to the conic chamber \mathfrak{c}' . We suppose also that the orthogonal projection of ξ_\pm on Δ are equal to ξ . We know that $P_{\mathfrak{c}_+}(\mu) - P_{\mathfrak{c}_-}(\mu)$ is equal to the μ -multiplicity of $RR_0^{\xi^+}(\mathbb{C}^d) - RR_0^{\xi^-}(\mathbb{C}^d)$. Proposition 1.3.23 tells us that

$$RR_0^{\xi^+}(\mathbb{C}^d) - RR_0^{\xi^-}(\mathbb{C}^d) = RR_\gamma^{\xi^-}(\mathbb{C}^d) - RR_{-\gamma}^{\xi^+}(\mathbb{C}^d),$$

where $\gamma \in \mathbb{R}^{>0}\beta$ is such that $\xi_- + \gamma = \xi_+ - \gamma = \xi$. The localization (1.44) gives then

$$RR_\gamma^{\xi^-}(\mathbb{C}^d) - RR_{-\gamma}^{\xi^+}(\mathbb{C}^d) = RR_0^\xi((\mathbb{C}^d)^\beta) \otimes \left([\wedge_{\mathbb{C}}^\bullet \bar{N}]_\beta^{-1} - [\wedge_{\mathbb{C}}^\bullet \bar{N}]_{-\beta}^{-1} \right). \quad (1.123)$$

Let $P_{\mathfrak{c}'} : \Lambda^* \cap \Delta \rightarrow \mathbb{Z}$ be the periodic polynomial map which coincides with the vector partition function $N_{R \cap \Delta}$ on $\mathfrak{c}' \cap \Lambda^*$. If we work with the vector space $(\mathbb{C}^d)^\beta$ equipped with the hamiltonian action of T/T_Δ , (1.46) gives the following equality in $R^{-\infty}(T/T_\Delta)$

$$RR_0^\xi((\mathbb{C}^d)^\beta) = \sum_{\gamma \in \Lambda^* \cap \Delta} P_{\mathfrak{c}'}(\gamma) \mathbb{C}_\gamma. \quad (1.124)$$

A straightforward computation gives

$$[\wedge_{\mathbb{C}}^{\bullet} \overline{N}]_{\beta}^{-1} = (-1)^{r^{-}} \sum_{\mu \in \Lambda^*} N_{R'}(\mu + \delta^{-}) \mathbb{C}_{\mu} \tag{1.125}$$

and

$$[\wedge_{\mathbb{C}}^{\bullet} \overline{N}]_{-\beta}^{-1} = (-1)^{r^{+}} \sum_{\mu \in \Lambda^*} N_{-R'}(\mu + \delta^{+}) \mathbb{C}_{\mu}, \tag{1.126}$$

where $r^{\pm}, \delta^{\pm}, R'$ are defined in (1.106), (1.107) and (1.108). Since $N_{-R'}(\mu) = N_{R'}(-\mu)$, the equations (1.124), (1.125) and (1.126) show that the right hand side of (1.123) is equal to

$$\sum_{\mu \in \Lambda^*} \sum_{\gamma \in \Lambda^* \cap \Delta} D(\mu) P_{c'}(\gamma) \mathbb{C}_{\mu+\gamma}$$

with $D(\mu) = (-1)^{r^{-}} N_{R'}(\mu + \delta^{-}) - (-1)^{r^{+}} N_{R'}(-\mu - \delta^{+})$. Finally we have proved that $P_{c_+}(\mu) - P_{c_-}(\mu) = \sum_{\gamma \in \Lambda^* \cap \Delta} D(\mu - \gamma) P_{c'}(\gamma)$. \square

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