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Summary. In this article, we study the local invariants associated to the Hamiltonian action of a compact torus. Our main results are wall-crossing formulas between invariants attached to adjacent connected components of regular values of the moment map.

Key words: moment map, equivariant cohomology, geometric quantization, transversally elliptic.

1.1 Introduction

Let (M,Ω) be a compact symplectic manifold with the Hamiltonian action of a compact torus T, and moment map $\Phi : M \to \mathfrak{t}^*$. Let us assume that the action is *effective*. We are interested here in two global invariants:

1. the Duistermaat-Heckman measure $\mathrm{DH}(M)$ which is the pushforward by Φ of the Liouville volume form,
2. the Riemann-Roch characters $RR(M,L^{\otimes k}), k \geq 1$, which are virtual representations of T. Here the data (M,Ω,Φ) is prequantized by a Kostant-Souriau line bundle L.

Let $\Lambda^* \subset \mathfrak{t}^*$ be the weight lattice of T. For every couple $(\mu,k) \in \Lambda^* \times \mathbb{Z}^{>0}$, we denote by $\mathrm{m}(\mu,k) \in \mathbb{Z}$ the multiplicity of the weight μ in $RR(M,L^{\otimes k})$.

One stricking property of the moment map is that its image $\Phi(M)$ is a convex polytope in $\mathfrak{t}^*$. In fact, as noted for example in [17] or [20], each component of the set of regular values of Φ is either an open convex polytope contained in $\Phi(M)$, or the open subset $\mathfrak{c}_{\mathrm{ext}} = \mathfrak{t}^* \setminus \Phi(M)$.

Let us fix a connected component $\mathfrak{c}$ of regular values of Φ. A celebrated Theorem of Duistermaat and Heckman [15] tells us that the measure $\mathrm{DH}(M)$ is equal to a *polynomial* $\mathrm{DH}_{\mathfrak{c}}$ times a Lebesgue measure on the open subset $\mathfrak{c}$. Note that $\mathrm{DH}_{\mathfrak{c}_{\mathrm{ext}}}$ is the zero polynomial.

The "quantization commutes with reduction" Theorem [28, 29] shows that there exists a *periodic polynomial* $\mathrm{m}_{\mathfrak{c}} : \Lambda^* \times \mathbb{Z} \to \mathbb{Z}$ which coincides with the multiplicity map $\mathrm{m} : \Lambda^* \times \mathbb{Z}^{>0} \to \mathbb{Z}$ on the cone of $\mathfrak{t}^* \times \mathbb{R}$ generated by $\mathfrak{c} \times \{1\}$. The periodic polynomial $\mathrm{m}_{\mathfrak{c}}$ is defined by a Kawasaki-Riemann-Roch formula on a symplectic quotient $\mathcal{M}_a = \Phi^{-1}(a)/T$ where $a \in \mathfrak{c}$. As a corollary, we get that $\mathrm{DH}_{\mathfrak{c}}$ is the semi-classical limit of $\mathrm{m}_{\mathfrak{c}}$: one has

$$\lim_{k\to\infty} \frac{\mathrm{m}_{\mathfrak{c}}(k\mu, k)}{k^d} = \frac{1}{(2\pi)^d} \mathrm{DH}_{\mathfrak{c}}(\mu) \tag{1.1}$$

for every $\mu \in \Lambda^*$. Here $d = \frac{1}{2} \dim \mathcal{M}_a$.

We have seen that the *global* invariants $\mathrm{DH}(M)$, $RR(M, L^{\otimes k}), k \geq 1$ gives rises to a family of *local* invariants $\mathrm{DH}_{\mathfrak{c}}$, $\mathrm{m}_{\mathfrak{c}}$, where $\mathfrak{c}$ runs over the connected component of regular values of Φ.

This paper is concerned by the differences $\mathrm{DH}_{\mathfrak{c}_+} - \mathrm{DH}_{\mathfrak{c}_-}$ and $\mathrm{m}_{\mathfrak{c}_+} - \mathrm{m}_{\mathfrak{c}_-}$ when $\mathfrak{c}_\pm$ are two *adjacent* connected components of regular values of Φ. Let $\Delta \subset \mathfrak{t}^*$ be the hyperplane that separates $\mathfrak{c}_\pm$. Some continuity properties are known:

1. the polynomial $\mathrm{DH}_{\mathfrak{c}_+} - \mathrm{DH}_{\mathfrak{c}_-}$ is divisible by a certain power of the equation the hyperplane Δ (see [17] and [12]),
2. the periodic polynomial $\mathrm{m}_{\mathfrak{c}_+} - \mathrm{m}_{\mathfrak{c}_-}$ vanishes on

$$\{(\mu, k) \in \Lambda^* \times \mathbb{Z} \,|\, \mu \in k\Delta\}. \tag{1.2}$$

See [29].

In this paper, we compute explicitely the difference $\mathrm{DH}_{\mathfrak{c}_+} - \mathrm{DH}_{\mathfrak{c}_-}$, and we show that $\mathrm{m}_{\mathfrak{c}_+} - \mathrm{m}_{\mathfrak{c}_-}$ vanishes also on some translates of (1.2).

Let us introduce some notations. We denote by $T_\Delta \subset T$ the subtorus of dimension 1 that has for Lie algebra the one dimensional subspace $\mathfrak{t}_\Delta$ which is orthogonal to the direction of Δ. Let $\beta \in \mathfrak{t}_\Delta$ be the primitive element of the lattice $\ker(\exp : \mathfrak{t} \to T)$ which is pointing out of $\mathfrak{c}_-$.

We make the choice of a decomposition $T = T/T_\Delta \times T_\Delta$, where T/T_Δ denotes a subtorus de T. At the level of Lie algebras, we have then $\mathfrak{t} = (\mathfrak{t}/\mathfrak{t}_\Delta) \oplus \mathfrak{t}_\Delta$ and $\mathfrak{t}^* = (\mathfrak{t}/\mathfrak{t}_\Delta)^* \oplus \mathfrak{t}_\Delta^*$: hence $\xi + (\mathfrak{t}/\mathfrak{t}_\Delta)^* = \Delta$ for any $\xi \in \Delta$. We denote $\mathcal{S}(\mathfrak{t})$ the algebra of polynomials on the vector space $\mathfrak{t}^*$. We will consider the polynomial $\mathrm{DH}_{\mathfrak{c}_+} - \mathrm{DH}_{\mathfrak{c}_-} \in \mathcal{S}(\mathfrak{t})$ relatively to the decomposition

$$\mathcal{S}(\mathfrak{t}) = \bigoplus_{j\in\mathbb{N}} \mathcal{S}(\mathfrak{t}/\mathfrak{t}_\Delta)\beta^j.$$

Let us choose $\xi \in \Delta$ in the relative interior of $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-}$ in Δ. We consider the family $\mathcal{F}$ of connected components $Z \subset M^{T_\Delta}$ such that $\xi \in \Phi(Z) \subset \Delta$. It is easy to see that $\mathcal{F}$ does not depend of the choice of ξ : we have $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-} \subset \Phi(Z)$ for all $Z \in \mathcal{F}$. For each $Z \in \mathcal{F}$, we denote

$$\Phi_Z : Z \to (\mathfrak{t}/\mathfrak{t}_\Delta)^*$$

the restriction of the map $\Phi - \xi$ to the symplectic sub-manifold Z. The map Φ_Z is a moment map relative to the Hamiltonian action of T/T_Δ on Z. Let $\mathrm{DH}(Z)$ be Duistermaat-Heckman measure on $(\mathfrak{t}/\mathfrak{t}_\Delta)^*$ associated to the moment map Φ_Z. Since 0 is a regular value of Φ_Z, we may consider the Duistermaat-Heckman polynomial

$$\mathrm{DH}_0(Z) \in \mathcal{S}(\mathfrak{t}/\mathfrak{t}_\Delta)$$

such that $\mathrm{DH}(Z)(a') = \mathrm{DH}_0(Z)(a')da'$ for a' in a neighborhood of 0 in $(\mathfrak{t}/\mathfrak{t}_\Delta)^*$.

For $Z \in \mathcal{F}$, we consider the symplectic reduction

$$\mathcal{Z}_\xi = \Phi_Z^{-1}(0)/(T/T_\Delta),$$

and the normal bundle N_Z of Z in M. Let $2d_Z$ be the dimension of $\mathcal{Z}_\xi$ and $2r_Z$ be the (real) rank of N_Z. We prove in Section 1.2 the following.

Theorem A. *We have*

$$(\mathrm{DH}_{\mathfrak{c}_+} - \mathrm{DH}_{\mathfrak{c}_-})(a) = \sum_{Z \in \mathcal{F}} \mathbf{D}_Z(a - \xi), \quad a \in \mathfrak{t}^*$$

where each polynomial $\mathbf{D}_Z \in \mathcal{S}(\mathfrak{t})$ *admits the following decomposition*

$$\mathbf{D}_Z = \frac{\beta^{r_Z - 1}}{\det_Z^{1/2}(\frac{-\mathcal{L}_\beta}{2\pi})} \left(\frac{\mathrm{DH}_0(Z)}{(r_Z - 1)!} + \sum_{k=1}^{d_Z} \beta^k \mathrm{Q}_{Z,k} \right).$$

Each polynomial $\mathrm{Q}_{Z,k}$ *belongs to* $\mathcal{S}(\mathfrak{t}/\mathfrak{t}_\Delta)$ *and is of degree less than* $d_Z - k$. *The term* $\det_Z^{1/2}(\frac{-\mathcal{L}_\beta}{2\pi}) \in \mathbb{Z}$ *is the Pfaffian of the infinitesimal action of* $\frac{-\beta}{2\pi}$ *on the fibers of the normal bundle* N_Z.

Theorem **A** generalizes previous results of Guillemin-Lerman-Sternberg [17] and Brion-Procesi [12]. In Section 1.2.4 we give the precise definition of the polynomials $\mathrm{Q}_{Z,k}$.

Suppose now that M is prequantized by a Kostant-Souriau line bundle L. The hyperplane Δ is defined by the equation

$$\frac{\langle a, \beta \rangle}{2\pi} - r_\Delta = 0, \quad a \in \mathfrak{t}^*, \tag{1.3}$$

for some $r_\Delta \in \mathbb{Z}$. The bundle N_Z decomposes as the sum of two polarized sub-bundles $N_Z^{\pm,\beta}$. Let $s_Z^\pm \in \mathbb{N}$ be the absolute value of the trace of $\frac{1}{2\pi}\mathcal{L}_\beta$ on $N_Z^{\pm,\beta}$. Note that the integer $s_Z^+ + s_Z^-$ is larger than half of the codimension of Z in M.

We prove in Section 1.3.5 the following

Theorem B *Let* $s^{\pm} := \inf_{Z\in\mathcal{F}} s^{\pm}_Z$. *We have* $\mathrm{m}_{\mathfrak{c}_+}(\mu,k) = \mathrm{m}_{\mathfrak{c}_-}(\mu,k)$ *when*

$$-s^- < \frac{\langle \mu,\beta\rangle}{2\pi} - kr_{\Delta} < s^+. \tag{1.4}$$

Note that the symplectic orbifolds $\mathcal{Z}_\xi, Z\in\mathcal{F}$ are the connected component of the symplectic reduction

$$\mathcal{M}^{\Delta}_{\xi} := \left(\Phi^{-1}(\xi)\cap M^{T_\Delta}\right)/(T/T_\Delta).$$

We have the following refinement au Theorem **B**.

Theorem C *If* $\mathcal{M}^{\Delta}_{\xi}$ *is connected, the inequalities (1.4) are optimal, i.e. there exists* (μ,k) *such that* $\frac{\langle\mu,\beta\rangle}{2\pi} - kr_\Delta = \pm s^{\pm}$ *and* $\mathrm{m}_{\mathfrak{c}_+}(\mu,k) \neq \mathrm{m}_{\mathfrak{c}_-}(\mu,k)$.

In Section 1.4 we apply Theorem **B** to the particular cases where M is a integral coadjoint orbit of a compact Lie group G. In Section 1.4.4, we study more precisely the case $G=\mathrm{SU}(n)$: here our result precises some of the results of Billey-Guillemin-Rassart [10].

In Section 1.5, we obtain a strong version of Theorem **B** in the case of an action of a torus T on a complex vector spaces $\mathbb{C}^d$. The quantization of this action is in some sense the vector space $\mathrm{Pol}(\mathbb{C}^d)$ of complex polynomials on $\mathbb{C}^d$. The T-multiplicities of $\mathrm{Pol}(\mathbb{C}^d)$ are given by a partition function $N_R : \Lambda^*\to\mathbb{N}$. It was observed in [13, 35] that there exists a finite decomposition of the vector space $\mathfrak{t}^*$ in *conic chambers* such that N_R is periodic polynomial on each piece.

Let $\mathfrak{c}_\pm$ be two adjacents chambers, and let $P_{\mathfrak{c}\pm}$ be the corresponding periodic polynomials computing N_R on each chambers. The main result of Section 1.5 is the formula (1.109) which depicts the periodic polynomial $P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-}$ as a convolution of distributions. Recently[1], Boyal-Vergne [11] and De Concini-Procesi-Vergne [14] proposed differents proofs of this formula.

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Notations

Throughout the paper T will denote a compact, connected abelian Lie group, and $\mathfrak{t}$ its Lie algebra. The integral lattice $\Lambda\subset\mathfrak{t}$ is defined as the kernel of $\exp:\mathfrak{t}\to T$, and the real weight lattice $\Lambda^*\subset\mathfrak{t}^*$ is defined by : $\Lambda^* := \hom(\Lambda, 2\pi\mathbb{Z})$. Every $\mu\in\Lambda^*$ defines a 1-dimensional T-representation, denoted by $\mathbb{C}_\mu$, where $t=\exp X$ acts by $t^\mu := e^{i\langle\mu,X\rangle}$. We denote by $R(T)$ the ring of characters of finite-dimensional T-representations. We denote by $R^{-\infty}(T)$ the set of generalized characters of T. An element $\chi\in R^{-\infty}(T)$ is of

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the form $\chi = \sum_{\mu\in\Lambda^*} a_\mu \mathbb{C}_\mu$, where $\mu \mapsto a_\mu, \Lambda^* \to \mathbb{Z}$ has at most polynomial growth.

The symplectic manifolds are oriented by their Liouville volume forms. If (Z, o_Z) is an oriented submanifold of an oriented manifold (M, o_M), we take on the fibers of the normal bundle N of Z in M, the orientation o_N satisfying $o_M = o_Z \cdot o_N$.

1.2 Duistermaat-Heckman measures

Let (M, Ω) be a symplectic manifold of dimension $2n$ equipped with an Hamiltonian action of a torus T, with Lie algebra $\mathfrak{t}$. The moment map $\Phi : M \to \mathfrak{t}^*$ satisfies the relations $\Omega(X_M, -) + d\langle\Phi, X\rangle = 0$, $X \in \mathfrak{t}$. We assume in this section that Φ is *proper*, and that the generic stabiliser Γ_M of T on M is *finite*.

The Duistermaat-Heckman measure $\mathrm{DH}(M)$ is defined as the pushforward by Φ of the Liouville volume form $\frac{\Omega^n}{n!}$ on M. For every $f \in \mathcal{C}^\infty(\mathfrak{t}^*)$ with compact support one has $\int_{\mathfrak{t}^*} \mathrm{DH}(M)(a) f(a) = \int_M f(\Phi)\frac{\Omega^n}{n!}$. In other terms

$$\mathrm{DH}(M)(a) = \int_M \delta(a - \Phi)\frac{\Omega^n}{n!}, \quad a \in \mathfrak{t}^*.$$

We can define $\mathrm{DH}(M)$ in terms of equivariant forms as follows. Let $\mathcal{A}(M)$ be the space of differential forms on M with complex coefficients. We denote by $\mathcal{A}^{-\infty}_{temp}(\mathfrak{t}, M)$ the space of tempered generalized functions over $\mathfrak{t}$ with values in $\mathcal{A}(M)$, and by $\mathcal{M}^{-\infty}_{temp}(\mathfrak{t}^*, M)$ the space of tempered distributions over $\mathfrak{t}^*$ with values in $\mathcal{A}(M)$. Let $\mathcal{F} : \mathcal{A}^{-\infty}_{temp}(\mathfrak{t}, M) \to \mathcal{M}^{-\infty}_{temp}(\mathfrak{t}^*, M)$ be the Fourier transform normalized by the condition that $\mathcal{F}(X \mapsto e^{i\langle\xi, X\rangle})$ is equal to the Dirac distribution $a \mapsto \delta(a - \xi)$.

Let $\Omega_\mathfrak{t}(X) = \Omega - \langle\Phi, X\rangle$ be the equivariant symplectic form. We have then $\mathcal{F}(e^{-i\Omega_\mathfrak{t}}) = e^{-i\Omega}\delta(a - \Phi)$ and so

$$\mathrm{DH}(M) = (i)^n \int_M \mathcal{F}(e^{-i\Omega_\mathfrak{t}}). \tag{1.5}$$

1.2.1 Equivariant cohomology and localization

We first recall the Cartan model of equivariant cohomology with polynomial coefficients and the extension to generalized coefficients defined by Kumar and Vergne [26]. We give after a brief account to the method of localization developped in [30, 31],

Let M be a manifold provided with an action of a compact connected Lie group K with Lie algebra $\mathfrak{k}$. Let $d : \mathcal{A}(M) \to \mathcal{A}(M)$ be the exterior differentiation. Let $\mathcal{A}_c(M)$ be the sub-algebra of compactly supported differential forms. If V is a vector field on M we denote by $c(V) : \mathcal{A}(M) \to \mathcal{A}(M)$ the

contraction by V. The action of K on M gives a morphism $X \to X_M$ from $\mathfrak{k}$ to the Lie algebra of vector fields on M.

We consider the space of K-equivariant maps $\mathfrak{k} \to \mathcal{A}(M),\ X \mapsto \eta(X)$, equipped with the derivation $(D\eta)(X) := (d - c(X_M))(\eta(X)),\ X \in \mathfrak{k}$. Since $D^2 = 0$, one can define the cohomology space $\ker D/\mathrm{Im}D$. The Cartan model [7, 21] considers *polynomial* maps and the associated cohomology is denoted $\mathcal{H}_K^*(M)$. Kumar and Vergne [26] studied the cohomology spaces $\mathcal{H}_K^{\pm\infty}(M)$ obtained by taking $\mathcal{C}^{\pm\infty}$ maps. Recall the construction $\mathcal{H}_K^{-\infty}(M)$.

The space $\mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}(M))$ of generalized functions on $\mathfrak{k}$ with values in the space $\mathcal{A}(M)$ is, by definition, the space $\mathrm{Hom}(m_c(\mathfrak{k}), \mathcal{A}(M))$ of continuous $\mathbb{C}$-linear maps from the space $m_c(\mathfrak{k})$ of smooth compactly supported densities on $\mathfrak{k}$ to the space $\mathcal{A}(M)$, both endowed with the $\mathcal{C}^\infty$-topologies. We define $\mathcal{A}_K^{-\infty}(M) := \mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}(M))^K$ as the space of K-equivariant $\mathcal{C}^{-\infty}$-maps from $\mathfrak{k}$ to $\mathcal{A}(M)$. The differential D defined on $\mathcal{C}^\infty(\mathfrak{k}, \mathcal{A}(M))$ admits a natural extension to $\mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}(M))$ and $D^2 = 0$ on $\mathcal{A}_K^{-\infty}(M)$ [26]. The cohomology associated to $(\mathcal{A}_K^{-\infty}(M), D)$ is called the K-equivariant cohomology with generalized coefficients and is denoted by $\mathcal{H}_K^{-\infty}(M)$. The subspace $\mathcal{A}_{K,c}^{-\infty}(M) := \mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}_c(M))^K$ is stable under the differential D, and we denote by $\mathcal{H}_{K,c}^{-\infty}(M)$ the associated cohomology. When M is oriented, the integration over M gives rise to a map $\int_M : \mathcal{H}_{K,c}^{-\infty}(M) \to \mathcal{C}^{-\infty}(\mathfrak{k})^K$.

Localization procedure. Let λ be a K-invariant 1-form on M and let

$$\Phi_\lambda : M \to \mathfrak{k}^* \tag{1.6}$$

be the K-equivariant map defined by $\langle \Phi_\lambda(m), X\rangle = \lambda(X_M)_m$: then $D\lambda(X) = d\lambda - \langle \Phi_\lambda, X\rangle$. The localization procedure developped in [30, 31] is based on the existence of an inverse $[D\lambda]^{-1}$ of the K-equivariant form $D\lambda$. It is an equivariantly closed element of $\mathcal{A}_K^{-\infty}(M - \Phi_\lambda^{-1}(0))$ defined by the integral

$$[D\lambda]^{-1}(X) = i\int_0^\infty e^{-i\,t\,D\lambda(X)} dt. \tag{1.7}$$

An open subset $\mathcal{U} \subset M$ is called *adapted to* λ if $\mathcal{U}$ is K-invariant and if $(\partial\mathcal{U}) \cap \Phi_\lambda^{-1}(0) = \emptyset$. In [31], we associate to an open subset $\mathcal{U}$ adapted to λ, the following equivariantly closed form with generalized coefficients

$$\mathrm{P}_\lambda^{\mathcal{U}} = \chi^{\mathcal{U}} + d\chi^{\mathcal{U}}[D\lambda]^{-1}\lambda\ . \tag{1.8}$$

Here $\chi^{\mathcal{U}} \in \mathcal{C}^\infty(M)$ is a K-invariant function supported in $\mathcal{U}$ which is equal to 1 in a neighborhood of $\mathcal{U} \cap \Phi_\lambda^{-1}(0)$. The cohomology class defined by $\mathrm{P}_\lambda^{\mathcal{U}}$ in $\mathcal{H}_K^{-\infty}(M)$ does not depend of $\chi^{\mathcal{U}}$. In particular $\mathrm{P}_\lambda^{\mathcal{U}} = 0$ in $\mathcal{H}_K^{-\infty}(M)$ if $\mathcal{U} \cap \Phi_\lambda^{-1}(0) = \emptyset$. If $\mathcal{U} \cap \Phi_\lambda^{-1}(0)$ is compact, we take $\chi^{\mathcal{U}}$ with compact support, then $\mathrm{P}_\lambda^{\mathcal{U}}$ defines a cohomology class in $\mathcal{H}_{K,c}^{-\infty}(M)$.

1.2.2 Localization of DH(M)

We come back to the situation of a Hamiltonian action of a torus T on a symplectic manifold (M,ω). We need two auxilliary data : a T-invariant Riemannian metric on M denoted $(\cdot,\cdot)_M$, and a scalar product $(\cdot,\cdot)$ on $\mathfrak{t}^*$ which induces an identification $\mathfrak{t}^* \simeq \mathfrak{t}$.

Let $\mathcal{H}$ be the Hamiltonian vector field of the function $\frac{-1}{2}\|\Phi\|^2 : M \to \mathbb{R}$: for $m \in M$ we have $\mathcal{H}_m = (\Phi(m))_M|_m$. Then for every $\xi \in \mathfrak{t}^*$, the Hamiltonian vector field of $\frac{-1}{2}\|\Phi-\xi\|^2$ is $\mathcal{H}-\xi_M$, and we consider the following T-invariant 1-form

$$\lambda_\xi = (\mathcal{H} - \xi_M, \cdot)_M \tag{1.9}$$

with corresponding map $\Phi_{\lambda_\xi} : M \to \mathfrak{t}^*$ (see (1.6)). Here $\Phi_{\lambda_\xi}^{-1}(0)$ coincides with the subset $\mathrm{Cr}(\|\Phi-\xi\|^2) \subset M$ of critical points of the function $\|\Phi-\xi\|^2$, and $m \in \mathrm{Cr}(\|\Phi-\xi\|^2)$ if and only if $(\Phi(m)-\xi)_M$ vanishes at m [30, 31].

Definition 1.2.1 *Let $\mathrm{P}_\xi \in \mathcal{H}_{T,c}^{-\infty}(M)$ be the cohomology class defined by $\mathrm{P}_{\lambda_\xi}^{\mathcal{U}}$, where $\mathcal{U}$ is a T-invariant relatively compact neighborhood of $\Phi^{-1}(\xi)$ such that $\overline{\mathcal{U}} \cap \mathrm{Cr}(\|\Phi-\xi\|^2) = \Phi^{-1}(\xi)$.*

The cohomology class P_ξ will be used to localized the Duitermaat-Heckman measure. For every $\xi \in \mathfrak{t}^*$, we define the distribution $\mathrm{DH}_\xi(M)$ by

$$\mathrm{DH}_\xi(M) = (i)^n \mathcal{F}\left(\int_M \mathrm{P}_\xi e^{-i\Omega_\iota}\right). \tag{1.10}$$

Here we can put the Fourier transform outside the integral because P_ξ is compactly supported on M. For any $\xi \in \mathfrak{t}^*$, let $r_\xi > 0$ be the smallest non-zero critical value of the function $\|\Phi-\xi\|^2$. As a particular case of Proposition 3.8 in [31], we have

Proposition 1.2.2 *Let ξ be any point in $\mathfrak{t}^*$. The following equality of distributions on $\mathfrak{t}^*$*

$$\mathrm{DH}(M) = \mathrm{DH}_\xi(M)$$

holds in the open ball $B(\xi, r_\xi) \subset \mathfrak{t}^*$.

We will now use the last Proposition, first to recover the classical result of Duistermaat and Heckman [15] concerning the polynomial behaviour of $\mathrm{DH}(M)$ on the open subset of regular values of Φ. After we determine the difference taken by $\mathrm{DH}(M)$ between two adjacent regions of regular values.

1.2.3 Polynomial behaviour

We recall now the computation of the cohomology class P_ξ when ξ is a regular value of Φ, that is given in [30][Section 6] for the torus case (see [31] [Section 3.1] for the case of Hamiltonian action of a compact Lie group). First recall the following basic result which shows that $\xi \mapsto \mathrm{DH}_\xi(M)$ is locally constant on the open subset of regular values of Φ.

Lemma 1.2.3 ([33]) *If ξ and ξ' belong to the same connected component of regular values of Φ, we have $\mathrm{P}_\xi = \mathrm{P}_{\xi'}$ in $\mathcal{H}^{-\infty}_{T,c}(M)$.*

If we combine Lemma 1.2.3 with Proposition 1.2.2, we see that for any connected component $\mathfrak{c}$ of regular values of Φ, we have

$$\mathrm{DH}(M)(a) = \mathrm{DH}_\xi(M)(a), \quad a \in \mathfrak{c},$$

for any $\xi \in \mathfrak{c}$. We have to compute $\mathrm{DH}_\xi(M)$ when ξ a regular value of Φ.

We consider the T-principal bundle $\Phi^{-1}(\xi) \to \mathcal{M}_\xi := \Phi^{-1}(\xi)/T$ with curvature form $\omega_\xi \in \mathcal{H}^2(\mathcal{M}_\xi) \otimes \mathfrak{t}$. The orbifold $\mathcal{M}_\xi$ carries a canonical symplectic 2-form Ω_ξ. We denote

$$\mathbf{Kir}_\xi : \mathcal{H}^\infty_T(M) \to \mathcal{H}^*(\mathcal{M}_\xi)$$

the Kirwan morphism. For any $\psi \in \mathcal{C}^\infty(\mathfrak{t})$ and $\eta \in \mathcal{H}^\infty_T(M)$ we have $\mathbf{Kir}_\xi(\eta\psi) = \mathbf{Kir}_\xi(\eta)\psi(\omega_\xi)$, where the characteristic class $\psi(\omega_\xi)$ is the value of the differential operator $e^{\omega_\xi(\frac{\partial}{\partial X}|_0)}$ against ψ. After [31][Prop. 3.11], we know that the integral

$$\int_{\mathfrak{t}} \int_M \mathrm{P}_\xi(X)\eta(X)\psi(X)dX$$

is equal to

$$\frac{(-2i\pi)^{\dim T}\mathrm{vol}(T, dX)}{|\Gamma_M|} \int_{\mathcal{M}_\xi} \mathbf{Kir}_\xi(\eta)\psi(\omega_\xi) \tag{1.11}$$

for every equivariant class $\eta \in \mathcal{H}^\infty_T(M)$. Here $\mathrm{vol}(T, dX)$ is the volume of T for the Haar mesure compatible with dX, and $|\Gamma_M|$ is the cardinal of Γ_M (Note that the generic stabilizer of T on $\Phi^{-1}(\xi)$ is Γ_M). In other words, for every $\eta \in \mathcal{H}^\infty_T(M)$ we have the following equality of generalized functions on $\mathfrak{t}^*$ supported at 0

$$\int_M \mathrm{P}_\xi(X)\eta(X) = \frac{(-2i\pi)^{\dim T}}{|\Gamma_M|} \int_{\mathcal{M}_\xi} \mathbf{Kir}_\xi(\eta) e^{\omega_\xi(\frac{\partial}{\partial X}|_0)}\mathrm{vol}(T, -). \tag{1.12}$$

For $\eta = e^{-i\Omega_\mathfrak{t}}$ we have $\mathbf{Kir}_\xi(\eta) = e^{-i(\Omega_\xi - \langle \xi, \omega_\xi\rangle)}$, and a small computation shows that

$$\mathcal{F}\left(e^{\omega_\xi(\frac{\partial}{\partial X}|_0)}\mathrm{vol}(T, -)\right)(a) = e^{-i\langle a, \omega_\xi\rangle}\frac{da}{(2\pi)^{\dim T}}, \quad a \in \mathfrak{t}^*. \tag{1.13}$$

where da is the Lebesgue measure on $\mathfrak{t}^*$ normalized by the condition: $\mathrm{vol}(T, dX) = 1$ for the Lebesgue measure dX on $\mathfrak{t}$ which is dual to da.

Finally (1.10), (1.12) and (1.13) give

$$\begin{aligned} \mathrm{DH}_\xi(M)(a) &= \frac{(i)^d}{|\Gamma_M|} \int_{\mathcal{M}_\xi} e^{-i(\Omega_\xi + \langle a - \xi, \omega_\xi\rangle)}\, da \\ &= \frac{1}{|\Gamma_M|} \int_{\mathcal{M}_\xi} \frac{(\Omega_\xi + \langle a - \xi, \omega_\xi\rangle)^d}{d!}\, da, \end{aligned} \tag{1.14}$$

where $2d = \dim \mathcal{M}_\xi$.

Definition 1.2.4 *For any connected component $\mathfrak{c}$ of regular values of Φ we denote $\mathrm{DH}_{\mathfrak{c}}$ the polynomial function $a \mapsto \frac{1}{|\Gamma_M|}\int_{\mathcal{M}_\xi} \frac{(\Omega_\xi + \langle a-\xi, \omega_\xi\rangle)^d}{d!}$, where ξ is any point of $\mathfrak{c}$.*

With the help of Proposition 1.2.2 we recover the classical result of Duistermaat and Heckman [15] that says that the measure $\mathrm{DH}(M)$ is locally polynomial[2] on the open subset of regular values of Φ, and it's value at a regular element ξ is equal to the symplectic volume of the reduce space $\mathcal{M}_\xi$ (times $|\Gamma_M|^{-1}$) . More precisely we have shown that for a connected component $\mathfrak{c}$ of regular values of Φ we have

$$\mathrm{DH}(M)(a) = \mathrm{DH}_{\mathfrak{c}}(a)da, \qquad a \in \mathfrak{c}. \tag{1.15}$$

1.2.4 Wall-crossing formulas

Consider now two connected regions $\mathfrak{c}_\pm$ of regular values of Φ separated by an hyperplane $\Delta \subset \mathfrak{t}^*$. In this section we compute the polynomial $\mathrm{DH}_{\mathfrak{c}_+} - \mathrm{DH}_{\mathfrak{c}_-}$. It generalizes previous results of Guillemin-Lerman-Sternberg [17] and Brion-Procesi [12].

Let ξ_+, ξ_- be respectively two elements of $\mathfrak{c}_+$ and $\mathfrak{c}_-$. We know from (1.2.2), (1.14) and Definition (1.2.4) that

$$(\mathrm{DH}_{\mathfrak{c}_+} - \mathrm{DH}_{\mathfrak{c}_-})(a)da = (i)^n \mathcal{F}\left(\int_M (\mathrm{P}_{\xi_+} - \mathrm{P}_{\xi_-})e^{-i\Omega_{\mathfrak{t}}}\right)(a), \quad a \in \mathfrak{t}^*. \tag{1.16}$$

We recall now the computation of the cohomology class $\mathrm{P}_{\xi_+} - \mathrm{P}_{\xi_-} \in \mathcal{H}^{-\infty}_{T,c}(M)$ done in [33]. We use the notation defined in the introduction.

Definition 1.2.5 *We denote M^Δ the union of the connected component Z of the fixed point set M^{T_Δ} for which we have $\Phi(Z) \subset \Delta$. Let M^Δ_o be the T-invariant open subset of M^Δ where T/T_Δ acts locally freely.*

For a connected component $Z \subset M^\Delta$, one has either $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-} \subset \Phi(Z)$ or $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-} \cap \Phi(Z) = \emptyset$. It is due to the fact that for any ξ in relative interior of $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-}$ in Δ, and any $m \in \Phi^{-1}(\xi)$ the stabilizer $\mathfrak{t}_m \subset \mathfrak{t}$ is either equal to $\mathfrak{t}_\Delta$ or reduced to $\{0\}$.

The symplectic manifold M^Δ carries a Hamiltonian action of T/T_Δ with moment map $\Phi|_{M^\Delta} : M^\Delta \to \Delta$ equal to the restriction of Φ on M^Δ.

Let ξ be a point in the relative interior of $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-}$ in Δ. From the previous discussion, we knows that ξ is a regular value of $\Phi|_{M^\Delta}$, i.e. $\Phi^{-1}(\xi) \cap M^{T_\Delta}$ is a submanifold of M^Δ_o. Following Definition 1.2.1 we associate to ξ the cohomology class

$$\mathrm{P}^\Delta_\xi \in \mathcal{H}^{-\infty}_{T/T_\Delta,c}(M^\Delta_o).$$

[2] It is a polynomial times a Lebesgue measure on $\mathfrak{t}^*$.

Let $\mathcal{H}^*(M_o^{\Delta})^{bas}$ be the sub-algebra of $\mathcal{H}^*(M_o^{\Delta})$ formed by the T-basic elements. Since the T_{Δ}-action on M_o^{Δ} is trivial we have a canonical product operation

$$\mathcal{H}^{-\infty}_{T/T_{\Delta},c}(M_o^{\Delta}) \times \mathcal{C}^{-\infty}(\mathfrak{t}_{\Delta}, \mathcal{H}^*(M_o^{\Delta})^{bas}) \xrightarrow{\wedge} \mathcal{H}^{-\infty}_{T,c}(M_o^{\Delta}). \tag{1.17}$$

Proposition 1.2.6 ([33]) *There exists a generalized function supported at* 0, $\delta^{\Delta} \in \mathcal{C}^{-\infty}(\mathfrak{t}_{\Delta}, \mathcal{H}^*(M_o^{\Delta})^{bas})$, *such that*

$$\mathrm{P}_{\xi^+} - \mathrm{P}_{\xi^-} = (i_{\Delta})_* \left(\mathrm{P}_{\xi}^{\Delta} \wedge \delta^{\Delta}\right) \qquad \text{in} \quad \mathcal{H}^{-\infty}_{T,c}(M).$$

Here $(i_{\Delta})_* : \mathcal{H}^{-\infty}_{T,c}(M_o^{\Delta}) \to \mathcal{H}^{-\infty}_{T,c}(M)$ *is the direct image map relative to the inclusion* $i_{\Delta} : M_o^{\Delta} \hookrightarrow M$.

We will now give the precise definition of δ^{Δ}. The decomposition $T = T_{\Delta} \times T/T_{\Delta}$ and the trivial action of T_{Δ} on M_o^{Δ} determine a canonical isomorphism

$$j_{\Delta} : \mathcal{H}^*_T(M_o^{\Delta}) \xrightarrow{\sim} \mathcal{S}(\mathfrak{t}^*_{\Delta}) \otimes \mathcal{H}^*_{T/T_{\Delta}}(M_o^{\Delta}),$$

where $\mathcal{S}(\mathfrak{t}^*_{\Delta})$ is the algebra of complex polynomial functions on $\mathfrak{t}_{\Delta}$. Since the T/T_{Δ}-action on M_o^{Δ} is locally free, we have the Chern-Weil isomorphism

$$\mathbf{cv}_{\Delta} : \mathcal{H}^*_{T/T_{\Delta}}(M_o^{\Delta}) \xrightarrow{\sim} \mathcal{H}^*(M_o^{\Delta})^{bas}.$$

Let N_{Δ} be the T-equivariant normal bundle of M^{Δ} in M, and let

$$\mathrm{Eul}(N_{\Delta}) \in \mathcal{H}^*_T(M^{\Delta})$$

be the T-equivariant Euler class of N_{Δ}. Now we consider the restriction of $\mathrm{Eul}(N_{\Delta})$ on the open subset $M_o^{\Delta} \subset M^{\Delta}$, that we look through the isomorphism $\mathbf{cv}_{\Delta} \circ j_{\Delta}$ as an element of $\mathcal{S}(\mathfrak{t}^*_{\Delta}) \otimes \mathcal{H}^*(M_o^{\Delta})^{bas}$ (for simplicity we keep the same notations $\mathrm{Eul}(N_{\Delta})$ for this element). Following [30], we define inverses $\mathrm{Eul}^{-1}_{\pm\beta}(N_{\Delta}) \in \mathcal{C}^{-\infty}(\mathfrak{t}_{\Delta}, \mathcal{H}^*(M_o^{\Delta})^{bas})$ by

$$\mathrm{Eul}^{-1}_{\pm\beta}(N_{\Delta})(X) = \lim_{s \to +\infty} \frac{1}{\mathrm{Eul}(N_{\Delta})(X \pm is\beta)}. \tag{1.18}$$

Here $\beta \in \mathfrak{t}_{\Delta}$ is chosen so that $\langle \xi^+ - \xi^-, \beta \rangle > 0$.

Definition 1.2.7 *The generalized function* $\delta^{\Delta} \in \mathcal{C}^{-\infty}(\mathfrak{t}_{\Delta}, \mathcal{H}^*(M_o^{\Delta})^{bas})$ *is defined by*

$$\delta^{\Delta} := \mathrm{Eul}^{-1}_{\beta}(N_{\Delta}) - \mathrm{Eul}^{-1}_{-\beta}(N_{\Delta}). \tag{1.19}$$

Since the polynomial $\mathrm{Eul}(N_{\Delta})$ is invertible in a smooth manner on $\mathfrak{t}_{\Delta} \setminus \{0\}$ the generalized function δ^{Δ} is supported at 0.

Let ξ be a point in the relative interior of $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-}$ in Δ. We consider the symplectic reduction

$$\mathcal{M}_\xi^\Delta := \Big(M^\Delta \cap \Phi^{-1}(\xi)\Big)/(T/T_\Delta).$$

If we restrict δ^Δ to the submanifold $M^\Delta \cap \Phi^{-1}(\xi)$ we get the generalized function

$$\delta_\xi^\Delta \in \mathcal{C}^{-\infty}(\mathfrak{t}_\Delta, \mathcal{H}^*(\mathcal{M}_\xi^\Delta)).$$

Now we are able to compute the right hand side of (1.16). Let $\omega_\xi^\Delta \in \mathcal{H}^2(\mathcal{M}_\xi^\Delta)\otimes\mathfrak{t}/\mathfrak{t}_\Delta$ be the curvature of the T/T_Δ-principal bundle $M^\Delta\cap\Phi^{-1}(\xi) \to \mathcal{M}_\xi^\Delta$. Let $|S_\xi^\Delta|$ be locally constant function on $M^\Delta \cap \Phi^{-1}(\xi)$ which is equal to the cardinal of the generic stabilizer of T/T_Δ. From (1.12) and Proposition 1.2.6 we have

$$\begin{aligned}
&\int_M (\mathrm{P}_{\xi_+} - \mathrm{P}_{\xi_-})(X) e^{-i\Omega_\mathfrak{t}(X)} \\
&= \int_{M_o^\Delta} \mathrm{P}_\xi^\Delta(X')\delta^\Delta(X'') e^{-i\Omega_\mathfrak{t}(X'+X'')} \\
&= \frac{(-2i\pi)^{\dim T - 1}}{|S_\xi^\Delta|} \int_{\mathcal{M}_\xi^\Delta} e^{\omega_\xi^\Delta(\frac{\partial}{\partial X'}|_0)} \mathrm{vol}(T/T_\Delta, -)\mathbf{Kir}_\xi^\Delta(e^{-i\Omega_\mathfrak{t}})(X'')\delta_\xi^\Delta(X'') \qquad (1.20)
\end{aligned}$$

In the last equation the notations are the following :

1. $X = X' + X''$ with $X' \in \mathfrak{t}/\mathfrak{t}_\Delta$ and $X'' \in \mathfrak{t}_\Delta$,
2. the Kirwan map $\mathbf{Kir}_\xi^\Delta : \mathcal{H}_T^\infty(M) \to \mathcal{C}^\infty(\mathfrak{t}_\Delta, \mathcal{H}^*(\mathcal{M}_\xi^\Delta))$ is the composition of the restriction $\mathcal{H}_T^\infty(M) \to \mathcal{H}_T^\infty(M^\Delta \cap \Phi^{-1}(\xi))$ with the Chern-Weil isomorphism $\mathcal{H}_T^\infty(M^\Delta \cap \Phi^{-1}(\xi)) \xrightarrow{\sim} \mathcal{C}^\infty(\mathfrak{t}_\Delta, \mathcal{H}^*(\mathcal{M}_\xi^\Delta))$.

A direct computation gives that $\mathbf{Kir}_\xi^\Delta(\Omega_\mathfrak{t})(X'') = \Omega_\xi^\Delta - \langle \xi, \omega_\xi^\Delta + X''\rangle$ where Ω_ξ^Δ is the induced symplectic form on the reduced space $\mathcal{M}_\xi^\Delta$. If we take the Fourier transform in (1.20) we get

$$\begin{aligned}
&(\mathrm{DH}_{\mathfrak{c}_+} - \mathrm{DH}_{\mathfrak{c}_-})(a)da \\
&= \frac{(i)^{n+1-\dim T}}{|S_\xi^\Delta|} \left(\int_{\mathcal{M}_\xi^\Delta} e^{-i(\Omega_\xi^\Delta + \langle a', \omega_\xi^\Delta\rangle)}\, da'\, \mathcal{F}_{\mathfrak{t}_\Delta}(\delta_\xi^\Delta)(a'') \right)(a-\xi), \\
&= \sum_{Z\in\mathcal{F}} \frac{(i)^{n+1-\dim T}}{|S_\xi^Z|} \left(\int_{\mathcal{Z}_\xi} e^{-i(\Omega_\xi^Z + \langle a', \omega_\xi^Z\rangle)}\, da'\, \mathcal{F}_{\mathfrak{t}_\Delta}(\delta_\xi^Z)(a'') \right)(a-\xi) \qquad (1.21)
\end{aligned}$$

where $a = a' + a''$ with $a' \in (\mathfrak{t}/\mathfrak{t}_\Delta)^*$ and $a'' \in (\mathfrak{t}_\Delta)^*$. In (1.21), we write $\int_{\mathcal{M}_\xi^\Delta} = \sum_{Z\in\mathcal{F}} \int_{\mathcal{Z}_\xi}$ where the sum is taken over the set $\mathcal{F}$ of connected components Z of M^Δ that intersects $\Phi^{-1}(\xi)$: we take then

$$\mathcal{Z}_\xi = \Big(Z \cap \Phi^{-1}(\xi)\Big)/(T/T_\Delta).$$

The 2-forms $\Omega_\xi^\Delta, \omega_\xi^\Delta$, the generic stabiliser S_ξ^Δ, the vector bundle N_Δ, the generalized function δ_ξ^Δ restrict to each component Z: we denote them respectively $\Omega_\xi^Z, \omega_\xi^Z$, S_ξ^Z, N_Z, δ_ξ^Z.

We recall now the computation of the Fourier tranform of the inverses $\mathrm{Eul}^{-1}_{\pm\beta}(N_Z) := \mathrm{Eul}^{-1}_{\pm\beta}(N_\Delta)|_Z$ that is given in [30][Proposition 4.8.]. We consider a T-invariant scalar product on the fibers of the bundle N_Δ. Let $R \in \mathcal{A}^2(M_o^\Delta, \mathrm{so}(N_\Delta))^{bas}$ be the curvature of a T-invariant and T/T_Δ-horizontal Euclidean connexion on N_Δ: we denote by $R^Z \in \mathcal{A}^2(Z, \mathrm{so}(N_Z))^{bas}$ the restriction of R to a component $Z \in \mathcal{F}$. The curvature commutes with the infinitesimal action $\mathcal{L}_X$ of $X \in \mathfrak{t}_\Delta$, and with the complex structure $J_\beta = \mathcal{L}_\beta(-\mathcal{L}_\beta^2)^{1/2}$ on N_Δ defined by $\beta \in \mathfrak{t}_\Delta$.

We denote by $S^\bullet$ the symmetric algebra of the complex vector bundle (N_Δ, J_β). We keep the same notation for the restriction of $S^\bullet$ on the submanifolds Z, $\Phi^{-1}(\xi) \cap M^\Delta$, and for the induced orbifold vector bundle on the reduced spaces $\mathcal{Z}_\xi$ and $\mathcal{M}_\xi^\Delta$. For each $k \in \mathbb{N}$, we denote by $\mathbf{Tr}_{S^k}$ the trace operator defined on the complex endomorphisms of S^k. For a complex endomorphism A of N_Δ, we denote by $A^{\otimes k}$ the induced endomorphism on S^k. For any $X \in \mathfrak{t}_\Delta$, the complex endomorphism $\mathcal{L}_X^{-1} R^Z$ is symmetric. Hence the trace $\mathbf{Tr}_{S^k}((\mathcal{L}_X^{-1} R^Z)^{\otimes k})$ is a basic *real* differential form of degree $2k$ on Z which does not depend of the choice of complex structures (J_β or $J_{-\beta}$).

Let $\beta^* \in \mathfrak{t}_\Delta^*$ the dual of $\beta \in \mathfrak{t}_\Delta$.

Proposition 1.2.8 ([30]) *For a smooth function f on $\mathfrak{t}_\Delta^*$ with compact support we have $\int_{\mathfrak{t}_\Delta^*} \mathcal{F}_{\mathfrak{t}_\Delta}(\mathrm{Eul}_\beta^{-1}(N_Z))(a'')f(a'') = \int_0^\infty P_Z(t) f(t\beta^*)dt$ where P_Z is the polynomial on $\mathbb{R}$ defined by:*

$$P_Z(t) = \frac{(2\pi i)^{r_Z}}{\det_Z^{1/2}(\mathcal{L}_\beta)} \left(\frac{t^{r_Z-1}}{(r_Z-1)!} + \sum_{k=1}^{dim(Z)/2} (i)^k \, \mathbf{Tr}_{S^k}((\mathcal{L}_\beta^{-1} R^Z)^{\otimes k}) \frac{t^{r_Z-1+k}}{(r_Z-1+k)!} \right). \tag{1.22}$$

Here $\det_Z^{1/2}(\mathcal{L}_\beta)$ is the Pfaffian of $\mathcal{L}_\beta$ on N_Z, and $r_Z = \mathrm{rk}_{\mathbb{C}}(N_Z)$.

One checks then that

$$\begin{aligned} \int_{\mathfrak{t}_\Delta^*} \mathcal{F}_{\mathfrak{t}_\Delta}(\mathrm{Eul}_{-\beta}^{-1}(N_Z))(a'')f(a'') &= \int_0^\infty -P_Z(-t) f(-t\beta^*)dt \\ &= -\int_{-\infty}^0 P_Z(t) f(t\beta^*)dt. \end{aligned}$$

Hence the distribution $\mathcal{F}_{\mathfrak{t}_\Delta}(\delta^Z)$ is equal to $P_Z(\beta)d\beta$. From now one we fix β as the primitive element of $\mathfrak{t}_\Delta \cap \Lambda$ which point out $\mathfrak{c}_-$. Then $d\beta$ and $d\beta^*$ are then (dual) Lebesgue measure on $\mathfrak{t}^*$ and $\mathfrak{t}$: we have $\mathrm{vol}(T_\Delta, d\beta^*) = 1$.

Let R_ξ^Z be the restriction of the curvature R^Z to the submanifold $Z \cap \Phi^{-1}(\xi)$. Since R^Z is T/T_Δ-basic, $\mathbf{Tr}_{S^k}((\mathcal{L}_\beta^{-1} R_\xi^Z)^{\otimes k})$ can be seen as a real differential form of degree $2k$ on the orbifold $\mathcal{Z}_\xi = (Z \cap \Phi^{-1}(\xi))/(T/T_\Delta)$.

Each connected component Z of M^Δ is a T/T_Δ Hamiltonian manifold: we take for moment map $\Phi_Z : Z \to (\mathfrak{t}/\mathfrak{t}_\Delta)^*$ the restriction of $\Phi - \xi$ to Z. Hence 0 is a regular value of Φ_Z. Let $\mathrm{DH}_0(Z)$ be the polynomial function on

$(\mathfrak{t}/\mathfrak{t}_\Delta)^* = \{a \in \mathfrak{t}^* \,|\, \langle \beta, a\rangle = 0\}$ such that $\mathrm{DH}(Z)(a') = \mathrm{DH}_0(Z)(a')da'$ near 0. Finally (1.21) together with the proposition 1.2.8 give the following

Theorem 1.2.9 *We have* $(\mathrm{DH}_{\mathfrak{c}_+} - \mathrm{DH}_{\mathfrak{c}_-})(a) = \sum_{Z\in\mathcal{F}} \mathbf{D}_Z(a-\xi)$, $a \in \mathfrak{t}^*$ *where each polynomial* $\mathbf{D}_Z \in \mathcal{S}(\mathfrak{t})$ *admits the following decomposition*

$$\mathbf{D}_Z = \frac{\beta^{r_Z-1}}{\det_Z^{1/2}(\frac{-\mathcal{L}_\beta}{2\pi})}\left(\frac{\mathrm{DH}_0(Z)}{(r_Z-1)!} + \sum_{k=1}^{d_Z}\beta^k \mathrm{Q}_{Z,k}\right). \tag{1.23}$$

The polynomials $\mathrm{Q}_{Z,k} \in \mathcal{S}(\mathfrak{t}/\mathfrak{t}_\Delta)$ *are defined by*

$$\mathrm{Q}_{Z,k}(a') = \frac{(-1)^k}{(r_Z-1+k)!|S_\xi^Z|}\int_{\mathcal{Z}_\xi} \frac{(\Omega_\xi^Z + \langle a', \omega_\xi^Z\rangle)^{d_Z-k}}{(d_Z-k)!}\mathbf{Tr}_{S^k}((\mathcal{L}_\beta^{-1}R_\xi^Z)^{\otimes k}). \tag{1.24}$$

Here $2d_Z = \dim \mathcal{Z}_\xi$ *and* $2r_Z = \dim M - \dim Z$.

Remark 1.2.10 • *The polynomial* $\mathrm{DH}_{\mathfrak{c}_+} - \mathrm{DH}_{\mathfrak{c}_-}$ *is divisible by the factor* $a \mapsto \langle a-\xi, \beta\rangle^{r-1}$ *with* $r = \inf_{Z\in\mathcal{F}} r_Z$. *If* $\Delta \cap \Phi(M)$ *is not a facet of the polytope* $\Phi(M)$ *we have* $r_Z \geq 2$ *for all connected component* $Z \in \mathcal{F}$, *hence* $r-1 \geq 1$.

• *Suppose now that* $\mathfrak{c}_-$ *is a connected component of regular values of* Φ *bording a facet* $\Phi(M)\cap\Delta$ *of the polytope* $\Phi(M)$. *Here* $Z = \Phi^{-1}(\Delta)$ *is a connected component of the fixed point set* M^{T_Δ}. *In this situation we have* $\mathrm{DH}_{\mathfrak{c}_-} = -\mathbf{D}_Z$ *where the polynomial* $\mathbf{D}_Z$ *is defined by (1.23).*

1.3 Quantum version of Duistermaat-Heckman measures

We suppose here that the Hamiltonian T-manifold (M, ω, Φ) is prequantized by a T-equivariant Hermitian line bundle L over M, which is equipped with an Hermitian connection ∇ satisfying the Kostant formula

$$\mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi, X\rangle, \quad X \in \mathfrak{t}. \tag{1.25}$$

The former equation implies that the first Chern class of L is equal to $\frac{\Omega}{2\pi}$. In this section we suppose that M is *compact* and we still assume that the generic stabiliser Γ_M of T on M is *finite*. The quantization of (M, Ω) is defined by the Riemann-Roch character $RR(M, L) \in R(T)$ which is compute with a T-equivariant almost complex stucture on M *compatible* with Ω [32]. For $k \geq 1$, we consider the tensor product $L^{\otimes k}$. Its Riemann-Roch character $RR(M, L^{\otimes k})$ decomposes as

$$RR(M, L^{\otimes k}) = \sum_{\mu\in\Lambda^*} \mathrm{m}(\mu, k)\,\mathbb{C}_\mu. \tag{1.26}$$

Let us recall the well-known properties of the map $\mathrm{m} : \Lambda^* \times \mathbb{Z}^{>0} \to \mathbb{Z}$. When $\frac{\mu}{k}$ is a regular value of Φ, the "Quantization commutes with Reduction Theorem" [28, 29] tell us that

$$\mathrm{m}(\mu,k) = RR(\mathcal{M}_{\frac{\mu}{k}}, \mathcal{L}^{\mu,k}) \tag{1.27}$$

where $\mathcal{L}^{\mu,k} = (L^{\otimes k}|_{\Phi^{-1}(\frac{\mu}{k})} \otimes \mathbb{C}_{-\mu})/T$ is an orbifold line bundle over the symplectic orbifold $\mathcal{M}_{\frac{\mu}{k}} = \Phi^{-1}(\frac{\mu}{k})/T$. In particular if $\frac{\mu}{k}$ does not belong to $\Phi(M)$ we have $\mathrm{m}(\mu,k) = 0$. When $\frac{\mu}{k} \in \Phi(M)$ is not necessarilly a regular value of Φ, one procceed by shift desingularization. If $\xi \in \Phi(M)$ is a regular value of Φ close enough to $\frac{\mu}{k}$ then (1.27) becomes

$$\mathrm{m}(\mu,k) = RR(\mathcal{M}_{\xi}, \mathcal{L}_{\xi}^{\mu,k}) \tag{1.28}$$

where $\mathcal{L}_{\xi}^{\mu,k} = (L^{\otimes k}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu})/T$ (for a proof see [29, 32]).

Definition 1.3.1 *A function $f : \Xi \to \mathbb{Z}$ defined over a lattice $\Xi \simeq \mathbb{Z}^r$ is called* periodic polynomial *if*

$$f(x) = \sum_{i=1}^{p} e^{i\frac{\langle \alpha_j, x\rangle}{N}} P_j(x), \quad x \in \Xi,$$

where $\alpha_1, \cdots, \alpha_p \in \Xi^*$, $N \geq 1$, and the functions $P_1, \cdots, P_p$ are polynomials with complex coefficients.

Remark 1.3.2 *Let $\mathcal{C}$ a cone with* non-empty *interior in the real vector space $\Xi \otimes_{\mathbb{Z}} \mathbb{R}$. Any* periodic-polynomial *function $f : \Xi \to \mathbb{Z}$ is completely determined by its restriction on $\mathcal{C} \cap \Xi$.*

Let $\mathfrak{c} \subset \mathfrak{t}^*$ be a connected component of regular values of Φ. In [29] Meinrenken an Sjamaar proved that there exits a periodic polynomial function $\mathrm{m}_{\mathfrak{c}} : \Lambda^* \times \mathbb{Z} \to \mathbb{Z}$ such that $\mathrm{m}_{\mathfrak{c}}(\mu,k) = \mathrm{m}(\mu,k)$ for every (μ,k) in the cone

$$\mathrm{Cone}(\mathfrak{c}) = \{(\xi,s) \in \mathfrak{t}^* \times \mathbb{R}^{>0} \,|\, \xi \in s \cdot \mathfrak{c}\}. \tag{1.29}$$

Consider now two adjacent connected regions $\mathfrak{c}_{\pm}$ of regular values of Φ separated by an hyperplane $\Delta \subset \mathfrak{t}^*$. When Δ does not contain a facet of the polytope $\Phi(M)$, Meinrenken an Sjamaar proved also that

$$\mathrm{m}_{\mathfrak{c}_+}(\mu,k) = \mathrm{m}_{\mathfrak{c}_-}(\mu,k) = \mathrm{m}(\mu,k) \tag{1.30}$$

for every $(\mu,k) \in \overline{\mathrm{Cone}(\mathfrak{c}_+)} \cap \overline{\mathrm{Cone}(\mathfrak{c}_-)} = \mathrm{Cone}(\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-}) \subset \mathrm{Cone}(\Delta)$.

The main objective of this section is to prove that (1.30) extends to a "strip" containing $\mathrm{Cone}(\Delta)$.

Let $\beta \in \Lambda$ be the primitive orthogonal vector to the hyperplane $\Delta \subset \mathfrak{t}^*$ which is pointing out of $\mathfrak{c}_-$. Then $\Delta = \{\xi \in \mathfrak{t}^* \,|\, \frac{\langle \xi,\beta\rangle}{2\pi} = r_\Delta\}$ for some $r_\Delta \in \mathbb{Z}$, $\mathrm{Cone}(\Delta) = \{(\xi,s) \in \mathfrak{t}^* \times \mathbb{R}^{\geq 0} \,|\, \frac{\langle \xi,\beta\rangle}{2\pi} - s r_\Delta = 0\}$ and $\mathfrak{c}_- \subset \{\xi \in \mathfrak{t}^* \,|\, \frac{\langle \xi,\beta\rangle}{2\pi} < r_\Delta\}$.

Let T_Δ be the subtorus of T generated by β. Let N_Δ be the normal vector bundle of M^{T_Δ} in M. The almost complex structure on M induces a complex structure J on the fibers of N_Δ. We have a decomposition $N_\Delta = \sum_s N_\Delta^s$ where $N_\Delta^s = \{v \in N_\Delta \,|\, \mathcal{L}_\beta v = s\, Jv\ \}$. We write $N_\Delta = N_\Delta^{+,\beta} \oplus N_\Delta^{-,\beta}$ where

$$N_\Delta^{\pm,\beta} = \sum_{\pm s>0} N_\Delta^s. \tag{1.31}$$

Definition 1.3.3 *For every connected component $Z \subset M^{T_\Delta}$ we define $s_Z^\pm \in \mathbb{N}$ respectively as the absolute value of the trace of $\frac{1}{2\pi}\mathcal{L}_\beta$ on $N_\Delta^{\pm,\beta}|_Z$.*

Note that $s_Z^+ + s_Z^-$ is larger than half of the codimension of Z in M. We prove in Section 1.3.5 the following

Theorem 1.3.4 *We have $\mathrm{m}_{\mathfrak{c}_+}(\mu,k) = \mathrm{m}_{\mathfrak{c}_-}(\mu,k)$ for all $(\mu,k) \in \Lambda^* \times \mathbb{Z}$ such that*

$$-s^- < \frac{\langle \mu, \beta\rangle}{2\pi} - k\, r_\Delta < s^+\ . \tag{1.32}$$

The number $s^-, s^+ \in \mathbb{N}$ are defined as follows. We take $s^\pm = \inf_Z s_Z^\pm$ where the minimum is taken over the connected components Z of M^{T_Δ} for which $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-} \subset \Phi(Z)$.

Similar results were obtained by Billey-Guillemin-Rassart [10] in the case where M is a coadjoint orbit of $\mathrm{SU}(n)$, and by Szenes-Vergne [36] in the case where M is a complex vector space. See Sections 1.4.4 and 1.5 where we study these two particular cases in details. In Proposition 1.3.25, we give also a criterium which says when the inequalities in (1.32) are optimal. This criterium is fullfilled when there is *only one* component Z of M^{T_Δ} such that $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-} \subset \Phi(Z)$. Then (1.32) is *optimal* and $s^+ + s^-$ is larger than half of the codimension of Z in M.

The following easy Lemma (see Lemma 7.3. of [32]) gives some basic informations about the integer $s_Z^\pm$.

Lemma 1.3.5 *Let (M, Ω, Φ) be a compact Hamiltonian T-manifold equipped with a T-invariant almost complex structure compatible with Ω. Consider a non-zero vector $\gamma \in \mathfrak{t}$ and let Z be a connected component of the fixed point set M^γ. Let N be the normal vector of Z in M and let $N^{-,\gamma}$ be the negative polarized normal bundle (see (1.31)). Then $N^{-,\gamma} = 0$ if and only if the function $\langle \Phi, \gamma\rangle : M \to \mathbb{R}$ takes its maximal value on Z.*

This Lemma insures that $s^\pm \geq 1$ in Theorem 1.3.4 when $\Delta \cap \Phi(M)$ is not a facet of the polytope $\Phi(M)$.

Consider the situation where $\Delta \cap \Phi(M)$ is a facet of the polytope $\Phi(M)$ so that $\mathfrak{c}_+ \cap \Phi(M) = \emptyset$: hence $\mathrm{m}_{\mathfrak{c}_+} = 0$. If we apply Lemma 1.3.5 with $\gamma = \beta$, one gets $N^{-,\beta} = 0$ and so $s^- = 0$. In this situation we get

Corollary 1.3.6 *Let $\mathfrak{c}_-$ be a connected component of regular values of Φ bording a facet $\Phi(M)\cap\Delta$ of the polytope $\Phi(M)$. Let $\beta\in\Lambda$ be the primitive orthogonal vector to the hyperplane $\Delta\subset\mathfrak{t}^*$ which is pointing out of $\mathfrak{c}_-$. Here $Z=\Phi^{-1}(\Delta)$ is a connected component of the fixed point set M^{T_Δ}. We have $\mathrm{m}_{\mathfrak{c}_-}(\mu,k)=0$ for all $(\mu,k)\in\Lambda^*\times\mathbb{Z}$ such that*

$$0<\frac{\langle\mu,\beta\rangle}{2\pi}-kr_\Delta<s_Z^+. \tag{1.33}$$

Here $s_Z^+\in\mathbb{N}$ is larger than half of the codimension of Z in M, and the inequalities (1.33) are optimal.

The rest of this section is dedicated to the proof of Theorem 1.3.4. We start by reviewing some of the results of [32].

1.3.1 Elliptic and transversally elliptic symbols

We work in the setting of a compact manifold M equipped with a smooth action of a torus T.

Let $p:\mathbf{T}M\to M$ be the projection, and let $(\cdot,\cdot)_M$ be a T-invariant Riemannian metric. If E^0,E^1 are T-equivariant vector bundles over M, a T-equivariant morphism $\sigma\in\Gamma(\mathbf{T}M,\hom(p^*E^0,p^*E^1))$ is called a *symbol*. The subset of all $(m,v)\in\mathbf{T}M$ where $\sigma(m,v):E^0_m\to E^1_m$ is not invertible is called the *characteristic set* of σ, and is denoted by $\mathrm{Char}(\sigma)$.

Let $\mathbf{T}_TM$ be the following subset of $\mathbf{T}M$:

$$\mathbf{T}_TM\ =\{(m,v)\in\mathbf{T}M,\ (v,X_M(m))_M=0\quad\text{for all } X\in\mathfrak{k}\}\,.$$

A symbol σ is *elliptic* if σ is invertible outside a compact subset of $\mathbf{T}M$ ($\mathrm{Char}(\sigma)$ is compact), and is *transversally elliptic* if the restriction of σ to $\mathbf{T}_TM$ is invertible outside a compact subset of $\mathbf{T}_TM$ ($\mathrm{Char}(\sigma)\cap\mathbf{T}_TM$ is compact). An elliptic symbol σ defines an element in the equivariant K-theory of $\mathbf{T}M$ with compact support, which is denoted by $\mathbf{K}_T(\mathbf{T}M)$, and the index of σ is a virtual finite dimensional representation of T [3, 4, 5, 6].

A *transversally elliptic* symbol σ defines an element of $\mathbf{K}_T(\mathbf{T}_TM)$, and the index of σ is defined as a trace class virtual representation of T (see [1] for the analytic index and [8, 9] for the cohomological one). Remark that any elliptic symbol of $\mathbf{T}M$ is transversally elliptic, hence we have a restriction map $\mathbf{K}_T(\mathbf{T}M)\to\mathbf{K}_T(\mathbf{T}_TM)$, and a commutative diagram

$$\begin{array}{ccc}\mathbf{K}_T(\mathbf{T}M) & \longrightarrow & \mathbf{K}_T(\mathbf{T}_TM)\\ \downarrow{\scriptstyle \mathrm{Index}_M^T} & & \downarrow{\scriptstyle \mathrm{Index}_M^T}\\ R(T) & \longrightarrow & R^{-\infty}(T)\,.\end{array} \tag{1.34}$$

Using the *excision property*, one can easily show that the index map $\mathrm{Index}_{\mathcal{U}}^T:\mathbf{K}_T(\mathbf{T}_T\mathcal{U})\to R^{-\infty}(T)$ is still defined when $\mathcal{U}$ is a T-invariant relatively compact open subset of a T-manifold (see [32][section 3.1]).

1.3.2 Localization of the Riemann-Roch character

We suppose now that the compact T-manifold M is equipped with a T-invariant almost complex structure J. Let us recall the definitions of the Thom symbol $\mathrm{Thom}(M,J)$ and of the Riemann-Roch character [32].

Consider a T-invariant Riemannian metric q on M such that J is orthogonal relatively to q, and let h be the Hermitian structure on $\mathbf{T}M$ defined by : $h(v,w) = q(v,w) - \imath q(Jv,w)$ for $v,w \in \mathbf{T}M$. The symbol

$$\mathrm{Thom}(M,J) \in \Gamma\left(M, \hom(p^*(\wedge_{\mathbb{C}}^{even}\mathbf{T}M),\, p^*(\wedge_{\mathbb{C}}^{odd}\mathbf{T}M))\right)$$

at $(m,v) \in \mathbf{T}M$ is equal to the Clifford map

$$\mathrm{Cl}_m(v) \;:\; \wedge_{\mathbb{C}}^{even}\mathbf{T}_mM \longrightarrow \wedge_{\mathbb{C}}^{odd}\mathbf{T}_mM, \tag{1.35}$$

where $\mathrm{Cl}_m(v).w = v \wedge w - c_h(v).w$ for $w \in \wedge_{\mathbb{C}}^{\bullet}\mathbf{T}_xM$. Here $c_h(v) : \wedge_{\mathbb{C}}^{\bullet}\mathbf{T}_mM \to \wedge^{\bullet-1}\mathbf{T}_mM$ denotes the contraction map relative to h. Since the map $\mathrm{Cl}_m(v)$ is invertible for all $v \neq 0$, the symbol $\mathrm{Thom}(M,J)$ is elliptic.

The Riemann-Roch character $RR(M,-) : \mathbf{K}_T(M) \to R(T)$ is defined by the following relation

$$RR(M,E) = \mathrm{Index}_M^T\left(\mathrm{Thom}(M,J) \otimes p^*E\right)\ . \tag{1.36}$$

The important point is that for any T-vector bundle E, $\mathrm{Thom}(M,J) \otimes p^*E$ corresponds to the *principal symbol* of the twisted Spin^c Dirac operator $\mathcal{D}_E$ [16], hence $RR(M,E) \in R(T)$ is also defined as the (analytical) index of the elliptic operator $\mathcal{D}_E$.

Consider now the case of a *compact* Hamiltonian T-manifold (M,ω,Φ). Here J is a T-invariant almost comlex structure compatible with Ω: $(v,w) \mapsto \Omega(v,Jw)$ defines a Riemannian metric on M. Like in Section 1.2.2, we make the choice of a scalar product $(\cdot,\cdot)$ on $\mathfrak{t}^*$ (which induces an identification $\mathfrak{t}^* \simeq \mathfrak{t}$) and we consider for any $\xi \in \mathfrak{t}^*$ the function $\frac{-1}{2} \parallel \Phi - \xi \parallel^2 : M \to \mathbb{R}$ and its Hamiltonian vector field $\mathcal{H} - \xi_M$.

Definition 1.3.7 *For any $\xi \in \mathfrak{t}^*$ and any T-invariant open subset $\mathcal{U} \subset M$ we define the symbol* $\mathrm{Thom}_\xi(\mathcal{U})$ *by the relation*

$$\mathrm{Thom}_\xi(\mathcal{U})(m,v) := \mathrm{Thom}(M,J)(m, v - (\mathcal{H} - \xi_M)(m)) \quad (m,v) \in \mathbf{T}\mathcal{U}$$

The characteristic set of $\mathrm{Thom}_\xi(\mathcal{U})$ corresponds to $\{(m,v) \in \mathbf{T}\mathcal{U},\ v = (\mathcal{H} - \xi_M)(m)\}$, the graph of the vector field $\mathcal{H} - \xi_M$ over $\mathcal{U}$. Since $\mathcal{H} - \xi_M$ belongs to the set of tangent vectors to the T-orbits, we have

$$\begin{aligned}\mathrm{Char}\left(\mathrm{Thom}_\xi(\mathcal{U})\right) \cap \mathbf{T}_T\mathcal{U} &= \{(m,0) \in \mathbf{T}\mathcal{U} \mid (\mathcal{H} - \xi_M)(m) = 0\} \\ &\cong \{m \in \mathcal{U},\ d \parallel \Phi - \xi \parallel_m^2 = 0\}\ .\end{aligned}$$

Therefore the symbol $\mathrm{Thom}_\xi(\mathcal{U})$ is transversally elliptic if and only if

$$\mathrm{Cr}(\parallel \Phi - \xi \parallel^2) \cup \partial\mathcal{U} = \emptyset. \tag{1.37}$$

Definition 1.3.8 *When (1.37) holds we say that the couple $(\mathcal{U},\xi)$ is* good.

Definition 1.3.9 *Let $(\mathcal{U},\xi)$ be a good couple. For any T-vector bundle $E\to M$, the tensor product $\mathrm{Thom}_\xi(\mathcal{U})\otimes p^*E$ belongs to $\mathbf{K}_T(\mathbf{T}_T\mathcal{U})$ and we denote by*

$$RR^\xi_{\mathcal{U}}(M,E)\in R^{-\infty}(T)$$

its index.

Proposition 1.3.10 *Let $(\mathcal{U},\xi)$ be a good couple.*

• *If $\mathcal{U}$ possess two T-invariant open subsets $\mathcal{U}^1,\mathcal{U}^2$ such that $\overline{\mathcal{U}^1}\cap\overline{\mathcal{U}^2}\cap\mathrm{Cr}(\parallel\Phi-\xi\parallel^2)=\emptyset$ and $(\mathcal{U}^1\cup\mathcal{U}^2)\cap\mathrm{Cr}(\parallel\Phi-\xi\parallel^2)=\mathcal{U}\cap\mathrm{Cr}(\parallel\Phi-\xi\parallel^2)$, then the couples $(\mathcal{U}^1,\xi)$ and $(\mathcal{U}^2,\xi)$ are good and*

$$RR^\xi_{\mathcal{U}}(M,-)=RR^\xi_{\mathcal{U}^1}(M,-)+RR^\xi_{\mathcal{U}^2}(M,-).$$

In particular $RR^\xi_{\mathcal{U}}(M,-)=RR^\xi_{\mathcal{U}^1}(M,-)$ if $\mathcal{U}^1$ is an open subset of $\mathcal{U}$ such that $\mathcal{U}^1\cap\mathrm{Cr}(\parallel\Phi-\xi\parallel^2)=\mathcal{U}\cap\mathrm{Cr}(\parallel\Phi-\xi\parallel^2)$.

• *If $\xi'\in\mathfrak{t}^*$ is close enough to ξ, then $(\mathcal{U},\xi')$ is good and*

$$RR^\xi_{\mathcal{U}}(M,-)=RR^{\xi'}_{\mathcal{U}}(M,-).$$

Proof. The first point is a direct consequence of the excision property (see Proposition 4.1. in [32]).

Let us prove the second point. Consider now the scalar product

$$\phi(s):=(\mathcal{H}-\xi^s_M,\mathcal{H}-\xi_M)_M$$

where $\xi^s=s\xi'+(1-s)\xi,\ s\in[0,1]:\phi(s)$ is a smooth function on M. We have $\phi(s)=\|\mathcal{H}-\xi_M\|^2+s((\xi-\xi')_M,\mathcal{H}-\xi_M)$ and then the following inequality holds on M

$$\phi(s)\geq\|\mathcal{H}-\xi_M\|^2\Big(\|\mathcal{H}-\xi_M\|-s\|\xi_M-\xi'_M\|\Big).\tag{1.38}$$

Since $\partial\mathcal{U}$ is compact we have the following inequalities on it: $\|\mathcal{H}-\xi_M\|\geq c_1>0$ and $\|X_M\|\leq c_2\|X\|$ for any $a\in\mathfrak{t}$. So (1.38) implies the following inequality on $\partial\mathcal{U}$:

$$\phi(s)\geq c_1(c_1-s\|\xi-\xi'\|)\quad\text{for}\quad s\in[0,1].$$

So if ξ' is close enough to ξ, we have $\|\mathcal{H}-\xi^s_M\|\geq c_3>0$ on $\partial\mathcal{U}$ for any $s\in[0,1]$. We have first prove that the couple $(\mathcal{U},\xi^s)$ is good for any $s\in[0,1]$. We see then that the familly of transversally elliptic symbols $\mathrm{Thom}_{\xi^s}(\mathcal{U})$, $s\in[0,1]$ defines an homotopy between $\mathrm{Thom}_\xi(\mathcal{U})$ and $\mathrm{Thom}_{\xi'}(\mathcal{U})$. Hence $\mathrm{Thom}_\xi(\mathcal{U})=\mathrm{Thom}_{\xi'}(\mathcal{U})$ in $\mathbf{K}_T(\mathbf{T}_T\mathcal{U})$. □

The first point of Proposition 1.3.10 shows that $RR^\xi_{\mathcal{U}}(M,-)$ depends closely of the intersection $\mathcal{U}\cap\mathrm{Cr}(\parallel\Phi-\xi\parallel^2)$. In particular $RR^\xi_{\mathcal{U}}(M,-)=0$ when $\mathcal{U}\cap\mathrm{Cr}(\parallel\Phi-\xi\parallel^2)=\emptyset$. Recall that

$$\mathrm{Cr}(\| \Phi - \xi \|^2) = \bigcup_{\gamma \in \mathcal{B}_\xi} M^\gamma \cap \Phi^{-1}(\gamma + \xi) \tag{1.39}$$

where $\mathcal{B}_\xi \subset \mathfrak{t}^*$ is a finite set [24].

Definition 1.3.11 *For any $\xi \in \mathfrak{t}^*$ and $\gamma \in \mathcal{B}_\xi$, we denote simply by*

$$RR^\xi_\gamma(M, -) : \mathbf{K}_T(M) \to R^{-\infty}(T)$$

the map $RR^\xi_{\mathcal{U}}(M, -)$, where $\mathcal{U}$ is a T-invariant open neighborhood of $M^\gamma \cap \Phi^{-1}(\gamma + \xi)$ such that $\mathrm{Cr}(\| \Phi - \xi \|^2) \cap \overline{\mathcal{U}} = M^\gamma \cap \Phi^{-1}(\gamma + \xi)$.

Proposition 1.3.10 insures that the maps $RR^\xi_\gamma(M, -)$ are well defined, and for any good couple $(\mathcal{U}, \xi)$ we have

$$RR^\xi_{\mathcal{U}}(M, -) = \sum_{\gamma \in \mathcal{B}_\xi \cap \Phi(\mathcal{U})} RR^\xi_\gamma(M, -). \tag{1.40}$$

If one takes $\mathcal{U} = M$, we have $RR^\xi_{\mathcal{U}}(M, -) = RR(M, -) = \sum_{\gamma \in \mathcal{B}_\xi} RR^\xi_\gamma(M, -)$ (see [32][Section 4]).

1.3.3 Periodic polynomial behaviour of the multiplicities

We suppose here that the Hamiltonian T-manifold (M, Ω, Φ) is prequantized by a T-complex line bundle L satisfying (1.25) for a suitable invariant connection. In this section we will characterize the periodic polynomial behaviour of the multiplicities $\mathrm{m}(\mu, k)$ with the help of the localized Riemann-Roch character $RR^\xi_0(M, -)$.

Let us introduce some vocabulary. We say that two generalized characters $\chi^\pm = \sum_{\mu \in \Lambda^*} a^\pm_\mu \, \mathbb{C}_\mu$ *coincide* on a region $D \subset \mathfrak{t}^*$, if $a^+_\mu = a^-_\mu$ for every $\mu \in D \cap \Lambda^*$. A generalized character $\chi = \sum_\mu a_\mu \, \mathbb{C}_\mu$ is *supported* on a region $D \subset \mathfrak{t}^*$ if $a_\mu = 0$ for $\mu \notin D$. A weight $\mu \in \Lambda^*$ *occurs* in $\chi = \sum_\mu a_\mu \, \mathbb{C}_\mu$ if $a_\mu \neq 0$.

For $\xi \in \mathfrak{t}^*$, we define $r_\xi > 0$ as the smallest non-zero critical value of the function $\| \Phi - \xi \|$, and we denote by $B(\xi, r_\xi)$ the open ball of center ξ and radius r_ξ.

Theorem 1.3.12 ([32]) *For any $\xi \in \mathfrak{t}^*$, the generalized character $RR^\xi_0(M, L^{\otimes k})$ coincides with $RR(M, L^{\otimes k})$ on the open ball $k \cdot B(\xi, r_\xi)$.*

The arguments of [32] for the proof of this Theorem will be needed another time, so we recall them. Let $\xi \in \mathfrak{t}^*$. We start with the decomposition

$$RR(M, L^{\otimes k}) = \sum_{\gamma \in \mathcal{B}_\xi} RR^\xi_\gamma(M, L^{\otimes k}). \tag{1.41}$$

We recall now, for a non-zero $\gamma \in \mathcal{B}_\xi$, the localization of the map RR^ξ_γ on the fixed point set M^γ [32].

Let N be the normal bundle of M^γ in M. The almost complex structure on M induces an almost complex struture on M^γ and a complex structure on the bundles N and $N_{\mathbb{C}} := N \otimes \mathbb{C}$. Following (1.31) we define the γ-polarized complex vector bundles $N^{+,\gamma}$ and $(N_{\mathbb{C}})^{+,\gamma}$.

The manifold M^γ is a symplectic submanifold of M equipped with an induced Hamiltonian action of T: its moment map is the restriction of Φ on M^γ. Following Definition 1.3.11, we have on M^γ a localized Riemann-Roch character $RR_\gamma^\xi(M^\gamma, -)$. On M^γ, the Hamiltonian vector fields of the functions $\| \Phi - \xi \|^2$ and $\| \Phi - (\xi + \gamma) \|^2$ coincide, hence

$$RR_\gamma^\xi(M^\gamma, -) = RR_0^{\xi+\gamma}(M^\gamma, -). \tag{1.42}$$

We prove in [32][Theorem 5.8.] that

$$RR_\gamma^\xi(M, E) = \sum_{k\in\mathbb{N}} (-1)^l RR_\gamma^\xi(M^\gamma, E|_{M^\gamma} \otimes \det(N^{+,\gamma}) \otimes S^k(N_{\mathbb{C}}^{+,\gamma})) \tag{1.43}$$

for every T-vector bundle E. Here l is the locally constant fonction on M^γ equal to the complex rank of $N^{+,\gamma}$.

Proposition 1.3.13 ([32], Section 5) *Let $\overline{N}$ be the T-vector bundle N with the opposite complex structure on the fibers. The sum $(-1)^l \sum_{k\in\mathbb{N}} \det(N^{+,\gamma}) \otimes S^k(N_{\mathbb{C}}^{+,\gamma})$ is an inverse of $\wedge_{\mathbb{C}}^\bullet \overline{N}$ that we denote $\left[\wedge_{\mathbb{C}}^\bullet \overline{N}\right]_\gamma^{-1}$.*

If we use the notations of Proposition 1.3.13 and (1.42), the localization (1.43) can be rewritten as

$$RR_\gamma^\xi(M, E) = RR_0^{\xi+\gamma}\left(M^\gamma, E|_{M^\gamma} \otimes \left[\wedge_{\mathbb{C}}^\bullet \overline{N}\right]_\gamma^{-1}\right). \tag{1.44}$$

Let $i : T_\gamma \hookrightarrow T$ be the inclusion of the subtorus generated by γ. Let F be a T-vector bundle on M^γ.

Lemma 1.3.14 ([32], Lemma 9.4.) *A weight $\mu \in \Lambda^*$ occurs in $RR_\gamma^\xi(M^\gamma, F)$ only if $i^*(\mu)$ occurs as a weight for the T_γ-action on the fibers of $F \otimes \left[\wedge_{\mathbb{C}}^\bullet \overline{N}\right]_\gamma^{-1}$.*

Since the T_γ weights on the bundles $N_{\mathbb{C}}^{+,\gamma}$ and $N^{+,\gamma}$ are polarized by γ, the localization (1.43) gives the following

Corollary 1.3.15 *For a non-zero $\gamma \in \mathcal{B}_\xi$, the generalized character $RR_\gamma^\xi(M, L^{\otimes k})$ is supported on the half space $\{a \in \mathfrak{t}^* \,|\, (\gamma, a - k(\xi + \gamma)) \geq 0\}$.*

Since the condition $(\gamma, a - k(\xi + \gamma)) \geq 0$ implies that $\| a - k\xi \| \geq k \| \gamma \| \geq k r_\xi$, the last proposition shows that every weights of the open ball $k \cdot B(\xi, r_\xi)$ does not occurs in $RR_\gamma^\xi(M, L^{\otimes k})$. This last remark together with (1.41) prove Theorem 1.3.12.

For the localized Riemann-Roch character $RR_0^\xi(M, -)$ we have the following Lemma which is very similar to Lemma 1.2.3.

Lemma 1.3.16 *Let $\mathfrak{c} \subset \mathfrak{t}^*$ be a connected component of regular values of Φ. For every $\xi, \xi' \in \mathfrak{c}$, we have $RR_0^{\xi}(M,-) = RR_0^{\xi'}(M,-)$.*

Proof. We have to show that the map $\xi \mapsto RR_0^{\xi}(M,-)$ is locally constant on $\mathfrak{c}$. Let $\xi \in \mathfrak{c}$ and take an open neigborhood $\mathcal{U}$ of $\Phi^{-1}(\xi)$ small enough such that the stabilizer $T_m = \{t \in T \,|\, t \cdot m = m\}$ is finite for every $m \in \overline{\mathcal{U}}$. We see then that $\overline{\mathcal{U}} \cap \mathrm{Cr}(\| \Phi - \xi' \|^2) = \Phi^{-1}(\xi')$ and $\partial\mathcal{U} \cap \mathrm{Cr}(\| \Phi - \xi' \|^2) = \emptyset$ if ξ' is close enough to ξ: hence $RR_0^{\xi'}(M,-) = RR_{\mathcal{U}}^{\xi'}(M,-)$ for ξ' close enough to ξ. The second point of Proposition 1.3.10 finishes the proof. □

When ξ is a regular value of Φ, the localized Riemann-Roch character $RR_0^{\xi}(M,-)$ as been computed in [32] as follows. Let $RR(\mathcal{M}_{\xi},-)$ be the Riemann-Roch map defined on the orbifold $\mathcal{M}_{\xi} = \Phi^{-1}(\xi)/T$ by means of an almost complex structure compatible with the induced symplectic structure. For every T-vector bundle $E \to M$ we define the following familly of orbifold vector bundles over $\mathcal{M}_{\xi}$:

$$\mathcal{E}_{\xi}^{\mu} := \Big(E|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu}\Big)/T, \quad \mu \in \Lambda^*. \tag{1.45}$$

For every T-vector bundle E on M, we proved in [32][Section 6.2.] the following equality in $R^{-\infty}(T)$

$$RR_0^{\xi}(M,E) = \sum_{\mu \in \Lambda^*} RR(\mathcal{M}_{\xi}, \mathcal{E}_{\xi}^{\mu})\, \mathbb{C}_{\mu}. \tag{1.46}$$

This decomposition was first obtained by Vergne [37] when T is the circle group and when M is Spin. The number $RR(\mathcal{M}_{\xi}, \mathcal{E}_{\xi}^{\mu}) \in \mathbb{Z}$ is then equal to the T-invariant part of the index $RR_0^{\xi}(M,E) \otimes \mathbb{C}_{-\mu}$.

Remark 1.3.17 *Let $t \to t^{\lambda}$ be a character of T. Suppose that a subgroup $H \subset T$ acts trivially on M and with the character $t \in H \to t^{\lambda}$ on the the fibers of the T-vector bundle E. Then H acts with the character $t \in H \to t^{\lambda-\mu}$ on $RR_0^{\xi}(M,E) \otimes \mathbb{C}_{-\mu}$, and then $RR(\mathcal{M}_{\xi}, \mathcal{E}_{\xi}^{\mu}) \neq 0$ only if $t^{\lambda-\mu} = 1$ for every $t \in H$. So the sum in (1.46) can be restricted to $\lambda + \Lambda_H^*$, where Λ_H^* is the sub-lattice of Λ^* formed by the element $\alpha \in \Lambda^*$ satisfying $t^{\alpha} = 1,\ \forall\ t \in H$.*

This remark applies also on the usual character $RR(M,E) = \sum_{\mu \in \Lambda^*} m_{\mu} \mathbb{C}_{\mu}$. The multiplicity $m_{\mu} \in \mathbb{Z}$ is equal to the (virtual) dimension of the T-invariant part of $RR(M,E) \otimes \mathbb{C}_{-\mu}$. With the same hypothesis than above we see that $m_{\mu} \neq 0$ only if $\mu \in \lambda + \Lambda_H^*$.

Let Γ_M be the generic stabilizer for the action of T on M. Consider a weight α_o such that Γ_M acts on the fibers of L with the character $t \mapsto t^{\alpha_o}$. We define the sublattice $\Xi(M,L) \subset \Lambda^* \times \mathbb{Z}$ by

$$\Xi(M,L) := \{(\mu,k) \in \Lambda^* \times \mathbb{Z} \mid k\alpha_o - \mu \in \Lambda_{\Gamma_M}^*\}. \tag{1.47}$$

We know then that $\mathrm{m}(\mu,k) = 0$ if $(\mu,k) \notin \Xi(M,L)$.

Proposition 1.3.18 *Let $\mathfrak{c}$ be a connected component of regular values of Φ and let* $\mathrm{Cone}(\mathfrak{c})$ *be the corresponding cone in $\mathfrak{t}^* \times \mathbb{R}^{>0}$ (see (1.29)). Let $\xi \in \mathfrak{c}$. For any $(\mu, k) \in \mathrm{Cone}(\mathfrak{c}) \cap \Xi(M, L)$ we have*

$$\mathrm{m}(\mu, k) = RR(\mathcal{M}_\xi, \mathcal{L}_\xi^{\mu,k}) \tag{1.48}$$

where

$$\mathcal{L}_\xi^{\mu,k} = (L^{\otimes k}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu})/T. \tag{1.49}$$

Proof. Let $(\mu, k) \in \mathrm{Cone}(\mathfrak{c})$ and let $\xi' = \frac{\mu}{k} \in \mathfrak{c}$. We known from Theorem 1.3.12 that the generalized character $RR_0^{\xi'}(M, L^{\otimes k})$ coincides with $RR(M, L^{\otimes k})$ on the open ball $k \cdot B(\xi', r_{\xi'}) = B(\mu, k r_{\xi'})$. So $\mathrm{m}(\mu, k)$ is equal to the μ-multiplicity in $RR_0^{\xi'}(M, L^{\otimes k})$. Take now any $\xi \in \mathfrak{c}$. We know after Lemma 1.3.16 that $RR_0^{\xi}(M, -) = RR_0^{\xi'}(M, -)$ and (1.46) shows that the μ-multiplicity in $RR_0^{\xi}(M, L^{\otimes k})$ is equal to $RR(\mathcal{M}_\xi, \mathcal{L}_\xi^{\mu,k})$. □

Definition 1.3.19 *Take $\xi \in \mathfrak{c}$. The map $\mathrm{m}_{\mathfrak{c}} : \Lambda^* \times \mathbb{Z} \to \mathbb{Z}$ is defined by the equation*

$$\mathrm{m}_{\mathfrak{c}}(\mu, k) = RR(\mathcal{M}_\xi, \mathcal{L}_\xi^{\mu,k}), \tag{1.50}$$

where $\mathcal{L}_\xi^{\mu,k}$ is the orbifold line bundle defined by (1.49).

In other words, the map $\mathrm{m}_{\mathfrak{c}}$ is defined by the following equality in $R^{-\infty}(T)$

$$\sum_{\mu \in \Lambda^*} \mathrm{m}_{\mathfrak{c}}(\mu, k)\ \mathbb{C}_\mu = RR_0^{\xi}(M, L^{\otimes k}).$$

for all $k \in \mathbb{Z}$. After remark 1.3.17, we know that $\mathrm{m}_{\mathfrak{c}}$ is supported on the sub-lattice $\Xi(M, L)$ defined in (1.47).

We will now exploit the Riemann-Roch for orbifold due to Kawasaki [23] to show that the map $\mathrm{m}_{\mathfrak{c}}$ is a periodic polynomial.

1.3.4 Riemann-Roch-Kawasaki theorem

First we recall how is defined the Riemann-Roch character $RR(\mathcal{M}_\xi, \mathcal{E}_\xi)$ when ξ is a regular value of Φ, and $\mathcal{E}_\xi = E_{|\Phi^{-1}(\xi)}/T$ is the reduction of a complex T-vector bundle E over M. The number $RR(\mathcal{M}_\xi, \mathcal{E}_\xi) \in \mathbb{Z}$ is defined has the T-invariant part of the index of a transversally elliptic operator D_E on $\Phi^{-1}(\xi)$. Since the index of D_E depend only of the class of its symbol $\sigma(D_E)$ in $K_T(\mathbf{T}_T\Phi^{-1}(\xi))$, it is enough to define the transversally elliptic symbol $\sigma(D_E)$. Since the action of T on $\Phi^{-1}(\xi)$ is locally free, $V := \mathbf{T}_T\Phi^{-1}(\xi)$ is a vector bundle. It carries a canonical symplectic structure on the fibers and we choose any compatible complex structure making V into a Hermitian vector bundle. At $(m, v) \in \mathbf{T}\Phi^{-1}(\xi)$, the map $\sigma(D_E)(m, v)$ is the Clifford action

$$\mathrm{Cl}_m(v_1) \otimes Id_{E_m}\ :\ (\wedge_{\mathbb{C}}^{even} V_m) \otimes E_m \longrightarrow (\wedge_{\mathbb{C}}^{odd} V_m) \otimes E_m.$$

where $v_1 \in V_m$ is the V-component of the vector $v \in \mathbf{T}_m \Phi^{-1}(\xi)$. We explain now the formula of Kawasaki for $RR(\mathcal{M}_\xi, \mathcal{E}_\xi)$ when $\xi \in \Phi(M)$ is a regular value of Φ [23].

Let $\mathbb{F}$ be the collection of the finite subgroup of T which are stabilizer of points in M. Consider the orbit type stratification of $\Phi^{-1}(\xi)$ and denote by $\mathbb{S}_\xi$ the set of its orbit type strata. Each statum S is a connected component of the smooth submanifold

$$\Phi^{-1}(\xi)_{H_S} := \{m \in \Phi^{-1}(\xi) \,|\, \mathrm{Stab}_T(m) = H_S\}. \tag{1.51}$$

for a unique $H_S \in \mathbb{F}$. The orbifold $\mathcal{M}_\xi$ decomposes as a disjoint union $\cup_{S \in \mathbb{S}_\xi} S/T$ of smooth components, and each quotient $\overline{S}/T$ is a suborbifold of $\mathcal{M}_\xi$. The generic stabilizer Γ_M of T on M is also the generic stabilizer of T on the fiber[3] $\Phi^{-1}(\xi)$, and is associated to an open and dense stratum S_{max}.

Suppose that $E \to M$ is an Hermitian T-vector bundle. On each suborbifold $\overline{S}/T$, we get the orbifold complex vector bundle

$$\mathcal{E}_S := E_{|\overline{S}}/T. \tag{1.52}$$

We define twisted characteristic classes $\mathrm{Ch}^-(\mathcal{E}_S)$ and $D^-(\mathcal{E}_S)$ by

$$\mathrm{Ch}^\gamma(\mathcal{E}_S) := \mathbf{Tr}\left(\gamma^{\mathcal{E}_S} \cdot e^{\frac{i}{2\pi}R(\mathcal{E}_S)}\right), \quad \gamma \in H_S, \tag{1.53}$$

and

$$D^\gamma(\mathcal{E}_S) := \det\left(1 - (\gamma^{\mathcal{E}_S})^{-1} \cdot e^{-\frac{i}{2\pi}R(\mathcal{E}_S)}\right), \quad \gamma \in H_S. \tag{1.54}$$

Here $R(\mathcal{E}_S) \in \mathcal{A}^2(\overline{S}/T, \mathrm{End}(\mathcal{E}_S))$ is the curvature of an horizontal Hermitian connection on $E_{|\overline{S}}$, and $\gamma \mapsto \gamma^{\mathcal{E}_S}$ is the linear action of H_S on the fibers of $E_{|\overline{S}}$.

Let N_S be the normal bundle of $\overline{S}$ in $\Phi^{-1}(\xi)$. The symplectic struture on M induces a symplectic form Ω_S on each suborbifold $\overline{S}/T$, and a symplectic structure on the fibers of the bundle N_S. Choose a compatible almost complex structure on $\overline{S}/T$, and a compatible complex structure on the fibers of N_S making the tangent bundle of $\overline{S}/T$ and $\mathcal{N}_S := N_S/T$ into Hermitian vector bundle. Consider a Hermitian connexion on $\mathbf{T}(\overline{S}/T)$, with curvature $R(\overline{S}/T)$, and let

$$\mathrm{Todd}(\overline{S}/T) = \det\left(\frac{(i/2\pi)R(\overline{S}/T)}{1 - e^{-(i/2\pi)R(\overline{S}/T)}}\right) \tag{1.55}$$

be the corresponding Todd forms. Like in (1.54), we associate to the complex orbifold vector bundle $\mathcal{N}_S$, the twisted form $D^-(\mathcal{N}_S)$ which is a map form H_S to $\mathcal{A}^{even}(\overline{S}/T)$. The 0-degree part of $D^\gamma(\mathcal{N}_S)$ is equal to $\det(1 - (\gamma^{\mathcal{N}_S})^{-1})$, hence $D^\gamma(\mathcal{N}_S)$ is invertible in $\mathcal{A}^{even}(\overline{S}/T)$ when γ belongs to

[3] Since a neighborhood of $\Phi^{-1}(\xi)$ in M is T-equivariantly diffeomorphic to $\Phi^{-1}(\xi) \times \mathfrak{t}^*$.

$$H_S^o = \{\gamma \in H_S \mid \det(1-(\gamma^{\mathcal{N}_S})^{-1}) \neq 0\}. \tag{1.56}$$

Note that H_S^o corresponds to the set of $\gamma \in H_S$ for which $\overline{S}$ is a connected component of $(\Phi^{-1}(\xi))^\gamma$.

Theorem 1.3.20 (Kawasaki) *The number $RR(\mathcal{M}_\xi, \mathcal{E}_\xi) \in \mathbb{Z}$ is given by the formula*

$$RR(\mathcal{M}_\xi, \mathcal{E}_\xi) = \sum_{S \in \mathbb{S}_\xi} \frac{1}{|H_S|} \sum_{\gamma \in H_S^o} \int_{\overline{S}/T} \frac{\mathrm{Todd}(\overline{S}/T)\mathrm{Ch}^\gamma(\mathcal{E}_S)}{D^\gamma(\mathcal{N}_S)}. \tag{1.57}$$

We exploit now Theorem 1.3.20 to show that the map $\mathrm{m}_{\mathfrak{c}} : \Lambda^* \times \mathbb{Z} \to \mathbb{Z}$ which is defined by (1.50) is periodic polynomial. We need the classical computation of the first Chern class of the line bundle

$$\mathcal{L}_S^{\mu,k} = (L^{\otimes k} \otimes \mathbb{C}_{-\mu})_{|\overline{S}}/T. \tag{1.58}$$

The curvature form $\omega_\xi \in H^2(\mathcal{M}_\xi) \otimes \mathfrak{t}$ of the principal T-bundle $\Phi^{-1}(\xi) \to \mathcal{M}_\xi$ restricts to a curvature form $\omega_S \in H^2(\overline{S}/T) \otimes \mathfrak{t}$ on each strata.

Lemma 1.3.21 *The first Chern class of the line bundle $\mathcal{L}_S^{\mu,k}$ is given by*

$$c_1(\mathcal{L}_S^{\mu,k}) = \frac{1}{2\pi}\Big(k\Omega_S - \langle k\xi - \mu, \omega_S\rangle\Big).$$

For a strata S, we consider $\alpha_S \in \Lambda^*$ such that $\gamma \in H_S \mapsto \gamma^{\alpha_S}$ corresponds to the action of H_S on the fibers of $L_{|\overline{S}}$. Finally we have the decomposition

$$\mathrm{m}_{\mathfrak{c}}(\mu, k) = \sum_{S \in \mathbb{S}_\xi} P_S(\mu, k), \tag{1.59}$$

where

$$P_S(\mu,k) = \frac{1}{|H_S|} \sum_{\gamma \in H_S^o} \gamma^{k\alpha_S - \mu} \int_{\overline{S}/T} \frac{\mathrm{Todd}(\overline{S}/T)}{D^\gamma(\mathcal{N}_S)}\, e^{\frac{1}{2\pi}\left(k\Omega_S - \langle k\xi-\mu,\omega_S\rangle\right)}. \tag{1.60}$$

When S is the principal open dense stratum S_{max} the map P_S is

$$P_{max}(\mu,k) = \frac{\sum_{\gamma \in \Gamma_M} \gamma^{k\alpha_o - \mu}}{|\Gamma_M|} \int_{\mathcal{M}_\xi} \mathrm{Todd}(\mathcal{M}_\xi) e^{\frac{1}{2\pi}(k\Omega_\xi - \langle k\xi - \mu, \omega_\xi\rangle)}. \tag{1.61}$$

The term $\frac{\sum_{\gamma \in \Gamma_M} \gamma^{k\alpha_o - \mu}}{|\Gamma_M|}$ is equal to 1 when (μ, k) belongs to the lattice $\Xi(M, L)$ (see (1.47)), and is equal to 0 in the other cases. From (1.60) we see that P_S is a periodic polynomial of degree less than $\frac{dim(\overline{S}/T)}{2}$, and for $S = S_{max}$ we have on $\Xi(M, L)$

$$P_{max}(\mu,k)=\frac{1}{(2\pi)^d}\int_{\mathcal{M}_\xi}\frac{(k\Omega_\xi-\langle k\xi-\mu,\omega_\xi\rangle)^d}{d!}+O(d-1) \tag{1.62}$$

where $d=\frac{\dim\mathcal{M}_\xi}{2}$ and $O(d-1)$ denotes a polynomial of degree less than $d-1$. If we use the polynomial $\mathrm{DH}_{\mathfrak{c}}$ defined in Section 1.2 we can conclude our computations with the following

Proposition 1.3.22 *The map* $\mathrm{m}_{\mathfrak{c}}$ *is a periodic polynomial of degree* $d=\frac{\dim\mathcal{M}_\xi}{2}$ *supported on* $\Xi(M,L)$. *For* $(\mu,k)\in\Xi(M,L)$ *we have*

$$\mathrm{m}_{\mathfrak{c}}(\mu,k)=|\Gamma_M|\frac{k^d}{(2\pi)^d}\mathrm{DH}_{\mathfrak{c}}(\frac{\mu}{k})\ +\ O(d-1),$$

where $O(d-1)$ *means a periodic polynomial of degree less than* $d-1$.

1.3.5 Wall-crossing formulas for the $\mathrm{m}_{\mathfrak{c}}$

Let $\mathfrak{c}_+$ and $\mathfrak{c}_-$ be two adjacent connected component of regular values of Φ separated by an hyperplane Δ. The aim of this section is to compute the periodic polynomial $\mathrm{m}_{\mathfrak{c}_+}-\mathrm{m}_{\mathfrak{c}_-}$.

We consider two points $\xi_\pm\in\mathfrak{c}_\pm$ such that $\xi=\frac{1}{2}(\xi_++\xi_-)$ belongs to the relative interior of $\overline{\mathfrak{c}_+}\cap\overline{\mathfrak{c}_-}$ in Δ. We suppose furthermore that $\xi^+-\xi^-$ is orthogonal to Δ. Using the identification $\mathfrak{t}^*\simeq\mathfrak{t}$ given by the scalar product the vector $\gamma=\frac{1}{2}(\xi_+-\xi_-)$, seen as a vector of $\mathfrak{t}_\Delta$, belongs[4] to $\mathbb{R}^{>0}\beta$. We noticed in Section 1.2.4 that for all $m\in\Phi^{-1}(\xi)$ the stabilizer $\mathfrak{t}_m$ is either equal to $\mathfrak{t}_\Delta$ or to $\{0\}$. Then there exists an open T-invariant neighborhood $\mathcal{U}$ of $\Phi^{-1}(\xi)$ in M such that for all $m\in\overline{\mathcal{U}}$ either $\mathfrak{t}_m:=\{0\}$, or $\mathfrak{t}_m=\mathfrak{t}_\Delta$ and $\Phi(m)\in\Delta$.

One see easily that the couple $(\mathcal{U},\xi)$ is good and the second point of Proposition 1.3.10 tells us that

$$RR^{\xi}_{\mathcal{U}}(M,-)=RR^{\xi_-}_{\mathcal{U}}(M,-)=RR^{\xi_+}_{\mathcal{U}}(M,-) \tag{1.63}$$

when $\xi_\pm$ are close enough to ξ. Since $\mathcal{U}\cap\mathrm{Cr}(\parallel\Phi-\xi\parallel^2)=\Phi^{-1}(\xi)$ we have $RR^{\xi}_{\mathcal{U}}(M,-)=RR^{\xi}_{0}(M,-)$. If $\xi_\pm$ are close enough to ξ we have

$$\mathcal{U}\cap\mathrm{Cr}(\parallel\Phi-\xi_\pm\parallel^2)=\Phi^{-1}(\xi_\pm)\bigcup M^\gamma\cap\Phi^{-1}(\xi). \tag{1.64}$$

The former decomposition is due to (1.39) and to the fact that the stabiliser of $\mathfrak{t}$ on $\mathcal{U}$ are either equal to $\mathfrak{t}_\Delta$ or to $\{0\}$. Notice that $\xi_-+\gamma=\xi_++\gamma=\xi$. The decomposition (1.64) gives

$$RR^{\xi_\pm}_{\mathcal{U}}(M,-)=RR^{\xi_\pm}_{0}(M,-)+RR^{\xi_\pm}_{\mp\gamma}(M,-), \tag{1.65}$$

where $RR^{\xi_-}_{\gamma}(M,-)$ (resp. $RR^{\xi_+}_{-\gamma}(M,-)$) is the Riemann-Roch character localized on $M^\gamma\cap\Phi^{-1}(\xi)$ by the vector field $\mathcal{H}-(\xi_-)_M$ (resp. $\mathcal{H}-(\xi_+)_M$). Now (1.63) and (1.65) prove the following

[4] β is the primitive vector of $\mathfrak{t}_\Delta\cap\Lambda$ pointing out of $\mathfrak{c}_-$

Proposition 1.3.23 *If $\xi_\pm$ are close enough to Δ, we have*

$$RR_0^{\xi_+}(M,-) - RR_0^{\xi_-}(M,-) = RR_\gamma^{\xi_-}(M,-) - RR_{-\gamma}^{\xi_+}(M,-).$$

We know from Proposition 1.3.18 that $\mathrm{m}_{\mathfrak{c}_\pm}(\mu,k)$ is equal to the μ-multiplicity of T in $RR_0^{\xi_\pm}(M,L^{\otimes k})$. Hence $\mathrm{m}_{\mathfrak{c}_+}(\mu,k) - \mathrm{m}_{\mathfrak{c}_-}(\mu,k)$ is equal to the μ-multiplicity of T in $RR_\gamma^{\xi_-}(M,L^{\otimes k}) - RR_{-\gamma}^{\xi_+}(M,L^{\otimes k})$.

Let N_Δ be the normal bundle of M^{T_Δ} in M, and let $\left[\wedge_{\mathbb{C}}^\bullet \overline{N_\Delta}\right]_{\pm\beta}^{-1}$ be the polarized inverses of $\wedge_{\mathbb{C}}^\bullet \overline{N_\Delta}$ (see Proposition 1.3.13). Since $\xi = \xi_+ - \gamma = \xi_- + \gamma$ and $\gamma \in \mathbb{R}^{>0}\beta$, the localization (1.44) gives

$$RR_\gamma^{\xi_-}(M,L^{\otimes k}) = \sum_{Z\in\mathcal{F}} RR_0^\xi\left(Z, L^{\otimes k}|_Z \otimes \left[\wedge_{\mathbb{C}}^\bullet \overline{N_Z}\right]_\beta^{-1}\right),$$
$$RR_{-\gamma}^{\xi_+}(M,L^{\otimes k}) = \sum_{Z\in\mathcal{F}} RR_0^\xi\left(Z, L^{\otimes k}|_Z \otimes \left[\wedge_{\mathbb{C}}^\bullet \overline{N_Z}\right]_{-\beta}^{-1}\right).$$

Finally $\mathrm{m}_{\mathfrak{c}_+}(\mu,k) - \mathrm{m}_{\mathfrak{c}_-}(\mu,k) = \sum_{Z\in\mathcal{F}} \mathcal{A}_Z(\mu,k)$ where $\mathcal{A}_Z(\mu,k)$ is equal to the μ-multiplicity of T in

$$RR_0^\xi\left(Z, L^{\otimes k}|_Z \otimes \left[\wedge_{\mathbb{C}}^\bullet \overline{N_Z}\right]_{-\beta}^{-1}\right) - RR_0^\xi\left(Z, L^{\otimes k}|_Z \otimes \left[\wedge_{\mathbb{C}}^\bullet \overline{N_Z}\right]_{\beta}^{-1}\right) \qquad (1.66)$$

Let $\beta' \in \mathfrak{t}_\Delta^* \cap \Lambda^*$ which is defined by the relation $\langle \beta', \beta\rangle = 2\pi$, so that $\Lambda_{\mathfrak{t}_\Delta}^* = \mathbb{Z}\beta'$. Concerning the T_Δ-weights we have

1. The T_Δ-weight on $L^{\otimes k}|_Z$ is equal to $kr_\Delta\beta'$.
2. The T_Δ-weight on $\det(N_Z^{+,\pm\beta})$ is $\pm s_Z^\pm \beta'$ where $s_Z^\pm \in \mathbb{N}$ is the absolute value of the trace of $\frac{1}{2\pi}\mathcal{L}_\beta$ on $N_Z^{+,\pm\gamma}$.
3. The T_Δ-weights on $S^{>0}(N_Z^{+,\beta})$ (resp. $S^{>0}(N_Z^{+,-\beta})$) are of the form $p\beta'$ with $p>0$ (resp. $p<0$).

Now Lemma 1.3.14 shows that if a weight μ occurs in $RR_0^\xi\left(Z, L^{\otimes k}|_Z \otimes \left[\wedge_{\mathbb{C}}^\bullet \overline{N_Z}\right]_{\beta}^{-1}\right)$ we have $i^*(\mu) = (kr_\Delta + s_Z^+ + p)\beta'$ with $p \geq 0$, and then

$$\frac{\langle \mu, \beta\rangle}{2\pi} - kr_\Delta \geq s_Z^+.$$

Similarly, if a weight μ occurs in $RR_0^\xi\left(Z, L^{\otimes k}|_Z \otimes \left[\wedge_{\mathbb{C}}^\bullet \overline{N_Z}\right]_{-\beta}^{-1}\right)$ we have

$$\frac{\langle \mu, \beta\rangle}{2\pi} - kr_\Delta \leq -s_Z^-.$$

Finally, $\mathcal{A}_Z(\mu,k) = 0$ when $-s_Z^- < \frac{\langle \mu,\beta\rangle}{2\pi} - kr_\Delta < s_Z^+$. We have proved the following

Theorem 1.3.24 *Let $s^{\pm} = \inf_Z s_Z^{\pm}$ where the infimum is taken over the connected components Z of M^{T_Δ} for which $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-} \subset \Phi(Z)$. For every $(\mu, k) \in \Lambda^* \times \mathbb{Z}$, we have $\mathrm{m}_{\mathfrak{c}_+}(\mu, k) = \mathrm{m}_{\mathfrak{c}_-}(\mu, k)$ if*

$$-s^- < \frac{\langle \mu, \beta \rangle}{2\pi} - k r_\Delta < s^+. \tag{1.67}$$

Sometimes the inequalities (1.67) are optimal.

Proposition 1.3.25 • *Consider the connected components $Z \in \mathcal{F}$ for which s_Z^+ is minimal. Among them consider the subset $\mathcal{F}_+$ where $\dim(Z)$ is maximal. If the integers $\mathrm{rk}_{\mathbb{C}}(N_Z^{+,\beta})$, $Z \in \mathcal{F}_+$ have the same parity, then the condition "$\frac{\langle \mu, \beta \rangle}{2\pi} - k r_\Delta < s^+$" is optimal in (1.67).*

• *In the same way, consider the connected components $Z \in \mathcal{F}$ for which s_Z^- is minimal. Among them consider the subset $\mathcal{F}_-$ where $\dim(Z)$ is maximal. If the integers $\mathrm{rk}_{\mathbb{C}}(N_Z^{+,\beta})$, $Z \in \mathcal{F}_+$ have the same parity, then the condition "$-s^- < \frac{\langle \mu, \beta \rangle}{2\pi} - k r_\Delta$" is optimal in (1.67).*

Remark 1.3.26 *The last Proposition applies when there is a* unique *connected component Z of M^{T_Δ} for which $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-} \subset \Phi(Z)$.*

Proof. We consider only the first point since the other point works similarly. We restrict our attention to the couples (μ, k) such that $\frac{\langle \mu, \beta \rangle}{2\pi} - k r_\Delta = s^+$. They are of the form

$$\mu = (k r_\Delta + s^+)\beta' + \mu_2 \tag{1.68}$$

with $\mu_2 \in \Lambda^*_{\mathfrak{t}/\mathfrak{t}_\Delta}$. Let us denote $D(\mu_2, k)$ the restriction of $\mathrm{m}_{\mathfrak{c}_+}(\mu, k) - \mathrm{m}_{\mathfrak{c}_-}(\mu, k)$ to the set of couples (μ, k) parametrized by (1.68). We want to prove that $D(\mu_2, k)$ is not identically equal to zero.

From the previous discussion one knows that

$$D(\mu_2, k) = \sum_{Z\,,\, s_Z^+ = s^+} (-1)^{\mathrm{rk}_{\mathbb{C}}(N_Z^{+,\beta})} D_Z(\mu_2, k), \tag{1.69}$$

where $D_Z(\mu_2, k)$ is the μ-multiplicity of T in

$$RR_0^\xi \left(Z, L^{\otimes k}|_Z \otimes \det(N_Z^{+,\beta}) \right).$$

Let us make few remarks concerning the maps $RR_0^\xi(Z, -) : \mathbf{K}_T(Z) \to R^{-\infty}(T)$. Since T_Δ acts trivially on Z, the decomposition $T = T/T_\Delta \times T_\Delta$ induces a canonical isomorphism $\mathbf{K}_T(Z) \simeq \mathbf{K}_{T/T_\Delta}(Z) \otimes R(T_\Delta)$: i.e. every T-equivariant vector bundle $E \to Z$ decomposes as

$$E = \sum_{\mu_1 \in \mathbb{Z}\beta'} E^{\mu_1} \otimes \mathbb{C}_{\mu_1}. \tag{1.70}$$

Here each E^{μ_1} is a T/T_Δ-equivariant vector bundle on Z and $\mathbb{C}_{\mu_1}$ denotes the one dimensional T_Δ-representation associated to $\mu_1 \in \Lambda^*_{\mathfrak{t}_\Delta}$.

For every T-equivariant vector bundle $E \to Z$, the character $RR_0^\xi(Z,E)$ is equal to the T-equivariant index of the T-transversally elliptic symbol $\mathrm{Thom}_\xi(\mathcal{V}) \otimes p^*(E)$, where $\mathcal{V}$ is a small neighborhood of $\Phi^{-1}(\xi) \cap Z$ in Z (see Definition 1.3.9). Since the T_Δ action is trivial on Z the symbol $\mathrm{Thom}_\xi(\mathcal{V})$ is also T/T_Δ transversally elliptic and the action of T_Δ is trivial on it. We have then

$$RR_0^\xi(Z,E) = \sum_{\mu_1 \in \mathbb{Z}\beta'} RR_0^\xi(Z, E^{\mu_1}) \otimes \mathbb{C}_{\mu_1}. \tag{1.71}$$

where the character $RR_0^\xi(Z,E^{\mu_1}) \in R^{-\infty}(T/T_\Delta)$ is computed by Theorem 1.46 applied to the Hamiltonian T/T_Δ-manifold Z. For every T-vector bundle $E \to Z$ we define the familly $\mathcal{E}_\xi^{\mu_1,\mu_2}, \mu_1 \in \mathbb{Z}\beta', \mu_2 \in \Lambda^*_{\mathfrak{t}/\mathfrak{t}_\Delta}$ of orbifold vector bundles over the reduced space $\mathcal{Z}_\xi = Z \cap \Phi^{-1}(\xi)/(T/T_\Delta)$ by

$$\mathcal{E}_\xi^{\mu_1,\mu_2} := (E^{\mu_1} \otimes \mathbb{C}_{-\mu_2})\,|_{\Phi^{-1}(\xi)\cap Z}/(T/T_\Delta). \tag{1.72}$$

Finally (1.46) and (1.71) give the following

$$\begin{aligned} RR_0^\xi(Z,E) &= \sum_{\mu_1 \in \mathbb{Z}\beta'} \sum_{\mu_2 \in \Lambda^*_{\mathfrak{t}/\mathfrak{t}_\Delta}} RR(\mathcal{Z}_\xi, \mathcal{E}_\xi^{\mu_1,\mu_2}) \otimes \underbrace{\mathbb{C}_{\mu_1}}_{\in R(T_\Delta)} \otimes \underbrace{\mathbb{C}_{\mu_2}}_{\in R(T/T_\Delta)} \\ &= \sum_{\mu \in \Lambda^*} RR(\mathcal{Z}_\xi, \mathcal{E}_\xi^{\mu_1,\mu_2})\ \mathbb{C}_\mu \quad . \end{aligned} \tag{1.73}$$

In (1.73) we write $\mu \in \Lambda^*$ as a sum of $\mu_1 \in \mathbb{Z}\beta'$ with $\mu_2 \in \Lambda^*_{\mathfrak{t}/\mathfrak{t}_\Delta}$ so that $\mathbb{C}_\mu \in R(T)$ is equal to the tensor product $\mathbb{C}_{\mu_1} \otimes \mathbb{C}_{\mu_1}$.

When the vector bundle $E \to Z$ is the line bundle $\mathbb{L} := L^{\otimes k}|_Z \otimes \det(N_Z^{+,\beta})$ we have $\mathbb{L} = \mathbb{L}^{j\beta'} \otimes \mathbb{C}_{j\beta'}$ for $j = kr_\Delta + s^+$. Finally, we have

$$D_Z(\mu_2, k) = RR(\mathcal{Z}_\xi, \mathbb{L}_\xi^{(kr_\Delta + s^+)\beta', \mu_2}).$$

Now we use the results of Section 1.3.4 to study the map

$$D_Z : \Lambda^*_{\mathfrak{t}/\mathfrak{t}_\Delta} \times \mathbb{Z} \longrightarrow \mathbb{Z}. \tag{1.74}$$

Let $\Gamma_Z \subset T/T_\Delta$ be the generic stabiliser of T/T_Δ on a component Z. Let $\alpha_Z, \delta_Z \in \Lambda^*_{\mathfrak{t}/\mathfrak{t}_\Delta}$ such that the action of Γ_Z on the fibers of $L_{|Z}$ and $\det(N_Z^{+,\beta})$ are respectively $t \to t^{\alpha_Z}$ and $t \to t^{\delta_Z}$. After Remark 1.3.17 we know that the map (1.74) is supported on the subset

$$\Xi_Z := \{(\mu_2, k) \in \Lambda^*_{\mathfrak{t}/\mathfrak{t}_\Delta} \times \mathbb{Z} \mid t^{k\alpha_Z + \delta_Z + \mu_2} = 1,\ \forall\, t \in \Gamma_Z\}. \tag{1.75}$$

The only difference with the computations done in Section 1.3.4 is the line bundle $\det(N_Z^{+,\beta})$. But this do not change the global behaviour of the map (1.74) on Ξ_Z: it is a periodic polynomial map of degree $d_Z = \dim(\mathcal{Z}_\xi)/2$ and we have

$$D_Z(\mu_2,k)=\frac{1}{(2\pi)^{d_Z}}\int_{Z_\xi}\frac{(k\Omega_{Z_\xi}-\langle k\xi-\mu_2,\omega_{Z_\xi}\rangle)^{d_Z}}{d_Z!}+O(d_Z-1) \tag{1.76}$$

for all $(\mu_2,k)\in\Xi_Z$.

Suppose now that all the signs $(-1)^{\mathrm{rk}_{\mathbb{C}}(N_Z^{+,\beta})}$ coincide when $Z\in\mathcal{F}_+$. From (1.69), we get that $D(\mu_2,k)$ does not vanish for large values of (μ_2,k). □

1.4 Multiplicities of group representations

Let K be a semi-simple compact Lie group with Lie algebra $\mathfrak{k}$, and let T be a maximal torus in K with Lie algebra $\mathfrak{t}$. In this section we denote $(-,-)$ the scalar product on $\mathfrak{k}$ induced by the Killing form, and we keep the same notation for the induced scalar products on $\mathfrak{t}^*$ and on $\mathfrak{t}$.

Let $\Lambda^*\subset\mathfrak{t}^*$ be the weight lattice, and let $\mathfrak{R}\subset\Lambda^*$ be the set of roots for the action of T on $\mathfrak{k}\otimes\mathbb{C}$: we denote $\Lambda^*_{\mathfrak{R}}$ the sub-lattice of Λ^* generated by $\mathfrak{R}$. We choose a system of positive roots $\mathfrak{R}^+\subset\mathfrak{R}$, and we denote $\mathfrak{t}^*_+$ the corresponding Weyl chamber.

The irreducible representations of K are parametrized by the set $\Lambda^*_+=\Lambda^*\cap\mathfrak{t}^*_+$. For $\lambda\in\Lambda^*_+$ we denote by V_λ the irreducible representation of K with heighest weight λ. Here we are interested in the T-multiplicities in $V_\lambda|_T$. Let $\mathrm{m}:\Lambda^*\times\Lambda^*_+\to\mathbb{N}$ be the map defined by

$$V_\lambda|_T=\sum_{\mu\in\Lambda^*}\mathrm{m}(\mu,\lambda)\ \mathbb{C}_\mu \tag{1.77}$$

for every $\lambda\in\Lambda^*_+$.

Definition 1.4.1 *For every $\lambda\in\Lambda^*_+$, we denote $\mathrm{m}^\lambda:\Lambda^*\times\mathbb{Z}^{>0}\to\mathbb{N}$ the map defined by $\mathrm{m}^\lambda(\mu,k)=\mathrm{m}(\mu,k\lambda)$. So $\mathrm{m}^\lambda(\mu,k)$ is equal to the multiplicity of $\mathbb{C}_\mu$ in $V_{k\lambda}|_T$.*

1.4.1 Borel-Weil Theorem

First we recall the realization of the K-representation V_λ given by the Borel-Weil Theorem. The coadjoint orbit $K\cdot\lambda$ is equipped with the Kirillov-Kostant-Souriau symplectic form Ω which is defined by:

$$\Omega(X_M,Y_M)_m=\langle m,[X,Y]\rangle,\quad\text{for}\quad m\in K\cdot\lambda\quad\text{and}\quad X,Y\in\mathfrak{k}. \tag{1.78}$$

The action of K on $K\cdot\lambda$ is Hamiltonian with moment map $K\cdot\lambda\hookrightarrow\mathfrak{k}^*$ equal to the inclusion. The action of T on $K\cdot\lambda$ is also Hamiltonian with moment map $\Phi:K\cdot\lambda\to\mathfrak{t}^*$ equal to the composition of the inclusion $K\cdot\lambda\hookrightarrow\mathfrak{k}^*$ with the projection map $\mathfrak{k}^*\to\mathfrak{t}^*$.

There exists a unique K-invariant complex structure on $K\cdot\lambda$ compatible with the symplectic form. In this situation the Kostant-Souriau prequantum line bundle over $K\cdot\lambda$ is

$$\mathbb{C}_{[\lambda]} = K \times_{K_\lambda} \mathbb{C}_\lambda.$$

Here we use the canonical identification $K/K_\lambda \simeq K\cdot\lambda,\ [k] \mapsto k\cdot\lambda$, where K_λ is the stabilizer of λ in K. The line bundle $\mathbb{C}_{[\lambda]}$ over the complex manifold $K\cdot\lambda$ carries a canonical holomorphic structure. If one work with the symplectic form $k\Omega$, for an integer $k \geq 1$, the corresponding Kostant-Souriau prequantum line bundle is $\mathbb{C}_{[\lambda]}^{\otimes k} = K \times_{K_\lambda} \mathbb{C}_{k\lambda} = \mathbb{C}_{[k\lambda]}$.

Let $\mathcal{H}^q(K\cdot\lambda, \mathbb{C}_{[\lambda]}^{\otimes k})$ be qth cohomology group of the sheaf of holomorphic section of $\mathbb{C}_{[\lambda]}^{\otimes k}$ over $K\cdot\lambda$. The Borel-Weil Theorem tells us that

$$\mathcal{H}^0(K\cdot\lambda, \mathbb{C}_{[\lambda]}^{\otimes k}) = V_{k\lambda} \tag{1.79}$$

and

$$\mathcal{H}^q(K\cdot\lambda, \mathbb{C}_{[\lambda]}^{\otimes k}) = 0 \quad \text{for} \quad q \geq 1. \tag{1.80}$$

If $RR^K(K\cdot\lambda, -) : \mathbf{K}_K(K\cdot\lambda) \to R(K)$ is the K-equivariant Riemann-Roch character defined by the compatible complex structure, (1.79) and (1.80) give

$$RR^K(K\cdot\lambda, \mathbb{C}_{[\lambda]}^{\otimes k}) = V_{k\lambda} \quad \text{in} \quad R(K). \tag{1.81}$$

Now if we denote by $RR(K\cdot\lambda, -) : \mathbf{K}_T(K\cdot\lambda) \to R(T)$ the T-equivariant Riemann-Roch character, we have $V_{k\lambda}|_T = RR(K\cdot\lambda, \mathbb{C}_{[\lambda]}^{\otimes k})$. The multiplicity fonction $\mathrm{m}^\lambda : \Lambda^*_+ \times \mathbb{N}^* \to \mathbb{N}$ is characterized by the relation

$$RR(K\cdot\lambda, \mathbb{C}_{[\lambda]}^{\otimes k}) = \sum_{\mu\in\Lambda^*} \mathrm{m}^\lambda(\mu, k)\ \mathbb{C}_\mu, \quad \text{in } R(T), \tag{1.82}$$

for $k \geq 1$.

The sub-lattice $\Lambda^*_{\mathfrak{R}}$ of Λ^* generated by the roots is characterized by the (finite) center $Z(K)$ of K as follows. For $\alpha \in \Lambda^*$ we have

$$\lambda \in \Lambda^*_{\mathfrak{R}} \iff t^\lambda = 1,\ \forall t \in Z(K), \tag{1.83}$$

and for $t \in T$ we have $t \in Z(K) \iff t^\lambda = 1,\ \forall \lambda \in \Lambda^*_{\mathfrak{R}}$. The finite abelian group $\Lambda^*/\Lambda^*_{\mathfrak{R}}$ is then naturally identified with the dual of $Z(K)$. We have the following well-known fact.

Lemma 1.4.2 *The mutiplicity map* m^λ *is supported on the sub-lattice* $\Xi_\lambda = \{(\mu, k) \in \Lambda^* \times \mathbb{Z} \,|\, \mu - k\lambda \in \Lambda^*_{\mathfrak{R}}\}$.

Proof. The center $Z(K)$ of K acts trivially on $K\cdot\lambda$ and with the character $t \in Z(K) \mapsto t^{k\lambda}$ on the fibers of the line bundle $\mathbb{C}_{[\lambda]}^{\otimes k}$. Since $\mathrm{m}^\lambda(\mu, k)$ is equal to the dimension of the T-invariant subspace of $RR(K\cdot\lambda, \mathbb{C}_{[\lambda]}^{\otimes k}) \otimes \mathbb{C}_{-\mu}$, we have following Lemma 1.3.17 that $\mathrm{m}^\lambda(\mu, k) \neq 0$ only if $t^{\mu-k\lambda} = 1, \forall t \in Z(K)$. We conclude then with (1.83). □

In this section we study the periodic polynomials

$$\mathrm{m}_{\mathfrak{c}}^{\lambda} : \Lambda^* \times \mathbb{Z} \longrightarrow \mathbb{Z}. \tag{1.84}$$

defined for every connected component $\mathfrak{c} \subset \mathfrak{t}^*$ of regular values of the moment map $\Phi : K \cdot \lambda \to \mathfrak{t}^*$. We know that $\mathrm{m}_{\mathfrak{c}}^{\lambda}$ is also supported on the sub-lattice Ξ_λ (see Section 1.3.3).

In order to apply Theorem 1.3.24 to the periodic polynomials $\mathrm{m}_{\mathfrak{c}}^{\lambda}$, we have to compute the critical values of the moment map $\Phi : K \cdot \lambda \to \mathfrak{t}^*$.

1.4.2 Critical points of $\Phi : K \cdot \lambda \to \mathfrak{t}^*$

Let $\{\alpha_1, \cdots, \alpha_{\dim T}\}$ be the simple roots of the set $\mathfrak{R}_+$ of positive weights. The fundamental weights $\varpi_k, 1 \leq k \leq \dim T$ are defined by the conditions

$$2\frac{(\varpi_i, \alpha_j)}{|\alpha_j|^2} = \delta_{i,j} \quad \text{for all} \quad 1 \leq i, j \leq \dim T. \tag{1.85}$$

Recall that the fundamental weights generate the lattice Λ^*_{alg} of *algebraic* integral element of $\mathfrak{t}^*$. We have $\Lambda^* \subset \Lambda^*_{alg}$ and equality holds only if K is simply-connected.

Let W be the Weyl group of (K, T). We will look at

$$\mathcal{G} = \{\sigma \cdot \varpi_i \mid \sigma \in W,\ 1 \leq i \leq \dim T\}. \tag{1.86}$$

as a subset of $\mathfrak{t}$ modulo the identification $\mathfrak{t} \simeq \mathfrak{t}^*$ given by the scalar product. The singular points of Φ have the following nice description. This result first appeared in Heckman's Thesis [22].

Proposition 1.4.3 ([22, 17]) *The critical points of $\Phi : K \cdot \lambda \to \mathfrak{t}^*$ is the union of the fixed points set $(K \cdot \lambda)^\beta,\ \beta \in \mathcal{G}$. For each $\beta \in \mathcal{G}$ we have*

$$(K \cdot \lambda)^\beta = \bigcup_{\sigma \in W} K^\beta \cdot \sigma\lambda,$$

where K^β is the stabilizer subgroup of β in K.

The fixed points of the action of T on $K \cdot \lambda$ characterize the image of Φ completely: $\Phi(K \cdot \lambda)$ is the convex polytope

$$\mathrm{conv}(W \cdot \lambda) := \text{ convex hull of } W \cdot \lambda. \tag{1.87}$$

This result was first proved by Kostant [25]. This is particular case of the convexity theorem of Atiyah, Guillemin and Sternberg [2, 18]. From Proposition 1.4.3, we know that the singular values of $\Phi : K \cdot \lambda \to \mathfrak{t}^*$ are the convex polytopes

$$\mathrm{conv}(W^\beta \cdot \sigma\lambda), \quad \beta \in \mathcal{F},\ \sigma \in W/W^\beta, \tag{1.88}$$

where W^β is the stabilizer[5] of β in W, i.e. W^β is the Weyl group of (K^β, T). Each convex polytope $\mathrm{conv}(W^\beta \cdot \sigma\lambda)$ lies in the hyperplane

[5] When $\beta = \varpi_i$, we denote W^i the stabilizer of ϖ_i in W.

$$\Delta_{\beta,\sigma} = \{\xi \in \mathfrak{t}^* \mid (\xi - \sigma\lambda, \beta) = 0\}. \tag{1.89}$$

We will use the following

Lemma 1.4.4 • *$K^\beta \cdot \sigma\lambda = K^\beta \cdot \sigma'\lambda$ if and only if $\sigma\lambda \in W^\beta \sigma'\lambda$,*
• *$\mathrm{conv}(W^\beta \cdot \sigma\lambda) \cap \mathrm{conv}(W^\beta \cdot \sigma'\lambda) \neq \emptyset$ if and only if $\Delta_{\beta,\sigma} = \Delta_{\beta,\sigma'}$.*

Proof. The first point follows from the fact that the intersection of a coadjoint orbit $K^\beta \cdot \mu$, $\mu \in \mathfrak{t}^*$ with $\mathfrak{t}^*$ is equal to $W^\beta \cdot \mu$.

It is sufficient to prove the second point for $\beta = \varpi_i$. The half-line $\mathbb{R}^{>0}\varpi_i$ is an edge of the Weyl chamber. It is well known that the following vector subspaces coincides:

- the line $\mathbb{R}\varpi_i$,
- the vector sub-space of K^{ϖ_i}-invariant element of $\mathfrak{k}^*$,
- the vector sub-space of W^i-invariant element of $\mathfrak{t}^*$.

Each convex polytope $\mathrm{conv}(W^i \cdot \sigma\lambda)$ contains the W^i-invariant element

$$\frac{1}{|W^i|} \sum_{\tau \in W^i} \tau \cdot \sigma\lambda$$

which is equal to the intersection of the hyperplane $\Delta_{\beta,\sigma}$ with the line $\mathbb{R}\varpi_i$. Hence, if $\Delta_{\beta,\sigma} = \Delta_{\beta,\sigma'}$, the intersection $\mathrm{conv}(W^\beta \cdot \sigma\lambda) \cap \mathrm{conv}(W^\beta \cdot \sigma'\lambda)$ contains a W^i-invariant element, and then is not empty.

Definition 1.4.5 *An element $\lambda \in \Lambda^*_+$ is* generic *if for every fundamental root ϖ_i and any $\sigma, \sigma' \in W$, we have*

$$\Delta_{\beta,\sigma} \neq \Delta_{\beta,\sigma} \tag{1.90}$$

each times the submanifolds $K^\beta \cdot \sigma\lambda$ and $K^\beta \cdot \sigma'\lambda$ are not equal.

This condition of genericity imposes that $(\sigma\lambda, \varpi_i) \neq (\sigma'\lambda, \varpi_i)$ when $\sigma\lambda \notin W^i\sigma'\lambda$.

Example 1.4.6 *Consider the case of* $\mathrm{SU}(4)$. *Take the coadjoint orbit trough $\lambda = (2,1,-1,-2)$, and σ, σ' such that $\sigma\lambda = (2,-2,1,-1)$ and $\sigma'\lambda = (1,-1,2,-2)$. Take the fundamental weight $\varpi_2 = \frac{1}{2}(1,1,-1,-1)$. In this case λ is not "generic" since $\sigma\lambda \notin W^i\sigma'\lambda$ but $(\sigma\lambda, \varpi_2) = 0 = (\sigma'\lambda, \varpi_2)$.*

1.4.3 Main theorems

Let $\mathfrak{c}_+$ and $\mathfrak{c}_-$ be two adjacent connected components of regular values of $\Phi : K \cdot \lambda \to \mathfrak{t}^*$. The intersection $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_+}$ is contained in an hyperplane orthogonal to $\beta \in \mathcal{F}$.

Definition 1.4.7 *Let $\mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ be the set of all $[\sigma] \in W/W^\beta$ such that the convex polytope $\mathrm{conv}(W^\beta \cdot \sigma\lambda)$ contains $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_+}$.*

The following

$$\bigcup_{[\sigma]\in\mathcal{A}(\mathfrak{c}_+,\mathfrak{c}_-)} K^\beta \cdot \sigma\lambda$$

is the union of the connected components of $(K \cdot \lambda)^\beta$ that intersect $\Phi^{-1}(\xi)$ when $\xi \in \overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_+}$.

Remark 1.4.8 *When λ is a regular element of $\mathfrak{t}^*$, all polytopes $\mathrm{conv}(W^\beta \cdot \sigma\lambda)$ are of codimension 1. When λ is "generic" (see Def. 1.4.5), the set $\mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ is reduced to one element.*

The multiplicity function $\mathrm{m}^\lambda : \Lambda^* \times \mathbb{Z}^{>0} \to \mathbb{N}$ is invariant under the action of the Weyl group: $\mathrm{m}^\lambda(\sigma\mu, k) = \mathrm{m}^\lambda(\mu, k)$ for every $\sigma \in W$. The set of connected component of regular values of Φ is also invariant under the action of W.

So, for the rest of this section we restrict our attention to case where $\mathfrak{c}_+$ and $\mathfrak{c}_-$ are separated by an hyperplane orthogonal to a fundamental weight $\beta = \varpi_i$: the vector ϖ_i is pointing out of $\mathfrak{c}_-$. We denote K^i the stabilizer of ϖ_i in K.

Consider $[\sigma] \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ and let $K^i \cdot \sigma\lambda$ be the corresponding connected component of $(K \cdot \lambda)^\beta$. The tangent space of $K \cdot \lambda$ at $\sigma\lambda$ is the following $K^{\sigma\lambda}$-module

$$\mathbf{T}_{\sigma\lambda}(K \cdot \lambda) = \sum_{(\alpha,\sigma\lambda)>0} \mathfrak{k}_\alpha \tag{1.91}$$

where $\mathfrak{k}_\alpha \subset \mathfrak{k} \otimes \mathbb{C}$ is the one-dimensional complex subspace associated to the weight $\alpha \in \mathfrak{R}$. In the same way, the tangent space of $K^i \cdot \sigma\lambda$ at $\sigma\lambda$ is the $K^i \cap K^{\sigma\lambda}$-module defined by

$$\mathbf{T}_{\sigma\lambda}(K^i \cdot \sigma\lambda) = \sum_{\substack{(\alpha,\sigma\lambda)>0 \\ (\alpha,\varpi_i)=0}} \mathfrak{k}_\alpha \tag{1.92}$$

Finally the normal bundle of $K^i \cdot \sigma\lambda \simeq K^i/(K^i \cap K^{\sigma\lambda})$ in $K \cdot \lambda$ is $\mathcal{N}_{\sigma,i} = K^i \times_{K^i \cap K^{\sigma\lambda}} N_{\sigma,i}$ where

$$N_{\sigma,i} = \sum_{\substack{(\alpha,\sigma\lambda)>0 \\ (\alpha,\varpi_i)\neq 0}} \mathfrak{k}_\alpha \tag{1.93}$$

For an element $\mu \in \mathfrak{t}^*$, we have $\mu = \sum_{i=1}^{\dim T} [\mu]_k \, \alpha_k$ where

$$[\mu]_k = 2\frac{(\varpi_k, \mu)}{|\alpha_k|^2} \in \mathbb{R}.$$

Note that $[\mu]_k \in \mathbb{Z}$ when μ belongs to the lattice $\Lambda^*_{\mathfrak{R}}$.

Definition 1.4.9 *For $[\sigma] \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ we define the positive integers*

$$s^{\pm}_{\sigma,i} = \pm \sum_{\substack{(\alpha,\sigma\lambda)>0 \\ \pm(\alpha,\varpi_i)>0}} [\alpha]_i.$$

Note that $s^+_{\sigma,i} + s^-_{\sigma,i}$ is larger than half of the codimension of $K^i \cdot \sigma\lambda$ in $K \cdot \lambda$.

Theorem 1.4.10 *Let $\mathfrak{c}_+$ and $\mathfrak{c}_-$ be two adjacent connected component of regular values of $\Phi : K \cdot \lambda \to \mathfrak{t}^*$ separated by an hyperplane orthogonal to a fundamental weight ϖ_i: we denote r_i the commum value $[\xi]_i$ for all ξ in this hyperplane. Let $\mathrm{m}^\lambda_{\mathfrak{c}_\pm} : \Lambda^* \times \mathbb{Z} \longrightarrow \mathbb{Z}$ be the corresponding periodic polynomials which are supported on the sub-lattice $\Xi_\lambda := \{(\mu, k) \,|\, \mu \in k\lambda + \Lambda^*_{\mathfrak{R}}\}$.*

For all $(\mu, k) \in \Xi_\lambda$, we have $\mathrm{m}^\lambda_{\mathfrak{c}_+}(\mu, k) = \mathrm{m}^\lambda_{\mathfrak{c}_-}(\mu, k)$ when the integer $[\mu]_i - kr_i$ satisfies

$$-s^-_i < [\mu]_i - kr_i < s^+_i. \tag{1.94}$$

Here the positive integer $s^\pm_i$ are defined by

$$s^\pm_i = \inf_{[\sigma]\in\mathcal{A}(\mathfrak{c}_+,\mathfrak{c}_-)} s^\pm_{\sigma,i}. \tag{1.95}$$

When $\mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ is reduced to one element σ, for example if λ is "generic", the integer $s^+_i + s^-_i$ is larger than half of the codimension of $K^i \cdot \sigma\lambda$ in $K \cdot \lambda$.

Another way to express the result of Theorem 1.4.10 is to introduce like in [36] the convex polytope

$$\square(\mathfrak{c}_+, \mathfrak{c}_-) = \bigcap_{\sigma\in\mathcal{A}(\mathfrak{c}_+,\mathfrak{c}_-)} \left(\sum_{(\alpha,\sigma\lambda)>0} [0, 1[\, \alpha \right). \tag{1.96}$$

Let Δ be the hyperplane which separates $\mathfrak{c}_+$ and $\mathfrak{c}_-$. Equation (1.94) is equivalent to saying that

$$\mathrm{m}^\lambda_{\mathfrak{c}_+}(\mu, k) = \mathrm{m}^\lambda_{\mathfrak{c}_-}(\mu, k) \quad \text{if} \quad \mu \in k\Delta + \square(\mathfrak{c}_+, \mathfrak{c}_-). \tag{1.97}$$

Corollary 1.4.11 *Let $\mathfrak{c}$ be a connected component of regular values of Φ which is bording a facet of the polytope $\Phi(K \cdot \lambda)$ orthogonal to the fundamental weight ϖ_i: the facet is $\mathrm{conv}(W^i \cdot \sigma\lambda)$ for a unique $\sigma \in W/W^i$. We suppose that ϖ_i is pointing out of $\mathfrak{c}$. We denote r_i the commum value $[\xi]_i$ for all ξ in the facet. For all $(\mu, k) \in \Xi_\lambda$, we have $\mathrm{m}^\lambda_{\mathfrak{c}}(\mu, k) = 0$ when the integer $[\mu]_i - kr_i$ satisfies*

$$-s^-_{\sigma,i} < [\mu]_i - kr_i < s^+_{\sigma,i}. \tag{1.98}$$

Proof. Theorem 1.4.10 is a direct consequence of Theorem 1.3.24. The main difference between them is the decomposition of the lattice supporting the periodic polynomials. In the former we use the decomposition $\Lambda^* = \Lambda^*_{\mathfrak{t}_\Delta} \oplus \Lambda^*_{\mathfrak{t}/\mathfrak{t}_\Delta}$ associated to the choice of a subtorus T/T_Δ. Here we use the decomposition $\Lambda^*_{\mathfrak{R}} = \mathbb{Z}\alpha_i \oplus \sum_{k\neq i}\mathbb{Z}\alpha_k$. Note first that for $(\mu,k) \in \Xi_\lambda$, we have $\mu - \sigma\lambda \in \Lambda^*_{\mathfrak{R}}$ and then $[\mu-\sigma\lambda]_i = [\mu]_i - kr_i$ is an integer.

We start like after Proposition 1.3.23: $\mathrm{m}^\lambda_{\mathfrak{c}_+}(\mu,k) - \mathrm{m}^\lambda_{\mathfrak{c}_-}(\mu,k)$ is equal to the μ-mutiplicity in $\sum_{\sigma\in\mathcal{A}(\mathfrak{c}_+,\mathfrak{c}_-)} A^-_\sigma - A^+_\sigma$ where

$$A^\pm_\sigma = RR^\xi_0\left(K^i\cdot\sigma\lambda, \mathbb{C}^{\otimes k}_{[\lambda]} \otimes \left[\wedge^\bullet_{\mathbb{C}}\overline{\mathcal{N}_{\sigma,i}}\right]^{-1}_{\mp\varpi_i}\right) \tag{1.99}$$

Here ξ belongs to the relative interior of $\overline{\mathfrak{c}_+}\cap\overline{\mathfrak{c}_+}$, the line bundle $\mathbb{C}^{\otimes k}_{[\lambda]}$ is equal to $K^i\times_{K^i\cap K^{\sigma\lambda}}\mathbb{C}_{k\sigma\lambda}$ and $\left[\wedge^\bullet_{\mathbb{C}}\overline{\mathcal{N}_{\sigma,i}}\right]^{-1}_{\pm\varpi_i}$ corresponds to $(-1)^{\mathrm{rk}_{\mathbb{C}}(N^\pm_{\sigma,i})}$ times

$$K^i\times_{K^i\cap K^{\sigma\lambda}}\left(\det(N^\pm_{\sigma,i})\otimes S^\bullet((N_{\sigma,i}\otimes\mathbb{C})^\pm)\right),$$

with

$$N^\pm_{\sigma,i} = \sum_{\substack{(\alpha,\sigma\lambda)>0\\ \pm(\alpha,\varpi_i)>0}} \mathfrak{k}_\alpha\ ,$$

and

$$(N_{\sigma,i}\otimes\mathbb{C})^\pm = \sum_{\substack{(\alpha,\sigma\lambda)\neq 0\\ \pm(\alpha,\varpi_i)>0}} \mathfrak{k}_\alpha.$$

Now we can apply Remark 1.3.17 with the subgroup $H\subset T$ equal to the center $Z(K^i)$ of K^i: an element $\gamma\in\Lambda^*$ belong to $\sum_{k\neq i}\mathbb{Z}\alpha_k$ if and only if $t^\gamma = 1$ for all $t\in Z(K^i)$.

The group $Z(K^i)$ acts trivially on the manifolds $K^i\cdot\sigma\lambda$, and with the characters associated to the weights

$$k\sigma\lambda + \sum_{\substack{(\alpha,\sigma\lambda)>0\\ (\alpha,\varpi_i)>0}}\alpha+\delta \quad\text{with}\quad (\delta,\varpi_i)\geq 0$$

on the bundle $\mathbb{C}^{\otimes k}_{[\lambda]}\otimes\left[\wedge^\bullet_{\mathbb{C}}\overline{\mathcal{N}_{\sigma,i}}\right]^{-1}_{\varpi_i}$, and with the characters associated to the weights

$$k\sigma\lambda + \sum_{\substack{(\alpha,\sigma\lambda)>0\\ (\alpha,\varpi_i)<0}}\alpha+\delta \quad\text{with}\quad (\delta,\varpi_i)\leq 0$$

on the bundle $\mathbb{C}^{\otimes k}_{[\lambda]}\otimes\left[\wedge^\bullet_{\mathbb{C}}\overline{\mathcal{N}_{\sigma,i}}\right]^{-1}_{-\varpi_i}$. Now the μ-multiplicity in $A^\pm_\sigma$ is not equal to 0 only if

$$k\sigma\lambda + \sum_{\substack{(\alpha,\sigma\lambda)>0\\ \pm(\alpha,\varpi_i)>0}}\alpha+\delta-\mu\in\sum_{k\neq i}\mathbb{Z}\alpha_k \quad\text{with}\quad \pm(\delta,\varpi_i)\geq 0. \tag{1.100}$$

Condition (1.100) implies that $[\mu]_i \geq k[\sigma\lambda]_i + s^+_{\sigma,i}$ or $[\mu]_i \leq k[\sigma\lambda]_i - s^-_{\sigma,i}$. Finally we have prove that $\mathrm{m}^\lambda_{\mathfrak{c}_+}(\mu,k) = \mathrm{m}^\lambda_{\mathfrak{c}_-}(\mu,k)$ if

$$-s^-_{\sigma,i} < [\mu]_i - k[\sigma\lambda]_i < s^+_{\sigma,i}$$

for all $\sigma \in \mathcal{A}(\mathfrak{c}_+,\mathfrak{c}_-)$. □

1.4.4 The case of $\mathrm{SU}(n)$

Let T be the maximal torus of $\mathrm{SU}(n)$ consisting of the diagonal matrices. The dual $\mathfrak{t}^*$ can be identified with the subspace $x_1+\cdots+x_n=0$ of $\mathbb{R}^n$. The roots are $\mathfrak{R} = \{e_i - e_j \,|\, 1 \leq i \neq j \leq n\}$ and we will choose the positives ones to be $\mathfrak{R}^+ = \{e_i - e_j \,|\, 1 \leq i < j \leq n\}$. The simple roots are then $\alpha_i = e_i - e_{i+1}$, for $1 \leq i \leq n-1$, and for these simple roots, the fundamental weights are

$$\varpi_k = \frac{1}{n}(\underbrace{n-k, n-k, \cdots, n-k}_{k \text{ times}}, \underbrace{-k, -k, \cdots, -k}_{n-k \text{ times}}), \quad 1 \leq k \leq n-1. \quad (1.101)$$

Consider now the coadjoint orbit O_λ for $\lambda \in \mathfrak{t}^*$. Let $\Phi : O_\lambda \to \mathfrak{t}^*$ the moment map associated to the Hamiltonian action of T on O_λ. The center of $\mathrm{SU}(n)$, that we denote $\mathbb{Z}_n$ corresponds to the set of matrices zI with $z^n = 1$. Recall the following well-known fact.

Lemma 1.4.12 *Let ξ be a regular value of $\Phi : O_\lambda \to \mathfrak{t}^*$. Then for every $m \in \Phi^{-1}(\xi)$ the stabilizer subgroup $T_m := \{t \in T \,|\, t \cdot m = m\}$ is equal to $\mathbb{Z}_n$.*

Proof. Since ξ is a regular value, we know that T_m is finite for every $m \in \Phi^{-1}(\xi)$. The dual of the Lie algebra $\mathfrak{su}(n)$ decomposes as $\mathfrak{su}(n)^* = \mathfrak{t}^* \oplus \sum_{\alpha\in\mathfrak{R}^+} \mathfrak{su}(n)^*_\alpha$ where $\mathfrak{su}(n)^*_\alpha \simeq \mathbb{C}_{-\alpha}$ as T-module. For $m \in \Phi^{-1}(\xi)$, we have $m = m_0 + \sum_{\alpha\in\mathfrak{R}^+} m_\alpha$ with $m_\alpha \in \mathfrak{su}(n)^*_\alpha$, and then $T_m = \cap_{m_\alpha\neq 0} \ker(t \mapsto t^\alpha)$. So the lattice Λ^*_m generated by the set $\{\alpha \in \mathfrak{R}^+ \,|\, m_\alpha \neq 0\}$ is a subgroup of $\Lambda^*_{\mathfrak{R}}$ with $\Lambda^*_{\mathfrak{R}}/\Lambda^*_m$ finite. We have to show that $\Lambda^*_m = \Lambda^*_{\mathfrak{R}}$. For this purpose we introduce the following equivalence relation on $\{1,\dots,n\}$:

$$i \sim j \Longleftrightarrow e_i - e_j \in \Lambda^*_m.$$

Suppose that $\{1,\dots,n\}/\sim$ is not reduced to a point: let C_1 and C_2 be two distinct equivalent classes and let $\beta = (\beta_1,\dots,\beta_n)$ be the element of $\mathfrak{t}^*$ defined by: $\beta_i = \frac{1}{|C_1|}$ if $i \in C_1$, $\beta_i = \frac{-1}{|C_2|}$ if $i \in C_2$, and $\beta_i = 0$ in the other cases. We see then that $(\beta,\alpha) = 0$ for all $\alpha \in \Lambda^*_m$: it is in contradiction with the fact that $\Lambda^*_{\mathfrak{R}}/\Lambda^*_m$ is finite. We have proved that $e_i - e_j \in \Lambda^*_m$ for all $i,j \in \{1,\dots,n\}$. □

We are in the particularly nice situation where the symplectic reduction $(O_\lambda)_\xi = \Phi^{-1}(\xi)/T$ is a smooth manifold for any regular value ξ.

Suppose now that λ is a positive weight, and let $\mathfrak{c}$ a connected component of regular values of $\Phi : O_\lambda \to \mathfrak{t}^*$. We know that $\mathrm{m}^\lambda_{\mathfrak{c}} : \Lambda^* \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is supported on the sub-lattice $\Xi_\lambda := \{(\mu,k) \,|\, \mu \in k\lambda + \Lambda^*_{\mathfrak{R}}\}$.

Corollary 1.4.13 *The map* $\mathrm{m}_{\mathfrak{c}}^{\lambda} : \Xi_\lambda \longrightarrow \mathbb{Z}$ *is a* polynomial *of degree* $\frac{(n-1)(n-2)}{2} - d_\lambda$*, where* d_λ *is the number of positive roots orthogonal to* λ*..*

Proof. Take $\xi \in \mathfrak{c}$. Following Proposition 1.3.18, the periodic-polynomial $\mathrm{m}_{\mathfrak{c}}^{\lambda}$ is defined by $\mathrm{m}_{\mathfrak{c}}^{\lambda}(\mu, k) = RR((O_\lambda)_\xi, \mathcal{L}_{\xi,\mu}^k)$ for all $(\mu, k) \in \Xi_\lambda$. Here $(O_\lambda)_\xi = \Phi^{-1}(\xi)/T$ is a *smooth* manifold, and the line bundle $\mathcal{L}_{\xi,\mu}^k = (L^{\otimes k}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu})/T$ is also *smooth* since the center $\mathbb{Z}_n$ acts trivially on $L^{\otimes k}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu}$. Now the Atiyah-Singer integral formula for the Riemann-Roch number $RR((O_\lambda)_\xi, \mathcal{L}_{\xi,\mu}^k)$ shows that $\mathrm{m}_{\mathfrak{c}}^{\lambda}$ is a polynomial of degree $\frac{\dim(O_\lambda)_\xi}{2} = \frac{\dim O_\lambda}{2} - (n-1) = \frac{(n-1)(n-2)}{2} - d_\lambda$. □

Now we rewrite Theorem 1.4.10 for the group $\mathrm{SU}(n)$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a positive weight and let $\mathfrak{c}_+$ and $\mathfrak{c}_-$ be two adjacent connected components of regular values of $\Phi : O_\lambda \to \mathfrak{t}^*$ separated by an hyperplane orthogonal to a fundamental weight ϖ_i: the vector ϖ_i is pointing out of $\mathfrak{c}_-$, and let $(\varpi_i, \xi) - r_i = 0$ be the equation of this hyperplane. We consider the linear map

$$Q(\xi, t) := (\varpi_i, \xi) - t r_i.$$

The hyperplane $\{Q = 0\} \subset \mathfrak{t}^* \times \mathbb{R}$ separates $\mathrm{Cone}(\mathfrak{c}_+)$ and $\mathrm{Cone}(\mathfrak{c}_-)$.

The conditions $(e_k - e_l, \sigma\lambda) > 0$ and $(e_k - e_l, \varpi_i) > 0$ are respectively equivalent to $\lambda_{\sigma(k)} > \lambda_{\sigma(l)}$ and $k \leq i < l$. For $\mathrm{SU}(n)$, the number $[\alpha]_i$ is equal to $0, 1$ or -1 for any roots α and any $i = 1, \cdots, n-1$. Hence for every $\sigma \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$, the integers $s_{\sigma,i}^-, s_{\sigma,i}^+ \geq 0$ introduced in Definition (1.4.9) are equal to

$$s_{\sigma,i}^+ = \mathrm{rk}_{\mathbb{C}}(N_{\sigma,i}^+) = \sharp\{k \leq i < l \quad \text{such that} \quad \lambda_{\sigma(k)} > \lambda_{\sigma(l)}\}, \tag{1.102}$$

$$s_{\sigma,i}^- = \mathrm{rk}_{\mathbb{C}}(N_{\sigma,i}^-) = \sharp\{k \leq i < l \quad \text{such that} \quad \lambda_{\sigma(k)} < \lambda_{\sigma(l)}\}, \tag{1.103}$$

and the sum $s_{\sigma,i}^+ + s_{\sigma,i}^-$ is equal to half of the codimension of $K^i \cdot \sigma\lambda$ in $K \cdot \lambda$, that is $s_{\sigma,i}^+ + s_{\sigma,i}^- = i(n-i) - \dim(K^{\sigma\lambda}/K^i \cap K^{\sigma\lambda})/2$.

Now we precise the results of [10].

Theorem 1.4.14 • *The polynomial* $\mathrm{m}_{\mathfrak{c}_-}^{\lambda} - \mathrm{m}_{\mathfrak{c}_+}^{\lambda} : \Xi_\lambda \to \mathbb{Z}$ *is divible by the linear factors*

$$(Q - s_i^- + 1), (Q - s_i^- + 2), \ldots, Q, \ldots, (Q + s_i^+ - 2), (Q + s_i^+ - 1),$$

where $s_i^\pm = \inf_{[\sigma] \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)} s_{\sigma,i}^\pm$.

• *The linear factors* $(Q - s_i^-)$ *and* $(Q - s_i^+)$ do not divide $\mathrm{m}_{\mathfrak{c}_-}^{\lambda} - \mathrm{m}_{\mathfrak{c}_+}^{\lambda}$.

Proof. The first part is the translation of Theorem 1.4.10. We have just to prove that the linear factors $(Q - s_i^-)$ and $(Q - s_i^+)$ do not divide $\mathrm{m}_{\mathfrak{c}_-}^{\lambda} - \mathrm{m}_{\mathfrak{c}_+}^{\lambda}$. This point is a direct application of Proposition 1.3.25. The only fact we use here is that $\mathrm{rk}_{\mathbb{C}}(N_{\sigma,i}^\pm) = s_{\sigma,i}^\pm$. So the number $\mathrm{rk}_{\mathbb{C}}(N_{\sigma,i}^\pm)$ is constant for all $\sigma \in \mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ for which $s_{\sigma,i}^\pm = s_i^\pm$. □

We rewrite now Theorem 1.4.14 in the particular case where $\mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ contains just one element: it happens when λ is a "generic" positive weight (see Definition 1.4.5), or when $\mathfrak{c}_+$ does not intersect $\Phi(O_\lambda)$. Here a positive weight $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ is "generic" if for every couple of permutations σ, σ' and any $k = 1, \cdots, n-1$, we have

$$\sum_{i=1}^{k} \lambda_{\sigma(i)} \neq \sum_{i=1}^{k} \lambda_{\sigma'(i)}$$

when $(\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(n)}) \notin \mathfrak{S}_k \times \mathfrak{S}_{n-k}(\lambda_{\sigma'(1)}, \cdots, \lambda_{\sigma'(n)})$.

Corollary 1.4.15 *Let λ be a* regular *weight. Let $\mathfrak{c}_+$ and $\mathfrak{c}_-$ be two adjacent connected components of regular values of $\Phi : O_\lambda \to \mathfrak{t}^*$ and suppose that $\mathcal{A}(\mathfrak{c}_+, \mathfrak{c}_-)$ contains just one element σ. Then the polynomial $\mathrm{m}^\lambda_{\mathfrak{c}_-} - \mathrm{m}^\lambda_{\mathfrak{c}_+} : \Xi_\lambda \to \mathbb{Z}$ is divible by the $i(n-i)$ linear factors*

$$(Q - s_i^- + 1), (Q - s_i^- + 2), \ldots, q, \ldots, (Q + s_i^+ - 2), (Q + s_i^+ - 1),$$

where $s_i^\pm = s^\pm_{\sigma,i}$ are defined by (1.102) and (1.103). Moreover the linear factors $(Q - s_i^-)$ and $(Q - s_i^+)$ do not divide $\mathrm{m}^\lambda_{\mathfrak{c}_-} - \mathrm{m}^\lambda_{\mathfrak{c}_+}$.

1.5 Vector partition functions

Let T be a torus with Lie algebra $\mathfrak{t}$ and let $\Lambda^* \subset \mathfrak{t}^*$ be the weight lattice. Let $R = \{\alpha_1, \ldots, \alpha_d\}$ be a subset of not necessarily distinct elements of Λ^* which are in an open halfspace of $\mathfrak{t}^*$. We associate with the collection R a function

$$N_R : \Lambda^* \longrightarrow \mathbb{N}$$

called the *vector partition function* associated to R. By definition, for a weight μ, the value $N_R(\mu)$ is the number of solutions of the equation

$$\sum_{j=1}^{d} k_j \alpha_j = \mu, \quad k_j \in \mathbb{Z}^{\geq 0}, \quad j = 1, \ldots, d. \tag{1.104}$$

Let $C(R) \subset \mathfrak{t}^*$ be the closed convex cone generated by the elements of R, and denote by $\Lambda^*_R \subset \Lambda^*$ the sublattice generated by R. Obviously, $N_R(\mu)$ vanishes if μ does not belong to $C(R) \cap \Lambda^*_R$.

Suppose now that R generates the vector space $\mathfrak{t}^*$. Following [36], we will call a vector *singular* with respect to R if it is in a cone $C(\nu)$ generated by a subset $\nu \subset R$ of cardinality strictly less than $\dim T$. The connected components of $\mathfrak{t}^* \setminus \{singular\ vectors\}$ are called *conic chambers*. The periodic polynomial behavior of N_R on closures of conic chambers of the cone $C(R)$ is proved in [35]. We have the following refinement due to Szenes and Vergne [36]. Let us introduce the convex polytope

$$\square(R) = \sum_{j=1}^{d} [0,1]\alpha_j. \tag{1.105}$$

We remark that $\mathfrak{c} - \square(R)$ is a neighborhood of $\overline{\mathfrak{c}}$ for any conic chamber $\mathfrak{c}$ of the cone $C(R)$. We have the following qualitative result.

Theorem 1.5.1 ([36]) *Let $\mathfrak{c}$ be a conic chamber of the cone $C(R)$. There exists a periodic polynomial $P_{\mathfrak{c}}$ on Λ^* such that for each $\mu \in \mathfrak{c} - \square(R)$, we have*

$$N_R(\mu) = P_{\mathfrak{c}}(\mu).$$

In Section 1.5.4 we will give another proof of Theorem 1.5.1.

Let $\mathfrak{c}_\pm \subset \mathfrak{t}^*$ be two adjacent conic chambers separated by the hyperplane $\Delta = \{\xi \in \mathfrak{t}^* \mid \langle \xi, \beta \rangle = 0\}$. Here $\beta \in \mathfrak{t}$ is chosen so that $\mathfrak{c}_\pm \subset \{\xi \in \mathfrak{t}^* \mid \pm\langle \xi, \beta \rangle > 0\}$. The aim of this Section is to give a wall-crossing formula for the periodic polynomial $P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-}$.

Note that the vector space Δ is generated by $R \cap \Delta$. We polarize the elements of R that are outside Δ. We define

$$R' = \{\epsilon_j \alpha_j \mid \langle \alpha_j, \beta \rangle \neq 0 \text{ and } \epsilon_j = \text{sign } \langle \alpha_j, \beta \rangle\}, \tag{1.106}$$

$$\delta^{\pm} = \sum_{\pm\langle \alpha_j, \beta \rangle > 0} \alpha_j \ , \tag{1.107}$$

and

$$r^{\pm} = \sharp\{j \mid \pm \langle \alpha_j, \beta \rangle > 0\}. \tag{1.108}$$

We now look at the vector space Δ equipped with the subset $R \cap \Delta \subset \Lambda^* \cap \Delta$ which lie enterely in an open halfspace: let $N_{R\cap\Delta} : \Lambda^* \cap \Delta \to \mathbb{N}$ be the corresponding vector partition function. It is easy to see that $\overline{\mathfrak{c}_+} \cap \overline{\mathfrak{c}_-}$ is contained in the closure of a conic chamber $\mathfrak{c}' \subset \Delta$ relative to $R \cap \Delta$. Following Proposition 1.5.1 there exists a periodic polynomial $P_{\mathfrak{c}'}$ on $\Lambda^* \cap \Delta$ such that for each $\mu \in \overline{\mathfrak{c}'} \cap \Lambda^*$, we have

$$N_{R\cap\Delta}(\gamma) = P_{\mathfrak{c}'}(\gamma).$$

Let $N_{R'} : \Lambda^* \to \mathbb{N}$ be the vector partition function associated to the polarized set of weight R' (see (1.106)). The main result of this Section is the following

Theorem 1.5.2 *The periodic polynomial $P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-} : \Lambda^* \to \mathbb{Z}$ satisfies*

$$P_{\mathfrak{c}_+}(\mu) - P_{\mathfrak{c}_-}(\mu) = \sum_{\gamma \in \Lambda^* \cap \Delta} D(\mu - \gamma) P_{\mathfrak{c}'}(\gamma), \quad \mu \in \Lambda^*, \tag{1.109}$$

where $D : \Lambda^* \to \mathbb{Z}$ is defined by

$$D(\mu) = (-1)^{r^-} N_{R'}(\mu + \delta^-) - (-1)^{r^+} N_{R'}(-\mu - \delta^+).$$

The proof of Theorem 1.5.2 will be given in Section 1.5.5.

Corollary 1.5.3 *$P_{\mathfrak{c}_+}(\mu) = P_{\mathfrak{c}_-}(\mu)$ for all the weights $\mu \in \Lambda^*$ satisfying the condition*

$$-\langle \delta^+, \beta\rangle < \langle \mu, \beta\rangle < -\langle \delta^-, \beta\rangle.$$

The former ineqalities are optimal since

$$\left(P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-}\right)(-\delta^- + \gamma) = (-1)^{r^-} P_{\mathfrak{c}'}(\gamma)$$

and

$$\left(P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-}\right)(-\delta^+ + \gamma) = (-1)^{1+r^+} P_{\mathfrak{c}'}(\gamma)$$

for all $\gamma \in \Lambda^* \cap \Delta$.

Proof. In (1.109), the term $D(\mu-\gamma)P_{\mathfrak{c}'}(\gamma)$ does not vanish only if $\mu - \gamma \in -\delta^- + C(R')$ or $-(\mu-\gamma) \in \delta^+ + C(R')$ for some $\gamma \in C(R \cap \Delta)$. These two conditions impose respectively that $\langle \mu, \beta\rangle \geq -\langle \delta^-, \beta\rangle$ and $\langle \mu, \beta\rangle \leq -\langle \delta^+, \beta\rangle$. If one take $\mu = -\delta^- + \gamma$ with $\gamma \in \Lambda^* \cap \Delta$, (1.109) becomes $(P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-})(-\delta^- + \gamma) = \sum_{\gamma' \in \Lambda^* \cap \Delta} D(-\delta^- + \gamma - \gamma')P_{\mathfrak{c}'}(\gamma')$ with

$$D(-\delta^- + \gamma - \gamma') = (-1)^{r^-} N_{R'}(\gamma - \gamma') - (-1)^{r^+} N_{R'}(\delta^- - \delta^+ - \gamma + \gamma').$$

Since the cone $C(R')$ intersects Δ only at $\{0\}$, $N_{R'}(\gamma - \gamma') = 0$ if $\gamma \neq \gamma'$. Since $\langle \delta^- - \delta^+, \beta\rangle < 0$ we always have $N_{R'}(\delta^- - \delta^+ - \gamma + \gamma') = 0$. We get finally that $(P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-})(-\delta^- + \gamma) = (-1)^{r^-} P_{\mathfrak{c}'}(\gamma)$. One can show in the same way that $(P_{\mathfrak{c}_+} - P_{\mathfrak{c}_-})(-\delta^+ + \gamma) = -(-1)^{r^+} P_{\mathfrak{c}'}(\gamma)$. □

1.5.1 Quantization of $\mathbb{C}^d$

We consider the complex vector space $\mathbb{C}^d$ equipped with the canonical symplectic form $\Omega = \frac{i}{2}\sum_{i=1}^d dz_j \wedge d\overline{z}_j$. The standard complex struture J on $\mathbb{C}^d$ is compatible with Ω. Let T be a torus, let $\alpha_j \in \mathfrak{t}^*$, $j = 1, \ldots, d$ be weights of T, and let T acts on $\mathbb{C}^d$ as

$$t \cdot (z_1, \ldots, z_d) = (t^{-\alpha_1} z_1, \ldots, t^{-\alpha_d} z_d). \tag{1.110}$$

The action of T preserve the symplectic form Ω and the moment map associated with this action is

$$\Phi(z) = \frac{1}{2}\sum_{i=1}^d |z_j|^2 \alpha_j. \tag{1.111}$$

The pre-quantization data $(L, \langle,\rangle, \nabla)$ on the Hamiltonian T-manifold $(\mathbb{C}^d, \Omega, \Phi)$ is a trivial line bundle L with a trivial action of T equipped with the Hermitian structure $\langle s, s'\rangle_z = e^{\frac{-|z|^2}{2}} s\overline{s'}$ and the Hermitian connexion $\nabla = d - \theta$ where $\theta = \frac{1}{2}\sum_{i=1}^d \overline{z}_j dz_j$.

The quantization of the Hamiltonian T-manifold $(\mathbb{C}^d, \Omega)$, that we denote $\mathcal{Q}^T(\mathbb{C}^d)$, is the *Bargman space* of entire holomorphic functions on $\mathbb{C}^d$ which are $\mathcal{L}^2$ integrable with respect to the Gaussian measure $e^{\frac{-|z|^2}{2}}\Omega^d$.

We suppose now that the set of weights $R = \{\alpha_1, \ldots, \alpha_d\}$ is *polarized* by $\eta \in \mathfrak{t}$, which means that $\langle \alpha_j, \eta \rangle > 0$ for all j. The T-representation $\mathcal{Q}^T(\mathbb{C}^d)$ is then admissible and we have the following equality in $R^{-\infty}(T)$:

$$\mathcal{Q}^T(\mathbb{C}^d) = \sum_{\mu \in \Lambda^*} N_R(\mu)\, \mathbb{C}_\mu, \tag{1.112}$$

where $N_R : \Lambda^* \to \mathbb{N}$ is the vector partition function associated to R. In other words, the generalized character of $\mathcal{Q}^T(\mathbb{C}^d)$ coincides with the generalized character of the symmetric algebra $S^\bullet(\overline{\mathbb{C}^d})$, where $\overline{\mathbb{C}^d}$ means $\mathbb{C}^d$ with the opposite complex structure.

For the remaining part of Section 1.5, we assume that the set of weights $R = \{\alpha_1, \ldots, \alpha_d\}$ is *polarized*, and *generates* the vector space $\mathfrak{t}^*$. The first assumption is equivalent to the fact that the moment map $\Phi : \mathbb{C}^d \to \mathfrak{t}^*$ is *proper*, and the second assumption is equivalent to the fact that the generic stabiliser of T on $\mathbb{C}^d$ is *finite*. Notice that the vectors of $\mathfrak{t}^*$ which are *singular* with respect to R correspond to the *singular values* of Φ.

In the next section we will show that $\mathcal{Q}^T(\mathbb{C}^d)$, viewed as an element of $R^{-\infty}(T)$, can be realized as the index of transversally elliptic symbols on $\mathbb{C}^d$. After we will apply the techniques developped in Section 1.3. The main difference here is that we work with the *non-compact* manifold $\mathbb{C}^d$.

1.5.2 Transversally elliptic symbols on $\mathbb{C}^d$

Let $p : \mathbf{T}\mathbb{C}^d \to \mathbb{C}^d$ be the canonical projection. We consider the Thom symbol

$$\mathrm{Thom}(\mathbb{C}^d) \in \Gamma\left(\mathbf{T}\mathbb{C}^d, \hom(p^*(\wedge_{\mathbb{C}}^{\mathrm{even}}\mathbf{T}\mathbb{C}^d), p^*(\wedge_{\mathbb{C}}^{\mathrm{odd}}\mathbf{T}\mathbb{C}^d))\right)$$

associated to the standard Hermitian structure on $\mathbb{C}^d$. Obviously the symbol $\mathrm{Thom}(\mathbb{C}^d)$ is not elliptic since its characteristic set is equal to the zero section in $\mathbf{T}\mathbb{C}^d$ (hence is not compact).

Now we deform the symbol $\mathrm{Thom}(\mathbb{C}^d)$ in order to obtain transversally elliptic symbols. Since $\mathbb{C}^d$ can be realized as an open subset of a compact T-manifold we have a well defined index map

$$\mathrm{Index}^T_{\mathbb{C}^d} : \mathbf{K}_T(\mathbf{T}_T\mathbb{C}^d) \longrightarrow R^{-\infty}(T).$$

Definition 1.5.4 *For any $\eta \in \mathfrak{t}$, we define the symbol* $\mathrm{Thom}^\eta(\mathbb{C}^d)$ *by*

$$\mathrm{Thom}^\eta(\mathbb{C}^d)(z, v) = \mathrm{Thom}(\mathbb{C}^d)(z, v - \eta_{\mathbb{C}^d}(z)), \quad (z, v) \in \mathbf{T}\mathbb{C}^d,$$

where $\eta_{\mathbb{C}^d}$ is the vector field on $\mathbb{C}^d$ generated by η.

The symbols $\mathrm{Thom}^\eta(\mathbb{C}^d)$ were studied in [32]. It is easy to see that $\mathrm{Thom}^\eta(\mathbb{C}^d)$ is tranversally elliptic if and only if the vector subspace $(\mathbb{C}^d)^\eta$ is reduced to $\{0\}$, i.e. if $\langle \alpha_j, \eta\rangle \neq 0$ for all $j = 1, \dots, d$. We prove in Proposition 5.4. of [32] that

$$\mathrm{Index}^T_{\mathbb{C}^d}\left(\mathrm{Thom}^\eta(\mathbb{C}^d)\right) = S^\bullet(\overline{\mathbb{C}^d}) \quad \text{in} \quad R^{-\infty}(T), \tag{1.113}$$

when $\langle \alpha_j, \eta\rangle > 0$ for all $j = 1, \dots, d$.

In order to compute the multiplicities $N_R(\mu)$ of $\mathcal{Q}^T(\mathbb{C}^d)$ we introduce the following tranversally elliptic symbols. Take a scalar product $b(\cdot,\cdot)$ on $\mathfrak{t}^*$, and denote by $\xi \mapsto \xi^b, \mathfrak{t}^* \simeq \mathfrak{t}$ the induced isomorphism. For each $\xi \in \mathfrak{t}^*$, the Hamiltonian vector field of the function $\frac{-1}{2}\|\Phi - \xi\|^2_b$ is the vector field

$$z \mapsto \Big((\Phi(z) - \xi)^b\Big)_{\mathbb{C}^d}(z).$$

that we denote $\mathcal{H}^b - \xi^b_{\mathbb{C}^d}$.

Definition 1.5.5 *For any $\xi \in \mathfrak{t}^*$, and any scalar product $b(\cdot,\cdot)$ on $\mathfrak{t}^*$, we define the symbol $\mathrm{Thom}_{\xi,b}(\mathbb{C}^d)$ by*

$$\mathrm{Thom}_{\xi,b}(\mathbb{C}^d)(z,v) = \mathrm{Thom}(\mathbb{C}^d)(z, v - (\mathcal{H}^b - \xi^b_{\mathbb{C}^d})(z)), \quad (z,v) \in \mathbf{T}\mathbb{C}^d.$$

Let $\mathrm{Char}(\mathrm{Thom}_{\xi,b}(\mathbb{C}^d)) \subset \mathbf{T}\mathbb{C}^d$ be the characterictic set of $\mathrm{Thom}_{\xi,b}(\mathbb{C}^d)$. We know that $\mathrm{Char}(\mathrm{Thom}_{\xi,b}(\mathbb{C}^d)) \cap \mathbf{T}_T\mathbb{C}^d$ is equal to the critical set $\mathrm{Cr}(\|\Phi - \xi\|^2_b)$ of the function $\|\Phi - \xi\|^2_b : \mathbb{C}^d \to \mathbb{R}$ (see Section 1.3.2). A straightforward computation gives that $z \in \mathrm{Cr}(\|\Phi - \xi\|^2_b)$ if and only if

$$b(\Phi(z) - \xi, \alpha_j)\, z_j = 0 \quad \text{for all} \quad j = 1, \dots, d. \tag{1.114}$$

The former relations implies in particular that $b(\Phi(z) - \xi, \Phi(z)) = \frac{1}{2}\sum_j b(\Phi(z) - \xi, \alpha_j)\,|z_j|^2 = 0$. Hence $\|\Phi(z)\|^2_b = b(\Phi(z), \xi)$ which implies

$$\|\Phi(z)\|_b \leq \|\xi\|_b. \tag{1.115}$$

Take now $\eta \in \mathfrak{t}$ such that $\langle \alpha_j, \eta\rangle > 0$ for all j, and let $\eta_b \in \mathfrak{t}^*$ such that $(\eta_b)^b = \eta$. We have then

$$C_\eta \|z\|^2 \leq \langle \Phi(z), \eta\rangle = b(\Phi(z), \eta_b) \leq \|\Phi(z)\|_b \|\eta_b\|_b \tag{1.116}$$

where $C_\eta = \frac{1}{2}\inf_j \langle \alpha_j, \eta\rangle$, and $z \mapsto \|z\|^2$ is the usual hermitian form on $\mathbb{C}^d$. With (1.114) and (1.116) we get the following

Lemma 1.5.6 *The critical set $\mathrm{Cr}(\|\Phi - \xi\|^2_b) \subset \mathbb{C}^d$ is contained in the ball of radius*

$$\frac{\|\xi\|_b\, \|\eta_b\|_b}{C_\eta},$$

where $\eta \in \mathfrak{t}$ is such that $C_\eta = \frac{1}{2}\inf_j \langle \alpha_j, \eta\rangle > 0$.

We have then proved that the symbols $\mathrm{Thom}_{\xi,b}(\mathbb{C}^d)$ are transversally elliptic.

Proposition 1.5.7 *The class of the transversally elliptic symbol* $\mathrm{Thom}_{\xi,b}(\mathbb{C}^d)$ *in* $\mathbf{K}_T(\mathbf{T}_T\mathbb{C}^d)$ *does not depend of the data* ξ, b*, and is equal to the class defined by* $\mathrm{Thom}^\eta(\mathbb{C}^d)$ *where* $\eta \in \mathfrak{t}$ *is chosen so that* $\langle \alpha_j, \eta\rangle > 0$ *for all* j*.*

Proof. After Lemma 1.5.6, we know that for any scalar product $b(\cdot,\cdot)$ on $\mathfrak{t}^*$, the characteristic set of $\mathrm{Thom}_{0,b}(\mathbb{C}^d)$ intersects $\mathbf{T}_T\mathbb{C}^d$ at $\{0\}$. If b_0 and b_1 are two scalar products on $\mathfrak{t}^*$ we consider the family $b_t = tb_1 + (1-t)b_0,\ 0 \le t \le 1$, of scalar products on $\mathfrak{t}^*$. Hence $\mathrm{Thom}_{0,b_t}(\mathbb{C}^d)$, $t \in [0,1]$, defines an homotopy of transversally elliptic symbols. We have proved that $\mathrm{Thom}_{0,b_0}(\mathbb{C}^d) = \mathrm{Thom}_{0,b_1}(\mathbb{C}^d)$ in $\mathbf{K}_T(\mathbf{T}_T\mathbb{C}^d)$ for any $\xi \in \mathfrak{t}^*$.

Fix now the scalar product b and an element $\xi \in \mathfrak{t}^*$. For any $t \in [0,1]$ the characteristic set of $\mathrm{Thom}_{t\xi,b}(\mathbb{C}^d)$ intersects $\mathbf{T}_T\mathbb{C}^d$ in the ball of radius

$$\frac{\|\xi\|_b\,\|\eta_b\|_b}{C_\eta}.$$

Hence $\mathrm{Thom}_{t\xi,b}(\mathbb{C}^d)$, $t \in [0,1]$, defines an homotopy of transversally elliptic symbols: $\mathrm{Thom}_{\xi,b}(\mathbb{C}^d) = \mathrm{Thom}_{0,b}(\mathbb{C}^d)$ in $\mathbf{K}_T(\mathbf{T}_T\mathbb{C}^d)$. We have proved that the class of the transversally elliptic symbol $\mathrm{Thom}_{\xi,b}(\mathbb{C}^d)$ in $\mathbf{K}_T(\mathbf{T}_T\mathbb{C}^d)$ does not depend of the data ξ, b.

Since the weights α_j lie enterely in an open halfspace of $\mathfrak{t}^*$, there exists a scalar product $b_+(\cdot,\cdot)$ on $\mathfrak{t}^*$ for which we have

$$b_+(\alpha_i, \alpha_j) > 0$$

for all $i, j = 1, \dots, d$. Let $\mathcal{H}^{b_+}$ be the Hamiltonian vector field of the function $\frac{-1}{2}\|\Phi\|^2_{b_+}$, and let $\eta_{\mathbb{C}^d}$ be the vector field on $\mathbb{C}^d$ generated by $\eta \in \mathfrak{t}$ such that $\langle \alpha_j, \eta\rangle > 0$ for all j. A straightforward computation gives that

$$(\mathcal{H}^{b_+}(z), \eta_{\mathbb{C}^d}(z)) > 0 \tag{1.117}$$

for all non zero $z \in \mathbb{C}^d$. Consider now the following familly of symbols on $\mathbb{C}^d$

$$\sigma_t(z,v) = \mathrm{Thom}(\mathbb{C}^d)(z, v - (t\mathcal{H}^{b_+} + (1-t)\eta_{\mathbb{C}^d})(z)), \quad (z,v) \in \mathbf{T}\mathbb{C}^d.$$

so that $\sigma_0 = \mathrm{Thom}^\eta(\mathbb{C}^d)$ and $\sigma_1 = \mathrm{Thom}_{0,b_+}(\mathbb{C}^d)$. The inequality (1.117) shows that $\mathrm{Char}(\sigma_t) \cap \mathbf{T}_T\mathbb{C}^d = \{0\}$ for all $t \in [0,1]$. Hence σ_t, $t \in [0,1]$, defines an homotopy of transversally elliptic symbols: $\mathrm{Thom}^\eta(\mathbb{C}^d) = \mathrm{Thom}_{0,b_+}(\mathbb{C}^d)$ in $\mathbf{K}_T(\mathbf{T}_T\mathbb{C}^d)$. □

For the remaining part of this paper we fix a scalar product on $\mathfrak{t}^*$, and we consider the family of transversally elliptic symbols $\mathrm{Thom}_\xi(\mathbb{C}^d)$, $\xi \in \mathfrak{t}^*$ (to simplify, we do not mention the scalar product in the notation). Proposition 1.5.7 and (1.113) imply the following

Proposition 1.5.8 *For every* $\xi \in \mathfrak{t}^*$, $\mathcal{Q}^T(\mathbb{C}^d)$ *is equal to the generalized character* $\mathrm{Index}^T_{\mathbb{C}^d}\left(\mathrm{Thom}_\xi(\mathbb{C}^d)\right)$.

Now we apply the techniques developped in Section 1.3 in order to compute the multiplicities of $\mathrm{Index}^T_{\mathbb{C}^d}\big(\mathrm{Thom}_\xi(\mathbb{C}^d)\big)$.

1.5.3 Localization in a non-compact setting

Like in Section 1.3.2 we start with the

Definition 1.5.9 *For any $\xi \in \mathfrak{t}^*$ and any T-invariant* relatively compact *open subset $\mathcal{U} \subset \mathbb{C}^d$ we define the symbol $\mathrm{Thom}_\xi(\mathcal{U})$ by the relation*

$$\mathrm{Thom}_\xi(\mathcal{U})(z,v) := \mathrm{Thom}(\mathbb{C}^d)(z, v - (\mathcal{H} - \xi_{\mathbb{C}^d})(z)) \quad (z,v) \in \mathbf{T}\mathcal{U}.$$

The symbol $\mathrm{Thom}_\xi(\mathcal{U})$ is transversally elliptic when $\mathrm{Cr}(\| \Phi - \xi \|^2) \cap \partial\mathcal{U} = \emptyset$ (the couple $(\mathcal{U}, \xi)$ is called *good*) and we denote by

$$RR^\xi_{\mathcal{U}}(\mathbb{C}^d) \in R^{-\infty}(T)$$

its index. Proposition 1.3.10 is still valid here. In particular, for a good couple $(\mathcal{U}, \xi)$, we have $RR^{\xi'}_{\mathcal{U}}(\mathbb{C}^d) = RR^\xi_{\mathcal{U}}(\mathbb{C}^d)$ if ξ' is close enough to ξ. Consider now the decomposition

$$\mathrm{Cr}(\| \Phi - \xi \|^2) = \bigcup_{\gamma \in \mathcal{B}_\xi} (\mathbb{C}^d)^\gamma \cap \Phi^{-1}(\gamma + \xi).$$

Here $\mathcal{B}_\xi \subset \mathfrak{t}^*$ is finite set since $\mathbb{C}^d$ has a finite number of stabilizer. Since $0 \in (\mathbb{C}^d)^\gamma$ and $z \mapsto \langle \Phi(z), \gamma \rangle$ is constant on $(\mathbb{C}^d)^\gamma$, we have

$$(\gamma + \xi, \gamma) = 0 \tag{1.118}$$

for all $\gamma \in \mathcal{B}_\xi$.

Definition 1.5.10 *For any $\xi \in \mathfrak{t}^*$ and $\gamma \in \mathcal{B}_\xi$, we denote simply by*

$$RR^\xi_\gamma(\mathbb{C}^d) \in R^{-\infty}(T)$$

the generalized character $RR^\xi_{\mathcal{U}}(\mathbb{C}^d)$, where $\mathcal{U}$ is a T-invariant relatively compact open neighborhood of $(\mathbb{C}^d)^\gamma \cap \Phi^{-1}(\gamma + \xi)$ such that $\mathrm{Cr}(\| \Phi - \xi \|^2) \cap \overline{\mathcal{U}} = (\mathbb{C}^d)^\gamma \cap \Phi^{-1}(\gamma + \xi)$.

Since $RR^\xi_{\mathbb{C}^d}(\mathbb{C}^d)$ is equal to $\mathcal{Q}^T(\mathbb{C}^d)$ (see Proposition 1.5.8), part a) of Proposition 1.3.10 insures that we have the decomposition

$$\mathcal{Q}^T(\mathbb{C}^d) = \sum_{\gamma \in \mathcal{B}_\xi} RR^\xi_\gamma(\mathbb{C}^d).$$

Let $\mathfrak{c} \subset \mathfrak{t}^*$ be a conic chamber of the cone $C(R)$, and take ξ in $\mathfrak{c}$. Then ξ is a regular value of the moment map $\Phi : \mathbb{C}^d \to \mathfrak{t}^*$ defined in (1.111). Let Ω_ξ be the symplectic structure on the orbifold $(\mathbb{C}^d)_\xi = \Phi^{-1}(\xi)/T$ that is induced

from Ω. The orbifold $(\mathbb{C}^d)_\xi$ is also equipped with a *complex structure* J_ξ that is induced from the standard complex structure on $\mathbb{C}^d$, in such a way that the orbifold $((\mathbb{C}^d)_\xi, \Omega_\xi, J_\xi))$ is a Kähler orbifold. If ξ belongs to the lattice Λ^*, the reduced space $(\mathbb{C}^d)_\xi$ is the Kähler toric variety corresponding to the polytope $\{s \in (\mathbb{R}^{\geq 0})^d \mid \sum s_j \alpha_j = \xi\}$ of $\mathbb{R}^d$. For every $\mu \in \Lambda$ we consider the holomorphic orbifold line bundle

$$\mathcal{L}_{\xi,\mu} = (\Phi^{-1}(\xi) \times \mathbb{C}_{-\mu})/T$$

on $(\mathbb{C}^d)_\xi$.

Definition 1.5.11 *The periodic polynomial $P_{\mathfrak{c}} : \Lambda^* \to \mathbb{Z}$ associated to the conic chamber $\mathfrak{c}$ is given by*

$$P_{\mathfrak{c}}(\mu) = RR((\mathbb{C}^d)_\xi, \mathcal{L}_{\xi,\mu}), \tag{1.119}$$

where the right hand side is the Riemann-Roch number associated to the holomorphic orbifold line bundle $\mathcal{L}_{\xi,\mu}$.

Another way to define the periodic polynomial $P_{\mathfrak{c}}$ is to consider the generalized character $RR_0^\xi(\mathbb{C}^d)$ for $\xi \in \mathfrak{c}$: here $\gamma = 0$ parametrizes the component $\Phi^{-1}(\xi)$ of $\mathrm{Cr}(\| \Phi - \xi \|^2)$. Following (1.46) we have

$$RR_0^\xi(\mathbb{C}^d) = \sum_{\mu \in \Lambda^*} P_{\mathfrak{c}}(\mu)\, \mathbb{C}_\mu \quad \text{in} \quad R^{-\infty}(T). \tag{1.120}$$

After Lemma 1.3.16, we know that $RR_0^{\xi'}(\mathbb{C}^d) = RR_0^\xi(\mathbb{C}^d)$ when ξ, ξ' are two elements of $\mathfrak{c}$: hence the polynomial $P_{\mathfrak{c}}$ does not depend of the choice of ξ in $\mathfrak{c}$.

1.5.4 Proof of Theorem 1.5.1

Consider a weight $\mu \in (\mathfrak{c} - \square(R)) \cap \Lambda^*$ of the form $\mu = \xi' - \sum_j t_j \alpha_j$ with $\xi' \in \mathfrak{c}$ and $t_j \in [0,1]$. We start with the decomposition

$$\mathcal{Q}^T(\mathbb{C}^d) = \sum_{\gamma \in \mathcal{B}_{\xi'}} RR_\gamma^{\xi'}(\mathbb{C}^d).$$

Since $N_R(\mu)$ and $P_{\mathfrak{c}}(\mu)$ are respectively the multiplicity of $\mathbb{C}_\mu$ in $\mathcal{Q}^T(\mathbb{C}^d)$ and in $RR_0^{\xi'}(\mathbb{C}^d)$, the proof will be complete if we show that the multiplicity of $\mathbb{C}_\mu$ in $RR_\gamma^{\xi'}(\mathbb{C}^d)$ is equal to zero when $\gamma \neq 0$.

Consider a *non-zero* element γ in $\mathcal{B}_{\xi'}$. For the character $RR_\gamma^{\xi'}(\mathbb{C}^d)$ the localization (1.44) gives

$$RR_\gamma^{\xi'}(\mathbb{C}^d) = RR_0^{\xi'+\gamma}((\mathbb{C}^d)^\gamma) \otimes \left[\wedge_{\mathbb{C}}^\bullet \overline{N}\right]_\gamma^{-1}, \tag{1.121}$$

where $N = \sum_{(\alpha_j,\gamma)\neq 0} \mathbb{C}_{-\alpha_j}$ corresponds to the normal bundle of $(\mathbb{C}^d)^\gamma$ in $\mathbb{C}^d$. The inverse $\left[\wedge^\bullet_{\mathbb{C}}\overline{N}\right]^{-1}_\gamma$ is equal to $(-1)^l\mathbb{C}_{\delta(\gamma)}\otimes S^\bullet(N^{+,\gamma}_{\mathbb{C}})$ where

$$\delta(\gamma) = -\sum_{(\alpha_j,\gamma)<0}\alpha_j.$$

Since γ acts trivially on $(\mathbb{C}^d)^\gamma$ all the weights $\mu' \in \Lambda^*$ that appear in $RR_0^{\xi'+\gamma}((\mathbb{C}^d)^\gamma)$ satisfy $(\mu',\gamma)=0$. Since the weights of $N^{+,\gamma}_{\mathbb{C}}$ are polarized by γ, we see from (1.121) that all the weights $\mu' \in \Lambda^*$ that appear in $RR^{\xi'}_\gamma(\mathbb{C}^d)$ must satisfy

$$(\mu',\gamma) \geq (\delta(\gamma),\gamma). \tag{1.122}$$

Consider now the weight $\mu = \xi' - \sum_j t_j\alpha_j$. Since $\xi' \in \mathfrak{c}$, the equality (1.118) implies $(\xi',\gamma)<0$ and then

$$(\mu,\gamma) = \underbrace{(\xi',\gamma)}_{<0} + \underbrace{\sum_{(\alpha_j,\gamma)>0} -t_j(\alpha_j,\gamma)}_{\leq 0} - \sum_{(\alpha_j,\gamma)<0} t_j(\alpha_j,\gamma) < -\sum_{(\alpha_j,\gamma)<0}(\alpha_j,\gamma).$$

So we have proved that $(\mu,\gamma) < (\delta(\gamma),\gamma)$, hence the multiplicity of $\mathbb{C}_\mu$ in $RR^{\xi'}_\gamma(\mathbb{C}^d)$ is equal to zero. □

1.5.5 Proof of Theorem 1.5.2

Let $\mathfrak{c}_\pm$ be two adjacent conic chambers separated by the hyperplane $\Delta = \{\xi \in \mathfrak{t}^* \mid \langle \xi,\beta\rangle = 0\}$. Here β is pointing out $\mathfrak{c}_-$.

We consider two points $\xi_\pm \in \mathfrak{c}_\pm$ such that $\xi = \frac{1}{2}(\xi^+ + \xi^-) \in \Delta$ belongs to the conic chanber $\mathfrak{c}'$. We suppose also that the orthogonal projection of $\xi_\pm$ on Δ are equal to ξ. We know that $P_{\mathfrak{c}_+}(\mu) - P_{\mathfrak{c}_-}(\mu)$ is equal to the μ-mutiplicity of $RR_0^{\xi_+}(\mathbb{C}^d) - RR_0^{\xi_-}(\mathbb{C}^d)$. Proposition 1.3.23 tells us that

$$RR_0^{\xi_+}(\mathbb{C}^d) - RR_0^{\xi_-}(\mathbb{C}^d) = RR_\gamma^{\xi_-}(\mathbb{C}^d) - RR_{-\gamma}^{\xi_+}(\mathbb{C}^d),$$

where $\gamma \in \mathbb{R}^{>0}\beta$ is such that $\xi_- + \gamma = \xi_+ - \gamma = \xi$. The localization (1.44) gives then

$$RR_\gamma^{\xi_-}(\mathbb{C}^d) - RR_{-\gamma}^{\xi_+}(\mathbb{C}^d) = RR_0^\xi((\mathbb{C}^d)^\beta)\otimes\left(\left[\wedge^\bullet_{\mathbb{C}}\overline{N}\right]^{-1}_\beta - \left[\wedge^\bullet_{\mathbb{C}}\overline{N}\right]^{-1}_{-\beta}\right). \tag{1.123}$$

Let $P_{\mathfrak{c}'} : \Lambda^* \cap \Delta \to \mathbb{Z}$ be the periodic polynomial map which coincides with the vector partition function $N_{R\cap\Delta}$ on $\overline{\mathfrak{c}'}\cap\Lambda^*$. If we work with the vector space $(\mathbb{C}^d)^\beta$ equipped with the hamiltonian action of T/T_Δ, (1.46) gives the following equality in $R^{-\infty}(T/T_\Delta)$

$$RR_0^\xi((\mathbb{C}^d)^\beta) = \sum_{\gamma\in\Lambda^*\cap\Delta} P_{\mathfrak{c}'}(\gamma)\mathbb{C}_\gamma. \tag{1.124}$$

A straightforward computation gives

$$\left[\wedge_{\mathbb{C}}^{\bullet}\overline{N}\right]_{\beta}^{-1} = (-1)^{r^-}\sum_{\mu\in\Lambda^*} N_{R'}(\mu+\delta^-)\mathbb{C}_{\mu} \tag{1.125}$$

and

$$\left[\wedge_{\mathbb{C}}^{\bullet}\overline{N}\right]_{-\beta}^{-1} = (-1)^{r^+}\sum_{\mu\in\Lambda^*} N_{-R'}(\mu+\delta^+)\mathbb{C}_{\mu}, \tag{1.126}$$

where $r^{\pm}, \delta^{\pm}, R'$ are defined in (1.106), (1.107) and (1.108). Since $N_{-R'}(\mu) = N_{R'}(-\mu)$, the equations (1.124), (1.125) and (1.126) show that the right hand side of (1.123) is equal to

$$\sum_{\mu\in\Lambda^*}\sum_{\gamma\in\Lambda^*\cap\Delta} D(\mu)P_{\mathfrak{c}'}(\gamma)\mathbb{C}_{\mu+\gamma}$$

with $D(\mu) = (-1)^{r^-}N_{R'}(\mu+\delta^-) - (-1)^{r^+}N_{R'}(-\mu-\delta^+)$. Finally we have proved that $P_{\mathfrak{c}_+}(\mu) - P_{\mathfrak{c}_-}(\mu) = \sum_{\gamma\in\Lambda^*\cap\Delta} D(\mu-\gamma)P_{\mathfrak{c}'}(\gamma)$. □

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