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The Fourier transform of semi-simple coadjoint orbits

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Abstract

Let M be a closed coadjoint orbit of a real connected semi-simple Lie group G , and let $F_M \in C^{-\infty}(\mathfrak{g})^G$ be its Fourier transform. In this paper we compute the restriction of F_M to the Lie algebra \mathfrak{k} of a maximal compact subgroup K of G . Using a technique of localization in equivariant cohomology developed in [16, 17], we extend previous results by M. Duflou, G. Heckman, M. Vergne and I. Sengupta.

Contents

1	Coadjoint orbits of semi-simple Lie groups	1
2	Localization of the Fourier transform	5
2.1	Equivariant cohomology-Definitions	5
2.2	Critical points of $\ \mu_K\ ^2$	7
2.3	Localization on $\text{Cr}(\ \mu_K\ ^2)$	8
3	First reduction: Symplectic induction	10
4	Second reduction: Deformation procedure	14
5	Computation of $\int_{\tilde{Y}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$	19

1 Coadjoint orbits of semi-simple Lie groups

Let G be a connected, real, semi-simple Lie group with finite center. Let \mathfrak{g} be its Lie algebra, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of \mathfrak{g} . We denote by K (resp. θ) the compact connected subgroup of G with Lie algebra \mathfrak{k} (resp. Cartan involution).

Let M be an orbit of the coadjoint representation. It is a regularly embedded submanifold of \mathfrak{g}^* which carries a canonical symplectic 2-form Ω ; in particular the manifold M is of even dimension $2d$. We denote by $d\mathcal{L} := \frac{\Omega^d}{(2\pi)^{d!}}$

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the corresponding Liouville volume form on M . The action of G on M is Hamiltonian and the corresponding moment map $\mu_G : M \hookrightarrow \mathfrak{g}^*$ is the inclusion. Note that

$$\mu_G \text{ is proper} \iff M \text{ is closed in } \mathfrak{g}^*. \quad (1.1)$$

The induced action of K on M is Hamiltonian, and the moment map $\mu_K : M \rightarrow \mathfrak{k}^*$ is by definition the composition of μ_G with the projection $\mathfrak{g}^* \rightarrow \mathfrak{k}^*$. For $X \in \mathfrak{g}$, we write $X = X_1 + X_2$ with $X_1 \in \mathfrak{k}$ and $X_2 \in \mathfrak{p}$. The Killing form B defines K -invariant Euclidean structures on \mathfrak{k} and \mathfrak{p} such that

$$B(X, X) = -\|X_1\|^2 + \|X_2\|^2, \quad X \in \mathfrak{g}, \quad (1.2)$$

is a G -invariant quadratic form on \mathfrak{g} . Let $\|\cdot\|$ be the K -invariant Euclidean norm on \mathfrak{g} defined by the equation $\|X\|^2 := -B(X, \theta X) = \|X_1\|^2 + \|X_2\|^2$.

Remark 1.1 *The Killing form B provides a G -equivariant identification $\mathfrak{g} \cong \mathfrak{g}^*$, and also the following K -invariant identification $\mathfrak{k} \cong \mathfrak{k}^*$, $\mathfrak{p} \cong \mathfrak{p}^*$. Then we will only deal with adjoint orbits M of G . In this case the symplectic structure on M is defined by the equation $\Omega_m(X_M, Y_M) = -B(m, [X, Y])$ for $m \in M$ and $X, Y \in \mathfrak{g}$, and the moment map $\mu_K : M \rightarrow \mathfrak{k}$ is the restriction to M of the orthogonal projection $\mathfrak{g} \rightarrow \mathfrak{k}$, $X \mapsto X_1$.*

Let $a \in \mathbb{R}$ be the value of $X \mapsto B(X, X)$ on M . From the decomposition $\|X_1\|^2 = \frac{1}{2}(\|X_1\|^2 + \|X_2\|^2) + \frac{1}{2}(\|X_1\|^2 - \|X_2\|^2)$ and using (1.2) we see that $\|\mu_K\|^2 = \frac{1}{2}\|\mu_G\|^2 - \frac{1}{2}a$ holds on M . Hence the relation (1.1) becomes

$$\mu_K \text{ is proper} \iff M \text{ is closed in } \mathfrak{g}. \quad (1.3)$$

(We just use the fact that: μ_K is proper $\iff \|\mu_K\|^2$ is proper, and the same is true for μ_G .)

We denote respectively by $\mathcal{C}_{rd}^\infty(\mathfrak{g})$ and $\mathcal{C}_{cpt}^\infty(\mathfrak{g})$, the Schwartz space of smooth rapidly decreasing functions on \mathfrak{g} , and the space of smooth functions with compact support in \mathfrak{g} .

Assumption 1.2 *We suppose for the rest of this paper that M is a closed adjoint orbit in \mathfrak{g} .*

In this case the Liouville volume form $d\mathcal{L}$ defines a tempered positive measure on \mathfrak{g} : for every function $f \in \mathcal{C}_{rd}^\infty(\mathfrak{g})$, the integral $\int_M f(m)d\mathcal{L}(m)$ converges. Then we can define the Fourier transform of this measure

$$F_M(X) = (\imath)^d \int_M e^{-\imath B(m, X)} d\mathcal{L}(m), \quad X \in \mathfrak{g},$$

which is the generalized, tempered, and G -invariant function $F_M \in \mathcal{C}_{temp}^{-\infty}(\mathfrak{g})^G$ defined by the equation $\langle F_M(X), f(X)dX \rangle_{\mathfrak{g}} := (\imath)^d \int_M \left(\int_{\mathfrak{g}} e^{-\imath B(m, X)} f(X)dX \right) d\mathcal{L}(m)$, for every $f \in \mathcal{C}_{rd}^\infty(\mathfrak{g})$ (where dX is a Lebesgue measure on \mathfrak{g}).

The generalized function $F_M \in \mathcal{C}_{temp}^{-\infty}(\mathfrak{g})^G$ admits a restriction $F_M|_{\mathfrak{k}}$ to \mathfrak{k} (see section 1 of [8] for the notion of restriction). Moreover the generalized function $F_M|_{\mathfrak{k}} \in \mathcal{C}_{temp}^{-\infty}(\mathfrak{k})^K$ is determined by the equation $X \in \mathfrak{k}$, $F_M|_{\mathfrak{k}}(X) =$

$(i)^d \int_M e^{i(\mu_K(m), X)} d\mathcal{L}(m)$, where (\cdot, \cdot) is the K -invariant scalar product on \mathfrak{k} coming from the Killing form (see Proposition 5 of [8]). Using the K -equivariant symplectic 2-form $\Omega_{\mathfrak{k}}(X) := \Omega + (\mu_K, X)$, $X \in \mathfrak{k}$, the function $F_M|_{\mathfrak{k}} \in \mathcal{C}_{temp}^{-\infty}(\mathfrak{k})^K$ can be rewritten in the following way

$$F_M|_{\mathfrak{k}}(X) = \int_M e^{i\Omega_{\mathfrak{k}}(X)}, \quad X \in \mathfrak{k}$$

In this paper we shall compute the generalized functions $F_M|_{\mathfrak{k}} \in \mathcal{C}_{temp}^{-\infty}(\mathfrak{k})^K$ for every closed orbits $M \subset \mathfrak{g}$.

This computation has been already carried out in the *regular* case by J. Sengupta [20] (modulo a constant) and in the *elliptic* case by Duflo-Heckman-Vergne [7, 8]. Our method, which is closed to those of M. Duflo and M. Vergne in [8], uses the techniques of localization in equivariant cohomology developed in [16, 17].

Recall now the Rossmann formula. Suppose that G and K have the same rank, and let $T \subset K$ be a Cartan subgroup of K . We denote by W the associated Weyl group. Let \mathfrak{t} be the Lie algebra of T and \mathfrak{t}_r be the open subset of regular point: $X \in \mathfrak{t}_r$ iff the stabilizer of X in K is equal to the torus T .

Definition 1.3 *Let V be an oriented Euclidean space provided with an action $\rho : H \rightarrow SO(V)$ of a compact Lie group H that preserves the orientation o of V . Let \mathfrak{h} be the Lie algebra of H and we still denote by $\rho : \mathfrak{h} \rightarrow so(V)$ the morphism of Lie algebras. We denote by $\Pi_V(X) = \det_{V,o}^{1/2}(X)$, $X \in \mathfrak{h}$ the K -invariant polynomial square root of the polynomial function $X \mapsto \det_V(\rho(X))$ on \mathfrak{h} .*

Let H be a compact Lie group with Lie algebra \mathfrak{h} . For any Lebesgue measure dX on \mathfrak{h} , we denote by $\text{vol}(H, dX)$ the volume of H computed with the Haar measure compatible with dX .

For $\beta \in \mathfrak{k}$ we denote by \mathfrak{g}_β (resp. \mathfrak{k}_β and \mathfrak{p}_β) the subspace of \mathfrak{g} (resp. \mathfrak{k} and \mathfrak{p}) fixed by the adjoint action of β . Let K_β be the stabilizer of β in K : it is a connected subgroup with Lie algebra \mathfrak{k}_β . The vector spaces $\mathfrak{g}/\mathfrak{g}_\beta$ and $\mathfrak{k}/\mathfrak{k}_\beta$, considered as the tangent space at β of the orbits $G.\beta$ and $K.\beta$, are oriented by the respective symplectic structures. Using the decomposition $\mathfrak{g}/\mathfrak{g}_\beta = \mathfrak{k}/\mathfrak{k}_\beta \oplus \mathfrak{p}/\mathfrak{p}_\beta$, the space $\mathfrak{p}/\mathfrak{p}_\beta$ has an induced orientation. The space $\mathfrak{g}/\mathfrak{g}_\beta$ (resp. $\mathfrak{k}/\mathfrak{k}_\beta$ and $\mathfrak{p}/\mathfrak{p}_\beta$) carries a natural Euclidean structure and a K_β -action, coming from its identification with the orthogonal complement of \mathfrak{g}_β (resp. \mathfrak{k}_β and \mathfrak{p}_β) in \mathfrak{g} (resp. \mathfrak{k} and \mathfrak{p}). Following the Definition 1.3, this data define the K_β -invariant polynomial functions $\Pi_{\mathfrak{g}/\mathfrak{g}_\beta}$, $\Pi_{\mathfrak{k}/\mathfrak{k}_\beta}$, and $\Pi_{\mathfrak{p}/\mathfrak{p}_\beta}$ on \mathfrak{k}_β .

When β belongs to \mathfrak{t}_r the following proposition is due to Rossmann [18].

Proposition 1.4 *Let $\beta \in \mathfrak{t}$ and let $M = G.\beta$ be the associated (closed) adjoint orbit. The function F_M is analytic on $G.\mathfrak{t}_r$ (hence $F_M|_{\mathfrak{k}}$ is analytic on $K.\mathfrak{t}_r$), and for every $X \in \mathfrak{t}_r$ we have*

$$F_M(X) = (-2\pi)^{d_\beta} \sum_{W/W_\beta} \frac{e^{i(\beta, w.X)}}{\Pi_{\mathfrak{g}/\mathfrak{g}_\beta}(w.X)},$$

where $\Pi_{\mathfrak{g}/\mathfrak{g}_\beta}$ is the K_β -invariant polynomial defined at the Definition 1.3, $d_\beta = \frac{1}{2} \dim(M)$, and W_β is the subgroup of W that stabilizes β .

Apart from the proof of Rossmann [18], we can find another proofs of this Proposition in [4, 7, 22].

Notations : Let $F \in \mathcal{C}^{-\infty}(V)$ be a generalized function on a vector space V . For every test density $f(v)dv$ over V , we denote by $\langle F(v), f(v)dv \rangle_V \in \mathbb{C}$ the image (or the integral) of $f(v)dv$ by F .

Let's now give a global expression of $F_M|_{\mathfrak{k}}$. For a function $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k})^K$ supported in $K.t_r$ we get from Proposition 1.4

$$\begin{aligned} \langle F_M|_{\mathfrak{k}}(X), f(X)dX \rangle_{\mathfrak{k}} &= \text{cte} \langle \frac{1}{\Pi_{\mathfrak{g}/\mathfrak{g}_\beta}(Y)}, e^{\iota(\beta, Y)} f|_{\mathfrak{t}}(Y) \Pi_{\mathfrak{t}/\mathfrak{t}}^2(Y) dY \rangle_{\mathfrak{t}} \quad [1] \\ &= \text{cte} \langle \frac{1}{\Pi_{\mathfrak{p}/\mathfrak{p}_\beta}(Y)}, e^{\iota(\beta, Y)} f|_{\mathfrak{t}}(Y) \Pi_{\mathfrak{t}/\mathfrak{k}_\beta}(Y) \Pi_{\mathfrak{k}_\beta/\mathfrak{t}}^2(Y) dY \rangle_{\mathfrak{t}} \quad [2] \\ &= \text{cte}' \langle \frac{1}{\Pi_{\mathfrak{p}/\mathfrak{p}_\beta}(Z)}, e^{\iota(\beta, Z)} f|_{\mathfrak{k}_\beta}(Z) \Pi_{\mathfrak{t}/\mathfrak{k}_\beta}(Z) dZ \rangle_{\mathfrak{k}_\beta}, \quad [3] \end{aligned}$$

where $\text{cte} = \frac{\text{vol}(K, dX)}{\text{vol}(T, dY)} \frac{(-2\pi)^{d_\beta}}{|W_\beta|}$, and $\text{cte}' = (-2\pi)^{d_\beta} \frac{\text{vol}(K, dX)}{\text{vol}(K_\beta, dZ)}$. In the equality [1] we use the Weyl integration formula for $(\mathfrak{k}, \mathfrak{t})$ and the fact that $f|_{\mathfrak{t}}$ is W -invariant. The equality [2] comes from the equalities $\Pi_{\mathfrak{g}/\mathfrak{g}_\beta} = \Pi_{\mathfrak{t}/\mathfrak{k}_\beta} \Pi_{\mathfrak{p}/\mathfrak{p}_\beta}$ and $\Pi_{\mathfrak{t}/\mathfrak{t}} = \Pi_{\mathfrak{t}/\mathfrak{k}_\beta} \Pi_{\mathfrak{k}_\beta/\mathfrak{t}}$. In the last equality [3] we use the Weyl integration formula for $(\mathfrak{k}_\beta, \mathfrak{t})$. Note that $Z \rightarrow e^{\iota(\beta, Z)} f|_{\mathfrak{k}_\beta}(Z) \Pi_{\mathfrak{t}/\mathfrak{k}_\beta}(Z)$ and $Z \rightarrow \Pi_{\mathfrak{p}/\mathfrak{p}_\beta}(Z)$ are K_β -invariant functions on \mathfrak{k}_β .

The K_β -equivariant Euler form of the oriented K_β vector bundle $\mathfrak{p}/\mathfrak{p}_\beta \rightarrow \{0\}$ is equal to $Z \rightarrow \Pi_{\mathfrak{p}/\mathfrak{p}_\beta}(\frac{1}{-2\pi}Z)$. This equivariant polynomial has a tempered generalized inverse $\text{Eul}_\beta^{-1}(\mathfrak{p}/\mathfrak{p}_\beta)$ defined by

$$\text{Eul}_\beta^{-1}(\mathfrak{p}/\mathfrak{p}_\beta)(Z) = \lim_{s \rightarrow 0^+} \frac{1}{\Pi_{\mathfrak{p}/\mathfrak{p}_\beta}(\frac{1}{-2\pi}(Z + \iota s \beta))}, \quad Z \in \mathfrak{k}_\beta.$$

For these notions see section 4 of [16].

Finally we can state the global version of the 'Rossmann formula' that is obtained by Duflo-Vergne in [8] (with a different expression). For every function $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k})^K$, we have

$$\langle F_M|_{\mathfrak{k}}(X), f(X)dX \rangle_{\mathfrak{k}} = \text{cte} \langle \text{Eul}_\beta^{-1}(\mathfrak{p}/\mathfrak{p}_\beta)(Z), e^{\iota(\beta, Z)} f|_{\mathfrak{k}_\beta}(Z) \Pi_{\mathfrak{t}/\mathfrak{k}_\beta}(Z) dZ \rangle_{\mathfrak{k}_\beta}, \quad (1.4)$$

with $\text{cte} = (-2\pi)^{\dim(K/K_\beta)/2} \frac{\text{vol}(K, dX)}{\text{vol}(K_\beta, dZ)}$.

The central result of this paper is the following global expression of $F_M|_{\mathfrak{k}}$ for the case of general closed orbits. We come back in the general case of a real semisimple connected Lie group G with finite center, and we make no assumption related to the ranks of G and K . Let T be a Cartan subgroup of K with Lie algebra \mathfrak{t} , and let \mathfrak{t}_+ be a Weyl chamber of \mathfrak{t} .

Theorem 5.4 *Let M be a closed orbit in \mathfrak{g} . There exists a unique $(\beta_0, \beta_1) \in \mathfrak{t}_+ \times \mathfrak{p}$ with $[\beta_0, \beta_1] = 0$ such that $M = G \cdot (\beta_0 + \beta_1)$. Let K_β be the stabilizer of $\beta := \beta_0 + \beta_1$ in K , and let \mathfrak{k}_β be its Lie algebra. The generalized function $\text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0}) \in \mathcal{C}^{-\infty}(\mathfrak{k}_{\beta_0})^{K_{\beta_0}}$ admits a restriction to \mathfrak{k}_β , and for every function $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k})^K$ we have*

$$\langle F_M|_{\mathfrak{k}}(X), f(X)dX \rangle_{\mathfrak{k}} = \text{cte} \langle \text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0})(Z), e^{i(\beta_0, Z)} f|_{\mathfrak{k}_\beta}(Z) \Pi_{\mathfrak{k}/\mathfrak{k}_{\beta_0}}(Z) dZ \rangle_{\mathfrak{k}_\beta},$$

$$\text{with cte} = (-2\pi)^{\dim(K/K_{\beta_0})/2} (2\pi i)^{\dim(K_{\beta_0}/K_\beta)} \frac{\text{vol}(K, dX)}{\text{vol}(K_\beta, dZ)}.$$

Remark 1.5 *The result of Theorem 5.4 can be extended in the following way. Let $\alpha(X)$ be a closed K -equivariant form on M depending polynomially of $X \in \mathfrak{g}$ (see sub-section 2.1 for the definitions). If $\alpha(X)$ is régulière on M (for the notion of being régulière see pages 20-21 of [8]), then the integral*

$$\int_M \alpha(X) e^{i\Omega_{\mathfrak{k}}(X)}, \quad X \in \mathfrak{k},$$

defines a tempered measure on \mathfrak{k} , and we have

$$\langle \left(\int_M \alpha e^{i\Omega_{\mathfrak{k}}} \right)(X), f(X)dX \rangle_{\mathfrak{k}} = \text{cte} \langle \text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0})(Z), e^{i(\beta_0, Z)} r_\beta(\alpha)(Z) f|_{\mathfrak{k}_\beta}(Z) \Pi_{\mathfrak{k}/\mathfrak{k}_{\beta_0}}(Z) dZ \rangle_{\mathfrak{k}_\beta}$$

for every $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k})^K$. In this equality $r_\beta : \mathcal{A}_K^\infty(M) \rightarrow \mathcal{A}_{K_\beta}^\infty(\{\beta\}) = \mathcal{C}^\infty(\mathfrak{k}_\beta)^{K_\beta}$ is the restriction map to the point $\{\beta\} \subset M$.

Acknowledgement. I am grateful to Michel Brion for bringing me the reference [21] to my attention.

2 Localization of the Fourier transform

We take the same notations as before. Let M be a closed adjoint orbit of G in \mathfrak{g} , and consider the Hamiltonian action of the compact subgroup K on M . We know from (1.3) that the associated moment map $\mu_K : M \rightarrow \mathfrak{k}$ is proper (we make the identification $\mathfrak{k} \cong \mathfrak{k}^*$ via the Killing form, see Remark 1.1).

Consider the equivariant symplectic form $\Omega_{\mathfrak{k}}(X) := \Omega + (\mu_{\mathfrak{k}}, X)$, $X \in \mathfrak{k}$ defined on M . By definition of the moment map, the equivariant form $\Omega_{\mathfrak{k}}$ is closed: $\mathcal{D}(\Omega_{\mathfrak{k}})(X) = d(\mu_{\mathfrak{k}}, X) - c(X_M)\Omega = 0$, $X \in \mathfrak{k}$ (see sub-section 2.1 for the notations). And we know from the introduction that the generalized function $F_M|_{\mathfrak{k}}$ is given by the integral of the closed equivariant form $e^{i\Omega_{\mathfrak{k}}} \in \mathcal{A}_K^\infty(M)$ on M . Consider the function $\|\mu_K\|^2 : M \rightarrow \mathbb{R}$. In this section we show that the integral of $e^{i\Omega_{\mathfrak{k}}}$ can be localized on the set $\text{Cr}(\|\mu_K\|^2)$ of critical points of the function $\|\mu_K\|^2$.

First, we recall the definition and notations we use in the rest of this paper.

2.1 Equivariant cohomology-Definitions

Let M be a manifold provided with an action of a compact connected Lie group K with Lie algebra \mathfrak{k} . We denote by $\mathcal{A}^*(M)$ the algebra of differential forms on M (over \mathbb{C}), and by d the exterior differentiation. Let $\mathcal{A}_{cpt}^*(M)$ be the sub-algebra of compactly supported differential forms. If ξ is a vector field on M we denote by $c(\xi) : \mathcal{A}^*(M) \rightarrow \mathcal{A}^{*-1}(M)$ the contraction by ξ . The action of K on M gives a morphism $X \rightarrow X_M$ from \mathfrak{k} to the Lie algebra of vector fields on M .

We now recall the different de Rham complexes of K -equivariant forms on M . For more details see [2, 3, 8, 12].

Let $\mathcal{C}^\infty(\mathfrak{k}, \mathcal{A}^*(M))$ be the algebra of forms $\alpha(X)$ on M depending smoothly of $X \in \mathfrak{k}$. We note $\mathcal{A}_K^\infty(M)$ the sub-algebra of $\mathcal{C}^\infty(\mathfrak{k}, \mathcal{A}^*(M))$ consisting of the K -invariant elements: these elements are called the equivariant forms with \mathcal{C}^∞ -coefficients. Let $\mathcal{A}_K^*(M) \subset \mathcal{A}_K^\infty(M)$ be the sub-algebra of equivariant forms $\alpha(X)$ depending *polynomially* of $X \in \mathfrak{k}$. The differential \mathcal{D} on $\mathcal{A}_K^\infty(M)$ is given by the equation

$$\forall \alpha \in \mathcal{A}_K^\infty(M), \quad (\mathcal{D}\alpha)(X) := (d - c(X_M))(\alpha(X)), \quad X \in \mathfrak{k}.$$

We see that $\mathcal{A}_K^*(M)$ is stable under \mathcal{D} , and that $\mathcal{D}^2 = 0$ on $\mathcal{A}_K^\infty(M)$. The cohomologies associated to $(\mathcal{A}_K^*(M), \mathcal{D})$ and $(\mathcal{A}_K^\infty(M), \mathcal{D})$ are denoted respectively $\mathcal{H}_K^*(M)$ and $\mathcal{H}_K^\infty(M)$.

The algebra $\mathcal{A}_K^\infty(M)$ has a sub-algebra $\mathcal{A}_{K,cpt}^\infty(M) := \mathcal{C}^\infty(\mathfrak{k}, \mathcal{A}_{cpt}^*(M))^K$, stable under the differential \mathcal{D} . The cohomology associated to $(\mathcal{A}_{K,cpt}^\infty(M), \mathcal{D})$ is called the K -equivariant cohomology with compact support and is denoted by $\mathcal{H}_{K,cpt}^\infty(M)$.

For our purpose we need equivariant forms with generalized coefficients. For a more precise description see [12].

The space $\mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}^*(M))$ of generalized functions on \mathfrak{k} with values in the space $\mathcal{A}^*(M)$ is, by definition, the space $\text{Hom}(\mathfrak{m}_c(\mathfrak{k}), \mathcal{A}^*(M))$ of continuous \mathbb{C} -linear maps from the space $\mathfrak{m}_c(\mathfrak{k})$ of smooth compactly supported densities on \mathfrak{k} to the space $\mathcal{A}^*(M)$, both endowed with the \mathcal{C}^∞ -topologies. We define

$$\mathcal{A}_K^{-\infty}(M) := \mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}^*(M))^K$$

as the space of K -equivariant $\mathcal{C}^{-\infty}$ -maps from \mathfrak{k} to $\mathcal{A}^*(M)$. An element of the space $\mathcal{A}_K^{-\infty}(M)$ is called an equivariant form with generalized coefficients. The image of $\phi \in \mathfrak{m}_c(\mathfrak{k})$ under $\alpha \in \mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}^*(M))$ is a differential form on M denoted by $\langle \alpha, \phi \rangle_{\mathfrak{k}}$.

We see that $\mathcal{A}_K^\infty(M) \subset \mathcal{A}_K^{-\infty}(M)$ and we can also extend the differential \mathcal{D} to $\mathcal{A}_K^{-\infty}(M)$ [12]. Take a basis $\{E^1, \dots, E^p\}$ of \mathfrak{k} , with associated dual basis $\{E_1, \dots, E_p\}$. Let $\{X_1, \dots, X_p\}$ be the corresponding coordinate functions on \mathfrak{k} . For every $\gamma \in \mathcal{A}_K^{-\infty}(M)$,

$$\langle \mathcal{D}(\gamma), \phi \rangle_{\mathfrak{k}} := d \langle \gamma, \phi \rangle_{\mathfrak{k}} - \sum_{k=1}^p c(E_M^k) \langle \gamma, X_k \phi \rangle_{\mathfrak{k}} \quad \text{for every } \phi \in \mathfrak{m}_c(\mathfrak{k}). \quad (2.5)$$

Using the K -equivariance condition, we verify that $\mathcal{D}^2 = 0$ on $\mathcal{A}_K^{-\infty}(M)$. The cohomology associated to $(\mathcal{A}_K^{-\infty}(M), \mathcal{D})$ is called the K -equivariant cohomology with generalized coefficients and is denoted by $\mathcal{H}_K^{-\infty}(M)$. The sub-space $\mathcal{A}_{K,cpt}^{-\infty}(M) := \mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}_{cpt}^*(M))^K$ is stable under the differential \mathcal{D} , and we denote by $\mathcal{H}_{K,cpt}^{-\infty}(M)$ the associated cohomology.

Let H be a compact Lie group with Lie algebra \mathfrak{h} . For every $f \in \mathcal{C}^\infty(\mathfrak{h})$, we denote by \overline{f}^H the H -invariant function on \mathfrak{h} defined by the equation

$$\overline{f}^H(X) := \int_K f(h.X)dh, \quad X \in \mathfrak{h},$$

where dh is the normalized Haar measure on H ($\int_H dh = 1$). It defines the projection $f \mapsto \overline{f}^H$, $\mathcal{C}^\infty(\mathfrak{h}) \rightarrow \mathcal{C}^\infty(\mathfrak{h})^H$, and the same holds for the subspace $\mathcal{C}_{cpt}^\infty(\mathfrak{h})$ and $\mathcal{C}_{rd}^\infty(\mathfrak{h})$. Recall that every H -invariant generalized function $\phi \in \mathcal{C}^{-\infty}(\mathfrak{h})$ is completely determined by its values on the H -invariant densities of \mathfrak{h} , and moreover, for every $f \in \mathcal{C}_{cpt}^\infty(\mathfrak{h})$ we have

$$\langle \phi(X), f(X)dX \rangle_{\mathfrak{h}} = \langle \phi(X), \overline{f}^H(X)dX \rangle_{\mathfrak{h}}.$$

2.2 Critical points of $\|\mu_K\|^2$

In this sub-section we prove that the set $\text{Cr}(\|\mu_K\|^2)$ of critical points of the function $\|\mu_K\|^2$ is a K -orbit in M .

For a point $m \in M$ we decompose $m = x_m + y_m$ with $x_m = \mu_K(m) \in \mathfrak{k}$ and $y_m \in \mathfrak{p}$. By definition of the moment map we have $\frac{1}{2}d\|\mu_K\|_m^2 = (d\mu_K(m), \mu_K(m)) = \Omega((x_m)_M|_m, \cdot)$, $m \in M$. Then $d\|\mu_K\|_m^2 = 0$ iff $(x_m)_M|_m = 0$ or equivalently $[x_m, m] = [x_m, y_m] = 0$. We have shown the following

$$\text{Cr}(\|\mu_K\|^2) = \{m \in M \mid [x_m, y_m] = 0\}. \quad (2.6)$$

Proposition 2.1 *The critical points of $\|\mu_K\|^2$ form a K -orbit in M . In particular the points of $\text{Cr}(\|\mu_K\|^2)$ are the points where $\|\mu_K\|^2$ is minimum.*

To prove this proposition we consider the length function $\Psi : M \rightarrow \mathbb{R}$, $\Psi(m) = \|m\|^2$. We have already seen that $\|\mu_K\|^2 = \frac{1}{2}\Psi - \frac{a}{2}$ for some $a \in \mathbb{R}$. Then we can work with Ψ instead of $\|\mu_K\|^2$.

Proposition (2.1) is a consequence of the next Lemma which is due to P. Slodowy [21].

Lemma 2.2 *Let $m \in M$ be a critical point of Ψ , and $m' \in M$. Then either $\Psi(m') > \Psi(m)$ or $m' \in K.m$.*

Proof : Let $g \in G$ such that $m' = g.m$. Using the Cartan decomposition $G = K \exp(\mathfrak{p})$ we know that

$$m' = k \exp(X).m, \quad \text{with } k \in K \text{ and } X \in \mathfrak{p}.$$

We are now going to prove that $\Psi(m') > \Psi(m)$ or $\exp(X).m = m$. Consider the following function $\kappa(t) = \Psi(\exp(tX).m)$, $t \in \mathbb{R}$. The element X belongs to \mathfrak{p} , then the endomorphism $ad(X) : \mathfrak{g} \rightarrow \mathfrak{g}$, $Z \rightarrow [X, Z]$ is auto-adjoint relatively to the scalar product, hence diagonalizable. Let $\mathfrak{g} = \sum_a \mathfrak{g}_a$ be the decomposition into orthogonal subspaces of \mathfrak{k} where for $a \in \mathbb{R}$,

$$\mathfrak{g}_a = \{Y \in \mathfrak{g} \mid [X, Y] = aY\}.$$

If we write $m = \sum_a m_a$ with $m_a \in \mathfrak{g}_a$, we have $\kappa(t) = \|\sum_a e^{at}m_a\|^2 = \sum_a e^{2at}\|m_a\|^2$. This shows that $\kappa''(t) = 4\sum_{a \neq 0} a^2 e^{2at}\|m_a\|^2$ is a positive function on \mathbb{R} , and we have also $\kappa'(0) = 0$ (because m is a critical point of Ψ). The function κ is convex and the equality $\kappa'(0) = 0$ implies that either $\kappa(1) > \kappa(0)$ or κ is constant on the interval $[0, 1]$. In the first case we get that $\Psi(m') > \Psi(m)$. The second point imposes that $\kappa''(t) = 4\sum_{a \neq 0} a^2 e^{2at}\|m_a\|^2 = 0$ for every $0 < t < 1$. This means that $m_a = 0$ for every $a \neq 0$ or equivalently $[X, m] = 0$ (hence $\exp(X).m = m$). \square

This Lemma implies that every critical point of Ψ reaches the minimum of Ψ , and that two critical points of Ψ are in the same K -orbit. The properness of Ψ implies that $\Psi(M)$ is closed in \mathbb{R} , in particular the minimum of Ψ is reached and so $\text{Cr}(\|\mu_K\|^2) = \text{Cr}(\Psi) \neq \emptyset$. The proof of Proposition 2.1 is then completed.

Remark 2.3 *One could consider this length function Ψ in the case of a general orbit M . But from the results of P. Slodowy in [21], we know that $\text{Cr}(\Psi) = \emptyset$ when M is not closed.*

2.3 Localization on $\text{Cr}(\|\mu_K\|^2)$

The K -invariant scalar product on \mathfrak{g} defines a K -invariant Riemannian metric $(\cdot, \cdot)_M$ on the adjoint orbit M . Let $\lambda_K := (\mathcal{H}, \cdot)_M$ be the K -invariant one form on M , where \mathcal{H} is the Hamiltonian vector field associated to the function $\frac{1}{2}\|\mu_K\|^2$. More precisely, if we note $m = x_m + y_m$ with $x_m \in \mathfrak{k}$ and $y_m \in \mathfrak{p}$, we have $\mathcal{H}_m = -[x_m, y_m] \in \mathbf{T}_m M$ for every $m \in M$, and

$$\lambda_K|_m = -\left([x_m, y_m], \cdot\right)_M, \quad m \in M. \quad (2.7)$$

This 1-form was introduced by Witten in [24] to describe a Non-Abelian localization in equivariant cohomology. Now, this idea has been developed by the author in [16, 17]. In the case of an elliptic orbit $M = G.\beta$, with $\beta \in \mathfrak{k}$, Duflou and Vergne already used the 1-form λ_K in [8] (it was denoted by θ , see Proposition 35) to localize the integral defining $F_M|_{\mathfrak{k}}$ to the submanifold $K.\beta \subset M$. We are going to extend this localization for all closed orbits, using the technique of *partition of unity in equivariant cohomology* introduced in [16].

We note $\Phi_{\lambda_K} : M \rightarrow \mathfrak{k}$ the K -equivariant map defined by $(\Phi_{\lambda}(m), X) := \lambda(X_M)(m)$, $m \in M$, $X \in \mathfrak{k}$. By definition $X_M(m) = -[X, m]$ for every $X \in \mathfrak{k}$ and $(\Phi_{\lambda}(m), X) = ([x_m, y_m], [X, m])_M = -B([x_m, y_m], \theta[X, m])$. When $X \in \mathfrak{k}$, the \mathfrak{p} -part of $\theta[X, m]$ is $-[X, y_m]$ and hence $-B([x_m, y_m], \theta[X, m]) =$

$B([x_m, y_m], [X, y_m]) = -B([x_m, y_m], y_m, X)$. Finally we see that $\Phi_{\lambda_K} : M \rightarrow \mathfrak{k}$ is defined by

$$\Phi_{\lambda_K}(m) = [[x_m, y_m], y_m] \in \mathfrak{k} \quad \text{for every } m \in M.$$

The equality $(\Phi_{\lambda}(m), \mu_K)(m) = \|[x_m, y_m]\|^2$ shows that $\{\Phi_{\lambda_K} = 0\} = \{m \in M \mid [x_m, y_m] = 0\} = \text{Cr}(\|\mu_K\|^2)$, and we know from the last sub-section that the set $\text{Cr}(\|\mu_K\|^2)$ is a K -orbit.

We work now with the following data. Take a point $\beta \in \text{Cr}(\|\mu_K\|^2)$. Then the manifold M is of the form $M = G.\beta$ with $\beta = \beta_0 + \beta_1$, $\beta_0 \in \mathfrak{k}$, $\beta_1 \in \mathfrak{p}$, $[\beta_0, \beta_1] = 0$, and $\{\Phi_{\lambda_K} = 0\} = K.\beta$. If we make the choice of a Weyl chamber \mathfrak{t}^+ in \mathfrak{k} , we can take $\beta_0 \in \mathfrak{t}^+$.

Lemma 3.1 of [16] tells us that the equivariant form $\mathcal{D}\lambda_K(X) = d\lambda_K - (\Phi_{\lambda_K}, X)$ is invertible outside the submanifold $K.\beta$ in the space of generalized equivariant forms. For each K -invariant differential form χ_{ext} on M , equal to zero in a neighbourhood of $K.\beta$, we can define $\chi_{ext} \left(\int_0^\infty \iota e^{-t\mathcal{D}\lambda_K} dt \right) \in \mathcal{A}_K^{-\infty}(M)$, and this form satisfies the equality

$$\chi_{ext} \left(\int_0^\infty \iota e^{-t\mathcal{D}\lambda_K} dt \right) \mathcal{D}\lambda_K = \chi_{ext}$$

in $\mathcal{A}_K^{-\infty}(M)$.

Let $\chi \in \mathcal{C}^\infty(M)^K$ be a function with *compact support* on M , and equal to 1 in a neighbourhood of $K.\beta$. Then the differential form $d\chi$ is equal to 0 in a neighbourhood of $K.\beta$, and we can define the equivariant form with *compact support* on M

$$P^K = \chi + d\chi \left(\int_0^\infty \iota e^{-t\mathcal{D}\lambda_K} dt \right) \lambda_K \in \mathcal{A}_{K, cpt}^{-\infty}(M). \quad (2.8)$$

Recall Propositions 3.3 and 3.11 of [16].

Proposition 2.4 *The equivariant form P^K is closed, and we have the identity*

$$\mathbf{1}_M = P^K + \mathcal{D}(\delta), \quad (2.9)$$

where $\mathbf{1}_M$ is the constant function equal to 1 on M , and $\delta = (1-\chi) \left(\int_0^\infty \iota e^{-t\mathcal{D}\lambda_K} dt \right) \lambda_K$ is a generalized K -equivariant form. Moreover the cohomology class of P^K in $\mathcal{H}_{K, cpt}^{-\infty}(M)$ does not depend neither of the choice of the function χ nor of the choice of the Riemannian metric near $K.\beta$.

We use the phrase “partition of unity” to refer to the equality (2.9), and we will use it to decompose every closed form $\eta \in \mathcal{A}_K^\infty(M)$ in the following way $\eta = \eta P^K + \mathcal{D}(\eta\delta)$; a sum of an equivariant form ηP^K supported near $K.\beta$ and an exact equivariant form $\mathcal{D}(\eta\delta)$.

With the compactly supported equivariant form P^K we are going to localize the Fourier transform of M . The generalized equivariant form $e^{i\Omega_t} P^K$ is *tempered*. For every $f \in \mathcal{C}_{dr}^\infty(\mathfrak{k})$, the differential form $\langle e^{i\Omega_t(X)} P^K(X), f(X) dX \rangle_{\mathfrak{k}}$ is well defined, with compact support (included in the support of χ), and is given

$$\begin{aligned} \langle e^{i\Omega_t(X)} P^K(X), f(X) dX \rangle_{\mathfrak{k}}|_m &= \chi(m) e^{i\Omega_m} \widehat{f}(-\mu_K(m)) + \\ & d\chi(m) e^{i\Omega_m} \left(\int_0^\infty \iota e^{-t\lambda_K|_m} \widehat{f}(-t\Phi_{\lambda_K}(m) - \mu_K(m)) dt \right) \lambda_K|_m, \end{aligned} \quad (2.10)$$

where \widehat{f} is the Fourier transform of f relatively to dX . Note that the map $m \rightarrow \int_0^\infty \iota e^{-t\lambda_K|_m} \widehat{f}(-t\Phi_{\lambda_K}(m) - \mu_K(m)) dt$ is a well defined differential form on $M \setminus \{K.\beta\}$.

The integral of $e^{i\Omega_t} P^K$ on M defines a K -invariant tempered measure on \mathfrak{k} by the equation

$$\left\langle \left(\int_M e^{i\Omega_t} P^K \right) (X), f(X) dX \right\rangle_{\mathfrak{k}} := \int_M \langle e^{i\Omega_t(X)} P^K(X), f(X) dX \rangle_{\mathfrak{k}}, \quad f \in \mathcal{C}_{rd}^\infty(\mathfrak{k}).$$

Theorem 2.5 *We have the following equality of tempered measure on \mathfrak{k}*

$$\int_M e^{i\Omega_t} = \int_M e^{i\Omega_t} P^K.$$

Proof: For every *tempered measure* \mathfrak{D} on \mathfrak{k} we have the following property: if for every function $f \in \mathcal{C}_{cpt}^\infty(\mathfrak{k})$ with compact support we have $\langle \mathfrak{D}(X), \widehat{f}(X) dX \rangle_{\mathfrak{k}} = 0$ (where \widehat{f} is the Fourier transform of f relatively to dX), then the measure \mathfrak{D} is *identically equal to zero*.

Here we take $\mathfrak{D} := \int_M e^{i\Omega_t} - \int_M e^{i\Omega_t} P^K$. Using now the partition of unity (2.9), we see that $\langle \mathfrak{D}(X), \widehat{f}(X) dX \rangle_{\mathfrak{k}} = \int_M A_f$ with $A_f = \langle \mathcal{D}(e^{i\Omega_t} \delta)(X), \widehat{f}(X) dX \rangle_{\mathfrak{k}} \in \mathcal{A}^*(M)$. The theorem will be proved after showing that, for every function $f \in \mathcal{C}_{cpt}^\infty(\mathfrak{k})$, there exists a *compactly supported* differential form B_f on M such that $A_f - d(B_f) \in \mathcal{A}^{<\dim M}(M)$: the usual ‘Stokes’ argument implies that $\int_M A_f = \int_M d(B_f) = 0$

By definition of the differential \mathcal{D} , we have $A_f = \langle \mathcal{D}(e^{i\Omega_t} \delta)(X), \widehat{f}(X) dX \rangle_{\mathfrak{k}} = d(\langle e^{i\Omega_t(X)} \delta(X), \widehat{f}(X) dX \rangle_{\mathfrak{k}}) - \sum_{k=1}^p c(E_M^k) \langle e^{i\Omega_t(X)} \delta(X), X_k \widehat{f}(X) dX \rangle_{\mathfrak{k}}$, and we take $B_f := \langle e^{i\Omega_t(X)} \delta(X), \widehat{f}(X) dX \rangle_{\mathfrak{k}}$. For every $m \in M$ we have

$$B_f|_m = (2\pi)^{\dim K} (1 - \chi(m)) e^{i\Omega_m} \left(\int_0^\infty \iota e^{-t\lambda_K|_m} f(t\Phi_{\lambda_K}(m) + \mu_K(m)) dt \right) \lambda_K|_m, \quad (2.11)$$

since $\widehat{f}(-X) = (2\pi)^{\dim K} f(X)$, $X \in \mathfrak{k}$. From (2.11), we see that $B_f|_m = 0$ if $t\Phi_{\lambda_K}(m) + \mu_K(m)$ is not in the support of f for every $t > 0$. But

$$\|t\Phi_{\lambda_K} + \mu_K\|^2 = \|\mu_K\|^2 + t^2 \|\Phi_{\lambda_K}\|^2 + 2t(\Phi_{\lambda_K}, \mu_K) \geq \|\mu_K\|^2$$

because $(\Phi_{\lambda_K}, \mu_K) = \|\mathcal{H}\|_M^2 \geq 0$ on M . Let $r > 0$ such that the function f is supported in the ball $B(O, r)$ of radius r . Using the precedent inequalities we see finally that the differential form B_f is supported in $\mu_K^{-1}(B(O, r))$, hence is with compact support (μ_K being proper). \square

3 First reduction: Symplectic induction

In this subsection, we prove an induction formula for the equivariant form P^K similar to the induction we obtain in section 3 of [17]. For this purpose we use the *cross section Theorem* of Guillemin-Sternberg [10] (see Theorem 6.7), and also a Proposition of Duflo-Vergne [8] about generalized equivariant forms that admit a *restriction*.

Let T be a maximal torus of K with Lie algebra \mathfrak{t} , and let $W := W(K, T)$ be the Weyl group associated. We make the choice of a Weyl chamber \mathfrak{t}_+ in \mathfrak{t} . Recall that a closed adjoint orbit M in \mathfrak{g} is of the form $M = G.\beta$ with $\beta = \beta_0 + \beta_1$, $\beta_0 \in \mathfrak{t}_+$, $\beta_1 \in \mathfrak{p}$ and $[\beta_0, \beta_1] = 0$, and that the cohomology class of P^K does not depend on the choice of the K -invariant Riemannian metric in a neighbourhood of $K.\beta$.

Symplectic cross-section

Let σ be the unique open face of \mathfrak{t}_+ which contains β_0 . The stabilizer subgroup $K_\xi \subset K$ that does not depend on the choice of $\xi \in \sigma$ is denoted K_σ , and we denote by \mathfrak{k}_σ its Lie algebra (sometimes we use the different notations K_{β_0} , \mathfrak{k}_{β_0}).

Let U_σ be the K_σ -invariant open subset of \mathfrak{k}_σ defined by

$$U_\sigma := K_\sigma.\{y \in \mathfrak{t}_+ \mid K_y \subset K_\sigma\} = K_\sigma.\bigcup_{\sigma \subset \bar{\tau}} \tau, \quad (3.12)$$

where $\{\sigma \subset \bar{\tau}\}$ is the set of all faces τ of \mathfrak{t}_+ which contain σ in their closure. By construction, U_σ is a *slice* for the adjoint action at any $\xi \in \sigma$ (see Definition 3.1 of [13]). This means that the map $K \times U_\sigma \rightarrow \mathfrak{k}$, $(g, \xi) \rightarrow g.\xi$, factors through an inclusion $K \times_{K_\sigma} U_\sigma \hookrightarrow \mathfrak{k}$.

The *symplectic cross-section theorem* [10] asserts that the pre-image $\mathcal{Y}_\sigma = \mu_K^{-1}(U_\sigma)$ is a symplectic submanifold provided with an Hamiltonian action of the group K_σ . The restriction $\mu_K|_{\mathcal{Y}_\sigma}$ is a moment map for the action of K_σ on \mathcal{Y}_σ that we denote by μ_σ . In our case we just need the fact that $\mathfrak{k}_\sigma \oplus \mathfrak{p}$ intersects transversally M in a neighbourhood of $K_\sigma.\beta \subset \mathcal{Y}_\sigma \subset M \cap (\mathfrak{k}_\sigma \oplus \mathfrak{p})$.

Moreover, the set $K.\mathcal{Y}_\sigma$ is a K -invariant *open* neighbourhood of $K.\beta$ in M diffeomorphic to $K \times_{K_\sigma} \mathcal{Y}_\sigma$. Then, we can compute the equivariant form P^K on the manifold $\mathcal{M}_\sigma := K \times_{K_\sigma} \mathcal{Y}_\sigma$. We will denote by $\Omega_\sigma(Y) := \Omega|_{\mathcal{Y}_\sigma} + (\mu_\sigma, Y)$, $Y \in \mathfrak{k}_\sigma$, the corresponding K_σ -equivariant symplectic form on \mathcal{Y}_σ .

Induced metric on \mathcal{M}_σ

The quotient $\mathfrak{k}/\mathfrak{k}_\sigma$ is identified with the orthogonal complement $\mathfrak{k}_\sigma^\perp$ of \mathfrak{k}_σ in \mathfrak{k} . We denote respectively by $pr_{\mathfrak{k}/\mathfrak{k}_\sigma}$ and $pr_{\mathfrak{k}_\sigma}$ the orthogonal projections $\mathfrak{k} \rightarrow \mathfrak{k}_\sigma^\perp$ and $\mathfrak{k} \rightarrow \mathfrak{k}_\sigma$. From now on we fix a K_σ -invariant metric $(\cdot, \cdot)_{\mathcal{Y}_\sigma}$ on \mathcal{Y}_σ and we

associate to this metric a natural K -invariant Riemannian metric $(\cdot, \cdot)_{\mathcal{M}_\sigma}$ on $\mathcal{M}_\sigma := K \times_{K_\sigma} \mathcal{Y}_\sigma$ which is defined by

$$\|X_{\mathcal{M}_\sigma} + v_m\|_{\mathcal{M}_\sigma}^2 [k, m] := \|pr_{\mathfrak{k}/\mathfrak{k}_\sigma}(k^{-1}X)\|_{\mathfrak{k}}^2 + \|pr_{\mathfrak{k}_\sigma}(k^{-1}X)|_{\mathcal{Y}_\sigma}|_m + v_m\|_{\mathcal{Y}_\sigma}^2, \quad (3.13)$$

for $X \in \mathfrak{k}$ and $v_m \in \mathbf{T}_m M$. Here we use the identification between $\mathbf{T}_{[k, m]} \mathcal{M}_\sigma$ and $(\mathbf{T}_g K \times \mathbf{T}_m \mathcal{Y}_\sigma) / \sim$, (\sim is the relation of equivalence coming from the K_σ -orbits in $K \times \mathcal{Y}_\sigma$).

Let $\lambda_\sigma := (\mathcal{H}_\sigma, \cdot)_{\mathcal{Y}_\sigma}$ be the K_σ -invariant 1-form on \mathcal{Y}_σ , where \mathcal{H}_σ is the Hamiltonian vector field of $\frac{1}{2}\|\mu_\sigma\|^2$. A straightforward computation shows that $\text{Cr}(\|\mu_\sigma\|^2) = \text{Cr}(\|\mu_K\|^2) \cap \mathcal{Y}_\sigma = K_\sigma \cdot \beta$. Let $\tilde{\lambda}_K := (\mathcal{H}, \cdot)_{\mathcal{M}_\sigma}$ be the K -invariant 1-form on \mathcal{M}_σ defined with the induced metric $(\cdot, \cdot)_{\mathcal{M}_\sigma}$, where \mathcal{H} is still the Hamiltonian vector field of $\frac{1}{2}\|\mu_K\|^2$.

Using the definition of the induced metric $(\cdot, \cdot)_{\mathcal{M}_\sigma}$ we see that $i_\sigma^*(\tilde{\lambda}_K) = \lambda_\sigma$, where $i_\sigma : \mathcal{Y}_\sigma \hookrightarrow \mathcal{M}_\sigma$ denotes the K_σ -equivariant inclusion, and we also remark that

$$\left(\Phi_{\tilde{\lambda}_K}([k, m]), X \right)_\mathfrak{k} = \left(\Phi_{\lambda_{K_\sigma}}(m), pr_{\mathfrak{k}_\sigma}(k^{-1}X) \right)_\mathfrak{k}, \quad X \in \mathfrak{k}, [k, m] \in \mathcal{M}_\sigma, \quad (3.14)$$

where $pr_{\mathfrak{k}_\sigma} : \mathfrak{k} \rightarrow \mathfrak{k}_\sigma$ is the orthogonal projection. We will use these facts at Proposition 3.4.

Definition 3.1 *Let $\chi \in \mathcal{C}_{cpt}^\infty(\mathcal{M}_\sigma)^K$ be the function coming from a K_σ -invariant function χ_σ on \mathcal{Y}_σ , where χ_σ is equal to 1 in a neighbourhood of $\text{Cr}(\|\mu_\sigma\|^2) = K_\sigma \cdot \beta$. Then the function χ , that is equal to 1 in a neighbourhood of $\text{Cr}(\|\mu_K\|^2) = K \times_{K_\sigma} (K_\sigma \cdot \beta)$, defines the K -equivariant form*

$$\tilde{\mathbf{P}}^K = \chi + d\chi \left(\int_0^\infty ie^{-it} \mathcal{D}\tilde{\lambda}_K dt \right) \tilde{\lambda}_K \in \mathcal{A}_{K, cpt}^{-\infty}(\mathcal{M}_\sigma).$$

With the function χ_σ we define in the same way the K_σ -equivariant form

$$\mathbf{P}^{K_\sigma} = \chi_\sigma + d\chi_\sigma \left(\int_0^\infty ie^{-it} \mathcal{D}\lambda_\sigma dt \right) \lambda_\sigma \in \mathcal{A}_{K_\sigma, cpt}^{-\infty}(\mathcal{Y}_\sigma).$$

The inclusion $\mathcal{M}_\sigma \hookrightarrow M$ of an open subset defines a natural morphism $\mathcal{A}_{cpt}^*(\mathcal{M}_\sigma) \rightarrow \mathcal{A}_{cpt}^*(M)$ between the differential forms with compact support, and hence a morphism $j : \mathcal{A}_{K, cpt}^{-\infty}(\mathcal{M}_\sigma) \rightarrow \mathcal{A}_{K, cpt}^{-\infty}(M)$ between the K -equivariant forms with compact support. We know from Proposition 2.4, that

$$\mathbf{P}^K = j \left(\tilde{\mathbf{P}}^K \right) \quad \text{in } \mathcal{H}_{K, cpt}^{-\infty}(M).$$

Let $\pi : \mathcal{M}_\sigma \rightarrow K/K_\sigma$, $[k, y] \mapsto [k]$ be the projection map, and denote by $\int_{\text{Fiber}} := \pi_* : \mathcal{A}_{K, cpt}^{-\infty}(\mathcal{M}_\sigma) \rightarrow \mathcal{A}_K^{-\infty}(K/K_\sigma)$ the morphism of integration along the fiber of π . Then we have

$$\begin{aligned} F_M|_\mathfrak{k} &= \int_{\mathcal{M}_\sigma} e^{i\Omega_\mathfrak{k}} \tilde{\mathbf{P}}^K \\ &= \int_{K/K_\sigma} \left(\int_{\text{Fiber}} e^{i\Omega_\mathfrak{k}} \tilde{\mathbf{P}}^K \right). \end{aligned} \quad (3.15)$$

In the next paragraph we are going to show that the ‘computation’ of the closed equivariant form $\int_{\text{Fiber}} e^{i\Omega_{\mathfrak{k}}} \tilde{\mathbf{P}}^K \in \mathcal{A}_K^{-\infty}(K/K_\sigma)$ can be deduced from those of the generalized function $\int_{\mathcal{Y}_\sigma} e^{i\Omega_{\mathfrak{k}_\sigma}} \mathbf{P}^{K_\sigma} \in \mathcal{C}^{-\infty}(\mathfrak{k}_\sigma)^{K_\sigma}$.

Restriction of generalized equivariant forms

The manifold \mathcal{Y}_σ is oriented by its symplectic form, and $K \times_{K_\sigma} \mathcal{Y}_\sigma$ is oriented by the symplectic form on M . Hence, we get an orientation o for K/K_σ and we verify that this orientation coincides with those of Definition 1.3. This gives a polynomial square root $Y \rightarrow \Pi_{\mathfrak{k}/\mathfrak{k}_\sigma}(Y) := \det_{\mathfrak{k}/\mathfrak{k}_\sigma, o}^{1/2}(Y)$, $Y \in \mathfrak{k}_\sigma$ (we will use also the notation $\Pi_{\mathfrak{k}/\mathfrak{k}_{\beta_0}}$ for this polynomial). Note that $\Pi_{\mathfrak{k}/\mathfrak{k}_\sigma}$ never vanishes on the open subset U_σ .

We have a natural identification between $\mathcal{A}^*(K/K_\sigma)$ and $\mathcal{C}^\infty\left(K, (\wedge \mathfrak{k}^*)_{\text{hor}K_\sigma}\right)^{K_\sigma}$, where K_σ -invariants are taken with respect to the action of K_σ by right multiplication on K and coadjoint action on \mathfrak{k}^* . The restriction map to the neutral element $e \in K$, $\mathcal{A}^*(K/K_\sigma) \rightarrow (\wedge \mathfrak{k}^*)_{\text{hor}K_\sigma}$, $\alpha \rightarrow \alpha_e$, defines a morphism

$$\begin{aligned} \mathcal{A}_K^{-\infty}(K/K_\sigma) &\longrightarrow \mathcal{C}^{-\infty}\left(\mathfrak{k}, (\wedge \mathfrak{k}^*)_{\text{hor}K_\sigma}\right)^{K_\sigma} \\ \alpha(X) &\longmapsto \alpha_e(X). \end{aligned}$$

Let E_1, \dots, E_p be a basis of \mathfrak{k}^* , and let $\{E_I = E_{i_1} \wedge \dots \wedge E_{i_k}, I = [i_1 < i_2 < \dots < i_k] \subset [1, 2, \dots, p]\}$ be the corresponding basis of $\wedge \mathfrak{k}^*$. In particular, $E_\emptyset = 1$ generates $\mathbb{R} \subset \wedge \mathfrak{k}^*$. For each $\alpha \in \mathcal{A}_K^{-\infty}(K/K_\sigma)$, the form α_e can be decomposed relatively to the basis $\{E_I, I\}$: $\alpha_e = \sum_I (\alpha_e)_{[I]} E_I$ with $(\alpha_e)_{[I]} \in \mathcal{C}^{-\infty}(\mathfrak{k})$.

We say that α admits a restriction to \mathfrak{k}_σ if the wave front set of each component $(\alpha_e)_{[I]}$ is transverse to \mathfrak{k}_σ (see [8] for this notion). In this case, each generalized function $(\alpha_e)_{[I]}$ admits a restriction to \mathfrak{k}_σ . We can then define $r_{\mathfrak{k}_\sigma} \alpha := (\alpha_e)_{[\emptyset]}|_{\mathfrak{k}_\sigma}$ that is a generalized K_σ -invariant function on \mathfrak{k}_σ . This definition extends the usual restriction map $r_{\mathfrak{k}_\sigma} : \mathcal{A}_K^\infty(K/K_\sigma) \rightarrow \mathcal{C}^\infty(\mathfrak{k}_\sigma)^{K_\sigma}$.

Remark 3.2 *There is a basic way to know if $\alpha \in \mathcal{A}_K^{-\infty}(K/K_\sigma)$ admits a restriction to \mathfrak{k}_σ . Suppose there exist $\alpha^a \in \mathcal{A}_K^\infty(K/K_\sigma)$, $a > 0$, such that*

- $\lim_{a \rightarrow \infty} \alpha^a = \alpha$, and
- the restriction $\alpha_e^a|_{\mathfrak{k}_\sigma} \in \mathcal{C}^\infty(\mathfrak{k}_\sigma, \wedge \mathfrak{k}^*)$ converges in $\mathcal{C}^{-\infty}(\mathfrak{k}_\sigma, \wedge \mathfrak{k}^*)$ when $a \rightarrow \infty$. In particular $r_{\mathfrak{k}_\sigma} \alpha^a \in \mathcal{C}^\infty(\mathfrak{k}_\sigma)^{K_\sigma}$ converges in $\mathcal{C}^{-\infty}(\mathfrak{k}_\sigma)^{K_\sigma}$ when $a \rightarrow \infty$.

Then the equivariant form α admits a restriction to \mathfrak{k}_σ and we have $r_{\mathfrak{k}_\sigma} \alpha = \lim_{a \rightarrow \infty} r_{\mathfrak{k}_\sigma} \alpha^a$.

Recall Proposition 31 of [8].

Proposition 3.3 *Let $\alpha \in \mathcal{A}_K^{-\infty}(K/K_\sigma)$ be a closed equivariant form that admits a restriction to \mathfrak{k}_σ . Then for every $f \in \mathcal{C}_{\text{cpt}}^\infty(\mathfrak{k})$, we have*

$$\left\langle \left(\int_{K/K_\sigma} \alpha \right)(X), f(X) dX \right\rangle_{\mathfrak{k}} = \text{cte} \left\langle r_{\mathfrak{k}_\sigma} \alpha(Y), \Pi_{\mathfrak{k}/\mathfrak{k}_\sigma}(Y) \overline{f}^K|_{\mathfrak{k}_\sigma}(Y) dY \right\rangle_{\mathfrak{k}_\sigma},$$

with $\text{cte} = (-2\pi)^{\dim(K/K_\sigma)/2} \frac{\text{vol}(K, dX)}{\text{vol}(K_\sigma, dY)}$.

We can now state the following

Lemma 3.4 *The closed K -equivariant form $\int_{\text{Fiber}} e^{i\Omega_t} \tilde{\mathbf{P}}^K \in \mathcal{A}_K^{-\infty}(K/K_\sigma)$ admits a restriction to \mathfrak{k}_σ equal to $\int_{\mathcal{Y}_\sigma} e^{i\Omega_{\mathfrak{k}_\sigma}} \mathbf{P}^{K_\sigma} \in \mathcal{C}^{-\infty}(\mathfrak{k}_\sigma)^{K_\sigma}$.*

Proof: Here we are in the situation of Remark 3.2. We know already that $\tilde{\mathbf{P}}^K$ is the limit of the equivariant form $\tilde{\mathbf{P}}_a^K := \chi + d\chi \left(\int_0^a i e^{-itD\tilde{\lambda}_K} dt \right) \tilde{\lambda}_K \in \mathcal{A}_{K,\text{cpt}}^\infty(\mathcal{M}_\sigma)$, and so $\alpha := \int_{\text{Fiber}} e^{i\Omega_t} \tilde{\mathbf{P}}^K$ is the limit of the equivariant forms $\alpha^a := \int_{\text{Fiber}} e^{i\Omega_t} \tilde{\mathbf{P}}_a^K \in \mathcal{A}_K^\infty(K/K_\sigma)$. Consider the maps $\alpha_e^a|_{\mathfrak{k}_\sigma} \in \mathcal{C}^\infty(\mathfrak{k}_\sigma, \wedge \mathfrak{k}^*)^{K_\sigma}$. We have

$$\begin{aligned} \alpha_e^a|_{\mathfrak{k}_\sigma}(Y) &= \int_{\mathcal{Y}_\sigma} e^{i\Omega_e + i(\mu_\sigma, Y)} \tilde{\mathbf{P}}_{a,e}^K, \quad \text{with} \\ \tilde{\mathbf{P}}_{a,e}^K &= \chi_\sigma + d\chi_\sigma \left(\int_0^a i e^{-itd\tilde{\lambda}_K|_e} e^{it(\Phi_{\lambda_\sigma}, Y)} dt \right) \tilde{\lambda}_K|_e, \end{aligned}$$

where Ω_e , $\tilde{\lambda}_K|_e$ and $d\tilde{\lambda}_K|_e$ are in $\wedge \mathfrak{k}^* \otimes \mathcal{A}^*(\mathcal{Y}_\sigma)$. In these equalities we have used that $(\Phi_{\tilde{\lambda}_K}|_e, Y)_\mathfrak{k} = (\Phi_{\lambda_\sigma}, Y)_\mathfrak{k}$ for every $Y \in \mathfrak{k}_\sigma$ (see equation (3.14)). Since $d\chi_\sigma$ is equal to 0 in a neighbourhood of $\{\Phi_{\lambda_\sigma} = 0\}$, we see that $\alpha_e^a|_{\mathfrak{k}_\sigma}$ converge in $\mathcal{C}^{-\infty}(\mathfrak{k}_\sigma, \wedge \mathfrak{k}^*)^{K_\sigma}$, when $a \rightarrow \infty$.

If $i_\sigma : \mathcal{Y}_\sigma \hookrightarrow \mathcal{M}_\sigma$ denotes the K_σ -equivariant inclusion, we have $i_\sigma^*(\tilde{\lambda}_K) = \lambda_\sigma$, and $i_\sigma^*(\Omega) = \Omega_\sigma$ (see equation (3.14)). Then we see that the functions $r_{\mathfrak{k}_\sigma} \alpha^a = \int_{\mathcal{Y}_\sigma} e^{i\Omega_{\mathfrak{k}_\sigma}} \mathbf{P}_a^{K_\sigma}$ with $\mathbf{P}_a^{K_\sigma} = \chi_\sigma + d\chi_\sigma \left(\int_0^a i e^{-itD\lambda_\sigma} dt \right) \lambda_\sigma$ converge to $\int_{\mathcal{Y}_\sigma} e^{i\Omega_{\mathfrak{k}_\sigma}} \mathbf{P}^{K_\sigma}$ when $a \rightarrow \infty$. The proof is completed. \square

Proposition 3.5 *We have the following description of the generalized function $F_M|_{\mathfrak{k}}$. For every function $f \in \mathcal{C}_{\text{rd}}^\infty(\mathfrak{k})$, we have*

$$\langle F_M|_{\mathfrak{k}}(X), f(X)dX \rangle_{\mathfrak{k}} = \text{cte} \langle \left(\int_{\mathcal{Y}_\sigma} \mathbf{P}^{K_\sigma} \right)(Y), e^{i(\beta_0, Y)} \Pi_{\mathfrak{k}/\mathfrak{k}_\sigma}(Y) \bar{f}^K|_{\mathfrak{k}_\sigma}(Y) dY \rangle_{\mathfrak{k}_\sigma},$$

$$\text{with } \text{cte} = (-2\pi)^{\dim(K/K_\sigma)/2} \frac{\text{vol}(K, dX)}{\text{vol}(K_\sigma, dY)}.$$

Proof: Let \mathcal{U} be a K_σ -invariant tubular neighbourhood of $K_\sigma \cdot \beta$ in \mathcal{Y}_σ , and denote by $p : \mathcal{U} \rightarrow K_\sigma \cdot \beta$ the corresponding fibration. We denote by $i : K_\sigma \cdot \beta \hookrightarrow \mathcal{U}$ the 0-section of this fibration. We need to compute the restriction $i^*(\Omega_{\mathfrak{k}_\sigma})$ of the equivariant form $\Omega_{\mathfrak{k}_\sigma}$ on $K_\sigma \cdot \beta$. First we note that the inclusion $K_\sigma \cdot \beta \hookrightarrow \mathcal{Y}_\sigma$ is an *isotropic* embedding, because $\beta_0 = \mu_\sigma(K_\sigma \cdot \beta)$ is fixed by K_σ : for every $X, X' \in \mathfrak{k}_\sigma$, and $m \in K_\sigma \cdot \beta$ we have $\Omega_\sigma|_m(X_{\mathcal{Y}_\sigma}, X'_{\mathcal{Y}_\sigma}) = -(\beta_0, [X, X']) = 0$. Then $i^*(\Omega_\sigma) = 0$ and we have also $i^*(\mu_\sigma) = \beta_0$. Finally we get $i^*(\Omega_{\mathfrak{k}_\sigma})(Y) = (\beta_0, Y)$, $Y \in \mathfrak{k}_\sigma$.

The equivariant form \mathbf{P}^{K_σ} can be taken with (compact) support in \mathcal{U} , and we have

$$\begin{aligned} \left(\int_{\mathcal{Y}_\sigma} e^{i\Omega_{\mathfrak{k}_\sigma}} \mathbf{P}^{K_\sigma} \right)(Y) &= \int_{K_\sigma \cdot \beta} i^*(e^{i\Omega_{\mathfrak{k}_\sigma}})(Y) p_* \left(\mathbf{P}^{K_\sigma} \right)(Y) \\ &= e^{i(\beta_0, Y)} \left(\int_{\mathcal{Y}_\sigma} \mathbf{P}^{K_\sigma} \right)(Y). \end{aligned}$$

With this equality, the Proposition is just a consequence of Proposition 3.3, Lemma 3.4, and the equality (3.15). \square

4 Second reduction: Deformation procedure

We are now going to compute the generalized function $\int_{\mathcal{Y}_\sigma} P^{K_\sigma}$ by means of a deformation procedure (see Proposition 3.11 of [16] and section 2.3 of [17]).

We first recall briefly the context of our computation. Let M be a semi-simple orbit of the adjoint representation of a real semi-simple Lie group G . We have shown that $M = G.\beta$ with $\beta = \beta_0 + \beta_1$, $\beta_0 \in \mathfrak{k}$, $\beta_1 \in \mathfrak{p}$ and $[\beta_0, \beta_1] = 0$. In the last section we have denote by \mathfrak{k}_σ the subalgebra of \mathfrak{k} fixed by the adjoint action of β_0 . We have also defined a symplectic K_σ -manifold \mathcal{Y}_σ which is an open neighbourhood of $K_\sigma.\beta$ in $M \cap (\mathfrak{k}_\sigma \oplus \mathfrak{p})$. The action of K_σ is Hamiltonian and the moment map $\mu_\sigma : \mathcal{Y}_\sigma \rightarrow \mathfrak{k}_\sigma$ is the restriction to \mathcal{Y}_σ of the orthogonal projection $\mathfrak{g} \rightarrow \mathfrak{k}_\sigma$.

The equivariant form P^{K_σ} is supported in a (small) neighbourhood of $K_\sigma.\beta$ in \mathcal{Y}_σ , and to compute it we just need to describe a K_σ -invariant neighbourhood of $K_\sigma.\beta$ in \mathcal{Y}_σ .

We have already remarked that the inclusion $K_\sigma.\beta \hookrightarrow \mathcal{Y}_\sigma$ is an *isotropic* embedding. In fact $K_\sigma.\beta$ is the 0-level of the shifted (K_σ -invariant) moment map $\mu_\sigma - \beta_0$. Then to describe a K_σ -invariant neighbourhood of $K_\sigma.\beta$ in \mathcal{Y}_σ we can use the normal-form recipe of Marle, Guillemin and Sternberg.

First we can form, following Weinstein (see [11, 23]), the symplectic normal bundle

$$\mathcal{V}_{\beta_0} := \mathbf{T}(K_\sigma.\beta)^{\perp, \Omega_\sigma} / \mathbf{T}(K_\sigma.\beta), \quad (4.16)$$

where the orthogonal (${}^{\perp, \Omega_\sigma}$) is taken relatively to the symplectic 2-form Ω_σ . Let K_β be the subgroup of K_σ which stabilizes β , and let \mathfrak{k}_β be its Lie algebra (K_β is the subgroup of point $k \in K$ such that $k.\beta_0 = \beta_0$ and $k.\beta_1 = \beta_1$). We have

$$\mathcal{V}_{\beta_0} = K_\sigma \times_{K_\beta} V_{\beta_0}$$

where the vector space $V_{\beta_0} := \mathbf{T}_\beta(K_\sigma.\beta)^{\perp, \Omega_\sigma} / \mathbf{T}_\beta(K_\sigma.\beta)$ inherits a symplectic structure and an Hamiltonian action of the group K_β .

Lemma 4.1 *The vector space V_{β_0} is equal to $[\mathfrak{p}, \beta_0] \cong \mathfrak{p}/\mathfrak{p}_{\beta_0}$ and is equipped with the natural K_β -action. The moment map $\mu_\beta : V_{\beta_0} \rightarrow \mathfrak{k}_\beta$ associated to this action verifies*

$$\mu_\beta(y) = -\frac{1}{2} pr_{\mathfrak{k}_\beta} \left([y, [y, \beta_0]] \right), \quad y \in [\mathfrak{p}, \beta_0],$$

where $pr_{\mathfrak{k}_\beta} : \mathfrak{k}_\sigma \rightarrow \mathfrak{k}_\beta$ denotes the orthogonal projection. Moreover, $\mu_\beta(y) = 0$ iff $y = 0$.

Proof: The vector space $\mathbf{T}_\beta M = [\mathfrak{g}, \beta]$ carries the symplectic form Ω_β that is defined by the equation

$$\Omega_\beta([X, \beta], [Y, \beta]) := -B(\beta, [X, Y]), \quad \text{for } X, Y \in \mathfrak{g}, \quad (4.17)$$

where B is the Killing form on \mathfrak{g} . We see from (4.17) that the symplectic orthogonal $[\mathfrak{k}_\sigma, \beta_1]^{\perp, \Omega_\sigma}$ of the vector space $\mathbf{T}_\beta(K_\sigma \cdot \beta) = [\mathfrak{k}_\sigma, \beta_1]$ in $\mathbf{T}_\beta M$ is equal to the subspace B -orthogonal to \mathfrak{k}_σ . Then we have $V_{\beta_0} = (\mathfrak{k}_\sigma)^{\perp, B} \cap \mathbf{T}_\beta \mathcal{Y}_\sigma / [\mathfrak{k}_\sigma, \beta_1]$.

The subspace $[\mathfrak{g}, \beta]$ of \mathfrak{g} is θ -stable because $\ker(ad(\beta)) = \ker(ad(\theta\beta)) = \ker(ad(\beta_0)) \cap \ker(ad(\beta_1))$ is θ -stable and $[\mathfrak{g}, \beta] = \ker(ad(\beta))^{\perp, B}$. Hence we get

$$\begin{aligned} \mathbf{T}_\beta \mathcal{Y}_\sigma &= [\mathfrak{g}, \beta] \cap (\mathfrak{k}_\sigma \oplus \mathfrak{p}) \\ &= \underbrace{[\mathfrak{g}, \beta] \cap \mathfrak{k}_\sigma}_{\subset \mathfrak{k}} \oplus \underbrace{[\mathfrak{g}, \beta] \cap \mathfrak{p}}_{\subset \mathfrak{p}}. \end{aligned} \quad (4.18)$$

The \mathfrak{k} -part of $(\mathfrak{k}_\sigma)^{\perp, B} \cap \mathbf{T}_\beta \mathcal{Y}_\sigma$ is reduced to $\{0\}$ because it is included in $(\mathfrak{k}_\sigma)^{\perp, B} \cap \mathfrak{k}_\sigma = \{0\}$ (see (4.18)). Hence we have $(\mathfrak{k}_\sigma)^{\perp, B} \cap \mathbf{T}_\beta \mathcal{Y}_\sigma = [\mathfrak{g}, \beta] \cap \mathfrak{p} = [\mathfrak{k}, \beta_1] + [\mathfrak{p}, \beta_0]$ because \mathfrak{p} is B -orthogonal to \mathfrak{k}_σ . Using the decomposition $\mathfrak{k} = \mathfrak{k}_\sigma \oplus [\mathfrak{k}, \beta_0]$ we can finally write $(\mathfrak{k}_\sigma)^{\perp, B} \cap \mathbf{T}_\beta \mathcal{Y}_\sigma = [\mathfrak{k}_\sigma, \beta_1] \oplus [\mathfrak{p}, \beta_0]$ (the last sum is direct because the two members are B -orthogonal and B is positive definite on \mathfrak{p}). We proved finally that V_{β_0} is equal to $[\mathfrak{p}, \beta_0]$. The computation of μ_β is left to the reader. For the last point, we remark that $(pr_{\mathfrak{k}_\beta}(X), \beta_0)_\mathfrak{k} = (X, \beta_0)_\mathfrak{k}$ for every $X \in \mathfrak{k}_\sigma$, because $\beta_0 \in \mathfrak{k}_\beta$. It follows that $(\mu_\beta(y), \beta_0)_\mathfrak{k} = \frac{1}{2} \| [y, \beta_0] \|^2$ for every $y \in [\mathfrak{p}, \beta_0]$, and this proves the last assertion. \square

Consider now the following symplectic manifold

$$\tilde{\mathcal{Y}}_\sigma := \mathcal{V}_{\beta_0} \times \mathbf{T}(K_\sigma/K_\beta) = K_\sigma \times_{K_\beta} \left(\mathfrak{k}_\sigma/\mathfrak{k}_\beta \oplus \mathfrak{p}/\mathfrak{p}_{\beta_0} \right) \quad (4.19)$$

where the tangent bundle $\mathbf{T}(K_\sigma/K_\beta)$ is here naturally identified with the cotangent bundle $\mathbf{T}^*(K_\sigma/K_\beta)$ through the K_σ -invariant scalar product on $\mathfrak{k}_\sigma/\mathfrak{k}_\beta$ coming from the scalar product on \mathfrak{k} (after identification of $\mathfrak{k}_\sigma/\mathfrak{k}_\beta$ with the orthogonal complement of \mathfrak{k}_β in \mathfrak{k}_σ). The action of K_σ on $\tilde{\mathcal{Y}}_\sigma$ is Hamiltonian and the moment map $\tilde{\mu}_\sigma : \tilde{\mathcal{Y}}_\sigma \rightarrow \mathfrak{k}_\sigma$ is given by the equation

$$\tilde{\mu}_\sigma([k; x, y]) = \beta_0 + k \cdot (x + \mu_\beta(y)) \quad k \in K_\sigma, \quad x \in \mathfrak{k}_\sigma/\mathfrak{k}_\beta, \quad y \in \mathfrak{p}/\mathfrak{p}_{\beta_0}. \quad (4.20)$$

The *local normal form* Theorem, applied to the moment map $\mu_\sigma - \beta_0$, (see [19] Proposition 2.5) tells us that there exists a K_σ -Hamiltonian isomorphism

$$\Upsilon : \mathcal{U}_1 \xrightarrow{\sim} \mathcal{U}_2,$$

where \mathcal{U}_1 is a K_σ -invariant neighbourhood of $K_\sigma \cdot \beta$ in \mathcal{Y}_σ , and \mathcal{U}_2 is a K_σ -invariant neighbourhood of K_σ/K_β in $\tilde{\mathcal{Y}}_\sigma$. Furthermore the isomorphism, when restricted to $K_\sigma \cdot \beta$, corresponds to the natural isomorphism $K_\sigma \cdot \beta \xrightarrow{\sim} K_\sigma/K_\beta$.

Let $\tilde{\mathcal{H}}_\sigma$ be the Hamiltonian vector field on $\tilde{\mathcal{Y}}_\sigma$ of $\frac{1}{2} \|\tilde{\mu}_\sigma\|^2$. For every $[k; x, y] \in \tilde{\mathcal{Y}}_\sigma$ we have

$$\begin{aligned} \tilde{\mathcal{H}}_\sigma|_{[k; x, y]} &= \left(k \cdot \tilde{\mu}_\sigma([1; x, y]) \right)_{\tilde{\mathcal{Y}}_\sigma}|_{[k; x, y]} \\ &= \left. \frac{d}{dt} \right|_{t=0} [k \cdot e^{-tX}; x, y], \quad \text{with } X = \beta_0 + x + \mu_\beta(y) \\ &= -x_{K_\sigma, r}(k) - [\beta_0, y] - [\mu_\beta(y), x + y]. \end{aligned} \quad (4.21)$$

In the last equality, $x_{K_\sigma, r}$ is the vector field generated by the right action of $\{e^{tx}, t \in \mathbb{R}\}$ on K_σ .

Remark now that the action of K_β on V_{β_0} comes from a K_σ -action. Hence the vector bundle $\tilde{\mathcal{Y}}_\sigma \rightarrow \mathbf{T}(K_\sigma/K_\beta)$ is trivial through the isomorphism

$$\begin{aligned} \Xi & : \tilde{\mathcal{Y}}_\sigma \xrightarrow{\sim} \mathbf{T}(K_\sigma/K_\beta) \times V_{\beta_0} \\ [k; x, y] & \longmapsto ([k, x], k.y). \end{aligned} \quad (4.22)$$

We have a natural K_β -invariant Euclidean structure on $\mathfrak{k}_\sigma/\mathfrak{k}_\beta \oplus \mathfrak{p}/\mathfrak{p}_{\beta_0}$ coming from the Euclidean structure of \mathfrak{g} and we denote by $(\cdot, \cdot)_{\mathbf{T}(K_\sigma/K_\beta)}$ the induced K_σ -invariant Riemannian structure on $\mathbf{T}(K_\sigma/K_\beta)$ (see equation (3.13)).

Definition 4.2 Let θ_{β_0} be the K_σ -invariant 1-form on V_{β_0} given by $\theta_{\beta_0}|_y = -([\beta_0, y], dy)_{\mathfrak{p}}$, $y \in V_{\beta_0}$, and let γ_σ be the K_σ -invariant 1-form on $\mathbf{T}(K_\sigma/K_\beta)$ defined by the equation: $\gamma_\sigma|_{[k; x, y]} = -\left(x_{K_\sigma, r}(k), \cdot\right)_{\mathbf{T}(K_\sigma/K_\beta)}$, $[k; x] \in \mathbf{T}(K_\sigma/K_\beta)$.

We will denote by $\tilde{\lambda}_\sigma$ the K_σ -invariant 1-form on $\tilde{\mathcal{Y}}_\sigma$ defined below

$$\tilde{\lambda}_\sigma = \Xi^*(\gamma_\sigma + \theta_{\beta_0}).$$

We are going to prove that $\tilde{\lambda}_\sigma$ and λ_σ define (in cohomology) the same generalized equivariant form in the neighbourhood of K_σ/K_β (through the isomorphism Υ). First we compute the function $(\Phi_{\tilde{\lambda}_\sigma}, \tilde{\mu}_\sigma)$ near K_σ/K_β . We have

$$\begin{aligned} (\Phi_{\tilde{\lambda}_\sigma}, \tilde{\mu}_\sigma)([k; x, y]) &= \gamma_\sigma\left(X_{\mathbf{T}(K_\sigma/K_\beta)}\right)([k, x]) + \theta_{\beta_0}\left(X_{V_{\beta_0}}\right)(k.y) \quad \text{with } X = \tilde{\mu}_\sigma([k; x, y]) \\ &= \|x\|_{\mathfrak{k}}^2 + \left([\beta_0, k.y], [\tilde{\mu}_\sigma([k; x, y]), k.y]\right)_{\mathfrak{p}} \\ &= \|x\|_{\mathfrak{k}}^2 + \|[\beta_0, y]\|_{\mathfrak{p}}^2 + O(\|x, y\|^3). \end{aligned} \quad (4.23)$$

In the last equality we just use the fact that $\tilde{\mu}_\sigma([k; x, y]) = \beta_0 + O(\|x, y\|)$, then $\left([\beta_0, k.y], [\beta_0 + O(\|x, y\|), k.y]\right)_{\mathfrak{p}} = \|[\beta_0, k.y]\|_{\mathfrak{p}}^2 + O(\|x, y\|^3)$ and the K_σ -invariance of β_0 imposes $\|[\beta_0, k.y]\|_{\mathfrak{p}}^2 = \|[\beta_0, y]\|_{\mathfrak{p}}^2$.

From the equality (4.23), we know that there exists a K_σ -invariant neighbourhood $\tilde{\mathcal{V}}$ of the 0-section in $\tilde{\mathcal{Y}}_\sigma$ such that the function $(\Phi_{\tilde{\lambda}_\sigma}, \tilde{\mu}_\sigma)$ is *strictly positive* on $\tilde{\mathcal{V}} \setminus \{K_\sigma/K_\beta\}$ (Note that the vector $[\beta_0, y]$ is non-zero for each non-zero vector $y \in \mathfrak{p}/\mathfrak{p}_{\beta_0}$). In particular we get $\{\Phi_{\tilde{\lambda}_\sigma} = 0\} \cap \tilde{\mathcal{V}} = K_\sigma/K_\beta$.

Let $\tilde{\chi}_\sigma \in \mathcal{C}_{cpt}^\infty(\tilde{\mathcal{Y}}_\sigma)$ be a function supported on $\tilde{\mathcal{V}} \cap \mathcal{U}_2$, and equal to 1 in a neighbourhood of K_σ/K_β . With this function we define the equivariant form

$$\tilde{\mathbb{P}}^{K_\sigma} = \tilde{\chi}_\sigma + d\tilde{\chi}_\sigma \left(\int_0^\infty \iota e^{-t\mathcal{D}\tilde{\lambda}_\sigma} dt \right) \tilde{\lambda}_\sigma \in \mathcal{A}_{K_\sigma, cpt}^{-\infty}(\tilde{\mathcal{Y}}_\sigma).$$

The main point of our deformation procedure is the following

Lemma 4.3 *The generalized K_σ -equivariant forms P^{K_σ} , and $\Upsilon^* \left(\tilde{P}^{K_\sigma} \right)$, define the same cohomology class in $\mathcal{H}_{K_\sigma, \text{cpt}}^{-\infty}(\mathcal{Y}_\sigma)$. In particular we have the equality*

$$\int_{\mathcal{Y}_\sigma} P^{K_\sigma} = \int_{\tilde{\mathcal{Y}}_\sigma} \tilde{P}^{K_\sigma}$$

of K_σ -invariant tempered generalized functions on \mathfrak{k}_σ .

Proof: This is a consequence of Proposition 3.11 of [16], with the function $f := \mu_\sigma$ on the manifold $\mathcal{V} := \Upsilon^{-1}(\tilde{\mathcal{V}} \cap \mathcal{U}_2)$, with the 1-forms λ_σ and $\Upsilon^*(\tilde{\lambda}_\sigma)$. All the point is that the functions $(\Phi_{\Upsilon^*(\tilde{\lambda}_\sigma)}, \mu_\sigma) = (\Phi_{\tilde{\lambda}_\sigma}, \tilde{\mu}_\sigma) \circ \Upsilon$ and $(\Phi_{\lambda_\sigma}, \mu_\sigma) = \|\mathcal{H}_\sigma\|^2$ are strictly positive on $\mathcal{V} \setminus \{K_\sigma \cdot \beta\}$. This fact implies that the 1-forms λ_σ and $\Upsilon^*(\tilde{\lambda}_\sigma)$ define the same equivariant forms *in cohomology*. \square

Consider now the K_σ -equivariant form $e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$ on $\tilde{\mathcal{Y}}_\sigma$. First of all we note that the 1-form $\tilde{\lambda}_\sigma$ is linear in the variable $x \in \mathfrak{k}_\sigma/\mathfrak{k}_\beta$, and quadratic in the variable $y \in \mathfrak{p}/\mathfrak{p}_{\beta_0}$. Consider the corresponding map $\Phi_{\tilde{\lambda}_\sigma} : \tilde{\mathcal{Y}}_\sigma \rightarrow \mathfrak{k}_\sigma$. For every $Z \in \mathfrak{k}_\sigma$, $[k; , x, y] \in \tilde{\mathcal{Y}}_\sigma$ we have

$$\begin{aligned} \left(\Phi_{\tilde{\lambda}_\sigma}, Z \right)_\mathfrak{k} ([k; x, y]) &= \left(x, pr_{\mathfrak{k}_\sigma/\mathfrak{k}_\beta}(k^{-1} \cdot Z) \right)_\mathfrak{k} + \left([\beta_0, k \cdot y], [Z, k \cdot y] \right)_\mathfrak{p} \\ &= \left(x, k^{-1} \cdot Z \right)_\mathfrak{k} + \left([[\beta_0, k \cdot y], k \cdot y], Z \right)_\mathfrak{k} \quad [1] \\ &= \left(k(x + [[\beta_0, y], y]), Z \right)_\mathfrak{k}. \quad [2] \end{aligned}$$

For the point [1]: we first use the fact that $(X, pr_{\mathfrak{k}_\sigma/\mathfrak{k}_\beta}(X'))_\mathfrak{k} = (pr_{\mathfrak{k}_\sigma/\mathfrak{k}_\beta}(X), X')_\mathfrak{k}$, for $X, X' \in \mathfrak{k}_\sigma$, and we use also the identity $(a, [X, b])_\mathfrak{p} = ([a, b], X)_\mathfrak{k}$, for $a, b \in \mathfrak{p}$, and $X \in \mathfrak{k}$. The point [2] follows from the K_σ -invariance of β_0 . We have found the following expression: $\Phi_{\tilde{\lambda}_\sigma}([k; x, y]) = k(x + [[\beta_0, y], y])$, $[k; x, y] \in \tilde{\mathcal{Y}}_\sigma$.

Consider, for every $t > 0$, the following K_σ -equivariant contraction

$$\begin{aligned} \delta_t : \quad \tilde{\mathcal{Y}}_\sigma &\longrightarrow \tilde{\mathcal{Y}}_\sigma \\ [k; x, y] &\longmapsto \left[k, \frac{x}{t}, \frac{y}{\sqrt{t}} \right]. \end{aligned} \quad (4.24)$$

We have $t\delta_t^*(\tilde{\lambda}_\sigma) = \tilde{\lambda}_\sigma$ and $t\delta_t^*(\Phi_{\tilde{\lambda}_\sigma}) = \Phi_{\tilde{\lambda}_\sigma}$ for every $t > 0$. This gives

$$\delta_t^* \left(e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma} \right) = e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma} \quad \text{in } \mathcal{A}_{K_\sigma}^\infty(\tilde{\mathcal{Y}}_\sigma),$$

for every $t > 0$.

For every $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k}_\sigma)$, the differential form $\langle e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(Y)}, f(Y)dY \rangle_{\mathfrak{k}_\sigma}$ is the product of the differential form $e^{-\iota d\tilde{\lambda}_\sigma}$ which has a polynomial dependence in the direction of the fibers, with the function $\hat{f}(-\Phi_{\tilde{\lambda}_\sigma})$ which is rapidly decreasing along the fibers. The differential form $\langle e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(Y)}, f(Y)dY \rangle_{\mathfrak{k}_\sigma}$ is then integrable on $\tilde{\mathcal{Y}}_\sigma$. We denote by $\int_{\tilde{\mathcal{Y}}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$ the tempered K_σ -invariant function on \mathfrak{k}_σ defined by

$$\left\langle \left(\int_{\tilde{\mathcal{Y}}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma} \right) (Y), f(Y)dY \right\rangle_{\mathfrak{k}_\sigma} := \int_{\tilde{\mathcal{Y}}_\sigma} \langle e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(Y)}, f(Y)dY \rangle_{\mathfrak{k}_\sigma} \quad \text{for } f \in \mathcal{C}_{rd}^\infty(\mathfrak{k}_\sigma).$$

Proposition 4.4 *We have the following equality*

$$\int_{\tilde{\mathcal{Y}}_\sigma} \tilde{\mathbb{P}}^{K_\sigma} = \int_{\tilde{\mathcal{Y}}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$$

of K_σ -invariant tempered generalized functions on \mathfrak{k}_σ .

Proof : The generalized equivariant form $\tilde{\mathbb{P}}^{K_\sigma}$ is the limit of the smooth equivariant form

$$\begin{aligned} \tilde{\mathbb{P}}_a^{K_\sigma} &:= \tilde{\chi}_\sigma + d\tilde{\chi}_\sigma \left(\int_0^a \iota e^{-\iota t \mathcal{D}\tilde{\lambda}_\sigma} dt \right) \tilde{\lambda}_\sigma \\ &= \tilde{\chi}_\sigma e^{-\iota a \mathcal{D}\tilde{\lambda}_\sigma} + \mathcal{D} \left(\tilde{\chi}_\sigma \left(\int_0^a \iota e^{-\iota t \mathcal{D}\tilde{\lambda}_\sigma} dt \right) \tilde{\lambda}_\sigma \right), \end{aligned}$$

when $a \rightarrow \infty$. Note that the equivariant form $\tilde{\chi}_\sigma \left(\int_0^a \iota e^{-\iota t \mathcal{D}\tilde{\lambda}_\sigma} dt \right) \tilde{\lambda}_\sigma$ is with compact support on $\tilde{\mathcal{Y}}_\sigma$, then after integration on $\tilde{\mathcal{Y}}_\sigma$ we get

$$\int_{\tilde{\mathcal{Y}}_\sigma} \tilde{\mathbb{P}}_a^{K_\sigma} = \int_{\tilde{\mathcal{Y}}_\sigma} \tilde{\chi}_\sigma e^{-\iota a \mathcal{D}\tilde{\lambda}_\sigma} = \int_{\tilde{\mathcal{Y}}_\sigma} \tilde{\chi}_\sigma \circ \delta_a e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$$

where we make the change of variable δ_a in the last equality. From this, we see that $\int_{\tilde{\mathcal{Y}}_\sigma} \tilde{\mathbb{P}}_a^{K_\sigma} = \lim_{a \rightarrow \infty} \int_{\tilde{\mathcal{Y}}_\sigma} \tilde{\chi}_\sigma \circ \delta_a e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$. The function $\tilde{\chi}_\sigma$ is equal to 1 in a neighbourhood of the 0-section, then $\lim_{a \rightarrow \infty} \tilde{\chi}_\sigma \circ \delta_a = 1$, and we conclude with the ‘Lebesgue’ convergence argument. \square

5 Computation of $\int_{\tilde{\mathcal{Y}}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$

Note that the vector field on $\tilde{\mathcal{Y}}_\sigma$ generated by β_0 is identically equal to zero on $\mathbf{T}(K_\sigma/K_\beta)$ and on \mathcal{V}_{β_0} we have $\beta_0 \mathcal{V}_{\beta_0}|_{[k,y]} = -[\beta_0, y]$. With the K_σ -equivariant form

$$-\iota \mathcal{D}\tilde{\lambda}_\sigma(X + \iota s \beta_0) = -\iota \mathcal{D}\tilde{\lambda}_\sigma(X) - s \|\beta_0, y\|^2$$

depending of the parameter $s > 0$, we can compute the generalized function $\int_{\tilde{\mathcal{Y}}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$ as a limit of the generalized functions Λ_s on \mathfrak{k}_σ defined by the equation

$$\Lambda_s(X) = \int_{\tilde{\mathcal{Y}}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X + \iota s \beta_0)}, \quad s > 0.$$

The integration is well defined because $e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X + \iota s \beta_0)}|_{[k;x,y]} = e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X)}|_{[k;x,y]} e^{-s \|\beta_0, y\|^2}$ and the function $[k; x, y] \rightarrow e^{-s \|\beta_0, y\|^2}$ is bounded on $\tilde{\mathcal{Y}}_\sigma$. To compute Λ_s , first we can integrate $e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X + \iota s \beta_0)}$ on the fibers of the projection $\pi_1 : \tilde{\mathcal{Y}}_\sigma \rightarrow \mathbf{T}(K_\sigma/K_\beta)$. The term $e^{-s \|\beta_0, y\|^2}$ insures that the equivariant form $(\pi_1)_* \left(e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X + \iota s \beta_0)} \right)$ is smooth, and we have

$$(\pi_1)_* \left(e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X + \iota s \beta_0)} \right) = e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X)} (\pi_1)_* \circ \Xi^* \left(e^{-\iota \mathcal{D}\theta_{\beta_0}(X + \iota s \beta_0)} \right).$$

The map Ξ is a trivialization of the bundle $\pi_1 : \tilde{\mathcal{Y}}_\sigma \rightarrow \mathbf{T}(K_\sigma/K_\beta)$, then the equivariant form $(\pi_1)_* \circ \Xi^* \left(e^{-\imath \mathcal{D}\theta_{\beta_0}(X+\imath s\beta_0)} \right) \in \mathcal{A}_{K_\sigma}^\infty(\mathbf{T}(K_\sigma/K_\beta))$ is equal to the *function*

$$\int_{V_{\beta_0}} e^{-\imath \mathcal{D}\theta_{\beta_0}(X+\imath s\beta_0)}, \quad X \in \mathfrak{k}_\sigma. \quad (5.25)$$

The equivariant Euler form $\text{Eul}_o(\mathfrak{p}/\mathfrak{p}_{\beta_0})$ of the oriented K_σ -bundle $V_{\beta_0} = \mathfrak{p}/\mathfrak{p}_{\beta_0} \rightarrow \{0\}$ is equal to the a K_σ -invariant polynomial $X \in \mathfrak{k}_\sigma \rightarrow \Pi_{\mathfrak{p}/\mathfrak{p}_{\beta_0}} \left(\frac{-1}{2\pi} X \right)$ (for $\Pi_{\mathfrak{p}/\mathfrak{p}_{\beta_0}}$ see the Definition 1.3).

The integrals like (5.25) have been studied in section 4 of [16], where we prove in particular that

$$\int_{V_{\beta_0}} e^{-\imath \mathcal{D}\theta_{\beta_0}(X+\imath s\beta_0)} = \text{Eul}_o(\mathfrak{p}/\mathfrak{p}_{\beta_0})(X + \imath s\beta_0)^{-1} \quad (5.26)$$

for every $X \in \mathfrak{k}_\sigma$, $s > 0$. Hence, we have the following

Lemma 5.1 *The generalized function $\int_{\tilde{\mathcal{Y}}_\sigma} e^{-\imath \mathcal{D}\tilde{\lambda}_\sigma}$ is the limit of the generalized functions*

$$\Lambda_s(X) = \frac{1}{\text{Eul}_o(\mathfrak{p}/\mathfrak{p}_{\beta_0})(X + \imath s\beta_0)} \int_{\mathbf{T}(K_\sigma/K_\beta)} e^{-\imath \mathcal{D}\gamma_\sigma(X)}, \quad s > 0,$$

when $s \rightarrow 0^+$.

Then it remains to compute the generalized function $\int_{\mathbf{T}(K_\sigma/K_\beta)} e^{-\imath \mathcal{D}\gamma_\sigma(X)}$, $X \in \mathfrak{k}_\sigma$.

Proposition 5.2 *Let L be a compact Lie group and H a closed subgroup, with corresponding Lie algebras \mathfrak{l} , and \mathfrak{h} . We denote by $\lambda_{L/H}$ the (L -invariant) Liouville 1-form on $\mathbf{T}^*(L/H)$. For every $f \in C_{rd}^\infty(\mathfrak{l})$ the differential form $\langle e^{\imath \mathcal{D}\lambda_{L/H}(X)}, f(X)dX \rangle_{\mathfrak{l}}$ is integrable on $\mathbf{T}^*(L/H)$ and*

$$\int_{\mathbf{T}^*(L/H)} \langle e^{\imath \mathcal{D}\lambda_{L/H}(X)}, f(X)dX \rangle_{\mathfrak{l}} = \text{cte} \langle \mathbf{1}_{\mathfrak{h}}(Y), \overline{f}^L(Y)dY \rangle_{\mathfrak{h}},$$

with $\text{cte} = (2\imath\pi)^{\dim(L/H)} \frac{\text{vol}(L,dX)}{\text{vol}(H,dY)}$, and $\mathbf{1}_{\mathfrak{h}}$ is the constant function equal to 1 on \mathfrak{h} .

Remark 5.3 *A similar computation has been done by Witten in [24] (See equation (2.42)).*

Proof : First we parameterize the Liouville 1-form $\lambda_{L/H} \in \mathcal{A}^1(\mathbf{T}^*(L/H))$. Let \mathfrak{r} be a H -invariant complement of \mathfrak{h} in \mathfrak{l} : $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{r}$. Then the dual vector space \mathfrak{r}^* is naturally identified with the orthogonal (for the duality) \mathfrak{h}^\perp of \mathfrak{h} in \mathfrak{l}^* . The tangent bundle $\mathbf{T}(L/H)$ (resp. the cotangent bundle $\mathbf{T}^*(L/H)$) is naturally identified with the bundle $L \times_H \mathfrak{r}$ over L/H (resp. the bundle $L \times_H \mathfrak{r}^*$). Let E^i , $i = 1, \dots, \dim(G/H)$ be a basis of \mathfrak{r} , and we denote by E_i the corresponding dual basis of \mathfrak{r}^* . For every $X \in \mathfrak{l}$, let $X_{L,r}$ the vector field

on L generated by the right action of L on itself: $X_{L,r}|_l = \frac{d}{dt}\Big|_{t=0} l.e^{tX}$, $l \in L$. Let ω_i , $i = 1, \dots, \dim(G/H)$ be the (left) L -invariant 1 forms on L such that $\omega_i(X_{L,r})$ is a constant function on L equal to $\langle E_i, X \rangle$ for every $X \in \mathfrak{l}$. The space $\left(\mathcal{A}^1(L)^L \otimes \mathfrak{r}\right)_{H\text{-basic}}$ is a subspace of the space of (left) L -invariant one form on $L \times_H \mathfrak{r}^*$, where the H -invariant are taken with respect to the action of H by right multiplication on L , and adjoint action on \mathfrak{r} ; the L -invariant are taken with respect to the action of L by left multiplication on itself. The 1-form $\lambda_{L/H} \in \left(\mathcal{A}^1(L)^L \otimes \mathfrak{r}\right)_{H\text{-basic}}$ is defined by the equation

$$\lambda_{L/H}|_{[l,\xi]} := \sum_{i=1}^{\dim(\mathfrak{r})} \langle \xi, E^i \rangle \omega_i|_l, \quad [l, \xi] \in L \times_H \mathfrak{r}^*. \quad (5.27)$$

A straightforward computation gives $\mathcal{D}\lambda_{L/H}(X)|_{[l,\xi]} = \Omega_{L/H} + \langle l, \xi, X \rangle$, $[l, \xi] \in L \times_H \mathfrak{r}^*$, $X \in \mathfrak{l}$, where $\Omega_{L/H} = d\lambda_{L/H} = \sum_{i=1}^k dE^i \wedge \omega_i + E^i d\omega_i$ is the symplectic two form on $L \times_H \mathfrak{r}^* \cong \mathbf{T}^*(L/H)$. The corresponding Liouville volume form $\frac{\Omega_{L/H}^k}{k!}$ is equal to $\prod_{i=1}^k dE^i \wedge \omega_i$. Then, for every L -invariant function $f \in \mathcal{C}_{rd}^\infty(\mathfrak{l})$ we have

$$\begin{aligned} \int_{\mathbf{T}^*(L/H)} \langle e^{i\mathcal{D}\lambda_{L/H}(X)}, f(X)dX \rangle_{\mathfrak{l}} &= (i)^k \int_{L \times_H \mathfrak{r}^*} \frac{\Omega_{L/H}^k}{k!} \widehat{f}(-l, \xi) \quad [1] \\ &= (i)^k \int_{L/H} \Pi_i \omega_i \int_{\text{Fiber}} \widehat{f}(-l, \xi) \Pi_i dE^i. \quad [2] \end{aligned}$$

In [1], the function $\widehat{f} \in \mathcal{C}_{rd}^\infty(\mathfrak{r}^*)$ is the Fourier transform of f relatively to the measure dX , and $k = \dim(L/H)$. In [2], we have decomposed $\prod_{i=1}^k dE^i \wedge \omega_i$ in $\Pi_i \omega_i \wedge \Pi_i dE^i$, and the morphism $\int_{\text{Fiber}} : \mathcal{A}_{rd}^*(L \times_H \mathfrak{r}^*) \rightarrow \mathcal{A}^*(L/H)$ is the morphism of integration along the fiber of $L \times_H \mathfrak{r}^* \rightarrow L/H$.

The L -invariance of f imposes that $[l] \rightarrow \int_{\text{Fiber}} \widehat{f}(-l, \xi) \Pi_i dE^i$ is constant on L/H , and is equal to $\int_{\mathfrak{r}^*} \widehat{f}(-\xi) d\xi$, where $d\xi$ is the Lebesgue measure on \mathfrak{r}^* compatible with $\Pi_i dE^i$. The double Fourier integral gives

$$\int_{\mathfrak{r}^*} \widehat{f}(-\xi) d\xi = (2\pi)^k \int_{\mathfrak{h}} f(Y) dY,$$

where $\frac{dX}{dY}$ is the Lebesgue measure on \mathfrak{r} dual to $d\xi$. Finally, we have proved that

$$\int_{\mathbf{T}^*(L/H)} \langle e^{i\mathcal{D}\lambda_{L/H}(X)}, f(X)dX \rangle_{\mathfrak{l}} = (2i\pi)^k \text{vol}(L/H, \Pi_i \omega_i) \int_{\mathfrak{h}} f|_{\mathfrak{h}}(Y) dY.$$

But $\text{vol}(L/H, \Pi_i \omega_i) = \frac{\text{vol}(L, dX)}{\text{vol}(H, dY)}$, and Proposition is then proved. \square

We come back now at the previous situation. We verify that the 1-form $-\gamma_\sigma$ on $\mathbf{T}(K_\sigma/K_\beta)$ corresponds to the Liouville 1-form on $\mathbf{T}^*(K_\sigma/K_\beta)$, through the identification $\mathbf{T}^*(K_\sigma/K_\beta) \cong \mathbf{T}(K_\sigma/K_\beta)$.

The function defined over \mathfrak{k}_σ , $X \rightarrow \text{Eul}_o(\mathfrak{p}/\mathfrak{p}_{\beta_0})(X + \iota s\beta_0)^{-1}$, defines, when $s \rightarrow 0^+$, a generalized function in $\mathcal{C}^{-\infty}(V)$ on each vector subspace $V \subset \mathfrak{k}_\sigma$ that contains β_0 . We denote by $\text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0})$ this generalized function (whatever the vector subspace V is).

We can now state the Theorem.

Theorem 5.4 *Let M be a closed orbit in \mathfrak{g} . There exists a unique $(\beta_0, \beta_1) \in \mathfrak{k}_+ \times \mathfrak{p}$ with $[\beta_0, \beta_1] = 0$ such that $M = G \cdot (\beta_0 + \beta_1)$. Let K_β be the stabilizer of $\beta := \beta_0 + \beta_1$ in K , and let \mathfrak{k}_β be its Lie algebra. The generalized function $\text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0}) \in \mathcal{C}^{-\infty}(\mathfrak{k}_{\beta_0})^{K_{\beta_0}}$ admits a restriction to \mathfrak{k}_β , and for every function $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k})^K$ we have*

$$\langle F_M|_{\mathfrak{k}}(X), f(X)dX \rangle_{\mathfrak{k}} = \text{cte} \langle \text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0})(Z), e^{\iota(\beta_0, Z)} f|_{\mathfrak{k}_\beta}(Z) \Pi_{\mathfrak{k}/\mathfrak{k}_{\beta_0}}(Z) dZ \rangle_{\mathfrak{k}_\beta},$$

$$\text{with } \text{cte} = (-2\pi)^{\dim(K/K_{\beta_0})/2} (2\pi\iota)^{\dim(K_{\beta_0}/K_\beta)} \frac{\text{vol}(K, dX)}{\text{vol}(K_\beta, dZ)}.$$

Proof: Proposition 5.2 and Lemma 5.1 give

$$\langle \Lambda_s(X), f(X)dX \rangle_{\mathfrak{k}_\sigma} = \text{cte} \langle \frac{1}{\text{Eul}_o(\mathfrak{p}/\mathfrak{p}_{\beta_0})(Y + \iota s\beta_0)}, f|_{\mathfrak{k}_\beta}(Y) dY \rangle_{\mathfrak{k}_\beta}$$

for every function $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k}_\sigma)$ that is K_σ -invariant, and with $\text{cte} = (2\iota\pi)^{\dim(\mathfrak{k}_\sigma/\mathfrak{k}_\beta)} \frac{\text{vol}(K_\sigma, dX)}{\text{vol}(K_\beta, dY)}$.

After taking the limit $s \rightarrow 0^+$, we get from Lemma 5.1

$$\langle \left(\int_{\tilde{Y}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma} \right)(X), f(X)dX \rangle_{\mathfrak{k}_\sigma} = \text{cte} \langle \text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0})(Y), f|_{\mathfrak{k}_\beta}(Y) dY \rangle_{\mathfrak{k}_\beta}.$$

Now Proposition 3.5, Lemma 4.3, and Proposition 4.4 used with this last equality complete the proof. \square

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