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# The Fourier transform of semi-simple coadjoint orbits

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## Abstract

Let  $M$  be a closed coadjoint orbit of a real connected semi-simple Lie group  $G$ , and let  $F_M \in C^{-\infty}(\mathfrak{g})^G$  be its Fourier transform. In this paper we compute the restriction of  $F_M$  to the Lie algebra  $\mathfrak{k}$  of a maximal compact subgroup  $K$  of  $G$ . Using a technique of localization in equivariant cohomology developed in [16, 17], we extend previous results by M. Duflou, G. Heckman, M. Vergne and I. Sengupta.

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## 1 Coadjoint orbits of semi-simple Lie groups

Let  $G$  be a connected, real, semi-simple Lie group with finite center. Let  $\mathfrak{g}$  be its Lie algebra, and let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be a Cartan decomposition of  $\mathfrak{g}$ . We denote by  $K$  (resp.  $\theta$ ) the compact connected subgroup of  $G$  with Lie algebra  $\mathfrak{k}$  (resp. Cartan involution).

Let  $M$  be an orbit of the coadjoint representation. It is a regularly embedded submanifold of  $\mathfrak{g}^*$  which carries a canonical symplectic 2-form  $\Omega$ ; in particular the manifold  $M$  is of even dimension  $2d$ . We denote by  $d\mathcal{L} := \frac{\Omega^d}{(2\pi)^{d!}}$

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the corresponding Liouville volume form on  $M$ . The action of  $G$  on  $M$  is Hamiltonian and the corresponding moment map  $\mu_G : M \hookrightarrow \mathfrak{g}^*$  is the inclusion. Note that

$$\mu_G \text{ is proper} \iff M \text{ is closed in } \mathfrak{g}^*. \quad (1.1)$$

The induced action of  $K$  on  $M$  is Hamiltonian, and the moment map  $\mu_K : M \rightarrow \mathfrak{k}^*$  is by definition the composition of  $\mu_G$  with the projection  $\mathfrak{g}^* \rightarrow \mathfrak{k}^*$ . For  $X \in \mathfrak{g}$ , we write  $X = X_1 + X_2$  with  $X_1 \in \mathfrak{k}$  and  $X_2 \in \mathfrak{p}$ . The Killing form  $B$  defines  $K$ -invariant Euclidean structures on  $\mathfrak{k}$  and  $\mathfrak{p}$  such that

$$B(X, X) = -\|X_1\|^2 + \|X_2\|^2, \quad X \in \mathfrak{g}, \quad (1.2)$$

is a  $G$ -invariant quadratic form on  $\mathfrak{g}$ . Let  $\|\cdot\|$  be the  $K$ -invariant Euclidean norm on  $\mathfrak{g}$  defined by the equation  $\|X\|^2 := -B(X, \theta X) = \|X_1\|^2 + \|X_2\|^2$ .

**Remark 1.1** *The Killing form  $B$  provides a  $G$ -equivariant identification  $\mathfrak{g} \cong \mathfrak{g}^*$ , and also the following  $K$ -invariant identification  $\mathfrak{k} \cong \mathfrak{k}^*$ ,  $\mathfrak{p} \cong \mathfrak{p}^*$ . Then we will only deal with adjoint orbits  $M$  of  $G$ . In this case the symplectic structure on  $M$  is defined by the equation  $\Omega_m(X_M, Y_M) = -B(m, [X, Y])$  for  $m \in M$  and  $X, Y \in \mathfrak{g}$ , and the moment map  $\mu_K : M \rightarrow \mathfrak{k}$  is the restriction to  $M$  of the orthogonal projection  $\mathfrak{g} \rightarrow \mathfrak{k}$ ,  $X \mapsto X_1$ .*

Let  $a \in \mathbb{R}$  be the value of  $X \mapsto B(X, X)$  on  $M$ . From the decomposition  $\|X_1\|^2 = \frac{1}{2}(\|X_1\|^2 + \|X_2\|^2) + \frac{1}{2}(\|X_1\|^2 - \|X_2\|^2)$  and using (1.2) we see that  $\|\mu_K\|^2 = \frac{1}{2}\|\mu_G\|^2 - \frac{1}{2}a$  holds on  $M$ . Hence the relation (1.1) becomes

$$\mu_K \text{ is proper} \iff M \text{ is closed in } \mathfrak{g}. \quad (1.3)$$

(We just use the fact that:  $\mu_K$  is proper  $\iff \|\mu_K\|^2$  is proper, and the same is true for  $\mu_G$ .)

We denote respectively by  $\mathcal{C}_{rd}^\infty(\mathfrak{g})$  and  $\mathcal{C}_{cpt}^\infty(\mathfrak{g})$ , the Schwartz space of smooth rapidly decreasing functions on  $\mathfrak{g}$ , and the space of smooth functions with compact support in  $\mathfrak{g}$ .

**Assumption 1.2** *We suppose for the rest of this paper that  $M$  is a closed adjoint orbit in  $\mathfrak{g}$ .*

In this case the Liouville volume form  $d\mathcal{L}$  defines a tempered positive measure on  $\mathfrak{g}$ : for every function  $f \in \mathcal{C}_{rd}^\infty(\mathfrak{g})$ , the integral  $\int_M f(m)d\mathcal{L}(m)$  converges. Then we can define the Fourier transform of this measure

$$F_M(X) = (\imath)^d \int_M e^{-\imath B(m, X)} d\mathcal{L}(m), \quad X \in \mathfrak{g},$$

which is the generalized, tempered, and  $G$ -invariant function  $F_M \in \mathcal{C}_{temp}^{-\infty}(\mathfrak{g})^G$  defined by the equation  $\langle F_M(X), f(X)dX \rangle_{\mathfrak{g}} := (\imath)^d \int_M \left( \int_{\mathfrak{g}} e^{-\imath B(m, X)} f(X)dX \right) d\mathcal{L}(m)$ , for every  $f \in \mathcal{C}_{rd}^\infty(\mathfrak{g})$  (where  $dX$  is a Lebesgue measure on  $\mathfrak{g}$ ).

The generalized function  $F_M \in \mathcal{C}_{temp}^{-\infty}(\mathfrak{g})^G$  admits a restriction  $F_M|_{\mathfrak{k}}$  to  $\mathfrak{k}$  (see section 1 of [8] for the notion of restriction). Moreover the generalized function  $F_M|_{\mathfrak{k}} \in \mathcal{C}_{temp}^{-\infty}(\mathfrak{k})^K$  is determined by the equation  $X \in \mathfrak{k}$ ,  $F_M|_{\mathfrak{k}}(X) =$

$(i)^d \int_M e^{i(\mu_K(m), X)} d\mathcal{L}(m)$ , where  $(\cdot, \cdot)$  is the  $K$ -invariant scalar product on  $\mathfrak{k}$  coming from the Killing form (see Proposition 5 of [8]). Using the  $K$ -equivariant symplectic 2-form  $\Omega_{\mathfrak{k}}(X) := \Omega + (\mu_K, X)$ ,  $X \in \mathfrak{k}$ , the function  $F_M|_{\mathfrak{k}} \in \mathcal{C}_{temp}^{-\infty}(\mathfrak{k})^K$  can be rewritten in the following way

$$F_M|_{\mathfrak{k}}(X) = \int_M e^{i\Omega_{\mathfrak{k}}(X)}, \quad X \in \mathfrak{k}$$

In this paper we shall compute the generalized functions  $F_M|_{\mathfrak{k}} \in \mathcal{C}_{temp}^{-\infty}(\mathfrak{k})^K$  for every closed orbits  $M \subset \mathfrak{g}$ .

This computation has been already carried out in the *regular* case by J. Sengupta [20] (modulo a constant) and in the *elliptic* case by Duflo-Heckman-Vergne [7, 8]. Our method, which is closed to those of M. Duflo and M. Vergne in [8], uses the techniques of localization in equivariant cohomology developed in [16, 17].

Recall now the Rossmann formula. Suppose that  $G$  and  $K$  have the same rank, and let  $T \subset K$  be a Cartan subgroup of  $K$ . We denote by  $W$  the associated Weyl group. Let  $\mathfrak{t}$  be the Lie algebra of  $T$  and  $\mathfrak{t}_r$  be the open subset of regular point:  $X \in \mathfrak{t}_r$  iff the stabilizer of  $X$  in  $K$  is equal to the torus  $T$ .

**Definition 1.3** *Let  $V$  be an oriented Euclidean space provided with an action  $\rho : H \rightarrow SO(V)$  of a compact Lie group  $H$  that preserves the orientation  $o$  of  $V$ . Let  $\mathfrak{h}$  be the Lie algebra of  $H$  and we still denote by  $\rho : \mathfrak{h} \rightarrow so(V)$  the morphism of Lie algebras. We denote by  $\Pi_V(X) = \det_{V,o}^{1/2}(X)$ ,  $X \in \mathfrak{h}$  the  $K$ -invariant polynomial square root of the polynomial function  $X \mapsto \det_V(\rho(X))$  on  $\mathfrak{h}$ .*

*Let  $H$  be a compact Lie group with Lie algebra  $\mathfrak{h}$ . For any Lebesgue measure  $dX$  on  $\mathfrak{h}$ , we denote by  $\text{vol}(H, dX)$  the volume of  $H$  computed with the Haar measure compatible with  $dX$ .*

For  $\beta \in \mathfrak{k}$  we denote by  $\mathfrak{g}_\beta$  (resp.  $\mathfrak{k}_\beta$  and  $\mathfrak{p}_\beta$ ) the subspace of  $\mathfrak{g}$  (resp.  $\mathfrak{k}$  and  $\mathfrak{p}$ ) fixed by the adjoint action of  $\beta$ . Let  $K_\beta$  be the stabilizer of  $\beta$  in  $K$  : it is a connected subgroup with Lie algebra  $\mathfrak{k}_\beta$ . The vector spaces  $\mathfrak{g}/\mathfrak{g}_\beta$  and  $\mathfrak{k}/\mathfrak{k}_\beta$ , considered as the tangent space at  $\beta$  of the orbits  $G.\beta$  and  $K.\beta$ , are oriented by the respective symplectic structures. Using the decomposition  $\mathfrak{g}/\mathfrak{g}_\beta = \mathfrak{k}/\mathfrak{k}_\beta \oplus \mathfrak{p}/\mathfrak{p}_\beta$ , the space  $\mathfrak{p}/\mathfrak{p}_\beta$  has an induced orientation. The space  $\mathfrak{g}/\mathfrak{g}_\beta$  (resp.  $\mathfrak{k}/\mathfrak{k}_\beta$  and  $\mathfrak{p}/\mathfrak{p}_\beta$ ) carries a natural Euclidean structure and a  $K_\beta$ -action, coming from its identification with the orthogonal complement of  $\mathfrak{g}_\beta$  (resp.  $\mathfrak{k}_\beta$  and  $\mathfrak{p}_\beta$ ) in  $\mathfrak{g}$  (resp.  $\mathfrak{k}$  and  $\mathfrak{p}$ ). Following the Definition 1.3, this data define the  $K_\beta$ -invariant polynomial functions  $\Pi_{\mathfrak{g}/\mathfrak{g}_\beta}$ ,  $\Pi_{\mathfrak{k}/\mathfrak{k}_\beta}$ , and  $\Pi_{\mathfrak{p}/\mathfrak{p}_\beta}$  on  $\mathfrak{k}_\beta$ .

When  $\beta$  belongs to  $\mathfrak{t}_r$  the following proposition is due to Rossmann [18].

**Proposition 1.4** *Let  $\beta \in \mathfrak{t}$  and let  $M = G.\beta$  be the associated (closed) adjoint orbit. The function  $F_M$  is analytic on  $G.\mathfrak{t}_r$  (hence  $F_M|_{\mathfrak{k}}$  is analytic on  $K.\mathfrak{t}_r$ ), and for every  $X \in \mathfrak{t}_r$  we have*

$$F_M(X) = (-2\pi)^{d_\beta} \sum_{W/W_\beta} \frac{e^{i(\beta, w.X)}}{\Pi_{\mathfrak{g}/\mathfrak{g}_\beta}(w.X)},$$

where  $\Pi_{\mathfrak{g}/\mathfrak{g}_\beta}$  is the  $K_\beta$ -invariant polynomial defined at the Definition 1.3,  $d_\beta = \frac{1}{2} \dim(M)$ , and  $W_\beta$  is the subgroup of  $W$  that stabilizes  $\beta$ .

Apart from the proof of Rossmann [18], we can find another proofs of this Proposition in [4, 7, 22].

**Notations** : Let  $F \in \mathcal{C}^{-\infty}(V)$  be a generalized function on a vector space  $V$ . For every test density  $f(v)dv$  over  $V$ , we denote by  $\langle F(v), f(v)dv \rangle_V \in \mathbb{C}$  the image (or the integral) of  $f(v)dv$  by  $F$ .

Let's now give a global expression of  $F_M|_{\mathfrak{k}}$ . For a function  $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k})^K$  supported in  $K.t_r$  we get from Proposition 1.4

$$\begin{aligned} \langle F_M|_{\mathfrak{k}}(X), f(X)dX \rangle_{\mathfrak{k}} &= \text{cte} \langle \frac{1}{\Pi_{\mathfrak{g}/\mathfrak{g}_\beta}(Y)}, e^{\iota(\beta, Y)} f|_{\mathfrak{t}}(Y) \Pi_{\mathfrak{t}/\mathfrak{t}}^2(Y) dY \rangle_{\mathfrak{t}} \quad [1] \\ &= \text{cte} \langle \frac{1}{\Pi_{\mathfrak{p}/\mathfrak{p}_\beta}(Y)}, e^{\iota(\beta, Y)} f|_{\mathfrak{t}}(Y) \Pi_{\mathfrak{t}/\mathfrak{k}_\beta}(Y) \Pi_{\mathfrak{k}_\beta/\mathfrak{t}}^2(Y) dY \rangle_{\mathfrak{t}} \quad [2] \\ &= \text{cte}' \langle \frac{1}{\Pi_{\mathfrak{p}/\mathfrak{p}_\beta}(Z)}, e^{\iota(\beta, Z)} f|_{\mathfrak{k}_\beta}(Z) \Pi_{\mathfrak{t}/\mathfrak{k}_\beta}(Z) dZ \rangle_{\mathfrak{k}_\beta}, \quad [3] \end{aligned}$$

where  $\text{cte} = \frac{\text{vol}(K, dX)}{\text{vol}(T, dY)} \frac{(-2\pi)^{d_\beta}}{|W_\beta|}$ , and  $\text{cte}' = (-2\pi)^{d_\beta} \frac{\text{vol}(K, dX)}{\text{vol}(K_\beta, dZ)}$ . In the equality [1] we use the Weyl integration formula for  $(\mathfrak{k}, \mathfrak{t})$  and the fact that  $f|_{\mathfrak{t}}$  is  $W$ -invariant. The equality [2] comes from the equalities  $\Pi_{\mathfrak{g}/\mathfrak{g}_\beta} = \Pi_{\mathfrak{t}/\mathfrak{k}_\beta} \Pi_{\mathfrak{p}/\mathfrak{p}_\beta}$  and  $\Pi_{\mathfrak{t}/\mathfrak{t}} = \Pi_{\mathfrak{t}/\mathfrak{k}_\beta} \Pi_{\mathfrak{k}_\beta/\mathfrak{t}}$ . In the last equality [3] we use the Weyl integration formula for  $(\mathfrak{k}_\beta, \mathfrak{t})$ . Note that  $Z \rightarrow e^{\iota(\beta, Z)} f|_{\mathfrak{k}_\beta}(Z) \Pi_{\mathfrak{t}/\mathfrak{k}_\beta}(Z)$  and  $Z \rightarrow \Pi_{\mathfrak{p}/\mathfrak{p}_\beta}(Z)$  are  $K_\beta$ -invariant functions on  $\mathfrak{k}_\beta$ .

The  $K_\beta$ -equivariant Euler form of the oriented  $K_\beta$  vector bundle  $\mathfrak{p}/\mathfrak{p}_\beta \rightarrow \{0\}$  is equal to  $Z \rightarrow \Pi_{\mathfrak{p}/\mathfrak{p}_\beta}(\frac{1}{-2\pi}Z)$ . This equivariant polynomial has a tempered generalized inverse  $\text{Eul}_\beta^{-1}(\mathfrak{p}/\mathfrak{p}_\beta)$  defined by

$$\text{Eul}_\beta^{-1}(\mathfrak{p}/\mathfrak{p}_\beta)(Z) = \lim_{s \rightarrow 0^+} \frac{1}{\Pi_{\mathfrak{p}/\mathfrak{p}_\beta}(\frac{1}{-2\pi}(Z + \iota s \beta))}, \quad Z \in \mathfrak{k}_\beta.$$

For these notions see section 4 of [16].

Finally we can state the global version of the ‘Rossmann formula’ that is obtained by Duflo-Vergne in [8] (with a different expression). For every function  $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k})^K$ , we have

$$\langle F_M|_{\mathfrak{k}}(X), f(X)dX \rangle_{\mathfrak{k}} = \text{cte} \langle \text{Eul}_\beta^{-1}(\mathfrak{p}/\mathfrak{p}_\beta)(Z), e^{\iota(\beta, Z)} f|_{\mathfrak{k}_\beta}(Z) \Pi_{\mathfrak{t}/\mathfrak{k}_\beta}(Z) dZ \rangle_{\mathfrak{k}_\beta}, \quad (1.4)$$

with  $\text{cte} = (-2\pi)^{\dim(K/K_\beta)/2} \frac{\text{vol}(K, dX)}{\text{vol}(K_\beta, dZ)}$ .

The central result of this paper is the following global expression of  $F_M|_{\mathfrak{k}}$  for the case of general closed orbits. We come back in the general case of a real semisimple connected Lie group  $G$  with finite center, and we make no assumption related to the ranks of  $G$  and  $K$ . Let  $T$  be a Cartan subgroup of  $K$  with Lie algebra  $\mathfrak{t}$ , and let  $\mathfrak{t}_+$  be a Weyl chamber of  $\mathfrak{t}$ .

**Theorem 5.4** *Let  $M$  be a closed orbit in  $\mathfrak{g}$ . There exists a unique  $(\beta_0, \beta_1) \in \mathfrak{t}_+ \times \mathfrak{p}$  with  $[\beta_0, \beta_1] = 0$  such that  $M = G \cdot (\beta_0 + \beta_1)$ . Let  $K_\beta$  be the stabilizer of  $\beta := \beta_0 + \beta_1$  in  $K$ , and let  $\mathfrak{k}_\beta$  be its Lie algebra. The generalized function  $\text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0}) \in \mathcal{C}^{-\infty}(\mathfrak{k}_{\beta_0})^{K_{\beta_0}}$  admits a restriction to  $\mathfrak{k}_\beta$ , and for every function  $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k})^K$  we have*

$$\langle F_M|_{\mathfrak{k}}(X), f(X)dX \rangle_{\mathfrak{k}} = \text{cte} \langle \text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0})(Z), e^{i(\beta_0, Z)} f|_{\mathfrak{k}_\beta}(Z) \Pi_{\mathfrak{k}/\mathfrak{k}_{\beta_0}}(Z) dZ \rangle_{\mathfrak{k}_\beta},$$

$$\text{with cte} = (-2\pi)^{\dim(K/K_{\beta_0})/2} (2\pi i)^{\dim(K_{\beta_0}/K_\beta)} \frac{\text{vol}(K, dX)}{\text{vol}(K_\beta, dZ)}.$$

**Remark 1.5** *The result of Theorem 5.4 can be extended in the following way. Let  $\alpha(X)$  be a closed  $K$ -equivariant form on  $M$  depending polynomially of  $X \in \mathfrak{g}$  (see sub-section 2.1 for the definitions). If  $\alpha(X)$  is régulière on  $M$  (for the notion of being régulière see pages 20-21 of [8]), then the integral*

$$\int_M \alpha(X) e^{i\Omega_{\mathfrak{k}}(X)}, \quad X \in \mathfrak{k},$$

*defines a tempered measure on  $\mathfrak{k}$ , and we have*

$$\langle \left( \int_M \alpha e^{i\Omega_{\mathfrak{k}}} \right)(X), f(X)dX \rangle_{\mathfrak{k}} = \text{cte} \langle \text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0})(Z), e^{i(\beta_0, Z)} r_\beta(\alpha)(Z) f|_{\mathfrak{k}_\beta}(Z) \Pi_{\mathfrak{k}/\mathfrak{k}_{\beta_0}}(Z) dZ \rangle_{\mathfrak{k}_\beta}$$

*for every  $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k})^K$ . In this equality  $r_\beta : \mathcal{A}_K^\infty(M) \rightarrow \mathcal{A}_{K_\beta}^\infty(\{\beta\}) = \mathcal{C}^\infty(\mathfrak{k}_\beta)^{K_\beta}$  is the restriction map to the point  $\{\beta\} \subset M$ .*

**Acknowledgement.** I am grateful to Michel Brion for bringing me the reference [21] to my attention.

## 2 Localization of the Fourier transform

We take the same notations as before. Let  $M$  be a closed adjoint orbit of  $G$  in  $\mathfrak{g}$ , and consider the Hamiltonian action of the compact subgroup  $K$  on  $M$ . We know from (1.3) that the associated moment map  $\mu_K : M \rightarrow \mathfrak{k}$  is proper (we make the identification  $\mathfrak{k} \cong \mathfrak{k}^*$  via the Killing form, see Remark 1.1).

Consider the equivariant symplectic form  $\Omega_{\mathfrak{k}}(X) := \Omega + (\mu_{\mathfrak{k}}, X)$ ,  $X \in \mathfrak{k}$  defined on  $M$ . By definition of the moment map, the equivariant form  $\Omega_{\mathfrak{k}}$  is closed:  $\mathcal{D}(\Omega_{\mathfrak{k}})(X) = d(\mu_{\mathfrak{k}}, X) - c(X_M)\Omega = 0$ ,  $X \in \mathfrak{k}$  (see sub-section 2.1 for the notations). And we know from the introduction that the generalized function  $F_M|_{\mathfrak{k}}$  is given by the integral of the closed equivariant form  $e^{i\Omega_{\mathfrak{k}}} \in \mathcal{A}_K^\infty(M)$  on  $M$ . Consider the function  $\|\mu_K\|^2 : M \rightarrow \mathbb{R}$ . In this section we show that the integral of  $e^{i\Omega_{\mathfrak{k}}}$  can be localized on the set  $\text{Cr}(\|\mu_K\|^2)$  of critical points of the function  $\|\mu_K\|^2$ .

First, we recall the definition and notations we use in the rest of this paper.

## 2.1 Equivariant cohomology-Definitions

Let  $M$  be a manifold provided with an action of a compact connected Lie group  $K$  with Lie algebra  $\mathfrak{k}$ . We denote by  $\mathcal{A}^*(M)$  the algebra of differential forms on  $M$  (over  $\mathbb{C}$ ), and by  $d$  the exterior differentiation. Let  $\mathcal{A}_{cpt}^*(M)$  be the sub-algebra of compactly supported differential forms. If  $\xi$  is a vector field on  $M$  we denote by  $c(\xi) : \mathcal{A}^*(M) \rightarrow \mathcal{A}^{*-1}(M)$  the contraction by  $\xi$ . The action of  $K$  on  $M$  gives a morphism  $X \rightarrow X_M$  from  $\mathfrak{k}$  to the Lie algebra of vector fields on  $M$ .

We now recall the different de Rham complexes of  $K$ -equivariant forms on  $M$ . For more details see [2, 3, 8, 12].

Let  $\mathcal{C}^\infty(\mathfrak{k}, \mathcal{A}^*(M))$  be the algebra of forms  $\alpha(X)$  on  $M$  depending smoothly of  $X \in \mathfrak{k}$ . We note  $\mathcal{A}_K^\infty(M)$  the sub-algebra of  $\mathcal{C}^\infty(\mathfrak{k}, \mathcal{A}^*(M))$  consisting of the  $K$ -invariant elements: these elements are called the equivariant forms with  $\mathcal{C}^\infty$ -coefficients. Let  $\mathcal{A}_K^*(M) \subset \mathcal{A}_K^\infty(M)$  be the sub-algebra of equivariant forms  $\alpha(X)$  depending *polynomially* of  $X \in \mathfrak{k}$ . The differential  $\mathcal{D}$  on  $\mathcal{A}_K^\infty(M)$  is given by the equation

$$\forall \alpha \in \mathcal{A}_K^\infty(M), \quad (\mathcal{D}\alpha)(X) := (d - c(X_M))(\alpha(X)), \quad X \in \mathfrak{k}.$$

We see that  $\mathcal{A}_K^*(M)$  is stable under  $\mathcal{D}$ , and that  $\mathcal{D}^2 = 0$  on  $\mathcal{A}_K^\infty(M)$ . The cohomologies associated to  $(\mathcal{A}_K^*(M), \mathcal{D})$  and  $(\mathcal{A}_K^\infty(M), \mathcal{D})$  are denoted respectively  $\mathcal{H}_K^*(M)$  and  $\mathcal{H}_K^\infty(M)$ .

The algebra  $\mathcal{A}_K^\infty(M)$  has a sub-algebra  $\mathcal{A}_{K,cpt}^\infty(M) := \mathcal{C}^\infty(\mathfrak{k}, \mathcal{A}_{cpt}^*(M))^K$ , stable under the differential  $\mathcal{D}$ . The cohomology associated to  $(\mathcal{A}_{K,cpt}^\infty(M), \mathcal{D})$  is called the  $K$ -equivariant cohomology with compact support and is denoted by  $\mathcal{H}_{K,cpt}^\infty(M)$ .

For our purpose we need equivariant forms with generalized coefficients. For a more precise description see [12].

The space  $\mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}^*(M))$  of generalized functions on  $\mathfrak{k}$  with values in the space  $\mathcal{A}^*(M)$  is, by definition, the space  $\text{Hom}(\mathfrak{m}_c(\mathfrak{k}), \mathcal{A}^*(M))$  of continuous  $\mathbb{C}$ -linear maps from the space  $\mathfrak{m}_c(\mathfrak{k})$  of smooth compactly supported densities on  $\mathfrak{k}$  to the space  $\mathcal{A}^*(M)$ , both endowed with the  $\mathcal{C}^\infty$ -topologies. We define

$$\mathcal{A}_K^{-\infty}(M) := \mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}^*(M))^K$$

as the space of  $K$ -equivariant  $\mathcal{C}^{-\infty}$ -maps from  $\mathfrak{k}$  to  $\mathcal{A}^*(M)$ . An element of the space  $\mathcal{A}_K^{-\infty}(M)$  is called an equivariant form with generalized coefficients. The image of  $\phi \in \mathfrak{m}_c(\mathfrak{k})$  under  $\alpha \in \mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}^*(M))$  is a differential form on  $M$  denoted by  $\langle \alpha, \phi \rangle_{\mathfrak{k}}$ .

We see that  $\mathcal{A}_K^\infty(M) \subset \mathcal{A}_K^{-\infty}(M)$  and we can also extend the differential  $\mathcal{D}$  to  $\mathcal{A}_K^{-\infty}(M)$  [12]. Take a basis  $\{E^1, \dots, E^p\}$  of  $\mathfrak{k}$ , with associated dual basis  $\{E_1, \dots, E_p\}$ . Let  $\{X_1, \dots, X_p\}$  be the corresponding coordinate functions on  $\mathfrak{k}$ . For every  $\gamma \in \mathcal{A}_K^{-\infty}(M)$ ,

$$\langle \mathcal{D}(\gamma), \phi \rangle_{\mathfrak{k}} := d \langle \gamma, \phi \rangle_{\mathfrak{k}} - \sum_{k=1}^p c(E_M^k) \langle \gamma, X_k \phi \rangle_{\mathfrak{k}} \quad \text{for every } \phi \in \mathfrak{m}_c(\mathfrak{k}). \quad (2.5)$$

Using the  $K$ -equivariance condition, we verify that  $\mathcal{D}^2 = 0$  on  $\mathcal{A}_K^{-\infty}(M)$ . The cohomology associated to  $(\mathcal{A}_K^{-\infty}(M), \mathcal{D})$  is called the  $K$ -equivariant cohomology with generalized coefficients and is denoted by  $\mathcal{H}_K^{-\infty}(M)$ . The sub-space  $\mathcal{A}_{K,cpt}^{-\infty}(M) := \mathcal{C}^{-\infty}(\mathfrak{k}, \mathcal{A}_{cpt}^*(M))^K$  is stable under the differential  $\mathcal{D}$ , and we denote by  $\mathcal{H}_{K,cpt}^{-\infty}(M)$  the associated cohomology.

Let  $H$  be a compact Lie group with Lie algebra  $\mathfrak{h}$ . For every  $f \in \mathcal{C}^\infty(\mathfrak{h})$ , we denote by  $\overline{f}^H$  the  $H$ -invariant function on  $\mathfrak{h}$  defined by the equation

$$\overline{f}^H(X) := \int_K f(h.X)dh, \quad X \in \mathfrak{h},$$

where  $dh$  is the normalized Haar measure on  $H$  ( $\int_H dh = 1$ ). It defines the projection  $f \mapsto \overline{f}^H$ ,  $\mathcal{C}^\infty(\mathfrak{h}) \rightarrow \mathcal{C}^\infty(\mathfrak{h})^H$ , and the same holds for the subspace  $\mathcal{C}_{cpt}^\infty(\mathfrak{h})$  and  $\mathcal{C}_{rd}^\infty(\mathfrak{h})$ . Recall that every  $H$ -invariant generalized function  $\phi \in \mathcal{C}^{-\infty}(\mathfrak{h})$  is completely determined by its values on the  $H$ -invariant densities of  $\mathfrak{h}$ , and moreover, for every  $f \in \mathcal{C}_{cpt}^\infty(\mathfrak{h})$  we have

$$\langle \phi(X), f(X)dX \rangle_{\mathfrak{h}} = \langle \phi(X), \overline{f}^H(X)dX \rangle_{\mathfrak{h}}.$$

## 2.2 Critical points of $\|\mu_K\|^2$

In this sub-section we prove that the set  $\text{Cr}(\|\mu_K\|^2)$  of critical points of the function  $\|\mu_K\|^2$  is a  $K$ -orbit in  $M$ .

For a point  $m \in M$  we decompose  $m = x_m + y_m$  with  $x_m = \mu_K(m) \in \mathfrak{k}$  and  $y_m \in \mathfrak{p}$ . By definition of the moment map we have  $\frac{1}{2}d\|\mu_K\|_m^2 = (d\mu_K(m), \mu_K(m)) = \Omega((x_m)_M|_m, \cdot)$ ,  $m \in M$ . Then  $d\|\mu_K\|_m^2 = 0$  iff  $(x_m)_M|_m = 0$  or equivalently  $[x_m, m] = [x_m, y_m] = 0$ . We have shown the following

$$\text{Cr}(\|\mu_K\|^2) = \{m \in M \mid [x_m, y_m] = 0\}. \quad (2.6)$$

**Proposition 2.1** *The critical points of  $\|\mu_K\|^2$  form a  $K$ -orbit in  $M$ . In particular the points of  $\text{Cr}(\|\mu_K\|^2)$  are the points where  $\|\mu_K\|^2$  is minimum.*

To prove this proposition we consider the length function  $\Psi : M \rightarrow \mathbb{R}$ ,  $\Psi(m) = \|m\|^2$ . We have already seen that  $\|\mu_K\|^2 = \frac{1}{2}\Psi - \frac{a}{2}$  for some  $a \in \mathbb{R}$ . Then we can work with  $\Psi$  instead of  $\|\mu_K\|^2$ .

Proposition (2.1) is a consequence of the next Lemma which is due to P. Slodowy [21].

**Lemma 2.2** *Let  $m \in M$  be a critical point of  $\Psi$ , and  $m' \in M$ . Then either  $\Psi(m') > \Psi(m)$  or  $m' \in K.m$ .*

*Proof* : Let  $g \in G$  such that  $m' = g.m$ . Using the Cartan decomposition  $G = K \exp(\mathfrak{p})$  we know that

$$m' = k \exp(X).m, \quad \text{with } k \in K \text{ and } X \in \mathfrak{p}.$$



We are now going to prove that  $\Psi(m') > \Psi(m)$  or  $\exp(X).m = m$ . Consider the following function  $\kappa(t) = \Psi(\exp(tX).m)$ ,  $t \in \mathbb{R}$ . The element  $X$  belongs to  $\mathfrak{p}$ , then the endomorphism  $ad(X) : \mathfrak{g} \rightarrow \mathfrak{g}$ ,  $Z \rightarrow [X, Z]$  is auto-adjoint relatively to the scalar product, hence diagonalizable. Let  $\mathfrak{g} = \sum_a \mathfrak{g}_a$  be the decomposition into orthogonal subspaces of  $\mathfrak{k}$  where for  $a \in \mathbb{R}$ ,

$$\mathfrak{g}_a = \{Y \in \mathfrak{g} \mid [X, Y] = aY\}.$$

If we write  $m = \sum_a m_a$  with  $m_a \in \mathfrak{g}_a$ , we have  $\kappa(t) = \|\sum_a e^{at}m_a\|^2 = \sum_a e^{2at}\|m_a\|^2$ . This shows that  $\kappa''(t) = 4\sum_{a \neq 0} a^2 e^{2at}\|m_a\|^2$  is a positive function on  $\mathbb{R}$ , and we have also  $\kappa'(0) = 0$  (because  $m$  is a critical point of  $\Psi$ ). The function  $\kappa$  is convex and the equality  $\kappa'(0) = 0$  implies that either  $\kappa(1) > \kappa(0)$  or  $\kappa$  is constant on the interval  $[0, 1]$ . In the first case we get that  $\Psi(m') > \Psi(m)$ . The second point imposes that  $\kappa''(t) = 4\sum_{a \neq 0} a^2 e^{2at}\|m_a\|^2 = 0$  for every  $0 < t < 1$ . This means that  $m_a = 0$  for every  $a \neq 0$  or equivalently  $[X, m] = 0$  (hence  $\exp(X).m = m$ ).  $\square$

This Lemma implies that every critical point of  $\Psi$  reaches the minimum of  $\Psi$ , and that two critical points of  $\Psi$  are in the same  $K$ -orbit. The properness of  $\Psi$  implies that  $\Psi(M)$  is closed in  $\mathbb{R}$ , in particular the minimum of  $\Psi$  is reached and so  $\text{Cr}(\|\mu_K\|^2) = \text{Cr}(\Psi) \neq \emptyset$ . The proof of Proposition 2.1 is then completed.

**Remark 2.3** *One could consider this length function  $\Psi$  in the case of a general orbit  $M$ . But from the results of P. Slodowy in [21], we know that  $\text{Cr}(\Psi) = \emptyset$  when  $M$  is not closed.*

### 2.3 Localization on $\text{Cr}(\|\mu_K\|^2)$

The  $K$ -invariant scalar product on  $\mathfrak{g}$  defines a  $K$ -invariant Riemannian metric  $(\cdot, \cdot)_M$  on the adjoint orbit  $M$ . Let  $\lambda_K := (\mathcal{H}, \cdot)_M$  be the  $K$ -invariant one form on  $M$ , where  $\mathcal{H}$  is the Hamiltonian vector field associated to the function  $\frac{1}{2}\|\mu_K\|^2$ . More precisely, if we note  $m = x_m + y_m$  with  $x_m \in \mathfrak{k}$  and  $y_m \in \mathfrak{p}$ , we have  $\mathcal{H}_m = -[x_m, y_m] \in \mathbf{T}_m M$  for every  $m \in M$ , and

$$\lambda_K|_m = -\left([x_m, y_m], \cdot\right)_M, \quad m \in M. \quad (2.7)$$

This 1-form was introduced by Witten in [24] to describe a Non-Abelian localization in equivariant cohomology. Now, this idea has been developed by the author in [16, 17]. In the case of an elliptic orbit  $M = G.\beta$ , with  $\beta \in \mathfrak{k}$ , Duflou and Vergne already used the 1-form  $\lambda_K$  in [8] (it was denoted by  $\theta$ , see Proposition 35) to localize the integral defining  $F_M|_{\mathfrak{k}}$  to the submanifold  $K.\beta \subset M$ . We are going to extend this localization for all closed orbits, using the technique of *partition of unity in equivariant cohomology* introduced in [16].

We note  $\Phi_{\lambda_K} : M \rightarrow \mathfrak{k}$  the  $K$ -equivariant map defined by  $(\Phi_{\lambda}(m), X) := \lambda(X_M)(m)$ ,  $m \in M$ ,  $X \in \mathfrak{k}$ . By definition  $X_M(m) = -[X, m]$  for every  $X \in \mathfrak{k}$  and  $(\Phi_{\lambda}(m), X) = ([x_m, y_m], [X, m])_M = -B([x_m, y_m], \theta[X, m])$ . When  $X \in \mathfrak{k}$ , the  $\mathfrak{p}$ -part of  $\theta[X, m]$  is  $-[X, y_m]$  and hence  $-B([x_m, y_m], \theta[X, m]) =$

$B([x_m, y_m], [X, y_m]) = -B([x_m, y_m], y_m, X)$ . Finally we see that  $\Phi_{\lambda_K} : M \rightarrow \mathfrak{k}$  is defined by

$$\Phi_{\lambda_K}(m) = [[x_m, y_m], y_m] \in \mathfrak{k} \quad \text{for every } m \in M.$$

The equality  $(\Phi_{\lambda_K}(m), \mu_K)(m) = \|[x_m, y_m]\|^2$  shows that  $\{\Phi_{\lambda_K} = 0\} = \{m \in M \mid [x_m, y_m] = 0\} = \text{Cr}(\|\mu_K\|^2)$ , and we know from the last sub-section that the set  $\text{Cr}(\|\mu_K\|^2)$  is a  $K$ -orbit.

We work now with the following data. Take a point  $\beta \in \text{Cr}(\|\mu_K\|^2)$ . Then the manifold  $M$  is of the form  $M = G.\beta$  with  $\beta = \beta_0 + \beta_1$ ,  $\beta_0 \in \mathfrak{k}$ ,  $\beta_1 \in \mathfrak{p}$ ,  $[\beta_0, \beta_1] = 0$ , and  $\{\Phi_{\lambda_K} = 0\} = K.\beta$ . If we make the choice of a Weyl chamber  $\mathfrak{t}^+$  in  $\mathfrak{k}$ , we can take  $\beta_0 \in \mathfrak{t}^+$ .

Lemma 3.1 of [16] tells us that the equivariant form  $\mathcal{D}\lambda_K(X) = d\lambda_K - (\Phi_{\lambda_K}, X)$  is invertible outside the submanifold  $K.\beta$  in the space of generalized equivariant forms. For each  $K$ -invariant differential form  $\chi_{ext}$  on  $M$ , equal to zero in a neighbourhood of  $K.\beta$ , we can define  $\chi_{ext} \left( \int_0^\infty \iota e^{-t\mathcal{D}\lambda_K} dt \right) \in \mathcal{A}_K^{-\infty}(M)$ , and this form satisfies the equality

$$\chi_{ext} \left( \int_0^\infty \iota e^{-t\mathcal{D}\lambda_K} dt \right) \mathcal{D}\lambda_K = \chi_{ext}$$

in  $\mathcal{A}_K^{-\infty}(M)$ .

Let  $\chi \in \mathcal{C}^\infty(M)^K$  be a function with *compact support* on  $M$ , and equal to 1 in a neighbourhood of  $K.\beta$ . Then the differential form  $d\chi$  is equal to 0 in a neighbourhood of  $K.\beta$ , and we can define the equivariant form with *compact support* on  $M$

$$P^K = \chi + d\chi \left( \int_0^\infty \iota e^{-t\mathcal{D}\lambda_K} dt \right) \lambda_K \in \mathcal{A}_{K, cpt}^{-\infty}(M). \quad (2.8)$$

Recall Propositions 3.3 and 3.11 of [16].

**Proposition 2.4** *The equivariant form  $P^K$  is closed, and we have the identity*

$$\mathbf{1}_M = P^K + \mathcal{D}(\delta), \quad (2.9)$$

where  $\mathbf{1}_M$  is the constant function equal to 1 on  $M$ , and  $\delta = (1-\chi) \left( \int_0^\infty \iota e^{-t\mathcal{D}\lambda_K} dt \right) \lambda_K$  is a generalized  $K$ -equivariant form. Moreover the cohomology class of  $P^K$  in  $\mathcal{H}_{K, cpt}^{-\infty}(M)$  does not depend neither of the choice of the function  $\chi$  nor of the choice of the Riemannian metric near  $K.\beta$ .

We use the phrase “partition of unity” to refer to the equality (2.9), and we will use it to decompose every closed form  $\eta \in \mathcal{A}_K^\infty(M)$  in the following way  $\eta = \eta P^K + \mathcal{D}(\eta\delta)$ ; a sum of an equivariant form  $\eta P^K$  supported near  $K.\beta$  and an exact equivariant form  $\mathcal{D}(\eta\delta)$ .

With the compactly supported equivariant form  $P^K$  we are going to localize the Fourier transform of  $M$ . The generalized equivariant form  $e^{i\Omega_t} P^K$  is *tempered*. For every  $f \in \mathcal{C}_{dr}^\infty(\mathfrak{k})$ , the differential form  $\langle e^{i\Omega_t(X)} P^K(X), f(X) dX \rangle_{\mathfrak{k}}$  is well defined, with compact support (included in the support of  $\chi$ ), and is given

$$\begin{aligned} \langle e^{i\Omega_t(X)} P^K(X), f(X) dX \rangle_{\mathfrak{k}}|_m &= \chi(m) e^{i\Omega_m} \widehat{f}(-\mu_K(m)) + \\ & d\chi(m) e^{i\Omega_m} \left( \int_0^\infty \iota e^{-t\lambda_K|_m} \widehat{f}(-t\Phi_{\lambda_K}(m) - \mu_K(m)) dt \right) \lambda_K|_m, \end{aligned} \quad (2.10)$$

where  $\widehat{f}$  is the Fourier transform of  $f$  relatively to  $dX$ . Note that the map  $m \rightarrow \int_0^\infty \iota e^{-t\lambda_K|_m} \widehat{f}(-t\Phi_{\lambda_K}(m) - \mu_K(m)) dt$  is a well defined differential form on  $M \setminus \{K.\beta\}$ .

The integral of  $e^{i\Omega_t} P^K$  on  $M$  defines a  $K$ -invariant tempered measure on  $\mathfrak{k}$  by the equation

$$\left\langle \left( \int_M e^{i\Omega_t} P^K \right) (X), f(X) dX \right\rangle_{\mathfrak{k}} := \int_M \langle e^{i\Omega_t(X)} P^K(X), f(X) dX \rangle_{\mathfrak{k}}, \quad f \in \mathcal{C}_{rd}^\infty(\mathfrak{k}).$$

**Theorem 2.5** *We have the following equality of tempered measure on  $\mathfrak{k}$*

$$\int_M e^{i\Omega_t} = \int_M e^{i\Omega_t} P^K.$$

*Proof:* For every *tempered measure*  $\mathfrak{D}$  on  $\mathfrak{k}$  we have the following property: if for every function  $f \in \mathcal{C}_{cpt}^\infty(\mathfrak{k})$  with compact support we have  $\langle \mathfrak{D}(X), \widehat{f}(X) dX \rangle_{\mathfrak{k}} = 0$  (where  $\widehat{f}$  is the Fourier transform of  $f$  relatively to  $dX$ ), then the measure  $\mathfrak{D}$  is *identically equal to zero*.

Here we take  $\mathfrak{D} := \int_M e^{i\Omega_t} - \int_M e^{i\Omega_t} P^K$ . Using now the partition of unity (2.9), we see that  $\langle \mathfrak{D}(X), \widehat{f}(X) dX \rangle_{\mathfrak{k}} = \int_M A_f$  with  $A_f = \langle \mathcal{D}(e^{i\Omega_t} \delta)(X), \widehat{f}(X) dX \rangle_{\mathfrak{k}} \in \mathcal{A}^*(M)$ . The theorem will be proved after showing that, for every function  $f \in \mathcal{C}_{cpt}^\infty(\mathfrak{k})$ , there exists a *compactly supported* differential form  $B_f$  on  $M$  such that  $A_f - d(B_f) \in \mathcal{A}^{<\dim M}(M)$ : the usual ‘Stokes’ argument implies that  $\int_M A_f = \int_M d(B_f) = 0$

By definition of the differential  $\mathcal{D}$ , we have  $A_f = \langle \mathcal{D}(e^{i\Omega_t} \delta)(X), \widehat{f}(X) dX \rangle_{\mathfrak{k}} = d(\langle e^{i\Omega_t(X)} \delta(X), \widehat{f}(X) dX \rangle_{\mathfrak{k}}) - \sum_{k=1}^p c(E_M^k) \langle e^{i\Omega_t(X)} \delta(X), X_k \widehat{f}(X) dX \rangle_{\mathfrak{k}}$ , and we take  $B_f := \langle e^{i\Omega_t(X)} \delta(X), \widehat{f}(X) dX \rangle_{\mathfrak{k}}$ . For every  $m \in M$  we have

$$B_f|_m = (2\pi)^{\dim K} (1 - \chi(m)) e^{i\Omega_m} \left( \int_0^\infty \iota e^{-t\lambda_K|_m} f(t\Phi_{\lambda_K}(m) + \mu_K(m)) dt \right) \lambda_K|_m, \quad (2.11)$$

since  $\widehat{f}(-X) = (2\pi)^{\dim K} f(X)$ ,  $X \in \mathfrak{k}$ . From (2.11), we see that  $B_f|_m = 0$  if  $t\Phi_{\lambda_K}(m) + \mu_K(m)$  is not in the support of  $f$  for every  $t > 0$ . But

$$\|t\Phi_{\lambda_K} + \mu_K\|^2 = \|\mu_K\|^2 + t^2 \|\Phi_{\lambda_K}\|^2 + 2t(\Phi_{\lambda_K}, \mu_K) \geq \|\mu_K\|^2$$

because  $(\Phi_{\lambda_K}, \mu_K) = \|\mathcal{H}\|_M^2 \geq 0$  on  $M$ . Let  $r > 0$  such that the function  $f$  is supported in the ball  $B(O, r)$  of radius  $r$ . Using the precedent inequalities we see finally that the differential form  $B_f$  is supported in  $\mu_K^{-1}(B(O, r))$ , hence is with compact support ( $\mu_K$  being proper).  $\square$

### 3 First reduction: Symplectic induction

In this subsection, we prove an induction formula for the equivariant form  $P^K$  similar to the induction we obtain in section 3 of [17]. For this purpose we use the *cross section Theorem* of Guillemin-Sternberg [10] (see Theorem 6.7), and also a Proposition of Duflo-Vergne [8] about generalized equivariant forms that admit a *restriction*.

Let  $T$  be a maximal torus of  $K$  with Lie algebra  $\mathfrak{t}$ , and let  $W := W(K, T)$  be the Weyl group associated. We make the choice of a Weyl chamber  $\mathfrak{t}_+$  in  $\mathfrak{t}$ . Recall that a closed adjoint orbit  $M$  in  $\mathfrak{g}$  is of the form  $M = G.\beta$  with  $\beta = \beta_0 + \beta_1$ ,  $\beta_0 \in \mathfrak{t}_+$ ,  $\beta_1 \in \mathfrak{p}$  and  $[\beta_0, \beta_1] = 0$ , and that the cohomology class of  $P^K$  does not depend on the choice of the  $K$ -invariant Riemannian metric in a neighbourhood of  $K.\beta$ .

#### Symplectic cross-section

Let  $\sigma$  be the unique open face of  $\mathfrak{t}_+$  which contains  $\beta_0$ . The stabilizer subgroup  $K_\xi \subset K$  that does not depend on the choice of  $\xi \in \sigma$  is denoted  $K_\sigma$ , and we denote by  $\mathfrak{k}_\sigma$  its Lie algebra (sometimes we use the different notations  $K_{\beta_0}$ ,  $\mathfrak{k}_{\beta_0}$ ).

Let  $U_\sigma$  be the  $K_\sigma$ -invariant open subset of  $\mathfrak{k}_\sigma$  defined by

$$U_\sigma := K_\sigma.\{y \in \mathfrak{t}_+ \mid K_y \subset K_\sigma\} = K_\sigma.\bigcup_{\sigma \subset \bar{\tau}} \tau, \quad (3.12)$$

where  $\{\sigma \subset \bar{\tau}\}$  is the set of all faces  $\tau$  of  $\mathfrak{t}_+$  which contain  $\sigma$  in their closure. By construction,  $U_\sigma$  is a *slice* for the adjoint action at any  $\xi \in \sigma$  (see Definition 3.1 of [13]). This means that the map  $K \times U_\sigma \rightarrow \mathfrak{k}$ ,  $(g, \xi) \rightarrow g.\xi$ , factors through an inclusion  $K \times_{K_\sigma} U_\sigma \hookrightarrow \mathfrak{k}$ .

The *symplectic cross-section theorem* [10] asserts that the pre-image  $\mathcal{Y}_\sigma = \mu_K^{-1}(U_\sigma)$  is a symplectic submanifold provided with an Hamiltonian action of the group  $K_\sigma$ . The restriction  $\mu_K|_{\mathcal{Y}_\sigma}$  is a moment map for the action of  $K_\sigma$  on  $\mathcal{Y}_\sigma$  that we denote by  $\mu_\sigma$ . In our case we just need the fact that  $\mathfrak{k}_\sigma \oplus \mathfrak{p}$  intersects transversally  $M$  in a neighbourhood of  $K_\sigma.\beta \subset \mathcal{Y}_\sigma \subset M \cap (\mathfrak{k}_\sigma \oplus \mathfrak{p})$ .

Moreover, the set  $K.\mathcal{Y}_\sigma$  is a  $K$ -invariant *open* neighbourhood of  $K.\beta$  in  $M$  diffeomorphic to  $K \times_{K_\sigma} \mathcal{Y}_\sigma$ . Then, we can compute the equivariant form  $P^K$  on the manifold  $\mathcal{M}_\sigma := K \times_{K_\sigma} \mathcal{Y}_\sigma$ . We will denote by  $\Omega_\sigma(Y) := \Omega|_{\mathcal{Y}_\sigma} + (\mu_\sigma, Y)$ ,  $Y \in \mathfrak{k}_\sigma$ , the corresponding  $K_\sigma$ -equivariant symplectic form on  $\mathcal{Y}_\sigma$ .

#### Induced metric on $\mathcal{M}_\sigma$

The quotient  $\mathfrak{k}/\mathfrak{k}_\sigma$  is identified with the orthogonal complement  $\mathfrak{k}_\sigma^\perp$  of  $\mathfrak{k}_\sigma$  in  $\mathfrak{k}$ . We denote respectively by  $pr_{\mathfrak{k}/\mathfrak{k}_\sigma}$  and  $pr_{\mathfrak{k}_\sigma}$  the orthogonal projections  $\mathfrak{k} \rightarrow \mathfrak{k}_\sigma^\perp$  and  $\mathfrak{k} \rightarrow \mathfrak{k}_\sigma$ . From now on we fix a  $K_\sigma$ -invariant metric  $(\cdot, \cdot)_{\mathcal{Y}_\sigma}$  on  $\mathcal{Y}_\sigma$  and we

associate to this metric a natural  $K$ -invariant Riemannian metric  $(\cdot, \cdot)_{\mathcal{M}_\sigma}$  on  $\mathcal{M}_\sigma := K \times_{K_\sigma} \mathcal{Y}_\sigma$  which is defined by

$$\|X_{\mathcal{M}_\sigma} + v_m\|_{\mathcal{M}_\sigma}^2 [k, m] := \|pr_{\mathfrak{k}/\mathfrak{k}_\sigma}(k^{-1}X)\|_{\mathfrak{k}}^2 + \|pr_{\mathfrak{k}_\sigma}(k^{-1}X)|_{\mathcal{Y}_\sigma}|_m + v_m\|_{\mathcal{Y}_\sigma}^2, \quad (3.13)$$

for  $X \in \mathfrak{k}$  and  $v_m \in \mathbf{T}_m M$ . Here we use the identification between  $\mathbf{T}_{[k, m]} \mathcal{M}_\sigma$  and  $(\mathbf{T}_g K \times \mathbf{T}_m \mathcal{Y}_\sigma) / \sim$ , ( $\sim$  is the relation of equivalence coming from the  $K_\sigma$ -orbits in  $K \times \mathcal{Y}_\sigma$ ).

Let  $\lambda_\sigma := (\mathcal{H}_\sigma, \cdot)_{\mathcal{Y}_\sigma}$  be the  $K_\sigma$ -invariant 1-form on  $\mathcal{Y}_\sigma$ , where  $\mathcal{H}_\sigma$  is the Hamiltonian vector field of  $\frac{1}{2}\|\mu_\sigma\|^2$ . A straightforward computation shows that  $\text{Cr}(\|\mu_\sigma\|^2) = \text{Cr}(\|\mu_K\|^2) \cap \mathcal{Y}_\sigma = K_\sigma \cdot \beta$ . Let  $\tilde{\lambda}_K := (\mathcal{H}, \cdot)_{\mathcal{M}_\sigma}$  be the  $K$ -invariant 1-form on  $\mathcal{M}_\sigma$  defined with the induced metric  $(\cdot, \cdot)_{\mathcal{M}_\sigma}$ , where  $\mathcal{H}$  is still the Hamiltonian vector field of  $\frac{1}{2}\|\mu_K\|^2$ .

Using the definition of the induced metric  $(\cdot, \cdot)_{\mathcal{M}_\sigma}$  we see that  $i_\sigma^*(\tilde{\lambda}_K) = \lambda_\sigma$ , where  $i_\sigma : \mathcal{Y}_\sigma \hookrightarrow \mathcal{M}_\sigma$  denotes the  $K_\sigma$ -equivariant inclusion, and we also remark that

$$\left( \Phi_{\tilde{\lambda}_K}([k, m]), X \right)_\mathfrak{k} = \left( \Phi_{\lambda_{K_\sigma}}(m), pr_{\mathfrak{k}_\sigma}(k^{-1}X) \right)_\mathfrak{k}, \quad X \in \mathfrak{k}, [k, m] \in \mathcal{M}_\sigma, \quad (3.14)$$

where  $pr_{\mathfrak{k}_\sigma} : \mathfrak{k} \rightarrow \mathfrak{k}_\sigma$  is the orthogonal projection. We will use these facts at Proposition 3.4.

**Definition 3.1** *Let  $\chi \in \mathcal{C}_{cpt}^\infty(\mathcal{M}_\sigma)^K$  be the function coming from a  $K_\sigma$ -invariant function  $\chi_\sigma$  on  $\mathcal{Y}_\sigma$ , where  $\chi_\sigma$  is equal to 1 in a neighbourhood of  $\text{Cr}(\|\mu_\sigma\|^2) = K_\sigma \cdot \beta$ . Then the function  $\chi$ , that is equal to 1 in a neighbourhood of  $\text{Cr}(\|\mu_K\|^2) = K \times_{K_\sigma} (K_\sigma \cdot \beta)$ , defines the  $K$ -equivariant form*

$$\tilde{\mathbf{P}}^K = \chi + d\chi \left( \int_0^\infty ie^{-it} \mathcal{D}\tilde{\lambda}_K dt \right) \tilde{\lambda}_K \in \mathcal{A}_{K, cpt}^{-\infty}(\mathcal{M}_\sigma).$$

With the function  $\chi_\sigma$  we define in the same way the  $K_\sigma$ -equivariant form

$$\mathbf{P}^{K_\sigma} = \chi_\sigma + d\chi_\sigma \left( \int_0^\infty ie^{-it} \mathcal{D}\lambda_\sigma dt \right) \lambda_\sigma \in \mathcal{A}_{K_\sigma, cpt}^{-\infty}(\mathcal{Y}_\sigma).$$

The inclusion  $\mathcal{M}_\sigma \hookrightarrow M$  of an open subset defines a natural morphism  $\mathcal{A}_{cpt}^*(\mathcal{M}_\sigma) \rightarrow \mathcal{A}_{cpt}^*(M)$  between the differential forms with compact support, and hence a morphism  $j : \mathcal{A}_{K, cpt}^{-\infty}(\mathcal{M}_\sigma) \rightarrow \mathcal{A}_{K, cpt}^{-\infty}(M)$  between the  $K$ -equivariant forms with compact support. We know from Proposition 2.4, that

$$\mathbf{P}^K = j \left( \tilde{\mathbf{P}}^K \right) \quad \text{in } \mathcal{H}_{K, cpt}^{-\infty}(M).$$

Let  $\pi : \mathcal{M}_\sigma \rightarrow K/K_\sigma$ ,  $[k, y] \mapsto [k]$  be the projection map, and denote by  $\int_{\text{Fiber}} := \pi_* : \mathcal{A}_{K, cpt}^{-\infty}(\mathcal{M}_\sigma) \rightarrow \mathcal{A}_K^{-\infty}(K/K_\sigma)$  the morphism of integration along the fiber of  $\pi$ . Then we have

$$\begin{aligned} F_M|_\mathfrak{k} &= \int_{\mathcal{M}_\sigma} e^{i\Omega_\mathfrak{k}} \tilde{\mathbf{P}}^K \\ &= \int_{K/K_\sigma} \left( \int_{\text{Fiber}} e^{i\Omega_\mathfrak{k}} \tilde{\mathbf{P}}^K \right). \end{aligned} \quad (3.15)$$

In the next paragraph we are going to show that the ‘computation’ of the closed equivariant form  $\int_{\text{Fiber}} e^{i\Omega_{\mathfrak{k}}} \tilde{\mathbf{P}}^K \in \mathcal{A}_K^{-\infty}(K/K_\sigma)$  can be deduced from those of the generalized function  $\int_{\mathcal{Y}_\sigma} e^{i\Omega_{\mathfrak{k}_\sigma}} \mathbf{P}^{K_\sigma} \in \mathcal{C}^{-\infty}(\mathfrak{k}_\sigma)^{K_\sigma}$ .

### Restriction of generalized equivariant forms

The manifold  $\mathcal{Y}_\sigma$  is oriented by its symplectic form, and  $K \times_{K_\sigma} \mathcal{Y}_\sigma$  is oriented by the symplectic form on  $M$ . Hence, we get an orientation  $o$  for  $K/K_\sigma$  and we verify that this orientation coincides with those of Definition 1.3. This gives a polynomial square root  $Y \rightarrow \Pi_{\mathfrak{k}/\mathfrak{k}_\sigma}(Y) := \det_{\mathfrak{k}/\mathfrak{k}_\sigma, o}^{1/2}(Y)$ ,  $Y \in \mathfrak{k}_\sigma$  (we will use also the notation  $\Pi_{\mathfrak{k}/\mathfrak{k}_{\beta_0}}$  for this polynomial). Note that  $\Pi_{\mathfrak{k}/\mathfrak{k}_\sigma}$  never vanishes on the open subset  $U_\sigma$ .

We have a natural identification between  $\mathcal{A}^*(K/K_\sigma)$  and  $\mathcal{C}^\infty\left(K, (\wedge \mathfrak{k}^*)_{\text{hor}K_\sigma}\right)^{K_\sigma}$ , where  $K_\sigma$ -invariants are taken with respect to the action of  $K_\sigma$  by right multiplication on  $K$  and coadjoint action on  $\mathfrak{k}^*$ . The restriction map to the neutral element  $e \in K$ ,  $\mathcal{A}^*(K/K_\sigma) \rightarrow (\wedge \mathfrak{k}^*)_{\text{hor}K_\sigma}$ ,  $\alpha \rightarrow \alpha_e$ , defines a morphism

$$\begin{aligned} \mathcal{A}_K^{-\infty}(K/K_\sigma) &\longrightarrow \mathcal{C}^{-\infty}\left(\mathfrak{k}, (\wedge \mathfrak{k}^*)_{\text{hor}K_\sigma}\right)^{K_\sigma} \\ \alpha(X) &\longmapsto \alpha_e(X). \end{aligned}$$

Let  $E_1, \dots, E_p$  be a basis of  $\mathfrak{k}^*$ , and let  $\{E_I = E_{i_1} \wedge \dots \wedge E_{i_k}, I = [i_1 < i_2 < \dots < i_k] \subset [1, 2, \dots, p]\}$  be the corresponding basis of  $\wedge \mathfrak{k}^*$ . In particular,  $E_\emptyset = 1$  generates  $\mathbb{R} \subset \wedge \mathfrak{k}^*$ . For each  $\alpha \in \mathcal{A}_K^{-\infty}(K/K_\sigma)$ , the form  $\alpha_e$  can be decomposed relatively to the basis  $\{E_I, I\}$ :  $\alpha_e = \sum_I (\alpha_e)_{[I]} E_I$  with  $(\alpha_e)_{[I]} \in \mathcal{C}^{-\infty}(\mathfrak{k})$ .

We say that  $\alpha$  admits a restriction to  $\mathfrak{k}_\sigma$  if the wave front set of each component  $(\alpha_e)_{[I]}$  is transverse to  $\mathfrak{k}_\sigma$  (see [8] for this notion). In this case, each generalized function  $(\alpha_e)_{[I]}$  admits a restriction to  $\mathfrak{k}_\sigma$ . We can then define  $r_{\mathfrak{k}_\sigma} \alpha := (\alpha_e)_{[\emptyset]}|_{\mathfrak{k}_\sigma}$  that is a generalized  $K_\sigma$ -invariant function on  $\mathfrak{k}_\sigma$ . This definition extends the usual restriction map  $r_{\mathfrak{k}_\sigma} : \mathcal{A}_K^\infty(K/K_\sigma) \rightarrow \mathcal{C}^\infty(\mathfrak{k}_\sigma)^{K_\sigma}$ .

**Remark 3.2** *There is a basic way to know if  $\alpha \in \mathcal{A}_K^{-\infty}(K/K_\sigma)$  admits a restriction to  $\mathfrak{k}_\sigma$ . Suppose there exist  $\alpha^a \in \mathcal{A}_K^\infty(K/K_\sigma)$ ,  $a > 0$ , such that*

- $\lim_{a \rightarrow \infty} \alpha^a = \alpha$ , and
- the restriction  $\alpha_e^a|_{\mathfrak{k}_\sigma} \in \mathcal{C}^\infty(\mathfrak{k}_\sigma, \wedge \mathfrak{k}^*)$  converges in  $\mathcal{C}^{-\infty}(\mathfrak{k}_\sigma, \wedge \mathfrak{k}^*)$  when  $a \rightarrow \infty$ . In particular  $r_{\mathfrak{k}_\sigma} \alpha^a \in \mathcal{C}^\infty(\mathfrak{k}_\sigma)^{K_\sigma}$  converges in  $\mathcal{C}^{-\infty}(\mathfrak{k}_\sigma)^{K_\sigma}$  when  $a \rightarrow \infty$ .

*Then the equivariant form  $\alpha$  admits a restriction to  $\mathfrak{k}_\sigma$  and we have  $r_{\mathfrak{k}_\sigma} \alpha = \lim_{a \rightarrow \infty} r_{\mathfrak{k}_\sigma} \alpha^a$ .*

Recall Proposition 31 of [8].

**Proposition 3.3** *Let  $\alpha \in \mathcal{A}_K^{-\infty}(K/K_\sigma)$  be a closed equivariant form that admits a restriction to  $\mathfrak{k}_\sigma$ . Then for every  $f \in \mathcal{C}_{\text{cpt}}^\infty(\mathfrak{k})$ , we have*

$$\left\langle \left( \int_{K/K_\sigma} \alpha \right)(X), f(X) dX \right\rangle_{\mathfrak{k}} = \text{cte} \left\langle r_{\mathfrak{k}_\sigma} \alpha(Y), \Pi_{\mathfrak{k}/\mathfrak{k}_\sigma}(Y) \overline{f}^K|_{\mathfrak{k}_\sigma}(Y) dY \right\rangle_{\mathfrak{k}_\sigma},$$

with  $\text{cte} = (-2\pi)^{\dim(K/K_\sigma)/2} \frac{\text{vol}(K, dX)}{\text{vol}(K_\sigma, dY)}$ .

We can now state the following

**Lemma 3.4** *The closed  $K$ -equivariant form  $\int_{\text{Fiber}} e^{i\Omega_t} \tilde{\mathbf{P}}^K \in \mathcal{A}_K^{-\infty}(K/K_\sigma)$  admits a restriction to  $\mathfrak{k}_\sigma$  equal to  $\int_{\mathcal{Y}_\sigma} e^{i\Omega_{\mathfrak{k}_\sigma}} \mathbf{P}^{K_\sigma} \in \mathcal{C}^{-\infty}(\mathfrak{k}_\sigma)^{K_\sigma}$ .*

*Proof:* Here we are in the situation of Remark 3.2. We know already that  $\tilde{\mathbf{P}}^K$  is the limit of the equivariant form  $\tilde{\mathbf{P}}_a^K := \chi + d\chi \left( \int_0^a i e^{-itD\tilde{\lambda}_K} dt \right) \tilde{\lambda}_K \in \mathcal{A}_{K,\text{cpt}}^\infty(\mathcal{M}_\sigma)$ , and so  $\alpha := \int_{\text{Fiber}} e^{i\Omega_t} \tilde{\mathbf{P}}^K$  is the limit of the equivariant forms  $\alpha^a := \int_{\text{Fiber}} e^{i\Omega_t} \tilde{\mathbf{P}}_a^K \in \mathcal{A}_K^\infty(K/K_\sigma)$ . Consider the maps  $\alpha_e^a|_{\mathfrak{k}_\sigma} \in \mathcal{C}^\infty(\mathfrak{k}_\sigma, \wedge \mathfrak{k}^*)^{K_\sigma}$ . We have

$$\begin{aligned} \alpha_e^a|_{\mathfrak{k}_\sigma}(Y) &= \int_{\mathcal{Y}_\sigma} e^{i\Omega_e + i(\mu_\sigma, Y)} \tilde{\mathbf{P}}_{a,e}^K, \quad \text{with} \\ \tilde{\mathbf{P}}_{a,e}^K &= \chi_\sigma + d\chi_\sigma \left( \int_0^a i e^{-itd\tilde{\lambda}_K|_e} e^{it(\Phi_{\lambda_\sigma}, Y)} dt \right) \tilde{\lambda}_K|_e, \end{aligned}$$

where  $\Omega_e$ ,  $\tilde{\lambda}_K|_e$  and  $d\tilde{\lambda}_K|_e$  are in  $\wedge \mathfrak{k}^* \otimes \mathcal{A}^*(\mathcal{Y}_\sigma)$ . In these equalities we have used that  $(\Phi_{\tilde{\lambda}_K}|_e, Y)_\mathfrak{k} = (\Phi_{\lambda_\sigma}, Y)_\mathfrak{k}$  for every  $Y \in \mathfrak{k}_\sigma$  (see equation (3.14)). Since  $d\chi_\sigma$  is equal to 0 in a neighbourhood of  $\{\Phi_{\lambda_\sigma} = 0\}$ , we see that  $\alpha_e^a|_{\mathfrak{k}_\sigma}$  converge in  $\mathcal{C}^{-\infty}(\mathfrak{k}_\sigma, \wedge \mathfrak{k}^*)^{K_\sigma}$ , when  $a \rightarrow \infty$ .

If  $i_\sigma : \mathcal{Y}_\sigma \hookrightarrow \mathcal{M}_\sigma$  denotes the  $K_\sigma$ -equivariant inclusion, we have  $i_\sigma^*(\tilde{\lambda}_K) = \lambda_\sigma$ , and  $i_\sigma^*(\Omega) = \Omega_\sigma$  (see equation (3.14)). Then we see that the functions  $r_{\mathfrak{k}_\sigma} \alpha^a = \int_{\mathcal{Y}_\sigma} e^{i\Omega_{\mathfrak{k}_\sigma}} \mathbf{P}_a^{K_\sigma}$  with  $\mathbf{P}_a^{K_\sigma} = \chi_\sigma + d\chi_\sigma \left( \int_0^a i e^{-itD\lambda_\sigma} dt \right) \lambda_\sigma$  converge to  $\int_{\mathcal{Y}_\sigma} e^{i\Omega_{\mathfrak{k}_\sigma}} \mathbf{P}^{K_\sigma}$  when  $a \rightarrow \infty$ . The proof is completed.  $\square$

**Proposition 3.5** *We have the following description of the generalized function  $F_M|_{\mathfrak{k}}$ . For every function  $f \in \mathcal{C}_{\text{rd}}^\infty(\mathfrak{k})$ , we have*

$$\langle F_M|_{\mathfrak{k}}(X), f(X) dX \rangle_{\mathfrak{k}} = \text{cte} \langle \left( \int_{\mathcal{Y}_\sigma} \mathbf{P}^{K_\sigma} \right)(Y), e^{i(\beta_0, Y)} \Pi_{\mathfrak{k}/\mathfrak{k}_\sigma}(Y) \bar{f}^K|_{\mathfrak{k}_\sigma}(Y) dY \rangle_{\mathfrak{k}_\sigma},$$

$$\text{with } \text{cte} = (-2\pi)^{\dim(K/K_\sigma)/2} \frac{\text{vol}(K, dX)}{\text{vol}(K_\sigma, dY)}.$$

*Proof:* Let  $\mathcal{U}$  be a  $K_\sigma$ -invariant tubular neighbourhood of  $K_\sigma \cdot \beta$  in  $\mathcal{Y}_\sigma$ , and denote by  $p : \mathcal{U} \rightarrow K_\sigma \cdot \beta$  the corresponding fibration. We denote by  $i : K_\sigma \cdot \beta \hookrightarrow \mathcal{U}$  the 0-section of this fibration. We need to compute the restriction  $i^*(\Omega_{\mathfrak{k}_\sigma})$  of the equivariant form  $\Omega_{\mathfrak{k}_\sigma}$  on  $K_\sigma \cdot \beta$ . First we note that the inclusion  $K_\sigma \cdot \beta \hookrightarrow \mathcal{Y}_\sigma$  is an *isotropic* embedding, because  $\beta_0 = \mu_\sigma(K_\sigma \cdot \beta)$  is fixed by  $K_\sigma$ : for every  $X, X' \in \mathfrak{k}_\sigma$ , and  $m \in K_\sigma \cdot \beta$  we have  $\Omega_\sigma|_m(X_{\mathcal{Y}_\sigma}, X'_{\mathcal{Y}_\sigma}) = -(\beta_0, [X, X']) = 0$ . Then  $i^*(\Omega_\sigma) = 0$  and we have also  $i^*(\mu_\sigma) = \beta_0$ . Finally we get  $i^*(\Omega_{\mathfrak{k}_\sigma})(Y) = (\beta_0, Y)$ ,  $Y \in \mathfrak{k}_\sigma$ .

The equivariant form  $\mathbf{P}^{K_\sigma}$  can be taken with (compact) support in  $\mathcal{U}$ , and we have

$$\begin{aligned} \left( \int_{\mathcal{Y}_\sigma} e^{i\Omega_{\mathfrak{k}_\sigma}} \mathbf{P}^{K_\sigma} \right)(Y) &= \int_{K_\sigma \cdot \beta} i^*(e^{i\Omega_{\mathfrak{k}_\sigma}})(Y) p_* \left( \mathbf{P}^{K_\sigma} \right)(Y) \\ &= e^{i(\beta_0, Y)} \left( \int_{\mathcal{Y}_\sigma} \mathbf{P}^{K_\sigma} \right)(Y). \end{aligned}$$

With this equality, the Proposition is just a consequence of Proposition 3.3, Lemma 3.4, and the equality (3.15).  $\square$

## 4 Second reduction: Deformation procedure

We are now going to compute the generalized function  $\int_{\mathcal{Y}_\sigma} \mathbf{P}^{K_\sigma}$  by means of a deformation procedure (see Proposition 3.11 of [16] and section 2.3 of [17]).

We first recall briefly the context of our computation. Let  $M$  be a semi-simple orbit of the adjoint representation of a real semi-simple Lie group  $G$ . We have shown that  $M = G.\beta$  with  $\beta = \beta_0 + \beta_1$ ,  $\beta_0 \in \mathfrak{k}$ ,  $\beta_1 \in \mathfrak{p}$  and  $[\beta_0, \beta_1] = 0$ . In the last section we have denote by  $\mathfrak{k}_\sigma$  the subalgebra of  $\mathfrak{k}$  fixed by the adjoint action of  $\beta_0$ . We have also defined a symplectic  $K_\sigma$ -manifold  $\mathcal{Y}_\sigma$  which is an open neighbourhood of  $K_\sigma.\beta$  in  $M \cap (\mathfrak{k}_\sigma \oplus \mathfrak{p})$ . The action of  $K_\sigma$  is Hamiltonian and the moment map  $\mu_\sigma : \mathcal{Y}_\sigma \rightarrow \mathfrak{k}_\sigma$  is the restriction to  $\mathcal{Y}_\sigma$  of the orthogonal projection  $\mathfrak{g} \rightarrow \mathfrak{k}_\sigma$ .

The equivariant form  $\mathbf{P}^{K_\sigma}$  is supported in a (small) neighbourhood of  $K_\sigma.\beta$  in  $\mathcal{Y}_\sigma$ , and to compute it we just need to describe a  $K_\sigma$ -invariant neighbourhood of  $K_\sigma.\beta$  in  $\mathcal{Y}_\sigma$ .

We have already remarked that the inclusion  $K_\sigma.\beta \hookrightarrow \mathcal{Y}_\sigma$  is an *isotropic* embedding. In fact  $K_\sigma.\beta$  is the 0-level of the shifted ( $K_\sigma$ -invariant) moment map  $\mu_\sigma - \beta_0$ . Then to describe a  $K_\sigma$ -invariant neighbourhood of  $K_\sigma.\beta$  in  $\mathcal{Y}_\sigma$  we can use the normal-form recipe of Marle, Guillemin and Sternberg.

First we can form, following Weinstein (see [11, 23]), the symplectic normal bundle

$$\mathcal{V}_{\beta_0} := \mathbf{T}(K_\sigma.\beta)^{\perp, \Omega_\sigma} / \mathbf{T}(K_\sigma.\beta), \quad (4.16)$$

where the orthogonal ( $^{\perp, \Omega_\sigma}$ ) is taken relatively to the symplectic 2-form  $\Omega_\sigma$ . Let  $K_\beta$  be the subgroup of  $K_\sigma$  which stabilizes  $\beta$ , and let  $\mathfrak{k}_\beta$  be its Lie algebra ( $K_\beta$  is the subgroup of point  $k \in K$  such that  $k.\beta_0 = \beta_0$  and  $k.\beta_1 = \beta_1$ ). We have

$$\mathcal{V}_{\beta_0} = K_\sigma \times_{K_\beta} V_{\beta_0}$$

where the vector space  $V_{\beta_0} := \mathbf{T}_\beta(K_\sigma.\beta)^{\perp, \Omega_\sigma} / \mathbf{T}_\beta(K_\sigma.\beta)$  inherits a symplectic structure and an Hamiltonian action of the group  $K_\beta$ .

**Lemma 4.1** *The vector space  $V_{\beta_0}$  is equal to  $[\mathfrak{p}, \beta_0] \cong \mathfrak{p}/\mathfrak{p}_{\beta_0}$  and is equipped with the natural  $K_\beta$ -action. The moment map  $\mu_\beta : V_{\beta_0} \rightarrow \mathfrak{k}_\beta$  associated to this action verifies*

$$\mu_\beta(y) = -\frac{1}{2} pr_{\mathfrak{k}_\beta} \left( [y, [y, \beta_0]] \right), \quad y \in [\mathfrak{p}, \beta_0],$$

where  $pr_{\mathfrak{k}_\beta} : \mathfrak{k}_\sigma \rightarrow \mathfrak{k}_\beta$  denotes the orthogonal projection. Moreover,  $\mu_\beta(y) = 0$  iff  $y = 0$ .

*Proof:* The vector space  $\mathbf{T}_\beta M = [\mathfrak{g}, \beta]$  carries the symplectic form  $\Omega_\beta$  that is defined by the equation

$$\Omega_\beta([X, \beta], [Y, \beta]) := -B(\beta, [X, Y]), \quad \text{for } X, Y \in \mathfrak{g}, \quad (4.17)$$



where  $B$  is the Killing form on  $\mathfrak{g}$ . We see from (4.17) that the symplectic orthogonal  $[\mathfrak{k}_\sigma, \beta_1]^{\perp, \Omega_\sigma}$  of the vector space  $\mathbf{T}_\beta(K_\sigma \cdot \beta) = [\mathfrak{k}_\sigma, \beta_1]$  in  $\mathbf{T}_\beta M$  is equal to the subspace  $B$ -orthogonal to  $\mathfrak{k}_\sigma$ . Then we have  $V_{\beta_0} = (\mathfrak{k}_\sigma)^{\perp, B} \cap \mathbf{T}_\beta \mathcal{Y}_\sigma / [\mathfrak{k}_\sigma, \beta_1]$ .

The subspace  $[\mathfrak{g}, \beta]$  of  $\mathfrak{g}$  is  $\theta$ -stable because  $\ker(ad(\beta)) = \ker(ad(\theta\beta)) = \ker(ad(\beta_0)) \cap \ker(ad(\beta_1))$  is  $\theta$ -stable and  $[\mathfrak{g}, \beta] = \ker(ad(\beta))^{\perp, B}$ . Hence we get

$$\begin{aligned} \mathbf{T}_\beta \mathcal{Y}_\sigma &= [\mathfrak{g}, \beta] \cap (\mathfrak{k}_\sigma \oplus \mathfrak{p}) \\ &= \underbrace{[\mathfrak{g}, \beta] \cap \mathfrak{k}_\sigma}_{\subset \mathfrak{k}} \oplus \underbrace{[\mathfrak{g}, \beta] \cap \mathfrak{p}}_{\subset \mathfrak{p}}. \end{aligned} \quad (4.18)$$

The  $\mathfrak{k}$ -part of  $(\mathfrak{k}_\sigma)^{\perp, B} \cap \mathbf{T}_\beta \mathcal{Y}_\sigma$  is reduced to  $\{0\}$  because it is included in  $(\mathfrak{k}_\sigma)^{\perp, B} \cap \mathfrak{k}_\sigma = \{0\}$  (see (4.18)). Hence we have  $(\mathfrak{k}_\sigma)^{\perp, B} \cap \mathbf{T}_\beta \mathcal{Y}_\sigma = [\mathfrak{g}, \beta] \cap \mathfrak{p} = [\mathfrak{k}, \beta_1] + [\mathfrak{p}, \beta_0]$  because  $\mathfrak{p}$  is  $B$ -orthogonal to  $\mathfrak{k}_\sigma$ . Using the decomposition  $\mathfrak{k} = \mathfrak{k}_\sigma \oplus [\mathfrak{k}, \beta_0]$  we can finally write  $(\mathfrak{k}_\sigma)^{\perp, B} \cap \mathbf{T}_\beta \mathcal{Y}_\sigma = [\mathfrak{k}_\sigma, \beta_1] \oplus [\mathfrak{p}, \beta_0]$  (the last sum is direct because the two members are  $B$ -orthogonal and  $B$  is positive definite on  $\mathfrak{p}$ ). We proved finally that  $V_{\beta_0}$  is equal to  $[\mathfrak{p}, \beta_0]$ . The computation of  $\mu_\beta$  is left to the reader. For the last point, we remark that  $(pr_{\mathfrak{k}_\beta}(X), \beta_0)_\mathfrak{k} = (X, \beta_0)_\mathfrak{k}$  for every  $X \in \mathfrak{k}_\sigma$ , because  $\beta_0 \in \mathfrak{k}_\beta$ . It follows that  $(\mu_\beta(y), \beta_0)_\mathfrak{k} = \frac{1}{2} \| [y, \beta_0] \|^2$  for every  $y \in [\mathfrak{p}, \beta_0]$ , and this proves the last assertion.  $\square$

Consider now the following symplectic manifold

$$\tilde{\mathcal{Y}}_\sigma := \mathcal{V}_{\beta_0} \times \mathbf{T}(K_\sigma/K_\beta) = K_\sigma \times_{K_\beta} \left( \mathfrak{k}_\sigma/\mathfrak{k}_\beta \oplus \mathfrak{p}/\mathfrak{p}_{\beta_0} \right) \quad (4.19)$$

where the tangent bundle  $\mathbf{T}(K_\sigma/K_\beta)$  is here naturally identified with the cotangent bundle  $\mathbf{T}^*(K_\sigma/K_\beta)$  through the  $K_\sigma$ -invariant scalar product on  $\mathfrak{k}_\sigma/\mathfrak{k}_\beta$  coming from the scalar product on  $\mathfrak{k}$  (after identification of  $\mathfrak{k}_\sigma/\mathfrak{k}_\beta$  with the orthogonal complement of  $\mathfrak{k}_\beta$  in  $\mathfrak{k}_\sigma$ ). The action of  $K_\sigma$  on  $\tilde{\mathcal{Y}}_\sigma$  is Hamiltonian and the moment map  $\tilde{\mu}_\sigma : \tilde{\mathcal{Y}}_\sigma \rightarrow \mathfrak{k}_\sigma$  is given by the equation

$$\tilde{\mu}_\sigma([k; x, y]) = \beta_0 + k \cdot (x + \mu_\beta(y)) \quad k \in K_\sigma, \quad x \in \mathfrak{k}_\sigma/\mathfrak{k}_\beta, \quad y \in \mathfrak{p}/\mathfrak{p}_{\beta_0}. \quad (4.20)$$

The *local normal form* Theorem, applied to the moment map  $\mu_\sigma - \beta_0$ , (see [19] Proposition 2.5) tells us that there exists a  $K_\sigma$ -Hamiltonian isomorphism

$$\Upsilon : \mathcal{U}_1 \xrightarrow{\sim} \mathcal{U}_2,$$

where  $\mathcal{U}_1$  is a  $K_\sigma$ -invariant neighbourhood of  $K_\sigma \cdot \beta$  in  $\mathcal{Y}_\sigma$ , and  $\mathcal{U}_2$  is a  $K_\sigma$ -invariant neighbourhood of  $K_\sigma/K_\beta$  in  $\tilde{\mathcal{Y}}_\sigma$ . Furthermore the isomorphism, when restricted to  $K_\sigma \cdot \beta$ , corresponds to the natural isomorphism  $K_\sigma \cdot \beta \xrightarrow{\sim} K_\sigma/K_\beta$ .

Let  $\tilde{\mathcal{H}}_\sigma$  be the Hamiltonian vector field on  $\tilde{\mathcal{Y}}_\sigma$  of  $\frac{1}{2} \|\tilde{\mu}_\sigma\|^2$ . For every  $[k; x, y] \in \tilde{\mathcal{Y}}_\sigma$  we have

$$\begin{aligned} \tilde{\mathcal{H}}_\sigma|_{[k; x, y]} &= \left( k \cdot \tilde{\mu}_\sigma([1; x, y]) \right)_{\tilde{\mathcal{Y}}_\sigma}|_{[k; x, y]} \\ &= \left. \frac{d}{dt} \right|_{t=0} [k \cdot e^{-tX}; x, y], \quad \text{with } X = \beta_0 + x + \mu_\beta(y) \\ &= -x_{K_\sigma, r}(k) - [\beta_0, y] - [\mu_\beta(y), x + y]. \end{aligned} \quad (4.21)$$

In the last equality,  $x_{K_\sigma, r}$  is the vector field generated by the right action of  $\{e^{tx}, t \in \mathbb{R}\}$  on  $K_\sigma$ .

Remark now that the action of  $K_\beta$  on  $V_{\beta_0}$  comes from a  $K_\sigma$ -action. Hence the vector bundle  $\tilde{\mathcal{Y}}_\sigma \rightarrow \mathbf{T}(K_\sigma/K_\beta)$  is trivial through the isomorphism

$$\begin{aligned} \Xi & : \tilde{\mathcal{Y}}_\sigma \xrightarrow{\sim} \mathbf{T}(K_\sigma/K_\beta) \times V_{\beta_0} \\ [k; x, y] & \longmapsto ([k, x], k.y). \end{aligned} \quad (4.22)$$

We have a natural  $K_\beta$ -invariant Euclidean structure on  $\mathfrak{k}_\sigma/\mathfrak{k}_\beta \oplus \mathfrak{p}/\mathfrak{p}_{\beta_0}$  coming from the Euclidean structure of  $\mathfrak{g}$  and we denote by  $(\cdot, \cdot)_{\mathbf{T}(K_\sigma/K_\beta)}$  the induced  $K_\sigma$ -invariant Riemannian structure on  $\mathbf{T}(K_\sigma/K_\beta)$  (see equation (3.13)).

**Definition 4.2** Let  $\theta_{\beta_0}$  be the  $K_\sigma$ -invariant 1-form on  $V_{\beta_0}$  given by  $\theta_{\beta_0}|_y = -([\beta_0, y], dy)_{\mathfrak{p}}$ ,  $y \in V_{\beta_0}$ , and let  $\gamma_\sigma$  be the  $K_\sigma$ -invariant 1-form on  $\mathbf{T}(K_\sigma/K_\beta)$  defined by the equation:  $\gamma_\sigma|_{[k; x, y]} = -\left(x_{K_\sigma, r}(k), \cdot\right)_{\mathbf{T}(K_\sigma/K_\beta)}$ ,  $[k; x] \in \mathbf{T}(K_\sigma/K_\beta)$ .

We will denote by  $\tilde{\lambda}_\sigma$  the  $K_\sigma$ -invariant 1-form on  $\tilde{\mathcal{Y}}_\sigma$  defined below

$$\tilde{\lambda}_\sigma = \Xi^*\left(\gamma_\sigma + \theta_{\beta_0}\right).$$

We are going to prove that  $\tilde{\lambda}_\sigma$  and  $\lambda_\sigma$  define (in cohomology) the same generalized equivariant form in the neighbourhood of  $K_\sigma/K_\beta$  (through the isomorphism  $\Upsilon$ ). First we compute the function  $(\Phi_{\tilde{\lambda}_\sigma}, \tilde{\mu}_\sigma)$  near  $K_\sigma/K_\beta$ . We have

$$\begin{aligned} (\Phi_{\tilde{\lambda}_\sigma}, \tilde{\mu}_\sigma)([k; x, y]) & = \gamma_\sigma\left(X_{\mathbf{T}(K_\sigma/K_\beta)}\right)([k, x]) + \theta_{\beta_0}\left(X_{V_{\beta_0}}\right)(k.y) \quad \text{with } X = \tilde{\mu}_\sigma([k; x, y]) \\ & = \|x\|_{\mathfrak{k}}^2 + \left([\beta_0, k.y], [\tilde{\mu}_\sigma([k; x, y]), k.y]\right)_{\mathfrak{p}} \\ & = \|x\|_{\mathfrak{k}}^2 + \|[\beta_0, y]\|_{\mathfrak{p}}^2 + O(\|x, y\|^3). \end{aligned} \quad (4.23)$$

In the last equality we just use the fact that  $\tilde{\mu}_\sigma([k; x, y]) = \beta_0 + O(\|x, y\|)$ , then  $\left([\beta_0, k.y], [\beta_0 + O(\|x, y\|), k.y]\right)_{\mathfrak{p}} = \|[\beta_0, k.y]\|_{\mathfrak{p}}^2 + O(\|x, y\|^3)$  and the  $K_\sigma$ -invariance of  $\beta_0$  imposes  $\|[\beta_0, k.y]\|_{\mathfrak{p}}^2 = \|[\beta_0, y]\|_{\mathfrak{p}}^2$ .

From the equality (4.23), we know that there exists a  $K_\sigma$ -invariant neighbourhood  $\tilde{\mathcal{V}}$  of the 0-section in  $\tilde{\mathcal{Y}}_\sigma$  such that the function  $(\Phi_{\tilde{\lambda}_\sigma}, \tilde{\mu}_\sigma)$  is *strictly positive* on  $\tilde{\mathcal{V}} \setminus \{K_\sigma/K_\beta\}$  (Note that the vector  $[\beta_0, y]$  is non-zero for each non-zero vector  $y \in \mathfrak{p}/\mathfrak{p}_{\beta_0}$ ). In particular we get  $\{\Phi_{\tilde{\lambda}_\sigma} = 0\} \cap \tilde{\mathcal{V}} = K_\sigma/K_\beta$ .

Let  $\tilde{\chi}_\sigma \in \mathcal{C}_{cpt}^\infty(\tilde{\mathcal{Y}}_\sigma)$  be a function supported on  $\tilde{\mathcal{V}} \cap \mathcal{U}_2$ , and equal to 1 in a neighbourhood of  $K_\sigma/K_\beta$ . With this function we define the equivariant form

$$\tilde{\mathbb{P}}^{K_\sigma} = \tilde{\chi}_\sigma + d\tilde{\chi}_\sigma \left( \int_0^\infty \iota e^{-t\mathcal{D}\tilde{\lambda}_\sigma} dt \right) \tilde{\lambda}_\sigma \in \mathcal{A}_{K_\sigma, cpt}^{-\infty}(\tilde{\mathcal{Y}}_\sigma).$$

The main point of our deformation procedure is the following

**Lemma 4.3** *The generalized  $K_\sigma$ -equivariant forms  $P^{K_\sigma}$ , and  $\Upsilon^* \left( \tilde{P}^{K_\sigma} \right)$ , define the same cohomology class in  $\mathcal{H}_{K_\sigma, \text{cpt}}^{-\infty}(\mathcal{Y}_\sigma)$ . In particular we have the equality*

$$\int_{\mathcal{Y}_\sigma} P^{K_\sigma} = \int_{\tilde{\mathcal{Y}}_\sigma} \tilde{P}^{K_\sigma}$$

of  $K_\sigma$ -invariant tempered generalized functions on  $\mathfrak{k}_\sigma$ .

*Proof*: This is a consequence of Proposition 3.11 of [16], with the function  $f := \mu_\sigma$  on the manifold  $\mathcal{V} := \Upsilon^{-1}(\tilde{\mathcal{V}} \cap \mathcal{U}_2)$ , with the 1-forms  $\lambda_\sigma$  and  $\Upsilon^*(\tilde{\lambda}_\sigma)$ . All the point is that the functions  $(\Phi_{\Upsilon^*(\tilde{\lambda}_\sigma)}, \mu_\sigma) = (\Phi_{\tilde{\lambda}_\sigma}, \tilde{\mu}_\sigma) \circ \Upsilon$  and  $(\Phi_{\lambda_\sigma}, \mu_\sigma) = \|\mathcal{H}_\sigma\|^2$  are strictly positive on  $\mathcal{V} \setminus \{K_\sigma \cdot \beta\}$ . This fact implies that the 1-forms  $\lambda_\sigma$  and  $\Upsilon^*(\tilde{\lambda}_\sigma)$  define the same equivariant forms *in cohomology*.  $\square$

Consider now the  $K_\sigma$ -equivariant form  $e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$  on  $\tilde{\mathcal{Y}}_\sigma$ . First of all we note that the 1-form  $\tilde{\lambda}_\sigma$  is linear in the variable  $x \in \mathfrak{k}_\sigma/\mathfrak{k}_\beta$ , and quadratic in the variable  $y \in \mathfrak{p}/\mathfrak{p}_{\beta_0}$ . Consider the corresponding map  $\Phi_{\tilde{\lambda}_\sigma} : \tilde{\mathcal{Y}}_\sigma \rightarrow \mathfrak{k}_\sigma$ . For every  $Z \in \mathfrak{k}_\sigma$ ,  $[k; , x, y] \in \tilde{\mathcal{Y}}_\sigma$  we have

$$\begin{aligned} \left( \Phi_{\tilde{\lambda}_\sigma}, Z \right)_\mathfrak{k} ([k; x, y]) &= \left( x, pr_{\mathfrak{k}_\sigma/\mathfrak{k}_\beta}(k^{-1} \cdot Z) \right)_\mathfrak{k} + \left( [\beta_0, k \cdot y], [Z, k \cdot y] \right)_\mathfrak{p} \\ &= \left( x, k^{-1} \cdot Z \right)_\mathfrak{k} + \left( [[\beta_0, k \cdot y], k \cdot y], Z \right)_\mathfrak{k} \quad [1] \\ &= \left( k(x + [[\beta_0, y], y]), Z \right)_\mathfrak{k}. \quad [2] \end{aligned}$$

For the point [1]: we first use the fact that  $(X, pr_{\mathfrak{k}_\sigma/\mathfrak{k}_\beta}(X'))_\mathfrak{k} = (pr_{\mathfrak{k}_\sigma/\mathfrak{k}_\beta}(X), X')_\mathfrak{k}$ , for  $X, X' \in \mathfrak{k}_\sigma$ , and we use also the identity  $(a, [X, b])_\mathfrak{p} = ([a, b], X)_\mathfrak{k}$ , for  $a, b \in \mathfrak{p}$ , and  $X \in \mathfrak{k}$ . The point [2] follows from the  $K_\sigma$ -invariance of  $\beta_0$ . We have found the following expression:  $\Phi_{\tilde{\lambda}_\sigma}([k; x, y]) = k(x + [[\beta_0, y], y])$ ,  $[k; x, y] \in \tilde{\mathcal{Y}}_\sigma$ .

Consider, for every  $t > 0$ , the following  $K_\sigma$ -equivariant contraction

$$\begin{aligned} \delta_t : \quad \tilde{\mathcal{Y}}_\sigma &\longrightarrow \tilde{\mathcal{Y}}_\sigma \\ [k; x, y] &\longmapsto \left[ k, \frac{x}{t}, \frac{y}{\sqrt{t}} \right]. \end{aligned} \quad (4.24)$$

We have  $t\delta_t^*(\tilde{\lambda}_\sigma) = \tilde{\lambda}_\sigma$  and  $t\delta_t^*(\Phi_{\tilde{\lambda}_\sigma}) = \Phi_{\tilde{\lambda}_\sigma}$  for every  $t > 0$ . This gives

$$\delta_t^* \left( e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma} \right) = e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma} \quad \text{in } \mathcal{A}_{K_\sigma}^\infty(\tilde{\mathcal{Y}}_\sigma),$$

for every  $t > 0$ .

For every  $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k}_\sigma)$ , the differential form  $\langle e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(Y)}, f(Y)dY \rangle_{\mathfrak{k}_\sigma}$  is the product of the differential form  $e^{-\iota d\tilde{\lambda}_\sigma}$  which has a polynomial dependence in the direction of the fibers, with the function  $\hat{f}(-\Phi_{\tilde{\lambda}_\sigma})$  which is rapidly decreasing along the fibers. The differential form  $\langle e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(Y)}, f(Y)dY \rangle_{\mathfrak{k}_\sigma}$  is then integrable on  $\tilde{\mathcal{Y}}_\sigma$ . We denote by  $\int_{\tilde{\mathcal{Y}}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$  the tempered  $K_\sigma$ -invariant function on  $\mathfrak{k}_\sigma$  defined by

$$\left\langle \left( \int_{\tilde{\mathcal{Y}}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma} \right) (Y), f(Y)dY \right\rangle_{\mathfrak{k}_\sigma} := \int_{\tilde{\mathcal{Y}}_\sigma} \langle e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(Y)}, f(Y)dY \rangle_{\mathfrak{k}_\sigma} \quad \text{for } f \in \mathcal{C}_{rd}^\infty(\mathfrak{k}_\sigma).$$

**Proposition 4.4** *We have the following equality*

$$\int_{\tilde{\mathcal{Y}}_\sigma} \tilde{\mathbb{P}}^{K_\sigma} = \int_{\tilde{\mathcal{Y}}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$$

of  $K_\sigma$ -invariant tempered generalized functions on  $\mathfrak{k}_\sigma$ .

*Proof* : The generalized equivariant form  $\tilde{\mathbb{P}}^{K_\sigma}$  is the limit of the smooth equivariant form

$$\begin{aligned} \tilde{\mathbb{P}}_a^{K_\sigma} &:= \tilde{\chi}_\sigma + d\tilde{\chi}_\sigma \left( \int_0^a \iota e^{-\iota t \mathcal{D}\tilde{\lambda}_\sigma} dt \right) \tilde{\lambda}_\sigma \\ &= \tilde{\chi}_\sigma e^{-\iota a \mathcal{D}\tilde{\lambda}_\sigma} + \mathcal{D} \left( \tilde{\chi}_\sigma \left( \int_0^a \iota e^{-\iota t \mathcal{D}\tilde{\lambda}_\sigma} dt \right) \tilde{\lambda}_\sigma \right), \end{aligned}$$

when  $a \rightarrow \infty$ . Note that the equivariant form  $\tilde{\chi}_\sigma \left( \int_0^a \iota e^{-\iota t \mathcal{D}\tilde{\lambda}_\sigma} dt \right) \tilde{\lambda}_\sigma$  is with compact support on  $\tilde{\mathcal{Y}}_\sigma$ , then after integration on  $\tilde{\mathcal{Y}}_\sigma$  we get

$$\int_{\tilde{\mathcal{Y}}_\sigma} \tilde{\mathbb{P}}_a^{K_\sigma} = \int_{\tilde{\mathcal{Y}}_\sigma} \tilde{\chi}_\sigma e^{-\iota a \mathcal{D}\tilde{\lambda}_\sigma} = \int_{\tilde{\mathcal{Y}}_\sigma} \tilde{\chi}_\sigma \circ \delta_a e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$$

where we make the change of variable  $\delta_a$  in the last equality. From this, we see that  $\int_{\tilde{\mathcal{Y}}_\sigma} \tilde{\mathbb{P}}_a^{K_\sigma} = \lim_{a \rightarrow \infty} \int_{\tilde{\mathcal{Y}}_\sigma} \tilde{\chi}_\sigma \circ \delta_a e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$ . The function  $\tilde{\chi}_\sigma$  is equal to 1 in a neighbourhood of the 0-section, then  $\lim_{a \rightarrow \infty} \tilde{\chi}_\sigma \circ \delta_a = 1$ , and we conclude with the ‘Lebesgue’ convergence argument.  $\square$

## 5 Computation of $\int_{\tilde{\mathcal{Y}}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$

Note that the vector field on  $\tilde{\mathcal{Y}}_\sigma$  generated by  $\beta_0$  is identically equal to zero on  $\mathbf{T}(K_\sigma/K_\beta)$  and on  $\mathcal{V}_{\beta_0}$  we have  $\beta_0 \mathcal{V}_{\beta_0}|_{[k,y]} = -[\beta_0, y]$ . With the  $K_\sigma$ -equivariant form

$$-\iota \mathcal{D}\tilde{\lambda}_\sigma(X + \iota s \beta_0) = -\iota \mathcal{D}\tilde{\lambda}_\sigma(X) - s \|\beta_0, y\|^2$$

depending of the parameter  $s > 0$ , we can compute the generalized function  $\int_{\tilde{\mathcal{Y}}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma}$  as a limit of the generalized functions  $\Lambda_s$  on  $\mathfrak{k}_\sigma$  defined by the equation

$$\Lambda_s(X) = \int_{\tilde{\mathcal{Y}}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X + \iota s \beta_0)}, \quad s > 0.$$

The integration is well defined because  $e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X + \iota s \beta_0)}|_{[k;x,y]} = e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X)}|_{[k;x,y]} e^{-s \|\beta_0, y\|^2}$  and the function  $[k; x, y] \rightarrow e^{-s \|\beta_0, y\|^2}$  is bounded on  $\tilde{\mathcal{Y}}_\sigma$ . To compute  $\Lambda_s$ , first we can integrate  $e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X + \iota s \beta_0)}$  on the fibers of the projection  $\pi_1 : \tilde{\mathcal{Y}}_\sigma \rightarrow \mathbf{T}(K_\sigma/K_\beta)$ . The term  $e^{-s \|\beta_0, y\|^2}$  insures that the equivariant form  $(\pi_1)_* \left( e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X + \iota s \beta_0)} \right)$  is smooth, and we have

$$(\pi_1)_* \left( e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X + \iota s \beta_0)} \right) = e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma(X)} (\pi_1)_* \circ \Xi^* \left( e^{-\iota \mathcal{D}\theta_{\beta_0}(X + \iota s \beta_0)} \right).$$

The map  $\Xi$  is a trivialization of the bundle  $\pi_1 : \tilde{\mathcal{Y}}_\sigma \rightarrow \mathbf{T}(K_\sigma/K_\beta)$ , then the equivariant form  $(\pi_1)_* \circ \Xi^* \left( e^{-\imath \mathcal{D}\theta_{\beta_0}(X+\imath s\beta_0)} \right) \in \mathcal{A}_{K_\sigma}^\infty(\mathbf{T}(K_\sigma/K_\beta))$  is equal to the function

$$\int_{V_{\beta_0}} e^{-\imath \mathcal{D}\theta_{\beta_0}(X+\imath s\beta_0)}, \quad X \in \mathfrak{k}_\sigma. \quad (5.25)$$

The equivariant Euler form  $\text{Eul}_o(\mathfrak{p}/\mathfrak{p}_{\beta_0})$  of the oriented  $K_\sigma$ -bundle  $V_{\beta_0} = \mathfrak{p}/\mathfrak{p}_{\beta_0} \rightarrow \{0\}$  is equal to the a  $K_\sigma$ -invariant polynomial  $X \in \mathfrak{k}_\sigma \rightarrow \Pi_{\mathfrak{p}/\mathfrak{p}_{\beta_0}} \left( \frac{-1}{2\pi} X \right)$  ( for  $\Pi_{\mathfrak{p}/\mathfrak{p}_{\beta_0}}$  see the Definition 1.3).

The integrals like (5.25) have been studied in section 4 of [16], where we prove in particular that

$$\int_{V_{\beta_0}} e^{-\imath \mathcal{D}\theta_{\beta_0}(X+\imath s\beta_0)} = \text{Eul}_o(\mathfrak{p}/\mathfrak{p}_{\beta_0})(X + \imath s\beta_0)^{-1} \quad (5.26)$$

for every  $X \in \mathfrak{k}_\sigma$ ,  $s > 0$ . Hence, we have the following

**Lemma 5.1** *The generalized function  $\int_{\tilde{\mathcal{Y}}_\sigma} e^{-\imath \mathcal{D}\tilde{\lambda}_\sigma}$  is the limit of the generalized functions*

$$\Lambda_s(X) = \frac{1}{\text{Eul}_o(\mathfrak{p}/\mathfrak{p}_{\beta_0})(X + \imath s\beta_0)} \int_{\mathbf{T}(K_\sigma/K_\beta)} e^{-\imath \mathcal{D}\gamma_\sigma(X)}, \quad s > 0,$$

when  $s \rightarrow 0^+$ .

Then it remains to compute the generalized function  $\int_{\mathbf{T}(K_\sigma/K_\beta)} e^{-\imath \mathcal{D}\gamma_\sigma(X)}$ ,  $X \in \mathfrak{k}_\sigma$ .

**Proposition 5.2** *Let  $L$  be a compact Lie group and  $H$  a closed subgroup, with corresponding Lie algebras  $\mathfrak{l}$ , and  $\mathfrak{h}$ . We denote by  $\lambda_{L/H}$  the ( $L$ -invariant) Liouville 1-form on  $\mathbf{T}^*(L/H)$ . For every  $f \in C_{rd}^\infty(\mathfrak{l})$  the differential form  $\langle e^{\imath \mathcal{D}\lambda_{L/H}(X)}, f(X)dX \rangle_{\mathfrak{l}}$  is integrable on  $\mathbf{T}^*(L/H)$  and*

$$\int_{\mathbf{T}^*(L/H)} \langle e^{\imath \mathcal{D}\lambda_{L/H}(X)}, f(X)dX \rangle_{\mathfrak{l}} = \text{cte} \langle \mathbf{1}_{\mathfrak{h}}(Y), \overline{f}^L(Y)dY \rangle_{\mathfrak{h}},$$

with  $\text{cte} = (2\imath\pi)^{\dim(L/H)} \frac{\text{vol}(L,dX)}{\text{vol}(H,dY)}$ , and  $\mathbf{1}_{\mathfrak{h}}$  is the constant function equal to 1 on  $\mathfrak{h}$ .

**Remark 5.3** *A similar computation has been done by Witten in [24] (See equation (2.42)).*

*Proof* : First we parameterize the Liouville 1-form  $\lambda_{L/H} \in \mathcal{A}^1(\mathbf{T}^*(L/H))$ . Let  $\mathfrak{r}$  be a  $H$ -invariant complement of  $\mathfrak{h}$  in  $\mathfrak{l}$ :  $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{r}$ . Then the dual vector space  $\mathfrak{r}^*$  is naturally identified with the orthogonal (for the duality)  $\mathfrak{h}^\perp$  of  $\mathfrak{h}$  in  $\mathfrak{l}^*$ . The tangent bundle  $\mathbf{T}(L/H)$  (resp. the cotangent bundle  $\mathbf{T}^*(L/H)$ ) is naturally identified with the bundle  $L \times_H \mathfrak{r}$  over  $L/H$  (resp. the bundle  $L \times_H \mathfrak{r}^*$ ). Let  $E^i$ ,  $i = 1, \dots, \dim(G/H)$  be a basis of  $\mathfrak{r}$ , and we denote by  $E_i$  the corresponding dual basis of  $\mathfrak{r}^*$ . For every  $X \in \mathfrak{l}$ , let  $X_{L,r}$  the vector field

on  $L$  generated by the right action of  $L$  on itself:  $X_{L,r}|_l = \frac{d}{dt}\Big|_{t=0} l.e^{tX}$ ,  $l \in L$ . Let  $\omega_i$ ,  $i = 1, \dots, \dim(G/H)$  be the (left)  $L$ -invariant 1 forms on  $L$  such that  $\omega_i(X_{L,r})$  is a constant function on  $L$  equal to  $\langle E_i, X \rangle$  for every  $X \in \mathfrak{l}$ . The space  $\left(\mathcal{A}^1(L)^L \otimes \mathfrak{r}\right)_{H\text{-basic}}$  is a subspace of the space of (left)  $L$ -invariant one form on  $L \times_H \mathfrak{r}^*$ , where the  $H$ -invariant are taken with respect to the action of  $H$  by right multiplication on  $L$ , and adjoint action on  $\mathfrak{r}$ ; the  $L$ -invariant are taken with respect to the action of  $L$  by left multiplication on itself. The 1-form  $\lambda_{L/H} \in \left(\mathcal{A}^1(L)^L \otimes \mathfrak{r}\right)_{H\text{-basic}}$  is defined by the equation

$$\lambda_{L/H}|_{[l,\xi]} := \sum_{i=1}^{\dim(\mathfrak{r})} \langle \xi, E^i \rangle \omega_i|_l, \quad [l, \xi] \in L \times_H \mathfrak{r}^*. \quad (5.27)$$

A straightforward computation gives  $\mathcal{D}\lambda_{L/H}(X)|_{[l,\xi]} = \Omega_{L/H} + \langle l, \xi, X \rangle$ ,  $[l, \xi] \in L \times_H \mathfrak{r}^*$ ,  $X \in \mathfrak{l}$ , where  $\Omega_{L/H} = d\lambda_{L/H} = \sum_{i=1}^k dE^i \wedge \omega_i + E^i d\omega_i$  is the symplectic two form on  $L \times_H \mathfrak{r}^* \cong \mathbf{T}^*(L/H)$ . The corresponding Liouville volume form  $\frac{\Omega_{L/H}^k}{k!}$  is equal to  $\prod_{i=1}^k dE^i \wedge \omega_i$ . Then, for every  $L$ -invariant function  $f \in \mathcal{C}_{rd}^\infty(\mathfrak{l})$  we have

$$\begin{aligned} \int_{\mathbf{T}^*(L/H)} \langle e^{i\mathcal{D}\lambda_{L/H}(X)}, f(X) dX \rangle_{\mathfrak{l}} &= (i)^k \int_{L \times_H \mathfrak{r}^*} \frac{\Omega_{L/H}^k}{k!} \widehat{f}(-l, \xi) \quad [1] \\ &= (i)^k \int_{L/H} \Pi_i \omega_i \int_{\text{Fiber}} \widehat{f}(-l, \xi) \Pi_i dE^i. \quad [2] \end{aligned}$$

In [1], the function  $\widehat{f} \in \mathcal{C}_{rd}^\infty(\mathfrak{r}^*)$  is the Fourier transform of  $f$  relatively to the measure  $dX$ , and  $k = \dim(L/H)$ . In [2], we have decomposed  $\prod_{i=1}^k dE^i \wedge \omega_i$  in  $\Pi_i \omega_i \wedge \Pi_i dE^i$ , and the morphism  $\int_{\text{Fiber}} : \mathcal{A}_{rd}^*(L \times_H \mathfrak{r}^*) \rightarrow \mathcal{A}^*(L/H)$  is the morphism of integration along the fiber of  $L \times_H \mathfrak{r}^* \rightarrow L/H$ .

The  $L$ -invariance of  $f$  imposes that  $[l] \rightarrow \int_{\text{Fiber}} \widehat{f}(-l, \xi) \Pi_i dE^i$  is constant on  $L/H$ , and is equal to  $\int_{\mathfrak{r}^*} \widehat{f}(-\xi) d\xi$ , where  $d\xi$  is the Lebesgue measure on  $\mathfrak{r}^*$  compatible with  $\Pi_i dE^i$ . The double Fourier integral gives

$$\int_{\mathfrak{r}^*} \widehat{f}(-\xi) d\xi = (2\pi)^k \int_{\mathfrak{h}} f(Y) dY,$$

where  $\frac{dX}{dY}$  is the Lebesgue measure on  $\mathfrak{r}$  dual to  $d\xi$ . Finally, we have proved that

$$\int_{\mathbf{T}^*(L/H)} \langle e^{i\mathcal{D}\lambda_{L/H}(X)}, f(X) dX \rangle_{\mathfrak{l}} = (2i\pi)^k \text{vol}(L/H, \Pi_i \omega_i) \int_{\mathfrak{h}} f|_{\mathfrak{h}}(Y) dY.$$

But  $\text{vol}(L/H, \Pi_i \omega_i) = \frac{\text{vol}(L, dX)}{\text{vol}(H, dY)}$ , and Proposition is then proved.  $\square$

We come back now at the previous situation. We verify that the 1-form  $-\gamma_\sigma$  on  $\mathbf{T}(K_\sigma/K_\beta)$  corresponds to the Liouville 1-form on  $\mathbf{T}^*(K_\sigma/K_\beta)$ , through the identification  $\mathbf{T}^*(K_\sigma/K_\beta) \cong \mathbf{T}(K_\sigma/K_\beta)$ .

The function defined over  $\mathfrak{k}_\sigma$ ,  $X \rightarrow \text{Eul}_o(\mathfrak{p}/\mathfrak{p}_{\beta_0})(X + \iota s \beta_0)^{-1}$ , defines, when  $s \rightarrow 0^+$ , a generalized function in  $\mathcal{C}^{-\infty}(V)$  on each vector subspace  $V \subset \mathfrak{k}_\sigma$  that contains  $\beta_0$ . We denote by  $\text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0})$  this generalized function (whatever the vector subspace  $V$  is).

We can now state the Theorem.

**Theorem 5.4** *Let  $M$  be a closed orbit in  $\mathfrak{g}$ . There exists a unique  $(\beta_0, \beta_1) \in \mathfrak{k}_+ \times \mathfrak{p}$  with  $[\beta_0, \beta_1] = 0$  such that  $M = G \cdot (\beta_0 + \beta_1)$ . Let  $K_\beta$  be the stabilizer of  $\beta := \beta_0 + \beta_1$  in  $K$ , and let  $\mathfrak{k}_\beta$  be its Lie algebra. The generalized function  $\text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0}) \in \mathcal{C}^{-\infty}(\mathfrak{k}_{\beta_0})^{K_{\beta_0}}$  admits a restriction to  $\mathfrak{k}_\beta$ , and for every function  $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k})^K$  we have*

$$\langle F_M|_{\mathfrak{k}}(X), f(X)dX \rangle_{\mathfrak{k}} = \text{cte} \langle \text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0})(Z), e^{\iota(\beta_0, Z)} f|_{\mathfrak{k}_\beta}(Z) \Pi_{\mathfrak{k}/\mathfrak{k}_{\beta_0}}(Z) dZ \rangle_{\mathfrak{k}_\beta},$$

$$\text{with } \text{cte} = (-2\pi)^{\dim(K/K_{\beta_0})/2} (2\pi\iota)^{\dim(K_{\beta_0}/K_\beta)} \frac{\text{vol}(K, dX)}{\text{vol}(K_\beta, dZ)}.$$

*Proof:* Proposition 5.2 and Lemma 5.1 give

$$\langle \Lambda_s(X), f(X)dX \rangle_{\mathfrak{k}_\sigma} = \text{cte} \langle \frac{1}{\text{Eul}_o(\mathfrak{p}/\mathfrak{p}_{\beta_0})(Y + \iota s \beta_0)}, f|_{\mathfrak{k}_\beta}(Y) dY \rangle_{\mathfrak{k}_\beta}$$

for every function  $f \in \mathcal{C}_{rd}^\infty(\mathfrak{k}_\sigma)$  that is  $K_\sigma$ -invariant, and with  $\text{cte} = (2\iota\pi)^{\dim(\mathfrak{k}_\sigma/\mathfrak{k}_\beta)} \frac{\text{vol}(K_\sigma, dX)}{\text{vol}(K_\beta, dY)}$ .

After taking the limit  $s \rightarrow 0^+$ , we get from Lemma 5.1

$$\langle \left( \int_{\tilde{Y}_\sigma} e^{-\iota \mathcal{D}\tilde{\lambda}_\sigma} \right)(X), f(X)dX \rangle_{\mathfrak{k}_\sigma} = \text{cte} \langle \text{Eul}_{\beta_0}^{-1}(\mathfrak{p}/\mathfrak{p}_{\beta_0})(Y), f|_{\mathfrak{k}_\beta}(Y) dY \rangle_{\mathfrak{k}_\beta}.$$

Now Proposition 3.5, Lemma 4.3, and Proposition 4.4 used with this last equality complete the proof.  $\square$

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