The moment map and equivariant cohomology with generalized coefficients

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Abstract

Let M be a symplectic manifold acted on by a compact Lie group G in a Hamiltonian fashion, with *proper* moment map. In this situation we introduce a pushforward morphism $\mathcal{P} : \mathcal{H}^*_G(M) \longrightarrow \mathcal{M}^{-\infty}(\mathfrak{g}^*)^G$, from the equivariant cohomology of M to the space of G-invariant distributions on $\mathfrak{g}^*$, which gives rise to symplectic invariants, in particular the pushforward of the Liouville measure. For the study of this pushforward morphism we make an intensive use of equivariant forms with generalized coefficients.

Contents

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1 Introduction

Let M be a manifold provided with an action of a compact connected Lie group G with Lie algebra $\mathfrak{g}$; one may then define the equivariant cohomology $\mathcal{H}^*_G(M)$ of M. Suppose now that the manifold is symplectic, with symplectic 2-form Ω, and the action of G is Hamiltonian. Let $\mu : M \to \mathfrak{g}^*$ be the moment map associated to this action, and denote by $dm_L := \frac{\Omega^n}{n!}$, $\dim M = 2n$, the Liouville volume form on M.

The purpose of this paper is the study of a 'pushforward' morphism

$$\mathcal{P} : \mathcal{H}^*_G(M) \longrightarrow \mathcal{M}^{-\infty}(\mathfrak{g}^*)^G, \tag{1.1}$$

where $\mathcal{M}^{-\infty}(\mathfrak{g}^*)^G$ is the space of G-invariant distributions on $\mathfrak{g}^*$. The morphism $\mathcal{P}$, defined under the hypothesis that μ *is proper*, produces some interesting symplectic invariants, in particular the pushforward of the Liouville form $\mu_*(dm_L)$: for the class $\mathbf{1} \in \mathcal{H}^*_G(M)$, we have

$$\mathcal{P}(\mathbf{1}) = \frac{1}{\mathfrak{i}^n}\mu_*(dm_L).$$

Our study of the morphism $\mathcal{P}$ fits into the numerous studies of the symplectic invariant $\mu_*(dm_L)$ (and its relation with localization in equivariant cohomology) that started some years ago with the Duistermaat-Heckman formula [2, 4, 9, 10, 13, 14, 16, 21, 23, 33]. Nevertheless our method is new, and works particularly well in the case where M is *non-compact* and G is *(any) connected compact* Lie group (in the previous references, the study is often restricted to the action of a torus). The central idea, which is due to Witten [36], is to localize the integration of closed equivariant forms on the set $\mathrm{Cr}(\|\mu\|^2)$ of critical points of the function $\|\mu\|^2 : M \to \mathbb{R}$ (where $\|\cdot\|^2$ denotes a G-invariant Euclidean norm on $\mathfrak{g}^*$).

We will first produce some specific closed equivariant forms with generalized coefficients which completely describe the *local and global* behavior of the distributions $\mathcal{P}(\alpha)$, $\alpha \in \mathcal{H}^*_G(M)$. Next the computations of these equivariant forms are handled. In this way, we are for example able to describe the behavior of $\mu_*(dm_L)$ near 0 even if 0 is not a regular value of the moment map.

For this task, we will exploit the results of a previous paper [31], and we will make an essential use of *equivariant forms with generalized coefficients*, with their properties of *induction* and *restriction*, that have been introduced and developed by M. Duflo, S. Kumar and M. Vergne in [11, 26].

We now turn to a more extensive introduction of our subject.

Section 2 is devoted to the explanation of the tools coming from equivariant cohomology. We refine the technique related to the *partition of unity*, a notion that was introduced by the author in [31] (see in particular the "deformation process"), and we recall the definitions and properties concerning the induction and restriction of equivariant forms with generalized coefficients.

Let $\Omega_{\mathfrak{g}}(X) = \Omega + \langle \mu, X\rangle,\ X \in \mathfrak{g}$ be the equivariant symplectic 2-form. For every equivariant form $\alpha(X) \in \mathcal{A}_G^*(M)$ depending polynomially on $X \in \mathfrak{g}$, the Fourier transform $\mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})$ belongs to the space $\mathcal{M}^{-\infty}(\mathfrak{g}^*, M)^G$ of G-invariant distributions on $\mathfrak{g}^*$ with values in the algebra $\mathcal{A}^*(M)$ of differential forms on M. Furthermore, for every smooth function f on $\mathfrak{g}^*$ with *compact support*, the differential form $\int_{\mathfrak{g}^*} \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})(\xi) f(\xi)$ has a *compact support* on M, and then is integrable (see Definition 2.10). For instance, if $\alpha(X) = P(X)\eta$ where $X \to P(X)$ is polynomial on $\mathfrak{g}$ and $\eta \in \mathcal{A}^*(M)$ is a differential form on M, we have $\mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) = \eta e^{\mathrm{i}\Omega}\mathcal{F}(P(-)e^{\mathrm{i}\langle \mu, -\rangle})$, and $\int_{\mathfrak{g}^*} \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})(\xi) f(\xi) = \eta e^{\mathrm{i}\Omega} \int_{\mathfrak{g}^*} \mathcal{F}(P(-)e^{\mathrm{i}\langle \mu, -\rangle})(\xi) f(\xi) = \eta e^{\mathrm{i}\Omega}[P(-\mathfrak{i}\partial_\xi) f](\mu)$ is a differential form on M supported on $\mu^{-1}(\mathrm{support}\{f\})$, hence with compact support on M for every function f with compact support since μ is *proper* (for the definition of the differential operator $P(-\mathfrak{i}\partial_\xi)$ see (2.13)).

Then, for every equivariant form $\alpha \in \mathcal{A}_G^*(M)$ the integral

$$\tilde{\mathcal{P}}(\alpha) := \int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) \tag{1.2}$$

defines a G-invariant distribution on $\mathfrak{g}^*$: for $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$ with compact support, the value taken by $\tilde{\mathcal{P}}(\alpha)$ on f is the integral on M of the differential form $\int_{\mathfrak{g}^*} \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})(\xi) f(\xi)$. Next we show that the map $\tilde{\mathcal{P}} : \mathcal{A}_G^*(M) \to \mathcal{M}^{-\infty}(\mathfrak{g}^*)^G$ factors in cohomology and induces the pushforward map $\mathcal{P} : \mathcal{H}_G^*(M) \to \mathcal{M}^{-\infty}(\mathfrak{g}^*)^G$. In fact, the space $\mathcal{M}^{-\infty}(\mathfrak{g}^*, M)^G$ carries a derivative $\widetilde{\mathcal{D}}$, with $\widetilde{\mathcal{D}}^2 = 0$, and for the 'localizations' of $\mathcal{P}(\alpha)$ we will work directly on the complex $(\mathcal{M}^{-\infty}(\mathfrak{g}^*, M)^G, \widetilde{\mathcal{D}})$.

We end up Section 2 with a first illustration of the usefulness of our method. With the assumption that the group G is a torus T and that a component $\langle \mu, \beta\rangle$ of the moment map is *proper*, we obtain easily a refinement of the Berline-Vergne localization that extends previous results by Prato-Wu and Guillemin-Lerman-Sternberg [14, 33]. For every $\alpha \in \mathcal{H}_T^*(M)$, we have a locally finite decomposition $\mathcal{P}(\alpha) = \sum_{F \subset M^T} \mathfrak{D}_F^\alpha$, where each distribution $\mathfrak{D}_F^\alpha$ is *tempered* and depends only on the values of α, Ω and μ on the connected component F of fixed point set M^T.

In Section 3, we are interested in the *local behavior* of the distributions $\mathcal{P}(\alpha),\ \alpha \in \mathcal{H}_G^*(M)$. Concerning the behavior near 0, we first show that there exists a compactly supported equivariant form P_λ^0 which completely describes the pushforward $\mathcal{P}(\alpha)$ near 0 (see Proposition 3.8). When 0 is a regular value of μ, the computation of P_λ^0 leads to a new proof of the Jeffrey-Kirwan-Witten formula. Concerning the behavior near a coadjoint orbit $\mathcal{O} = G.\xi$, we show in the same way that the pushforward is completely determined by a compactly supported equivariant form $\mathrm{P}_{\lambda\xi}^{\mathcal{O}}$ (see Proposition 3.17). This fact is crucial to prove an induction formula which generalizes previous results of [9, 35]. This induction holds in the *entire* slice U_σ through the point ξ of $\mathcal{O}$ (see Proposition 3.13), and shows that the pushforwards $\mathcal{P}(\alpha)$ are *analytic* on the open subset of regular values of μ.

In Section 4, our aim is to compute the equivariant forms P_λ^0, $\mathrm{P}_{\lambda\xi}^{\mathcal{O}}$ when 0 or ξ are not necessarily regular values of μ. Our purpose is also a *global* formula

for the pushforwards $\mathcal{P}(\alpha)$. We show that the distribution $\mathcal{P}(\alpha)$ is always a locally finite sum

$$\mathcal{P}(\alpha) = \sum_{\beta \in \mathcal{B}_G} \mathfrak{D}_\beta^\alpha, \tag{1.3}$$

in *tempered* distributions, where $\mathcal{B}_G \subset \mathfrak{g}^*$ indexes a decomposition $\mathrm{Cr}(\|\mu\|^2) = \cup_{\beta \in \mathcal{B}_G} C_\beta^G$ with $\mu(C_\beta^G) = G.\beta$. Moreover, each tempered distributions $\mathfrak{D}_\beta^\alpha$ is supported on $\{\xi \in \mathfrak{g}^*,\ \|\xi\| \geq \|\beta\|\}$, and is described by means of an equivariant form P_β^G which is supported in a compact neighborhood of the component C_β^G of $\mathrm{Cr}(\|\mu\|^2)$ (see equation (4.45)). We will see that, under suitable conditions, the tempered distributions $\mathfrak{D}_\beta^\alpha$ can be computed explicitly.

One interesting example is the case where M is a *closed* coadjoint orbit of a connected semi-simple real Lie group S. Let G be a maximal compact connected subgroup of S. The action of G on M is Hamiltonian and the moment map $\mu : M \to \mathfrak{g}^*$ is *proper*. One can show that $\int_M e^{\mathrm{i}\Omega_\mathfrak{g}}$ defines a *tempered* generalized function on $\mathfrak{g}$ (see [10]): thus the pushforward $\mathcal{P}(\mathbf{1})$ is equal to $\mathcal{F}(\int_M e^{\mathrm{i}\Omega_\mathfrak{g}})$. On the other hand we prove in [32] that the set $\mathrm{Cr}(\|\mu\|^2)$ is just a G-orbit in M (in particular $\mathcal{B}_G$ is reduced to $\{\beta_0\}$): the pushforward $\mathcal{P}(\mathbf{1})$ is then equal to $\mathfrak{D}_{\beta_0}^{\mathbf{1}}$ (see (1.3)). Using this technique we were able in [32] to compute the generalized function $\int_M e^{\mathrm{i}\Omega_\mathfrak{g}}$, extending by this way previous results of Duflo-Heckman-Vergne [9, 10] and Sengupta [34].

The central result of the section 4 is a *symplectic induction formula* (see Theorem 4.5) that provides a tool to compute the generalized equivariant forms P_λ^0, $\mathrm{P}_{\lambda\xi}^{\mathcal{O}}$, and P_β^G: these G-equivariant forms are induced by generalized T-equivariant forms (where T is a maximal torus of G), and we show in sub-section 4.3 how to compute these T-equivariant forms using a shift of the moment map (see Definition 4.12).

In Section 5, we determine, for a Hamiltonian action of a compact Lie group on a *compact* manifold, the global behavior of the *partition functions* in a way conjectured by Witten [36]. This extends previous results by Jeffrey-Kirwan [23]. In particular we prove the following statement.

Corollary 5.2 *There exist smooth functions $h_\beta : \mathbb{R} \to \mathbb{R}$, such that*

$$\int_M e^{-\frac{\|\mu\|^2}{2u}} dm_L = (u)^{-\frac{dim G}{2}} \sum_{\beta \in \mathcal{B}_G} e^{-\frac{\|\beta\|^2}{2u}} h_\beta(\sqrt{u}), \quad u > 0,$$

where dm_L is the Liouville measure $\frac{\Omega^n}{n!}$. Moreover each function h_β is uniquely determined by the local behavior of Ω and μ near the component C_β^G of $\mathrm{Cr}(\|\mu\|^2)$, and the function h_0 that corresponds to the set $C_0^G = \mu^{-1}(0)$ is always polynomial *(note that we make no assumption on the regularity of 0 relative to the moment map).*

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2 Partition of unity in equivariant cohomology

2.1 Definitions and notations

Let M be a manifold provided with an action of a compact connected Lie group G with Lie algebra $\mathfrak{g}$. We denote by $\mathcal{A}^*(M)$ the algebra of differential forms on M (over $\mathbb{C}$), and by d the exterior differentiation. Let $\mathcal{A}^*_{cpt}(M)$ be the sub-algebra of compactly supported differential forms. If ξ is a vector field on M we denote by $c(\xi) : \mathcal{A}^*(M) \to \mathcal{A}^{*-1}(M)$ the contraction by ξ. The action of G on M gives a morphism $X \to X_M$ from $\mathfrak{g}$ to the Lie algebra of vector fields on M.

We now recall the different de Rham complexes of G-equivariant forms on M. For more details see [3, 5, 10, 26].

Let $\mathcal{C}^\infty(\mathfrak{g}, \mathcal{A}^*(M))$ be the algebra of forms $\alpha(X)$ on M depending smoothly on $X \in \mathfrak{g}$. We note $\mathcal{A}^\infty_G(M)$ the sub-algebra of $\mathcal{C}^\infty(\mathfrak{g}, \mathcal{A}^*(M))$ consisting of the G-invariant elements: these elements are called equivariant forms with $\mathcal{C}^\infty$ coefficients. Let $\mathcal{A}^*_G(M) \subset \mathcal{A}^\infty_G(M)$ be the sub-algebra of equivariant forms $\alpha(X)$ depending *polynomially* on $X \in \mathfrak{g}$. The differential $\mathcal{D}$ on $\mathcal{A}^\infty_G(M)$ is given by the equation

$$\forall\, \alpha \in \mathcal{A}^\infty_G(M), \quad (\mathcal{D}\alpha)(X) := (d - c(X_M))(\alpha(X)),\ X \in \mathfrak{g}.$$

We see that $\mathcal{A}^*_G(M)$ is stable under $\mathcal{D}$, and that $\mathcal{D}^2 = 0$ on $\mathcal{A}^\infty_G(M)$. The cohomologies associated to $(\mathcal{A}^*_G(M), \mathcal{D})$ and $(\mathcal{A}^\infty_G(M), \mathcal{D})$ are denoted respectively $\mathcal{H}^*_G(M)$ and $\mathcal{H}^\infty_G(M)$.

The algebra $\mathcal{A}^\infty_G(M)$ has a sub-algebra $\mathcal{A}^\infty_{G,cpt}(M) := \mathcal{C}^\infty(\mathfrak{g}, \mathcal{A}^*_{cpt}(M))^G$, stable under the differential $\mathcal{D}$. The cohomology associated to $(\mathcal{A}^\infty_{G,cpt}(M), \mathcal{D})$ is called the G-equivariant cohomology with compact support and is denoted by $\mathcal{H}^\infty_{G,cpt}(M)$.

For our purpose we need equivariant forms with generalized coefficients. For a more precise description see [26].

The space $\mathcal{C}^{-\infty}(\mathfrak{g}, \mathcal{A}^*(M))$ of generalized functions on $\mathfrak{g}$ with values in the space $\mathcal{A}^*(M)$ is, by definition, the space $\mathrm{Hom}(m_c(\mathfrak{g}), \mathcal{A}^*(M))$ of continuous $\mathbb{C}$-linear maps from the space $m_c(\mathfrak{g})$ of smooth compactly supported densities on $\mathfrak{g}$ to the space $\mathcal{A}^*(M)$, both endowed with the $\mathcal{C}^\infty$-topologies. We define

$$\mathcal{A}^{-\infty}_G(M) := \mathcal{C}^{-\infty}(\mathfrak{g}, \mathcal{A}^*(M))^G$$

as the space of G-equivariant $\mathcal{C}^{-\infty}$-maps from $\mathfrak{g}$ to $\mathcal{A}^*(M)$. An element of the space $\mathcal{A}^{-\infty}_G(M)$ is called an equivariant form with generalized coefficients. The image of $\phi \in m_c(\mathfrak{g})$ under $\alpha \in \mathcal{C}^{-\infty}(\mathfrak{g}, \mathcal{A}^*(M))$ is a differential form on M denoted by $< \alpha, \phi >_{\mathfrak{g}}$.

We see that $\mathcal{A}^\infty_G(M) \subset \mathcal{A}^{-\infty}_G(M)$ and we can also extend the differential $\mathcal{D}$ to $\mathcal{A}^{-\infty}_G(M)$ [26]. Take a basis $\{E^1, \cdots, E^p\}$ of $\mathfrak{g}$, and $\{E_1, \cdots, E_p\}$ the associated dual basis of $\mathfrak{g}^*$. Let $\{X_1, \cdots, X_p\}$ be the corresponding coordinate functions on $\mathfrak{g}$. For every $\gamma \in \mathcal{A}^{-\infty}_G(M)$,

$$< \mathcal{D}(\gamma), \phi >_{\mathfrak{g}} := d < \gamma, \phi >_{\mathfrak{g}} - \sum_{k=1}^{p} c(E^k_M) < \gamma, X_k \phi >_{\mathfrak{g}} \quad \text{for every} \quad \phi \in m_c(\mathfrak{g}).$$

We verify that $\mathcal{D}^2$ vanishes on $\mathcal{A}_G^{-\infty}(M)$. The cohomology associated to $(\mathcal{A}_G^{-\infty}(M), \mathcal{D})$ is called the G-equivariant cohomology with generalized coefficients and is denoted by $\mathcal{H}_G^{-\infty}(M)$. The sub-space $\mathcal{A}_{G,cpt}^{-\infty}(M) := \mathcal{C}^{-\infty}(\mathfrak{g}, \mathcal{A}_{cpt}^*(M))^G$ is stable under the differential $\mathcal{D}$, and we denote by $\mathcal{H}_{G,cpt}^{-\infty}(M)$ the associated cohomology.

Let $\mathcal{M}^{-\infty}(\mathfrak{g}^*, M)$ be the space of distributions on $\mathfrak{g}^*$ with values in the space $\mathcal{A}^*(M)$. More precisely, it is the space $\mathrm{Hom}(\mathcal{C}_{cpt}^\infty(\mathfrak{g}^*), \mathcal{A}^*(M))$ of continuous $\mathbb{C}$-linear maps from the space of smooth compactly supported functions $\mathcal{C}_{cpt}^\infty(\mathfrak{g}^*)$ to the space $\mathcal{A}^*(M)$, both endowed with the $\mathcal{C}^\infty$-topologies. We can also define a derivation $\widetilde{\mathcal{D}}$ on $\mathcal{M}^{-\infty}(\mathfrak{g}^*, M)$ in the following way. For every $\gamma \in \mathcal{M}^{-\infty}(\mathfrak{g}^*, M)$, the distribution $\widetilde{\mathcal{D}}(\gamma)$ is defined by the equation

$$< \widetilde{\mathcal{D}}(\gamma), f >_{\mathfrak{g}^*} := d < \gamma, f >_{\mathfrak{g}^*} + \mathfrak{i} \sum_{k=1}^{p} c(E_M^k) < \gamma, \partial_{E_k} f >_{\mathfrak{g}^*},$$

for every $f \in \mathcal{C}_{cpt}^\infty(\mathfrak{g}^*)$. In this equation we denote by $\partial_\xi : \mathcal{C}_{cpt}^\infty(\mathfrak{g}^*) \to \mathcal{C}_{cpt}^\infty(\mathfrak{g}^*)$ the partial derivative associated to every $\xi \in \mathfrak{g}^*$. One can easily check that $\widetilde{\mathcal{D}}^2 = 0$ on the subspace $\mathcal{M}^{-\infty}(\mathfrak{g}^*, M)^G$ of G-invariant distributions.

For every open subset $\mathcal{U}$ of $\mathfrak{g}^*$, we denote by $\mathcal{M}^{-\infty}(\mathcal{U})$ the space of distributions on $\mathcal{U}$ with complex values. If $\mathcal{U}$ is a G-invariant subset we can consider the subspace $\mathcal{M}^{-\infty}(\mathcal{U})^G$ of G-invariant distributions. Recall that the inclusion $\mathcal{C}_{cpt}^\infty(\mathcal{U})^G \hookrightarrow \mathcal{C}_{cpt}^\infty(\mathfrak{g}^*)^G$ induces the restriction map $\mathcal{M}^{-\infty}(\mathfrak{g}^*)^G \to \mathcal{M}^{-\infty}(\mathcal{U})^G$.

For any compact Lie group G, and any Lebesgue measure dX on its Lie algebra, we denote by $\mathrm{vol}(G, dX)$ the volume of G for the Haar measure on G compatible with dX.

2.2 Partition of unity

For more details see Section 3 of [31].

Let $\lambda \in \mathcal{A}^1(M)$ be a G-invariant 1-form on M. We note $\Phi_\lambda : M \to \mathfrak{g}^*$ the G-equivariant map defined by

$$\langle \Phi_\lambda(m), X \rangle := \lambda_m(X_M(m)), \quad m \in M,\ X \in \mathfrak{g}.$$

Lemma 3.1 of [31] tells us that the equivariant form $\mathcal{D}\lambda(X) = d\lambda - \langle \Phi_\lambda, X \rangle$ is invertible outside $\{\Phi_\lambda = 0\}$ in the space of generalized equivariant forms. For each G-invariant differential form χ_{ext} on M, equal to zero in a neighborhood of $\{\Phi_\lambda = 0\}$, we can define $\chi_{ext}\left(\int_0^\infty \mathfrak{i} e^{-\mathfrak{i}t\,\mathcal{D}\lambda} dt\right) \in \mathcal{A}_G^{-\infty}(M)$, and this equivariant form satisfies

$$\chi_{ext}\left(\int_0^\infty \mathfrak{i} e^{-\mathfrak{i}t\,\mathcal{D}\lambda} dt\right) \mathcal{D}\lambda = \chi_{ext}$$

in $\mathcal{A}_G^{-\infty}(M)$.

Let $\chi \in \mathcal{C}^\infty(M)^G$, equal to 1 in a neighborhood of $\{\Phi_\lambda = 0\}$. Then the form $d\chi$ is equal to 0 in a neighborhood of $\{\Phi_\lambda = 0\}$, and we can define the equivariant form

$$\mathrm{P}_\lambda = \chi + d\chi \left(\int_0^\infty \mathrm{i} e^{-\mathrm{i}t\,\mathcal{D}\lambda} dt \right) \lambda \ \in \ \mathcal{A}_G^{-\infty}(M). \tag{2.4}$$

Recall Proposition 3.3 of [31].

Proposition 2.1 *The equivariant form* P_λ *is closed, and we have the identity*

$$\mathbf{1}_M = \mathrm{P}_\lambda + \mathcal{D}(\delta) \tag{2.5}$$

where $\delta = (1-\chi)(\int_0^\infty \mathrm{i} e^{-\mathrm{i}t\,\mathcal{D}\lambda} dt)\lambda \in \mathcal{A}_G^{-\infty}(M)$ *and* $\mathbf{1}_M$ *is the constant function equal to* 1 *on* M.

We use the phrase "partition of unity" to refer to the equality (2.5).

In practice we will decompose the equivariant form P_λ from a partition of $\{\Phi_\lambda = 0\}$ into closed subsets.

A subset $C \subset M$ is called a *component* of $\{\Phi_\lambda = 0\}$ if there exists a G-invariant neighborhood $\mathcal{U}$ of C in M such that $\overline{\mathcal{U}} \cap \{\Phi_\lambda = 0\} = C$. These conditions imply that $\partial\mathcal{U} \cap \{\Phi_\lambda = 0\} = \emptyset$ $(\star)$.

Definition 2.2 *A* G*-invariant open set* $\mathcal{U}$ *of* M *which satisfies* $(\star)$ *will be call* good *for the* 1*-form* λ*: it intersects* $\{\Phi_\lambda = 0\}$ *in the interior of* $\overline{\mathcal{U}}$.

If we start with a *good* open set $\mathcal{U}$ for the 1-form λ, the intersection $\overline{\mathcal{U}} \cap \{\Phi_\lambda = 0\}$ is a *component* (perhaps empty !) of $\{\Phi_\lambda = 0\}$.

Definition 2.3 *Let* $\mathcal{U}$ *be a* good *open set for the* 1*-form* λ*, and* $C = \overline{\mathcal{U}} \cap \{\Phi_\lambda = 0\}$ *be the corresponding* component *of* $\{\Phi_\lambda = 0\}$. *We denote by* $\mathrm{P}_\lambda^{\mathcal{U}}$ *(or* P_λ^{C}*) the equivariant form*

$$\mathrm{P}_\lambda^{\mathcal{U}} = \chi_{\mathcal{U}} + d\chi_{\mathcal{U}} \left(\int_0^\infty \mathrm{i} e^{-\mathrm{i}t\,\mathcal{D}\lambda} dt \right) \lambda \ ,$$

where $\chi_{\mathcal{U}}$ *is a* G*-invariant real function on* M*, equal to* 1 *in a neighborhood of* C*, and with support in* $\mathcal{U}$.

Remark 2.4 *If the open set* $\mathcal{U}$ *is relatively compact in* M*, then the equivariant form* $\mathrm{P}_\lambda^{\mathcal{U}}$ *has a compact support on* M.

Remark 2.5 *The cohomology class of* $\mathrm{P}_\lambda^{\mathcal{U}}$ *does not depend on the choice of the function* $\chi_{\mathcal{U}}$.

Consider an open covering $\{\Phi_\lambda = 0\} \subset \cup_i \mathcal{U}_i$, where the open sets $\mathcal{U}_i$ are G-invariant. Suppose furthermore that $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ if $i \neq j$. Then the open sets $\mathcal{U}_i$ are good for the 1-form λ and

$$\mathrm{P}_\lambda = \sum_i \ \mathrm{P}_\lambda^{\mathcal{U}_i} \ . \tag{2.6}$$

The preceding sum is well defined, even if it is infinite, because the equivariant forms $\mathrm{P}_\lambda^{\mathcal{U}_i}$ have *disjoint* supports. In practice, the equation (2.6) enables us to study the equivariant form P_λ in the neighborhood of each component of $\{\Phi_\lambda = 0\}$.

2.3 Deformation process

We will reformulate Proposition 3.11 of [31] in a more general context .

Consider two G-invariant 1-forms λ_0, λ_1 on M, and a G-invariant open set $\mathcal{U}$ of M.

Proposition 2.6 *Suppose there exist a smooth map $f : M \to \mathfrak{g}$ and a real $\rho > 0$ such that the functions, $l_0 = \langle \Phi_{\lambda_0}, f\rangle$ and $l_1 = \langle \Phi_{\lambda_1}, f\rangle$, are bounded from below by ρ on $\partial\mathcal{U}$. Then, the open set $\mathcal{U}$ is good for λ_0 and λ_1 (see Definition 2.2), and*

$$\mathrm{P}^{\mathcal{U}}_{\lambda_1} = \mathrm{P}^{\mathcal{U}}_{\lambda_0} \qquad \text{in} \qquad \mathcal{H}^{-\infty}_G(M) \ .$$

Proof : By an averaging over G, we can suppose that the map f is G-equivariant. Then the real functions l_0 and l_1 are G-invariant. Consider the G-invariant 1-forms $\lambda_s = s\lambda_1 + (1-s)\lambda_0$, $s \in [0,1]$. The equality $\langle \Phi_{\lambda_s}, f\rangle = sl_1 + (1-s)l_0$ shows that

$$\{\Phi_{\lambda_s} = 0\} \subset \{l_0 l_1 \leq 0\} \qquad \text{for} \quad 0 \leq s \leq 1 \ .$$

The open set $\mathcal{U}$ is good for every 1-form λ_s, because the function $l_0 l_1$ is strictly positive on $\partial\mathcal{U}$. Consider a (smooth) cut-off function $g : \mathbb{R} \to \mathbb{R}^+$, equal to 1 on the interval $\{x \in \mathbb{R},\ x \leq \rho^2/3\}$ and equal to 0 on $\{x \in \mathbb{R},\ x \geq \rho^2/2\}$.

Let χ be the G-invariant function on M, equal to $g(l_0 l_1)$ in $\mathcal{U}$, and equal to 0 outside $\mathcal{U}$. It is a well defined smooth function because $g(l_0 l_1)$ is identically 0 in a neighborhood of $\partial\mathcal{U}$. It follows that the sets $\{\Phi_{\lambda_s} = 0\} \cap \mathcal{U}$ are included in the interior of $\{\chi = 1\}$ for every $s \in [0,1]$.

Then, we can define the equivariant forms $\mathrm{P}^{\mathcal{U}}_{\lambda_s} = \chi + d\chi(\int_0^\infty \mathrm{i} e^{-\mathrm{i}t\,\mathcal{D}\lambda_s} dt)\lambda_s$ for every $s \in [0,1]$. If we take the derivative with respect to s, we have

$$\frac{\partial \mathrm{P}^{\mathcal{U}}_{\lambda_s}}{\partial s} = d\chi\Big(\int_0^\infty t\ e^{-\mathrm{i}t\,\mathcal{D}\lambda_s} dt\Big)\mathcal{D}(\lambda_1 - \lambda_0)\lambda_s + d\chi\Big(\int_0^\infty \mathrm{i} e^{-\mathrm{i}t\,\mathcal{D}\lambda_s} dt\Big)(\lambda_1 - \lambda_0) \ .$$

The first term of the RHS of this equation is equal to

$$\mathcal{D}\left(-d\chi\Big(\int_0^\infty t\ e^{-\mathrm{i}t\,\mathcal{D}\lambda_s} dt\Big)(\lambda_1 - \lambda_0)\lambda_s\right) + d\chi\Big(\int_0^\infty t\ e^{-\mathrm{i}t\,\mathcal{D}\lambda_s} dt\Big)\mathcal{D}\lambda_s(\lambda_1 - \lambda_0) \ ,$$

and an integration by part shows that $d\chi(\int_0^\infty t\ \ e^{-\mathrm{i}t\,\mathcal{D}\lambda_s} dt)\mathcal{D}\lambda_s = -d\chi(\int_0^\infty \mathrm{i} e^{-\mathrm{i}t\,\mathcal{D}\lambda_s} dt)$ $(\star)$. More precisely, we have $d\chi(\int_0^\infty t\, e^{-\mathrm{i}t\,\mathcal{D}\lambda_s} dt) = \lim_{a\to\infty} d\chi\Big(\int_0^a t\, e^{-\mathrm{i}t\,\mathcal{D}\lambda_s} dt\Big)$ and an integration by part (of smooth equivariant forms) gives $(\int_0^a t\, e^{-\mathrm{i}t\,\mathcal{D}\lambda_s} dt)\mathcal{D}\lambda_s = \mathrm{i}a\, e^{-\mathrm{i}a\,\mathcal{D}\lambda_s} - \mathrm{i}\int_0^a e^{-\mathrm{i}t\,\mathcal{D}\lambda_s} dt$, $\forall a \in \mathbb{R}$. Taking now the limit $a \to \infty$, we get the equation $(\star)$: we just use that $\lim_{a\to\infty} a\, e^{-\mathrm{i}a\,\mathcal{D}\lambda_s} d\chi = 0$ in $\mathcal{A}^{-\infty}_G(M)$ because $d\chi = 0$ in a neighborhood of $\{\Phi_{\lambda_s} = 0\}$.

Thus we find

$$\frac{\partial \mathrm{P}^{\mathcal{U}}_{\lambda_s}}{\partial s} = \mathcal{D}\left(d\chi\Big(\int_0^\infty t\ e^{-\mathrm{i}t\,\mathcal{D}\lambda_s} dt\Big)\lambda_0\lambda_1\right) \ .$$

After integrating this last equation on $[0,1]$, we get $\mathrm{P}^{\mathcal{U}}_{\lambda_1} - \mathrm{P}^{\mathcal{U}}_{\lambda_0} = \mathcal{D}(\theta)$, with $\theta \in \mathcal{A}^{-\infty}_G(M)$. $\square$

Remark 2.7 *If the open set $\mathcal{U}$ is relatively compact in M, Proposition 2.6 holds if the function l_0 and l_1 are strictly positive on $\partial\mathcal{U}$. In this case, the equality $\mathrm{P}^{\mathcal{U}}_{\lambda_1} = \mathrm{P}^{\mathcal{U}}_{\lambda_0}$ holds in $\mathcal{H}^{-\infty}_{G,cpt}(M)$.*

2.4 Generic form near a smooth component

In this section, we want to give a generic form of P^C_λ near a *smooth* component C of $\{\Phi_\lambda = 0\}$. Then a good neighborhood of C in M is modeled on the normal bundle $N_C := (\mathbf{T}\, M|_C)/\,\mathbf{T}\, C$.

We work now with the following data. Let $q : \mathcal{V} \to C$ be a G-equivariant real vector bundle on a *compact* G-manifold C. We denote by $i : C \hookrightarrow \mathcal{V}$ the 0-section of $\mathcal{V}$. We suppose that the fibers of $\mathcal{V}$ are *oriented*. Let $\lambda \in \mathcal{A}^1(\mathcal{V})$ be a G-invariant 1-form on $\mathcal{V}$ such that $\{\Phi_\lambda = 0\} = C$. Denote by P^C_λ the generalized equivariant form defined by λ in a neighborhood of the 0-section in $\mathcal{V}$ (see Definition 2.3).

Integration along the fibers of $\mathcal{V}$ defines a morphism

$$\int_{\mathcal{V}/C} : \mathcal{H}^{-\infty}_{G,cpt}(\mathcal{V}) \longrightarrow \mathcal{H}^{-\infty}_G(C)$$

that can be extended to equivariant forms which are *rapidly decreasing in $\mathfrak{g}$-mean*. Recall Definition 2 of [7].

Definition 2.8 *The differential form $\alpha \in \mathcal{C}^{-\infty}(\mathfrak{g}, \mathcal{A}^*(\mathcal{V}))$ is said to be rapidly decreasing in $\mathfrak{g}$-mean if, for every test function f on $\mathfrak{g}$, the differential form $< \alpha(X), f(X)dX >_{\mathfrak{g}}$ on $\mathcal{V}$ is rapidly decreasing along the fibers, as well as all its derivatives, and such that, moreover, the map $f \to < \alpha(X), f(X)dX >_{\mathfrak{g}}$ is continuous, with respect to the natural semi-norms on the space of rapidly decreasing differential forms.*

We denote by $\mathcal{A}^{-\infty}_{G,mean-rapid}(\mathcal{V})$ the space of G-equivariant differential forms on $\mathcal{V}$ which are rapidly decreasing in $\mathfrak{g}$-mean. The equivariant coboundary $\mathcal{D}$ maps $\mathcal{A}^{-\infty}_{G,mean-rapid}(\mathcal{V})$ into itself. We denote by $\mathcal{H}^{-\infty}_{G,mean-rapid}(\mathcal{V})$ the corresponding cohomology.

For every $\alpha \in \mathcal{A}^{-\infty}_{G,mean-rapid}(\mathcal{V})$ the integration $\int_{\mathcal{V}/C} \alpha$ along the fibers of $\mathcal{V}$ defines an equivariant form of $\mathcal{A}^{-\infty}_G(C)$ by

$$< \Big(\int_{\mathcal{V}/C} \alpha \Big)(X), f(X)dX >_{\mathfrak{g}} := \int_{\mathcal{V}/C} < \alpha(X), f(X)dX >_{\mathfrak{g}} \in \mathcal{A}^*(C)$$

for every test density $f(X)dX$ on $\mathfrak{g}$. The integration $\int_{\mathcal{V}/C}$ commutes with $\mathcal{D}$ and so defines a morphism

$$\int_{\mathcal{V}/C} : \mathcal{H}^{-\infty}_{G,mean-rapid}(\mathcal{V}) \longrightarrow \mathcal{H}^{-\infty}_G(C)\ .$$

Let $\mathrm{Thom}_G(\mathcal{V}) \in \mathcal{H}^\infty_{G,cpt}(\mathcal{V})$ be the G-equivariant Thom form with compact support of the G-vector bundle $\mathcal{V}$ (for an explicit description see [11, 29]).

Proposition 2.9 *Suppose that the 1-form λ is homogeneous on the fibers of $\mathcal{V}$, with a strictly positive degree of homogeneity. Then the closed equivariant form $e^{-\mathrm{i}\,\mathcal{D}\lambda}$ belongs to $\mathcal{A}^{-\infty}_{G,mean-rapid}(\mathcal{V})$ and we have the following equality in $\mathcal{H}^{-\infty}_{G}(C)$*

$$\int_{\mathcal{V}/C} \mathrm{P}^{C}_{\lambda} = \int_{\mathcal{V}/C} e^{-\mathrm{i}\,\mathcal{D}\lambda}\ .$$

Hence, for every closed form $\eta \in \mathcal{A}^{\infty}_{G}(\mathcal{V})$, we have

$$\eta\ \mathrm{P}^{C}_{\lambda} = q^{*}\Big(i^{*}(\eta)\int_{\mathcal{V}/C} e^{-\mathrm{i}\,\mathcal{D}\lambda}\Big)\,\mathrm{Thom}_{G}(\mathcal{V}) \quad \text{in} \quad \mathcal{H}^{-\infty}_{G,cpt}(\mathcal{V}). \tag{2.7}$$

Proof: If the 1-form λ is homogeneous, the map $\Phi_{\lambda} : \mathcal{V} \to \mathfrak{g}^{*}$ is homogeneous on the fibers of $\mathcal{V}$, and the fact that $\{\Phi_{\lambda} = 0\} = C$ implies $(\star)$ $\|\Phi_{\lambda}(m,v)\| \geq c\|v\|^{k},\ (m,v) \in\ \mathcal{V}$, for some $c > 0$, where $k > 0$ is the degree of homogeneity of λ (where $\|\cdot\|$ is a norm on the bundle $\mathcal{V}$). Then for every test density $f(X)dX$ on $\mathfrak{g}$, the differential form $< e^{-\mathrm{i}\,\mathcal{D}\lambda}(X), f(X)dX >_{\mathfrak{g}} = e^{-\mathrm{i}\,d\lambda}\hat{f}(-\Phi_{\lambda})$ is rapidly decreasing along the fibers of $\mathcal{V}$ ($\hat{f}$ is the Fourier transform of f relatively to dX). We have shown that the closed equivariant form $e^{-\mathrm{i}\,\mathcal{D}\lambda}$ belongs to $\mathcal{A}^{-\infty}_{G,mean-rapid}(\mathcal{V})$.

Let $\chi \in \mathcal{C}^{\infty}_{cpt}(\mathcal{V})$ be a G-invariant function, equal to 1 in a neighborhood of C. We define the following G-equivariant forms on $\mathcal{V}$

$$\alpha_{0} = \chi\Big(\int_{0}^{1} e^{-\mathrm{i}\,t\,\mathcal{D}\lambda}dt\Big)\mathrm{i}\lambda \quad \in \quad \mathcal{A}^{\infty}_{G,cpt}(\mathcal{V}),$$

and

$$\alpha_{1} = (1-\chi)\Big(\int_{1}^{\infty} e^{-\mathrm{i}\,t\,\mathcal{D}\lambda}dt\Big)\mathrm{i}\lambda \quad \in \quad \mathcal{A}^{-\infty}_{G}(\mathcal{V}).$$

The equivariant form α_1 is well defined because $1-\chi = 0$ in a neighborhood of C. We verify that $\mathcal{D}\alpha_0 = d\chi\Big(\int_0^1 e^{-\mathrm{i}\,t\,\mathcal{D}\lambda}dt\Big)\mathrm{i}\lambda - \chi(e^{-\mathrm{i}\,\mathcal{D}\lambda} - 1)$ and $\mathcal{D}\alpha_1 = -d\chi\Big(\int_1^{\infty} e^{-\mathrm{i}\,t\,\mathcal{D}\lambda}dt\Big)\mathrm{i}\lambda + (1-\chi)e^{-\mathrm{i}\,\mathcal{D}\lambda}$. Hence we have

$$\mathrm{P}^{C}_{\lambda} - e^{-\mathrm{i}\,\mathcal{D}\lambda} = \mathcal{D}(\alpha_0 - \alpha_1)\ . \tag{2.8}$$

To conclude the proof, we just need to show that $\int_{\mathcal{V}/C}(\alpha_0 - \alpha_1)$ exists and defines an equivariant form in $\mathcal{A}^{-\infty}_{G}(C)$. This is clear for $\int_{\mathcal{V}/C}\alpha_0$ because α_0 has a compact support on $\mathcal{V}$.

For every test density $f(X)dX$ on $\mathfrak{g}$ we have

$$< \alpha_1(X), f(X)dX >_{\mathfrak{g}}|_{v} = (1-\chi(v))\Big(\int_{1}^{\infty} e^{-\mathrm{i}\,t\,d\lambda_v}\hat{f}(-t\,\Phi_{\lambda}(v))dt\Big)\mathrm{i}\lambda_{v}, \quad v\ \in\ \mathcal{V}\,.$$

With the inequality $(\star)$, we get, for any integer N, the following: $|\hat{f}(-t\,\Phi_{\lambda}(v))| \leq c_N(1+t\|v\|^{k})^{-N},\ t > 0,\ v \in \mathcal{V}$. Each component of the differential form $(t,v) \to e^{-\mathrm{i}\,t\,d\lambda_v}\lambda_v$ is bounded by a function of the form $P(t,\|v\|)$, where P is a

polynomial in both variables. Take now a polynomial $P(x,y)$ with total degree less than n. Then for every $v \in \mathcal{V}$ with $\|v\| \geq 1$, we have for N big enough

$$\Big| \int_1^\infty \frac{P(t,\|v\|)}{(1+t\|v\|^k)^N} dt \Big| \leq cst\, \|v\|^n \int_{\|v\|^k}^\infty \frac{t^n}{(1+t)^N} dt \leq cst' \frac{1}{(1+\|v\|^k)^{N-2n-1}} \ .$$

We have finally shown that the differential form $< \alpha_1(X), f(X)dX >_{\mathfrak{g}}$ is rapidly decreasing along the fibers of $\mathcal{V}$. We see in the same way that all the derivatives of $< \alpha_1(X), f(X)dX >_{\mathfrak{g}}$ are also rapidly decreasing along the fibers, and so that the G-equivariant form α_1 on $\mathcal{V}$ is rapidly decreasing in $\mathfrak{g}$-mean. We can now integrate along the fiber the equality (2.8), to get

$$\int_{\mathcal{V}/C} \mathrm{P}_\lambda^C - \int_{\mathcal{V}/C} e^{-\mathrm{i}\mathcal{D}\lambda} = \mathcal{D}\Big(\int_{\mathcal{V}/C} \alpha_0 - \alpha_1 \Big) \quad \text{in} \quad \mathcal{A}_G^{-\infty}(C) \ . \tag{2.9}$$

The Thom isomorphism tells us that for every $\eta \in \mathcal{H}_{G,cpt}^{-\infty}(\mathcal{V})$ we have $\eta = q^*(\int_{\mathcal{V}/C} \eta)\, \mathrm{Thom}_G(\mathcal{V})$. Using now the fact that for every $\eta \in \mathcal{H}_G^{-\infty}(\mathcal{V})$, we have $\int_{\mathcal{V}/C}(\eta.\, \mathrm{Thom}_G(\mathcal{V})) = i^*(\eta)$, the equality (2.7) results from (2.9). $\square$

2.5 Fourier transform and pushforward

The idea of this section goes back to the paper [35] (see section 1.1).

We denote by $\mathcal{A}_{temp}^{-\infty}(\mathfrak{g}, M)$ the space of tempered generalized functions over $\mathfrak{g}$ with values in $\mathcal{A}^*(M)$, and by $\mathcal{M}_{temp}^{-\infty}(\mathfrak{g}^*, M)$ the space of tempered distributions over $\mathfrak{g}^*$ with values in $\mathcal{A}^*(M)$. Let $\mathcal{F} : \mathcal{A}_{temp}^{-\infty}(\mathfrak{g}, M) \to \mathcal{M}_{temp}^{-\infty}(\mathfrak{g}^*, M)$ the Fourier transform which assigns to each $\phi \in \mathcal{A}_{temp}^{-\infty}(\mathfrak{g}, M)$ the tempered distribution $\mathcal{F}(\phi)$ such that

$$\int_{\mathfrak{g}^*} e^{\mathrm{i}\langle \xi, X\rangle} \mathcal{F}(\phi)(\xi) = \phi(X). \tag{2.10}$$

We verify that the Fourier transform commutes with the differential $\mathcal{D}$ and $\widetilde{\mathcal{D}}$. For every $\phi \in \mathcal{A}_{temp}^{-\infty}(\mathfrak{g}, M)$, we have

$$\mathcal{F} \circ \mathcal{D}(\phi) = \widetilde{\mathcal{D}} \circ \mathcal{F}(\phi) \ . \tag{2.11}$$

We say that a distribution $\gamma \in \mathcal{M}^{-\infty}(\mathfrak{g}^*, M)$ has a compact support *in* $\mathfrak{g}^*$-*mean* on M, if for every function $f \in \mathcal{C}_{cpt}^\infty(\mathfrak{g}^*)$, the differential form $< \gamma, f >_{\mathfrak{g}^*}$ has a compact support on M.

Definition 2.10 *For every $\gamma \in \mathcal{M}^{-\infty}(\mathfrak{g}^*, M)$ with a compact support* in $\mathfrak{g}^*$-mean *on M, we denote $\int_M \gamma$ the distribution over $\mathfrak{g}^*$ defined by*

$$\forall\, f \in \mathcal{C}_{cpt}^\infty(\mathfrak{g}^*), \quad < \int_M \gamma, f >_{\mathfrak{g}^*} := \int_M < \gamma, f >_{\mathfrak{g}^*} \ .$$

Lemma 2.11 *If $\gamma \in \mathcal{M}^{-\infty}(\mathfrak{g}^*, M)$ has a compact support* in $\mathfrak{g}^*$-mean *on M, the distribution $\widetilde{\mathcal{D}}(\gamma)$ has also a compact support* in $\mathfrak{g}^*$-mean *on M and we have*

$$\int_M \widetilde{\mathcal{D}}(\gamma) = 0 \ . \tag{2.12}$$

Proof : For $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{g}^*)$ we have, $\int_M < \widetilde{\mathcal{D}}(\gamma), f >_{\mathfrak{g}^*} = \int_M d < \gamma, f >_{\mathfrak{g}^*} + \mathfrak{i} \sum_{k=1}^p \int_M c(E_M^k) < \gamma, \partial_{E_k} f >_{\mathfrak{g}^*}$. By "Stokes" argument the first term of the RHS is 0, and each integral $\int_M c(E_M^k) < \gamma, \partial_{E_k} f >_{\mathfrak{g}^*}$ is also 0 because the differential form $c(E_M^k) < \gamma, \partial_{E_k} f >_{\mathfrak{g}^*}$ does not have components of maximal degree. □

Suppose that the manifold M is equipped with a symplectic 2-form Ω, and that the action of G is Hamiltonian with *proper* moment map $\mu : M \to \mathfrak{g}^*$. Consider the equivariant symplectic 2-form $\Omega_{\mathfrak{g}}(X) = \Omega + \langle \mu, X \rangle,\ X \in \mathfrak{g}$. By definition of the moment map μ, the equivariant form $\Omega_{\mathfrak{g}}$ is closed.

For every polynomial $P \in S(\mathfrak{g}^*)$ on $\mathfrak{g}$, we shall associate the differential operator $P(-\mathfrak{i}\partial_\xi)$ (with constant coefficients) on $\mathcal{C}^\infty(\mathfrak{g}^*)$ that satisfies

$$P(X) \int_{\mathfrak{g}^*} e^{-\mathfrak{i}\langle \xi, X \rangle} f(\xi) d\xi = \int_{\mathfrak{g}^*} e^{-\mathfrak{i}\langle \xi, X \rangle} [P(-\mathfrak{i}\partial_\xi) f](\xi) d\xi, \quad X \in \mathfrak{g}, \tag{2.13}$$

for every function $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$ with compact support. Naturally, this notation extends for every $\alpha \in \mathcal{A}^*_G(M) \subset S(\mathfrak{g}^*) \otimes \mathcal{A}^*(M)$; we denote by $\alpha(-\mathfrak{i}\partial_\xi)$ the corresponding differential operator (with values in $\mathcal{A}^*(M)$).

Lemma 2.12 *For every equivariant form $\alpha(X) \in \mathcal{A}^*_G(M)$ with* polynomial dependence *in $X \in \mathfrak{g}$, the form $\alpha(X) e^{\mathfrak{i}\Omega_{\mathfrak{g}}(X)}$ belongs to $\mathcal{A}^{-\infty}_{temp}(\mathfrak{g}, M)$, and its Fourier transform $\mathcal{F}(\alpha e^{\mathfrak{i}\Omega_{\mathfrak{g}}})$ has a compact support* in $\mathfrak{g}^*$-mean *on M. The distribution $\int_M \mathcal{F}(\alpha e^{\mathfrak{i}\Omega_{\mathfrak{g}}})$ is G-invariant. When α is closed, the distribution $\int_M \mathcal{F}(\alpha e^{\mathfrak{i}\Omega_{\mathfrak{g}}})$ depends only on the cohomology class of α.*

Proof : Using the equation (2.13), we see that

$$m \in M, \quad < \mathcal{F}(\alpha e^{\mathfrak{i}\Omega_{\mathfrak{g}}}), f >_{\mathfrak{g}^*} |_m = e^{\mathfrak{i}\Omega_m} [\alpha(-\mathfrak{i}\partial_\xi) f](\mu(m)),$$

for every $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{g}^*)$. Then, the differential form $< \mathcal{F}(\alpha e^{\mathfrak{i}\Omega_{\mathfrak{g}}}), f >_{\mathfrak{g}^*}$ is supported in $\mu^{-1}(support\{f\})$. Finally the properness of μ insures that the distribution $\mathcal{F}(\alpha e^{\mathfrak{i}\Omega_{\mathfrak{g}}})$ has a compact support *in* $\mathfrak{g}^*$*-mean* on M. As the form $\alpha(X) e^{\mathfrak{i}\Omega_{\mathfrak{g}}(X)}$ is G-equivariant, the Fourier transform $\mathcal{F}(\alpha e^{\mathfrak{i}\Omega_{\mathfrak{g}}})$ is also G-equivariant, and its integral is a G-invariant distribution on $\mathfrak{g}^*$. Suppose now that α is *exact*: $\alpha = \mathcal{D}\alpha'$ with $\alpha' \in \mathcal{A}^*_G(M)$. Then $\int_M \mathcal{F}(\alpha e^{\mathfrak{i}\Omega_{\mathfrak{g}}}) = \int_M \widetilde{\mathcal{D}}(\gamma)$ where $\gamma = \mathcal{F}(\alpha' e^{\mathfrak{i}\Omega_{\mathfrak{g}}})$ has a compact support in $\mathfrak{g}^*$-mean. It follows from Lemma 2.11 that $\int_M \mathcal{F}(\alpha e^{\mathfrak{i}\Omega_{\mathfrak{g}}})$ is identically equal to 0. □

After Lemma 2.12 one can define the *pushforward morphism*

$$\begin{array}{rccc} \mathcal{P} : & \mathcal{H}^*_G(M) & \longrightarrow & \mathcal{M}^{-\infty}(\mathfrak{g}^*)^G \\ & \alpha & \longmapsto & \displaystyle\int_M \mathcal{F}(\alpha e^{\mathfrak{i}\Omega_{\mathfrak{g}}}). \end{array} \tag{2.14}$$

Note that $\mathcal{P}$ is a morphism of $S(\mathfrak{g}^*)^G$-modules: for any $q \in S(\mathfrak{g}^*)^G$ and $\alpha \in \mathcal{H}^*_G(M)$ we have $\mathcal{P}(q\alpha)(\xi) = q(\mathfrak{i}\partial_\xi)\,(\mathcal{P}(\alpha))\,(\xi),\ \xi \in \mathfrak{g}^*$. We remark also that $\mathfrak{i}^n \mu_*(dm_L) = \mathcal{P}(\mathbf{1}) = \int_M \mathcal{F}(e^{\mathfrak{i}\Omega_{\mathfrak{g}}})$, with $\dim M = 2n$. The rest of this article is devoted to the study of the G-invariant distributions $\mathcal{P}(\alpha),\ \alpha \in \mathcal{H}^*_G(M)$.

When the integral $\int_M \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}$ defines a tempered generalized function on $\mathfrak{g}$, we can write $\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) = \mathcal{F}(\int_M \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})$. It is the case, for example, when α has a compact support on M. One interesting example is the case where M is a closed coadjoint orbit of a connected semi-simple real Lie group S. Let G be a maximal compact connected subgroup of S. The action of G on M is Hamiltonian and one can show that $\int_M e^{\mathrm{i}\Omega_{\mathfrak{g}}}$ defines a tempered distribution on $\mathfrak{g}$ (see [10, 32] for more details).

The distribution $\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})$ has a useful property of *locality*. For every G-invariant open subset $\mathcal{U}$ of $\mathfrak{g}^*$, we denote by $\Phi \to \Phi|_{\mathcal{U}}$ the restriction map $\mathcal{M}^{-\infty}(\mathfrak{g}^*)^G \to \mathcal{M}^{-\infty}(\mathcal{U})^G$. Here we have the following equality in $\mathcal{M}^{-\infty}(\mathcal{U})^G$

$$\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})\Big|_{\mathcal{U}} = \int_{\mu^{-1}(\mathcal{U})} \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) \ . \tag{2.15}$$

One can check, as we did in Lemma 2.12, that the restriction $\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})|_{\mathcal{U}}$ depends only on the cohomology class of α in $\mu^{-1}(\mathcal{U})$.

2.6 Induction and restriction of generalized equivariant forms

In this section we give a brief review of results of Kumar and Vergne [26], that we will use intensively in the rest of this paper.

Let G be a *connected* compact Lie group, and H a *connected* Lie subgroup of G such that G and H have the same rank (in the following sections we take $H := T$ a maximal torus of G). We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the respective Lie algebras. We fix from now on an orientation o on $\mathfrak{g}/\mathfrak{h}$ (hence the homogeneous space G/H is oriented) which determines a polynomial square root $Y \to \Pi_{\mathfrak{g}/\mathfrak{h}}(Y) := \det^{1/2}_{\mathfrak{g}/\mathfrak{h},o}(Y)$ of the polynomial function $Y \to \det_{\mathfrak{g}/\mathfrak{h}}(ad(Y))$ on $\mathfrak{h}$. The equality of the ranks insures that $\Pi_{\mathfrak{g}/\mathfrak{h}}$ is non zero.

Let M be a H-manifold, and consider the associated G-manifold $G \times_H M$. Kumar and Vergne (see section 5 of [26]) define a morphism

$$Ind_{G/H} : \mathcal{H}^{-\infty}_{H,*}(M) \longrightarrow \mathcal{H}^{-\infty}_{G,*}(G \times_H M) \tag{2.16}$$

where $*$ means that we can choose $*$ = "compact support" or $*$ = "general support". This map is in fact an *isomorphism* (for a proof see Theorem 52 of [26]). We recall briefly the definition of this induction morphism.

For every H-manifold M, we first define a G-equivariant map

$$\Theta : \mathcal{C}^\infty\Big(G, \mathcal{A}^*(M)\Big)^H \longrightarrow \mathcal{A}^*(G \times_H M), \tag{2.17}$$

where G acts by left translations on itself, and H acts on G by right translation.

For the algebra $\mathcal{A}^*(G \times_H M)$ we have the natural identifications

$$\begin{aligned} \mathcal{A}^*(G \times_H M) &= \mathcal{A}^*(G \times M)_{H-basic} \\ &= \mathcal{C}^\infty\Big(G, [\wedge \mathfrak{g}^* \otimes \mathcal{A}^*(M)]_{H-hor}\Big)^H \ . \end{aligned}$$

where H-invariants are taken with respect to the action of H by right multiplication on G, left action on M and adjoint action on $\mathfrak{g}^*$ (here $\mathfrak{g}^*$ corresponds to the space of (real) 1-forms on G invariant by left translation).

Let $\mathfrak{r}$ be a H-invariant subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$. Then we have the inclusions $\mathfrak{h}^* \hookrightarrow \mathfrak{g}^* \hookrightarrow \mathcal{A}^*(G)$ and $\wedge^{max}\mathfrak{r}^* \hookrightarrow [\wedge\mathfrak{g}^*]_{H-basic} \hookrightarrow \mathcal{A}^*(G/H)$. Let ν be the unique element of $\wedge^{max}\mathfrak{r}^*$ such that

- ν is compatible with the orientation of $\mathfrak{g}/\mathfrak{h}$, and
- $\int_{G/H} \nu = 1$.

We denote by $h_H : \mathcal{A}^*(M) \to [\wedge\mathfrak{h}^* \otimes \mathcal{A}^*(M)]_{H-hor}$ the horizontal projection (see Definition 28 of [26]). Using the inclusion $\wedge\mathfrak{r}^* \hookrightarrow [\wedge\mathfrak{g}^*]_{H-hor}$ the data (h_H, ν) permits to define the H-equivariant map

$$h_H(-) \wedge \nu : \mathcal{A}^*(M) \longrightarrow \Big[\wedge \mathfrak{g}^* \otimes \mathcal{A}^*(M) \Big]_{H-hor} .$$

After tensoring by $\mathcal{C}^\infty(G)$ we extend this map in the G-H-equivariant map

$$\tilde{\Theta} : \mathcal{C}^\infty\Big(G, \mathcal{A}^*(M)\Big) \longrightarrow \mathcal{C}^\infty\Big(G, [\wedge\mathfrak{g}^* \otimes \mathcal{A}^*(M)]_{H-hor}\Big).$$

The map Θ is by definition the restriction of $\tilde{\Theta}$ to the subspace $\mathcal{C}^\infty\Big(G, \mathcal{A}^*(M)\Big)^H$ of H-invariant elements.

Definition 2.13 *Let $\Phi \in \mathcal{A}_H^{-\infty}(M)$. The equivariant form $Ind_{G/H}(\Phi) \in \mathcal{A}_G^{-\infty}(G \times_H M)$ is defined by the following equation: for every $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{g})$,*

$$< Ind_{G/H}(\Phi)(X), f(X)dX >_{\mathfrak{g}} = \mathrm{cst}\, \Theta\Big(g \mapsto < \Phi(Y), f(Ad(g)Y)dY >_{\mathfrak{h}} \Big),$$

with $\mathrm{cst} = \frac{\mathrm{vol}_{(G,dX)}}{\mathrm{vol}_{(H,dY)}}$.

The map $Ind_{G/H} : \mathcal{A}_H^{-\infty}(M) \to \mathcal{A}_G^{-\infty}(G \times_H M)$ commutes with the equivariant differentials and we still denote $Ind_{G/H}$ the map in cohomology.

Note that if M is a point, the map $\Theta : \mathcal{C}^\infty(G/H) \to \mathcal{A}^*(G/H)$ is the multiplication by ν. Then for every $\Phi \in \mathcal{C}^{-\infty}(\mathfrak{h})^H$ the differential form $< Ind_{G/H}(\Phi)(X), f(X)dX >_{\mathfrak{g}} \in \mathcal{A}^*(G/H)$ is given by

$$< Ind_{G/H}(\Phi)(X), f(X)dX >_{\mathfrak{g}} |_g = \mathrm{cst.} < \Phi(Y), f(Ad(g)Y)dY >_{\mathfrak{h}} \nu|_g, \quad (2.18)$$

for $g \in G/H$.

Let us now recall a result of Duflo and Vergne about generalized equivariant forms which admit a 'restriction' (see Proposition 31 of [10] and Proposition 55 of [26]).

Using the identification of $\mathcal{A}^*(G \times_H M)$ with $\mathcal{C}^\infty\Big(G, [\wedge\mathfrak{g}^* \otimes \mathcal{A}^*(M)]_{H-hor}\Big)^H$, the restriction map to the neutral element $e \in G$, $\mathcal{A}^*(G \times_H M) \to [\wedge\mathfrak{g}^* \otimes \mathcal{A}^*(M)]_{H-hor}$, $\alpha \to \alpha_e$, defines a morphism

$$\begin{array}{rcl} \mathcal{A}_G^{-\infty}(G \times_H M) & \longrightarrow & \mathcal{C}^{-\infty}\Big(\mathfrak{g}\, , [\wedge\mathfrak{g}^* \otimes \mathcal{A}^*(M)]_{H-hor}\Big)^H \\ \alpha(X) & \longmapsto & \alpha_e(X)\ . \end{array}$$

Let $E_1, \cdots, E_p$ be a basis of $\mathfrak{g}^*$, and let $\{E_I = E_{i_1} \wedge \cdots \wedge E_{i_k}, I = [i_1 < i_2 < \cdots < i_k] \subset [1, 2, \ldots, p]\}$ be the corresponding basis of $\wedge\mathfrak{g}^*$. In particular, $E_\emptyset = 1$ generates $\mathbb{R} \subset \wedge\mathfrak{g}^*$. For each $\alpha \in \mathcal{A}_G^{-\infty}(G \times_H M)$, the form α_e can be decomposed relatively to the basis $\{E_I, I\}$: $\alpha_e = \sum_I (\alpha_e)_{[I]} E_I$ with $(\alpha_e)_{[I]} \in \mathcal{C}^{-\infty}\Big(\mathfrak{g}\ , \mathcal{A}^*(M)\Big)$.

We say that α_e admits a restriction to $\mathfrak{h}$ if each component $(\alpha_e)_{[I]}$ admits a restriction to $\mathfrak{h}$ (see [10] for this notion). We can then define $r_\mathfrak{h}(\alpha) := (\alpha_e)_{[\emptyset]}|_\mathfrak{h}$ that is a generalized H-equivariant form on M. This definition extends the usual restriction map $\mathcal{A}_G^\infty(G \times_H M) \to \mathcal{A}_H^\infty(M)$.

Proposition 2.14 *Let $\alpha \in \mathcal{A}_{G,*}^{-\infty}(G \times_H M)$ be a closed equivariant form on $G \times_H M$. Assume that α admits a restriction to $\mathfrak{h}$. Then*

$$Ind_{G/H}\Big(\Pi_{\mathfrak{g}/\mathfrak{h}}\, r_\mathfrak{h}(\alpha)\Big) = (-2\pi)^{-dim(G/H)/2}\,\alpha \quad \text{in} \quad \mathcal{H}_{G,*}^{-\infty}(G \times_H M)\ .$$

Remark 2.15 *There is a basic way to know if $\alpha \in \mathcal{A}_{G,*}^{-\infty}(G \times_H M)$ admits a restriction to $\mathfrak{h}$. Suppose there exist $\alpha^a \in \mathcal{A}_{G,*}^\infty(G \times_H M)$, $a > 0$, such that*

- $\lim_{a\to\infty} \alpha^a = \alpha$, and

- the (usual) restriction $\alpha_e^a|_\mathfrak{h} \in \mathcal{C}^\infty(\mathfrak{h}, \wedge\mathfrak{g}^* \otimes \mathcal{A}^*(M))$ converges in $\mathcal{C}^{-\infty}(\mathfrak{h}, \wedge\mathfrak{g}^* \otimes \mathcal{A}^*(M))$ when $a \to \infty$. In particular $r_\mathfrak{h}(\alpha^a) \in \mathcal{A}_{H,*}^\infty(M)$ converges in $\mathcal{A}_{H,*}^{-\infty}(M)$ when $a \to \infty$.

Then the equivariant form α admits a restriction to $\mathfrak{h}$ and we have $r_\mathfrak{h}(\alpha) = \lim_{a\to\infty} r_\mathfrak{h}(\alpha^a)$.

If the H-action on M comes from a G-action, we have a natural projection map $\pi : G \times_H M \to M$, $[g, m] \mapsto gm$, which makes $G \times_H M$ fibered over M with (oriented) fiber G/H. The integration along the fiber, $\int_{G/H}$, defines a morphism

$$\int_{G/H} : \mathcal{H}_{G,*}^{-\infty}(G \times_H M) \longrightarrow \mathcal{H}_{G,*}^{-\infty}(M)\ .$$

Definition 2.16 *For every G-manifold M, the induction map*

$$\mathrm{ind}_H^G : \mathcal{H}_{H,*}^{-\infty}(M) \longrightarrow \mathcal{H}_{G,*}^{-\infty}(M)$$

is by definition the composition of $Ind_{G/H} : \mathcal{H}_{H,*}^{-\infty}(M) \longrightarrow \mathcal{H}_{G,*}^{-\infty}(G \times_H M)$ with the morphism $\int_{G/H} : \mathcal{H}_{G,*}^{-\infty}(G \times_H M) \longrightarrow \mathcal{H}_{G,*}^{-\infty}(M)$.

For every $f \in \mathcal{C}^\infty(\mathfrak{g})$, we denote by $\overline{f}^G$ the G-invariant function on $\mathfrak{g}$ defined by the equation

$$\overline{f}^G(X) := \int_G f(Ad(g)X)dg, \quad X \in \mathfrak{g}, \tag{2.19}$$

where dg is the normalized Haar measure on G ($\int_G dg = 1$).

When the manifold M is a point the induction map $\mathrm{ind}_H^G : \mathcal{C}^{-\infty}(\mathfrak{h})^H \to \mathcal{C}^{-\infty}(\mathfrak{g})^G$ is defined by

$$\begin{aligned} < \mathrm{ind}_H^G(\Phi)(X), f(X)dX >_{\mathfrak{g}} &= \frac{\mathrm{vol}(G, dX)}{\mathrm{vol}(H, dY)} \int_{G/H} < \Phi(Y), f(Ad(g)Y)dY >_{\mathfrak{h}} \nu|_g \\ &= \frac{\mathrm{vol}(G, dX)}{\mathrm{vol}(H, dY)} < \Phi(Y), \overline{f}^G|_{\mathfrak{h}}(Y)dY >_{\mathfrak{h}}, \end{aligned} \tag{2.20}$$

for every $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{g})$. The first equality comes from (2.18), and the second one comes from the H-invariance of Φ.

Using the Weyl integration formula, we see in particular that $\mathrm{ind}_H^G(\Pi^2_{\mathfrak{g}/\mathfrak{h}})$ is equal to the constant function $\frac{|W_{\mathfrak{g}}|}{|W_{\mathfrak{h}}|}$, where $W_{\mathfrak{g}}$ and $W_{\mathfrak{h}}$ are the respective Weyl groups for G and H.

The induction map ind_H^G is functorial in the following way. Let $\psi : M \to B$ be a G-equivariant fibration of G-manifolds, and suppose that the fibers are oriented. We get a map in cohomology $\psi_* : \mathcal{H}^{-\infty}_{G,cpt}(M) \to \mathcal{H}^{-\infty}_{G,cpt}(B)$ which corresponds to the integration along the fibers of ψ. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}^{-\infty}_{H,cpt}(M) & \xrightarrow{\mathrm{ind}_H^G} & \mathcal{H}^{-\infty}_{G,cpt}(M) \\ \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \psi_*} \\ \mathcal{H}^{-\infty}_{H,cpt}(B) & \xrightarrow{\mathrm{ind}_H^G} & \mathcal{H}^{-\infty}_{G,cpt}(B). \end{array} \tag{2.21}$$

The diagram (2.21) will be frequently used in the case where B is a point, and M is oriented. Then the map ψ_* is the integration map $\int_M$, and the diagram (2.21) becomes

$$\begin{array}{ccc} \mathcal{H}^{-\infty}_{H,cpt}(M) & \xrightarrow{\mathrm{ind}_H^G} & \mathcal{H}^{-\infty}_{G,cpt}(M) \\ \downarrow{\scriptstyle \int_M} & & \downarrow{\scriptstyle \int_M} \\ \mathcal{C}^{-\infty}(\mathfrak{h})^H & \xrightarrow{\mathrm{ind}_H^G} & \mathcal{C}^{-\infty}(\mathfrak{g})^G. \end{array} \tag{2.22}$$

In the case where M is just a H-manifold we have also a relation between the induction map ind_H^G (defined for a point) and $Ind_{G/H}$ (defined between M and $\mathcal{M} := G \times_H M$). The following commutative diagram summarizes this relation

$$\begin{array}{ccc} \mathcal{H}^{-\infty}_{H,cpt}(M) & \xrightarrow{Ind_{G/H}} & \mathcal{H}^{-\infty}_{G,cpt}(\mathcal{M}) \\ \Big\downarrow{\scriptstyle \int_M} & & \Big\downarrow{\scriptstyle \int_{\mathcal{M}}} \\ \mathcal{C}^{-\infty}(\mathfrak{h})^H & \xrightarrow{\mathrm{ind}_H^G} & \mathcal{C}^{-\infty}(\mathfrak{g})^G. \end{array} \tag{2.23}$$

Using the commutativity of the diagram (2.23), and Proposition 2.14, we get

Corollary 2.17 *Let $\alpha \in \mathcal{A}^{-\infty}_{G,cpt}(\mathcal{M})$ be a closed equivariant form on $\mathcal{M} := G \times_H M$. Assume that α admits a restriction to $\mathfrak{h}$. We have then the following equality in $\mathcal{C}^{-\infty}(\mathfrak{g})^G$:*

$$\int_{\mathcal{M}} \alpha = (-2\pi)^{dim(G/H)/2}\ \mathrm{ind}_H^G\Big(\Pi_{\mathfrak{g}/\mathfrak{h}} \int_M r_{\mathfrak{h}}(\alpha)\Big).$$

(See (2.20) for the definition of the induction map $\mathrm{ind}_H^G : \mathcal{C}^{-\infty}(\mathfrak{h})^H \to \mathcal{C}^{-\infty}(\mathfrak{g})^G$.)

Induced metric on $G \times_H M$

Let $\|\cdot\|$ be a G-invariant Euclidean structure on $\mathfrak{g}$. We denote respectively by $pr_{\mathfrak{g}/\mathfrak{h}}$ and $pr_{\mathfrak{h}}$ the orthogonal projections $\mathfrak{g} \to \mathfrak{h}^\perp$ and $\mathfrak{g} \to \mathfrak{h}$. Let $(.,.)_M$ a H-invariant Riemannian metric on M. Associated to this metric on M we have a natural G-invariant Riemannian metric on $\mathcal{M} := G \times_H M$ which is defined by the following equation

$$\|X_{\mathcal{M}} + v_m\|^2_{\mathcal{M}}[g,m] := \|pr_{\mathfrak{g}/\mathfrak{h}}(g^{-1}X)\|^2_{\mathfrak{g}} + \|pr_{\mathfrak{h}}(g^{-1}X)_M|_m + v_m\|^2_M\ , \tag{2.24}$$

for $X \in \mathfrak{g}$ and $v_m \in \mathbf{T}_m M$. Here we use the identification between $\mathbf{T}_{[g,m]}\mathcal{M}$ and $(\mathbf{T}_g G \times \mathbf{T}_m M)\Big/ \sim$, ($\sim$ is the relation of equivalence coming from the H-orbits in $G \times M$).

This induced metric on $\mathcal{M}$ will be used in the following sections.

2.7 The pushforward via the Berline-Vergne localization

Here we extend the calculus of section 5.2 of [31] to the case of (possibly) non-compact manifolds. In this situation we will see the usefulness of the techniques developed in this section. We consider the case of a Hamiltonian action of a (compact) torus on a symplectic manifold (M, Ω). Let $\mu : M \to \mathfrak{t}^*$ be the corresponding moment map. We work with the following assumption.

Assumption 2.18 *We suppose the existence of $\beta \in \mathfrak{t}$ verifying the following conditions:*

- $M^\beta = M^T$.

- the function $\langle \mu, \beta \rangle : M \to \mathbb{R}$ is proper and bounded from below.$(\star)$

Remark 2.19 *In [33], E. Prato and S. Wu take the same condition* ($\star$)*, but suppose furthermore that* M^T *is finite. The regular elliptic orbits of simple real Lie groups* S*, equipped with the Hamiltonian action of a Cartan subgroup* T *of* S *satisfy these assumptions (see section 4 of [33]).*

Let $(.,.)_M$ be a T-invariant Riemannian product on M. The partition of unity in equivariant cohomology $\mathbf{1}_M = \mathrm{P}_\lambda + \mathcal{D}(\delta)$ (see subsection 2.2) is defined here with the T-invariant 1-form

$$\lambda = (\beta_M, .)_M \ .$$

The partition of unity gives for every closed form $\alpha \in \mathcal{A}_T^*(M)$ the decomposition $\alpha e^{\mathrm{i}\Omega_\mathfrak{t}} = \mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{t}} + \mathcal{D}(\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{t}})$, where the equivariant forms $\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}$ and $\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{t}}$ are tempered.

Proposition 2.20 *Let* $\alpha(X)$ *be a closed equivariant form depending* polynomially *on* $X \in \mathfrak{t}$*. The distributions* $\mathcal{F}(\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{t}})$ *and* $\mathcal{F}(\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{t}})$ *have a compact support* in $\mathfrak{t}^*$-mean *on* M*, and we have the equality of distributions over* $\mathfrak{t}^*$

$$\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) = \int_M \mathcal{F}(\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) \ .$$

Proof : Take $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{t}^*)$ with support in a ball of radius A. We have

$$\begin{aligned} &< \mathcal{F}(\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}), f >_{\mathfrak{t}^*} |_m = \chi(m) < \mathcal{F}(\alpha e^{\mathrm{i}\Omega_\mathfrak{t}}), f >_{\mathfrak{t}^*} |_m \\ &\quad + d\chi(m) \left(\int_0^\infty \mathrm{i} e^{-\mathrm{i}t\, d\lambda} e^{\mathrm{i}\Omega} \left[\alpha(-\mathrm{i}\partial_\xi) f\right](t\Phi_\lambda(m) + \mu(m))\, dt \right) \lambda_{|m} \ . \quad (2.25) \end{aligned}$$

We already known that the first term of the RHS of this equality has a compact support. For every $m \in M$, we have the following inequalities

$$\|t\Phi_\lambda(m) + \mu(m)\| \geq \|\beta\|^{-1} \langle t\Phi_\lambda(m) + \mu(m), \beta\rangle \geq \|\beta\|^{-1} \langle \mu(m), \beta\rangle,$$

because $\langle \Phi_\lambda, \beta\rangle = \|\beta_M\|^2 \geq 0$. These inequalities show that the second term of (2.25) is supported in $\langle \mu, \beta\rangle^{-1}([-\infty, \|\beta\| A])$ and so is compactly supported (see Assumption 2.18). We use the same argument to show that $\mathcal{F}(\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{t}})$ has a compact support *in* $\mathfrak{t}^*$*-mean* on M. If we apply equality (2.12) to the distribution $\mathcal{F} \circ \mathcal{D}(\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) = \widetilde{\mathcal{D}} \circ \mathcal{F}(\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{t}})$ we find that

$$\begin{aligned} \int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) &= \int_M \mathcal{F}(\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) + \int_M \mathcal{F}(\mathcal{D}(\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{t}})) \\ &= \int_M \mathcal{F}(\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) \ . \end{aligned}$$

□

The moment map being proper, each connected component $F \subset M^T$ is compact and we decompose the equivariant form

$$\mathrm{P}_\lambda = \sum_{F \subset M^T} \mathrm{P}_F \ ,$$

where the equivariant form $\mathrm{P}_F = \chi_F + d\chi_F(\int_0^\infty \mathrm{i}e^{-\mathrm{i}t\,\mathcal{D}\lambda}dt)\lambda$ is supported in a arbitrary small neighborhood of F. Let $0 < \varepsilon << 1$, and for every $F \subset M^T$ take χ_F supported in $\{m \in M,\ \|\mu(m) - \mu(F)\| \leq \varepsilon\}$. Using the equality (2.25), we see that for $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{t}^*)$ supported in a ball of radius A, the differential form $< \mathcal{F}(\mathrm{P}_F \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}), f >_{\mathfrak{t}^*} \neq 0$ only if $\langle \mu(F), \beta\rangle \leq \|\beta\|(A+\varepsilon)$ $(*)$. The properness of $\langle \mu, \beta\rangle$ tells us that for every A, there exist a finite number of $F \subset M^T$ verifying $(*)$. Finally we see that for every $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{t}^*)$, we have $< \mathcal{F}(\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}), f >_{\mathfrak{t}^*} = \sum_{F\subset M^T} < \mathcal{F}(\mathrm{P}_F \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}), f >_{\mathfrak{t}^*}$, where the sum of the RHS is in fact *finite*. Hence we have a locally finite decomposition

$$\begin{aligned}\int_M \mathcal{F}(\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) &= \sum_{F\subset M^T} \int_M \mathcal{F}(\mathrm{P}_F \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) \\ &= \sum_{F\subset M^T} \mathcal{F}\left(\int_M \mathrm{P}_F \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}\right).\end{aligned}$$

In the second equality we can intertwine the integral and the Fourier transform because each equivariant form $\mathrm{P}_F \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}$ has a compact support on M. Let N_F be the normal bundle of F in M. In Section 4 of [31], we have defined an inverse $\mathrm{Eul}_\beta^{-1}(N_F) \in \mathcal{H}_T^{-\infty}(F)$ of the equivariant Euler form $\mathrm{Eul}(N_F)$. Corollary 3.10 and Theorem 5.1 of [31] show that, for every closed form $\eta \in \mathcal{A}_T^\infty(M)$

$$\int_M \mathrm{P}_F \eta = \int_F i_F^*(\eta)\, \mathrm{Eul}_\beta^{-1}(N_F)\ ,$$

where $i_F : F \to M$ denotes the inclusion. In Section 4 of [31], we show that the Fourier transform of $\mathrm{Eul}_\beta^{-1}(N_F)$ is a locally polynomial density supported by the half space $\{\xi \in \mathfrak{t}^*,\ \langle \xi, \beta\rangle \geq 0\}$, with values in the characteristic classes of N_F.

Theorem 2.21 *For every closed equivariant form $\alpha \in \mathcal{A}_T^*(M)$, the distribution $\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_\mathfrak{t}})$ is a locally finite sum $\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) = \sum_{F\subset M^T} \mathfrak{D}_F^\alpha$ where the distributions*

$$\mathfrak{D}_F^\alpha := \delta_{\mu(F)} * \mathcal{F}\left(\int_F e^{\mathrm{i}\Omega_F} i_F^*(\alpha)\, \mathrm{Eul}_\beta^{-1}(N_F)\right)$$

are tempered *and are supported by the half-space $\{\xi \in \mathfrak{t}^*,\ \langle \xi - \mu(F), \beta\rangle \geq 0\}$. Here, $\delta_{\mu(F)}$ denotes the Dirac measure in $\mu(F) \in \mathfrak{t}^*$, and "$*$" denotes the convolution product.*

If we take $\alpha = 1$ in the last theorem we have the following decomposition of $\mu_*(dm_L)$.

Corollary 2.22 *We have the following equality of locally polynomial measures on $\mathfrak{t}^*$:*

$$\mu_*(dm_L) = \sum_{F\subset M^T} dm_F(\xi),$$

where each locally polynomial measure $dm_F = \frac{1}{\mathrm{i}^n}\delta_{\mu(F)} * \int_F e^{\mathrm{i}\Omega_F} \mathcal{F}(\mathrm{Eul}_\beta^{-1}(N_F))$ is tempered *and supported by the half-space $\{\xi \in \mathfrak{t}^*,\ \langle \xi - \mu(F), \beta\rangle \geq 0\}$.*

3 Local behaviour of the pushforward

In this section we assume that a symplectic manifold M is provided with a Hamiltonian action of a compact connected Lie group G with Lie algebra $\mathfrak{g}$. We denote by Ω the symplectic form on M, and suppose that the moment map relative to the G-action, $\mu : M \to \mathfrak{g}^*$, is *proper*. Our purpose is to describe the local behaviour of the distributions $\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})$ that we have introduced in the previous section.

With the use of the partition of unity, we show, in the first subsection, that there exists a closed equivariant form $\mathrm{P}^0_\lambda \in \mathcal{A}^{-\infty}_{G,cpt}(M)$ equal to 1 in a neighborhood of $\mu^{-1}(0)$ such that $\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) = \mathcal{F}(\int_M \mathrm{P}^0_\lambda \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})$ in a neighborhood of 0. When 0 is a *regular* value of μ, the computation of P^0_λ leads to a new proof of the Jeffrey-Kirwan-Witten formula.

For the study of the pushforward near a coadjoint orbit $\mathcal{O} = G.\xi$, we prove in the second subsection an induction formula that holds in the entire *slice* through ξ. The proof use the fact that there exists a closed equivariant form $\mathrm{P}^{\mathcal{O}}_{\lambda\xi} \in \mathcal{A}^{-\infty}_{G,cpt}(M)$ equal to 1 in a neighborhood of $\mu^{-1}(\mathcal{O})$ such that $\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) = \mathcal{F}\Big(\int_M \mathrm{P}^{\mathcal{O}}_{\lambda\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}\Big)$ in a neighborhood of $\mathcal{O}$.

In the next chapter we will show that, under suitable conditions, the equivariant forms P^0_λ and $\mathrm{P}^{\mathcal{O}}_{\lambda\xi}$ are induced by T-equivariant forms that have the good property to be *computable* in terms of equivariant Thom forms supported in the neighborhood of submanifolds, and inverse of equivariant Euler forms (see sub-section 4.3).

3.1 Jeffrey-Kirwan-Witten formula in the non-compact setting

If 0 is a regular value of the moment map μ, we shall consider the symplectic orbifold $M_{red} := \mu^{-1}(0)/G$ with the symplectic form Ω_{red}, called the Marsden-Weinstein reduction of M.

Let $\mathcal{H}^\infty_G(M)$ be the equivariant cohomological algebra, with $\mathcal{C}^\infty$ coefficients, of the manifold M. The Kirwan map $k_o : \mathcal{H}^\infty_G(M) \to \mathcal{H}(M_{red})$ is the composition of the restriction map from $\mathcal{H}^\infty_G(M)$ to $\mathcal{H}^\infty_G(\mu^{-1}(0))$ and the natural isomorphism from $\mathcal{H}^\infty_G(\mu^{-1}(0))$ to $\mathcal{H}(M_{red})$. In fact the (usual) Kirwan map $k'_o : \mathcal{H}^*_G(M) \to \mathcal{H}(M_{red})$ factors through the map k_o: we have $k'_o = k_o \circ \tau$ where $\tau : \mathcal{H}^*_G(M) \to \mathcal{H}^\infty_G(M)$ is the morphism of extension of coefficients. When M is *compact*, the map k'_o is surjective (see [25]), and thus k_o is also *surjective*.

Let $\Omega_{\mathfrak{g}}(X) := \Omega + \langle \mu, X\rangle,\ X \in \mathfrak{g}$ be the equivariant symplectic form. The Fourier transform is defined by the equation (2.10).

Theorem 3.1 (Jeffrey-Kirwan-Witten) *Suppose that M is compact, and 0 is a regular value of μ. Let $\eta(X) := \alpha(X)e^{\mathrm{i}\Omega_{\mathfrak{g}}(X)} \in \mathcal{H}^\infty_G(M)$ where $\alpha(X)$ is a closed equivariant form depending polynomially on $X \in \mathfrak{g}$. Consider the Fourier transform $\mathcal{F}(\int_M \eta)$ of $\int_M \eta$. Near 0, the generalized density $\mathcal{F}(\int_M \eta)$ is a polynomial density $P(\xi)d\xi$ and*

$$P(0) = (\mathrm{i})^{dimG} \frac{\mathrm{vol}(G, dX)}{|S_o|} \int_{M_{red}} k_o(\eta)\ ,$$

where $|S_o|$ is the cardinal of the generic stabilizer of G on $\mu^{-1}(0)$. In this formula, dX is the Euclidean measure on $\mathfrak{g}$ dual of $d\xi$, $\mathrm{vol}(G, dX)$ is the volume of G for the Haar measure on G compatible with dX.

When the manifold is not compact the integral $\int_M \alpha e^{\mathrm{i}\Omega_\mathfrak{g}}$ does not have a meaning in general, but as we saw in the last chapter, the map μ being proper, we can define the distribution $\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_\mathfrak{g}})$ and we are going to study it.

Choose a G-invariant Euclidean norm $\|.\|$ on $\mathfrak{g}^*$. Following Witten [36], we consider the G-invariant 1-form λ defined below.

Definition 3.2 *Let λ be the following G-invariant 1-form*

$$\lambda := (\mathcal{H}, .)_M \ ,$$

where $\mathcal{H}$ is the Hamiltonian vector field associated to the function $\frac{1}{2}\|\mu\|^2$, and $(.,.)_M$ is a G-invariant Riemannian metric on M.

We have

$$\{\Phi_\lambda = 0\} = \mathrm{Cr}(\|\mu\|^2) \ ,$$

because the vector field $\mathcal{H}$ is always tangent to the G-orbits in M.

Remark 3.3 *For every element $c \in \mathcal{Z}(\mathfrak{g})$ in the center of $\mathfrak{g}$, we can translate the moment map by c, and consider the new moment map $\mu - c$. In the same way we define the G-invariant 1-form $\lambda^c := (\mathcal{H}^c, \cdot)_M$, where $\mathcal{H}^c$ is the Hamiltonian vector field associated to the function $\frac{1}{2}\|\mu - c\|^2$, and we still have $\{\Phi_{\lambda^c} = 0\} = \mathrm{Cr}(\|\mu - c\|^2)$. Proposition 3.4 and Corollary 3.6 remain true if we consider λ^c instead of λ.*

Following subsection 2.2 (*see section 3 of [31]*), we define, with the 1-form λ, a partition of unity in equivariant cohomology: $\mathbf{1}_M = \mathrm{P}_\lambda + \mathcal{D}(\delta)$. For every closed form $\alpha \in \mathcal{A}_G^*(M)$, the partition of unity decomposes the equivariant form $\alpha e^{\mathrm{i}\Omega_\mathfrak{g}}$ in the following way

$$\alpha e^{\mathrm{i}\Omega_\mathfrak{g}} = \mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{g}} + \mathcal{D}\Big(\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{g}}\Big) \ .$$

Proposition 3.4 *Let $\alpha(X) \in \mathcal{A}_G^*(M)$ be an equivariant form depending* polynomially *on $X \in \mathfrak{g}$. The equivariant forms $\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{g}}$ and $\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{g}}$ are tempered and their Fourier transform $\mathcal{F}(\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{g}})$ and $\mathcal{F}(\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{g}})$ have a compact support* in $\mathfrak{g}^*$-mean *on M.*

Proof : Take $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$ with support in the ball $B(o, A)$ of radius A. We have

$$\begin{aligned} < \mathcal{F}(\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{g}}), f >_{\mathfrak{g}^*} |_m \ = \ \chi(m) < \mathcal{F}(\alpha e^{\mathrm{i}\Omega_\mathfrak{g}}), f >_{\mathfrak{g}^*} |_m \quad + \\ d\chi(m) \left(\int_0^\infty \mathrm{i} e^{-\mathrm{i}t\, d\lambda} e^{\mathrm{i}\Omega} \left[\alpha(-\mathrm{i}\partial_\xi) f\right](t\Phi_\lambda(m) + \mu(m))\, dt \right) \lambda_{|m} \ . \end{aligned} \tag{3.26}$$

We know from Lemma 2.12 that the first term of the RHS of this equality is a differential form with compact support on M. We have the following inequalities

$$\|t\Phi_\lambda + \mu\|^2 = \|\mu\|^2 + t^2 \|\Phi_\lambda\|^2 + 2t\langle \Phi_\lambda, \mu \rangle \geq \|\mu\|^2 \ , \tag{3.27}$$

because $\langle \Phi_\lambda, \mu \rangle = \|\mathcal{H}\|_M^2 \geq 0$ on M. The RHS of (3.26) is zero at $m \in M$ if $\mu(m)$ and $t\Phi_\lambda(m) + \mu(m)$ are not in the support of f, for every $t \geq 0$. Using the inequality (3.27), we see finally that the support of $< \mathcal{F}(\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{g}}), f >_{\mathfrak{g}^*}$ is included in $\mu^{-1}(B(o, A))$, and so is compact. We prove in the same way that $\mathcal{F}(\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{g}})$ has a compact support *in* $\mathfrak{g}^*$*-mean* on M. $\square$

Remark 3.5 *If $\mathcal{U}$ is a good open set for the* 1*-form λ, Proposition 3.4 remains true for the equivariant form* $\mathrm{P}_\lambda^{\mathcal{U}}$.

Now, if we use the fact that $\mathcal{F} \circ \mathcal{D}(\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{g}}) = \widetilde{\mathcal{D}} \circ \mathcal{F}(\delta\alpha e^{\mathrm{i}\Omega_\mathfrak{g}})$, Lemma 2.11 gives

Corollary 3.6 *Let $\alpha(X)$ be a closed equivariant form depending* polynomially *on $X \in \mathfrak{g}$. We have the following equality of G-invariant distributions on $\mathfrak{g}^*$*

$$\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_\mathfrak{g}}) = \int_M \mathcal{F}(\mathrm{P}_\lambda \alpha e^{\mathrm{i}\Omega_\mathfrak{g}}) \ .$$

Recall Lemma 12 of [35]

Lemma 3.7 *Let R be the largest number u such that all $\xi \in \mathfrak{g}^*,\ \|\xi\|^2 < u$ are regular values of μ. Then R is also the smallest non-zero critical value of the function $\|\mu\|^2$.*

Take two real numbers $\varepsilon,\ r$ such that $0 < \varepsilon < r < R$. The equivariant form P_λ can be decomposed relatively to $M_\varepsilon = \{m \in M, \|\mu(m)\| < \varepsilon\}$ and $\mathcal{U}_{out} = \{m \in M, \|\mu(m)\| > r\}$. The component of $\mathrm{Cr}(\|\mu\|^2)$ included in M_ε is $\mu^{-1}(0)$.

To simplify the notation, we will write P_λ^0 for the equivariant form $\mathrm{P}_\lambda^{M_\varepsilon}$ and P_λ^{out} for the equivariant form $\mathrm{P}_\lambda^{\mathcal{U}_{out}}$ (see Definition 2.3). We have decomposed the equivariant form $\mathrm{P}_\lambda = \mathrm{P}_\lambda^0 + \mathrm{P}_\lambda^{out}$, and we will study separately the distributions $\int_M \mathcal{F}(\mathrm{P}_\lambda^0 \alpha e^{\mathrm{i}\Omega_\mathfrak{g}})$ and $\int_M \mathcal{F}(\mathrm{P}_\lambda^{out} \alpha e^{\mathrm{i}\Omega_\mathfrak{g}})$.

Proposition 3.8 *The support of the distribution $\int_M \mathcal{F}(\mathrm{P}_\lambda^{out} \alpha e^{\mathrm{i}\Omega_\mathfrak{g}})$ is contained in $\{\xi \in \mathfrak{g}^*,\ \|\xi\| \geq R\}$, where the constant R has been defined at Lemma 3.7. In particular, for every closed equivariant form $\alpha(X)$ depending* polynomially *on $X \in \mathfrak{g}$, the following equality of G-invariant distributions on $\mathfrak{g}^*$*

$$\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_\mathfrak{g}}) = \mathcal{F}\Big(\int_M \mathrm{P}_\lambda^0 \alpha e^{\mathrm{i}\Omega_\mathfrak{g}} \Big) \tag{3.28}$$

holds in the open ball $B_o(o, R)$.

Proof : The equivariant form P_λ^{out} has been defined with a choice of real r such that $0 < \varepsilon < r < R$ and a function $\chi_{out} \in \mathcal{C}^\infty(M)^G$ with support contained in $\{m \in M,\ \|\mu(m)\| > r\}$. But the distribution $\int_M \mathcal{F}(\mathrm{P}_\lambda^{out} \alpha e^{\mathrm{i}\Omega_\mathfrak{g}})$ does not depend on this choice. If we take another r' and χ'_{out} which define $\mathrm{P}_\lambda^{out'}$, we have $\mathrm{P}_\lambda^{out} - \mathrm{P}_\lambda^{out'} = \mathcal{D}(\delta')$ (see Proposition 3.3 of [31]) where the

generalized form $\mathcal{F}(\delta'\alpha e^{i\Omega_{\mathfrak{g}}})$ has a compact support *in* $\mathfrak{g}^*$*-mean* on M, hence $\int_M \mathcal{F}(\mathrm{P}_\lambda^{out}\alpha e^{i\Omega_{\mathfrak{g}}}) = \int_M \mathcal{F}(\mathrm{P}_\lambda^{out'}\alpha e^{i\Omega_{\mathfrak{g}}})$.

Take $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{g}^*)$ with support in a ball $B(o,A)$ of radius A, and compute the differential form $< \mathcal{F}(\mathrm{P}_\lambda^{out}\alpha e^{i\Omega_{\mathfrak{g}}}), f >_{\mathfrak{g}^*}$ as in the equation (3.26). This differential form is not equal to 0 at a point $m \in M$, only if

- $\chi_{out}(m) \neq 0$ and $\mu(m)$ is in the support of f,

or

- $d\chi_{out}(m) \neq 0$ and $t\,\Phi_\lambda(m) + \mu(m)$ is in the support of f for some $t \geq 0$.

Because $\|t\,\Phi_\lambda + \mu\| \geq \|\mu\|$ for every $t \geq 0$, both of these conditions imply that $support\{\chi_{out}\} \cap \mu^{-1}\Big(B(o,A)\Big)$ is not empty: in particular $A > r$. This shows that the differential form $< \mathcal{F}(\mathrm{P}_\lambda^{out}\alpha e^{i\Omega_{\mathfrak{g}}}), f >_{\mathfrak{g}^*} = 0$ if the support of f is included in $\{\xi \in \mathfrak{g}^*,\ \|\xi\| < r\}$. We have finally shown that the support of the distribution $\int_M \mathcal{F}(\mathrm{P}_\lambda^{out}\alpha e^{i\Omega_{\mathfrak{g}}})$ is included in $\{\xi \in \mathfrak{g}^*,\ \|\xi\| \geq r\}$ for every $r < R$. We know from Corollary 3.6 that $\int_M \mathcal{F}(\alpha e^{i\Omega_{\mathfrak{g}}}) = \int_M \mathcal{F}(\mathrm{P}_\lambda^0\alpha e^{i\Omega_{\mathfrak{g}}}) + \int_M \mathcal{F}(\mathrm{P}_\lambda^{out}\alpha e^{i\Omega_{\mathfrak{g}}})$ in $\mathcal{M}^{-\infty}(\mathfrak{g}^*)^G$. Finally we see that $\int_M \mathcal{F}(\alpha e^{i\Omega_{\mathfrak{g}}}) = \mathcal{F}(\int_M \mathrm{P}_\lambda^0\alpha e^{i\Omega_{\mathfrak{g}}})$ in the ball $B_o(o,R)$ (we have interchanged the integral $\int_M$ and the Fourier transform because the equivariant form P_λ^0 has a compact support on M). $\square$

We show in Section 4 how to compute the equivariant form P_λ^0, through an induction, in the case where 0 is not a regular value of the moment map. Now we compute the distribution $\mathcal{F}(\int_M \mathrm{P}_\lambda^0\alpha e^{i\Omega_{\mathfrak{g}}})$, when 0 is a regular value of the map $\mu : M \to \mathfrak{g}^*$, and show that is in fact a polynomial density.

Since μ is regular at 0, there exists, for $\varepsilon > 0$ small enough, a G-equivariant diffeomorphism

$$\psi : M_{2\varepsilon} \xrightarrow{\sim} \mathcal{W}\ , \tag{3.29}$$

where $\mathcal{W}$ is an open neighborhood of $\mu^{-1}(0)$ in $\mu^{-1}(0) \times \mathfrak{g}^*$, and $pr_2 \circ \psi = \mu$ (pr_2 is the projection map $\mu^{-1}(0) \times \mathfrak{g}^* \to \mathfrak{g}^*$).

Let $\sigma \in \mathcal{A}^1(\mu^{-1}(0)) \otimes \mathfrak{g}$ be a connection form for the principal bundle $\mu^{-1}(0) \to \mu^{-1}(0)/G = M_{red}$. We define a G-invariant 1-form γ on $\mu^{-1}(0) \times \mathfrak{g}^*$ in the following way

$$(m,\xi) \in \mu^{-1}(0) \times \mathfrak{g}^*, \quad \gamma_{|(m,\xi)} := \langle \sigma_{|m}, \xi \rangle\ . \tag{3.30}$$

We see that $\{\Phi_\gamma = 0\} = \mu^{-1}(0)$, and we define the equivariant form P_γ with compact support in $\{(m,\xi) \in \mu^{-1}(0) \times \mathfrak{g}^*,\ \|\xi\| < \varepsilon\}$.

Lemma 3.9 *The equivariant forms* P_λ^0 *and* $(\psi)^*(\mathrm{P}_\gamma)$ *are equal in* $\mathcal{H}^{-\infty}_{G,cpt}(M)$.

Proof : This is just an application of Proposition 2.6, with the open set $\mathcal{U} = M_\varepsilon$ and the function $f = \mu$ (where we have identified $\mathfrak{g}$ and $\mathfrak{g}^*$ via the scalar product). The functions $\langle \Phi_\lambda, \mu \rangle = \|\mathcal{H}\|^2$ and $\langle \Phi_{(\psi)^*(\gamma)}, \mu \rangle = \|\mu\|^2$ are strictly positive on $\partial\mathcal{U}$, hence are bounded from below by some $\rho > 0$ because $\partial\mathcal{U}$ is compact. Finally

$$\mathrm{P}_\lambda^0 = \mathrm{P}_\lambda^{M_\varepsilon} = \mathrm{P}^{M_\varepsilon}_{(\psi)^*(\gamma)} = (\psi)^*(\mathrm{P}_\gamma) \quad \text{in} \quad \mathcal{H}^{-\infty}_{G,cpt}(M).$$

$\square$

Via the diffeomorphism $\psi : M_{2\varepsilon} \to \mathcal{W} \subset \mu^{-1}(0) \times \mathfrak{g}^*$, we have

$$\begin{aligned} \int_M \mathrm{P}^0_\lambda \, \alpha e^{\mathfrak{i}\Omega_\mathfrak{g}} &= \int_{\mu^{-1}(0)\times\mathfrak{g}^*} \mathrm{P}_\gamma \, (\psi^{-1})^* \Big(\alpha e^{\mathfrak{i}\Omega_\mathfrak{g}} \Big) \\ &= \int_{\mu^{-1}(0)} i^*(\alpha e^{\mathfrak{i}\Omega_\mathfrak{g}}) \int_{\mathfrak{g}^*} \mathrm{P}_\gamma \ , \end{aligned} \tag{3.31}$$

where $i : \mu^{-1}(0) \to M$ is the inclusion. The equality (3.31) is due to the fact that the equivariant form P_γ has a compact support on $\mu^{-1}(0) \times \mathfrak{g}^*$. The 1-form γ is homogeneous along the fibers of $\mathfrak{g}^*$, then Proposition 2.9 gives

$$\int_{\mathfrak{g}^*} \mathrm{P}_\gamma = \int_{\mathfrak{g}^*} e^{-\mathfrak{i}\,\mathcal{D}\gamma} \quad \text{in} \quad \mathcal{H}_G^{-\infty}(\mu^{-1}(0)) \ . \tag{3.32}$$

Decompose the connection form with respect to the basis $E^1, E^2, \cdots, E^p$ of $\mathfrak{g}$: $\sigma = \sum_k \sigma_k E^k$. Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$. For every $P \in S(\mathfrak{g})$ we note $P(\partial_X|_0)$ the linear map

$$P(\partial_X|_0) : \mathcal{C}^\infty(\mathfrak{g}) \longrightarrow \mathbb{C}$$

defined by the equation $< P(\partial_X|_0), f >= \big(P(\frac{\partial}{\partial X}) f\big)\,(X)_{|X=0}$. Let $\omega = \sum_k \omega_k E^k$ be the curvature of σ: $\omega \in \mathcal{A}^2(\mu^{-1}(0)) \otimes \mathfrak{g}$. The element $e^\omega \in \mathcal{A}^{even}(\mu^{-1}(0)) \otimes S(\mathfrak{g})$ defines the map

$$e^{\omega(\partial_X|_0)} : \mathcal{C}^\infty(\mathfrak{g}) \longrightarrow \mathcal{A}^{even}(\mu^{-1}(0))$$

that we will note $f(\omega) :=< e^{\omega(\partial_X|_0)}, f >$.

Orientation: The manifold $N = \mu^{-1}(0) \times \mathfrak{g}^*$ is oriented by the symplectic form of M via the isomorphism ψ (equation (3.29)). The orbifold M_{red} is oriented by its symplectic form. We choose the orientation $o(\mu^{-1}(0))$ of $\mu^{-1}(0)$ given by the relation $o(\mu^{-1}(0)) = o(M_{red}) \wedge \sigma_p \wedge \cdots \wedge \sigma_1$. We can verify that $o(N) = o(\mu^{-1}(0)) \wedge dE^1 \wedge \cdots \wedge dE^p$. This means that $\mathfrak{g}^*$ is oriented by the volume form $dE^1 \wedge \cdots \wedge dE^p$.

Proposition 3.10 *For every $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{g})$, we have*

$$< \int_{\mathfrak{g}^*} e^{-\mathfrak{i}\,\mathcal{D}\gamma(X)}, f(X)dX >_\mathfrak{g} = (2\mathfrak{i}\pi)^p \, f(\omega) \wedge \frac{\sigma_p \wedge \cdots \wedge \sigma_1}{\mathrm{vol}(G)} \, \mathrm{vol}(G, dX) \ ,$$

where $\mathrm{vol}(G, dX)$ *is the volume of G for the Haar measure on G compatible with dX, and* $\mathrm{vol}(G) = \mathrm{vol}(G, dE_1 \cdots dE_p)$.

Proof : This computation is done (with different notations) at Proposition 80 of [26]. □

In another way

$$\int_{\mathfrak{g}^*} e^{-\mathfrak{i}\,\mathcal{D}\gamma} = (2\mathfrak{i}\pi)^p \, e^{\omega(\partial_X|_0)} \wedge \frac{\sigma_p \wedge \cdots \wedge \sigma_1}{\mathrm{vol}(G)} \, \mathrm{vol}(G, -) \ .$$

Using Propositions 3.9 and 3.10, and the equality (3.32), we can state the following

Proposition 3.11 *The generalized function* $\int_M \mathrm{P}^0_\lambda\, \alpha e^{\mathfrak{i}\Omega_\mathfrak{g}}$ *is supported at* 0 *and we have the following equality*

$$\int_M \mathrm{P}^0_\lambda\, \alpha e^{\mathfrak{i}\Omega_\mathfrak{g}} = (2\mathfrak{i}\pi)^p \left(\int_{\mu^{-1}(0)} k_0(\alpha e^{\mathfrak{i}\Omega_\mathfrak{g}}) e^{\omega(\partial_X|_0)} \frac{\sigma_p \wedge \cdots \wedge \sigma_1}{\mathrm{vol}(G)} \right) \mathrm{vol}(G,-)\ .$$

In particular, $< \int_M \mathrm{P}^0_\lambda \alpha e^{\mathfrak{i}\Omega_\mathfrak{g}}, f(X)dX >_\mathfrak{g} = \frac{(2\mathfrak{i}\pi)^p}{|S_o|} \left(\int_{M_{red}} k_0(\alpha e^{\mathfrak{i}\Omega_\mathfrak{g}}) f(\omega) \right) \mathrm{vol}(G, dX)$, *for every* $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{g})^G$.

We have $\mathcal{F}\Big(e^{\omega(\partial_X|_0)}\,\mathrm{vol}(G,-)\Big)(\xi) = e^{-\mathfrak{i}\langle\omega,\,\xi\rangle}\, d\xi^{can}/(2\pi)^p$, where $p = dim G$, and $d\xi^{can}$ is the Euclidean measure on $\mathfrak{g}^*$ such that its dual measure dX^{can} on $\mathfrak{g}$ verifies $\mathrm{vol}(G, dX^{can}) = 1$. Corollary 3.6, Propositions 3.8 and 3.11 give the following

Theorem 3.12 *Suppose that* 0 *is a regular value of the moment map. Let* $\alpha \in \mathcal{A}^*_G(M)$ *be a closed equivariant form. Let* R *be the smallest non zero critical value of* $\|\mu\|^2$. *On the open ball* $B_o(o,R)$ *the distribution* $\int_M \mathcal{F}(\alpha e^{\mathfrak{i}\Omega_\mathfrak{g}})$ *is a polynomial density, given by the equation*

$$\int_M \mathcal{F}(\alpha e^{\mathfrak{i}\Omega_\mathfrak{g}}) = (\mathfrak{i})^p \left(\int_{\mu^{-1}(0)} k_0(\alpha e^{\mathfrak{i}\Omega_\mathfrak{g}}) e^{-\mathfrak{i}\langle\omega,\,\xi\rangle} \frac{\sigma_p \cdots \sigma_1}{\mathrm{vol}(G)} \right) d\xi^{can}\ .$$

3.2 Local induction

For every sub-algebra $\mathfrak{h}$ of $\mathfrak{g}$, we denote by $\mathcal{F}_\mathfrak{h} : \mathcal{C}^{-\infty}_{temp}(\mathfrak{h}) \to \mathcal{M}^{-\infty}_{temp}(\mathfrak{h}^*)$ the corresponding Fourier transform (see equation (2.10)). From now on, we fix a G-invariant scalar product $(\cdot,\cdot)$ on $\mathfrak{g}^*$. For notational convenience, we shall identify, for every sub-algebra $\mathfrak{h}$, $\mathfrak{h} \simeq \mathfrak{h}^*$ under the invariant inner product $(\cdot,\cdot)$.

Now we want to study the behaviour of the pushforward $\int_M \mathcal{F}_\mathfrak{g}(\alpha e^{\mathfrak{i}\Omega_\mathfrak{g}})$ near a co-adjoint orbit $\mathcal{O}$ of $\mathfrak{g}^*$ by an induction which is closed to those defined by Vergne in [35] (see section 1.4.) and by Duflo-Heckman-Vergne in [9]. Here, our induction works even if $\mathcal{O}$ is not included in the set of regular values of μ, and moreover, it can be performed on the *entire* neighborhood $G \times_{G_\sigma} U_\sigma$ of $\mathcal{O}$, where U_σ is a *slice* at a point $\xi \in \mathcal{O}$. For this purpose we use the cross section theorem of Guillemin-Sternberg [18] (see Theorem 26.7).

Let T be a maximal torus of G with Lie algebra $\mathfrak{t}$, and let $W := W(G,T)$ be the Weyl group associated. We make a choice of a Weyl chamber $\mathfrak{t}^*_+$ in $\mathfrak{t}^*$. Let ξ be the unique point in $\mathfrak{t}^*_+$ such that $\mathcal{O} = G.\xi$, and let σ be the unique open face of $\mathfrak{t}^*_+$ which contains ξ. The stabilizer subgroup $G^{\xi'} \subset G$ does not depend on the choice of $\xi' \in \sigma$, and is denoted G_σ. Let $\mathfrak{g}_\sigma$ be the Lie algebra of G_σ, and let $\mathfrak{g}^*_\sigma$ be the dual vector space.

Following Guillemin-Sternberg [18], we introduce the following G_σ-invariant open subset of $\mathfrak{g}^*_\sigma$:

$$U_\sigma := G_\sigma.\{y \in \mathfrak{t}^*_+ \mid G_y \subset G_\sigma\} = G_\sigma. \bigcup_{\sigma \subset \bar{\tau}} \tau\ . \tag{3.33}$$

Here $\{\sigma \subset \bar{\tau}\}$ is the set of all faces τ of $\mathfrak{t}^*_+$ which contain σ in their closure. By construction, U_σ is a *slice* for the coadjoint action at any $\xi \in \sigma$ (see Definition 3.1 of [27]). This means that the map $G \times U_\sigma \to \mathfrak{g}^*$, $(g, \xi) \mapsto g.\xi$, factors through an inclusion $G \times_{G_\sigma} U_\sigma \hookrightarrow \mathfrak{g}^*$.

The *symplectic cross-section theorem* [18] asserts that the pre-image $Y_\sigma = \mu^{-1}(U_\sigma)$ is a symplectic submanifold provided with a Hamiltonian action of the group G_σ. The restriction $\mu|_{Y_\sigma}$ is a moment map for the action of G_σ on Y_σ that we denote by μ_σ. Moreover, the set $\mathcal{M}_\sigma := G \cdot Y_\sigma = \mu^{-1}(G \times_{G_\sigma} U_\sigma)$ is a G-invariant *open* subset of M *diffeomorphic* to $G \times_{G_\sigma} Y_\sigma$ (see Remark 3.5 of [27]). The manifold Y_σ is oriented by its symplectic form, and $G \times_{G_\sigma} Y_\sigma$ is oriented by the symplectic form on M. Hence, we get an orientation o for G/G_σ which determines a polynomial square root $Y \to \Pi_{\mathfrak{g}/\mathfrak{g}_\sigma}(Y) := \det^{1/2}_{\mathfrak{g}/\mathfrak{g}_\sigma, o}(Y)$, $Y \in \mathfrak{g}_\sigma$. Note that $\Pi_{\mathfrak{g}/\mathfrak{g}_\sigma}$ never vanishes on the open subset U_σ.

Let $\alpha \in \mathcal{A}^*_G(M)$ be a closed equivariant form. The restriction of the distribution $\int_M \mathcal{F}_{\mathfrak{g}}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})$ to the neighborhood $G \times_{G_\sigma} U_\sigma$ of $\mathcal{O}$ is equal to $\int_{\mathcal{M}_\sigma} \mathcal{F}_{\mathfrak{g}}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) \in \mathcal{M}^{-\infty}(G \times_{G_\sigma} U_\sigma)^G$ (see equation (2.15)), but in order to simplify the notation we will not distinguish between $\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) \in \mathcal{M}^{-\infty}(\mathfrak{g})^G$, and its restriction to $G \times_{G_\sigma} U_\sigma$.

Descent of distributions on $\mathfrak{g}$

We know that we have a one to one correspondence $\Phi \to \Phi_{[U_\sigma]}$ from $\mathcal{M}^{-\infty}(G \times_{G_\sigma} U_\sigma)^G$ to $\mathcal{M}^{-\infty}(U_\sigma)^{G_\sigma}$ (see [7], section 5.1) where $\Phi_{[U_\sigma]}$ is defined by

$$\int_{G \times_{G_\sigma} U_\sigma} f(X)\Phi(X) = \int_{U_\sigma} f|_{\mathfrak{g}_\sigma}(Y)\, \Pi^2_{\mathfrak{g}/\mathfrak{g}_\sigma}(Y)\, \Phi_{[U_\sigma]}(Y)\ ,$$

where, for every $f \in \mathcal{C}^\infty_{cpt}(G \times_{G_\sigma} U_\sigma)^G$, we denote by $f|_{\mathfrak{g}_\sigma} \in \mathcal{C}^\infty_{cpt}(U_\sigma)^{G_\sigma}$ the (usual) restriction to $\mathfrak{g}_\sigma$.

The 2-form Ω on M restricts to a symplectic 2-form $\Omega|_{Y_\sigma}$ on the submanifold Y_σ, and we denote by $\Omega_{\mathfrak{g}_\sigma}(Y) := \Omega|_{Y_\sigma} + \langle \mu_\sigma, Y \rangle$, $Y \in \mathfrak{g}_\sigma$, the corresponding G_σ-equivariant symplectic form. We denote by $\alpha \mapsto r_{\mathfrak{g}_\sigma}(\alpha)$, $\mathcal{H}^\infty_{G,cpt}(M) \to \mathcal{H}^\infty_{G_\sigma,cpt}(Y_\sigma)$ the morphism of restriction for equivariant forms with $\mathcal{C}^\infty$ coefficients.

We can now state the main result of this section.

Proposition 3.13 *Let $\alpha(X)$ be a closed G-equivariant form on M depending* polynomially *on $X \in \mathfrak{g}$. We have the following equality of G_σ-invariant distributions on U_σ*

$$\left(\int_M \mathcal{F}_{\mathfrak{g}}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})\right)_{[U_\sigma]} = \mathrm{cst}\ \frac{1}{\Pi_{\mathfrak{g}/\mathfrak{g}_\sigma}} \int_{Y_\sigma} \mathcal{F}_{\mathfrak{g}_\sigma}\left(r_{\mathfrak{g}_\sigma}(\alpha) e^{\mathrm{i}\Omega_{\mathfrak{g}_\sigma}}\right) \tag{3.34}$$

with $\mathrm{cst} = (-\mathrm{i})^{dim(G/G_\sigma)/2}\, \mathrm{vol}(G/G_\sigma)$. *Here, the volume* $\mathrm{vol}(G/G_\sigma)$ *is computed with a measure $dE^1 \cdots dE^{2r}$, where $E^1, \ldots, E^{2r}$ is an orthonormal basis of the orthogonal complement $\mathfrak{g}^\perp_\sigma$ of $\mathfrak{g}_\sigma$ in $\mathfrak{g}$.*

Corollary 3.14 *For every closed form $\alpha \in \mathcal{A}^*_G(M)$, the corresponding G-invariant distribution $\int_M \mathcal{F}_{\mathfrak{g}}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})$ is an* analytic density *on each connected component of the open set of regular values of the moment map.*

This Corollary which is an easy consequence of Propositions 3.11 and 3.13 was already obtained in [35] (when M is compact).

For $\alpha = 1$, one can prove Proposition 3.13 using arguments similar to those of [9] (see equation (8)). For a general closed form $\alpha \in \mathcal{A}^*_G(M)$, we reduce the proof of this Proposition to the proof of Lemma 3.15.

If we see M as the manifold $Y_{\{0\}}$, and $\mathfrak{g}$ as the open subset $U_{\{0\}}$ corresponding to the minimal facet $\{0\}$ of $\mathfrak{t}^*_+$, we can rewrite the equation (3.34) for two facets τ, σ of $\mathfrak{t}^*_+$ such that $\sigma \subset \bar{\tau}$. Then, Y_τ is a G_τ-Hamiltonian sub-manifold of Y_σ, G_τ is a (connected) subgroup of G_σ, and the set $G_\sigma \times_{G_\tau} U_\tau$ is an open subset of U_σ. We denote by $\Phi \to \Phi_{[\sigma,\tau]}, \quad \mathcal{M}^{-\infty}(U_\sigma)^{G_\sigma} \to \mathcal{M}^{-\infty}(U_\tau)^{G_\tau}$, the morphism equal to the composition of the restriction map $\mathcal{M}^{-\infty}(U_\sigma)^{G_\sigma} \to \mathcal{M}^{-\infty}(G_\sigma \times_{G_\tau} U_\tau)^{G_\sigma}$ with the (iso-)morphism of descent $\Phi \to \Phi_{[U_\tau]}, \quad \mathcal{M}^{-\infty}(G_\sigma \times_{G_\tau} U_\tau)^{G_\sigma} \to \mathcal{M}^{-\infty}(U_\tau)^{G_\tau}$. By definition we have the composition law $(\Phi_{[\sigma,\tau]})_{[\tau,\varrho]} = \Phi_{[\sigma,\varrho]}$.

Proposition 3.13 applied to (σ, τ) says that, for every closed equivariant form $\alpha \in \mathcal{A}^*_{G_\sigma}(Y_\sigma)$, we have the following equality of G_τ-invariant distributions on U_τ,

$$\left(\int_{Y_\sigma} \mathcal{F}_{\mathfrak{g}_\sigma}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}_\sigma}})\right)_{[\sigma,\tau]} = c(\sigma,\tau)\ \frac{1}{\Pi_{\mathfrak{g}_\sigma/\mathfrak{g}_\tau}} \int_{Y_\tau} \mathcal{F}_{\mathfrak{g}_\tau}\left(r_{\mathfrak{g}_\tau}(\alpha) e^{\mathrm{i}\Omega_{\mathfrak{g}_\tau}}\right)\ , \qquad (3.35)$$

with $c(\sigma,\tau) = (-\mathrm{i})^{dim(G_\sigma/G_\tau)/2}\,\mathrm{vol}(G_\sigma/G_\tau)$.

Lemma 3.15 *The equality (3.34) holds on a neighborhood of σ.*

Lemma 3.15 $\Longrightarrow$ Proposition 3.13

Assuming Lemma 3.15, we have to show that the equality (3.34) holds in a neighborhood of the facets $\tau,\ \sigma \subset \bar{\tau}$. Or equivalently that

$$\left(\left(\int_M \mathcal{F}_{\mathfrak{g}}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})\right)_{[\{0\},\sigma]}\right)_{[\sigma,\tau]} = \frac{c(\{0\},\sigma)}{\Pi_{\mathfrak{g}/\mathfrak{g}_\sigma}} \left(\int_{Y_\sigma} \mathcal{F}_{\mathfrak{g}_\sigma}\left(r_{\mathfrak{g}_\sigma}(\alpha) e^{\mathrm{i}\Omega_{\mathfrak{g}_\sigma}}\right)\right)_{[\sigma,\tau]} \qquad (3.36)$$

holds in a neighborhood of τ in U_τ. If we apply Lemma 3.15 to (σ,τ), the equality (3.35) holds in a neighborhood of τ. Then the equality (3.36) is equivalent to

$$\left(\int_M \mathcal{F}_{\mathfrak{g}}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})\right)_{[\{0\},\tau]} = \frac{c(\{0\},\tau)}{\Pi_{\mathfrak{g}_\sigma/\mathfrak{g}_\tau}} \int_{Y_\tau} \mathcal{F}_{\mathfrak{g}_\tau}\left(r_{\mathfrak{g}_\tau}(\alpha) e^{\mathrm{i}\Omega_{\mathfrak{g}_\tau}}\right) \qquad (3.37)$$

in a neighborhood of τ. Here we use the relations $c(\{0\},\sigma)c(\sigma,\tau) = c(\{0\},\tau)$ and $\Pi_{\mathfrak{g}/\mathfrak{g}_\sigma}.\Pi_{\mathfrak{g}_\sigma/\mathfrak{g}_\tau} = \Pi_{\mathfrak{g}/\mathfrak{g}_\tau}$. But Lemma 3.15 applied to $(\{0\},\tau)$ tells us exactly that (3.37) holds in a neighborhood of τ. □

Proof of Lemma 3.15

The rest of this subsection is devoted to the proof of Lemma 3.15.

Take now a point $\xi \in \sigma$. We have to prove that (3.34) holds in a neighborhood of the orbit $\mathcal{O} := G.\xi$. For this we define the following 1-form on the G-invariant open subset $\mathcal{M}_\sigma = G \times_{G_\sigma} Y_\sigma$ of M.

Let $\mu^\xi : \mathcal{M}_\sigma \to \mathfrak{g}^*$ be the G-equivariant map defined by the equation

$$\mu^\xi([g,m]) := g.(\mu_\sigma(m) - \xi), \quad [g,m] \ \in \ \mathcal{M}_\sigma \ , \tag{3.38}$$

and denote by $\mathcal{H}^\xi$ the corresponding vector field on $\mathcal{M}_\sigma$: $\mathcal{H}^\xi|_{[g,m]} := \mu^\xi([g,m])_{\mathcal{M}_\sigma}|_{[g,m]}$ for $[g,m] \in \mathcal{M}_\sigma$.

Definition 3.16 *We denote by λ^ξ the following G-invariant 1-form on $\mathcal{M}_\sigma$:*

$$\lambda^\xi := (\mathcal{H}^\xi, \cdot)_M \ .$$

A straightforward computation shows that $\{\Phi_{\lambda^\xi} = 0\} = G \times_{G_\sigma} \mathrm{Cr}(\|\mu_\sigma - \xi\|^2)$, and in particular $\{\Phi_{\lambda^\xi} = 0\}$ contains $\mu^{-1}(\mathcal{O}) = G \times_{G_\sigma} \mu_\sigma^{-1}(\xi)$.

Let P_{λ^ξ} be the G-equivariant form on $\mathcal{M}_\sigma$ defined with the 1-form λ^ξ (see Definition 2.4). As in subsection 3.1 we have the decomposition $\mathrm{P}_{\lambda^\xi} = \mathrm{P}^{\mathcal{O}}_{\lambda^\xi} + \mathrm{P}^{out}_{\lambda^\xi}$ where $\mathrm{P}^{\mathcal{O}}_{\lambda^\xi} \in \mathcal{A}^{-\infty}_{G,cpt}(\mathcal{M}_\sigma)$ is supported in a small neighborhood of $\mu^{-1}(\mathcal{O})$. Because $\mathcal{M}_\sigma$ is an open subset of the manifold M, we shall consider $\mathrm{P}^{\mathcal{O}}_{\lambda^\xi}$ as an element of $\mathcal{A}^{-\infty}_{G,cpt}(M)$.

Proposition 3.17 *Near $\mathcal{O}$ we have the equality of G-invariant distributions*

$$\int_M \mathcal{F}_{\mathfrak{g}}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) = \mathcal{F}_{\mathfrak{g}}\Big(\int_M \mathrm{P}^{\mathcal{O}}_{\lambda^\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}} \Big) \ .$$

Proof : The proof follows the same line than those of subsection 3.1. Let $r > 0$ such that the closed ball $B_c(\xi, r) := \{x \in \mathfrak{g}^*_\sigma, \ \|x - \xi\| \leq r\}$ is included in U_σ. Then $\mu^{-1}(G.B_c(\xi, r))$ is a compact neighborhood of $\mu^{-1}(\mathcal{O})$ in $\mathcal{M}_\sigma$. Let $\mathcal{U}^\xi \subset \mathfrak{g}^*$ be the open subset $G.B_o(\xi, r)$ (where $B_o(\xi, r) := \{x \in \mathfrak{g}^*_\sigma, \ \|x - \xi\| < r\}$) and take $\mathrm{P}^{\mathcal{O}}_{\lambda^\xi}$ with support in $\mu^{-1}(\mathcal{U}^\xi)$.

With the 1-form λ^ξ, we have a partition of unity $\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}} = \mathrm{P}^{\mathcal{O}}_{\lambda^\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}} + \mathrm{P}^{out}_{\lambda^\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}} + \mathcal{D}(\delta\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})$ of G-equivariant differential forms on $\mathcal{M}_\sigma$. Like in Proposition 3.4, we show that, for each $A \in \{\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}, \mathrm{P}^{\mathcal{O}}_{\lambda^\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}, \mathrm{P}^{out}_{\lambda^\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}, \delta\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}\}$ and each function $f \in \mathcal{C}^\infty_{cpt}(\mathcal{U}^\xi)$, the differential form $< \mathcal{F}_{\mathfrak{g}}(A), f >_{\mathfrak{g}^*} \in \mathcal{A}^*(\mathcal{M}_\sigma)$ has a *compact* support in $\mathcal{M}_\sigma$ (in fact included in $\mu^{-1}(\mathcal{U}^\xi)$). Lemma 2.11 gives, like in Corollary 3.6, the following equality in $\mathcal{M}^{-\infty}(\mathcal{U}^\xi)^G$

$$\int_M \mathcal{F}_{\mathfrak{g}}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) = \int_M \mathcal{F}_{\mathfrak{g}}(\mathrm{P}^{\mathcal{O}}_{\lambda^\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) + \int_M \mathcal{F}_{\mathfrak{g}}(\mathrm{P}^{out}_{\lambda^\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) \ .$$

Now, Proposition 3.8 tells us that the distribution $\int_M \mathcal{F}_{\mathfrak{g}}(\mathrm{P}^{out}_{\lambda^\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) \in \mathcal{M}^{-\infty}(\mathcal{U}^\xi)^G$ is equal to zero in a neighborhood of $\mathcal{O}$. The G-equivariant form $\mathrm{P}^{\mathcal{O}}_{\lambda^\xi}$ has a compact support on M, which implies $\int_M \mathcal{F}_{\mathfrak{g}}(\mathrm{P}^{\mathcal{O}}_{\lambda^\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) = \mathcal{F}_{\mathfrak{g}}(\int_M \mathrm{P}^{\mathcal{O}}_{\lambda^\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})$. □

We will show now that the closed G-equivariant form $\mathrm{P}^{\mathcal{O}}_{\lambda^\xi}$ is induced by a closed G_σ-equivariant form on Y_σ.

Fix a G_σ-invariant metric $(\cdot,\cdot)_{Y_\sigma}$ on Y_σ and denote by $(\cdot,\cdot)_{\mathcal{M}_\sigma}$ the induced G-invariant Riemannian metric on the manifold $\mathcal{M}_\sigma := G \times_{G_\sigma} Y_\sigma$ (see equation (2.24)).

Definition 3.18 *Let $\lambda^\xi_\sigma := (\mathcal{H}^\xi_\sigma, \cdot)_{Y_\sigma}$ be the G_σ-invariant 1-form on Y_σ, where $\mathcal{H}^\xi_\sigma$ is the Hamiltonian vector field of $\frac{1}{2}\|\mu_\sigma - \xi\|^2$. Let P^σ_ξ be the G_σ-equivariant form on Y_σ defined with the 1-form λ^ξ_σ in a neighborhood of the component $\mu_\sigma^{-1}(\xi)$ of $\mathrm{Cr}(\|\mu_\sigma - \xi\|^2)$.*

Remark 3.19 *From now on, the equivariant form $\mathrm{P}^{\mathcal{O}}_{\lambda^\xi}$ will be defined with the 1-form $(\mathcal{H}^\xi, \cdot)_{\mathcal{M}_\sigma}$. Using Proposition 2.6, we know that this modification does not change the class of $\mathrm{P}^{\mathcal{O}}_{\lambda^\xi}$. For convenience, we will still denote by λ^ξ the 1-form $(\mathcal{H}^\xi, \cdot)_{\mathcal{M}_\sigma}$.*

The inclusion $\mathcal{M}_\sigma \hookrightarrow M$ of an open subset defines a natural morphism $\mathcal{A}^*_{cpt}(\mathcal{M}_\sigma) \to \mathcal{A}^*_{cpt}(M)$ on the differential forms with compact support. We denote by

$$Ind_{G/G_\sigma} : \mathcal{H}^{-\infty}_{G_\sigma,cpt}(Y_\sigma) \longrightarrow \mathcal{H}^{-\infty}_{G,cpt}(M) \tag{3.39}$$

the morphism equal to the composition of the induction map $\mathcal{H}^{-\infty}_{G_\sigma,cpt}(Y_\sigma) \to \mathcal{H}^{-\infty}_{G,cpt}(\mathcal{M}_\sigma)$ (see subsection 2.6) with the natural morphism $\mathcal{H}^{-\infty}_{G,cpt}(\mathcal{M}_\sigma) \to \mathcal{H}^{-\infty}_{G,cpt}(M)$.

Proposition 3.20 *The closed G-equivariant form $\mathrm{P}^{\mathcal{O}}_{\lambda^\xi} \in \mathcal{A}^{-\infty}_{G,cpt}(\mathcal{M}_\sigma)$ admits a restriction to $\mathfrak{g}_\sigma$ equal to $\mathrm{P}^\sigma_\xi \in \mathcal{A}^{-\infty}_{G_\sigma,cpt}(Y_\sigma)$. This gives the following equality*

$$Ind_{G/G_\sigma}\Big(\Pi_{\mathfrak{g}/\mathfrak{g}_\sigma}\, \mathrm{P}^\sigma_\xi\, r_{\mathfrak{g}_\sigma}(\eta)\Big) = (-2\pi)^{-dim(G/G_\sigma)/2}\, \mathrm{P}^{\mathcal{O}}_{\lambda^\xi}\eta \quad \text{in} \quad \mathcal{H}^{-\infty}_{G,cpt}(M)\ , \tag{3.40}$$

for every $\eta \in \mathcal{H}^\infty_G(M)$. In particular, we get

$$\int_M \mathrm{P}^{\mathcal{O}}_{\lambda^\xi}\eta = (-2\pi)^{dim(G/G_\sigma)/2}\ \mathrm{ind}^G_{G_\sigma}\Big(\Pi_{\mathfrak{g}/\mathfrak{g}_\sigma}\int_{Y_\sigma}\mathrm{P}^\sigma_\xi\, r_{\mathfrak{g}_\sigma}(\eta)\Big), \tag{3.41}$$

for every $\eta \in \mathcal{H}^\infty_G(M)$. (See equation (2.20) for the definition of the induction map $\mathrm{ind}^G_{G_\sigma} : \mathcal{C}^{-\infty}(\mathfrak{g}_\sigma)^{G_\sigma} \to \mathcal{C}^{-\infty}(\mathfrak{g})^G$.)

Proof : If $i_\sigma : Y_\sigma \hookrightarrow \mathcal{M}_\sigma$ denotes the G_σ-equivariant inclusion, we see that $i^*_\sigma(\lambda^\xi) = \lambda^\xi_\sigma$. Let $\chi \in \mathcal{C}^\infty_{cpt}(\mathcal{M}_\sigma)^G$ be the function coming from a G_σ-invariant function χ' on Y_σ, where χ' is with compact support, equal to 1 in a neighborhood of $\mu_\sigma^{-1}(\xi)$, and $support\{\chi'\} \cap \mathrm{Cr}(\|\mu_\sigma - \xi\|^2) = \mu_\sigma^{-1}(\xi)$. Then the function χ, which verifies $support\{\chi\} \cap \{\Phi_{\lambda^\xi} = 0\} = G \times_{G_\sigma} \mu_\sigma^{-1}(\xi)$, defines the closed G-equivariant form $Q := \chi + d\chi\left(\int_0^\infty \mathrm{i}e^{-\mathrm{i}t\,\mathcal{D}\lambda^\xi}dt\right)\lambda^\xi$ which is equal to $\mathrm{P}^{\mathcal{O}}_{\lambda^\xi}$ in $\mathcal{H}^{-\infty}_{G,cpt}(M)$. We are now in the situation of the Remark 2.15. We have $Q = \lim_{a\to\infty} Q^a$ with $Q^a := \chi + d\chi\left(\int_0^a \mathrm{i}e^{-\mathrm{i}t\,\mathcal{D}\lambda^\xi}dt\right)\lambda^\xi \in \mathcal{A}^\infty_{G,cpt}(\mathcal{M}_\sigma)$. We verify that the restriction $Q^a_e|_{\mathfrak{g}_\sigma}$ converges in $\mathcal{C}^{-\infty}(\mathfrak{g}_\sigma, \wedge\mathfrak{g}^* \otimes \mathcal{A}^*(Y_\sigma))$ when $a \to \infty$. In particular, the equivariant forms $r_{\mathfrak{g}_\sigma}(Q^a) = \chi' + d\chi'\left(\int_0^a \mathrm{i}e^{-\mathrm{i}t\,\mathcal{D}\lambda^\xi_\sigma}dt\right)\lambda^\xi_\sigma \in \mathcal{A}^\infty_{G_\sigma,cpt}(Y_\sigma)$ converge to P^σ_ξ when $a \to \infty$.

This shows that the generalized G-equivariant form Q admits a restriction to M equal to P^σ_ξ, and the equality (3.40) is just a consequence of Proposition

2.14. Using now Corollary 2.17, the equality (3.41] is just a consequence of (3.40).□

We fix an orthonormal basis $E^1, \dots, E^{2r}, \dots, E^p$ of $\mathfrak{g}$ such that $E^{2r+1} \dots, E^p$ is a basis of $\mathfrak{g}_\sigma$. Let $dX = dE^1 \cdots dE^p$ and $dY = dE^{2r+1} \cdots dE^p$ be the corresponding normalized Lebesgue measure on $\mathfrak{g}$ and $\mathfrak{g}_\sigma$. For every $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{g})$ we denote by $\mathcal{F}^{dX}_{\mathfrak{g}}(f)(X') := \frac{1}{(2\pi)^{dimG}} \int_{\mathfrak{g}} e^{-\mathrm{i}(X,X')} f(X) dX$ the Fourier transform of f relatively to dX (and in the same way $\mathcal{F}^{dY}_{\mathfrak{g}_\sigma}$). Our Lemma follows from the relation of Harish-Chandra

$$\Pi_{\mathfrak{g}/\mathfrak{g}_\sigma} \mathcal{F}^{dX}_{\mathfrak{g}}(f)|_{\mathfrak{g}_\sigma} = (-2\mathrm{i}\pi)^{-r} \mathcal{F}^{dY}_{\mathfrak{g}_\sigma}(\Pi_{\mathfrak{g}/\mathfrak{g}_\sigma} f|_{\mathfrak{g}_\sigma}), \quad \text{for every } f \in \mathcal{C}^\infty_{cpt}(\mathfrak{g})^G, \ (3.42)$$

where $r = \dim(G/G_\sigma)/2$ (see Lemma 15 in [20] or Lemma 3.4.[1] in [23]).The formula 3.42 is usually stated and proved when $\mathfrak{g}_\sigma = \mathfrak{t}$, but one can see easily that this case implies the others.

Take now $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{g}^*)^G$ with support in a small neighborhood of the orbit $\mathcal{O} = G.\xi$. Finally, we have the following equalities (after identification of $\mathfrak{g}$ and $\mathfrak{g}^*$; $\mathfrak{g}_\sigma$ and $\mathfrak{g}^*_\sigma$).

$$\begin{aligned}
&< \int_M \mathcal{F}_{\mathfrak{g}}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}),\ f >_{\mathfrak{g}} = < \mathcal{F}_{\mathfrak{g}}\Big(\int_M \mathrm{P}^{\mathcal{O}}_{\lambda\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}} \Big),\ f >_{\mathfrak{g}} \quad [1] \\
&= \ < \int_M \mathrm{P}^{\mathcal{O}}_{\lambda\xi} \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}, \mathcal{F}^{dX}_{\mathfrak{g}}(f) dX >_{\mathfrak{g}} \\
&= \ (-2\pi)^r \frac{\mathrm{vol}(G)}{\mathrm{vol}(G_\sigma)} < \int_{Y_\sigma} \mathrm{P}^{\sigma}_{\xi}\, r_{\mathfrak{g}_\sigma}(\alpha)\, e^{\mathrm{i}\Omega_{\mathfrak{g}_\sigma}}, \Pi_{\mathfrak{g}/\mathfrak{g}_\sigma} \mathcal{F}^{dX}_{\mathfrak{g}}(f)|_{\mathfrak{g}_\sigma} dY >_{\mathfrak{g}_\sigma} \quad [2] \\
&= \ (-\mathrm{i})^r \frac{\mathrm{vol}(G)}{\mathrm{vol}(G_\sigma)} < \mathcal{F}_{\mathfrak{g}_\sigma}\Big(\int_{Y_\sigma} \mathrm{P}^{\sigma}_{\xi}\, r_{\mathfrak{g}_\sigma}(\alpha)\, e^{\mathrm{i}\Omega_{\mathfrak{g}_\sigma}} \Big), \Pi_{\mathfrak{g}/\mathfrak{g}_\sigma} f|_{\mathfrak{g}_\sigma} >_{\mathfrak{g}_\sigma} \quad [3] \\
&= \ (-\mathrm{i})^r \frac{\mathrm{vol}(G)}{\mathrm{vol}(G_\sigma)} < \int_{Y_\sigma} \mathcal{F}_{\mathfrak{g}_\sigma}\Big(r_{\mathfrak{g}_\sigma}(\alpha)\, e^{\mathrm{i}\Omega_{\mathfrak{g}_\sigma}} \Big), \Pi_{\mathfrak{g}/\mathfrak{g}_\sigma} f|_{\mathfrak{g}_\sigma} >_{\mathfrak{g}_\sigma}, \quad [4]
\end{aligned}$$

where $\mathrm{vol}(G) = \mathrm{vol}(G, dX)$ and $\mathrm{vol}(G_\sigma) = \mathrm{vol}(G_\sigma, dY)$. The point [1] is due to Proposition 3.17; the point [2] is just the equation (3.41); the point [3] comes from the relation of Harish-Chandra (see (3.42)) and the point [4] is due to Proposition 3.8 (applied to the manifold Y_σ with the Hamiltonian G_σ-action at the point ξ belonging to the center of $\mathfrak{g}_\sigma$). This last equality shows that the equality (3.34) holds in a neighborhood of $\mathcal{O}$, and hence that Lemma 3.15 has been proved. □

Two kinds of things remain to be done. First we have to compute the equivariant forms P^0_λ, P^σ_ξ in the general case where 0 or ξ are not necessarily regular values of the moment map. On the other hand, it will be interesting to have a *global* expression for the pushforward. For this we have to compute the term $\int_M \mathcal{F}(\mathrm{P}^{out}_\lambda \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}})$. In general, one of the main difficulties in these computations is the non-smoothness of some components of $\mathrm{Cr}(\|\mu\|^2)$.

[1]The reader should note that the factor $(-\mathrm{i})^{-\dim(G/T)/2}$ is missing in the RHS of equation 3.10 and in the formula of Lemma 3.4. of [23].

Under suitable conditions we will bypass this difficulty by showing that we can reduce our problem to a maximal torus of G. In Section 4.3, we will make the computations in the case of a Hamiltonian action of a torus. In this case, we can perturb the moment map with a constant ε: $\mu_\varepsilon := \mu - \varepsilon$. For generic ε, the set of critical point $\mathrm{Cr}(\|\mu_\varepsilon\|^2)$ is smooth. Moreover for small ε, the perturbation does not affect the cohomology classes of the equivariant forms $\mathrm{P}_\lambda^{\mathcal{U}}$.

4 Reduction to the maximal torus

The notations are the same as before.

Let T be a maximal torus of G, and $\mathfrak{t}$ its Lie algebra. Let $W = N_{_G}(T)/T$ be the Weyl group associated. We have a natural decomposition $\mathfrak{g}^* = \mathfrak{t}^* \oplus \mathfrak{t}^\perp$, where $\mathfrak{t}^*$ is identified with the T-invariant subspace of $\mathfrak{g}^*$, and $\mathfrak{t}^\perp$ is the orthogonal (for the duality) of $\mathfrak{t}$. We fix a Weyl chamber $\mathfrak{t}^*_+$ in $\mathfrak{t}^*$, and denote by $\Pi_{\mathfrak{g}/\mathfrak{t}}$ the associated polynomial on $\mathfrak{t}$ equal to the product of the positive roots. This choice of a Weyl chamber induces an orientation on the manifold G/T. We denote by $\mathcal{F}_{\mathfrak{g}}$ (resp. $\mathcal{F}_{\mathfrak{t}}$) the Fourier transform for the tempered G-equivariant forms (resp. T-equivariant forms).

Notations: *Using the isomorphism $\mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}$ coming from the G-invariant scalar product on $\mathfrak{g}^*$, we will make no difference between $\mathfrak{g}^*$ and $\mathfrak{g}$, $\mathfrak{t}^*$ and $\mathfrak{t}$, in our notations.*

We will denote by $\mu_{_G} : M \to \mathfrak{g}^*$ the moment map associated with the Hamiltonian G-action. The action of T is also Hamiltonian, with moment map $\mu_{_T} : M \to \mathfrak{t}^*$ equal to the composition of $\mu_{_G}$ with the (orthogonal) projection $\mathfrak{g}^* \to \mathfrak{t}^*$.

We will denote by $\lambda_{_T}$ and $\lambda_{_G}$ the 1-forms defined respectively with the Hamiltonian vector fields of $\|\mu_{_T}\|^2$ and $\|\mu_{_G}\|^2$ (see Definition 3.2).

Kirwan [25] gives the following description of the critical points of $\|\mu_{_G}\|^2$. Let $\mathcal{B}_{_G}$ be the collection of points of $\mathfrak{t}^*_+$ defined by the relation $\beta \in \mathcal{B}_{_G} \iff \{\beta \in \mathfrak{t}^*_+ \text{ and } M^\beta \cap \mu_{_G}^{-1}(\beta) \neq \emptyset\}$. We have

$$\mathrm{Cr}(\|\mu_{_G}\|^2) = G\left(\bigcup_{\beta \in \mathcal{B}_{_G}} M^\beta \cap \mu_{_G}^{-1}(\beta)\right) , \tag{4.43}$$

where M^β denotes the submanifold $\{m \in M,\ \beta_M(m) = 0\}$. For the critical points of $\|\mu_{_T}\|^2$ we have

$$\mathrm{Cr}(\|\mu_{_T}\|^2) = \bigcup_{\beta \in \mathcal{B}_{_T}} M^\beta \cap \mu_{_T}^{-1}(\beta) ,$$

where $\mathcal{B}_{_T}$ is the subset of $\mathfrak{t}$ defined by the relation $\beta \in \mathcal{B}_{_T} \iff \beta \in \mathfrak{t}$ and $M^\beta \cap \mu_{_T}^{-1}(\beta) \neq \emptyset$.

The Weyl group acts on $\mathrm{Cr}(\|\mu_{_T}\|^2)$ in the following way: $w.\Big(M^\beta \cap \mu_{_T}^{-1}(\beta)\Big) = M^{w\beta} \cap \mu_{_T}^{-1}(w\beta)$ for every $w \in W$. Thus, W acts on $\mathcal{B}_{_T}$, and we have $\mathcal{B}_{_T} = W.(\mathcal{B}_{_T} \cap \mathfrak{t}^*_+)$. The inclusions $\mu_{_G}^{-1}(\beta) \subset \mu_{_T}^{-1}(\beta),\ \beta \in \mathfrak{t}^*$ show that $\mathcal{B}_{_G} \subset \mathcal{B}_{_T} \cap \mathfrak{t}^*_+$.

Assumption 4.1 *We suppose in this section that the moment map $\mu_{_T}$ is proper (Therefore $\mu_{_G}$ is also proper).*

These properness conditions give the following

Lemma 4.2 *For each $\beta \in \mathcal{B}_{_G}$, $C^G_\beta := G\Big(M^\beta \cap \mu_{_G}^{-1}(\beta)\Big)$ is a non-empty compact component of $\mathrm{Cr}(\|\mu_{_G}\|^2)$. In the same way, for each $\beta \in \mathcal{B}_{_T}$, $C^T_\beta := M^\beta \cap \mu_{_T}^{-1}(\beta)$ is a non-empty compact component of $\mathrm{Cr}(\|\mu_{_T}\|^2)$.*

For each $\beta \in \mathcal{B}_{_G}$, we denote by

$$\mathrm{P}^G_\beta \in \mathcal{A}^{-\infty}_{G,cpt}(M) \tag{4.44}$$

the closed equivariant form $\mathrm{P}^{C^G_\beta}_{\lambda_G}$ defined with the 1-form λ_G in a neighborhood of C^G_β (see Definition 2.3). The equivariant forms P^0_λ, P^{out}_λ of Section 3.1 correspond here, respectively to P^G_0 and to $\sum_{\beta\neq 0}\mathrm{P}^G_\beta$ (where the sum is taken on $\mathcal{B}_{_G}$).

In the same way, for each $\beta \in \mathcal{B}_{_T}$, we denote by

$$\mathrm{P}^T_\beta \in \mathcal{A}^{-\infty}_{T,cpt}(M)$$

the closed equivariant form $\mathrm{P}^{C^T_\beta}_{\lambda_T}$ defined with the 1-form λ_T in a neighborhood of C^T_β.

Remark 4.3 *As an application of Proposition 2.6, we see that the cohomology classes of P^T_β and P^G_β do not depend on the choice of the G-(or T-)invariant Riemannian metric on M.*

We have now a global expression for the pushforward.

Proposition 4.4 *For every closed form $\alpha \in \mathcal{A}^*_G(M)$ we have the following equality in $\mathcal{M}^{-\infty}(\mathfrak{g}^*)^G$*

$$\int_M \mathcal{F}_{\mathfrak{g}}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}) = \sum_{\beta\in\mathcal{B}_{_G}} \mathcal{F}_{\mathfrak{g}}\Big(\int_M \mathrm{P}^G_\beta \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}\Big)\ , \tag{4.45}$$

where the distribution $\mathcal{F}_{\mathfrak{g}}\Big(\int_M \mathrm{P}^G_\beta \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}\Big)$ are tempered. *The same holds for the T-equivariant forms. For every closed form $\alpha \in \mathcal{A}^*_T(M)$, we have $\int_M \mathcal{F}_{\mathfrak{t}}(\alpha e^{\mathrm{i}\Omega_{\mathfrak{t}}}) = \sum_{\beta\in\mathcal{B}_{_T}} \mathcal{F}_{\mathfrak{t}}\Big(\int_M \mathrm{P}^T_\beta \alpha e^{\mathrm{i}\Omega_{\mathfrak{t}}}\Big)$ in $\mathcal{M}^{-\infty}(\mathfrak{t}^*)$.*

Proof : Because $\mu_{_G}$ is proper, we see that the intersections of $\mathcal{B}_{_G}$ with the compact subsets of $\mathfrak{g}^*$ are *finite*. Like in Proposition 3.8, we can show that, for every $\beta \in \mathcal{B}_{_G}$, the distribution $\int_M \mathcal{F}_{\mathfrak{g}}(\alpha\, \mathrm{P}^G_\beta e^{\mathrm{i}\Omega_{\mathfrak{g}}})$ is supported outside the ball $B(o, \|\beta\|)$. Then for every $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{g}^*)$ the terms $< \int_M \mathcal{F}_{\mathfrak{g}}(\alpha\, \mathrm{P}^G_\beta e^{\mathrm{i}\Omega_{\mathfrak{g}}}), f >_{\mathfrak{g}}$ are non zero for a finite number of β in $\mathcal{B}_{_G}$. This fact insures that the RHS

of (4.45) is a well defined distribution on $\mathfrak{g}^*$ (we have $\int_M \mathcal{F}_{\mathfrak{g}}\Big(\mathrm{P}^G_\beta \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}\Big) = \mathcal{F}_{\mathfrak{g}}\Big(\int_M \mathrm{P}^G_\beta \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}\Big)$ because P^G_β has a compact support on M). Using the decomposition $\mathrm{P}_{\lambda_G} = \sum_\beta \mathrm{P}^G_\beta$, Corollary 3.6 finally gives the equality (4.45). The generalized function $\int_M \mathrm{P}^G_\beta \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}$ is tempered because the equivariant form $\mathrm{P}^G_\beta \alpha e^{\mathrm{i}\Omega_{\mathfrak{g}}}$ is tempered and with compact support on M. $\square$

4.1 Symplectic induction

The action of the Weyl group W on the algebra $\mathcal{A}^\infty_{T,cpt}(M)$ extends naturally to an action on $\mathcal{A}^{-\infty}_{T,cpt}(M)$ which commutes with the differential $\mathcal{D}$. For every $\gamma \in \mathcal{A}^{-\infty}_{T,cpt}(M)$ and $w \in W$, the equivariant form $w.\gamma$ is defined by the equation

$$< w.\gamma, \phi >_{\mathfrak{t}} := w \cdot_1 < \gamma, w^{-1} \cdot_2 \phi >_{\mathfrak{t}} \quad \text{for every} \quad \phi \in m_c(\mathfrak{t}),$$

where $\cdot_1$ denotes the action of W on $\mathcal{A}^*(M)$ and $\cdot_2$ denotes the action of W on $m_c(\mathfrak{t})$. We have defined a closed equivariant form P^T_β for every $\beta \in \mathcal{B}_T$, and we know that W acts on $\mathcal{B}_T$. A straightforward computation shows that

$$w.\mathrm{P}^T_\beta = \mathrm{P}^T_{w.\beta} \quad \text{in} \quad \mathcal{H}^{-\infty}_{T,cpt}(M), \tag{4.46}$$

for every $w \in W$ and $\beta \in \mathcal{B}_T$.

Let $\mathrm{ind}^G_T : \mathcal{H}^{-\infty}_{T,*}(M) \longrightarrow \mathcal{H}^{-\infty}_{G,*}(M)$ be the induction defined in subsection 2.6. For every $\gamma \in \mathcal{H}^{-\infty}_{T,*}(M)$ and $w \in W$, we have $\mathrm{ind}^G_T(w.\gamma) = \mathrm{ind}^G_T(\gamma)$. Moreover this map induces an isomorphism between $\mathcal{H}^{-\infty}_{T,*}(M)^W$ and $\mathcal{H}^{-\infty}_{G,*}(M)$ (see Theorem 74 of [26]). So, $\mathrm{ind}^G_T(\gamma) = 0$ in $\mathcal{H}^{-\infty}_{G,*}(M)$ if and only if $\sum_{w\in W} w.\gamma = 0$ in $\mathcal{H}^{-\infty}_{T,*}(M)$.

For every $\beta \in \mathcal{B}_T$, we denote by W_β the subgroup of elements of W that leave β fixed, and by $|W_\beta|$ its cardinal. We can now write the main theorem of this section.

Theorem 4.5 *Let M be a Hamiltonian G-manifold such that the moment map μ_T relative to the action of the maximal torus T is* proper. *The equivariant forms $\mathrm{P}^T_\beta \in \mathcal{A}^{-\infty}_{T,cpt}(M),\ \beta \in \mathcal{B}_T$, and $\mathrm{P}^G_\beta \in \mathcal{A}^{-\infty}_{G,cpt}(M),\ \beta \in \mathcal{B}_G$, are related by the following equalities in cohomology. For every $\beta \in \mathcal{B}_G$,*

$$\mathrm{ind}^G_T\Big(\Pi^2_{\mathfrak{g}/\mathfrak{t}} \mathrm{P}^T_\beta\Big) = |W_\beta|\ \mathrm{P}^G_\beta \quad \text{in} \quad \mathcal{H}^{-\infty}_{G,cpt}(M)\ .$$

Otherwise, for $\beta \in \mathcal{B}_T$ such that $\beta \notin W.\mathcal{B}_G$, we have

$$\mathrm{ind}^G_T\Big(\Pi_{\mathfrak{g}/\mathfrak{t}} \mathrm{P}^T_\beta\Big) = \mathrm{ind}^G_T\Big(\Pi^2_{\mathfrak{g}/\mathfrak{t}} \mathrm{P}^T_\beta\Big) = 0 \quad \text{in} \quad \mathcal{H}^{-\infty}_{G,cpt}(M). \tag{4.47}$$

Note that the equality $\mathrm{ind}^G_T\Big(\Pi_{\mathfrak{g}/\mathfrak{t}} \mathrm{P}^T_\beta\Big) = 0$, which is equivalent to $\Pi_{\mathfrak{g}/\mathfrak{t}} \sum_{w\in W} (-1)^w \mathrm{P}^T_{w.\beta} = 0$, is *trivial* if β is not in the *interior* of $\mathfrak{t}^*_+$. For instance, the equality $\mathrm{ind}^G_T\Big(\Pi_{\mathfrak{g}/\mathfrak{t}} \mathrm{P}^T_0\Big) = 0$ always holds because P^T_0 is W-invariant.

The identity $\mathrm{ind}_T^G\Big(\Pi^2_{\mathfrak{g}/\mathfrak{t}}\,\mathrm{P}^T_\beta\Big) = 0$ is more interesting, and is equivalent to the following:

$$\Pi^2_{\mathfrak{g}/\mathfrak{t}} \sum_{\beta'\in W.\beta} \mathrm{P}^T_{\beta'} = 0 \quad \text{in} \quad \mathcal{H}^{-\infty}_{T,cpt}(M).$$

Using the commutativity of the following diagram (see (2.22))

$$\begin{array}{ccc} \mathcal{H}^{-\infty}_{T,cpt}(M) & \xrightarrow{\mathrm{ind}_T^G} & \mathcal{H}^{-\infty}_{G,cpt}(M) \\ \Big\downarrow{\scriptstyle \int_M} & & \Big\downarrow{\scriptstyle \int_M} \\ \mathcal{C}^{-\infty}(\mathfrak{t}) & \xrightarrow{\mathrm{ind}_T^G} & \mathcal{C}^{-\infty}(\mathfrak{g})^G. \end{array} \tag{4.48}$$

and the equation (2.20) we get

Corollary 4.6 *1) For every $\beta \in \mathcal{B}_G$, $\eta \in \mathcal{H}^\infty_G(M)$ and $f \in \mathcal{C}^\infty_{cpt}(\mathfrak{g})^G$ we have*

$$< \Big(\int_M \eta\, \mathrm{P}^G_\beta\Big)(X), f(X)dX >_{\mathfrak{g}} = \mathrm{cst} \ < \Big(\int_M \eta|_{\mathfrak{t}} \mathrm{P}^T_\beta\Big)(Y), \Pi^2_{\mathfrak{g}/\mathfrak{t}}(Y) f|_{\mathfrak{t}}(Y)dY >_{\mathfrak{t}},$$

with $\mathrm{cst} = |W_\beta| \frac{\mathrm{vol}(G,dX)}{\mathrm{vol}(T,dY)}$, *where* $\mathcal{H}^\infty_G(M) \to \mathcal{H}^\infty_T(M)$, $\eta \mapsto \eta|_{\mathfrak{t}}$, *is the restriction morphism.*

2) For every $\beta \in \mathcal{B}_T$ such that $\beta \notin W.\mathcal{B}_G$, we have

$$\Pi^2_{\mathfrak{g}/\mathfrak{t}}(Y) \sum_{\beta'\in W.\beta} \Big(\int_M \eta\, \mathrm{P}^T_{\beta'}\Big)(Y) = 0 \quad \text{in} \quad \mathcal{C}^{-\infty}(\mathfrak{t}) \ ,$$

for every $\eta \in \mathcal{H}^\infty_T(M)$.

Before going into the proof, we can play a little 'game' to understand the role of the polynomial $\Pi^2_{\mathfrak{g}/\mathfrak{t}}$ in Theorem 4.5. Suppose the manifold compact, and write the partition of unity relatively to the 1-forms λ_G and λ_T:

- $1_M = \sum_{\beta\in\mathcal{B}_T} \mathrm{P}^T_\beta$ in $\mathcal{H}^{-\infty}_T(M)$, and
- $1_M = \sum_{\beta\in\mathcal{B}_G} \mathrm{P}^G_\beta$ in $\mathcal{H}^{-\infty}_G(M)$.

Now take the first partition of unity, multiply by $\Pi^2_{\mathfrak{g}/\mathfrak{t}}$, and take the image by the induction map. We get

$$\mathrm{ind}_T^G\Big(\Pi^2_{\mathfrak{g}/\mathfrak{t}} 1_M\Big) = \sum_{\beta\in\mathcal{B}_T\setminus W.\mathcal{B}_G} \mathrm{ind}_T^G\Big(\Pi^2_{\mathfrak{g}/\mathfrak{t}}\,\mathrm{P}^T_\beta\Big) + \sum_{\beta\in W.\mathcal{B}_G} \mathrm{ind}_T^G\Big(\Pi^2_{\mathfrak{g}/\mathfrak{t}}\,\mathrm{P}^T_\beta\Big).$$

Now using Theorem 4.5 and the equality (4.50), we see that this last equation is in fact the second partition of unity. The term $\mathrm{ind}_T^G(\Pi^2_{\mathfrak{g}/\mathfrak{t}} 1_M)$ is equal to $|W|\,1_M$, the terms of the first sum of the RHS are equal to zero, and for every $\beta \in \mathcal{B}_G$ we have $\sum_{\beta'\in W.\beta} \mathrm{ind}_T^G(\Pi^2_{\mathfrak{g}/\mathfrak{t}}\,\mathrm{P}^T_{\beta'}) = \frac{|W|}{|W_\beta|}\mathrm{ind}_T^G(\Pi^2_{\mathfrak{g}/\mathfrak{t}}\,\mathrm{P}^T_\beta) = |W|\,\mathrm{P}^G_\beta$.

4.2 Proof of Theorem 4.5

The natural projection map $\pi : G \times_T M \to M$, $[g,m] \mapsto gm$ makes $\mathcal{M} := G \times_T M$ fibered over M with oriented fiber G/T. The Weyl group W acts on $\mathcal{M}$ in the following way: $w.[g,m] := [g.w^{-1}, w.m]$ for $w \in W$. The integration on the fiber, denoted by $\int_{G/T}$, defines a morphism

$$\int_{G/T} : \mathcal{H}^{-\infty}_{G,*}(G \times_T M) \longrightarrow \mathcal{H}^{-\infty}_{G,*}(M) \ .$$

Remark 4.7 *This morphism induces an isomorphism between the W-anti-invariant part of $\mathcal{H}^{-\infty}_{G,*}(G \times_T M)$ with $\mathcal{H}^{-\infty}_{G,*}(M)$ (see section 8 of [26]).*

Let $\mathcal{W}_{\mathfrak{g}/\mathfrak{t}} \in \mathcal{A}^*_G(G/T)$ be the closed equivariant form equal to the image under the Chern-Weil homomorphism of the W-anti-invariant polynomial function $(-2\pi)^{-dim(G/T)/2}\Pi_{\mathfrak{g}/\mathfrak{t}}(X)$, $X \in \mathfrak{t}$. Let $p : \mathcal{M} \to G/T$ be the projection map $[g,m] \mapsto [g]$. We will also denote by $\mathcal{W}_{\mathfrak{g}/\mathfrak{t}}$ the pullback $p^*(\mathcal{W}_{\mathfrak{g}/\mathfrak{t}}) \in \mathcal{A}^*_G(\mathcal{M})$.

The form $\mathcal{W}_{\mathfrak{g}/\mathfrak{t}} \in \mathcal{A}^*_G(\mathcal{M})$ is closed, *W-anti-invariant*, and we have

$$\int_{G/T} \mathcal{W}_{\mathfrak{g}/\mathfrak{t}} = |W|\, 1_M \quad \text{in} \quad \mathcal{A}^*_G(M), \tag{4.49}$$

where $|W|$ is the cardinal of W. By definition this form restricts to $(-2\pi)^{-dim(G/T)/2}\Pi_{\mathfrak{g}/\mathfrak{t}}$ on M (see section 8 of [26]). Proposition 2.14 insures that $Ind_{G/T}(\Pi^2_{\mathfrak{g}/\mathfrak{t}} 1_M) = \mathcal{W}_{\mathfrak{g}/\mathfrak{t}}$ and after integration on the fibers of π we get

$$\mathrm{ind}^G_T\left(\Pi^2_{\mathfrak{g}/\mathfrak{t}} 1_M\right) = |W|\, 1_M \quad \text{in} \quad \mathcal{H}^{-\infty}_G(M). \tag{4.50}$$

• For each $\beta \in \mathcal{B}_G$, we are going to prove that the G-equivariant form P^G_β is induced by the T-equivariant form $\Pi^2_{\mathfrak{g}/\mathfrak{t}}\mathrm{P}^T_\beta$, through the use of a G-equivariant form $\mathrm{P}^{\mathcal{M}}_\beta$ on $G \times_T M$ (see Definition 4.9). Our proof is in two steps:

(1) First we show that $|W_\beta|^{-1}\sum_{w\in W} w.\mathrm{P}^{\mathcal{M}}_\beta = \pi^*(\mathrm{P}^G_\beta)$ in $\mathcal{H}^{-\infty}_{G,cpt}(\mathcal{M})$ (see Proposition 4.11 and equation (4.52)).

(1) Next we show that $\mathrm{P}^{\mathcal{M}}_\beta$ admits a restriction to $\mathfrak{t}$, and that $r_{\mathfrak{t}}(\mathrm{P}^{\mathcal{M}}_\beta) = \mathrm{P}^T_\beta$ in $\mathcal{H}^{-\infty}_{T,cpt}(M)$ (see Proposition 4.10).

The form $\sum_{w\in W} w.\mathrm{P}^{\mathcal{M}}_\beta \mathcal{W}_{\mathfrak{g}/\mathfrak{t}} = \sum_{w\in W}(-1)^w w.\left(\mathrm{P}^{\mathcal{M}}_\beta \mathcal{W}_{\mathfrak{g}/\mathfrak{t}}\right)$ is equal to the W-anti-invariant part of the form $|W|\,\mathrm{P}^{\mathcal{M}}_\beta \mathcal{W}_{\mathfrak{g}/\mathfrak{t}}$. With Remark 4.7 and equation (4.49), the first point gives

$$\int_{G/T} \mathcal{W}_{\mathfrak{g}/\mathfrak{t}}\mathrm{P}^{\mathcal{M}}_\beta = |W_\beta|\ \mathrm{P}^G_\beta \ .$$

The second point shows that $r_{\mathfrak{t}}(\mathcal{W}_{\mathfrak{g}/\mathfrak{t}}\mathrm{P}^{\mathcal{M}}_\beta) = (-2\pi)^{-dim(G/T)/2}\Pi_{\mathfrak{g}/\mathfrak{t}}\mathrm{P}^T_\beta$. Combining these two equalities with Proposition 2.14, we found finally

$$\int_{G/T} \circ\, Ind_{G/T}\left(\Pi^2_{\mathfrak{g}/\mathfrak{t}}\mathrm{P}^T_\beta\right) = |W_\beta|\ \mathrm{P}^G_\beta \ .$$

• For every $\beta \in \mathcal{B}_{_T} \backslash W.\mathcal{B}_{_G}$, we will show that there exists an *exact* equivariant form $Q_\beta \in \mathcal{A}^{-\infty}_{G,cpt}(\mathcal{M})$ that admits a restriction to M equal to $\mathrm{P}^{_T}_\beta$ (see Proposition 4.10). Proposition 2.14, applied to the equalities $r_{\mathfrak{t}}(Q_\beta) = \mathrm{P}^{_T}_\beta$ and $Q_\beta \equiv 0$, gives $\mathrm{ind}^{_G}_{_T}(\Pi_{\mathfrak{g}/\mathfrak{t}} \mathrm{P}^{_T}_\beta) \equiv 0$. We have also $r_{\mathfrak{t}}(\mathcal{W}_{\mathfrak{g}/\mathfrak{t}} Q_\beta) = (-2\pi)^{-dim(G/T)/2} \Pi_{\mathfrak{g}/\mathfrak{t}} \mathrm{P}^{_T}_\beta$ and that gives $\mathrm{ind}^{_G}_{_T}(\Pi^2_{\mathfrak{g}/\mathfrak{t}} \mathrm{P}^{_T}_\beta) \equiv 0$. (Here '$\equiv$' means equality in cohomology).

Construction of $\mathrm{P}^{\mathcal{M}}_\beta$ and Q_β

Let $pr_{\mathfrak{g}/\mathfrak{t}}$ and $pr_{\mathfrak{t}}$ be respectively the orthogonal projections $\mathfrak{g} \to \mathfrak{t}^\perp$ and $\mathfrak{g} \to \mathfrak{t}$, and denote by $\mu_{_{G/T}}$ the map $pr_{\mathfrak{g}/\mathfrak{t}}(\mu_{_G})$. We make the choice of a G-invariant Riemannian metric $(.,.)_{_M}$ on M, and denote by$(.,.)_{_{\mathcal{M}}}$ the induced G-invariant Riemannian metric on $\mathcal{M}$ (see equation (2.24)).

Let $\tilde{\mu} : \mathcal{M} \to \mathfrak{g}$ be the equivariant map equal to the pullback of $\mu_{_G}$ by π, and denote by $\tilde{\mathcal{H}}$ the corresponding vector field on $\mathcal{M}$:

$$\tilde{\mathcal{H}}|_{[g,m]} = \mu_{_G}(gm)_{\mathcal{M}}|_{[g,m]}, \quad \text{for} \quad [g,m] \in \mathcal{M} \ .$$

Definition 4.8 *We denote by $\tilde{\lambda}$ the following G-invariant 1-form on $\mathcal{M}$:*

$$\tilde{\lambda} := (\tilde{\mathcal{H}}, \cdot)_{_{\mathcal{M}}} \ .$$

For every $X \in \mathfrak{g}$, we have

$$\langle \Phi_{\tilde{\lambda}}([g,m]), X \rangle = \langle \mu_{_{G/T}}(m), pr_{\mathfrak{g}/\mathfrak{t}}(g^{-1}X) \rangle + \langle \Phi_{\lambda_{_T}}(m), pr_{\mathfrak{t}}(g^{-1}X) \rangle \ , \qquad (4.51)$$

where $\lambda_{_T} = (\mathcal{H}_{_T}, \cdot)_{_M}$ and $\mathcal{H}_{_T}$ is the Hamiltonian vector field of $\frac{1}{2}\|\mu_{_T}\|^2$. In particular $\langle \Phi_{\tilde{\lambda}}, \tilde{\mu} \rangle([g,m]) = \|\mu_{_{G/T}}(m)\|^2 + \|\mathcal{H}_{_T}\|^2(m)$. This implies that $\Phi_{\tilde{\lambda}}([g,m]) = 0$ if and only if $\mu_{_{G/T}}(m) = 0$ and $\mathcal{H}_{_T}|_m = 0$. In other words

$$\{\Phi_{\tilde{\lambda}} = 0\} = G \times_{_T} \Big(\mathrm{Cr}(\|\mu_{_T}\|^2) \cap \mu_{_G}^{-1}(\mathfrak{t}) \Big) \ .$$

But $\mathrm{Cr}(\|\mu_{_T}\|^2) \cap \mu_{_G}^{-1}(\mathfrak{t}) = \mathrm{Cr}(\|\mu_{_G}\|^2) \cap \mu_{_G}^{-1}(\mathfrak{t})$, then we can parameterize $\{\Phi_{\tilde{\lambda}} = 0\}$ by $W.\mathcal{B}_{_G}$

$$\{\Phi_{\tilde{\lambda}} = 0\} = \bigcup_{\beta \in W.\mathcal{B}_{_G}} \Big(G \times_{_T} (M^\beta \cap \mu_{_G}^{-1}(\beta)) \Big) \ .$$

Definition 4.9 *For every $\beta \in W.\mathcal{B}_{_G}$, let $\mathrm{P}^{\mathcal{M}}_\beta := \mathrm{P}^{C}_{\tilde{\lambda}} \in \mathcal{A}^{-\infty}_{G,cpt}(\mathcal{M})$ be the closed G-equivariant form defined by the 1-form $\tilde{\lambda}$ in a neighborhood of the component $C = G \times_{_T} (M^\beta \cap \mu_{_G}^{-1}(\beta))$ of $\{\Phi_{\tilde{\lambda}} = 0\}$.*

Proposition 4.10 *For every $\beta \in \mathcal{B}_{_T}$, there exists a closed equivariant form $Q_\beta \in \mathcal{A}^{-\infty}_{G,cpt}(\mathcal{M})$ that admits a restriction to M equal to $\mathrm{P}^{_T}_\beta$: in other word $r_{\mathfrak{t}}(Q_\beta) = \mathrm{P}^{_T}_\beta$.*

Moreover if β belongs to $W.\mathcal{B}_{_G}$, we have $Q_\beta = \mathrm{P}^{\mathcal{M}}_\beta$ in $\mathcal{H}^{-\infty}_{G,cpt}(\mathcal{M})$. In the other case ($\beta \notin W.\mathcal{B}_{_G}$), we have $Q_\beta = 0$ in $\mathcal{H}^{-\infty}_{G,cpt}(\mathcal{M})$.

Proof : If $i : M \hookrightarrow \mathcal{M}$ denotes the inclusion, a straightforward computation shows that $i^*(\tilde{\lambda}) = \lambda_{_T}$. Let $\chi_\beta \in \mathcal{C}^\infty_{cpt}(\mathcal{M})^G$ be the function coming from a T-invariant function χ'_β on M, where χ'_β is equal to 1 in a neighborhood of $C^T_\beta = M^\beta \cap \mu_{_T}^{-1}(\beta)$, and $support\{\chi'_\beta\} \cap \mathrm{Cr}(\|\mu_{_T}\|^2) = C^T_\beta$. Then the function χ_β which satisfies $support\{\chi_\beta\} \cap \{\Phi_{\tilde{\lambda}} = 0\} = G \times_T (M^\beta \cap \mu_{_G}^{-1}(\beta))$ $(\star)$ defines the G-equivariant form

$$Q_\beta := \chi_\beta + d\chi_\beta \left(\int_0^\infty \mathrm{i} e^{-\mathrm{i}t\,\mathcal{D}\tilde{\lambda}} dt \right) \tilde{\lambda}.$$

(For the equality $(\star)$, we have just used the trivial implication $\beta \neq \beta'$ in $\mathfrak{t} \Longrightarrow \mu_{_G}^{-1}(\beta') \cap \mu_{_T}^{-1}(\beta) = \emptyset$). We are now in the situation of the Remark 2.15. We have $Q_\beta = \lim_{a\to\infty} Q^a_\beta$ with $Q^a_\beta := \chi_\beta + d\chi_\beta \left(\int_0^a \mathrm{i} e^{-\mathrm{i}t\,\mathcal{D}\tilde{\lambda}} dt \right) \tilde{\lambda} \in \mathcal{A}^\infty_{G,cpt}(\mathcal{M})$. We verify that the restrictions $(Q^a_\beta)_e|_{\mathfrak{h}} \in \mathcal{C}^\infty(\mathfrak{h}, \wedge\mathfrak{g}^* \otimes \mathcal{A}^*_{cpt}(M))$ converge in $\mathcal{C}^{-\infty}(\mathfrak{h}, \wedge\mathfrak{g}^* \otimes \mathcal{A}^*_{cpt}(M))$ when $a \to \infty$. In particular, we have $r_{\mathfrak{t}}(Q^a_\beta) = \chi'_\beta + d\chi'_\beta \left(\int_0^a \mathrm{i} e^{-\mathrm{i}t\,\mathcal{D}\lambda_{_T}} dt \right) \lambda_{_T} \in \mathcal{A}^\infty_{T,cpt}(M)$, and we see that the limit $\lim_{a\to\infty} r_{\mathfrak{t}}(Q^a_\beta)$ is equal to $\mathrm{P}^{_T}_\beta$. This shows that the generalized G-equivariant form Q_β admits a restriction to $\mathfrak{t}$ which is equal to $\mathrm{P}^{_T}_\beta$. If β belongs to $W.\mathcal{B}_{_G}$, the equivariant form Q_β is by definition equal to $\mathrm{P}^{\mathcal{M}}_\beta$ in cohomology. In the other case ($\beta \notin W.\mathcal{B}_{_G}$), we have from $(\star)$ $support\{\chi_\beta\} \cap \{\Phi_{\tilde{\lambda}} = 0\} = \emptyset$. Then $\chi_\beta(\int_0^\infty \mathrm{i} e^{-\mathrm{i}t\,\mathcal{D}\tilde{\lambda}})\tilde{\lambda}$ is a well defined G-equivariant form on $\mathcal{M}$, with compact support, such that

$$Q_\beta = \mathcal{D}\left(\chi_\beta(\int_0^\infty \mathrm{i} e^{-\mathrm{i}t\,\mathcal{D}\tilde{\lambda}} dt) \tilde{\lambda} \right).$$

□

The proof is completed in the case $\beta \in \mathcal{B}_{_T} \setminus W.\mathcal{B}_{_G}$. We fix now an element β in $\mathcal{B}_{_G}$ and take a G-invariant neighborhood $\mathcal{U}_\beta$ of $C^G_\beta = G(M^\beta \cap \mu_{_G}^{-1}(\beta))$ such that $\overline{\mathcal{U}_\beta} \cap \mathrm{Cr}(\|\mu_{_G}\|^2) = C^G_\beta$. Let $\mathcal{V}_\beta$ be the open subset $G \times_T \mathcal{U}_\beta$.

Proposition 4.11 *The open set $\mathcal{V}_\beta$ is good for $\tilde{\lambda}$, and*

$$\mathrm{P}^{\mathcal{V}_\beta}_{\tilde{\lambda}} = \pi^*\left(\mathrm{P}^{\mathcal{U}_\beta}_{\lambda_{_G}} \right) \quad \text{in} \quad \mathcal{H}^{-\infty}_{G,cpt}(\mathcal{M}) \ .$$

Proof: By definition, we have $\pi^*\left(\mathrm{P}^{\mathcal{U}_\beta}_{\lambda_{_G}} \right) = \mathrm{P}^{\mathcal{V}_\beta}_{\pi^*(\lambda_{_G})}$. The proof of this proposition will be an application of Proposition 2.6 with the 1-forms $\tilde{\lambda}$, $\pi^*(\lambda_{_G})$, and the function $f := \tilde{\mu}$ on the open set $\mathcal{V}_\beta$. We have $\langle \Phi_{\tilde{\lambda}}, \tilde{\mu} \rangle([g,m]) = \|\mu_{G/T}(m)\|^2 + \|\mathcal{H}_{_T}\|^2(m)$ which is strictly positive on $\partial\mathcal{V}_\beta$ (because $\|\mu_{G/T}\|^2 = \|\mathcal{H}_{_T}\|^2 = 0 \Longrightarrow \|\mathcal{H}_{_G}\|^2 = 0$). We see that $\langle \Phi_{\pi^*\lambda_{_G}}, \tilde{\mu} \rangle = \|\mathcal{H}_{_G}\|^2 \circ \pi$ is also strictly positive on $\partial\mathcal{V}_\beta$. □

By definition we have $\mathrm{P}^{\mathcal{U}_\beta}_{\lambda_{_G}} = \mathrm{P}^{_G}_\beta$. We have now just to verify that

$$\mathrm{P}^{\mathcal{V}_\beta}_{\tilde{\lambda}} = \sum_{\beta' \in W.\beta} \mathrm{P}^{\mathcal{M}}_{\beta'} = |W_\beta|^{-1} \sum_{w \in W} \mathrm{P}^{\mathcal{M}}_{w.\beta} = |W_\beta|^{-1} \sum_{w \in W} w.\mathrm{P}^{\mathcal{M}}_\beta \ . \tag{4.52}$$

This first equality comes from the decomposition $\mathcal{V}_\beta \cap \{\Phi_{\tilde{\lambda}} = 0\} = \cup_{\beta' \in W.\beta}\Big(G \times_T (M^{\beta'} \cap \mu_G^{-1}(\beta'))\Big)$. The third equality is due to the fact that for every $w \in W$, we have $\mathrm{P}^{\mathcal{M}}_{w.\beta} = w.\mathrm{P}^{\mathcal{M}}_{\beta}$. $\square$

4.3 The Pushforward in the torus case

Let (M, Ω) be a symplectic manifold provided with a Hamiltonian action of a torus T with Lie algebra $\mathfrak{t}$. The moment map $\mu_{_T} : M \to \mathfrak{t}^*$ is supposed *proper*. We recall that the pushforward is given by

$$(\mu_{_T})_*(dm_L) = \frac{1}{\mathfrak{i}^n} \int_M \mathcal{F}\Big(e^{\mathfrak{i}\Omega_\mathfrak{t}}\Big), \quad dimM = 2n.$$

Let $\alpha \in \mathcal{A}^*_T(M)$ be a closed equivariant form. We know from Proposition 4.4 that $\int_M \mathcal{F}(\alpha e^{\mathfrak{i}\Omega_\mathfrak{t}}) = \sum_{\beta \in \mathcal{B}_{_T}} \mathfrak{D}^\alpha_\beta$ where each tempered distribution $\mathfrak{D}^\alpha_\beta$ is given by

$$\mathfrak{D}^\alpha_\beta = \mathcal{F}\Big(\int_M \mathrm{P}^T_\beta \, \alpha e^{\mathfrak{i}\Omega_\mathfrak{t}}\Big) \, . \tag{4.53}$$

Each T-equivariant form P^T_β is supported in the neighborhood of the set C^T_β which is *not smooth* in general. Then, in order to compute the distribution $\mathfrak{D}^\alpha_\beta$, we modify the moment map to $\mu_\varepsilon := \mu_{_T} - \varepsilon, \ \varepsilon \in \mathfrak{t}^*$.

Let $\|.\|$ be a scalar product on $\mathfrak{t}^*$. As in the last section, consider the T-invariant 1-form λ^ε defined below.

Definition 4.12 *For $\varepsilon \in \mathfrak{t}^*$, let λ^ε be the following T-invariant 1-form*

$$\lambda^\varepsilon := (\mathcal{H}^\varepsilon, .)_M \ ,$$

where $\mathcal{H}^\varepsilon$ is the Hamiltonian vector field associated to the function $\frac{1}{2}\|\mu_\varepsilon\|^2$, and $(.,.)_M$ is a T-invariant Riemannian metric on M.

We know that $\{\Phi_{\lambda^\varepsilon} = 0\} = \mathrm{Cr}(\|\mu_\varepsilon\|^2)$ and we will show that for "generic" $\varepsilon \in \mathfrak{t}^*$, the set $\mathrm{Cr}(\|\mu_\varepsilon\|^2)$ is a smooth submanifold such that the quotient $\mathrm{Cr}(\|\mu_\varepsilon\|^2)\Big/T$ is a symplectic orbifold. We rewrite the result of subsection 6.1.2 of [31], in a more general context of a (possibly) non-compact manifold.

Let $\mathcal{I}$ be the set of subgroups of T stabilizer of points in M. For $l \in \mathcal{I}$, we denote by T_l the corresponding subgroup. For each $l \in \mathcal{I}$, we denote by $Z^k_l, \ k \in I_l$ the connected components of M^{T_l} that have T_l for generic stabilizer. The subsets Z^k_l are T-invariant symplectic submanifolds of M, and the convexity theorem of Atiyah-Guillemin-Sternberg (in the version of Condeveaux-Dazord-Molino [8, 22]) insures that the $\mathrm{P}^k_l = \mu(Z^k_l)$ are polyhedral subsets of $\mathfrak{t}^*$. We have $(\overrightarrow{\mathrm{P}^k_l})^\perp = \mathrm{Lie}(T_l)$, for all k. In this equality, $\overrightarrow{\mathrm{P}^k_l}$ is the subspace of $\mathfrak{t}^*$ generated by $\{a - b |\ a, b \in \mathrm{P}^k_l\}$, and $^\perp$ corresponds to the orthogonal for the duality between $\mathfrak{t}$ and $\mathfrak{t}^*$.

We denote by $\mathcal{B} := \{\mathrm{Aff}(\mathrm{P}_l^k)|l,k\}$ the set of the affine subspaces of $\mathfrak{t}^*$ generated by the polyhedral subsets P_l^k. For each $\Delta \in \mathcal{B}$, we note T_Δ the sub-torus of T with Lie algebra $(\overrightarrow{\Delta})^\perp$.

Let $j : \mathfrak{t}^* \to \mathfrak{t}$ be the isomorphism induced by the scalar product on $\mathfrak{t}^*$. For each $\varepsilon \in \mathfrak{t}^*$, we consider the function $\|\mu_\varepsilon\|^2 : M \to \mathbb{R}$. Propositions 6.8 and 6.9 of [31] remain true.

Proposition 4.13 *For every $\varepsilon \in \mathfrak{t}^*$, the critical points of $\|\mu_\varepsilon\|^2$ can be written on the form*

$$(\star) \qquad \mathrm{Cr}(\|\mu_\varepsilon\|^2) = \bigcup_{\Delta\in\mathcal{B}} M^{T_\Delta} \cap \mu_{_T}^{-1}(\beta(\varepsilon,\Delta)),$$

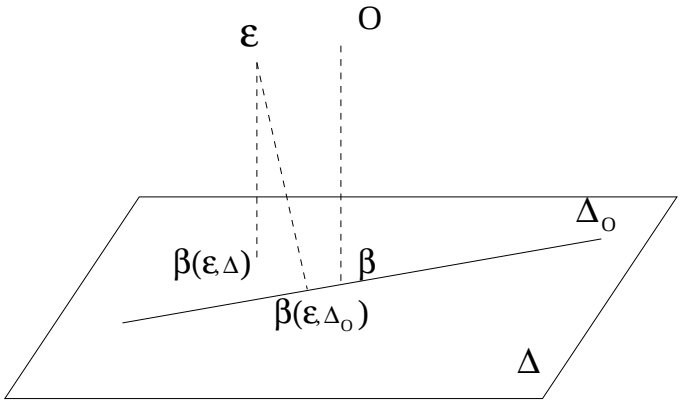
where $\beta(\varepsilon,\Delta)$ is the orthogonal projection of ε on Δ. There exists a dense subset $\mathcal{W}$ of $\mathfrak{t}^*$, such that for every $\varepsilon \in \mathcal{W}$ the set $\mathrm{Cr}(\|\mu_\varepsilon\|^2)$ is a smooth submanifold of M, the union $(\star)$ is disjoint, and the group T/T_Δ acts locally freely on $C_\Delta^\varepsilon = M^{T_\Delta} \cap \mu_{_T}^{-1}(\beta(\varepsilon,\Delta))$.

Remark 4.14 *When the manifold is compact the subset $\mathcal{W}$ of "generic" ε is the complement of a finite number of affine sub-spaces. In the general case, $\mathcal{W}$ is the complement of (at most) a countable number of affine sub-spaces. Thus the set $\mathcal{W}$ is dense.*

Remark 4.15 *Each subset C_Δ^ε is compact because $\mu_{_T}$ is proper. When $\varepsilon \in \mathcal{W}$, the sub-manifolds C_Δ^ε, $\Delta \in \mathcal{B}$ have a finite number of connected components.*

Fortunately we can recover the equivariant form P_β^T, $\beta \in \mathcal{B}_{_T}$ after this perturbation with ε. For $\beta \in \mathcal{B}_{_T}$, and $\Delta \in \mathcal{B}$ such that $\beta = \beta(0,\Delta)$ (in other words β is equal to the orthogonal projection of 0 on the affine subspace Δ), we have $\|\beta - \beta(\varepsilon,\Delta)\| \leq \|\varepsilon\|$ for every $\varepsilon \in \mathfrak{t}^*$ (see Picture 1). Let $\varepsilon_\beta = \min_{\beta'\neq\beta} \|\beta' - \beta\|$ be the minimal distance between β and the other points of $\mathcal{B}_{_T}$, and let $\mathcal{U}_\beta$ be the open neighborhood of C_β^T defined by

$$\mathcal{U}_\beta := \mu_{_T}^{-1}\Big(\xi \in \mathfrak{t}^*,\ \|\beta - \xi\| < \frac{\varepsilon_\beta}{2}\Big).$$



Picture 1

Lemma 4.16 *For $\varepsilon \in \mathfrak{t}^*$ small enough we have*

$$\mathrm{P}_\beta^T = \mathrm{P}_{\lambda^\varepsilon}^{\mathcal{U}_\beta} \quad \text{in} \quad \mathcal{H}_{T,cpt}^{-\infty}(M) \ .$$

Proof: By definition we have $\mathrm{P}_\beta^T = \mathrm{P}_{\lambda_T}^{\mathcal{U}_\beta}$. Consider the functions $(\Phi_{\lambda^\varepsilon}, \mu_\varepsilon) = \|\mathcal{H}^\varepsilon\|_M^2$ and $(\Phi_{\lambda_T}, \mu_\varepsilon) = \|\mathcal{H}\|_M^2 - (\mathcal{H}, \varepsilon_M)_M$. For $\|\varepsilon\| < \varepsilon_\beta/2$, the functions $\|\mathcal{H}^\varepsilon\|_M^2$ are strictly positive on $\partial\mathcal{U}_\beta$. Let $\rho > 0$ be the minimal value of $\|\mathcal{H}\|_M^2$ on $\partial\mathcal{U}_\beta$. Then for ε small enough, the function $|(\mathcal{H}, \varepsilon_M)_M|$ is bounded by $\rho/2$ on $\partial\mathcal{U}_\beta$. Finally for $\varepsilon \in \mathfrak{t}^*$ small enough, we can apply Proposition 2.6 to the 1-forms λ^ε and λ_T, on the open set $\mathcal{U}_\beta$, with the function $f = \mu_\varepsilon$. □

For $\varepsilon \in \mathcal{W}$ and $\Delta \in \mathcal{B}$, let

$$\mathrm{P}_\Delta^\varepsilon := \mathrm{P}_{\lambda^\varepsilon}^{C_\Delta^\varepsilon} \quad \in \ \mathcal{A}_{T,cpt}^{-\infty}(M)$$

be the closed equivariant form defined with the 1-form λ^ε in a neighborhood of C_Δ^ε (see Definition 2.3). If $C_\Delta^\varepsilon = \emptyset$ the equivariant form $\mathrm{P}_\Delta^\varepsilon$ is identically equal to 0. For $\varepsilon \in \mathcal{W}$ small enough, the submanifold C_Δ^ε is contained in $\mathcal{U}_\beta$ if and only if $\beta(0,\Delta) = \beta$, and we have the decomposition $\mathrm{P}_{\lambda^\varepsilon}^{\mathcal{U}_\beta} = \sum_{\beta(0,\Delta)=\beta} \mathrm{P}_\Delta^\varepsilon$. Lemma 4.16 gives

$$\mathrm{P}_\beta^T = \sum_{\beta(0,\Delta)=\beta} \mathrm{P}_\Delta^\varepsilon \quad \text{in } \mathcal{H}_{T,cpt}^{-\infty}(M), \quad \text{for} \quad \varepsilon \in \mathcal{W}, \ \|\varepsilon\| << 1. \tag{4.54}$$

Remark 4.17 *For $\varepsilon \in \mathcal{W}$ small, the submanifold $\cup_{\beta(0,\Delta)=\beta} C_\Delta^\varepsilon$ can be understood as a desingularization of C_β^T.*

Finally we have proved the following

Proposition 4.18 *Let $\alpha \in \mathcal{A}_T^*(M)$ be a closed equivariant form. We have the following equality of distributions on $\mathfrak{t}^*$: $\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) = \sum_{\beta \in \mathcal{B}_T} \mathfrak{D}_\beta^\alpha$, where the tempered distribution $\mathfrak{D}_\beta^\alpha$ can be decomposed in the following way*

$$\mathfrak{D}_\beta^\alpha = \sum_{\beta(0,\Delta)=\beta} \mathfrak{D}_{\Delta,\varepsilon}^\alpha, \quad \text{for} \quad \varepsilon \in \mathcal{W}, \ \|\varepsilon\| << 1,$$

with $\mathfrak{D}_{\Delta,\varepsilon}^\alpha = \mathcal{F}\left(\int_M \mathrm{P}_\Delta^\varepsilon \, \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}\right)$.

Now, we recall the computation of the generalized functions $\int_M \mathrm{P}_\Delta^\varepsilon \, \alpha e^{\mathrm{i}\Omega_\mathfrak{t}}$ for $\varepsilon \in \mathcal{W}$ sufficiently close to 0 (see subsection 6.1.2 of [31]). The equivariant form $\mathrm{P}_\Delta^\varepsilon$ has a compact support in a tubular neighborhood of the submanifold C_Δ^ε, then

$$\int_M \mathrm{P}_\Delta^\varepsilon \, \alpha e^{\mathrm{i}\Omega_\mathfrak{t}} = \int_{C_\Delta^\varepsilon} i_\Delta^*(\alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) \wedge \Lambda_\Delta \ , \tag{4.55}$$

where

$$\Lambda_\Delta = \int_{N_\Delta/C_\Delta^\varepsilon} \mathrm{P}_\Delta^\varepsilon \quad \text{in} \quad \mathcal{H}_T^{-\infty}(C_\Delta^\varepsilon)$$

is the integral of $\mathrm{P}_\Delta^\varepsilon$ along the fibers of the normal bundle N_Δ of C_Δ^ε in M. The map $i_\Delta : C_\Delta^\varepsilon \to M$ denotes the inclusion.

For every $\Delta \in \mathcal{B}$, we choose a subspace $\mathfrak{t}/\mathfrak{t}_\Delta$ of $\mathfrak{t}$, such that there is a decomposition $T = T_\Delta \times T/T_\Delta$, where T/T_Δ is a subtorus of T with Lie algebra $\mathfrak{t}/\mathfrak{t}_\Delta$.

Because the group T/T_Δ acts locally freely on the submanifold C_Δ^ε we can form the quotient $\mathcal{M}_\Delta^\varepsilon := C_\Delta^\varepsilon/(T/T_\Delta)$ which is a symplectic V-manifold. We have the Kirwan morphism in equivariant cohomology $k_\Delta : \mathcal{H}_T^\infty(M) \to \mathcal{H}_{T_\Delta}^\infty(\mathcal{M}_\Delta^\varepsilon)$, defined as the composition of the restriction morphism $\mathcal{H}_T^\infty(M) \to \mathcal{H}_T^\infty(C_\Delta^\varepsilon)$ with the Chern-Weil isomorphism $\mathcal{H}_T^\infty(C_\Delta^\varepsilon) \to \mathcal{H}_{T_\Delta}^\infty(\mathcal{M}_\Delta^\varepsilon)$.

For every connected component F of C_Δ^ε, let $S^\Delta(F)$ be the generic stabilizer of T/T_Δ over F, and denote by $|S^\Delta(F)|$ its cardinal. The map $F \to |S^\Delta(F)|$ defines a locally constant function over $\mathcal{M}_\Delta^\varepsilon$ denoted $|S^\Delta|$.

Let $\beta_\Delta := j(\beta(\varepsilon, \Delta) - \varepsilon) \in \mathfrak{t}_\Delta$ and N_Δ be the normal bundle of M^{T_Δ} in M restricted to C_Δ^ε. We have a T_Δ-equivariant V-vector bundle $\mathcal{E}_\Delta := N_\Delta/(T/T_\Delta) \to \mathcal{M}_\Delta^\varepsilon$. We associate to this data the equivariant form with generalized coefficients $\mathrm{Eul}_{\beta_\Delta}^{-1}(\mathcal{E}_\Delta) \in \mathcal{H}_{T_\Delta}^{-\infty}(\mathcal{M}_\Delta^\varepsilon)$ (see Définitions 4.5, 6.8 of [31]), which is an inverse of the T_Δ-equivariant Euler form of the V-vector bundle $\mathcal{E}_\Delta$.

Take a connection form σ_Δ for the V-principal bundle $C_\Delta^\varepsilon \to \mathcal{M}_\Delta^\varepsilon$. The associated curvature ω_Δ determines the equivariant form with generalized coefficients $\delta(X_2 - \omega_\Delta) \in \mathcal{H}_{T/T_\Delta}^{-\infty}(\mathcal{M}_\Delta^\varepsilon)$ by

$$< \delta(X_2 - \omega_\Delta), f(X_2) dX_2 >_{\mathfrak{t}/\mathfrak{t}_\Delta} = f(\omega_\Delta) \operatorname{vol}(T/T_\Delta, dX_2),$$

for every function $f \in \mathcal{C}^\infty(\mathfrak{t}/\mathfrak{t}_\Delta)$. In other words $\delta(X_2 - \omega_\Delta) = e^{\omega_\Delta(\partial_{X_2}|_0)} \operatorname{vol}(T/T_\Delta, -)$. (see Définitions 6.19 of [31]).

The decomposition $\mathfrak{t} = \mathfrak{t}_\Delta \oplus \mathfrak{t}/\mathfrak{t}_\Delta$ gives the bilinear map $(\eta, \nu) \mapsto \eta \diamond \nu$, $\mathcal{H}_{T_\Delta}^{-\infty}(\mathcal{M}_\Delta^\varepsilon) \times \mathcal{H}_{T/T_\Delta}^{-\infty}(\mathcal{M}_\Delta^\varepsilon) \to \mathcal{H}_T^{-\infty}(\mathcal{M}_\Delta^\varepsilon)$.

We can now state Proposition 6.20 of [31].

Proposition 4.19 *For every $\eta \in \mathcal{H}_T^\infty(M)$, we have the following equality in $\mathcal{H}_T^{-\infty}(C_\Delta^\varepsilon)$*

$$\Big(i_\Delta^*(\eta) \wedge \Lambda_\Delta\Big)(X) = (2\pi\mathrm{i})^{dim\,\Delta} \Big(k_\Delta(\eta)\, \mathrm{Eul}_{\beta_\Delta}^{-1}(\mathcal{E}_\Delta)\Big)(X_1) \diamond \delta(X_2 - \omega_\Delta) \wedge \frac{\sigma_{r_2} \wedge \cdots \wedge \sigma_1}{\operatorname{vol}(T/T_\Delta)} \,,$$

with $X = X_1 + X_2$, $X_1 \in \mathfrak{t}_\Delta$, and $X_2 \in \mathfrak{t}/\mathfrak{t}_\Delta$.

For $\Delta \in \mathcal{B}$ such that $\beta = \beta(0, \Delta)$, we have $k_\Delta(\Omega_\mathfrak{t})(X_1) = \Omega_\Delta + \langle \beta, X_1 \rangle$, $X_1 \in \mathfrak{t}_\Delta$, with $\Omega_\Delta \in \mathcal{H}^2(\mathcal{M}_\Delta^\varepsilon)$. Using Proposition 4.19, the equality (4.55) gives

$$\begin{aligned}
&\left(\int_M \mathrm{P}_\Delta^\varepsilon\, \alpha e^{\mathrm{i}\Omega_\mathfrak{t}} \right)(X) \\
&= \ \mathrm{cst} \int_{C_\Delta^\varepsilon} k_\Delta(\alpha\, e^{\mathrm{i}\Omega_\mathfrak{t}})\, \mathrm{Eul}_{\beta_\Delta}^{-1}(\mathcal{E}_\Delta)(X_1) \diamond \delta(X_2 - \omega_\Delta) \Pi \sigma_k \\
&= \ \mathrm{cst}\ e^{\mathrm{i}\langle \beta, X_1 \rangle} \int_{\mathcal{M}_\Delta^\varepsilon} \frac{e^{\mathrm{i}\Omega_\Delta}}{|S^\Delta|} (k_\Delta(\alpha)\, \mathrm{Eul}_{\beta_\Delta}^{-1}(\mathcal{E}_\Delta))(X_1) \diamond \delta(X_2 - \omega_\Delta) \ , \quad (4.56)
\end{aligned}$$

with $X = X_1 + X_2$, $X_1 \in \mathfrak{t}_\Delta$, $X_2 \in \mathfrak{t}/\mathfrak{t}_\Delta$, and $\mathrm{cst} = (2\pi\mathfrak{i})^{dim\Delta}\,\mathrm{vol}(T/T_\Delta)^{-1}$.

Note that for each closed form $\alpha \in \mathcal{A}_T^*(M)$, $X_1 \to k_\Delta(\alpha)(X_1)$ is just a polynomial function on $\mathfrak{t}_\Delta$ with values in $\mathcal{H}^*(\mathcal{M}_\Delta^\varepsilon)$. Taking now the Fourier transform, we find

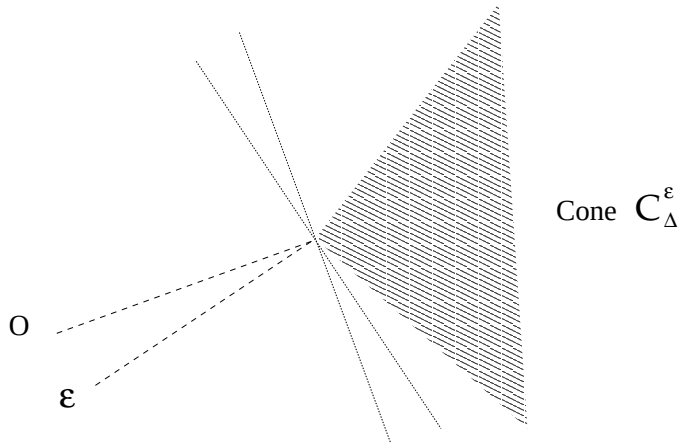
$$\mathfrak{D}^\alpha_{\Delta,\varepsilon}(\xi) = (\mathfrak{i})^{dim\Delta}\int_{\mathcal{M}_\Delta^\varepsilon}\frac{e^{\mathfrak{i}\Omega_\Delta}}{|S^\Delta|}k_\Delta(\alpha)(\mathfrak{i}\partial_{\xi_1})\Big(\mathcal{F}_{\mathfrak{t}_\Delta}(\mathrm{Eul}^{-1}_{\beta_\Delta}(\mathcal{E}_\Delta))\Big)(\xi_1-\beta)\,e^{-\mathfrak{i}\langle\xi_2,\,\omega_\Delta\rangle}d\xi_2^{can}\,, \tag{4.57}$$

with $\xi = \xi_1 + \xi_2$, $\xi_1 \in \mathfrak{t}_\Delta^*$, $\xi \in (\mathfrak{t}/\mathfrak{t}_\Delta)^*$. Here $d\xi_2^{can}$ is the Euclidean measure on $(\mathfrak{t}/\mathfrak{t}_\Delta)^*$ such that its dual measure dX_2^{can} verifies $\mathrm{vol}(T/T_\Delta, dX_2^{can}) = 1$.

Let $\alpha_1^+, \cdots, \alpha_l^+$ be the weights for the action of T_Δ on the fibers of $\mathcal{E}_\Delta$ that are oriented by $\beta(\varepsilon,\Delta) - \varepsilon$ (in other words $(\alpha_k^+, \beta(\varepsilon,\Delta) - \varepsilon) > 0$, $\forall k$). After Proposition 4.8 of [31], we know that $\mathcal{F}_{\mathfrak{t}_\Delta}(\mathrm{Eul}^{-1}_{\beta_\Delta}(\mathcal{E}_\Delta))$ is of the form

$$\mathcal{F}_{\mathfrak{t}_\Delta}(\mathrm{Eul}^{-1}_{\beta_\Delta}(\mathcal{E}_\Delta))(\xi_1) = \sum_{I=\{i_1,\cdots,i_p\}} R_I\ (H_{\alpha_1^+})^{i_1} * \cdots * (H_{\alpha_l^+})^{i_l}(\xi_1), \tag{4.58}$$

where $R_I \in \mathcal{H}^*(\mathcal{M}_\Delta^\varepsilon)$, the sum $\sum_I$ is finite, $H_{\alpha_k^+}$ is the Heaviside measure associated to α_k^+, and $*$ is the convolution product. In particular $\mathcal{F}_{\mathfrak{t}_\Delta}(\mathrm{Eul}^{-1}_{\beta_\Delta}(\mathcal{E}_\Delta))$ is a locally polynomial measure, with values in $\mathcal{H}^*(\mathcal{M}_\Delta^\varepsilon)$, supported by the cone $\mathfrak{C}_\Delta^\varepsilon = \mathbb{R}^+\alpha_1^+ + \cdots + \mathbb{R}^+\alpha_l^+$ that is *strictly* included in the half-space $\{\xi \in \mathfrak{t}_\Delta^*,\ (\xi, \beta(\varepsilon,\Delta) - \varepsilon) \geq 0\}$. Here, the *strict* inclusion means that for some $c > 0$ the inequality $(\xi, \beta(\varepsilon,\Delta) - \varepsilon) \geq c\|\xi\|$ holds on $\mathfrak{C}_\Delta^\varepsilon$ (see Picture 2). Then, for ε small enough, the cone $\mathfrak{C}_\Delta^\varepsilon$ is *strictly* included in the half-space $\{\xi \in \mathfrak{t}_\Delta^*,\ (\xi,\beta) \geq 0\}$ with $\beta = \beta(0,\Delta)$.



Picture 2

From the integral (4.57) and the equality (4.58) we see that there exist a finite collection of polynomials $P_j \in S(\mathfrak{t}_\Delta^*)$, $Q_j \in S(\mathfrak{t}/\mathfrak{t}_\Delta)$ such that

$$\mathfrak{D}^\alpha_{\Delta,\varepsilon}(\xi) = \sum_{\{i_1,\cdots,i_p\},j} P_j(\partial_{\xi_1})\left((H_{\alpha_1^+})^{i_1} * \cdots * (H_{\alpha_l^+})^{i_l}\right)(\xi_1-\beta)\,Q_j(\xi_2)d\xi_2, \tag{4.59}$$

with $\xi = \xi_1 + \xi_2$, $\xi_1 \in \mathfrak{t}^*_\Delta$, $\xi \in (\mathfrak{t}/\mathfrak{t}_\Delta)^*$. Since $(\mathfrak{t}/\mathfrak{t}_\Delta)^* = (\mathfrak{t}_\Delta)^\perp$ and $j(\beta) \in \mathfrak{t}_\Delta$ for $\beta(0,\Delta) = \beta$, we see that the distribution $\mathfrak{D}^\alpha_{\Delta,\varepsilon}$ is supported by the half-space $\{\xi \in \mathfrak{t}^*,\ (\xi - \beta, \beta) \geq 0\}$ with $\beta = \beta(0,\Delta)$.

Remark 4.20 *In the case where $\beta = 0$, the "half-space" $\{\xi \in \mathfrak{t}^*,\ (\xi - \beta, \beta) \geq 0\}$ corresponds in fact to all the vector space $\mathfrak{t}^*$.*

We can now state the

Theorem 4.21 *Let $\alpha \in \mathcal{A}^*_T(M)$ be a closed equivariant form. We have the following equality of distribution of $\mathfrak{t}^*$,*

$$\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) = \sum_{\beta \in \mathcal{B}_T} \mathfrak{D}^\alpha_\beta \qquad \text{with} \quad \mathfrak{D}^\alpha_\beta = \sum_{\beta(0,\Delta)=\beta} \mathfrak{D}^\alpha_{\Delta,\varepsilon},$$

where the distributions $\mathfrak{D}^\alpha_{\Delta,\varepsilon}$ are given by the equality (4.59). In particular each distribution $\mathfrak{D}^\alpha_\beta$ is tempered and supported by the half-space $\{\xi \in \mathfrak{t}^*,\ (\xi - \beta, \beta) \geq 0\}$.

Remark 4.22 *We see in particular that the support of the distribution $\mathfrak{D}^\alpha_\beta$ does not meet a small neighborhood of 0 if $\beta \neq 0$. This means that $\int_M \mathcal{F}(\alpha e^{\mathrm{i}\Omega_\mathfrak{t}}) = \mathfrak{D}^\alpha_0$ in a neighborhood of 0. We know from Theorem 3.12 that $\mathfrak{D}^\alpha_0$ is a polynomial measure if 0 is a regular value of μ. In general $\mathfrak{D}^\alpha_0$ will be the sum of a locally polynomial measure with distributions supported by linear subspaces of $\mathfrak{t}^*$.*

In particular this Theorem gives

Corollary 4.23 *The pushforward of the Liouville measure $\mu_{T*}(dm_L)$ can be decomposed in the following way*

$$\mu_{T*}(dm_L)(\xi) = \sum_{\beta \in \mathcal{B}_T} dm_\beta(\xi),$$

where each dm_β is a locally polynomial measure supported by the half-plane $\{\xi \in \mathfrak{t}^*,\ (\xi - \beta, \beta) \geq 0\}$.

5 Witten partition functions

Let M be a compact symplectic manifold equipped with a Hamiltonian action of a compact connected Lie group G, with Lie algebra $\mathfrak{g}$. We denote by Ω the symplectic form on M and by $\mu_G : M \to \mathfrak{g}^*$ the moment map of this action. Let $\|.\|$ be the norm induced by a G-invariant scalar product on $\mathfrak{g}$. Let $\mathcal{B}_G$ be the indexing set of a decomposition of $\mathrm{Cr}(\|\mu_G\|^2)$ (for the definition, see (4.43)).

In [36], Witten introduced the "partition functions"

$$Z^\alpha(u) = \int_{M \times \mathfrak{g}} \alpha(Y) e^{\mathrm{i}\Omega_\mathfrak{g}(Y)} e^{-u\|Y\|^2/2} \frac{dY}{\mathrm{vol}(G, dY)}, \quad u > 0, \tag{5.60}$$

where $\alpha(Y)$ is a closed equivariant form, polynomial in the variable $Y \in \mathfrak{g}$, and where $\Omega_{\mathfrak{g}}(Y) = \Omega + \langle \mu_{_G}, Y \rangle$ is the equivariant symplectic form.

Under the hypothesis that G acts freely on $\mu^{-1}(0)$, Witten proposed a formula of the form $Z^{\alpha}(u) = \sum_N Z_N^{\alpha}(u)$, where "$N$" ranges over the components of the subset $\mathrm{Cr}(\|\mu_{_G}\|^2)$ of critical points of the square of the moment map. In this formula, each function $Z_N^{\alpha}(u)$ should only depend on the values of α, Ω, $\mu_{_G}$ in a neighborhood of N. Witten determined the function $Z^{\alpha}_{\mu_{_G}^{-1}(0)}$, which is in fact polynomial, and gives for the other terms a bound on the form $Z_N^{\alpha}(u) = O(e^{-k_N/u})$, $u > 0$, with $k_N > 0$. Witten's formula for $Z^{\alpha}_{\mu_{_G}^{-1}(0)}$ has been proved by Kalkman [24], Wu [37] in the case of circle actions and by Jeffrey-Kirwan [23] in the general case.

The partition of unity defined with the G-invariant 1-form $\lambda_{_G}$ gives automatically the decomposition $Z^{\alpha}(u) = \sum_{\beta \in \mathcal{B}_{_G}} Z_{\beta}^{\alpha}(u)$, with

$$Z_{\beta}^{\alpha}(u) = \int_{M \times \mathfrak{g}} \mathrm{P}_{\beta}^{G}(Y) \alpha(Y) e^{\mathrm{i}\Omega_{\mathfrak{g}}(Y)} e^{-u\|Y\|^2/2} \frac{dY}{\mathrm{vol}(G, dY)}, \quad u > 0 \ . \tag{5.61}$$

(see section 4 and Definition 3.2). Each function Z_{β}^{α}, $\beta \in \mathcal{B}_{_G}$ is determined by the behaviour of α, Ω and $\mu_{_G}$ in a small neighborhood of the component $C_{\beta}^{G} = G\Big(M^{\beta} \cap \mu_{_G}^{-1}(\beta)\Big)$ of $\mathrm{Cr}(\|\mu_{_G}\|^2)$.

This section is devoted to the computation of the terms Z_{β}^{α}, $\beta \in \mathcal{B}_{_G}$ of this formula. *We do not require that 0 is a regular value for the moment map* $\mu_{_G}$.

A function $h : \mathbb{R} \to \mathbb{C}$ is called *slowly increasing* if it is bounded by a polynomial at infinity. The rest of this section is devoted to the proof of the following Theorem.

Theorem 5.1 *For every closed equivariant form $\alpha(X) \in \mathcal{A}_{G}^{*}(M)$ with polynomial dependence in $X \in \mathfrak{g}$, let $Z_{\beta}^{\alpha} : \mathbb{R}_{+}^{*} \to \mathbb{C}$ be the function defined by the equation (5.61). Denote by N_{α} the integer $d_{\alpha} + \dim G$, where d_{α} is the polynomial degree of α.*

For each $\beta \in \mathcal{B}_{_G}$, there exists a smooth function $h_{\beta}^{\alpha} : \mathbb{R} \to \mathbb{C}$ which is slowly increasing *such that*

$$Z_{\beta}^{\alpha}(u) = u^{-N_{\alpha}} \, e^{-\frac{\|\beta\|^2}{2u}} \, h_{\beta}^{\alpha}(\sqrt{u}) \quad \text{for every } u > 0.$$

Moreover the function h_0^{α} is polynomial and, if G acts locally freely on $\mu_{_G}^{-1}(0)$, the function $Z_0^{\alpha}(u) = u^{-N_{\alpha}} \, h_0^{\alpha}(\sqrt{u})$ is polynomial in u.

Corollary 5.2 *There exist smooth,* slowly increasing *functions $h_{\beta} : \mathbb{R} \to \mathbb{R}$, such that*

$$\int_M e^{-\frac{\|\mu\|^2}{2u}} dm_L = u^{-\frac{dimG}{2}} \sum_{\beta \in \mathcal{B}_{_G}} e^{-\frac{\|\beta\|^2}{2u}} h_{\beta}(\sqrt{u}), \quad u > 0,$$

where dm_L is the Liouville measure $\frac{\Omega^n}{n!}$. Moreover each function h_{β} is uniquely determined by the local behaviour of Ω and μ near the component C_{β}^{G} of $\mathrm{Cr}(\|\mu\|^2)$,

and the function h_0 that corresponds to the set $C_0^G = \mu_G^{-1}(0)$ is always polynomial *(note that we make no assumption on the regularity of* 0 *relative to the moment map).*

Proof of the Theorem :

We suppose first that the action of G *quasi-effective* (i.e. there exists no Lie-subgroup of G of dimension bigger than 1 with a trivial action on M). The action of T is also *quasi-effective*, and then, the moment map $\mu_{_T}$ is a submersion almost everywhere.

Theorem 4.5 tells us that, for every $\beta \in \mathcal{B}_{_G}$, the equivariant form P_β^G is induced by $\Pi^2_{\mathfrak{g}/\mathfrak{t}}\mathrm{P}_\beta^T$. In Proposition 4.18 we have shown that $\mathrm{P}_\beta^T = \sum_{\beta(0,\Delta)=\beta}\mathrm{P}_\Delta^\varepsilon$ for $\varepsilon \in \mathcal{W}$ close to 0. Then, with Corollary 4.6, we see that the function Z_β^α is the sum (modulo some multiplicative constants) of the functions $Z_{\Delta,\beta}^\alpha$, $\beta(0,\Delta)=\beta$ given by the equation

$$Z_{\Delta,\beta}^\alpha(u) = \int_{M\times\mathfrak{t}} \mathrm{P}_\Delta^\varepsilon(Y)\Pi^2_{\mathfrak{g}/\mathfrak{t}}(Y)\alpha(Y)e^{\mathrm{i}\Omega_{\mathfrak{g}}(Y)}e^{-u\|Y\|^2/2}\,dY, \quad u>0\ .$$

Following equality 4.56 we get

$$Z_{\Delta,\beta}^\alpha(u) = (2\mathrm{i}\pi)^{dim\Delta}\,\mathrm{vol}(T_\Delta,dX_1)^{-1}\times \int_{\mathcal{M}_\Delta^\varepsilon\times\mathfrak{t}_\Delta}\frac{1}{|S^\Delta|}k_\Delta(\alpha\Pi^2_{\mathfrak{g}/\mathfrak{t}}e^{\mathrm{i}\Omega_{\mathfrak{t}}})(X_1)\,\mathrm{Eul}_{\beta_\Delta}^{-1}(\mathcal{E}_\Delta)(X_1)e^{-u\|X_1+\omega_\Delta\|^2/2}dX_1.$$

<u>• $\Delta=\mathfrak{t}^*$</u> : The action of T is quasi-effective, hence the polytope $\mu_T(M)$ generates $\mathfrak{t}^*\in\mathcal{B}$. The only $\beta\in\mathcal{B}_{_G}$ with $\beta(0,\mathfrak{t}^*)=\beta$ is 0, and the term $Z_{\mathfrak{t}^*,0}^\alpha(u)$ is the polynomial given by

$$Z_{\mathfrak{t}^*,0}^\alpha(u) = \frac{(2\mathrm{i}\pi)^{dimT}}{|S^\varepsilon|}\int_{\mathcal{M}_\varepsilon} k_\varepsilon(\alpha\Pi^2_{\mathfrak{g}/\mathfrak{t}}e^{\mathrm{i}\Omega_{\mathfrak{t}}})e^{-u\|\omega_\varepsilon\|^2/2}\ , \tag{5.62}$$

where $\mathcal{M}_\varepsilon$ is the Marsden-Weinstein reduction of the manifold M, with respect to T at $\varepsilon\in\mathfrak{t}^*$.

<u>• $\Delta\neq\mathfrak{t}^*$</u> : We want now to give for $\Delta\neq\mathfrak{t}^*$ a simple expression of the functions $Z_{\Delta,\beta}^\alpha$.

The scalar product $(\cdot,\cdot)$ on $\mathfrak{t}^*$ determines a linear isomorphism $\mathfrak{t}^*\to\mathfrak{t}, v\to\tilde{v}$. For $\Delta\in\mathcal{B}$ such that $\beta=\beta(0,\Delta)$ we have $k_\Delta(\Omega_{\mathfrak{t}})=\Omega_\Delta+\langle\beta,X_1\rangle=\Omega_\Delta+(\tilde{\beta},X_1)$, with $\Omega_\Delta\in\mathcal{H}^2(\mathcal{M}_\Delta^\varepsilon)$. For $\Delta\neq\mathfrak{t}^*$ and $C_\Delta^\varepsilon\neq\emptyset$, we define

$$\alpha_\Delta(u,X_1) := \frac{(2\mathrm{i}\pi)^{dimT}}{|S^\varepsilon|}e^{\mathrm{i}\Omega_\Delta}k_\Delta(\alpha\Pi^2_{\mathfrak{g}/\mathfrak{t}})(X_1)e^{-u\|\omega_\Delta\|^2/2}e^{-u(X_1,\omega_\Delta)}\,\mathrm{vol}(T_\Delta,dX_1)^{-1},$$

for $X_1\in\mathfrak{t}_\Delta$. The differential forms $\alpha_\Delta(u,X_1)$ have a polynomial dependence with respect to each parameter $u>0$ and $X_1\in\mathfrak{t}_\Delta$. We have

$$\begin{aligned} Z_{\Delta,\beta}^\alpha(u) &= \int_{\mathcal{M}_\Delta^\varepsilon\times\mathfrak{t}_\Delta}\mathrm{Eul}_{\beta_\Delta}^{-1}(\mathcal{E}_\Delta)(X_1)\alpha_\Delta(u,X_1)\,e^{\mathrm{i}(\tilde{\beta},X_1)}\,e^{-u\|X_1\|^2/2}dX_1 \\ &= \int_{\mathcal{M}_\Delta^\varepsilon\times\mathfrak{t}_\Delta^*}\mathcal{F}_{\mathfrak{t}_\Delta}\Big(\mathrm{Eul}_{\beta_\Delta}^{-1}(\mathcal{E}_\Delta)\Big)(\xi)\,\mathcal{F}_{\mathfrak{t}_\Delta}^{-1}\Big(\alpha_\Delta(u,X_1)\,e^{\mathrm{i}(\tilde{\beta},X_1)}\,e^{-u\|X_1\|^2/2}dX_1\Big)(\xi)\ , \end{aligned}$$

where $\mathcal{F}_{\mathfrak{t}_\Delta}$ is the Fourier transform on $\mathfrak{t}_\Delta$. The Fourier transform of a Gaussian function gives

$$\begin{aligned}\mathcal{F}_{\mathfrak{t}_\Delta}^{-1}\Big(\alpha_\Delta(u,X_1)\,e^{\mathrm{i}(\tilde{\beta},X_1)}\,e^{-u\|X_1\|^2/2}dX_1\Big)(\xi) &= (\frac{2\pi}{u})^{\frac{dimT_\Delta}{2}}\alpha_\Delta(u,-\mathrm{i}\partial_\xi)\left(e^{-\|\beta+\xi\|^2/2u}\right)\\ &= (\frac{2\pi}{u})^{\frac{dimT_\Delta}{2}}\,u^{-d_\alpha^\Delta}\,\tilde{\alpha}_\Delta(u,\xi)\,e^{-\|\beta+\xi\|^2/2u}\ ,\end{aligned}$$

where d_α^Δ is the (maximal) polynomial degree of the equivariant form $k_\Delta(\alpha\Pi^2_{\mathfrak{g}/\mathfrak{t}})$, and $\tilde{\alpha}_\Delta(u,\xi)$ is a differential form which depends polynomially on the parameters $u>0$ and $X_1\in\mathfrak{t}_\Delta$. We get finally

$$Z^\alpha_{\Delta,\beta}(u)=(\frac{2\pi}{u})^{\frac{dimT_\Delta}{2}}\,u^{-d_\alpha^\Delta}\int_{\mathcal{M}^\varepsilon_\Delta\times\mathfrak{t}^*_\Delta}\mathcal{F}\Big(\mathrm{Eul}^{-1}_{\beta_\Delta}(\mathcal{E}_\Delta)\Big)(\xi)\,\tilde{\alpha}_\Delta(u,\xi)\,e^{-\|\beta+\xi\|^2/2u}\ . \tag{5.63}$$

For $\varepsilon\in\mathfrak{t}^*$ sufficiently close to 0, the density $\mathcal{F}(\mathrm{Eul}^{-1}_{\beta_\Delta}(\mathcal{E}_\Delta))$ is of the form $P_\Delta(\xi)d\xi$, where P_Δ is a locally polynomial function supported by a cone $\mathfrak{C}^\varepsilon_\Delta$ *strictly* included in the half-plane $\{\xi\in\mathfrak{t}^*_\Delta|\ (\xi,\beta)\geq 0\}$. There are two cases.

1) $\beta=0$

We make the change of variable $\xi=\sqrt{u}.\xi'$ in the integral (5.63). We find

$$Z^\alpha_{\Delta,0}(u)=(2\pi)^{\frac{dimT_\Delta}{2}}\,u^{-d_\alpha^\Delta}\int_{\mathcal{M}^\varepsilon_\Delta\times\mathfrak{t}^*_\Delta}P_\Delta(\sqrt{u}.\xi')\,\tilde{\alpha}_\Delta(u,\sqrt{u}.\xi')\,e^{-\|\xi'\|^2/2}d\xi'\ ,$$

which is a polynomial in the variable $\sqrt{u}$, times $u^{-d_\alpha^\Delta}$. The polynomial $\Pi^2_{\mathfrak{g}/\mathfrak{t}}$ is of degree $\dim(G/T)$, then for every Δ, the integer d_α^Δ is bounded by $N_\alpha=d_\alpha+\dim G$.

2) $\beta\neq 0$

After the change of variable $\xi=u.\xi'$ in the integral (5.63), we find

$$Z^\alpha_{\Delta,\beta}(u)=(2\pi u)^{\frac{dimT_\Delta}{2}}\,u^{-d_\alpha^\Delta}\,e^{-\|\beta\|^2/2u}\int_{\mathcal{M}^\varepsilon_\Delta\times\mathfrak{t}^*_\Delta}P_\Delta(u.\xi')\,\tilde{\alpha}_\Delta(u,u.\xi')\,e^{-u\|\xi'\|^2/2}e^{-(\beta,\xi')}d\xi'\ .$$

Because of the *strict* inclusion of the cone $\mathfrak{C}^\varepsilon_\Delta$ in the half-space $\{\xi\in\mathfrak{t}^*_\Delta|\ (\xi,\beta)\geq 0\}$, there exists a constant $c>0$ such that $(\beta,\xi')\geq c\,\|\xi'\|$ for every $\xi'\in\mathfrak{C}^\varepsilon_\Delta$. Then, the term $e^{-(\beta,\xi')}$ insures, for every $u\in\mathbb{R}$, the convergence of the integral

$$h^\beta_\Delta(u)=(2\pi)^{\frac{dimT_\Delta}{2}}\int_{\mathcal{M}^\varepsilon_\Delta\times\mathfrak{t}^*_\Delta}P_\Delta(u^2.\xi')\,\tilde{\alpha}_\Delta(u^2,u^2.\xi')\,e^{-\|u.\xi'\|^2/2}e^{-(\beta,\xi')}d\xi'\ . \tag{5.64}$$

Furthermore we see from the integral (5.64) that the function h^β_Δ is *smooth* and even over $\mathbb{R}$.

Conclusion: For $\beta=0$, the function

$$Z^\alpha_0(u)=Z^\alpha_{\mathfrak{t}^*,0}(u)+\sum_{0\in\Delta}Z^\alpha_{\Delta,0}(u)$$

is a polynomial in $\sqrt{u}$ times u^{-N_α}. For $\beta \neq 0$, the function Z_β^α is of the form

$$Z_\beta^\alpha(u) = u^{-N_\alpha}\, e^{-\|\beta\|^2/2u}\Big(\sum_{\beta(0,\Delta)=\beta} u^{\frac{n_\Delta}{2}} h_\Delta^\beta(\sqrt{u}) \Big),$$

where the h_Δ^β are smooth, and $n_\Delta = 2(N_\alpha - d_\alpha^\Delta) + dim T_\Delta \geq dim T_\Delta \geq 1$. From the equation (5.64), we see that the functions h_Δ^β are slowly increasing.

When 0 is a regular value of μ_G, the function Z_0^α can be computed directly with Proposition 3.11. In this case,

$$Z_0^\alpha(u) = \frac{(2\mathrm{i}\pi)^p}{|S_o|}\left(\int_{M_{red}} k_0(\alpha e^{\mathrm{i}\Omega_\mathfrak{g}}) e^{-u\|\omega\|^2} \right), \quad u > 0,$$

is polynomial in the variable u (for the notations see subsection 3.1).

In the case of a general action of G on M, let S be the connected component of the group $\cap_{m\in M} \mathrm{Stab}_G(m)$. The group S is a normal subgroup of G with a trivial action on M, hence we have an induced Hamiltonian action of G/S on M which is furthermore *quasi-effective*. We see that the computation of the functions Z_β^α reduces to the computations of the related functions for the G/S-action. The details are left to the reader. □

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