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# A note on robust Nash equilibria in games with uncertainties

Vianney Perchet \*

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## Abstract

In this short note, we investigate extensions of Nash equilibria when players have some uncertainties upon their payoffs mappings, the behavior (or the type, number or any other characteristics) of their opponents. These solutions are qualified either as robust, ambiguous, partially specified or with uncertainty aversion, depending on the context. We provide a simple necessary and sufficient condition that guarantees their existence and we show that this is actually a selection of conjectural (or self-confirming) equilibria. We finally conclude by how this concept can and should be defined in games with partial monitoring in order to preserve existence properties.

## Introduction

Nash equilibria are " $N$ -tuple of [mixed] strategies, one for each player, [that] may be regarded as a point in the product space obtained by multiplying the  $N$  strategy spaces of the players" [13], "such that each player's mixed strategy maximizes his payoff if the strategies of the others are held fixed. Thus each player's strategy is optimal against those of the others" [14]. It is therefore implicitly assumed that each player knows not only his own payoff mapping, but also his set of opponents and strategies they are going to play. However, in some cases (which can rely on empirical or intuitive results as in Ellsberg's paradox [6]), players do not have perfect knowledge of their own payoff mapping (or preferences can not be represented by such mappings) or they might have only partial information upon their opponents or their action set as in games on internet.

Several authors have modeled this and different, yet quite related, notions have emerged. Among them, we can cite *robustness* [1] – when payoff mappings are unknown (the name refers to robust optimization [5]) – or *uncertainty aversion* [10], *ambiguity* [3], *partially specified probabilities* [11] – when strategies actually played at an equilibrium is not perfectly known to players (their only knowledge is that they must belong to some given sets). A common feature of these concepts is that they all rely on maximization of a criterion with a non-unique prior, following [8]. Another solution is to assume that,

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in presence of uncertainties, players formulate a *conjecture* upon their opponents, and then maximize payoffs with respect to this conjecture. This is exactly the basic ideas behind *conjectural* and *self-confirming* equilibria [4, 7, 9].

The main focus of this note is to formalize and unify in a general framework different notions of Nash equilibria with uncertainties (yet we kept the name of robust Nash equilibria) and to provide a simple necessary and sufficient condition guaranteeing their existence. We also show that this is in fact a selection of conjectural equilibria and we describe how it should be defined in games with partial monitoring [12]. Results are, as often as possible, based on simple and illustrating examples.

## 1 Robust Nash equilibria

We consider  $N$ -player games where  $\mathcal{X}_n \subset \mathbb{R}^{A_n}$ , action set of player  $n \in \mathcal{N} := \{1, \dots, N\}$ , is a compact and convex set and his payoff mapping  $u_n : \mathcal{X} \rightarrow \mathbb{R}$ , with  $\mathcal{X} = \prod_{m \in \mathcal{N}} \mathcal{X}_m$ , is multilinear. In particular  $u_n(\cdot, x_{-n})$  is linear for every  $x_{-n} \in \mathcal{X}_{-n} := \prod_{m \neq n} \mathcal{X}_m$ .

We first recall some basic facts on Nash equilibria. Define, for every  $n \in \mathcal{N}$ , the best reply correspondence  $\text{BR}_n$  from  $\mathbb{R}^{A_n}$  to  $\mathcal{X}_n$  by  $\text{BR}_n(U_n) := \arg \max_{x_n \in \mathcal{X}_n} \langle x_n, U_n \rangle$ . Then  $x^* = (x_1^*, \dots, x_n^*) \in \mathcal{X}$  is a Nash equilibrium iff there exists, for every  $n \in \mathcal{N}$ ,  $U_n \in \mathbb{R}^{A_n}$  satisfying  $x_n^* \in \text{BR}_n(U_n)$  and  $U_n = u_n(\cdot, x_{-n}^*)$ . As it is a fixed point of  $x \mapsto \prod_{n \in \mathcal{N}} \text{BR}_n(u_n(\cdot, x_{-n}))$ , its existence is ensured by Kakutani's theorem.

Best replies are extended with *uncertainties*, following [8], into

$$\text{BR}_n : \mathcal{P}(\mathbb{R}^{A_n}) \rightrightarrows \mathcal{X}_n \quad \text{with} \quad \text{BR}_n(\mathcal{U}_n) = \arg \max_{x_n \in \mathcal{X}_n} \inf_{U_n \in \mathcal{U}_n} \langle x_n, U_n \rangle,$$

where  $\mathcal{P}(\mathbb{R}^{A_n})$  is the family of subsets of  $\mathbb{R}^{A_n}$ . This is well-defined since  $x \mapsto \inf_{U_n \in \mathcal{U}_n} \langle x, U_n \rangle$  is concave and upper semi-continuous hence maxima are attained.

**Remark 1** *Evaluation of payoff decreases with respect to uncertainties, represented by the subset  $\mathcal{U}_n$ . Indeed, if  $\mathcal{V}_n \subset \mathcal{U}_n$ , then  $\inf_{V_n \in \mathcal{V}_n} \langle x_n, V_n \rangle \geq \inf_{U_n \in \mathcal{U}_n} \langle x_n, U_n \rangle$ , for every  $x_n \in \mathcal{X}_n$ . This is referred as "uncertainty aversion" of players, see for instance [8]: the more information a player has, the more he values his payoff.*

No assumptions are made on origins or structure of uncertainties; they are represented, for every  $n \in \mathcal{N}$ , by a given mapping  $\Phi_n : \mathcal{X}_{-n} \rightrightarrows \mathbb{R}^{A_n}$ . The usual framework, called *full monitoring case*, corresponds to  $\Phi_n(x_{-n}) = \{u_n(\cdot, x_{-n})\}$ .

**Definition 1**  $x^* = (x_1^*, \dots, x_n^*) \in \mathcal{X}$  is a robust Nash equilibrium iff there exists, for every  $n \in \mathcal{N}$ ,  $\mathcal{U}_n \subset \mathbb{R}^{A_n}$  satisfying  $x_n^* \in \text{BR}_n(\mathcal{U}_n)$  and  $\mathcal{U}_n = \Phi_n(x_{-n}^*)$ .

Existence of Robust Nash equilibria is ensured under a mild regularity assumption.

**Proposition 1** *If every  $\Phi_n$  is continuous, there exist robust Nash equilibria.*

**Proof:** Robust Nash equilibria are fixed points of the correspondence defined, for every  $x \in \mathcal{X}$ , by  $\text{BR}[\Phi(x)] = \prod_{n \in \mathcal{N}} \text{BR}_n[\Phi_n(x_{-n})]$ , which is always a compact non-empty convex subset of  $\mathcal{X}$ . If  $\Phi$  is continuous then  $\text{BR}[\Phi(\cdot)]$  has a closed graph, hence by Kakutani's theorem, it has fixed points that are Nash equilibria.  $\square$

A key property is that existence of Nash equilibria is not implied only by either upper nor lower semi-continuity of  $\Phi$ , as illustrated in the following Example 1.

**Example 1** Consider the following bi-matrix game, whose unique Nash equilibrium in full monitoring is  $(x^*, y^*) = (1/2T + 1/2B, 2/3L + 1/3R)$ , defined by

$$\begin{array}{c} \begin{array}{cc} & \begin{array}{c} L \\ R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{array}{|c|c|} \hline 1;0 & 0;1 \\ \hline 0;1 & 2;0 \\ \hline \end{array} \end{array} & \begin{array}{l} \mathcal{X}_1 = \Delta(\{T, B\}), \mathcal{X}_2 = \Delta(\{L, R\}), \Phi_2(x) = \{u_2(x, \cdot)\}, \\ \Phi_1(y^*) = \{u_1(\cdot, y); y \in \mathcal{X}_2\} \text{ and } \Phi_1(y) = \{u_1(\cdot, y)\} \text{ otherwise.} \end{array} \end{array}$$

$\Phi_2$  is continuous and  $\Phi_1$  upper semi-continuous, yet no robust Nash equilibrium exists.

If  $\Phi_1$  is modified into  $\Phi'_1(R) = \{u_1(\cdot, R)\}$  and  $\Phi'_1(y) = \{u_1(\cdot, y); y \in \mathcal{X}_2\}$  otherwise, then  $\Phi'_1$  is lower semi-continuous and no robust Nash equilibrium exists.

Precedent notions of equilibria corresponds to specific structures of  $\Phi_n$ : for instance,  $\Phi_n(x_{-n}) = \{u(\cdot, x_{-n}); u \in \mathbf{U}\}$  where  $\mathbf{U}$  is some given convex family of possible payoff mappings in [1], or  $\Phi_n(x_{-n}) = \{u_n(\cdot, q_{-n}); q_{-n} \in \prod_{m \neq n} \mathcal{X}_m[x_m]\}$  where  $\mathcal{X}_m[x_m] \subset \mathcal{X}_m$  is defined by a small number of linear (in  $x_{-m}$ ) mappings [11].

## 2 Selection of conjectural equilibria

*Conjectural, self-confirming* or *subjective* equilibria [4, 7, 9] can be related to robust Nash equilibria. Recall that  $x^* \in \mathcal{X}$  is a conjectural equilibrium of a game with uncertainties if, for every  $n \in \mathcal{N}$ , there exists a conjecture  $V_n^*$ , i.e. an element of the convex hull of  $\Phi(x_{-n}^*)$ , denoted by  $\text{co}(\Phi(x_{-n}^*))$ , such that  $x_n^*$  is a best reply to  $\{V_n^*\}$ , see e.g. [2].

Sets of conjectural equilibria can be very large and even equal to  $\mathcal{X}$ . This is the case in every game *in the dark* (without strictly dominated strategies) i.e., as soon as  $\Phi_n(\cdot) = \{u_n(\cdot, x_{-n}); x_{-n} \in \mathcal{X}_{-n}\}$ . On the other hand, in those games, there are generically only one robust Nash equilibrium: each player plays his maxmin strategy.

**Proposition 2** A robust Nash equilibrium is a conjectural equilibrium.

**Proof** Any robust Nash equilibrium  $x^*$  satisfies, by linearity of  $\langle x, \cdot \rangle$ ,

$$x_n^* \in \arg \max_{x \in \mathcal{X}_n} \min_{U_n \in \Phi_n(x_{-n}^*)} \langle x, U_n \rangle = \arg \max_{x \in \mathcal{X}_n} \min_{V_n \in \text{co}(\Phi_n(x_{-n}^*))} \langle x, V_n \rangle$$

So  $x_n^*$  is an optimal strategy in the zero-sum game with action sets  $\mathcal{X}_n$ ,  $\text{co}(\Phi_n(x_{-n}^*))$  and payoff  $\langle x, V_n \rangle$ . It remains to let  $V_n^*$  be any optimal strategy of the second player.  $\square$

Nash equilibria with uncertainties, as defined in [10], is a pair  $(x^*, \mathcal{U})$  such that  $x_n^* \in \mathcal{U}_n$  and  $u_n(\cdot, x_{-n}^*) \in \mathcal{U}_n$ . This is more related to conjectural than to robust equilibria (except that a conjecture is not some probability upon a set but the whole set).

### 3 Equilibria of games with partial monitoring

We consider finite games, where sets of *pure and mixed action* of player  $n$  are respectively  $\mathcal{A}_n$  and  $\mathcal{X}_n = \Delta(\mathcal{A}_n)$ , with *partial monitoring*: players do not observe actions of their opponents but they receive messages, see [12]. Formally, there exist a convex compact set of messages  $\mathcal{H}$  and signaling mappings  $H_n$  from  $\mathcal{A} := \prod_{n \in \mathcal{N}} \mathcal{A}_n$  into  $\mathcal{H}$ , extended multi-linearly to  $\mathcal{X}$ . Given  $a \in \mathcal{A}$ , player  $n$  receives the message  $H_n(a)$ .

No matter his choice of actions, player  $n$  cannot distinguish between  $x_{-n}$  and  $x'_{-n}$  in  $\mathcal{X}_{-n}$  that satisfy  $H_n(a, x_{-n}) = H_n(a, x'_{-n})$  for every  $a \in \mathcal{A}_n$ . We define the *maximal informative mapping*  $\mathbf{H}_n : \mathcal{X}_n \rightarrow \mathcal{H}^{A_n}$  by:

$$\forall x_{-n} \in \mathcal{X}_{-n}, \mathbf{H}_n(x_{-n}) = \left[ H_n(a, x_{-n}) \right]_{a \in \mathcal{A}_n} \in \mathcal{H}^{A_n}.$$

These mappings induce naturally the correspondences  $\Phi_n : \mathcal{X}_{-n} \rightrightarrows \mathbb{R}^{A_n}$  defined by:

$$\Phi_n(x_{-n}) := \{u_n(\cdot, x'_{-n}) \in \mathbb{R}^{A_n}; \mathbf{H}_n(x'_{-n}) = \mathbf{H}_n(x_{-n})\}. \quad (1)$$

**Definition 2**  $x^* \in \mathcal{X}$  is a Nash equilibrium of a game with partial monitoring  $\mathbf{H}$  iff it is a robust Nash equilibrium, with uncertainties  $\Phi_n$  defined by Equation (1).

$\mathbf{H}_n$  and  $u_n$  are continuous, so  $\Phi$  is continuous and Nash equilibria always exist.

**Example 2** Consider the game with payoffs given by the left matrix and  $\mathcal{H} = \{a, b, c\}$ . Player 2 has full monitoring, so  $H_2$  is not represented, and  $H_1$  is the right matrix:

$$u_1; u_2 = \begin{array}{c} T \\ B \end{array} \begin{array}{|c|c|c|} \hline L & M & R \\ \hline 2; 0 & 1; 0 & 1; 2 \\ \hline 0; 1 & 2; 0 & 2; 2 \\ \hline \end{array} \quad \text{and} \quad H_1 = \begin{array}{c} T \\ B \end{array} \begin{array}{|c|c|c|} \hline L & M & R \\ \hline a & a & b \\ \hline c & c & c \\ \hline \end{array}$$

Actions  $L$  and  $M$  are undistinguishable so, for every  $\lambda \in [0, 1]$  and  $\eta \in [0, \lambda]$ :

$$\begin{aligned} \Phi_1(\lambda L + (1 - \lambda)R) &= \Phi_1(\lambda M + (1 - \lambda)R) = \Phi_1(\eta L + (\lambda - \eta)M + (1 - \lambda)R) \\ &= \left\{ (1 + \gamma, 2 - 2\gamma); \gamma \in [0, \lambda] \right\}, \end{aligned}$$

where  $(1 + \gamma, 2 - 2\gamma)$  are respective payoffs of  $T$  and  $B$  for some  $\gamma$ . As a consequence:

$$\text{BR}_1\left(\Phi_1(\lambda L + (1 - \lambda)R)\right) = \begin{cases} \{2/3T + 1/3B\} & \text{if } \lambda < 1/3 \\ \{B\} & \text{if } \lambda > 1/3 \\ \Delta(\{T, B\}) & \text{if } \lambda = 1/3 \end{cases},$$

and, since  $R$  is a strictly dominating strategy, the only Nash equilibrium is  $(B, R)$ .

One may object that, given  $x^* = (x_n^*, x_{-n}^*)$ , there might exist  $a_n \in \mathcal{A}_n$  such that  $x_n^*[a_n]$ , the weight put by  $x_n^*$  on  $a_n$ , is zero. So, player  $n$  cannot observe  $H_n(a_n, x_{-n}^*)$  nor

compute  $\mathbf{H}_n(x_n^*)$ , as in Example 2. So we should consider instead of  $\Phi_n$  the following correspondence  $\widehat{\Phi}_n : \mathcal{X} \rightrightarrows \mathbb{R}^{A_n}$  defined by

$$\widehat{\Phi}_n(x) = \{u_n(\cdot, x'_{-n}) \in \mathbb{R}^{A_n}; H_n(a_n, x'_{-n}) = H_n(a_n, x_{-n}), \forall a_n \in \mathcal{A}_n \text{ s.t. } x_n[a_n] > 0\}.$$

We also recall that, for some  $\varepsilon > 0$ ,  $x^* \in \mathcal{X}$  is an  $\varepsilon$ -equilibria if for every  $n \in \mathcal{N}$

$$\inf_{U_n \in \mathcal{U}_n} \langle x_n^*, U_n \rangle \geq \sup_{x_n \in \mathcal{X}_n} \inf_{U_n \in \mathcal{U}_n} \langle x_n, U_n \rangle - \varepsilon \text{ with } \mathcal{U}_n = \widehat{\Phi}_n(x^*).$$

**Proposition 3** *With respect to  $\widehat{\Phi}$ , there exist games that do not have equilibria or such that any perturbation of equilibria is not an  $\varepsilon$ -equilibrium*

**Proof:** In Example 2, if  $(B, R)$  is played, the only message received is  $c$  so  $\widehat{\Phi}(B, R) = \{(1 + \gamma, 2 - 2\gamma); \gamma \in [0, 1]\}$ . Its best reply is  $T$ ; yet, for every  $\delta \in (0, 1]$ , best reply to  $\widehat{\Phi}(\delta T + (1 - \delta)B, R) = \{(1, 2)\}$  is  $B$ . So this game has no equilibria.

For the second part of the proposition, consider the following two players game:

$$(u_1, u_2) = \begin{array}{c} T \\ B \end{array} \begin{array}{cc} L & R \\ \hline -1; 0 & 1; 1 \\ \hline 0; 0 & 0; 1 \end{array} \quad \text{and} \quad H_1 = \begin{array}{c} T \\ B \end{array} \begin{array}{cc} L & R \\ \hline a & b \\ \hline c & c \end{array}$$

$(B, R)$  is an equilibrium since  $\widehat{\Phi}(B, R) = \{(\lambda, 0); \lambda \in [-1, 1]\}$ . However, for every  $\delta > 0$ ,  $\widehat{\Phi}(\delta T + (1 - \delta)B, R) = \{(1, 0)\}$ , so  $\delta T + (1 - \delta)B$  is not an  $\varepsilon$ -equilibrium.  $\square$

Random messages can be embedded into this framework. Assume that there exists a finite set  $\mathcal{S}$  and given  $a \in \mathcal{A}$ , player  $n$  receives the signal  $s \in \mathcal{S}$  of law  $s_n(a) \in \Delta(\mathcal{S})$ . We define  $\mathcal{H} := \Delta(\mathcal{S})$  and  $\mathbf{H}_n(a_{-n}) := [s_n(a, a_{-n})]_{a \in \mathcal{A}_n}$ . Although  $\mathbf{H}_n(a_{-n})$  is a vector of laws, unbiased estimators can estimate it at an arbitrarily small cost, see e.g. [15].

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