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# Limiting spectral distribution of large sample covariance matrices associated to a class of stationary processes

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**Abstract**

In this paper we derive an extension of the Marčenko-Pastur theorem to a large class of weak dependent sequences of real random variables having only moment of order 2. Under a mild dependence condition that is easily verifiable in many situations, we derive that the limiting spectral distribution of the associated sample covariance matrix is characterised by an explicit equation for its Stieltjes transform, depending on the spectral density of the underlying process. Applications to linear processes, functions of linear processes and ARCH models are given.

*Key words*: Sample covariance matrices, weak dependence, Lindeberg method, Marčenko-Pastur distributions, limiting spectral distribution.
*Mathematical Subject Classification* (2010): 60F99, 60G10, 62E20.

## 1 Introduction

A typical object of interest in many fields is the sample covariance matrix $\mathbf{B}_n = n^{-1}\sum_{j=1}^n \mathbf{X}_j^T\mathbf{X}_j$ where $(\mathbf{X}_j)$, $j = 1, \ldots, n$, is a sequence of $N = N(n)$-dimensional real-valued row random vectors. The interest in studying the spectral properties of such matrices has emerged from multivariate statistical inference since many test statistics can be expressed in terms of functionals of their eigenvalues. The study of the empirical distribution function (e.d.f.) $F^{\mathbf{B}_n}$ of the eigenvalues of $\mathbf{B}_n$ goes back to Wishart 1920's, and the spectral analysis of large-dimensional sample covariance matrices has been actively developed since the remarkable work of Marčenko and Pastur (1967) stating that if $\lim_{n\to\infty} N/n = c \in (0, \infty)$, and all the coordinates of all the vectors $\mathbf{X}_j$'s are i.i.d. (independent identically distributed), centered and in $\mathbb{L}^2$, then, with probability one, $F^{\mathbf{B}_n}$ converges in distribution to a non-random distribution (the original Marčenko-Pastur's theorem is stated for random variables having moment of order four, for the proof under moment of order two only, we refer to Yin (1986)).

Since the Marčenko-Pastur's pioneering paper, there has been a large amount of work aiming at relaxing the independence structure between the coordinates of the $\mathbf{X}_j$'s. Yin (1986) and Silverstein (1995) considered a linear transformation of independent random variables which leads to the study of the empirical spectral distribution of random matrices of the form $\mathbf{B}_n = n^{-1}\sum_{j=1}^n \Gamma_N^{1/2}\mathbf{Y}_j^T\mathbf{Y}_j\Gamma_N^{1/2}$ where $\Gamma_N$ is an $N\times N$ non-negative definite Hermitian random matrix, independent of the $\mathbf{Y}_j$'s which are i.i.d and such that all their coordinates are i.i.d. In the later paper, it is shown that if $\lim_{n\to\infty} N/n = c \in (0, \infty)$, $\Gamma_N$ is bounded in spectral norm, and $F^{\Gamma_N}$ converges almost surely in distribution to a non-random probability distribution function (p.d.f.) $H$ on $[0, \infty)$, then, almost surely, $F^{\mathbf{B}_n}$ converges in distribution to a (non-random) p.d.f. $F$ that is characterized in terms of its Stieltjes transform which satisfies a certain equation. Some further investigations on the model above mentioned can be found Silverstein and Bai (1995) and Pan (2010).

A natural question is then to wonder if other possible correlation patterns of coordinates can be considered, in such a way that, almost surely (or in probability), $F^{\mathbf{B}_n}$ still converges in distribution to a non-random p.d.f. The recent work by Bai and Zhou (2008) is in this

direction. Assuming that the the $\mathbf{X}_j$'s are i.i.d. and a very general dependence structure of their coordinates, they derive the limiting spectral distribution (LSD) of $\mathbf{B}_n$. Their result has various applications. In particular, in case where the $\mathbf{X}_j$'s are independent copies of $\mathbf{X} = (X_1, \ldots, X_N)$ where $(X_k)_{k\in\mathbb{Z}}$ is a stationary linear process with centered i.i.d. innovations, applying their Theorem 1.1, they prove that, almost surely, $F^{\mathbf{B}_n}$ converges in distribution to a non-random p.d.f. $F$, provided that $\lim_{n\to\infty} N/n = c \in (0,\infty)$, the coefficients of the linear process are absolutely summable and the innovations have a moment of order four (see their Theorem 2.5). For this linear model, let us mention that in a recent paper, Yao (2012) shows that the Stieltjes transform of the limiting p.d.f. $F$ satisfies an explicit equation that depends on $c$ and on the spectral density of the underlying linear process. Still in the context of the linear model described above but, relaxing the equidistribution assumption on the innovations, and using a different approach than the one considered in the papers by Bai and Zhou (2008) and by Yao (2012), Pfaffel and Schlemm (2011) also derive the LSD of $\mathbf{B}_n$ still assuming moments of order four for the innovations plus a polynomial decay of the coefficients of the underlying linear process.

In this work, we extend such Marčenko-Pastur type theorems along another direction. We shall assume that the $\mathbf{X}_j$'s are independent copies of $\mathbf{X} = (X_1, \ldots, X_N)$ where $(X_k)_{k\in\mathbb{Z}}$ is a stationary process of the form $X_k = g(\cdots, \varepsilon_{k-1}, \varepsilon_k)$ where the $\varepsilon_k$'s are i.i.d. real valued random variables and $g : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ is a measurable function such that $X_k$ is a proper centered random variable. Assuming that $X_0$ has a moment of order two only, and imposing a dependence condition expressed in terms of conditional expectation, we prove that if $\lim_{n\to\infty} N/n = c \in (0,\infty)$, then almost surely, $F^{\mathbf{B}_n}$ converges in distribution to a non-random p.d.f. $F$ whose Stieltjes transform satisfies an explicit equation that depends on $c$ and on the spectral density of the underlying stationary process $(X_k)_{k\in\mathbb{Z}}$ (see our Theorem 2.1). The imposed dependence condition is directly related to the physical mechanisms of the underlying process, and is easy verifiable in many situations. For instance, when $(X_k)_{k\in\mathbb{Z}}$ is a linear process with i.i.d. innovations, our dependence condition is satisfied, and then our Theorem 2.1 applies, as soon as the coefficients of the linear process are absolutely summable and the innovations have a moment of order two only, which improves Theorem 2.5 in Bai and Zhou (2008) and Theorem 1.1 in Yao (2012). Other models, such as functions of linear processes and ARCH models, for which our Theorem 2.1 applies, are given in Section 3.

Let us now give an outline of the method used to prove our Theorem 2.1. Since the $\mathbf{X}_j$'s are independent, the result will follow if we can prove that the expectation of the Stieltjes transform of $F^{\mathbf{B}_n}$, say $S_{F^{\mathbf{B}_n}}(z)$, converges to the Stieltjes transform of $F$, say $S(z)$, for any complex number $z$ with positive imaginary part. With this aim, we shall consider a sample covariance matrix $\mathbf{G}_n = n^{-1}\sum_{j=1}^n \mathbf{Z}_j^T\mathbf{Z}_j$ where the $\mathbf{Z}_j$'s are independent copies of $\mathbf{Z} = (Z_1, \ldots Z_N)$ where $(Z_k)_{k\in\mathbb{Z}}$ is a sequence of Gaussian random variables having the same covariance structure as the underlying process $(X_k)_{k\in\mathbb{Z}}$. The $\mathbf{Z}_j$'s will be assumed to be independent of the $\mathbf{X}_j$'s. Using the Gaussian structure of $\mathbf{G}_n$, the convergence of $\mathbb{E}\big(S_{F^{\mathbf{G}_n}}(z)\big)$ to $S(z)$ will follow by Theorem 1.1 in Silverstein (1995). The main step of the proof is then to show that the difference between the expectations of the Stieltjes transform of $F^{\mathbf{B}_n}$ and that of $F^{\mathbf{G}_n}$ converges to zero. This will be achieved by approximating first $(X_k)_{k\in\mathbb{Z}}$ by an $m$-dependent sequence of random variables that are bounded. This leads to a new sample covariance matrix $\bar{\mathbf{B}}_n$. We then handle the difference between $\mathbb{E}\big(S_{F^{\bar{\mathbf{B}}_n}}(z)\big)$ and $\mathbb{E}\big(S_{F^{\mathbf{G}_n}}(z)\big)$ with the help of the so-called Lindeberg method used in the multidimensional case. Lindeberg method is known to be an efficient tool to derive limit theorems and, from our knowledge, it has been used for the first time in the context of random matrices by Chatterjee (2006). With the help of this method, he proved the LSD of Wigner matrices associated to exchangeable random variables.

The paper is organized as follows: in Section 2, we precise the model and state the LSD result for the sample covariance matrix associated to the underlying process. Applications to linear processes, functions of linear processes and ARCH models are given in Section 3. Section

4 is devoted to the proof of the main result, whereas some technical tools are stated and proved in Appendix.

Here is some notation used all along the paper. The notation $[x]$ is used to denote the integer part of any real $x$. For any non-negative integer $q$, the notation $\mathbf{0}_q$ means a row vector of size $q$. For a matrix $A$, we denote by $A^T$ its transpose matrix, by $\mathrm{Tr}(A)$ its trace, by $\|A\|$ its spectral norm, and by $\|A\|_2$ its Hilbert-Schmidt norm (also called the Frobenius norm). We shall also use the notation $\|X\|_r$ for the $\mathbb{L}^r$-norm ($r \geq 1$) of a real valued random variable $X$. For any square matrix $A$ of order $N$ with only real eigenvalues, the empirical spectral distribution of $A$ is defined as

$$F^A(x) = \frac{1}{N}\sum_{k=1}^{N} \mathbb{1}_{\{\lambda_k \leq x\}}\,,$$

where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of $A$. The Stieltjes transform of $F^A$ is given by

$$S_{F^A}(z) = \int \frac{1}{x-z} dF^A(x) = \frac{1}{N}\mathrm{Tr}(A - z\mathbf{I})^{-1}\,,$$

where $z = u + iv \in \mathbb{C}^+$ (the set of complex numbers with positive imaginary part), and $\mathbf{I}$ is the identity matrix.

## 2 Main result

We consider a stationary causal process $(X_k)_{k\in\mathbb{Z}}$ defined as follows: let $(\varepsilon_k)_{k\in\mathbb{Z}}$ be a sequence of i.i.d. real-valued random variables and let $g : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ be a measurable function such that, for any $k \in \mathbb{Z}$,

$$X_k = g(\xi_k) \quad \text{with} \quad \xi_k := (\ldots, \varepsilon_{k-1}, \varepsilon_k) \tag{2.1}$$

is a proper random variable, $\mathbb{E}(g(\xi_k)) = 0$ and $\|g(\xi_k)\|_2 < \infty$.

The framework (2.1) is very general and it includes many widely used linear and nonlinear processes. We refer to the papers by Wu (2005, 2011) for many examples of stationary processes that are of form (2.1). Following Priestley (1988) and Wu (2005), $(X_k)_{k\in\mathbb{Z}}$ can be viewed as a physical system with $\xi_k$ (respectively $X_k$) being the input (respectively output) and $g$ being the transform or data-generating mechanism.

For $n$ a positive integer, we consider $n$ independent copies of the sequence $(\varepsilon_k)_{k\in\mathbb{Z}}$ that we denote by $(\varepsilon_k^{(i)})_{k\in\mathbb{Z}}$ for $i = 1, \ldots, n$. Setting $\xi_k^{(i)} = \left(\ldots, \varepsilon_{k-1}^{(i)}, \varepsilon_k^{(i)}\right)$ and $X_k^{(i)} = g(\xi_k^{(i)})$, it follows that $(X_k^{(1)})_{k\in\mathbb{Z}}, \ldots, (X_k^{(n)})_{k\in\mathbb{Z}}$ are $n$ independent copies of $(X_k)_{k\in\mathbb{Z}}$. Let now $N = N(n)$ be a sequence of positive integers, and define for any $i \in \{1, \ldots, n\}$, $\mathbf{X}_i = \left(X_1^{(i)}, \ldots, X_N^{(i)}\right)$. Let

$$\mathcal{X}_n = (\mathbf{X}_1^T | \ldots | \mathbf{X}_n^T) \quad \text{and} \quad \mathbf{B}_n = \frac{1}{n}\mathcal{X}_n\mathcal{X}_n^T\,. \tag{2.2}$$

$\mathbf{B}_n$ will be referred as the sample covariance matrix associated to $(X_k)_{k\in\mathbb{Z}}$. To derive the limiting spectral distribution of $\mathbf{B}_n$, we need to impose some dependence structure on $(X_k)_{k\in\mathbb{Z}}$. With this aim, we introduce the projection operator: for any $k$ and $j$ belonging to $\mathbb{Z}$, let

$$P_j(X_k) = \mathbb{E}(X_k|\xi_j) - \mathbb{E}(X_k|\xi_{j-1})\,.$$

We state now our main result.

**Theorem 2.1** *Let $(X_k)_{k\in\mathbb{Z}}$ be defined in (2.1) and $\mathbf{B}_n$ by (2.2). Assume that*

$$\sum_{k\geq 0} \|P_0(X_k)\|_2 < \infty\,, \tag{2.3}$$

*and that* $c(n) = N/n \to c \in (0, \infty)$. *Then, with probability one,* $F^{\mathbf{B}_n}$ *tends to a probability distribution, whose Stieltjes transform* $S = S(z)$ $(z \in \mathbb{C}^+)$ *satisfies the equation*

$$z = -\frac{1}{\underline{S}} + \frac{c}{2\pi} \int_0^{2\pi} \frac{1}{\underline{S} + \big(2\pi f(\lambda)\big)^{-1}} d\lambda \,, \tag{2.4}$$

*where* $\underline{S}(z) := -(1-c)/z + cS(z)$ *and* $f(\cdot)$ *is the spectral density of* $(X_k)_{k\in\mathbb{Z}}$.

**Remark 2.2** *Under the condition* (2.3), *the series* $\sum_{k\geq 0} \big|\mathrm{Cov}(X_0, X_k)\big|$ *is finite (see for instance the inequality* (4.59)*). Therefore* (2.3) *implies that the spectral density* $f(\cdot)$ *of* $(X_k)_{k\in\mathbb{Z}}$ *exists, is continuous and bounded on* $[0, 2\pi)$.

Let us mention that the condition (2.3) is referred in the literature as the Hannan-Heyde condition and is known to be sufficient for the validity of the central limit theorem for the partial sums (normalized by $\sqrt{n}$) associated to an adapted regular stationary process in $\mathbb{L}^2$. As we shall see in the next section, the quantity $\|P_0(X_k)\|_2$ can be computed in many situations including non linear models. We would like to mention that the condition (2.3) is weaker that the 2-strong stability condition introduced by Wu (2005, Definition 3) that involves a coupling coefficient.

# 3 Applications

In this section, we give two different classes of models for which the condition (2.3) is satisfied and then for which our Theorem 2.1 applies. Other classes of models, including non linear time series such as iterative Lipschitz models, that are of the form (2.1) and for which the quantity $\|P_0(X_k)\|_2$ can be computed may be found in Wu (2011).

## 3.1 Functions of linear processes

In this section, we shall focus on functions of real-valued linear processes. Define

$$X_k = h\Big(\sum_{i\geq 0} a_i \varepsilon_{k-i}\Big) - \mathbb{E}\Big(h\Big(\sum_{i\geq 0} a_i \varepsilon_{k-i}\Big)\Big) \,, \tag{3.1}$$

where $(a_i)_{i\in\mathbf{Z}}$ be a sequence of real numbers in $\ell^1$ and $(\varepsilon_i)_{i\in\mathbb{Z}}$ is a sequence of i.i.d. real-valued random variables in $\mathbb{L}^1$. We shall give sufficient conditions in terms of the regularity of the function $h$, for the condition (2.3) to be satisfied.

Denote by $w_h(\cdot)$ the modulus of continuity of the function $h$ on $\mathbb{R}$, that is:

$$w_h(t) = \sup_{|x-y|\leq t} |h(x) - h(y)| \,.$$

**Corollary 3.1** *Assume that*

$$\sum_{k\geq 0} \|w_h(|a_k \varepsilon_0|)\|_2 < \infty \,, \tag{3.2}$$

*or*

$$\sum_{k\geq 1} \frac{\big\|w_h\big(\sum_{\ell\geq 0} |a_{k+\ell}||\varepsilon_{-\ell}|\big)\big\|_2}{k^{1/2}} < \infty \,. \tag{3.3}$$

*Then, provided that* $c(n) = N/n \to c \in (0, \infty)$, *the conclusion of Theorem 2.1 holds for* $F^{\mathbf{B}_n}$ *where* $\mathbf{B}_n$ *is the sample covariance matrix of dimension* $N$ *defined by* (2.2) *and associated to* $(X_k)_{k\in\mathbf{Z}}$ *defined by (3.1).*

**Example 1.** Assume that $h$ is $\gamma$-Hölder with $\gamma \in ]0,1]$, that is: there is a positive constant $C$ such that $w_h(t) \le C|t|^\gamma$. Assume that

$$\sum_{k\ge 0} |a_k|^\gamma < \infty \text{ and } \mathbb{E}(|\varepsilon_0|^{(2\gamma)\vee 1}) < \infty\,,$$

then the condition (3.2) is satisfied and the conclusion of Corollary 3.1 holds. In particular, when $h$ is the identity, which corresponds to the fact that $X_k$ is a causal linear process, the conclusion of Corollary 3.1 holds as soon as $\sum_{k\ge 0}|a_k| < \infty$ and $\varepsilon_0$ belongs to $\mathbb{L}^2$. This improves Theorem 2.5 in Bai and Zhou (2008) and Theorem 1 in Yao (2012) that require $\varepsilon_0$ to be in $\mathbb{L}^4$.

**Example 2.** Assume $\|\varepsilon_0\|_\infty \le M$ where $M$ is a finite positive constant, and that $a_k = \rho^k$ where $\rho \in (0,1)$, then the condition (3.3) is satisfied and the conclusion of Corollary 3.1 holds as soon as

$$\sum_{k\ge 1} \frac{w_h\big(\rho^k M(1-\rho)^{-1}\big)}{k^{1/2}} < \infty\,. \tag{3.4}$$

In particular the result applies to the case of the Bernoulli shift, that is $a_i = 2^{-i}$ and $\varepsilon_0$ is such that $\mathbb{P}(\varepsilon_0 = 1) = \mathbb{P}(\varepsilon_0 = 0) = 1/2$. In such a case, the condition (3.4) is equivalent to:

$$\int_0^1 \frac{w_h(t)}{t\sqrt{|\log t|}} dt < \infty\,.$$

For instance if $w_h(t) \le C|\log t|^{-\alpha}$ with $\alpha > 1/2$ near zero, then the above condition is satisfied.

**Proof of Corollary 3.1.** To prove the corollary, it suffices to show that the condition (2.3) is satisfied as soon as (3.2) or (3.3) holds. Let $(\varepsilon_k^*)_{k\in\mathbb{Z}}$ be an independent copy of $(\varepsilon_k)_{k\in\mathbb{Z}}$. Denoting by $\mathbb{E}_\varepsilon(\cdot)$ the conditional expectation with respect to $\varepsilon = (\varepsilon_k)_{k\in\mathbb{Z}}$, we have that, for any $k \ge 0$,

$$\begin{aligned}\|P_0(X_k)\|_2 &= \Big\|\mathbb{E}_\varepsilon\Big(h\Big(\sum_{i=0}^{k-1} a_i\varepsilon^*_{k-i} + \sum_{i\ge k} a_i\varepsilon_{k-i}\Big) - h\Big(\sum_{i=0}^{k} a_i\varepsilon^*_{k-i} + \sum_{i\ge k+1} a_i\varepsilon_{k-i}\Big)\Big)\Big\|_2\,.\\ &\le \|w_h\big(|a_k(\varepsilon_0 - \varepsilon_0^*)|\big)\|_2\end{aligned}$$

Next, by the subadditivity of $w_h(\cdot)$, $w_h(|a_k(\varepsilon_0 - \varepsilon_0^*)|) \le w_h(|a_k\varepsilon_0|) + w_h(|a_k\varepsilon_0^*|)$. Whence, $\|P_0(X_k)\|_2 \le 2\|w_h(|a_k\varepsilon_0|)\|_2$. This proves that the condition (2.3) is satisfied under (3.2).

We prove now that if (3.3) holds then so does the condition (2.3). By the computations page 1615 in Peligrad and Utev (2006),

$$\sum_{k\ge 1}\|P_0(X_k)\|_2 \le 3\sum_{\ell\ge 1}\ell^{-1/2}\|\mathbb{E}(X_\ell|\xi_0)\|_2\,. \tag{3.5}$$

With the same notations as before, we have that, for any $\ell \ge 0$,

$$\mathbb{E}(X_\ell|\xi_0) = \mathbb{E}_\varepsilon\Big(h\Big(\sum_{i=0}^{\ell-1} a_i\varepsilon^*_{\ell-i} + \sum_{i\ge \ell} a_i\varepsilon_{\ell-i}\Big) - h\Big(\sum_{i\ge 0} a_i\varepsilon^*_{\ell-i}\Big)\Big)\,.$$

Hence, for any non-negative integer $\ell$,

$$\|\mathbb{E}(X_\ell|\xi_0)\|_2 \le \Big\|w_h\Big(\sum_{i\ge\ell}|a_i(\varepsilon_{\ell-i} - \varepsilon^*_{\ell-i})|\Big)\Big\|_2 \le 2\Big\|w_h\Big(\sum_{i\ge\ell}|a_i||\varepsilon_{\ell-i}|\Big)\Big\|_2\,,$$

where we have used the subadditivity of $w_h(\cdot)$ for the last inequality. Therefore, starting from (3.5) and considering the last inequality, we derive that

$$\sum_{k\ge 1}\|P_0(X_k)\|_2 \le 6\sum_{\ell\ge 1}\frac{\Big\|w_h\Big(\sum_{i\ge\ell}|a_i||\varepsilon_{\ell-i}|\Big)\Big\|_2}{\ell^{1/2}}\,,$$

which shows that (2.3) holds as soon as (3.3) does. □

### 3.2 ARCH models

Let $(\varepsilon_k)_{k\in\mathbb{Z}}$ be an i.i.d. sequence of zero mean real-valued random variables such that $\|\varepsilon_0\|_2 = 1$. We consider the following ARCH($\infty$) model described by Giraitis *et al.* (2000):

$$Y_k = \sigma_k \varepsilon_k \quad \text{where} \quad \sigma_k^2 = c + \sum_{j\geq 1} c_j Y_{k-j}^2 \,, \tag{3.6}$$

where $c \geq 0$, $c_j \geq 0$ and $\sum_{j\geq 1} c_j < 1$. Such models are encountered when the volatility $(\sigma_k^2)_{k\in\mathbb{Z}}$ is unobserved. In that case, the process of interest is $(Y_k^2)_{k\in\mathbb{Z}}$ and, in what follows, we consider the process $(X_k)_{k\in\mathbb{Z}}$ defined, for any $k \in \mathbb{Z}$, by:

$$X_k = Y_k^2 - \mathbb{E}(Y_k^2) \quad \text{where } Y_k \text{ is defined in (3.6).} \tag{3.7}$$

Notice that, under the above conditions, there exists a unique stationary solution to (3.6) that satisfies (see Giraitis *et al.* (2000)):

$$\sigma_k^2 = c + c \sum_{\ell=1}^{\infty} \sum_{j_1,\ldots,j_\ell=1}^{\infty} c_{j_1} \ldots c_{j_\ell} \varepsilon_{k-j_1}^2 \ldots \varepsilon_{k-(j_1+\cdots+j_\ell)}^2 \,. \tag{3.8}$$

**Corollary 3.2** *Assume that $\varepsilon_0$ belongs to $\mathbb{L}^4$ and that*

$$\|\varepsilon_0\|_4^2 \sum_{j\geq 1} c_j < 1 \text{ and } \sum_{j\geq n} c_j = O(n^{-b}) \text{ for } b > 1/2 \,. \tag{3.9}$$

*Then, provided that $c(n) = N/n \to c \in (0,\infty)$, the conclusion of Theorem 2.1 holds for $F^{\mathbf{B}_n}$ where $\mathbf{B}_n$ is the sample covariance matrix of dimension $N$ defined by* (2.2) *and associated to $(X_k)_{k\in\mathbb{Z}}$ defined by (3.7).*

**Proof of Corollary 3.2.** To prove the corollary, it suffices to show that the condition (2.3) is satisfied as soon as (3.9) is. With this aim, let us notice that, for any integer $n \geq 1$,

$$\begin{aligned}
\|\mathbb{E}(X_n|\xi_0)\|_2 &= \|\varepsilon_0\|_4^2 \|\mathbb{E}(\sigma_n^2|\xi_0) - \mathbb{E}(\sigma_n^2)\|_2 \\
&\leq 2c\|\varepsilon_0\|_4^2 \Big\| \sum_{\ell=1}^{\infty} \sum_{j_1,\ldots,j_\ell=1}^{\infty} c_{j_1} \ldots c_{j_\ell} \varepsilon_{n-j_1}^2 \ldots \varepsilon_{n-(j_1+\cdots+j_\ell)}^2 \mathbf{1}_{j_1+\cdots+j_\ell \geq n} \Big\|_2 \\
&\leq 2c\|\varepsilon_0\|_4^2 \sum_{\ell=1}^{\infty} \sum_{j_1,\ldots,j_\ell=1}^{\infty} \sum_{k=1}^{\ell} c_{j_1} \ldots c_{j_\ell} \mathbf{1}_{j_k \geq [n/\ell]} \|\varepsilon_0\|_4^{2\ell} \leq 2c\|\varepsilon_0\|_4^2 \sum_{\ell=1}^{\infty} \ell \kappa^{\ell-1} \sum_{k=[n/\ell]}^{\infty} c_k \,,
\end{aligned}$$

where $\kappa = \|\varepsilon_0\|_4^2 \sum_{j\geq 1} c_j$. So, under (3.9), $\|\mathbb{E}(X_n|\xi_0)\|_2 \ll n^{-b}$ which combined with (3.5) implies that the condition (2.3) holds as soon as $b > 1/2$. □

**Remark 3.3** *Notice that if we consider the sample covariance matrix associated to $(Y_k)_{k\in\mathbb{Z}}$ defined in* (3.6)*, then its LSD follows directly by Theorem 2.1 since $P_0(Y_k) = 0$, for $k \geq 1$.*

## 4 Proof of Theorem 2.1

To prove the theorem it suffices to show that for any $z \in \mathbb{C}^+$,

$$S_{F^{\mathbf{B}_n}}(z) \to S(z) \quad \text{almost surely.} \tag{4.1}$$

Since the columns of $\mathcal{X}_n$ are independent, by Step 1 of the proof of Theorem 1.1 in Bai and Zhou (2008), to prove (4.1), it suffices to show that, for any $z \in \mathbb{C}^+$,

$$\lim_{n\to\infty} \mathbb{E}\big(S_{F^{\mathbf{B}_n}}(z)\big) = S(z)\,, \tag{4.2}$$

where $S(z)$ satisfies the equation (2.4).

The proof of (4.2) being very technical, for reader convenience, let us describe the different steps leading to it. We shall consider a sample covariance matrix $\mathbf{G}_n := \frac{1}{n}\mathcal{Z}_n\mathcal{Z}_n^T$ (see (4.32)) such that the columns of $\mathcal{Z}_n$ are independent and the random variables in each column of $\mathcal{Z}_n$ form a sequence of Gaussian random variables whose covariance structure is the same as that of the sequence $(X_k)_{k\in\mathbb{Z}}$ (see Section 4.2). The aim will be then to prove that, for any $z \in \mathbb{C}^+$,

$$\lim_{n\to\infty} \big|\mathbb{E}\big(S_{F^{\mathbf{B}_n}}(z) - \mathbb{E}\big(S_{F^{\mathbf{G}_n}}(z)\big)\big| = 0\,, \tag{4.3}$$

and

$$\lim_{n\to\infty} \mathbb{E}\big(S_{F^{\mathbf{G}_n}}(z)\big) = S(z)\,. \tag{4.4}$$

The proof of (4.4) will be achieved in Section 4.4 with the help of Theorem 1.1 in Silverstein (1995) combined with arguments developed in the proof of Theorem 1 in Yao (2012). The proof of (4.3) will be divided in several steps. First, to "break" the dependence structure, we introduce a parameter $m$, and approximate $\mathbf{B}_n$ by a sample covariance matrix $\bar{\mathbf{B}}_n := \frac{1}{n}\bar{\mathcal{X}}_n\bar{\mathcal{X}}_n^T$ (see (4.16)) such that the columns of $\bar{\mathcal{X}}_n$ are independent and the random variables in each column of $\bar{\mathcal{X}}_n$ form of an $m$-dependent sequence of random variables bounded by $2M$, with $M$ a positive real (see Section 4.1). This approximation will be done in such a way that, for any $z \in \mathbb{C}^+$,

$$\lim_{m\to\infty} \limsup_{M\to\infty} \limsup_{n\to\infty} \Big|\mathbb{E}\big(S_{F^{\mathbf{B}_n}}(z)\big) - \mathbb{E}\big(S_{F^{\bar{\mathbf{B}}_n}}(z)\big)\Big| = 0\,. \tag{4.5}$$

Next, the sample Gaussian covariance matrix $\mathbf{G}_n$ is approximated by another sample Gaussian covariance matrix $\widetilde{\mathbf{G}}_n$ (see (4.34)), depending on the parameter $m$ and constructed from $\mathbf{G}_n$ by replacing some of the variables in each column of $\mathcal{Z}_n$ by zeros (see Section 4.2). This approximation will be done in such a way that, for any $z \in \mathbb{C}^+$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \Big|\mathbb{E}\big(S_{F^{\mathbf{G}_n}}(z)\big) - \mathbb{E}\big(S_{F^{\widetilde{\mathbf{G}}_n}}(z)\big)\Big| = 0\,. \tag{4.6}$$

In view of (4.5) and (4.6), the convergence (4.3) will then follow if we can prove that, for any $z \in \mathbb{C}^+$,

$$\lim_{m\to\infty} \limsup_{M\to\infty} \limsup_{n\to\infty} \Big|\mathbb{E}\big(S_{F^{\bar{\mathbf{B}}_n}}(z)\big) - \mathbb{E}\big(S_{F^{\widetilde{\mathbf{G}}_n}}(z)\big)\Big| = 0\,. \tag{4.7}$$

This will be achieved in Section 4.3 with the help of the Lindeberg method. The rest of this section is devoted to the proofs of the convergences (4.3)-(4.7).

In what follows, we shall use the notation $a \ll b$ to mean that there exists a finite positive constant $C$, not depending on $n$, $m$ and $M$, and such that $a \leq Cb$.

## 4.1 Approximation by a sample covariance matrix associated to an $m$-dependent sequence.

Let $N \geq 3$ and $m$ be a fixed positive integer less than $N/2$. Set

$$p = p_{N,m} = [N^{1/8}m^{-1/8}] \quad \text{and} \quad k_{N,m} = \left[\frac{N}{p+m}\right]\,, \tag{4.8}$$

where we recall that $[\,\cdot\,]$ denotes the integer part. Let $M$ be a fixed positive number that depends neither on $N$, nor on $n$, nor on $m$. Let $\varphi_M$ be the function defined by $\varphi_M(x) = (x \wedge M) \vee (-M)$. Now for any $k \in \mathbb{Z}$ and $i \in \{1, \ldots, n\}$ let

$$\widetilde{X}^{(i)}_{k,M,m} = \mathbb{E}\Big(\varphi_M(X^{(i)}_k)|\varepsilon^{(i)}_k, \ldots, \varepsilon^{(i)}_{k-m}\Big) \text{ and } \bar{X}^{(i)}_{k,M,m} = \widetilde{X}^{(i)}_{k,M,m} - \mathbb{E}\big(\widetilde{X}^{(i)}_{k,M,m}\big)\,. \tag{4.9}$$

In what follows, to soothe the notations, we shall write $\widetilde{X}^{(i)}_{k,m}$ and $\bar{X}^{(i)}_{k,m}$ instead of respectively $\widetilde{X}^{(i)}_{k,M,m}$ and $\bar{X}^{(i)}_{k,M,m}$, when no confusion is allowed. Notice that $\big(\bar{X}^{(1)}_{k,m}\big)_{k\in\mathbb{Z}}, \ldots, \big(\bar{X}^{(n)}_{k,m}\big)_{k\in\mathbb{Z}}$ are $n$ independent copies of the centered and stationary sequence $\big(\bar{X}_{k,m}\big)_{k\in\mathbb{Z}}$ defined by

$$\widetilde{X}_{k,m} = \mathbb{E}\Big(\varphi_M(X_k)|\varepsilon_k, \ldots, \varepsilon_{k-m}\Big) \text{ and } \bar{X}_{k,m} = \widetilde{X}_{k,m} - \mathbb{E}\big(\widetilde{X}_{k,m}\big)\ ,\ k \in \mathbb{Z}\,. \tag{4.10}$$

This implies in particular that: for any $i \in \{1, \ldots, n\}$ and any $k \in \mathbb{Z}$,

$$\|\bar{X}^{(i)}_{k,m}\|_\infty = \|\bar{X}_{k,m}\|_\infty \le 2M\,. \tag{4.11}$$

For any $i \in \{1, \ldots, n\}$, note that $\big(\bar{X}^{(i)}_{k,m}\big)_{k\in\mathbb{Z}}$ forms a $m$-dependent sequence, in the sense that $\bar{X}^{(i)}_{k,m}$ and $\bar{X}^{(i)}_{k',m}$ are independent if $|k - k'| > m$. We write now the interval $[1, N] \cap \mathbb{N}$ as a union of disjoint sets as follows:

$$[1, N] \cap \mathbb{N} = \bigcup_{\ell=1}^{k_{N,m}+1} I_\ell \cup J_\ell\,,$$

where, for $\ell \in \{1, \ldots, k_{N,m}\}$,

$$\begin{aligned} I_\ell &:= \big[(\ell-1)(p+m)+1\,,\,(\ell-1)(p+m)+p\big] \cap \mathbb{N}, \\ J_\ell &:= \Big[(\ell-1)(p+m)+p+1\,,\,\ell(p+m)\Big] \cap \mathbb{N}\,, \end{aligned} \tag{4.12}$$

and, for $\ell = k_{N,m} + 1$,

$$I_{k_{N,m}+1} = \big[k_{N,m}(p+m)+1\,,\,N\big] \cap \mathbb{N}\,,$$

and $J_{k_{N,m}+1} = \emptyset$. Note that $I_{k_{N,m}+1} = \emptyset$ if $k_{N,m}(p+m) = N$.

Let now $\big(\mathbf{u}^{(i)}_\ell\big)_{\ell\in\{1,\ldots,k_{N,m}\}}$ be the random vectors defined as follows. For any $\ell$ belonging to $\{1, \ldots, k_{N,m} - 1\}$,

$$\mathbf{u}^{(i)}_\ell = \Big(\big(\bar{X}^{(i)}_{k,m}\big)_{k\in I_\ell}, \mathbf{0}_m\Big)\,. \tag{4.13}$$

Hence, the dimension of the random vectors defined above is equal to $p+m$. Now, for $\ell = k_{N,m}$, we set

$$\mathbf{u}^{(i)}_{k_{N,m}} = \Big(\big(\bar{X}^{(i)}_{k,m}\big)_{k\in I_{k_{N,m}}}, \mathbf{0}_r\Big)\,, \tag{4.14}$$

where $r = m + N - k_{N,m}(p+m)$. This last vector is then of dimension $N - (k_{N,m} - 1)(p+m)$. Notice that the random vectors $\big(\mathbf{u}^{(i)}_\ell\big)_{1\le i\le n, 1\le \ell\le k_{N,m}}$ are mutually independent.

For any $i \in \{1, \ldots, n\}$, we define now row random vectors $\bar{\mathbf{X}}^{(i)}$ of dimension $N$ by setting

$$\bar{\mathbf{X}}^{(i)} = \big(\mathbf{u}^{(i)}_{j,\ell}\,,\,\ell = 1, \ldots, k_{N,m}\big)\,, \tag{4.15}$$

where the $\mathbf{u}^{(i)}_\ell$'s are defined in (4.13) and (4.14). Let

$$\bar{\mathcal{X}}_n = \big(\bar{\mathbf{X}}^{(1)T}|\ldots|\bar{\mathbf{X}}^{(n)T}\big) \text{ and } \bar{\mathbf{B}}_n = \frac{1}{n}\bar{\mathcal{X}}_n\bar{\mathcal{X}}_n^T\,. \tag{4.16}$$

In what follows, we shall prove the following proposition.

**Proposition 4.1** *For any $z \in \mathbb{C}^+$, the convergence* (4.5) *holds true with $\mathbf{B}_n$ and $\bar{\mathbf{B}}_n$ as defined in* (2.2) *and* (4.16) *respectively.*

To prove the proposition above, we start by noticing that, by integration by parts, for any $z = u + iv \in \mathbb{C}^+$,

$$\Big|\mathbb{E}\big(S_{F^{\mathbf{B}_n}}(z)\big) - \mathbb{E}\big(S_{F^{\bar{\mathbf{B}}_n}}(z)\big)\Big| \leq \mathbb{E}\Big|\int \frac{1}{x-z} dF^{\mathbf{B}_n}(x) - \int \frac{1}{x-z} dF^{\bar{\mathbf{B}}_n}(x)\Big|$$
$$= \mathbb{E}\Big|\int \frac{F^{\mathbf{B}_n}(x) - F^{\bar{\mathbf{B}}_n}(x)}{(x-z)^2} dx\Big| \leq \frac{1}{v^2}\,\mathbb{E}\int \big|F^{\mathbf{B}_n}(x) - F^{\bar{\mathbf{B}}_n}(x)\big| dx\,. \quad (4.17)$$

Now, $\int \big|F^{\mathbf{B}_n}(x) - F^{\bar{\mathbf{B}}_n}(x)\big| dx$ is nothing else but the Wasserstein distance of order 1 between the empirical measure of $\mathbf{B}_n$ and that of $\bar{\mathbf{B}}_n$. To be more precise, if $\lambda_1, \ldots, \lambda_N$ denote the eigenvalues of $\mathbf{B}_n$ in the non-increasing order, and $\bar{\lambda}_1, \ldots, \bar{\lambda}_N$ the ones of $\bar{\mathbf{B}}_n$, also in the non-increasing order, then, setting $\eta_n = \frac{1}{N}\sum_{k=1}^N \delta_{\lambda_k}$ and $\bar{\eta}_n = \frac{1}{N}\sum_{k=1}^N \delta_{\bar{\lambda}_k}$, we have that

$$\int \big|F^{\mathbf{B}_n}(x) - F^{\bar{\mathbf{B}}_n}(x)\big| dx = W_1(\eta_n, \bar{\eta}_n) = \inf \mathbb{E}|X - Y|\,,$$

where the infimum runs over the set of couples of random variables $(X, Y)$ on $\mathbb{R} \times \mathbb{R}$ such that $X \sim \eta_n$ and $Y \sim \bar{\eta}_n$. Arguing as in Remark 4.2.6 in Chafaï, Guédon, Lecué and Pajor (2012), we have that

$$W_1(\eta_n, \bar{\eta}_n) = \frac{1}{N} \min_{\pi \in \mathcal{S}_N} \sum_{k=1}^{N \wedge n} |\lambda_k - \bar{\lambda}_{\pi(k)}|\,,$$

where $\pi$ is a permutation belonging to the symmetric group $\mathcal{S}_N$ of $\{1, \ldots, N\}$. By standard arguments, involving the fact that if $x, y, u, v$ are real numbers such that $x \leq y$ and $u > v$, then $|x-u| + |y-v| \geq |x-v| + |y-u|$, we get that $\min_{\pi \in \mathcal{S}_N} \sum_{k=1}^{N\wedge n} |\lambda_k - \bar{\lambda}_{\pi(k)}| = \sum_{k=1}^{N\wedge n} |\lambda_k - \bar{\lambda}_k|$. Therefore,

$$W_1(\eta_n, \bar{\eta}_n) = \int \big|F^{\mathbf{B}_n}(x) - F^{\bar{\mathbf{B}}_n}(x)\big| dx = \frac{1}{N} \sum_{k=1}^{N \wedge n} |\lambda_k - \bar{\lambda}_k|\,. \quad (4.18)$$

Notice that $\lambda_k = s_k^2$ and $\bar{\lambda}_k = \bar{s}_k^2$ where the $s_k$'s (respectively the $\bar{s}_k$'s) are the singular values of the matrix $n^{-1/2}\mathcal{X}_n$ (respectively of $n^{-1/2}\bar{\mathcal{X}}_n$). Hence, by Cauchy-Schwarz's inequality,

$$\sum_{k=1}^{N\wedge n} |\lambda_k - \bar{\lambda}_k| \leq \Big(\sum_{k=1}^{N\wedge n} \big|s_k + \bar{s}_k\big|^2\Big)^{1/2} \Big(\sum_{k=1}^{N\wedge n} \big|s_k - \bar{s}_k\big|^2\Big)^{1/2}$$
$$\leq 2^{1/2}\Big(\sum_{k=1}^{N\wedge n} \big(s_k^2 + \bar{s}_k^2\big)\Big)^{1/2} \Big(\sum_{k=1}^{N\wedge n} \big|s_k - \bar{s}_k\big|^2\Big)^{1/2} \leq 2^{1/2}\Big(\mathrm{Tr}(\mathbf{B}_n) + \mathrm{Tr}(\bar{\mathbf{B}}_n)\Big)^{1/2} \Big(\sum_{k=1}^{N\wedge n} \big|s_k - \bar{s}_k\big|^2\Big)^{1/2}.$$

Next, by Hoffman-Wielandt's inequality (see e.g. Corollary 7.3.8 in Horn and Johnson (1985)),

$$\sum_{k=1}^{N\wedge n} \big|s_k - \bar{s}_k\big|^2 \leq n^{-1}\mathrm{Tr}\big((\mathcal{X}_n - \bar{\mathcal{X}}_n)(\mathcal{X}_n - \bar{\mathcal{X}}_n)^T\big)\,.$$

Therefore,

$$\sum_{k=1}^{N\wedge n} |\lambda_k - \bar{\lambda}_k| \leq 2^{1/2} n^{-1/2}\Big(\mathrm{Tr}(\mathbf{B}_n) + \mathrm{Tr}(\bar{\mathbf{B}}_n)\Big)^{1/2} \Big(\mathrm{Tr}\big((\mathcal{X}_n - \bar{\mathcal{X}}_n)(\mathcal{X}_n - \bar{\mathcal{X}}_n)^T\big)\Big)^{1/2}. \quad (4.19)$$

Starting from (4.17), considering (4.18) and (4.19), and using Cauchy-Schwarz's inequality, it follows that

$$\begin{aligned}&\Big|\mathbb{E}\big(S_{F^{\mathbf{B}_n}}(z)\big)-\mathbb{E}\big(S_{F^{\bar{\mathbf{B}}_n}}(z)\big)\Big|\\&\qquad\le\frac{2^{1/2}}{v^2}\frac{1}{Nn^{1/2}}\|\mathrm{Tr}(\mathbf{B}_n)+\mathrm{Tr}(\bar{\mathbf{B}}_n)\|_1^{1/2}\|\mathrm{Tr}\big((\mathcal{X}_n-\bar{\mathcal{X}}_n)(\mathcal{X}_n-\bar{\mathcal{X}}_n)^T\big)\|_1^{1/2}\,.\end{aligned}\tag{4.20}$$

By the definition of $\mathbf{B}_n$,

$$\frac{1}{N}\mathbb{E}\big(|\mathrm{Tr}(\mathbf{B}_n)|\big)=\frac{1}{nN}\sum_{i=1}^{n}\sum_{k=1}^{N}\big\|X_k^{(i)}\big\|_2^2=\|X_0\|_2^2\,,\tag{4.21}$$

where we have used that for each $i$, $\big(X_k^{(i)}\big)_{k\in\mathbb{Z}}$ is a copy of the stationary sequence $(X_k)_{k\in\mathbb{Z}}$. Now, setting

$$\mathcal{I}_{N,m}=\bigcup_{\ell=1}^{k_{N,m}}I_\ell\ \text{ and }\ \mathcal{R}_{N,m}=\{1,\dots,N\}\backslash\mathcal{I}_{N,m}\,,\tag{4.22}$$

recalling the definition (4.16) of $\bar{\mathbf{B}}_n$, using the stationarity of the sequence $(\bar{X}_{k,m}^{(i)})_{k\in\mathbb{Z}}$, and the fact that $\mathrm{card}(\mathcal{I}_{N,m})=pk_{N,m}\le N$, we get that

$$\frac{1}{N}\mathbb{E}\big(|\mathrm{Tr}(\bar{\mathbf{B}}_n)|\big)=\frac{1}{nN}\sum_{i=1}^{n}\sum_{k\in\mathcal{I}_{N,m}}\big\|\bar{X}_{k,m}^{(i)}\big\|_2^2\le\|\bar{X}_{0,m}\|_2^2\,.$$

Next,

$$\|\bar{X}_{0,m}\|_2\le 2\|\widetilde{X}_{0,m}\|_2\le 2\|\varphi_M(X_0)\|_2\le 2\|X_0\|_2\,.\tag{4.23}$$

Therefore,

$$\frac{1}{N}\mathbb{E}\big(|\mathrm{Tr}(\bar{\mathbf{B}}_n)|\big)\le 4\|X_0\|_2^2\,.\tag{4.24}$$

Now, by definition of $\mathcal{X}_n$ and $\bar{\mathcal{X}}_n$,

$$\begin{aligned}&\frac{1}{Nn}\mathbb{E}\big(|\mathrm{Tr}\big(\mathcal{X}_n-\bar{\mathcal{X}}_n\big)\big(\mathcal{X}_n-\bar{\mathcal{X}}_n\big)^T|\big)\\&\qquad=\frac{1}{nN}\sum_{i=1}^{n}\sum_{k\in\mathcal{I}_{N,m}}\big\|X_k^{(i)}-\bar{X}_{k,m}^{(i)}\big\|_2^2+\frac{1}{nN}\sum_{i=1}^{n}\sum_{k\in\mathcal{R}_{N,m}}\big\|X_k^{(i)}\big\|_2^2\,.\end{aligned}$$

Using stationarity, the fact that $\mathrm{card}(\mathcal{I}_{N,m})\le N$ and

$$\mathrm{card}(\mathcal{R}_{N,m})=N-pk_{N,m}\le\frac{Nm}{p+m}+p\,,\tag{4.25}$$

we get that

$$\frac{1}{Nn}\mathbb{E}\big(|\mathrm{Tr}\big(\mathcal{X}_n-\bar{\mathcal{X}}_n\big)\big(\mathcal{X}_n-\bar{\mathcal{X}}_n\big)^T|\big)\le\|X_0-\bar{X}_{0,m}\|_2^2+(mp^{-1}+pN^{-1})\|X_0\|_2^2\,.\tag{4.26}$$

Starting from (4.20), considering the upper bounds (4.21), (4.24) and (4.26), and since $p=[N^{1/8}m^{-1/8}]$, we derive that

$$\limsup_{n\to\infty}\Big|\mathbb{E}\big(S_{F^{\mathbf{B}_n}}(z)\big)-\mathbb{E}\big(S_{F^{\bar{\mathbf{B}}_n}}(z)\big)\Big|\ll\frac{1}{v^2}\|X_0-\bar{X}_{0,m}\|_2\,.$$

Therefore, Proposition 4.1 will follow if we can prove that

$$\lim_{m\to\infty} \limsup_{M\to\infty} \|X_0 - \bar{X}_{0,m}\|_2 = 0\,. \tag{4.27}$$

Let us introduce now the sequence $(X_{k,m})_{k\in\mathbb{Z}}$ defined as follows: for any $k \in \mathbb{Z}$,

$$X_{k,m} = \mathbb{E}\big(X_k|\varepsilon_k, \ldots, \varepsilon_{k-m}\big)\,. \tag{4.28}$$

With the above notation, we write that

$$\|X_0 - \bar{X}_{0,m}\|_2 \le \|X_0 - X_{0,m}\|_2 + \|X_{0,m} - \bar{X}_{0,m}\|_2\,.$$

Since $X_0$ is centered, so is $X_{0,m}$. Then $\|X_{0,m} - \bar{X}_{0,m}\|_2 = \|X_{0,m} - \mathbb{E}(X_{0,m}) - \bar{X}_{0,m}\|_2$. Therefore, recalling the definition (4.10) of $\bar{X}_{0,m}$, it follows that

$$\|X_{0,m} - \bar{X}_{0,m}\|_2 \le 2\|X_{0,m} - \widetilde{X}_{0,m}\|_2 \le 2\|X_0 - \varphi_M(X_0)\|_2 \le 2\|\big(|X_0| - M\big)_+\|_2\,. \tag{4.29}$$

Since $X_0$ belongs to $\mathbb{L}^2$, $\lim_{M\to\infty} \|\big(|X_0| - M\big)_+\|_2 = 0$. Therefore, to prove (4.27) (and then Proposition 4.1), it suffices to prove that

$$\lim_{m\to\infty} \|X_0 - X_{0,m}\|_2 = 0\,. \tag{4.30}$$

Since $(X_{0,m})_{m\ge0}$ is a martingale with respect to the increasing filtration $(\mathcal{G}_m)_{m\ge0}$ defined by $\mathcal{G}_m = \sigma(\varepsilon_{-m}, \ldots, \varepsilon_0)$, and is such that $\sup_{m\ge0} \|X_{0,m}\|_2 \le \|X_0\|_2 < \infty$, (4.30) follows by the martingale convergence theorem in $\mathbb{L}^2$ (see for instance Corollary 2.2 in Hall and Heyde (1980)). This ends the proof of Proposition 4.1. □

## 4.2 Construction of approximating sample covariance matrices associated to Gaussian random variables.

Let $(Z_k)_{k\in\mathbb{Z}}$ be a centered Gaussian process with real values, whose covariance function is given for any $k, \ell \in \mathbb{Z}$ by

$$\mathrm{Cov}(Z_k, Z_\ell) = \mathrm{Cov}(X_k, X_\ell)\,. \tag{4.31}$$

For $n$ a positive integer, we consider $n$ independent copies of the Gaussian process $(Z_k)_{k\in\mathbb{Z}}$ that are in addition independent of $(X_k^{(i)})_{k\in\mathbb{Z}, i\in\{1,\ldots,n\}}$. We shall denote these copies by $(Z_k^{(i)})_{k\in\mathbb{Z}}$ for $i = 1, \ldots, n$. For any $i \in \{1, \ldots, n\}$, define $\mathbf{Z}_i = \big(Z_1^{(i)}, \ldots, Z_N^{(i)}\big)$. Let $\mathcal{Z}_n = (\mathbf{Z}_1^T|\ldots|\mathbf{Z}_n^T)$ be the matrix whose columns are the $\mathbf{Z}_i^T$'s and consider its associated sample covariance matrix

$$\mathbf{G}_n = \frac{1}{n}\mathcal{Z}_n\mathcal{Z}_n^T\,. \tag{4.32}$$

For $k_{N,m}$ given in (4.8), we define now random vectors $\big(\mathbf{v}_\ell^{(i)}\big)_{\ell\in\{1,\ldots,k_{N,m}\}}$ as follows. They are defined as the random vectors $\big(\mathbf{u}_\ell^{(i)}\big)_{\ell\in\{1,\ldots,k_{N,m}\}}$ defined in (4.13) and (4.14), but by replacing each $\bar{X}_{k,m}^{(i)}$ by $Z_k^{(i)}$. For any $i \in \{1, \ldots, n\}$, we then define the random vectors $\widetilde{\mathbf{Z}}^{(i)}$ of dimension $N$, as follows:

$$\widetilde{\mathbf{Z}}^{(i)} = \big(\mathbf{v}_\ell^{(i)}\,,\, \ell = 1, \ldots, k_{N,m}\big)\,. \tag{4.33}$$

Let now

$$\widetilde{\mathcal{Z}}_n = \big(\widetilde{\mathbf{Z}}^{(1)T}|\ldots|\widetilde{\mathbf{Z}}^{(n)T}\big) \quad\text{and}\quad \widetilde{\mathbf{G}}_n = \frac{1}{n}\widetilde{\mathcal{Z}}_n\widetilde{\mathcal{Z}}_n^T\,. \tag{4.34}$$

In what follows, we shall prove the following proposition.

**Proposition 4.2** *For any $z \in \mathbb{C}^+$, the convergence* (4.6) *holds true with* $\mathbf{G_n}$ *and* $\widetilde{\mathbf{G}}_n$ *as defined in* (4.32) *and* (4.34) *respectively.*

To prove the proposition above, we start by noticing that, for any $z = u + iv \in \mathbb{C}^+$,

$$\begin{aligned}\Big|\mathbb{E}\big(S_{F^{\mathbf{G}_n}}(z)\big) - \mathbb{E}\big(S_{F^{\widetilde{\mathbf{G}}_n}}(z)\big)\Big| &\leq \mathbb{E}\Big|\int \frac{1}{x-z} dF^{\mathbf{G}_n}(x) - \int \frac{1}{x-z} dF^{\widetilde{\mathbf{G}}_n}(x)\Big| \\ &\leq \mathbb{E}\Big|\int \frac{F^{\mathbf{G}_n}(x) - F^{\widetilde{\mathbf{G}}_n}}{(x-z)^2} dx\Big| \leq \frac{\pi \left\|F^{\mathbf{G}_n} - F^{\widetilde{\mathbf{G}}_n}\right\|_\infty}{v}\,.\end{aligned}$$

Hence, by Theorem A.44 in Bai and Silverstein (2010),

$$\Big|\mathbb{E}\big(S_{F^{\mathbf{G}_n}}(z)\big) - \mathbb{E}\big(S_{F^{\widetilde{\mathbf{G}}_n}}(z)\big)\Big| \leq \frac{\pi}{vN}\mathrm{rank}\big(\mathcal{Z}_n - \widetilde{\mathcal{Z}}_n\big)\,.$$

By definition of $\mathcal{Z}_n$ and $\widetilde{\mathcal{Z}}_n$, $\mathrm{rank}\big(\mathcal{Z}_n - \widetilde{\mathcal{Z}}_n\big) \leq \mathrm{card}(\mathcal{R}_{N,m})$, where $\mathcal{R}_{N,m}$ is defined in (4.22). Therefore, using (4.25), we get that, for any $z = u + iv \in \mathbb{C}^+$,

$$\Big|\mathbb{E}\big(S_{F^{\mathbf{G}_n}}(z)\big) - \mathbb{E}\big(S_{F^{\widetilde{\mathbf{G}}_n}}(z)\big)\Big| \leq \frac{\pi}{vN}\Big(\frac{mN}{p+m} + p\Big)\,,$$

which converges to zero as $n \to \infty$, since $p = [N^{1/8} m^{-1/8}]$. This ends the proof of Proposition 4.2. □

## 4.3 Approximation of $\mathbb{E}\big(S_{F^{\bar{\mathbf{B}}_n}}(z)\big)$ by $\mathbb{E}\big(S_{F^{\widetilde{\mathbf{G}}_n}}(z)\big)$.

In this section, we shall prove the following proposition.

**Proposition 4.3** *Under the assumptions of Theorem 2.1, for any $z \in \mathbb{C}^+$, the convergence* (4.7) *holds true with* $\bar{\mathbf{B}}_n$ *and* $\widetilde{\mathbf{G}}_n$ *as defined in* (4.16) *and* (4.34) *respectively.*

With this aim, we shall use the Lindeberg method that is based on telescoping sums. In order to develop it, we first give the following definition:

**Definition 4.1** *Let $x$ be a vector of $\mathbb{R}^{N\times n}$ with coordinates*

$$x = \big(x^{(1)}, \ldots, x^{(n)}\big) \text{ where for any } i \in \{1,\ldots,n\},\ x^{(i)} = \big(x_k^{(i)},\ k \in \{1,\ldots,N\}\big)\,.$$

*Let $z \in \mathbb{C}^+$ and $f := f_z$ be the function defined from $\mathbb{R}^{N\times n}$ to $\mathbb{C}$ by*

$$f(x) = \frac{1}{N}\mathrm{Tr}\big(A(x) - z\mathbf{I}\big)^{-1} \quad \text{where} \quad A(x) = \frac{1}{n}\sum_{k=1}^{n}(x^{(k)})^T x^{(k)}\,, \tag{4.35}$$

*and $\mathbf{I}$ is the identity matrix.*

The function $f$, as defined above, admits partial derivatives of all orders. Indeed, let $u$ be one of the coordinates of the vector $x$ and $A_u = A(x)$ the matrix-valued function of the scalar $u$. Then, setting $G_u = \big(A_u - z\mathbf{I}\big)^{-1}$ and differentiating both sides of the equality $G_u(A_u - z\mathbf{I}) = \mathbf{I}$, it follows that

$$\frac{dG}{du} = -G\frac{dA}{du}G\,, \tag{4.36}$$

(see the equality (17) in Chatterjee (2006)). Higher-order derivatives may be computed by applying repeatedly the above formula. Upper bounds for some partial derivarives up to the fourth order are given in Appendix.

Now, using Definition 4.1 and the notations (4.15) and (4.33), we get that, for any $z \in \mathbb{C}^+$,

$$\mathbb{E}\big(S_{F^{\bar{\mathbf{B}}_n}}(z)\big) - \mathbb{E}\big(S_{F^{\tilde{\mathbf{G}}_n}}(z)\big) = \mathbb{E}f\big(\bar{\mathbf{X}}^{(1)}, \ldots, \bar{\mathbf{X}}^{(n)}\big) - \mathbb{E}f\big(\widetilde{\mathbf{Z}}^{(1)}, \ldots, \widetilde{\mathbf{Z}}^{(n)}\big)\,. \tag{4.37}$$

To continue the development of the Lindeberg method, we introduce additional notations. For any $i \in \{1, \ldots, n\}$ and $k_{N,m}$ given in (4.8), we define random vectors $\big(\mathbf{U}_\ell^{(i)}\big)_{\ell \in \{1,\ldots,k_{N,m}\}}$ of dimension $N \times n$ as follows. For any $\ell \in \{1, \ldots, k_{N,m}\}$,

$$\mathbf{U}_\ell^{(i)} = \Big(\mathbf{0}_{(i-1)N}\,,\, \mathbf{0}_{(\ell-1)(p+m)}\,,\, \mathbf{u}_\ell^{(i)}\,,\, \mathbf{0}_{r_\ell}, \mathbf{0}_{(n-i)N}\Big)\,, \tag{4.38}$$

where the $\mathbf{u}_\ell^{(i)}$'s are defined in (4.13) and (4.14), and

$$r_\ell = N - \ell(p+m) \;\text{ for } \ell \in \{1, \ldots, k_{N,m} - 1\}, \text{ and } \; r_{k_{N,m}} = 0\,. \tag{4.39}$$

Note that the vectors $\big(\mathbf{U}_\ell^{(i)}\big)_{1 \le i \le n, 1 \le \ell \le k_{N,m}}$ are mutually independent. Moreover, with the notations (4.38) and (4.15), the following relations hold. For any $i \in \{1, \ldots, n\}$,

$$\sum_{\ell=1}^{k_{N,m}} \mathbf{U}_\ell^{(i)} = \Big(\mathbf{0}_{N(i-1)}\,,\, \bar{\mathbf{X}}^{(i)}\,,\, \mathbf{0}_{(n-i)N}\Big) \;\text{ and }\; \sum_{i=1}^{n}\sum_{\ell=1}^{k_{N,m}} \mathbf{U}_\ell^{(i)} = \Big(\bar{\mathbf{X}}^{(1)}, \ldots, \bar{\mathbf{X}}^{(n)}\Big)\,, \tag{4.40}$$

where the $\bar{\mathbf{X}}^{(i)}$'s are defined in (4.15).

Now, for any $i \in \{1, \ldots, n\}$, we define random vectors $\big(\mathbf{V}_\ell^{(i)}\big)_{\ell \in \{1,\ldots,k_{N,m}\}}$ of dimension $N \times n$, as follows: for any $\ell \in \{1, \ldots, k_{N,m}\}$,

$$\mathbf{V}_\ell^{(i)} = \Big(\mathbf{0}_{(i-1)N}\,,\, \mathbf{0}_{(\ell-1)(p+m)}\,,\, \mathbf{v}_\ell^{(i)}\,,\, \mathbf{0}_{r_\ell}, \mathbf{0}_{(n-i)N}\Big)\,, \tag{4.41}$$

where $r_\ell$ is defined in (4.39) and the $\mathbf{v}_\ell^{(i)}$'s are defined in Section 4.2. With the notations (4.41) and (4.33), the following relations hold: for any $i \in \{1, \ldots, n\}$,

$$\sum_{\ell=1}^{k_{N,m}} \mathbf{V}_\ell^{(i)} = \Big(\mathbf{0}_{N(i-1)}\,,\, \widetilde{\mathbf{Z}}^{(i)}\,,\, \mathbf{0}_{N(n-i)}\Big) \;\text{ and }\; \sum_{i=1}^{n}\sum_{\ell=1}^{k_{N,m}} \mathbf{V}_\ell^{(i)} = \Big(\widetilde{\mathbf{Z}}^{(1)}, \ldots, \widetilde{\mathbf{Z}}^{(n)}\Big)\,, \tag{4.42}$$

where the $\widetilde{\mathbf{Z}}^{(i)}$'s are defined in (4.33). We define now, for any $i \in \{1, \ldots, n\}$,

$$\mathbf{S}_i = \sum_{s=1}^{i}\sum_{\ell=1}^{k_{N,m}} \mathbf{U}_\ell^{(s)} \;\text{ and }\; \mathbf{T}_i = \sum_{s=i}^{n}\sum_{\ell=1}^{k_{N,m}} \mathbf{V}_\ell^{(s)}\,, \tag{4.43}$$

and any $s \in \{1, \ldots, k_{N,m}\}$,

$$\mathbf{S}_s^{(i)} = \sum_{\ell=1}^{s} \mathbf{U}_\ell^{(i)} \;\text{ and }\; \mathbf{T}_s^{(i)} = \sum_{\ell=s}^{k_{N,m}} \mathbf{V}_\ell^{(i)}\,. \tag{4.44}$$

In all the notations above, we use the convention that $\sum_{k=r}^{s} = 0$ if $r > s$. Therefore, starting from (4.37), considering the relations (4.40) and (4.42), and using the notations (4.43) and (4.44), we successively get

$$\begin{aligned}
\mathbb{E}\big(S_{F^{\bar{\mathbf{B}}_n}}(z)\big) - \mathbb{E}\big(S_{F^{\tilde{\mathbf{G}}_n}}(z)\big) &= \sum_{i=1}^{n}\Big(\mathbb{E}f\big(\mathbf{S}_i + \mathbf{T}_{i+1}\big) - \mathbb{E}f\big(\mathbf{S}_{i-1} + \mathbf{T}_i\big)\Big) \\
&= \sum_{i=1}^{n}\sum_{s=1}^{k_{N,m}}\Big(\mathbb{E}f\big(\mathbf{S}_{i-1} + \mathbf{S}_s^{(i)} + \mathbf{T}_{s+1}^{(i)} + \mathbf{T}_{i+1}\big) - \mathbb{E}f\big(\mathbf{S}_{i-1} + \mathbf{S}_{s-1}^{(i)} + \mathbf{T}_s^{(i)} + \mathbf{T}_{i+1}\big)\Big)\,.
\end{aligned}$$

Therefore, setting for any $i \in \{1, \dots, n\}$ and any $s \in \{1, \dots, k_{N,m}\}$,

$$\mathbf{W}_s^{(i)} = \mathbf{S}_{i-1} + \mathbf{S}_s^{(i)} + \mathbf{T}_{s+1}^{(i)} + \mathbf{T}_{i+1}\,, \tag{4.45}$$

and

$$\widetilde{\mathbf{W}}_s^{(i)} = \mathbf{S}_{i-1} + \mathbf{S}_{s-1}^{(i)} + \mathbf{T}_{s+1}^{(i)} + \mathbf{T}_{i+1}\,, \tag{4.46}$$

we are lead to

$$\mathbb{E}\big(S_{F^{\bar{\mathbf{B}}_n}}(z)\big) - \mathbb{E}\big(S_{F^{\tilde{\mathbf{G}}_n}}(z)\big) = \sum_{i=1}^{n}\sum_{s=1}^{k_{N,m}} \Big(\mathbb{E}\big(\Delta_s^{(i)}(f)\big) - \mathbb{E}\big(\widetilde{\Delta}_s^{(i)}(f)\big)\Big)\,, \tag{4.47}$$

where

$$\Delta_s^{(i)}(f) = f\big(\mathbf{W}_s^{(i)}\big) - f\big(\widetilde{\mathbf{W}}_s^{(i)}\big) \text{ and } \widetilde{\Delta}_s^{(i)}(f) = f\big(\mathbf{W}_{s-1}^{(i)}\big) - f\big(\widetilde{\mathbf{W}}_s^{(i)}\big)\,.$$

In order to continue the multidimensional Lindeberg method, it is useful to introduce the following notations.

**Definition 4.2** *Let $d_1$ and $d_2$ be two positive integers. Let $A = (a_1, \dots, a_{d_1})$ and $B = (b_1, \dots, b_{d_2})$ be two real valued row vectors of respective dimensions $d_1$ and $d_2$. We define $A \otimes B$ as being the transpose of the Kronecker product of $A$ by $B$. Therefore*

$$A \otimes B = \begin{pmatrix} a_1 B^T \\ \vdots \\ a_{d_1} B^T \end{pmatrix} \in \mathbb{R}^{d_1 \times d_2}\,.$$

*For any positive integer $k$, the $k$-th transpose Kronecker power $A^{\otimes k}$ is then defined inductively by: $A^{\otimes 1} = A^T$ and $A^{\otimes k} = A \bigotimes \big(A^{\otimes(k-1)}\big)^T$.*

Notice that, here, $A \otimes B$ is not exactly the usual Kronecker product (or Tensor product) of $A$ by $B$ that rather produces a row vector. However, for later notation convenience, the notation below is useful.

**Definition 4.3** *Let $d$ be a positive integer. If $\nabla$ denotes the differentiation operator given by $\nabla = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\big)$ acting on the differentiable functions $h : \mathbb{R}^d \to \mathbb{R}$, we define, for any positive integer $k$, $\nabla^{\otimes k}$ in the same way as in Definition 4.2. If $h : \mathbb{R}^d \to \mathbb{R}$ is $k$-times differentiable, for any $x \in \mathbb{R}^d$, let $D^k h(x) = \nabla^{\otimes k} h(x)$, and for any row vector $Y$ of $\mathbb{R}^d$, we define $D^k h(x).Y^{\otimes k}$ as the usual scalar product in $\mathbb{R}^{d^k}$ between $D^k h(x)$ and $Y^{\otimes k}$. We write $Dh$ for $D^1 h$.*

Let $z = u + iv \in \mathbb{C}^+$. We start by analyzing the term $\mathbb{E}\big(\Delta_s^{(i)}(f)\big)$ in (4.47). By Taylor's integral formula,

$$\begin{aligned}\Big|\mathbb{E}\big(\Delta_s^{(i)}(f)\big) - \mathbb{E}\big(Df\big(\widetilde{\mathbf{W}}_s^{(i)}\big).\mathbf{U}_s^{(i)\otimes 1}\big) - \frac{1}{2}\mathbb{E}\big(D^2 f\big(\widetilde{\mathbf{W}}_s^{(i)}\big).\mathbf{U}_s^{(i)\otimes 2}\big)\Big| \\ \leq \Big|\mathbb{E}\int_0^1 \frac{(1-t)^2}{2} D^3 f\big(\widetilde{\mathbf{W}}_s^{(i)} + t\mathbf{U}_s^{(i)}\big).\mathbf{U}_s^{(i)\otimes 3} dt\Big|\,. \end{aligned} \tag{4.48}$$

Let us analyze the right-hand term of (4.48). Recalling the definition (4.38) of the $\mathbf{U}_s^{(i)}$'s, for any $t \in [0,1]$,

$$\begin{aligned}&\mathbb{E}\big|D^3 f\big(\widetilde{\mathbf{W}}_s^{(i)} + t\mathbf{U}_s^{(i)}\big).\mathbf{U}_s^{(i)\otimes 3}\big| \\ &\qquad \leq \sum_{k\in I_s}\sum_{\ell\in I_s}\sum_{j\in I_s} \mathbb{E}\Big(\Big|\frac{\partial^3 f}{\partial x_k^{(i)}\partial x_\ell^{(i)}\partial x_j^{(i)}}\big(\widetilde{\mathbf{W}}_s^{(i)} + t\mathbf{U}_s^{(i)}\big)\bar{X}_{k,m}^{(i)}\bar{X}_{\ell,m}^{(i)}\bar{X}_{j,m}^{(i)}\Big|\Big) \\ &\qquad \leq \sum_{k\in I_s}\sum_{\ell\in I_s}\sum_{j\in I_s} \Big\|\frac{\partial^3 f}{\partial x_k^{(i)}\partial x_\ell^{(i)}\partial x_j^{(i)}}\big(\widetilde{\mathbf{W}}_s^{(i)} + t\mathbf{U}_s^{(i)}\big)\Big\|_2 \big\|\bar{X}_{k,m}^{(i)}\bar{X}_{\ell,m}^{(i)}\bar{X}_{j,m}^{(i)}\big\|_2\,,\end{aligned}$$

where $I_s$ is defined in (4.12). Therefore, using (4.11), stationarity and (4.23), it follows that, for any $t \in [0,1]$,

$$\mathbb{E}\big|D^3 f\big(\widetilde{\mathbf{W}}_s^{(i)} + t\mathbf{U}_s^{(i)}\big).\mathbf{U}_s^{(i)\otimes 3}\big| \\ \leq 8M^2 \sum_{k\in I_s}\sum_{\ell\in I_s}\sum_{j\in I_s} \Big\| \frac{\partial^3 f}{\partial x_k^{(i)}\partial x_\ell^{(i)}\partial x_j^{(i)}}\big(\widetilde{\mathbf{W}}_s^{(i)} + t\mathbf{U}_s^{(i)}\big)\Big\|_2 \|X_0\|_2 \,.$$

Notice that by (4.43) and (4.44),

$$\widetilde{\mathbf{W}}_s^{(i)} + t\mathbf{U}_s^{(i)} = \big(\bar{\mathbf{X}}^{(1)}, \ldots, \bar{\mathbf{X}}^{(i-1)}, w^{(i)}(t), \widetilde{\mathbf{Z}}^{(i+1)}, \ldots, \widetilde{\mathbf{Z}}^{(n)}\big)\,, \tag{4.49}$$

where $w^{(i)}(t)$ is the row vector of dimension $N$ defined by

$$w^{(i)}(t) = \mathbf{S}_{s-1}^{(i)} + t\mathbf{U}_s^{(i)} + \mathbf{T}_{s+1}^{(i)} = \big(\mathbf{u}_1^{(i)}, \ldots, \mathbf{u}_{s-1}^{(i)}, t\mathbf{u}_s^{(i)}, \mathbf{v}_{s+1}^{(i)}, \ldots, \mathbf{v}_{k_{N,m}}^{(i)}\big)\,, \tag{4.50}$$

where the $\mathbf{u}_\ell^{(i)}$'s are defined in (4.13) and (4.14) whereas the $\mathbf{v}_\ell^{(i)}$'s are defined in Section 4.2. Therefore, by Lemma 5.1 of the Appendix, (4.11), and since $(Z_k^{(i)})_{k\in\mathbb{Z}}$ is distributed as the stationary sequence $(Z_k)_{k\in\mathbb{Z}}$, we infer that, for any $t \in [0,1]$,

$$\Big\| \frac{\partial^3 f}{\partial x_k^{(i)}\partial x_\ell^{(i)}\partial x_j^{(i)}}\big(\widetilde{\mathbf{W}}_s^{(i)} + t\mathbf{U}_s^{(i)}\big)\Big\|_2 \ll \frac{M + \|Z_0\|_2}{v^3 N^{1/2} n^2} + \frac{N^{1/2}(M^3 + \|Z_0\|_6^3)}{v^4 n^3}\,.$$

Now, since $Z_0$ is a Gaussian random variable, $\|Z_0\|_6^6 = 15\|Z_0\|_2^6$. Hence, by (4.31), $\|Z_0\|_2 = \|X_0\|_2$. Therefore, for any $t \in [0,1]$, we get that

$$\mathbb{E}\big|D^3 f\big(\widetilde{\mathbf{W}}_s^{(i)} + t\mathbf{U}_s^{(i)}\big).\mathbf{U}_s^{(i)\otimes 3}\big| \ll \frac{p^3(1+M^3)}{v^3(1\wedge v)N^{1/2}n^2}\,. \tag{4.51}$$

On an other hand, since for any $i \in \{1,\ldots,n\}$ and any $s \in \{1,\ldots,k_{N,m}\}$, $\mathbf{U}_s^{(i)}$ is a centered random vector independent of $\widetilde{\mathbf{W}}_s^{(i)}$, it follows that

$$\mathbb{E}\big(Df\big(\widetilde{\mathbf{W}}_s^{(i)}\big).\mathbf{U}_s^{(i)\otimes 1}\big) = 0 \;\text{ and }\; \mathbb{E}\big(D^2 f\big(\widetilde{\mathbf{W}}_s^{(i)}\big).\mathbf{U}_s^{(i)\otimes 2}\big) = \mathbb{E}\big(D^2 f\big(\widetilde{\mathbf{W}}_s^{(i)}\big)\big).\mathbb{E}\big(\mathbf{U}_s^{(i)\otimes 2}\big)\,. \tag{4.52}$$

Hence starting from (4.48), using (4.51), (4.52) and the fact that $pk_{N,m} \leq N$, we derive that

$$\sum_{i=1}^n \sum_{s=1}^{k_{N,m}} \Big|\mathbb{E}\big(\Delta_s^{(i)}(f)\big) - \frac{1}{2}\mathbb{E}\big(D^2 f\big(\widetilde{\mathbf{W}}_s^{(i)}\big)\big).\mathbb{E}\big(\mathbf{U}_s^{(i)\otimes 2}\big)\Big| \ll \frac{(1+M^5)N^{1/2}p^2}{v^3(1\wedge v)n}\,. \tag{4.53}$$

We analyze now the "Gaussian part" in (4.47), namely: $\mathbb{E}\big(\widetilde{\Delta}_s^{(i)}(f)\big)$. By Taylor's integral formula,

$$\Big|\mathbb{E}\big(\widetilde{\Delta}_s^{(i)}(f)\big) - \mathbb{E}\big(Df\big(\widetilde{\mathbf{W}}_s^{(i)}\big).\mathbf{V}_s^{(i)\otimes 1}\big) - \frac{1}{2}\mathbb{E}\big(D^2 f\big(\widetilde{\mathbf{W}}_s^{(i)}\big).\mathbf{V}_s^{(i)\otimes 2}\big)\Big| \\ \leq \Big|\mathbb{E}\int_0^1 \frac{(1-t)^2}{2} D^3 f\big(\widetilde{\mathbf{W}}_s^{(i)} + t\mathbf{V}_s^{(i)}\big).\mathbf{V}_s^{(i)\otimes 3} dt\Big|\,.$$

Proceeding as to get (4.53), we then infer that

$$\sum_{i=1}^n \sum_{s=1}^{k_{N,m}} \Big|\mathbb{E}\big(\widetilde{\Delta}_s^{(i)}(f)\big) - \mathbb{E}\big(Df\big(\widetilde{\mathbf{W}}_s^{(i)}\big).\mathbf{V}_s^{(i)\otimes 1}\big) - \frac{1}{2}\mathbb{E}\big(D^2 f\big(\widetilde{\mathbf{W}}_s^{(i)}\big).\mathbf{V}_s^{(i)\otimes 2}\big)\Big| \\ \ll \frac{(1+M^3)N^{1/2}p^2}{v^3(1\wedge v)n}\,. \tag{4.54}$$

We analyze now the terms $\mathbb{E}\big(Df\big(\widetilde{\mathbf{W}}_s^{(i)}\big).\mathbf{V}_s^{(i)\otimes 1}\big)$ in (4.54). Recalling the definition (4.41) of the $\mathbf{V}_s^{(i)}$'s, we write that

$$\mathbb{E}\big(Df\big(\widetilde{\mathbf{W}}_s^{(i)}\big).\mathbf{V}_s^{(i)\otimes 1}\big) = \sum_{j\in I_s} \mathbb{E}\left(\frac{\partial f}{\partial x_j^{(i)}}\big(\widetilde{\mathbf{W}}_s^{(i)}\big) Z_j^{(i)}\right),$$

where $I_s$ is defined in (4.12). To handle the terms in the right-hand side, we shall use the so-called Stein's identity for Gaussian vectors (see, for instance, Lemma 1 in Liu (1994)), as done by Neumann (2011) in the context of dependent real random variables: for $G = (G_1, \ldots, G_d)$ a centered Gaussian vector of $\mathbb{R}^d$ and any function $h : \mathbb{R}^d \to \mathbb{R}$ such that its partial derivatives exist almost everywhere and $\mathbb{E}\big|\frac{\partial h}{\partial x_i}(G)\big| < \infty$ for any $i = 1, \ldots, d$, the following identity holds true:

$$\mathbb{E}\big(G_i\, h(G)\big) = \sum_{\ell=1}^{d} \mathbb{E}\big(G_i G_\ell\big)\mathbb{E}\Big(\frac{\partial h}{\partial x_\ell}(G)\Big) \quad \text{for any } i \in \{1, \ldots, d\}. \tag{4.55}$$

Using (4.55) with $G = \big(\mathbf{T}_{s+1}^{(i)}, Z_j^{(i)}\big) \in \mathbb{R}^{n\times N} \times \mathbb{R}$, $h : \mathbb{R}^{n\times N} \times \mathbb{R} \to \mathbb{R}$ satisfying $h(x, y) = \frac{\partial f}{\partial x_j^{(i)}}(x)$ for any $(x, y) \in \mathbb{R}^{n\times N} \times \mathbb{R}$, and noticing that $G$ is independent of $\widetilde{\mathbf{W}}_s^{(i)} - \mathbf{T}_{s+1}^{(i)}$, we infer that, for any $j \in I_s$,

$$\mathbb{E}\left(\frac{\partial f}{\partial x_j^{(i)}}\big(\widetilde{\mathbf{W}}_s^{(i)}\big) Z_j^{(i)}\right) = \sum_{\ell=s+1}^{k_{N,m}} \sum_{k\in I_\ell} \mathbb{E}\left(\frac{\partial^2 f}{\partial x_k^{(i)} \partial x_j^{(i)}}\big(\widetilde{\mathbf{W}}_s^{(i)}\big)\right) \mathrm{Cov}(Z_k^{(i)}, Z_j^{(i)}).$$

Therefore,

$$\mathbb{E}\big(Df\big(\widetilde{\mathbf{W}}_s^{(i)}\big).\mathbf{V}_s^{(i)\otimes 1}\big) = \sum_{\ell=s+1}^{k_{N,m}} \sum_{k\in I_\ell} \sum_{j\in I_s} \mathbb{E}\left(\frac{\partial^2 f}{\partial x_k^{(i)} \partial x_j^{(i)}}\big(\widetilde{\mathbf{W}}_s^{(i)}\big)\right) \mathrm{Cov}(Z_k^{(i)}, Z_j^{(i)}).$$

From (4.49), (4.50) and Lemma 5.1 of the Appendix, we infer that, for any $k \in I_\ell$ and any $j \in I_s$,

$$\mathbb{E}\left(\frac{\partial^2 f}{\partial x_k^{(i)} \partial x_j^{(i)}}\big(\widetilde{\mathbf{W}}_s^{(i)}\big)\right) \ll \frac{1}{Nnv^2} + \frac{1}{n^2 v^3}\big(\|X_0\|_2^2 + \|Z_0\|_2^2\big) \ll \frac{1}{Nnv^2(1\wedge v)}. \tag{4.56}$$

Hence, using the fact that $\mathrm{Cov}(Z_k^{(i)}, Z_j^{(i)}) = \mathrm{Cov}(Z_k, Z_j)$ together with (4.31), we then derive that

$$\mathbb{E}\big(Df\big(\widetilde{\mathbf{W}}_s^{(i)}\big).\mathbf{V}_s^{(i)\otimes 1}\big) \ll \frac{1}{Nnv^2(1\wedge v)} \sum_{\ell=s+1}^{k_{N,m}} \sum_{k\in I_\ell} \sum_{j\in I_s} \big|\mathrm{Cov}(X_k, X_j)\big|. \tag{4.57}$$

By stationarity,

$$\sum_{k\in I_\ell} \sum_{j\in I_s} \big|\mathrm{Cov}(X_k, X_j)\big| = \sum_{j=1}^{p} \sum_{k=1}^{p} \big|\mathrm{Cov}(X_0, X_{k-j+(\ell-s)(p+m)})\big| \le p \sum_{k\in\mathcal{E}_{p,\ell}} \big|\mathrm{Cov}(X_0, X_k)\big|,$$

where $\mathcal{E}_{p,\ell} := \{1 - p + (\ell - s)(p + m), \ldots, p - 1 + (\ell - s)(p + m)\}$. Notice that since $m \ge 1$, $\mathcal{E}_{p,\ell} \cap \mathcal{E}_{p,\ell+2} = \emptyset$. Then, summing on $\ell$, and using the fact that $k_{N,m}(p + m) \le N$, we get that, for any $s \ge 1$,

$$\sum_{\ell=s+1}^{k_{N,m}} \sum_{k\in\mathcal{E}_{p,\ell}} \big|\mathrm{Cov}(X_0, X_k)\big| \le 2 \sum_{k=m+1}^{p+N-1} \big|\mathrm{Cov}(X_0, X_k)\big|.$$

So, overall,

$$\sum_{\ell=s+1}^{k_{N,m}} \sum_{k\in I_\ell} \sum_{j\in I_s} \big|\mathrm{Cov}(X_k, X_j)\big| \le 2p \sum_{k\ge m+1} \big|\mathrm{Cov}(X_0, X_k)\big| \,. \tag{4.58}$$

Since $\mathcal{F}_{-\infty} = \bigcap_{k\in\mathbb{Z}} \sigma(\xi_k)$ is trivial, for any $k \in \mathbb{Z}$, $\mathbb{E}(X_k|\mathcal{F}_{-\infty}) = \mathbb{E}(X_k) = 0$ a.s. Therefore, the following decomposition is valid: $X_k = \sum_{r=-\infty}^{k} P_r(X_k)$. Next, since $\mathbb{E}\big(P_i(X_0)P_j(X_k)\big) = 0$ if $i \neq j$, we get, by stationarity, that for any integer $k \ge 0$,

$$\big|\mathrm{Cov}(X_0, X_k)\big| = \Big| \sum_{r=-\infty}^{0} \mathbb{E}\Big(P_r(X_0)P_r(X_k)\Big)\Big| \le \sum_{r=0}^{\infty} \|P_0(X_r)\|_2 \|P_0(X_{k+r})\|_2 \,. \tag{4.59}$$

Therefore, starting from (4.58) and using (4.59), we get that, for any $s \ge 1$,

$$\sum_{\ell=s+1}^{k_{N,m}} \sum_{k\in I_\ell} \sum_{j\in I_s} \big|\mathrm{Cov}(X_k, X_j)\big| \le 2p \sum_{r\ge 0} \|P_0(X_r)\|_2 \sum_{k\ge m+1} \|P_0(X_k)\|_2 \,. \tag{4.60}$$

Therefore, starting from (4.57), considering (4.60) together with the condition (2.3) and the fact that $pk_{N,m} \le N$, we derive that

$$\sum_{i=1}^{n} \sum_{s=1}^{k_{N,m}} \big|\mathbb{E}\big(Df(\widetilde{\mathbf{W}}_s^{(i)}).\mathbf{V}_s^{(i)\otimes 1}\big)\big| \ll \frac{1}{v^2(1\wedge v)} \sum_{k\ge m+1} \|P_0(X_k)\|_2 \,. \tag{4.61}$$

We analyze now the terms of second order in (4.54), namely: $\mathbb{E}\big(D^2 f(\widetilde{\mathbf{W}}_s^{(i)}).\mathbf{V}_s^{(i)\otimes 2}\big)$. Recalling the definition (4.41) of the $\mathbf{V}_s^{(i)}$'s, we first write that

$$\mathbb{E}\big(D^2 f(\widetilde{\mathbf{W}}_s^{(i)}).\mathbf{V}_s^{(i)\otimes 2}\big) = \sum_{j_1\in I_s} \sum_{j_2\in I_s} \mathbb{E}\left(\frac{\partial^2 f}{\partial x_{j_1}^{(i)} \partial x_{j_2}^{(i)}}(\widetilde{\mathbf{W}}_s^{(i)}) Z_{j_1}^{(i)} Z_{j_2}^{(i)}\right) , \tag{4.62}$$

where $I_s$ is defined in (4.12). Using now (4.55) with $G = \big(\mathbf{T}_{s+1}^{(i)}, Z_{j_1}^{(i)}, Z_{j_2}^{(i)}\big) \in \mathbb{R}^{n\times N} \times \mathbb{R}^2$, $h : \mathbb{R}^{n\times N} \times \mathbb{R}^2 \to \mathbb{R}$ satisfying $h(x, y, z) = y \frac{\partial^2 f}{\partial x_{j_1}^{(i)} \partial x_{j_2}^{(i)}}(x)$ for any $(x, y, z) \in \mathbb{R}^{n\times N} \times \mathbb{R}^2$, and noticing that $G$ is independent of $\widetilde{\mathbf{W}}_s^{(i)} - \mathbf{T}_{s+1}^{(i)}$, we infer that, for any $j_1, j_2$ belonging to $I_s$,

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2 f}{\partial x_{j_1}^{(i)} \partial x_{j_2}^{(i)}}(\widetilde{\mathbf{W}}_s^{(i)}) Z_{j_1}^{(i)} Z_{j_2}^{(i)}\right) &= \mathbb{E}\left(\frac{\partial^2 f}{\partial x_{j_1}^{(i)} \partial x_{j_2}^{(i)}}(\widetilde{\mathbf{W}}_s^{(i)})\right) \mathbb{E}\big(Z_{j_1}^{(i)} Z_{j_2}^{(i)}\big) \\ &\quad + \sum_{k=s+1}^{k_{N,m}} \sum_{j_3\in I_k} \mathbb{E}\left(\frac{\partial^3 f}{\partial x_{j_3}^{(i)} \partial x_{j_1}^{(i)} \partial x_{j_2}^{(i)}}(\widetilde{\mathbf{W}}_s^{(i)}) Z_{j_1}^{(i)}\right) \mathbb{E}\big(Z_{j_3}^{(i)} Z_{j_2}^{(i)}\big) \,. \end{aligned} \tag{4.63}$$

Therefore, starting from (4.62) and using (4.63) combined with the definitions 4.2 and 4.3, it follows that

$$\begin{aligned}&\mathbb{E}\big(D^2 f(\widetilde{\mathbf{W}}_s^{(i)}).\mathbf{V}_s^{(i)\otimes 2}\big) \\ &\qquad = \mathbb{E}\big(D^2 f(\widetilde{\mathbf{W}}_s^{(i)})\big).\mathbb{E}\big(\mathbf{V}_s^{(i)\otimes 2}\big) + \sum_{k=s+1}^{k_{N,m}} \mathbb{E}\Big(D^3 f(\widetilde{\mathbf{W}}_s^{(i)}).\mathbf{V}_s^{(i)} \otimes \mathbb{E}\big(\mathbf{V}_k^{(i)} \otimes \mathbf{V}_s^{(i)}\big)\Big) \,. \end{aligned} \tag{4.64}$$

Next, with similar arguments, we infer that

$$\sum_{k=s+1}^{k_{N,m}} \mathbb{E}\Big(D^3 f(\widetilde{\mathbf{W}}_s^{(i)}).\mathbf{V}_s^{(i)} \otimes \mathbb{E}\big(\mathbf{V}_k^{(i)} \otimes \mathbf{V}_s^{(i)}\big)\Big) = \sum_{k=s+1}^{k_{N,m}} \sum_{\ell=s+1}^{k_{N,m}} \mathbb{E}\big(D^4 f(\widetilde{\mathbf{W}}_s^{(i)})\big).\mathbb{E}\big(\mathbf{V}_\ell^{(i)} \otimes \mathbf{V}_s^{(i)}\big) \otimes \mathbb{E}\big(\mathbf{V}_k^{(i)} \otimes \mathbf{V}_s^{(i)}\big)\,. \quad (4.65)$$

By the definition (4.41) of the $\mathbf{V}_\ell^{(i)}$'s, we first write that

$$\begin{aligned}
&\mathbb{E}\big(D^4 f(\widetilde{\mathbf{W}}_s^{(i)})\big).\mathbb{E}\big(\mathbf{V}_\ell^{(i)} \otimes \mathbf{V}_s^{(i)}\big) \otimes \mathbb{E}\big(\mathbf{V}_k^{(i)} \otimes \mathbf{V}_s^{(i)}\big) \\
&\quad = \sum_{j_1 \in I_\ell} \sum_{j_2 \in I_s} \sum_{j_3 \in I_k} \sum_{j_4 \in I_s} \mathbb{E}\left(\frac{\partial^4 f}{\partial x_{j_1}^{(i)} \partial x_{j_2}^{(i)} \partial x_{j_3}^{(i)} \partial x_{j_4}^{(i)}}(\widetilde{\mathbf{W}}_s^{(i)})\right) \mathrm{Cov}\big(Z_{j_1}^{(i)}, Z_{j_2}^{(i)}\big)\mathrm{Cov}\big(Z_{j_3}^{(i)}, Z_{j_4}^{(i)}\big) \\
&\quad = \sum_{j_1 \in I_\ell} \sum_{j_2 \in I_s} \sum_{j_3 \in I_k} \sum_{j_4 \in I_s} \mathbb{E}\left(\frac{\partial^4 f}{\partial x_{j_1}^{(i)} \partial x_{j_2}^{(i)} \partial x_{j_3}^{(i)} \partial x_{j_4}^{(i)}}(\widetilde{\mathbf{W}}_s^{(i)})\right) \mathrm{Cov}\big(X_{j_1}, X_{j_2}\big)\mathrm{Cov}\big(X_{j_3}, X_{j_4}\big)\,, \quad (4.66)
\end{aligned}$$

where for the last line, we have used that $(Z_k^{(i)})_{k\in\mathbb{Z}}$ is distributed as $(Z_k)_{k\in\mathbb{Z}}$ together with (4.31). From (4.49), (4.50), Lemma 5.1 of the Appendix, and the stationarity of the sequences $(\bar{X}_{k,m}^{(i)})_{k\in\mathbb{Z}}$ and $(Z_k^{(i)})_{k\in\mathbb{Z}}$, we infer that

$$\begin{aligned}
\mathbb{E}\left(\frac{\partial^4 f}{\partial x_{j_1}^{(i)} \partial x_{j_2}^{(i)} \partial x_{j_3}^{(i)} \partial x_{j_4}^{(i)}}(\widetilde{\mathbf{W}}_s^{(i)})\right) &\ll \frac{1}{Nn^2v^3} + \frac{1}{Nn^3v^4}\Big(\sum_{k=1}^N \|\bar{X}_{k,m}^{(i)}\|_2^2 + \sum_{k=1}^N \|Z_k^{(i)}\|_2^2\Big) \\
&\quad + \frac{1}{Nn^4v^5}\Big(\Big\|\sum_{k=1}^N \big(\bar{X}_{k,m}^{(i)}\big)^2\Big\|_2^2 + \Big\|\sum_{k=1}^N \big(Z_k^{(i)}\big)^2\Big\|_2^2\Big) \\
&\ll \frac{1}{n^2Nv^3(1\wedge v^2)}\left(1 + \frac{N\big(\|\bar{X}_{0,m}\|_2^2 + \|Z_0\|_2^2\big)}{n} + \frac{N^2\big(\|\bar{X}_{0,m}\|_4^4 + \|Z_0\|_4^4\big)}{n^2}\right).
\end{aligned}$$

By (4.11) and (4.23), $\|\bar{X}_{0,m}\|_4^4 \le (2M)^2\|\bar{X}_{0,m}\|_2^2 \le 16M^2\|X_0\|_2^2$. Moreover, $Z_0$ being a Gaussian random variable, $\|Z_0\|_4^4 = 3\|Z_0\|_2^4$. Hence, by (4.31), $\|Z_0\|_4^4 = 3\|X_0\|_2^4$ and $\|Z_0\|_2^2 = \|X_0\|_2^2$. Therefore,

$$\mathbb{E}\left(\frac{\partial^4 f}{\partial x_{j_1}^{(i)} \partial x_{j_2}^{(i)} \partial x_{j_3}^{(i)} \partial x_{j_4}^{(i)}}(\widetilde{\mathbf{W}}_s^{(i)})\right) \ll \frac{(1+M^2)}{n^2Nv^3(1\wedge v^2)}\,. \quad (4.67)$$

On the other hand, by using (4.60), we get that

$$\begin{aligned}
&\sum_{k=s+1}^{k_{N,m}} \sum_{\ell=s+1}^{k_{N,m}} \sum_{j_1 \in I_\ell} \sum_{j_2 \in I_s} \sum_{j_3 \in I_k} \sum_{j_4 \in I_s} \big|\mathrm{Cov}\big(X_{j_1}, X_{j_2}\big)\mathrm{Cov}\big(X_{j_3}, X_{j_4}\big)\big| \\
&\qquad \le 4p^2\Big(\sum_{r\ge 0} \|P_0(X_r)\|_2\Big)^2\Big(\sum_{k\ge m+1} \|P_0(X_k)\|_2\Big)^2\,. \quad (4.68)
\end{aligned}$$

Whence, starting from (4.65), using (4.66), and considering the upper bounds (4.67) and (4.68) together with the condition (2.3), we derive that

$$\sum_{k=s+1}^{k_{N,m}} \mathbb{E}\Big(D^3 f(\widetilde{\mathbf{W}}_s^{(i)}).\mathbf{V}_s^{(i)} \otimes \mathbb{E}\big(\mathbf{V}_k^{(i)} \otimes \mathbf{V}_s^{(i)}\big)\Big) \ll \frac{(1+M^2)p^2}{n^2Nv^3(1\wedge v^2)}\Big(\sum_{k\ge m+1} \|P_0(X_k)\|_2\Big)^2\,. \quad (4.69)$$

So, overall, starting from (4.64), considering (4.69) and using the fact that $pk_{N,m} \leq N$, we derive that

$$\Big| \sum_{i=1}^{n} \sum_{s=1}^{k_{N,m}} \mathbb{E}\big(D^2 f(\widetilde{\mathbf{W}}_s^{(i)}).\mathbf{V}_s^{(i)\otimes 2}\big) - \sum_{i=1}^{n} \sum_{s=1}^{k_{N,m}} \mathbb{E}\big(D^2 f(\widetilde{\mathbf{W}}_s^{(i)})\big).\mathbb{E}\big(\mathbf{V}_s^{(i)\otimes 2}\big) \Big| \ll \frac{(1+M^2)p}{nv^3(1\wedge v^2)} \Big( \sum_{k\geq m+1} \|P_0(X_k)\|_2 \Big)^2 . \quad (4.70)$$

Then starting from (4.47), and considering the upper bounds (4.53), (4.54), (4.61) and (4.70), we get that

$$\Big| \mathbb{E}\big(S_{F^{\bar{\mathbf{B}}_n}}(z)\big) - \mathbb{E}\big(S_{F^{\tilde{\mathbf{G}}_n}}(z)\big) \Big| \ll \frac{1}{2} \sum_{i=1}^{n} \sum_{s=1}^{k_{N,m}} \Big| \mathbb{E}\big(D^2 f(\widetilde{\mathbf{W}}_s^{(i)})\big).\Big( \mathbb{E}\big(\mathbf{U}_s^{(i)\otimes 2}\big) - \mathbb{E}\big(\mathbf{V}_s^{(i)\otimes 2}\big) \Big) \Big|$$
$$+ \frac{(1+M^5)N^{1/2}p^2}{v^3(1\wedge v)n} + \frac{1}{v^2(1\wedge v)} \sum_{k\geq m+1} \|P_0(X_k)\|_2 + \frac{(1+M^2)p}{nv^3(1\wedge v^2)} \Big( \sum_{k\geq m+1} \|P_0(X_k)\|_2 \Big)^2 . \quad (4.71)$$

Since $p \leq N^{1/8}$ and $c(n) = N/n \to c \in (0,\infty)$, it follows that

$$\lim_{n\to\infty} \frac{(1+M^5)N^{1/2}p^2}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{(1+M^2)\,p}{n} = 0\,.$$

On an other hand, by the condition (2.3), $\lim_{m\to\infty} \sum_{k\geq m+1} \|P_0(X_k)\|_2 = 0$. Therefore, Proposition 4.3 will follow if we can prove that, for any $z \in \mathbb{C}^+$,

$$\lim_{m\to\infty} \limsup_{M\to\infty} \limsup_{n\to\infty} \sum_{i=1}^{n} \sum_{s=1}^{k_{N,m}} \Big| \mathbb{E}\big(D^2 f(\widetilde{\mathbf{W}}_s^{(i)})\big).\Big( \mathbb{E}\big(\mathbf{U}_s^{(i)\otimes 2}\big) - \mathbb{E}\big(\mathbf{V}_s^{(i)\otimes 2}\big) \Big) \Big| = 0\,. \quad (4.72)$$

Using the fact that $(Z_k^{(i)})_{k\in\mathbb{Z}}$ is distributed as $(Z_k)_{k\in\mathbb{Z}}$ together with (4.31) and that $(\bar{X}_{k,m}^{(i)})_{k\in\mathbb{Z}}$ is distributed as $(\bar{X}_{k,m})_{k\in\mathbb{Z}}$, we first write that

$$\mathbb{E}\big(D^2 f(\widetilde{\mathbf{W}}_s^{(i)})\big).\Big( \mathbb{E}\big(\mathbf{U}_s^{(i)\otimes 2}\big) - \mathbb{E}\big(\mathbf{V}_s^{(i)\otimes 2}\big) \Big) = \sum_{k\in I_s} \sum_{\ell\in I_s} \mathbb{E}\left( \frac{\partial^2 f}{\partial x_k^{(i)} \partial x_\ell^{(i)}} (\widetilde{\mathbf{W}}_s^{(i)}) \right) \Big( \mathrm{Cov}\big(\bar{X}_{k,m}, \bar{X}_{\ell,m}\big) - \mathrm{Cov}\big(X_k, X_\ell\big) \Big)\,.$$

Hence, by using (4.56) and stationarity, we get that

$$\Big| \mathbb{E}\big(D^2 f(\widetilde{\mathbf{W}}_s^{(i)})\big).\Big( \mathbb{E}\big(\mathbf{U}_s^{(i)\otimes 2}\big) - \mathbb{E}\big(\mathbf{V}_s^{(i)\otimes 2}\big) \Big) \Big| \ll \frac{1}{Nnv^2(1\wedge v)} \sum_{\ell=1}^{p} \sum_{k=0}^{p-\ell} \big| \mathrm{Cov}\big(\bar{X}_{0,m}, \bar{X}_{k,m}\big) - \mathrm{Cov}\big(X_0, X_k\big) \big|\,. \quad (4.73)$$

To handle the right-hand side term, we first write that

$$\sum_{\ell=1}^{p} \sum_{k=0}^{p-\ell} \big| \mathrm{Cov}\big(\bar{X}_{0,m}, \bar{X}_{k,m}\big) - \mathrm{Cov}\big(X_0, X_k\big) \big| \leq p \sum_{k=0}^{p} \big| \mathrm{Cov}\big(\bar{X}_{0,m}, \bar{X}_{k,m}\big) - \mathrm{Cov}\big(X_{0,m}, X_{k,m}\big) \big|$$
$$+ p \sum_{k=0}^{p} \big| \mathrm{Cov}\big(X_{0,m}, X_{k,m}\big) - \mathrm{Cov}\big(X_0, X_k\big) \big|\,, \quad (4.74)$$

where $X_{0,m}$ and $X_{k,m}$ are defined in (4.28). Notice now that $\mathrm{Cov}\big(\bar{X}_{0,m}, \bar{X}_{k,m}\big) = \mathrm{Cov}\big(X_{0,m}, X_{k,m}\big) = 0$ if $k > m$. Therefore,

$$\sum_{k=0}^{p} \big|\mathrm{Cov}\big(\bar{X}_{0,m}, \bar{X}_{k,m}\big) - \mathrm{Cov}\big(X_{0,m}, X_{k,m}\big)\big| = \sum_{k=0}^{m\wedge p} \big|\mathrm{Cov}\big(\bar{X}_{0,m}, \bar{X}_{k,m}\big) - \mathrm{Cov}\big(X_{0,m}, X_{k,m}\big)\big| \,.$$

Next, using stationarity, the fact that the random variables are centered, (4.11) and (4.29), we get that

$$\begin{aligned}
&\big|\mathrm{Cov}\big(\bar{X}_{0,m}, \bar{X}_{k,m}\big) - \mathrm{Cov}\big(X_{0,m}, X_{k,m}\big)\big| \\
&\quad = \big|\mathrm{Cov}\big(\bar{X}_{0,m} - X_{0,m}, \bar{X}_{k,m}\big) + \mathrm{Cov}\big(X_{0,m} - \bar{X}_{0,m}, \bar{X}_{k,m} - X_{k,m}\big) + \mathrm{Cov}\big(\bar{X}_{0,m}, \bar{X}_{k,m} - X_{k,m}\big)\big| \\
&\quad \le 4M\|X_{0,m} - \bar{X}_{0,m}\|_1 + 4\big\|\big(|X_0| - M\big)_+\big\|_2^2 \,.
\end{aligned}$$

As to get (4.29), notice that $\|X_{0,m} - \bar{X}_{0,m}\|_1 \le 2\big\|\big(|X_0| - M\big)_+\big\|_1$. Moreover, $\big(|x| - M\big)_+ \le 2|x|\mathbf{1}_{|x|\ge M}$ which in turn implies that $M\big(|x| - M\big)_+ \le 2|x|^2\mathbf{1}_{|x|\ge M}$. So, overall,

$$\sum_{k=0}^{p} \big|\mathrm{Cov}\big(\bar{X}_{0,m}, \bar{X}_{k,m}\big) - \mathrm{Cov}\big(X_{0,m}, X_{k,m}\big)\big| \le 32\, m\mathbb{E}\big(X_0^2\mathbf{1}_{|X_0|\ge M}\big) \,. \tag{4.75}$$

We handle now the second term in the right-hand side of (4.74). Let $b(m)$ be an increasing sequence of positive integers such that $b(m) \to \infty$, $b(m) \le [m/2]$, and

$$\lim_{m\to\infty} b(m)\big\|X_0 - X_{0,[m/2]}\big\|_2^2 = 0 \,. \tag{4.76}$$

Notice that since (4.30) holds true, it is always possible to find such a sequence. Now, using (4.59),

$$\begin{aligned}
&\sum_{k=b(m)}^{p} \big|\mathrm{Cov}\big(X_{0,m}, X_{k,m}\big) - \mathrm{Cov}\big(X_0, X_k\big)\big| \\
&\qquad \le \sum_{k=b(m)}^{p}\sum_{r=0}^{\infty} \|P_0(X_{r,m})\|_2\|P_0(X_{k+r,m})\|_2 + \sum_{k=b(m)}^{p}\sum_{r=0}^{\infty} \|P_0(X_r)\|_2\|P_0(X_{k+r})\|_2 \,.
\end{aligned} \tag{4.77}$$

Recalling the definition (4.28) of the $X_{j,m}$'s, we notice that $P_0(X_{j,m}) = 0$ if $j \ge m+1$. Now, for any $j \in \{0, \dots, m\}$,

$$\begin{aligned}
\mathbb{E}(X_{j,m}|\xi_0) &= \mathbb{E}(\mathbb{E}(X_j|\varepsilon_j, \dots, \varepsilon_{j-m})|\xi_0) = \mathbb{E}(\mathbb{E}(X_j|\varepsilon_j, \dots, \varepsilon_{j-m})|\varepsilon_0, \dots, \varepsilon_{j-m}) \\
&= \mathbb{E}(X_j|\varepsilon_0, \dots, \varepsilon_{j-m}) = \mathbb{E}(\mathbb{E}(X_j|\xi_0)|\varepsilon_0, \dots, \varepsilon_{j-m}) \quad \text{a.s.}
\end{aligned}$$

Indeed, the two last equalities follow from the tower lemma, whereas, for the second one, we have used the following well known fact with $\mathcal{G}_1 = \sigma(\varepsilon_0, \dots, \varepsilon_{j-m})$, $\mathcal{G}_2 = \sigma(\varepsilon_k, k \le j-m-1)$ and $Y = X_{j,m}$: if $Y$ is an integrable random variable, and $\mathcal{G}_1$ and $\mathcal{G}_2$ are two $\sigma$-algebras such that $\sigma(Y) \vee \mathcal{G}_1$ is independent of $\mathcal{G}_2$, then

$$\mathbb{E}(Y|\mathcal{G}_1 \vee \mathcal{G}_2) = \mathbb{E}(Y|\mathcal{G}_1) \quad \text{a.s.} \tag{4.78}$$

Similarly, for any $j \in \{0, \dots, m-1\}$,

$$\mathbb{E}(X_{j,m}|\xi_{-1}) = \mathbb{E}(X_j|\varepsilon_{-1}, \dots, \varepsilon_{j-m}) = \mathbb{E}(\mathbb{E}(X_j|\xi_{-1})|\varepsilon_{-1}, \dots, \varepsilon_{j-m}) \quad \text{a.s.}$$

Then using the equality (4.78) with $\mathcal{G}_1 = \sigma(\varepsilon_{-1}, \dots, \varepsilon_{j-m})$ and $\mathcal{G}_2 = \sigma(\varepsilon_0)$, we get that, for any $j \in \{1, \dots, m-1\}$,

$$\mathbb{E}(X_{j,m}|\xi_{-1}) = \mathbb{E}(\mathbb{E}(X_j|\xi_{-1})|\varepsilon_0, \dots, \varepsilon_{j-m}) \quad \text{a.s.}$$

whereas $\mathbb{E}(X_{m,m}|\xi_{-1}) = 0$ a.s. So, finally, $\|P_0(X_{m,m})\|_2 = \|\mathbb{E}(X_m|\varepsilon_0)\|_2$, $\|P_0(X_{j,m})\|_2 = 0$ if $j \geq m+1$, and, for any $j \in \{1, \ldots, m-1\}$,

$$\begin{aligned}\|P_0(X_{j,m})\|_2 &= \|\mathbb{E}(X_{j,m}|\xi_0) - \mathbb{E}(X_{j,m}|\xi_{-1})\|_2 \\ &= \|\mathbb{E}\big(\mathbb{E}(X_j|\xi_0) - \mathbb{E}(X_j|\xi_{-1})|\varepsilon_0, \ldots, \varepsilon_{j-m}\big)\|_2 \leq \|P_0(X_j)\|_2 \,.\end{aligned}$$

Therefore, starting from (4.77), we infer that

$$\begin{aligned}&\sum_{k=b(m)}^{p} \big|\mathrm{Cov}\big(X_{0,m}, X_{k,m}\big) - \mathrm{Cov}\big(X_0, X_k\big)\big| \\ &\qquad\qquad \leq 2\|X_0\|_2\|\mathbb{E}(X_m|\varepsilon_0)\|_2 + 2\sum_{r=0}^{\infty}\|P_0(X_r)\|_2 \sum_{k\geq b(m)}\|P_0(X_k)\|_2 \,. \quad (4.79)\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sum_{k=0}^{b(m)} \big|\mathrm{Cov}\big(X_{0,m}, X_{k,m}\big) - \mathrm{Cov}\big(X_0, X_k\big)\big| \\ &\qquad\qquad \leq \sum_{k=0}^{b(m)} \big|\mathrm{Cov}\big(X_0 - X_{0,m}, X_{k,m}\big)\big| + \sum_{k=0}^{b(m)} \big|\mathrm{Cov}\big(X_0, X_k - X_{k,m}\big)\big| \,. \quad (4.80)\end{aligned}$$

Since the random variables are centered, $\mathrm{Cov}\big(X_0 - X_{0,m}, X_{k,m}\big) = \mathbb{E}\big(X_{k,m}(X_0 - X_{0,m})\big)$. Since $X_{k,m}$ is $\sigma(\varepsilon_{k-m}, \ldots, \varepsilon_k)$-measurable.

$$\mathbb{E}\big(X_{k,m}(X_0 - X_{0,m})\big) = \mathbb{E}\big(X_{k,m}\big(\mathbb{E}(X_0|\varepsilon_k, \ldots, \varepsilon_{k-m}) - \mathbb{E}(X_{0,m}|\varepsilon_k, \ldots, \varepsilon_{k-m})\big)\big) \,.$$

But, for any $k \in \{0, \ldots, m\}$, by using the equality (4.78) with $\mathcal{G}_1 = \sigma(\varepsilon_0, \ldots, \varepsilon_{k-m})$ and $\mathcal{G}_2 = \sigma(\varepsilon_k, \ldots, \varepsilon_1)$, it follows that

$$\mathbb{E}(X_{0,m}|\varepsilon_k, \ldots, \varepsilon_{k-m}\big) = \mathbb{E}(X_0|\varepsilon_0, \ldots, \varepsilon_{k-m}) \ \text{ a.s.} \quad (4.81)$$

and

$$\mathbb{E}(X_0|\varepsilon_k, \ldots, \varepsilon_{k-m}\big) = \mathbb{E}(X_0|\varepsilon_0, \ldots, \varepsilon_{k-m}) \ \text{ a.s.}$$

Whence,

$$\sum_{k=0}^{b(m)} \big|\mathrm{Cov}\big(X_0 - X_{0,m}, X_{k,m}\big)\big| = 0 \,. \quad (4.82)$$

To handle the second term in the right-hand side of (4.80), we start by writing that

$$\mathrm{Cov}\big(X_0, X_k - X_{k,m}\big) = \mathrm{Cov}\big(X_0 - X_{0,m}, X_k - X_{k,m}\big) + \mathrm{Cov}\big(X_{0,m}, X_k - X_{k,m}\big) \,. \quad (4.83)$$

Using the fact that the random variables are centered together with stationarity, we get that

$$\big|\mathrm{Cov}\big(X_0 - X_{0,m}, X_k - X_{k,m}\big)\big| \leq \|X_0 - X_{0,m}\|_2^2 \,. \quad (4.84)$$

On the other hand, noticing that $\mathbb{E}(X_k - X_{k,m}|\varepsilon_k, \ldots, \varepsilon_{k-m}) = 0$, and using the fact that the random variables are centered, and stationarity, it follows that

$$\begin{aligned}\big|\mathrm{Cov}\big(X_{0,m}, X_k - X_{k,m}\big)\big| &= \big|\mathbb{E}\big(\big(X_{0,m} - \mathbb{E}(X_{0,m}|\varepsilon_k, \ldots, \varepsilon_{k-m})\big)\big(X_k - X_{k,m}\big)\big)\big| \\ &\leq \|X_{0,m} - \mathbb{E}(X_{0,m}|\varepsilon_k, \ldots, \varepsilon_{k-m})\|_2\|X_0 - X_{0,m}\|_2 \,. \quad (4.85)\end{aligned}$$

Next, using (4.81), we get that, for any $k \in \{0, \ldots, m\}$,

$$\begin{aligned}\|X_{0,m} - \mathbb{E}(X_{0,m}|\varepsilon_k, \ldots, \varepsilon_{k-m})\|_2 &= \|X_{0,m} - \mathbb{E}(X_0|\varepsilon_0, \ldots, \varepsilon_{k-m})\|_2 \\ &= \|\mathbb{E}\big(X_0 - \mathbb{E}(X_0|\varepsilon_0, \ldots, \varepsilon_{k-m})|\varepsilon_0, \ldots, \varepsilon_{-m}\big)\|_2 \le \|X_0 - \mathbb{E}(X_0|\varepsilon_0, \ldots, \varepsilon_{k-m})\|_2 \,. \quad (4.86)\end{aligned}$$

Therefore, starting from (4.85), taking into account (4.86) and the fact that

$$\max_{0\le k\le [m/2]} \|X_0 - \mathbb{E}(X_0|\varepsilon_0, \ldots, \varepsilon_{k-m})\|_2 \le 2\|X_0 - \mathbb{E}(X_0|\varepsilon_0, \ldots, \varepsilon_{-[m/2]})\|_2 \,,$$

we get that

$$\max_{0\le k\le [m/2]} \big|\mathrm{Cov}\big(X_{0,m}, X_k - X_{k,m}\big)\big| \le 4\|X_0 - X_{0,[m/2]}\|_2^2 \,. \quad (4.87)$$

Starting from (4.83), gathering (4.84) and (4.87), and using the fact that $b(m) \le [m/2]$, we then derive that

$$\sum_{k=0}^{b(m)} \big|\mathrm{Cov}\big(X_0, X_k - X_{k,m}\big)\big| \le 4\, b(m)\|X_0 - X_{0,[m/2]}\|_2^2 \,,$$

which combined with (4.80) and (4.82) implies that

$$\sum_{k=0}^{b(m)} \big|\mathrm{Cov}\big(X_{0,m}, X_{k,m}\big) - \mathrm{Cov}\big(X_0, X_k\big)\big| \le 4\, b(m)\|X_0 - X_{0,[m/2]}\|_2^2 \,. \quad (4.88)$$

So, overall, starting from (4.74), gathering the upper bounds (4.75), (4.79) and (4.88), and taking into account the condition (2.3), we get that

$$\begin{aligned}&\sum_{\ell=1}^{p}\sum_{k=0}^{p-\ell} \big|\mathrm{Cov}\big(\bar{X}_{0,m}, \bar{X}_{k,m}\big) - \mathrm{Cov}\big(X_0, X_k\big)\big| \\ &\quad \ll p\, m\mathbb{E}\big(X_0^2 \mathbf{1}_{|X_0|\ge M}\big) + p\|\mathbb{E}(X_m|\varepsilon_0)\|_2 + p\sum_{k\ge b(m)} \|P_0(X_k)\|_2 + 4p\, b(m)\|X_0 - X_{0,[m/2]}\|_2^2 \,. \end{aligned} \quad (4.89)$$

Therefore, starting from (4.73), considering the upper bound (4.89) and using the fact that $pk_{N,m} \le N$, we finally obtain that

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{s=1}^{k_{N,m}} \Big|E\big(D^2 f\big(\widetilde{\mathbf{W}}_s^{(i)}\big)\big).\Big(\mathbb{E}\big(\mathbf{U}_s^{(i)\otimes 2}\big) - \mathbb{E}\big(\mathbf{V}_s^{(i)\otimes 2}\big)\Big)\Big| \\ &\ll \frac{1}{v^2(1\wedge v)}\Big(m\mathbb{E}\big(X_0^2\mathbf{1}_{|X_0|\ge M}\big) + \|\mathbb{E}(X_m|\varepsilon_0)\|_2 + \sum_{k\ge b(m)} \|P_0(X_k)\|_2 + b(m)\|X_0 - X_{0,[m/2]}\|_2^2\Big) \,. \end{aligned} \quad (4.90)$$

Letting first $M$ tend to infinity and using the fact that $X_0$ belongs to $\mathbb{L}^2$, the first term in the right-hand side is going to zero. Letting now $m$ tend to infinity the third term vanishes by the condition (2.3), whereas the last one goes to zero by taking into account (4.76). To show that the second term goes to zero as $m$ tends to infinity, we notice that, by stationarity, $\|\mathbb{E}(X_m|\varepsilon_0)\|_2 \le \|\mathbb{E}(X_m|\xi_0)\|_2 = \|\mathbb{E}(X_0|\xi_{-m})\|_2$. By the reverse martingale convergence theorem, setting $\mathcal{F}_{-\infty} = \bigcap_{k\in\mathbb{Z}} \sigma(\xi_k)$, $\lim_{m\to\infty} \mathbb{E}(X_0|\xi_{-m}) = \mathbb{E}(X_0|\mathcal{F}_{-\infty}) = 0$ a.s. (since $\mathcal{F}_{-\infty}$ is trivial and $\mathbb{E}(X_0) = 0$). So, since $X_0$ belongs to $\mathbb{L}^2$, $\lim_{m\to\infty} \|\mathbb{E}(X_m|\varepsilon_0)\|_2 = 0$. This ends the proof of (4.72) and then of Proposition 4.3. □

### 4.4 End of the proof of Theorem 2.1

According to Propositions 4.1, 4.2 and 4.3, the convergence (4.3) follows. Therefore, to end the proof of Theorem 2.1, it remains to show that (4.4) holds true with $\mathbf{G}_n$ defined in Section 4.2. This can be achieved by using Theorem 1.1 in Silverstein (1995) combined with arguments developed in the proof of Theorem 1 in Yao (2012) (see also Wang *et al.* (2011)). With this aim, we consider $(y_k)_{k\in\mathbb{Z}}$ a sequence of i.i.d. real valued random variables with law $\mathcal{N}(0,1)$, and $n$ independent copies of $(y_k)_{k\in\mathbb{Z}}$ that we denote by $(y_k^{(1)})_{k\in\mathbb{Z}},\dots,(y_k^{(n)})_{k\in\mathbb{Z}}$. For any $i\in\{1,\dots,n\}$, define $\mathbf{y}_i=\big(y_1^{(i)},\dots,y_N^{(i)}\big)$. Let $\mathcal{Y}_n=(\mathbf{y}_1^T|\dots|\mathbf{y}_n^T)$ be the matrix whose columns are the $\mathbf{y}_i^T$'s and consider its associated sample covariance matrix $\mathbf{Y}_n=\frac{1}{n}\mathcal{Y}_n\mathcal{Y}_n^T$. Let $\gamma(k)=\mathrm{Cov}(X_0,X_k)$ and note that, by (4.31), $\gamma(k)$ is also equal to $\mathrm{Cov}(Z_0,Z_k)=\mathrm{Cov}(Z_0^{(i)},Z_k^{(i)})$ for any $i\in\{1,\dots,n\}$. Set

$$\Gamma_N:=\big(\gamma_{j,k}\big)=\begin{pmatrix}\gamma(0) & \gamma(1) & \cdots & \gamma(N-1)\\ \gamma(1) & \gamma(0) & & \gamma(N-2)\\ \vdots & \vdots & \vdots & \vdots\\ \gamma(N-1) & \gamma(N-2) & \cdots & \gamma(0)\end{pmatrix}.$$

Note that $(\Gamma_N)$ is bounded in spectral norm. Indeed, by the Gerschgorin theorem, the largest eigenvalue of $\Gamma_N$ is not larger than $\gamma(0)+2\sum_{k\geq 1}|\gamma(k)|$ which, according to Remark 2.2, is finite. Note also that, the vector $(\mathbf{Z}_1,\dots,\mathbf{Z}_n)$ has the same distribution as $\big(\mathbf{y}_1\Gamma_N^{1/2},\dots,\mathbf{y}_n\Gamma_N^{1/2}\big)$ where $\Gamma_N^{1/2}$ is the symmetric non-negative square root of $\Gamma_N$ and the $\mathbf{Z}_i$'s are defined in Section 4.2. Therefore, for any $z\in\mathbb{C}^+$, $\mathbb{E}\big(S_{F^{\mathbf{G}_n}}(z)\big)=\mathbb{E}\big(S_{F^{\mathbf{A}_n}}(z)\big)$ where $\mathbf{A}_n=\Gamma_N^{1/2}\mathbf{Y}_n\Gamma_N^{1/2}$. The proof of (4.4) is then reduced to prove that, for any $z\in\mathbb{C}^+$,

$$\lim_{n\to\infty}\mathbb{E}\big(S_{F^{\mathbf{A}_n}}(z)\big)=S(z)\,, \tag{4.91}$$

where $S$ is defined in (2.4). According to Theorem 1.1 in Silverstein (1995), if one can show that

$$F^{\Gamma_N}\text{ converges to a probability distribution }H, \tag{4.92}$$

then (4.91) holds with $S$ satisfying the equation (1.4) in Silverstein (1995). Due to the Toeplitz form of $\Gamma_N$ and to the fact that $\sum_{k\geq 0}|\gamma(k)|<\infty$ (see Remark 2.2), the convergence (4.92) can be proved by taking into account the arguments developed in the proof of Theorem 1 of Yao (2012). Indeed, the fundamental eigenvalue distribution theorem of Szegö for Toeplitz forms allows to assert that the empirical spectral distribution of $\Gamma_N$ converges weakly to a non random distribution $H$ that is defined via the spectral density of $(X_k)_{k\in\mathbb{Z}}$ (see Relations (12) and (13) in Yao (2012)). To end the proof, it suffices to notice that the relation (1.4) in Silverstein (1995) combined with the relation (13) in Yao (2012) leads to (2.4). □

## 5 Appendix

In this section, we give some upper bounds for the partial derivatives of $f$ defined in (4.35).

**Lemma 5.1** *Let $x$ be a vector of $\mathbb{R}^{N\times n}$ with coordinates*

$$x=\big(x^{(1)},\dots,x^{(n)}\big)\ \textit{where for any } i\in\{1,\dots,n\},\ x^{(i)}=\big(x_k^{(i)},\,k\in\{1,\dots,N\}\big)\,.$$

*Let $z=u+\sqrt{-1}v\in\mathbb{C}^+$ and $f:=f_z$ be the function defined in (4.35). Then, for any $i\in\{1,\dots,n\}$ and any $j,k,\ell,m\in\{1,\dots,N\}$, the following inequalities hold true:*

$$\left|\frac{\partial^2 f}{\partial x_m^{(i)}\partial x_j^{(i)}}(x)\right|\leq\frac{8}{v^3n^2N}\sum_{r=1}^{N}\big|x_r^{(i)}\big|^2+\frac{2}{v^2nN}\,,$$

$$\left|\frac{\partial^3 f}{\partial x_\ell^{(i)}\partial x_m^{(i)}\partial x_j^{(i)}}(x)\right| \le \frac{48}{v^4n^3N}\left(\sum_{r=1}^N \left|x_r^{(i)}\right|^2\right)^{3/2} + \frac{24}{v^3n^2N}\left(\sum_{r=1}^N \left|x_r^{(i)}\right|^2\right)^{1/2},$$

*and*

$$\left|\frac{\partial^4 f}{\partial x_k^{(i)}\partial x_\ell^{(i)}\partial x_m^{(i)}\partial x_j^{(i)}}(x)\right| \le \frac{24\times 16}{v^5n^4N}\left(\sum_{r=1}^N \left|x_r^{(i)}\right|^2\right)^2 + \frac{36\times 8}{v^4n^3N}\sum_{r=1}^N \left|x_r^{(i)}\right|^2 + \frac{24}{v^3n^2N}.$$

**Proof.** Recall that $f(x) = \frac{1}{N}\mathrm{Tr}\big(A(x) - z\mathbf{I}\big)^{-1}$ where $A(x) = \frac{1}{n}\sum_{k=1}^n (x^{(k)})^T x^{(k)}$. To prove the lemma, we shall proceed as in Chatterjee (2006) (see the proof of its Theorem 1.3) but with some modifications since his computations are made in case where $A(x)$ is a Wigner matrix of order $N$.

Let $i \in \{1,\dots,n\}$ and consider for any $j,k \in \{1,\dots,N\}$, the notations $\partial_j$ instead of $\partial/\partial x_j^{(i)}$, $\partial^2_{jk}$ instead of $\partial^2/\partial x_j^{(i)}\partial x_k^{(i)}$ and so on. We shall also write $A$ instead of $A(x)$, $f$ instead of $f(x)$, and define $G = \big(A - z\mathbf{I}\big)^{-1}$.

Note that $\partial_j A$ is the matrix with $n^{-1}\big(x_1^{(i)},\dots,x_{j-1}^{(i)},2x_j^{(i)},x_{j+1}^{(i)},\dots,x_N^{(i)}\big)$ as the $j^{th}$ row, its transpose as the $j^{th}$ column, and zero otherwise. Thus, the Hilbert-Schmidt norm of $\partial_j A$ is bounded as follows:

$$\|\partial_j A\|_2 = \frac{1}{n}\Big(2\sum_{k=1,k\neq j}^N |x_k^{(i)}|^2 + 4|x_j^{(i)}|^2\Big)^{1/2} \le \frac{2}{n}\Big(\sum_{k=1}^N |x_k^{(i)}|^2\Big)^{1/2}. \tag{5.1}$$

Now, for any $m,j \in \{1,\dots,N\}$ such that $m \neq j$, $\partial^2_{mj}A$ has only two non-zero entries which are equal to $1/n$, whereas if $m = j$, it has only one non-zero entry which is equal to $2/n$. Hence,

$$\|\partial^2_{mj}A\|_2 \le \frac{2}{n}. \tag{5.2}$$

Finally, note that $\partial^3_{lmj}A \equiv 0$ for any $j,m,l \in \{1,\dots,N\}$.

Now, by using (4.36), it follows that, for any $j \in \{1,\dots,N\}$,

$$\partial_j f = -\frac{1}{N}\mathrm{Tr}(G(\partial_j A)G). \tag{5.3}$$

In what follows, the notations $\sum_{\{j',m'\}=\{j,m\}}$, $\sum_{\{j',m',\ell'\}=\{j,m,\ell\}}$ and $\sum_{\{j',m',\ell'k'\}=\{j,m,\ell,k\}}$ mean respectively the sum over all permutations of $\{j,m\}$, of $\{j,m,\ell\}$ and of $\{j,m,\ell,k\}$. Therefore the first sum consists of 2 terms, the second one of 6 terms and the last one of 24 terms. Starting from (5.3) and applying repeatedly (4.36), we then derive the following cumbersome formulas for the partial derivatives up to the order four: for any $j,m,\ell,k \in \{1,\dots,N\}$,

$$\partial^2_{mj}f = \frac{1}{N}\sum_{\{j',m'\}=\{j,m\}} \mathrm{Tr}\big(G(\partial_{j'}A)G(\partial_{m'}A)G\big) - \frac{1}{N}\mathrm{Tr}\big(G(\partial^2_{mj}A)G\big), \tag{5.4}$$

$$\begin{aligned}\partial^3_{\ell mj}f = &-\frac{1}{N}\sum_{\{j',m',\ell'\}=\{j,m,\ell\}} \mathrm{Tr}\big(G(\partial_{j'}A)G(\partial_{m'}A)G(\partial_{\ell'}A)G\big)\\ &+\frac{1}{N}\sum_{\{j',m'\}=\{j,m\}} \mathrm{Tr}\Big(G(\partial^2_{\ell j'}A)G(\partial_{m'}A)G + G(\partial_{j'}A)G(\partial^2_{\ell m'}A)G\Big)\\ &+\frac{1}{N}\mathrm{Tr}\big(G(\partial_\ell A)G(\partial^2_{mj}A)G\big) + \frac{1}{N}\mathrm{Tr}\big(G(\partial^2_{mj}A)G(\partial_\ell A)G\big),\end{aligned} \tag{5.5}$$

and

$$\partial^4_{k\ell mj} f := I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \,, \tag{5.6}$$

where

$$I_1 = \frac{1}{N} \sum_{\{j',m',\ell',k'\}=\{j,m,\ell,k\}} \mathrm{Tr}\big(G(\partial_{j'}A)G(\partial_{m'}A)G(\partial_{\ell'}A)G(\partial_{k'}A)G\big)\,,$$

$$I_2 = -\frac{1}{N} \sum_{\{j',m',\ell'\}=\{j,m,\ell\}} \Big(\mathrm{Tr}\big(G(\partial^2_{kj'}A)G(\partial_{m'}A)G(\partial_{\ell'}A)G\big) + \mathrm{Tr}\big(G(\partial_{j'}A)G(\partial^2_{km'}A)G(\partial_{\ell'}A)G\big) + \mathrm{Tr}\big(G(\partial_{j'}A)G(\partial_{m'}A)G(\partial^2_{k\ell'}A)G\big)\Big)\,,$$

$$\begin{aligned} I_3 = &-\frac{1}{N} \sum_{\{j',m'\}=\{j,m\}} \Big(\mathrm{Tr}\big(G(\partial^2_{\ell j'}A)G(\partial_k A)G(\partial_{m'}A)G\big) + \mathrm{Tr}\big(G(\partial^2_{\ell j'}A)G(\partial_{m'}A)G(\partial_k A)G\big)\Big) \\ &- \frac{1}{N} \sum_{\{j',m'\}=\{j,m\}} \Big(\mathrm{Tr}\big(G(\partial_k A)G(\partial^2_{\ell j'}A)G(\partial_{m'}A)G\big) + \mathrm{Tr}\big(G(\partial_{j'}A)G(\partial^2_{\ell m'}A)G(\partial_k A)G\big)\Big) \\ &- \frac{1}{N} \sum_{\{j',m'\}=\{j,m\}} \Big(\mathrm{Tr}\big(G(\partial_k A)G(\partial_{j'}A)G(\partial^2_{\ell m'}A)G\big) + \mathrm{Tr}\big(G(\partial_{j'}A)G(\partial_k A)G(\partial^2_{\ell m'}A)G\big)\Big)\,, \end{aligned}$$

$$I_4 = -\frac{1}{N} \sum_{\{k',\ell'\}=\{k,\ell\}} \Big(\mathrm{Tr}\big(G(\partial^2_{mj}A)G(\partial_{k'}A)G(\partial_{\ell'}A)G\big) + \mathrm{Tr}\big(G(\partial_{k'}A)G(\partial^2_{mj}A)G(\partial_{\ell'}A)G\big) + \mathrm{Tr}\big(G(\partial_{k'}A)G(\partial_{\ell'}A)G(\partial^2_{mj}A)G\big)\Big)\,,$$

$$I_5 = \frac{1}{N} \sum_{\{k',\ell'\}=\{k,\ell\}} \sum_{\{j',m'\}=\{j,m\}} \mathrm{Tr}\big(G(\partial^2_{\ell' j'}A)G(\partial^2_{k'm'}A)G\big)\,,$$

and

$$I_6 = \frac{1}{N}\mathrm{Tr}\big(G(\partial^2_{mj}A)G(\partial^2_{k\ell}A)G\big) + \frac{1}{N}\mathrm{Tr}\big(G(\partial^2_{k\ell}A)G(\partial^2_{mj}A)G\big)\,.$$

We start by giving an upper bound for $\partial^2_{mj} f$. Since the eigenvalues of $G^2$ are all bounded by $v^{-2}$, then so are its entries. Then, as $\mathrm{Tr}(G(\partial^2_{mj}A)G) = \mathrm{Tr}((\partial^2_{mj}A)G^2)$, it follows that

$$|\mathrm{Tr}(G(\partial^2_{mj}A)G)| = |\mathrm{Tr}((\partial^2_{mj}A)G^2)| \leq 2v^{-2}n^{-1}\,. \tag{5.7}$$

Next, to give an upper bound for $|\mathrm{Tr}\big(G(\partial_j A)G(\partial_m A)G\big)|$, it is useful to recall some properties of the Hilbert-Schmidt norm: Let $B = (b_{ij})_{1\leq i,j\leq N}$ and $C = (c_{ij})_{1\leq i,j\leq N}$ be two $N \times N$ complex matrices in $\mathcal{L}_2$, the set of Hilbert-Schmidt operators. Then
**(a)-** $|\mathrm{Tr}(BC)| \leq \|B\|_2 \|C\|_2$,
**(b)-** If $B$ admits a spectral decomposition, then $\max\{\|BC\|_2, \|CB\|_2\} \leq \|B\|.\|C\|_2$,
(see e.g. Wilkinson (1965) pages 55-58, for a proof of these facts).

Using the properties of the Hilbert-Schmidt norm recalled above, the fact that the eigenvalues of $G$ are all bounded by $v^{-1}$, and (5.1), we then derive that

$$\begin{aligned} |\mathrm{Tr}(G(\partial_j A)G(\partial_m A)G)| &\leq \|G(\partial_j A)G\|_2.\|(\partial_m A)G\|_2 \leq \|G\|.\|(\partial_j A)G\|_2.\|\partial_m A\|_2.\|G\| \\ &\leq \|G\|^3.\|\partial_j A\|_2.\|\partial_m A\|_2 \leq \frac{4}{v^3 n^2} \sum_{k=1}^N \left|x_k^{(i)}\right|^2. \end{aligned} \tag{5.8}$$

Starting from (5.4) and considering (5.7) and (5.8), the first inequality of Lemma 5.1 follows.

Next, using again the above properties (a) and (b), the fact that the eigenvalues of $G$ are all bounded by $v^{-1}$, (5.1) and (5.2), we get that

$$\begin{aligned}|\mathrm{Tr}(G(\partial_j A)G(\partial_m A)G(\partial_\ell A)G)| &\leq \|G(\partial_j A)G(\partial_m A)G\|_2.\|(\partial_\ell A)G\|_2 \\ &\leq \|G(\partial_j A)G(\partial_m A)\|_2.\|G\|^2.\|\partial_\ell A\|_2 \leq \|G(\partial_j A)\|_2.\|G(\partial_m A)\|_2.\|G\|^2.\|\partial_\ell A\|_2 \\ &\leq \|G\|^4.\|\partial_j A\|_2.\|\partial_m A\|_2.\|\partial_\ell A\|_2 \leq \frac{8}{v^4 n^3}\Big(\sum_{k=1}^N \big|x_k^{(i)}\big|^2\Big)^{3/2}, \end{aligned} \tag{5.9}$$

and

$$\begin{aligned}|\mathrm{Tr}(G(\partial^2_{\ell j} A)G(\partial_m A)G)| &\leq \|G(\partial^2_{\ell j} A)G\|_2.\|(\partial_m A)G\|_2 \leq \|G\|^2\|G(\partial^2_{\ell j} A)\|_2.\|\partial_m A\|_2 \\ &\leq \|G\|^3.\|\partial^2_{\ell j} A\|_2.\|\partial_m A\|_2 \leq \frac{4}{v^3 n^2}\Big(\sum_{k=1}^N \big|x_k^{(i)}\big|^2\Big)^{1/2}. \end{aligned} \tag{5.10}$$

The same last bound is obviously valid for $|\mathrm{Tr}(G(\partial_m A)G(\partial^2_{\ell j} A)G)|$. Hence, starting from (5.5) and considering (5.9) and (5.10), the second inequality of Lemma 5.1 follows.

It remains to prove the third inequality of Lemma 5.1. Using again the above properties (a) and (b), the fact that the eigenvalues of $G$ are all bounded by $v^{-1}$, (5.1) and (5.2), we infer that

$$|\mathrm{Tr}(G(\partial_j A)G(\partial_m A)G(\partial_\ell A)G(\partial_k A)G)| \leq \frac{16}{v^5 n^4}\Big(\sum_{k=1}^N \big|x_k^{(i)}\big|^2\Big)^2, \tag{5.11}$$

$$|\mathrm{Tr}(G(\partial^2_{\ell j} A)G(\partial_m A)G(\partial_k A)G)| \leq \frac{8}{v^4 n^3}\sum_{k=1}^N \big|x_k^{(i)}\big|^2, \tag{5.12}$$

and

$$|\mathrm{Tr}(G(\partial^2_{\ell j} A)G(\partial^2_{mk} A)G)| \leq \frac{4}{v^3 n^2}. \tag{5.13}$$

Clearly the bound (5.12) is also valid for the quantities $|\mathrm{Tr}(G(\partial_m A)G(\partial^2_{\ell j} A)G(\partial_k A)G)|$ and $|\mathrm{Tr}(G(\partial_m A)G(\partial_k A)G(\partial^2_{\ell j} A)G)|$. So, overall, starting from (5.6) and considering (5.11), (5.12) and (5.13), the third inequality of Lemma 5.1 follows. □

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