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Derived Hall algebras

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Abstract

The purpose of this work is to define a derived Hall algebra $\mathcal{DH}(T)$, associated to any dg-category T (under some finiteness conditions), generalizing the Hall algebra of an abelian category. Our main theorem states that $\mathcal{DH}(T)$ is associative and unital. When the associated triangulated category $[T]$ is endowed with a t-structure with heart $\mathcal{A}$, it is shown that $\mathcal{DH}(T)$ contains the usual Hall algebra $\mathcal{H}(\mathcal{A})$. We will also prove an explicit formula for the derived Hall numbers purely in terms of invariants of the triangulated category associated to T. As an example, we describe the derived Hall algebra of an hereditary abelian category.

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1 Introduction

Hall algebras are associative algebras associated to abelian categories (under some finiteness conditions). They appear, as well as various variants, in several mathematical contexts, in particular in the theory of quantum groups, and have been intensively studied (we refer to the survey paper [De-Xi] for an overview of the subject as well as for bibliographic references). Roughly speaking, the Hall algebra $\mathcal{H}(A)$ of an abelian category A, is generated as a vector space by the isomorphism classes of objects in A, and the product of two objects X and Y is defined as $\sum_{Z\in Iso(A)} g^Z_{X,Y} Z$, where the structure constants $g^Z_{X,Y}$ count a certain number of extensions $X \to Z \to Y$. If $\mathfrak{g}$ is a complex Lie algebra with simply laced Dynkin diagram, and A the abelian category of $\mathbb{F}_q$-representations of a quiver whose underlying graph is the Dynkin diagram of $\mathfrak{g}$, Ringel proved that the Hall algebra $\mathcal{H}(A)$ is closely related to $U_q(b^+)$, the positive part of the quantum enveloping algebra $U_q(\mathfrak{g})$ (see [Ri]). It seems to have been stressed by several authors that an extension of the construction $A \mapsto \mathcal{H}(A)$ to the derived category of A-modules could be useful to recover the full algebra $U_q(g)$. To this end, certain Hall numbers $g^Z_{X,Y}$, that are supposed to count triangles $X \longrightarrow Z \longrightarrow Y \longrightarrow X[1]$ in a triangulated category, have been introduced (see for example [De-Xi] §2.2). However, these Hall numbers do not provide an associative multiplication. Quoting the introduction of [Ka]:

Unfortunately, a direct mimicking of the Hall algebra construction, but with exact triangles replacing exact sequences, fails to give an associative mutiplication, even though the octohedral axiom looks like the right tool to establish associativity.

The purpose of this work is to define analogs of Hall algebras and Hall numbers, called *derived Hall algebra* and *derived Hall numbers*, associated with dg-categories T (instead of abelian or triangulated categories) and providing an associative and unital multiplication on the vector space generated by equivalence classes of perfect T^{op}-dg-modules (see [To] for definitions and notations). Our main theorem is the following.

Theorem 1.1 *Let T be a dg-category over some finite field $\mathbb{F}_q$, such that for any two objects x and y in T the graded vector space $H^*(T(x,y)) = Ext^*(x,y)$ is of finite total dimension. Let $\mathcal{DH}(T)$ be the $\mathbb{Q}$-vector space with basis $\{\chi_x\}$, where x runs through the set of equivalence classes of perfect T^{op}-dg-modules x. Then, there exists an associative and unital product*

$$\mu : \mathcal{DH}(T) \otimes \mathcal{DH}(T) \longrightarrow \mathcal{DH}(T),$$

with

$$\mu(\chi_x, \chi_y) = \sum_z g^z_{x,y} \chi_z,$$

and where the $g^z_{x,y}$ are rational numbers counting the virtual number of morphisms in T, $f : x' \to z$, such that x' is equivalent to x and the homotopy cofiber of f is equivalent to y.

Our construction goes as follows. To each dg-category T, satisfying the condition of the theorem, we associate a certain homotopy type $X^{(0)}(T)$, which is a classifying space of perfect T^{op}-dg-modules. The connected components of $X^{(0)}(T)$ are in one to one correspondence

with the quasi-isomorphism classes of perfect T^{op}-dg-modules. Furthermore, for any point x, corresponding to a T^{op}-dg-module, the homotopy groups of $X^{(0)}(T)$ at x are given by

$$\pi_1(X^{(0)}(T), x) = Aut(x) \qquad \pi_i(X^{(0)}(T), x) = Ext^{1-i}(x, x)\ \forall i > 1$$

where the automorphism and Ext groups are computed in the homotopy category of T^{op}-dg-modules. In the same way, we consider the simplicial set $X^{(1)}(T)$ which is a classifying space for morphisms between perfect T^{op}-dg-modules. We then construct a diagram of simplicial sets

$$\begin{array}{ccc} X^{(1)}(T) & \xrightarrow{\ c\ } & X^{(0)}(T) \\ {\scriptstyle s\times t}\big\downarrow & & \\ X^{(0)}(T) \times X^{(0)}(T), & & \end{array}$$

where c sends a morphism of T to its target, and $s \times t$ sends a morphism $u : x \to y$ of T to the pair consisting of x and the homotopy cofiber (or cone) of u. We note that the vector space $\mathcal{DH}(T)$, with basis $\{\chi_x\}$, can be identified with the set of $\mathbb{Q}$-valued functions with finite support on $\pi_0(X^{(0)}(T))$. Our multiplication is then defined as a certain convolution product

$$\mu := c_! \circ (s \times t)^* : \mathcal{DH}(T) \otimes \mathcal{DH}(T) \longrightarrow \mathcal{DH}(T).$$

We should mention that though the operation $(s \times t)^*$ is the naive pull-back of functions, the morphism $c_!$ is a fancy push-forward, defined in terms of the full homotopy types $X^{(0)}(T)$ and $X^{(1)}(T)$ and involving alternating products of the orders of their higher homotopy groups. The fact that this product is associative is then proved using a base change formula, and the fact that certain commutative squares involving $X^{(2)}(T)$, the classifying space of pairs of composable morphisms in T, are homotopy cartesian.

In the second part of the paper we will establish an explicit formula for the derived Hall numbers, purely in terms of the triangulated category of perfect dg-modules (see Prop. 5.1). This will imply an invariance statement for our derived Hall algebras under derived equivalences. This also implies the *heart property*, stating that for a given t-structure on the category of perfect T^{op}-dg-modules, the sub-space of $\mathcal{DH}(T)$ spanned by the objects belonging to the heart $\mathcal{C}$ is a sub-algebra isomorphic to the usual Hall algebra of $\mathcal{C}$ (see Prop. 6.1). When A is a finite dimensional algebra, considered as a dg-category BA, this allows us to identify the usual Hall algebra $\mathcal{H}(A)$ as the sub-algebra in $\mathcal{DH}(BA^{op})$ spanned by the non-dg A-modules. We will finish this work with a description by generators and relations of the derived Hall algebra of an abelian category of cohomological dimension ≤ 1, in terms of its underived Hall algebra (see Prop. 7.1).

Related and future works: As we mentioned, there have been a huge amount of works on Hall algebras, in which several variants of them appear (e.g. Ringel-Hall algebras). There is no doubt that one can also define derived versions of these variant notions, and studying these seems to me an important work to do, as this might throw new lights on quantum groups. However, the precise relations between derived Hall algebras and quantum groups are still unclear and require to be investivated in the future. In fact, it seems to be expected that the full quantum

enveloping algebra is related not to the derived category of representations of the associated quiver but rather to its 2-periodic version. A 2-periodic dg-category will never satisfy the finiteness condition needed in this work in order to construct the derived Hall algebra, and therefore it seems that though we think the present work is a step in the right direction it is probably not enough to answer the question of reconstructing the full quantum group as the Hall algebra of something.

G. Lusztig has given a very nice geometric construction of Hall algebras, using certain perverse sheaves on the moduli stacks of modules over associative algebras (see for example [Lu]). This approach gives a much more flexible theory, as for example it allows to replace the finite field by essentially any base, and has found beautiful applications to quantum groups (such as the construction of the so-called canonical bases). I am convinced that the derived Hall algebra $\mathcal{DH}(T)$ of a dg-category T also has a geometric interpretation in terms of complexes of l-adic sheaves on the moduli stack of perfect T^{op}-dg-modules (as constructed in [To-Va]). This geometric interpretation of the derived Hall algebras will be investigated in a future work.

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I also would like to thanks both referees for their comments.

Conventions and notations: We use model category theory from [Ho]. For a diagram $x \longrightarrow z \longleftarrow y$ in a model category M we denote by $x \times^h_z y$ the homotopy fiber product, and by $x \coprod^h_z y$ the homotopy push-out. For dg-categories, we refer to [Tab, To], and we will mainly keep the same notation and terminology as in [To]. Finally, for a finite set X we denote by $|X|$ the cardinality of X.

2 Functions on locally finite homotopy types

In this section, we will discuss some finiteness conditions on homotopy types, functions with finite supports on them and their transformations (push-forwards, pull-backs and base change).

Definition 2.1 *A homotopy type $X \in Ho(Top)$ is* locally finite*, if it satisfies the following two conditions.*

1. *For all base points $x \in X$, the group $\pi_i(X, x)$ is finite.*
2. *For all base point $x \in X$, there is n (depending on x) such that $\pi_i(X, x) = 0$ for all $i > n$.*

The full sub-category of $Ho(Top)$ consisting of all locally finite objects will be denoted by $Ho(Top)^{lf}$.

Definition 2.2 *Let $X \in Ho(Top)$. The $\mathbb{Q}$-vector space of* rational functions with finite support *on X is the $\mathbb{Q}$-vector space of functions on the set $\pi_0(X)$ with values in $\mathbb{Q}$ and with finite support. It is denoted by $\mathbb{Q}_c(X)$.*

Let $f : X \longrightarrow Y$ be a morphism in $Ho(Top)^{lf}$. We define a morphism

$$f_! : \mathbb{Q}_c(X) \longrightarrow \mathbb{Q}_c(Y)$$

in the following way. For $y \in \pi_0(Y)$, one sets $F_y \in Ho(Top)$ the homotopy fiber of f at the point y, and $i : F_y \longrightarrow X$ the natural morphism. We note that the long exact sequence in homotopy implies that for all $z \in \pi_0(F_y)$ the group $\pi_i(F_y, z)$ is finite and moreover zero for all i big enough. Furthermore, the fibers of the map $i : \pi_0(F_y) \longrightarrow \pi_0(X)$ are finite.

For a function $\alpha \in \mathbb{Q}_c(X)$, and $y \in \pi_0(Y)$, one sets

$$f_!(\alpha)(y) = \sum_{z \in \pi_0(F_y)} \alpha(i(z)).\prod_{i>0} |\pi_i(F_y, z)|^{(-1)^i}.$$

Note that as α has finite support, the sum in the definition of $f_!$ only has a finite number of non-zero terms, and thus is well defined.

Lemma 2.3 *For any morphism $f : X \longrightarrow Y$ in $Ho(Top)^{lf}$, any $\alpha \in \mathbb{Q}_c(X)$, and any $y \in \pi_0(Y)$, one has*

$$f_!(\alpha)(y) = \sum_{x \in \pi_0(X)/f(x)=y} \alpha(x).\prod_{i>0} \left(|\pi_i(X, x)|^{(-1)^i}.|\pi_i(Y, y)|^{(-1)^{i+1}} \right)$$

Proof: Let $y \in \pi_0(Y)$, $i : F_y \to X$ the homotopy fiber of f at y, and $z \in \pi_0(F_y)$. The long exact sequence in homotopy

$$\pi_{i+1}(Y, y) \longrightarrow \pi_i(F_y, z) \longrightarrow \pi_i(X, x) \longrightarrow \pi_i(Y, y) \longrightarrow \pi_{i-1}(F_y, z) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(Y, y) \longrightarrow \pi_0(F_y, z) \longrightarrow \pi_0(X, x) \longrightarrow \pi_0(Y, y),$$

implies that one has for any $x \in \pi_0(X)$ with $f(x) = y$

$$\prod_{i>0} \left(|\pi_i(X, x)|^{(-1)^i}.|\pi_i(Y, y)|^{(-1)^{i+1}} \right) = \sum_{z \in \pi_0(F_y)/i(z)=x} \prod_{i>0} |\pi_i(F_y, z)|^{(-1)^i}.$$

□

Corollary 2.4 *For any morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ in $Ho(Top)^{lf}$, one has $(g \circ f)_! = g_! \circ f_!$.*

Proof: Clearly follows from the lemma 2.3. □

Definition 2.5 *A morphism $f : X \longrightarrow Y$ in $Ho(Top)^{lf}$ is called* proper, *if for any $y \in \pi_0(Y)$ there is only a finite number of elements $x \in \pi_0(X)$ with $f(x) = y$.*

Note that a morphism $f : X \longrightarrow Y$ in $Ho(Top)^{lf}$ is proper if and only if for any $y \in \pi_0(Y)$ the homotopy fiber F_y of f at y is such that $\pi_0(F_y)$ is a finite set.

Let $f : X \longrightarrow Y$ be a proper morphism between locally finite homotopy types. We define the pull-back

$$f^* : \mathbb{Q}_c(Y) \longrightarrow \mathbb{Q}_c(X)$$

by the usual formula

$$f^*(\alpha)(x) = \alpha(f(x)),$$

for any $\alpha \in \mathbb{Q}_c(Y)$ and any $x \in \pi_0(X)$. Note that as the morphism f is proper $f^*(\alpha)$ has finite support, and thus that f^* is well defined.

Lemma 2.6 *Let*

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

be a homotopy cartesian square of locally finite homotopy types, with u being proper. Then u' is proper and furthermore one has

$$u^* \circ f_! = (f')_! \circ (u')^* : \mathbb{Q}_c(X) \longrightarrow \mathbb{Q}_c(Y').$$

Proof: The fact that u' is proper if u is so follows from the fact that the homotopy fiber of u' at a point $x \in \pi_0(X)$ is equivalent to the homotopy fiber of u at the point $f(x)$.

Let $y' \in \pi_0(Y')$, $y = u(y')$. Let us denote by F the homotopy fiber of f' at y'. Note that F is also the homotopy fiber of f at y. We denote by $i' : F \longrightarrow X'$ the natural morphism, and we let $i = u' \circ i' : F \longrightarrow X$. For $\alpha \in \mathbb{Q}_c(X)$, one has

$$(u^* f_!)(\alpha)(y') = f_!(\alpha)(y) = \sum_{z \in \pi_0(F)} \alpha(i(z)). \prod_{i>0} |\pi_i(F, z)|^{(-1)^i}$$

$$= \sum_{z \in \pi_0(F)} (u')^*(\alpha)(i'(z)). \prod_{i>0} |\pi_i(F, z)|^{(-1)^i} = (f')_!(u'^*(\alpha))(y').$$

□

3 Hall numbers and Hall algebras for dg-categories

Let $k = \mathbb{F}_q$ be a finite field with q elements. We refer to [Tab, To] for the notion of dg-categories over k, as well as their homotopy theory.

Definition 3.1 *A (small) dg-category T over k is* locally finite *if for any two objects x and y in T, the complex $T(x, y)$ is cohomologically bounded and with finite dimensional cohomology groups.*

Let T be a locally finite dg-category, and let us consider $M(T) := T^{op} - Mod$, the model category of dg-T^{op}-modules. This is a stable model category, whose homotopy category $Ho(M(T))$ is therefore endowed with a natural triangulated structure. We will denote by $[-,-]$ the Hom's in this triangulated category. Recall that an object of $M(T)$ is perfect if it belongs to the smallest sub-category of $Ho(M(T))$ containing the quasi-representable modules and which is stable by retracts, homotopy push-outs and homotopy pull-backs. Note that the perfect objects in $M(T)$ are also the compact objects in the triangulated category $Ho(M(T))$. The full sub-category of perfect objects in $M(T)$ will be denoted by $P(T)$. It is easy to see, as T is locally finite, that for any two objects x and y in $P(T)$, the k-vector spaces $[x, y[i]] =: Ext^i(x, y)$ are finite dimensional and are non-zero only for a finite number of indices $i \in \mathbf{Z}$.

We let I be the category with two objects 0 and 1 and a unique morphism $0 \to 1$, and we consider the category $M(T)^I$, of morphisms in $M(T)$, endowed with its projective model structure (for which equivalences and fibrations are defined levelwise). Note that cofibrant objects in $M(T)^I$ are precisely the morphisms $u : x \to y$, where x and y are cofibrant and u is a cofibration in $M(T)$.

There exist three left Quillen functors

$$s, c, t : M(T)^I \longrightarrow M(T)$$

defined, for any $u : x \to y$ object in $M(T)^I$, by

$$s(u) = x \qquad c(u) = y \qquad t(u) = y \coprod_x 0.$$

We now consider the diagram of left Quillen functors

$$\begin{array}{ccc} M(T)^I & \xrightarrow{\ c\ } & M(T) \\ {\scriptstyle s\times t}\big\downarrow & & \\ M(T) \times M(T). & & \end{array}$$

Restricting to the sub-categories of cofibrant and perfect objects, and of equivalences between them, one gets a diagram of categories and functors

$$\begin{array}{ccc} w(P(T)^I)^{cof} & \xrightarrow{\ c\ } & wP(T)^{cof} \\ {\scriptstyle s\times t}\big\downarrow & & \\ wP(T)^{cof} \times wP(T)^{cof}. & & \end{array}$$

Applying the nerve construction, one gets a diagram of homotopy types

$$\begin{array}{ccc} N(w(P(T)^I)^{cof}) & \xrightarrow{\ c\ } & N(wP(T)^{cof}) \\ {\scriptstyle s\times t}\big\downarrow & & \\ N(wP(T)^{cof}) \times N(wP(T)^{cof}). & & \end{array}$$

This diagram will be denoted by

$$\begin{array}{ccc} X^{(1)}(T) & \xrightarrow{\quad c \quad} & X^{(0)}(T) \\ {\scriptstyle s\times t}\big\downarrow & & \\ X^{(0)}(T)\times X^{(0)}(T). & & \end{array}$$

Lemma 3.2 *1. The homotopy types $X^{(1)}(T)$ and $X^{(0)}(T)$ are locally finite.*

2. The morphism $s\times t$ above is proper.

Proof: (1) For $x\in\pi_0(X^{(0)}(T))$, corresponding to a perfect T^{op}-module, we know by [D-K1] (see also [HAGII, Appendix A]) that $\pi_1(X^{(0)}(T),x)$ is a subset in $Ext^0(x,x)=[x,x]$, and that $\pi_i(X^{(0)}(T),x)=Ext^{1-i}(x,x)$ for all $i>1$. By assumption on T we know that these Ext groups are finite and zero for i big enough. This implies that $X^{(0)}(T)$ is locally finite. The same is true for $X^{(1)}(T)$, as it can be identified, up to equivalence, with $X^{(0)}(T')$, for $T'=T\otimes_k I_k$ (where I_k is the dg-category freely generated by the category I).

(2) Let (x,z) be a point in $X^{(0)}(T)\times X^{(0)}(T)$, and let F be the homotopy fiber of $s\times t$ at (x,z). It is not hard to check that $\pi_0(F)$ is in bijection with $Ext^1(z,x)$, and thus is finite by assumption on T. This shows that $s\times t$ is proper. $\square$

The space of functions with finite support on $X^{(0)}(T)$ will be denoted by

$$\mathcal{DH}(T):=\mathbb{Q}_c(X^{(0)}(T)).$$

Note that there exists a natural isomorphism

$$\begin{array}{ccc} \mathcal{DH}(T)\otimes\mathcal{DH}(T) & \simeq & \mathbb{Q}_c(X^{(0)}(T)\times X^{(0)}(T)) \\ f\otimes g & \mapsto & ((x,y)\mapsto f(x).g(y)) \end{array}.$$

Therefore, using our lemma 3.2 one gets a natural morphism

$$\mu:=c_!\circ(s\times t)^*:\mathcal{DH}(T)\otimes\mathcal{DH}(T)\longrightarrow\mathcal{DH}(T).$$

Let us now consider the set $\pi_0(X^{(0)}(T))$. This set is clearly in natural bijection with the set of isomorphism classes of perfect objects in $Ho(M(T))$. Therefore, the $\mathbb{Q}$-vector space $\mathcal{DH}(T)$ is naturally isomorphic to the free vector space over the isomorphism classes of perfect objects in $Ho(M(T))$. For an isomorphism class of a perfect T^{op}-module x, we simply denote by $\chi_x\in\mathcal{DH}(T)$ the corresponding characteristic function. When x runs throught the set of isomorphism classes of perfect objects in $Ho(M(T))$, the elements χ_x form a $\mathbb{Q}$-basis of $\mathcal{DH}(T)$.

Definition 3.3 *Let T be a locally finite dg-category over k.*

1. *The* derived Hall algebra of T *is $\mathcal{DH}(T)$ with the multiplication μ defined above.*

2. *Let x, y and z be three perfect T^{op}-modules. The* derived Hall number *of x and y along z is defined by*

$$g^z_{x,y}:=\mu(\chi_x,\chi_y)(z),$$

where χ_x and χ_y are the characteristic functions of x and y.

4 Associativity and unit

The main theorem of this work is the following.

Theorem 4.1 *Let T be a locally finite dg-category. Then the derived Hall algebra $\mathcal{DH}(T)$ is associative and unital.*

Proof: Let us start with the existence of a unit. Let $\mathbf{1} \in \mathcal{DH}(T)$ be the characteristic function of the zero T^{op}-module. We claim that $\mathbf{1}$ is a unit for the multiplication μ. Indeed, let $x \in \pi_0(X^{(0)}(T))$ and $\chi_x \in \mathcal{DH}(T)$ its characteristic function. The set $\pi_0(X^{(1)}(T))$ is identified with the isomorphism classes of perfect objects in $Ho(M(T)^I)$ (i.e. $\pi_0(X^{(1)}(T))$ is the set of equivalence classes of morphisms between perfect T^{op}-modules). Let us fix a morphism $u : y \to z$ between perfect T^{op}-modules, considered as an element in $\pi_0(X^{(1)}(T))$. The function $(s \times t)^*(\mathbf{1}, \chi_x)$ sends u to 1 if y is isomorphic to zero in $Ho(M(T))$ and x is isomorphic to z in $Ho(M(T))$, and to 0 otherwise. In other words, $(s \times t)^*(\mathbf{1}, \chi_x)$ is the characteristic function of the sub-set of $\pi_0(X^{(1)}(T))$ consisting of morphisms $0 \to z$ with $z \simeq x$ in $Ho(M(T))$. We let X be the sub-simplicial set of $X^{(1)}(T)$ which is the union of all connected component belongings to the support of $(s \times t)^*(\mathbf{1}, \chi_x)$. This simplicial set is connected, and by definition of the product μ, one has (see Lemma 2.3)

$$\mu(\mathbf{1}, \chi_x)(x) = \prod_{i>0} \left(|\pi_i(X)|^{(-1)^i} . |\pi_i(X^{(0)}(T), x)|^{(-1)^{i+1}} \right).$$

$$\mu(\mathbf{1}, \chi_x)(y) = 0 \; if \; y \neq x.$$

The left Quillen functor $c : M(T)^I \longrightarrow M$ is clearly fully faithful up to homotopy when restricted to the sub-category of morphisms $y \to z$ such that y is equivalent to 0. This implies that the induced morphism

$$c : X \longrightarrow X^{(0)}(T)$$

induces an isomorphism on all homotopy groups π_i for $i > 0$, identifying X with a connected component of $X^{(0)}(T)$. Therefore, one has $\mu(\mathbf{1}, \chi_x)(x) = 1$, showing that $\mu(\mathbf{1}, \chi_x) = \chi_x$. Essentially the same argument shows that $\mu(\chi_x, \mathbf{1}) = \chi_x$, proving that $\mathbf{1}$ is a unit for μ.

We now pass to the associativity of the multiplication. For sake of simplicity we first introduce several notations. Let $M(T)^{(1)}$ be the model category $M(T)^I$. We also let $M(T)^{(2)}$ be the model category of composable pairs of morphisms in M (i.e. objects in $M(T)^{(2)}$ are diagrams of objects in $M(T)$, $x \to y \to z$), again endowed with its projective model structure. The full sub-categories of perfect and cofibrant objects in $M(T)^{(i)}$ will be denoted by $(P(T)^{(i)})^{cof}$ (for $i = 1, 2$). Finally, the nerve of the sub-category of equivalences will simply be denoted by

$$X^{(i)} := N(w(P(T)^{(i)})^{cof}) \qquad i = 1, 2$$

$$X^{(0)} := N(wP(T)^{cof}).$$

We recall the existence of the three Quillen functors

$$s, t, c : M(T)^{(1)} \longrightarrow M(T)$$

and their induced morphisms on the corresponding locally finite homotopy types

$$s, t, c : X^{(1)} \longrightarrow X^{(0)}.$$

We introduce four left Quillen functors

$$f, g, h, k : M(T)^{(2)} \longrightarrow M(T)^{(1)}$$

defined, for an object $u = (x \to y \to z)$ in $M(T)^{(2)}$ by

$$f(u) = x \to y \qquad g(u) = y \to z \qquad\qquad h(u) = x \to z \qquad k(u) = y \coprod_x 0 \to Z \coprod_x 0.$$

These left Quillen functors induce four morphisms on the corresponding locally finite homotopy types

$$f, g, h, k : X^{(2)} \longrightarrow X^{(1)}.$$

We know consider the following two diagrams of simplicial sets

$$\begin{tikzcd}
X^{(2)} \arrow[r, "g"] \arrow[d, "f\times(t\circ k)"'] & X^{(1)} \arrow[r, "c"] \arrow[d, "s\times t"] & X^{(0)} \\
X^{(1)} \times X^{(0)} \arrow[r, "c\times id"] \arrow[d, "(s\times t)\times id"'] & X^{(0)} \times X^{(0)} & \\
(X^{(0)} \times X^{(0)}) \times X^{(0)} & &
\end{tikzcd}$$

$$\begin{tikzcd}
X^{(2)} \arrow[r, "h"] \arrow[d, "(s\circ f)\times k"'] & X^{(1)} \arrow[r, "c"] \arrow[d, "s\times t"] & X^{(0)} \\
X^{(0)} \times X^{(1)} \arrow[r, "id\times c"] \arrow[d, "id\times(s\times t)"'] & X^{(0)} \times X^{(0)} & \\
X^{(0)} \times (X^{(0)} \times X^{(0)}). & &
\end{tikzcd}$$

The left and top sides of these two diagrams are clearly equal as diagrams in $Ho(Top)^{lf}$ to

$$\begin{tikzcd}
X^{(2)} \arrow[r, "c\circ h"] \arrow[d, "(s\circ f)\times(s\circ k)\times(t\circ h)"'] & X^{(0)} \\
X^{(0)}. &
\end{tikzcd}$$

Therefore, lemma 2.6 would imply that

$$c_! \circ (s \times t)^* \circ (c \times id)_! \circ ((s \times t) \times id)^* = c_! \circ (s \times t)^* \circ (id \times c)_! \circ (id \times (s \times t))^*$$

if we knew that the two squares of the above two diagrams were homotopy cartesian. As this is clearly the associativity condition for the multiplication μ, we see that it only remains to prove that the two diagrams

$$\begin{tikzcd} X^{(2)} \arrow[r, "g"] \arrow[d, "f\times(t\circ k)"'] & X^{(1)} \arrow[d, "s\times t"] \\ X^{(1)}\times X^{(0)} \arrow[r, "c\times id"'] & X^{(0)}\times X^{(0)} \end{tikzcd}$$

$$\begin{tikzcd} X^{(2)} \arrow[r, "h"] \arrow[d, "(s\circ f)\times k"'] & X^{(1)} \arrow[d, "s\times t"] \\ X^{(0)}\times X^{(1)} \arrow[r, "id\times c"'] & X^{(0)}\times X^{(0)} \end{tikzcd}$$

are homotopy cartesian. This can be easily seen by using the relations between nerves of categories of equivalences and mapping spaces in model categories (see for example [HAGII, Appendix A]). As an example, let us describe a proof of the fact that the second diagram above is homotopy cartesian (the first one is even more easy to handle).

We start by a short digression on the notion of fiber product of model categories. Let

$$\begin{tikzcd} & M_1 \arrow[d, "F_1"] \\ M_2 \arrow[r, "F_2"'] & M_3 \end{tikzcd}$$

be a diagram of model categories and left Quillen functors (we will denote by G_i the right adjoint to F_i). We define a new model category M, called the *fiber product* of the diagram above and denoted by $M = M_1 \times_{M_3} M_2$, in the following way. The objects in M are 5-uples $(x_1, x_2, x_3; u, v)$, where x_i is an object in M_i, and

$$F_1(x_1) \xrightarrow{u} x_3 \xleftarrow{v} F_2(x_2)$$

are morphisms in M_3. Morphisms from $(x_1, x_2, x_3; u, v)$ to $(x'_1, x'_2, x'_3; u', v')$ in M are defined in the obvious manner, as families of morphisms $f_i : x_i \to x'_i$ in M_i, such that the diagram

$$\begin{tikzcd} F_1(x_1) \arrow[r, "u"] \arrow[d, "F_1(f_1)"'] & x_3 \arrow[d, "f_3"] & F_2(x_2) \arrow[l, "v"'] \arrow[d, "F_2(f_2)"] \\ F_1(x'_1) \arrow[r, "u'"'] & x'_3 & F_2(x'_2) \arrow[l, "v'"] \end{tikzcd}$$

commutes.

The model structure on M is taken to be the injective levelwise model structure, for which cofibrations (resp. equivalences) are morphisms f such that each f_i is cofibration (resp. an equivalence) in M_i (the fact that this model structure exists can be checked easily). It is important to note that an object $(x_1, x_2, x_3; u, v)$ in M is fibrant if and only if it satisfies the following two conditions

- The object x_i is fibrant in M_i.
- The morphisms
$$x_1 \longrightarrow G_1(x_3) \qquad x_2 \longrightarrow G_2(x_3),$$
adjoint to u and v, are fibrations (in M_1 and M_2 respectively).

Another important fact is that the mapping space of M, between two objects $(x_1, x_2, x_3; u, v)$ and $(x_1', x_2', x_3'; u', v')$ is naturally equivalent the following homotopy fiber product

$$\big(Map_{M_1}(x_1, x_1') \times Map_{M_2}(x_2, x_2')\big) \times^h_{Map_{M_3}(\mathbb{L}F_1(x_1), x_3') \times Map_{M_3}(\mathbb{L}F_2(x_2), x_3')} Map_{M_3}(x_3, x_3').$$

When the morphisms u, v, u' and v' are all equivalences and all objects x_i, x_i' are cofibrant, this homotopy fiber product simplifies as

$$Map_{M_1}(x_1, x_1') \times^h_{Map_{M_3}(x_3, x_3')} Map_{M_2}(x_2, x_2').$$

We now assume that there exists a diagram of model categories and left Quillen functors

$$\begin{tikzcd} M_0 \arrow[r, "H_1"] \arrow[d, "H_2"'] & M_1 \arrow[d, "F_1"] \\ M_2 \arrow[r, "F_2"'] & M_3, \end{tikzcd}$$

together with an isomorphism of functors

$$\alpha : F_1 \circ H_1 \Rightarrow F_2 \circ H_2.$$

We define a left Quillen functor

$$F : M_0 \longrightarrow M = M_1 \times_{M_3} M_2$$

by sending an object $x \in M$ to $(H_1(x), H_2(x), F_2(H_2(x)); u, v)$, where v is the identity of $F_2(H_2(x))$, and
$$u = \alpha_x : F_1(H_1(x)) \longrightarrow F_2(H_2(x))$$
is the isomorphism induced by α evaluated at x.

In the situation above, one can restrict to sub-categories of cofibrant objects and equivalences between them to get a diagram of categories

$$\begin{tikzcd} wM_0^{cof} \arrow[r, "H_1"] \arrow[d, "H_2"'] & wM_1^{cof} \arrow[d, "F_1"] \\ wM_2^{cof} \arrow[r, "F_2"'] & wM_3^{cof}, \end{tikzcd}$$

which is commutative up to the natural isomorphism α. Passing to the nerves, one gets a diagram of simplicial sets

$$\begin{array}{ccc} N(wM_0^{cof}) & \xrightarrow{h_1} & N(wM_1^{cof}) \\ {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ N(wM_2^{cof}) & \xrightarrow[f_2]{} & N(wM_3^{cof}), \end{array}$$

and α induces a natural simplicial homotopy between $f_1 \circ h_1$ and $f_2 \circ h_2$.

Lemma 4.2 *With the notations above, we assume the following assumption satisfied.*

1. *The left derived functor*
$$\mathbb{L}F : Ho(M_0) \longrightarrow Ho(M)$$
is fully faithful.

2. *If an object $(x_1, x_2, x_3; u, v) \in M$ is such that the x_i are cofibrant and the morphisms u and v are equivalences, then it lies in the essential image of $\mathbb{L}F$.*

Then, the homotopy commutative square

$$\begin{array}{ccc} N(wM_0^{cof}) & \xrightarrow{h_1} & N(wM_1^{cof}) \\ {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ N(wM_2^{cof}) & \xrightarrow[f_2]{} & N(wM_3^{cof}), \end{array}$$

is homotopy cartesian.

Proof of the lemma: This is a very particular case of the strictification theorem of [H-S], which is reproduced in [HAGII, Appendix B]. We will recall the proof of the particular case of our lemma (the general case can be proved in the same way, but surjectivity on π_0 is more involved).

We consider the induced morphism

$$N(wM_0^{cof}) \longrightarrow N(wM_1^{cof}) \times^h_{N(wM_3^{cof})} N(wM_2^{cof}).$$

In order to prove that this morphism induces a surjective morphism on π_0 it is well known to be enough to prove the corresponding statement but replacing all the simplicial sets $N(wM_i^{cof})$ by their 1-truncation. Passing to the fundamental groupoids, which are the sub-categories of isomorphisms in $Ho(M_i)$, we see that it is enough to show that the natural functor of categories

$$Ho(M_0) \longrightarrow Ho(M_1) \times^h_{Ho(M_3)} Ho(M_2)$$

is essentially surjective. But, an object in the right hand side can be represented by two cofibrant objects $x_1 M_1^{cof}$, $x_2 \in M_2^{cof}$, and an isomorphism $u : F_1(x_1) \simeq F_2(x_2)$. The isomorphism u can be itself represented by a string of equivalences in M_3

$$F_1(x_1) \longrightarrow x_3 \longleftarrow F_2(x_2)\ ,$$

where $F_2(x_2) \to x_3$ is a fibrant model in M_3. This data represents an object in M, which by our condition (2) lifts to an object in $Ho(M_0)$. This shows that

$$Ho(M_0) \longrightarrow Ho(M_1) \times^h_{Ho(M_3)} Ho(M_2)$$

is essentially surjective, and thus that

$$N(wM_0^{cof}) \longrightarrow N(wM_1^{cof}) \times^h_{N(wM_3^{cof})} N(wM_2^{cof})$$

is surjective up to homotopy.

It remains to show that the morphism

$$N(wM_0^{cof}) \longrightarrow N(wM_1^{cof}) \times^h_{N(wM_3^{cof})} N(wM_2^{cof})$$

induces an injective morphism in π_0 and isomorphisms on all π_i for all base points. Equivalently, it remains to show that for two points x and y, the induced morphism on the path spaces

$$\Omega_{(x,y)}N(wM_0^{cof}) \longrightarrow \Omega_{(h_1(x),h_1(y))}N(wM_1^{cof}) \times^h_{\Omega_{(f_1(h_1(x)),f_1(h_1(y)))}N(wM_3^{cof})} \Omega_{(h_2(x),h_2(y))}N(wM_2^{cof})$$

is a homotopy equivalence. Using the relations between path spaces of nerves of equivalences in model categories and mapping spaces, as recalled e.g. in [HAGII, Appendix A], we see that the above morphism is equivalent to

$$Map^{eq}_{M_0}(x,y) \longrightarrow Map^{eq}_{M_1}(\mathbb{L}F_1(x),\mathbb{L}F_1(y)) \times^h_{Map^{eq}_{M_3}(\mathbb{L}H_1(\mathbb{L}F_1(x)),\mathbb{L}H_1(\mathbb{L}F_1(y)))} Map^{eq}_{M_2}(\mathbb{L}F_2(x),\mathbb{L}F_2(y))$$

(here Map^{eq} denotes the sub-simplicial sets of equivalences in the mapping space Map). It is easy to see that the right hand side is naturally equivalent to $Map^{eq}_M(\mathbb{L}F(x),\mathbb{L}F(y))$, and furthermore that the above morphism is equivalent to the morphism

$$Map^{eq}_{M_0}(x,y) \longrightarrow Map^{eq}_M(\mathbb{L}F(x),\mathbb{L}F(y))$$

induced by the left Quillen functor F. The fact that this morphism is an equivalence simply follows from our condition (1). $\square$

We now come back to the proof of the theorem. We need to show that

$$\begin{tikzcd} X^{(2)} \arrow[r, "h"] \arrow[d, "(s\circ f)\times k"'] & X^{(1)} \arrow[d, "s\times t"] \\ X^{(0)} \times X^{(1)} \arrow[r, "id\times c"'] & X^{(0)} \times X^{(0)} \end{tikzcd}$$

is homotopy cartesian. Simplifying the factor $X^{(0)}$, it is enough to show that the square

$$\begin{tikzcd} X^{(2)} \arrow[r, "h"] \arrow[d, "k"'] & X^{(1)} \arrow[d, "t"] \\ X^{(1)} \arrow[r, "c"'] & X^{(0)} \end{tikzcd}$$

is homotopy cartesian. For this we use lemma 4.2 applied to the following diagram of model categories and left Quillen functors

$$\begin{array}{ccc} M(T)^{(2)} & \xrightarrow{h} & M(T)^{(1)} \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle t} \\ M(T)^{(1)} & \xrightarrow[c]{} & M(T). \end{array}$$

We let M be the fiber product model category

$$M := M(T)^{(1)} \times_{M(T)} M(T)^{(1)},$$

and let us consider the induced left Quillen functor

$$F : M(T)^{(2)} \longrightarrow M$$

and its right adjoint

$$G : M \longrightarrow M(T)^{(2)}.$$

The objects in the category M are of the form $(f : a \to b, g : a' \to b', a_0; u, v)$, where f and g are morphisms in $M(T)$, a_0 is an object in $M(T)$, and

$$b/a \xrightarrow{u} a_0 \xleftarrow{v} b'$$

are morphisms in $M(T)$, where we have denoted symbolically by b/a the cofiber of the morphism f.

The functor $F : M(T)^{(2)} \longrightarrow M$ sends an object $x \longrightarrow y \longrightarrow z$, to

$$(x \to z, y/x \to z/x, z/x; id, id)$$

(note that the natural isomorphism α here is trivial). The right adjoint G to F sends $(a \to b, a' \to b', a_0; u, v)$ to $a \longrightarrow b \times_{a_0} a' \longrightarrow b \times_{a_0} b'$.

To finish the proof of theorem 4.1 let us check that the left derived functor

$$\mathbb{L}F : Ho(M(T)^{(2)}) \longrightarrow Ho(M)$$

does satisfy the two assumptions of lemma 4.2. For this, we let $A := x \longrightarrow y \longrightarrow z$ be an object in $M(T)^{(2)}$, and we consider the adjunction morphism $A \longrightarrow \mathbb{L}G \circ \mathbb{R}F(A)$. To see that it is an equivalence, it is enough to describe the values of this morphism levelwise, which are the three natural morphisms in $Ho(M(T))$

$$id : x \to x \qquad y \longrightarrow z \times^h_{(z \coprod^h_x 0)} (y \coprod^h_x 0) \qquad id : y \to y.$$

The two identities are of course isomorphisms, and the one in the middle is an isomorphism because $M(T)$ is a stable model category in the sense of [Ho, §7].

In the same way, let $B := (f : a \to b, g : a' \to b', a_0; u, v)$ be an object in M, which is cofibrant (i.e. f and g are cofibrations between cofibrants objects, and a_0 is cofibrant) and such that u and v are equivalences in $M(T)$. We consider the adjunction morphism in $Ho(M)$

$$\mathbb{L}F \circ \mathbb{R}G(B) \longrightarrow B$$

which we need to prove is an isomorphism. For this we evaluate this morphism at the various objects a, b, a', b' and a_0 to get the following morphisms in $Ho(M(T))$

$$id : a \to a \qquad id : b \to b \qquad (b \times^h_{a_0} a') \coprod_a^h 0 \longrightarrow a' \qquad (b \times^h_{a_0} b') \coprod_a^h 0 \to b' \qquad (b \times^h_{a_0} b') \coprod_a^h 0 \to a_0.$$

These morphisms are all isomorphisms in $Ho(M(T))$ as both morphisms u and v are themselves isomorphisms and moreover because $M(T)$ is a stable model category. This finishes the proof of the theorem 4.1. $\square$

5 A formula for the derived Hall numbers

The purpose of this section is to provide a formula for the numbers $g^z_{x,y}$, purely in terms of the triangulated category $Ho(M(T))$, of T^{op}-dg-modules.

Let us assume that T is a locally finite dg-category over $\mathbb{F}_q$, and recall that $Ho(M(T))$ has a natural triangulated structure whose triangles are the images of the fibration sequences (see [Ho, §7]). We will denote by $[-,-]$ the set of morphisms in $Ho(M(T))$. The shift functor on $Ho(M(T))$, given by the suspension, will be denoted by

$$\begin{array}{ccc} Ho(M(T)) & \longrightarrow & Ho(M(T)) \\ x & \mapsto & x[1] := * \coprod_x^h *. \end{array}$$

For any two objects x and y in $Ho(M(T))$, we will use the following standard notations

$$Ext^i(x,y) := [x, y[i]] \ \forall\, i \in \mathbb{Z}.$$

Let x, y and z be three perfect T^{op}-dg-modules. Let us denote by $[x,z]_y$ the sub-set of $[x,z]$ consisting of morphisms $f : x \to z$ whose cone is isomorphic to y in $Ho(M(T))$. The group $Aut(x)$, of automorphisms of x in $Ho(M(T))$ acts on the set $[x,z]_y$. For any $f \in [x,z]_y$ the stabilizer of f will be denoted by $Aut(f/z) \subset Aut(x)$. Finally, we will denote by $[x,z]_y/Aut(x)$ the set of orbits of the action of $Aut(x)$ on $[x,z]_y$.

Proposition 5.1 *With the notation above, one has*

$$\begin{aligned} g^z_{x,y} &= \sum_{f \in [x,z]_y/Aut(x)} |Aut(f/z)|^{-1}. \prod_{i>0} |Ext^{-i}(x,z)|^{(-1)^i}.|Ext^{-i}(x,x)|^{(-1)^{i+1}} \\ &= \frac{|[x,z]_y|. \prod_{i>0} |Ext^{-i}(x,z)|^{(-1)^i}}{|Aut(x)|. \prod_{i>0} |Ext^{-i}(x,x)|^{(-1)^i}}, \end{aligned}$$

where as usual $|A|$ denotes the order of the finite set A.

Proof: We consider the morphism

$$c : X^{(1)}(T) \longrightarrow X^{(0)}(T),$$

as well as its homotopy fiber F^z at the point z. The simplicial set F^z is easily seen to be naturally equivalent to the nerve of the category of equivalences in $P(T)/z$, i.e. the nerve of the category of quasi-isomorphisms between perfect T^{op}-dg-modules over z.

We denote by $F^z_{x,y}$ the full sub-simplicial set of F^z consisting of all connected components corresponding to objects $u : x' \to z$ in $P(T)/z$ such that x' is quasi-isomorphic to x, and the homotopy cofiber of u is quasi-isomorphic to y. The simplicial set $F^z_{x,y}$ is locally finite and furthermore $\pi_0(F^z_{x,y})$ is finite. By definition one has

$$g^z_{x,y} = \sum_{(u:x'\to z)\in\pi_0(F^z_{x,y})} \prod_{i>0} |\pi_i(F^z_{x,y}, u)|^{(-1)^i}.$$

It is easily seen that $\pi_0(F^z_{x,y})$ is in bijection with the set $[x, z]_y/Aut(x)$. Therefore, in order to prove the first equality of proposition 5.1 it only remains to show that for a given morphism $u : x \to z$, considered as point in $F^z_{x,y}$, one has

$$\prod_{i>0} |\pi_i(F^z_{x,y}, u)|^{(-1)^i} = |Aut(f/z)|^{-1}.\prod_{i>0} |Ext^{-i}(x, z)|^{(-1)^i}.|Ext^{-i}(x, x)|^{(-1)^{i+1}}.$$

For this, we first notice that there exists homotopy cartesian squares

$$\begin{array}{ccc} Map_{M(T)/z}(x, x) & \longrightarrow & Map_{M(T)}(x, x) \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & Map_{M(T)}(x, z), \end{array}$$

where the bottom horizontal map sends the point $*$ to the given morphism $u : x \to z$. Therefore, using the relations between mapping spaces and nerves of equivalences in model categories (see [HAGII, Appendix B]), one finds a long exact sequence of finite groups

$$Ext^{-i-1}(x, x) \longrightarrow Ext^{-i-1}(x, z) \longrightarrow \pi_{i+1}(F^z_{x,y}, u) \longrightarrow Ext^{-i}(x, x) \longrightarrow Ext^{-i}(x, z) \longrightarrow \cdots$$

$$\cdots \longrightarrow Ext^{-1}(x, x) \longrightarrow Ext^{-1}(x, z) \longrightarrow \pi_1(F^z_{x,y}, u) \longrightarrow Aut(f/z) \longrightarrow 0.$$

This implies the wanted formula

$$\prod_{i>0} |\pi_i(F^z_{x,y}, u)|^{(-1)^i} = |Aut(f/z)|^{-1}.\prod_{i>0} |Ext^{-i}(x, z)|^{(-1)^i}.|Ext^{-i}(x, x)|^{(-1)^{i+1}},$$

and thus the first equality of the proposition. Finally, the second equality follows from the general fact that for a finite group G acting on a finite set X, one has

$$\frac{|X|}{|G|} = \sum_{x\in X/G} |G_x|^{-1},$$

where G_x is the stabilizer of the point x. $\square$

For any dg-category T let us denote by $Ho(P(T))$ the full sub-category of $Ho(M(T))$ consisting of perfect T^{op}-dg-modules. It is a thick sub-category of $Ho(M(T))$, and thus inherits a natural triangulated structure.

Corollary 5.2 *Let T and T' be two locally finite dg-categories over $\mathbb{F}_q$. Any triangulated equivalence*

$$F : Ho(P(T)) \longrightarrow Ho(P(T'))$$

induces in a functorial way an isomorphism

$$F_* : \mathcal{DH}(T) \simeq \mathcal{DH}(T')$$

such that $F_*(\chi_x) = \chi_{F(x)}$.

Proof: Follows from the formula of proposition 5.1. $\square$

Corollary 5.2 implies in particular that the group of self equivalences of the triangulated category $Ho(P(T))$ acts on the algebra $\mathcal{DH}(T)$.

Remark 5.3 *Corollary 5.2 suggests that one can define the derived Hall algebra of any triangulated category (under the correct finiteness condition), simply by defining the derived Hall numbers using the formula of proposition 5.1. I do not know if this provides an associative algebra or not. However, theorem 4.1 implies this is the case for triangulated categories having a dg-model.*

6 The heart property

In this section we examine the behavior of the derived Hall algebra $\mathcal{DH}(T)$, of a dg-category T endowed with a t-structure, and the usual Hall algebra of the heart.

Let T be a locally finite dg-category. We consider $Ho(P(T)) \subset Ho(M(T))$ the full subcategory of perfect objects. It is a thick triangulated subcategory of $Ho(M(T))$. We assume that $Ho(P(T))$ is endowed with a t-structure, compatible with its triangulated structure, and we let $\mathcal{C}$ be the heart of the t-structure. Recall (e.g. from [De-Xi]) that we can construct a Hall algebra $\mathcal{H}(\mathcal{C})$ of the abelian category $\mathcal{C}$.

Proposition 6.1 *The Hall algebra $\mathcal{H}(\mathcal{C})$ is naturally isomorphic to the sub-algebra of $\mathcal{DH}(T)$ spanned by the χ_x where x lies in the heart A.*

Proof: We start show that given x, y and z three objects in T with x and y in $\mathcal{C}$, then

$$\left(g^z_{x,y} \neq 0\right) \Rightarrow z \in \mathcal{C}.$$

Indeed, if $g^z_{x,y}$ is non zero then there exists a triangle $x \to z \to y \to x[1]$ in $[T]$. As x and y lie in the heart $\mathcal{C}$ then so does z.

It only remains to show that for x, y and z in $\mathcal{C}$, our derived Hall numbers $g^z_{x,y}$ coincide with the usual ones. Using the explicit formula Prop. 5.1 we find

$$g^z_{x,y} = \frac{|[x,z]_y|}{|Aut(x)|},$$

which is equal to the number of sub-objects $x' \subset z$ such that x' is isomorphic to x and z/x' is isomorphic to y. □

Corollary 6.2 *Let A be a finite dimensional k-algebra, of finite global cohomological dimension. We let BA be the dg-category having a unique object and A as its endomorphism ring. Then, the Hall algebra $\mathcal{H}(A)$ is isomorphic to the sub-algebra of $\mathcal{DH}(BA^{op})$ generated by the χ_x's, for x a (non-dg) A-module.*

Proof: As A is of finite global cohomological dimension, an A-module M is finite dimension as a k-vector space if and only if it is a perfect complex of A-modules. Therefore, the triangulated category $Ho(P(BA^{op}))$ is nothing else than the bounded derived category of finite dimensional A-modules. We can therefore apply our theorem 4.1 to the standard t-structure on $Ho(P(BA^{op}))$, whose heart is precisely the abelian category of finite dimensional A-modules. □

An important example of application of theorem 4.1 is when T is the dg-category of perfect complexes on some smooth projective variety X over $k = \mathbb{F}_q$. This provides an associative algebra $\mathcal{DH}(X) := \mathcal{DH}(T)$, containing the Hall algebra of the category of coherent sheaves on X.

7 Derived Hall algebra of hereditary abelian categories

In this last section we will describe the derived Hall algebra of an hereditary (i.e. of cohomological dimension ≤ 1) abelian category A, in terms of its underived Hall algebra.

We let T be a locally finite dg-category over $\mathbb{F}_q$, and we assume that we are given a t-structure on the triangulated category $Ho(P(T))$, as in the previous section. We let $\mathcal{C}$ be the heart of this t-structure, and we will assume that $\mathcal{C}$ is an hereditary abelian category, or in other words that for any x and y in $\mathcal{C}$ we have $Ext^i(x,y) = 0$ for all $i > 1$ (the ext groups here are computed in $\mathcal{C}$). We will also assume that the natural triangulated functor $D^b(\mathcal{C}) \longrightarrow Ho(P(T))$ (see [B-B-D]) is an equivalence, or equivalently that the t-structure is bounded and that for any x and y in the heart we have $[x, y[i]] = 0$ for any $i > 1$ (here $[-,-]$ denotes the set of morphisms in $Ho(P(T))$). According to [De-Xi] we can construct a Hall algebra $\mathcal{H}(\mathcal{C})$, and our purpose is to describe the full derived Hall algebra $\mathcal{DH}(T)$ by generators and relations in terms of $\mathcal{H}(\mathcal{C})$.

For this, we need to introduce the following notations. For two objects x and y in $\mathcal{C}$, we set as usual

$$\langle x, y\rangle := Dim_{\mathbb{F}_q} Hom(x,y) - Dim_{\mathbb{F}_q} Ext^1(x,y).$$

For four objects x, y, c and k in $\mathcal{C}$, we define a rational number $\gamma_{x,y}^{k,c}$ as follows. We first consider $V(k,y,x,c)$, the subset of $Hom(k,y) \times Hom(y,x) \times Hom(x,c)$ consisting of all exact sequences

$$0 \longrightarrow k \longrightarrow y \longrightarrow x \longrightarrow c \longrightarrow 0.$$

The set $V(k,y,x,c)$ is finite and we set

$$\gamma_{x,y}^{k,c} := \frac{|V(k,y,x,c)|}{|Aut(x)|.|Aut(y)|}.$$

Recall finally that for three objects x, y and z in $\mathcal{C}$ we define $g_{x,y}^{z}$ as being the number of sub-objects in $x' \subset z$ such that x' is isomorphic to x and z/x' is isomorphic to y. By our proposition 6.1, this number $g_{x,y}^{z}$ is also the derived Hall number of the corresponding T^{op}-perfect dg-modules.

Proposition 7.1 *The $\mathbb{Q}$-algebra $\mathcal{DH}(T)$ is isomorphic to the associative and unital algebra generated by the set*

$$\{Z_x^{[n]}\}_{x \in \pi_0(\mathcal{C}), n \in \mathbb{Z}},$$

where $\pi_0(\mathcal{C})$ denotes the set of isomorphism classes of objects in $\mathcal{C}$, and with the following relations

1.

$$Z_x^{[n]}.Z_y^{[n]} = \sum_{z \in \pi_0(\mathcal{C})} g_{x,y}^{z}.Z_z^{[n]}$$

2.

$$Z_x^{[n]}.Z_y^{[n+1]} = \sum_{c,k \in \pi_0(\mathcal{C})} \gamma_{x,y}^{k,c}.q^{-\langle c,k \rangle}.Z_k^{[n+1]}.Z_c^{[n]}$$

3. For $n - m < -1$

$$Z_x^{[n]}.Z_y^{[m]} = q^{(-1)^{n-m}.\langle x,y \rangle}.Z_y^{[m]}.Z_x^{[n]}.$$

Proof: We let $\pi_0(P(T))$ be the set of isomorphism classes of objects in $Ho(P(T))$, and for $x \in \pi_0(P(T))$ we let $\chi_x \in \mathcal{DH}(T)$ the characteristic function of x. In the same way, for $x \in \pi_0(\mathcal{C})$ we denote by $\chi_x \in \mathcal{H}(\mathcal{C})$ the corresponding element. By our assumption on T, any object in $Ho(P(T))$ can be written as a direct sum $\oplus_n x_n[n]$, where x_n lies in the heart $\mathcal{C}$.

We construct a morphism of vector spaces

$$\Theta : B \longrightarrow \mathcal{DH}(T),$$

where B is the algebra generated by the $Z_x^{[n]}$ as described in the proposition, by setting

$$\Theta(Z_x^{[n]}) := \chi_{x[n]}.$$

We claim that Θ is an isomorphism of algebras.

To prove that Θ is a morphism of algebras we need to check that the corresponding relations in $\mathcal{DH}(T)$ are satisfied. In other words we have to prove the following three relations are satisfied

1.
$$\chi_{x[n]}\cdot\chi_{y[n]} = \sum_{z\in\pi_0(\mathcal{C})} g^z_{x,y}\cdot\chi_{z[n]}$$

2.
$$\chi_{x[n]}\cdot\chi_{y[n+1]} = \sum_{c,k\in\pi_0(\mathcal{C})} \gamma^{k,c}_{x,y}\cdot q^{-\langle c,k\rangle}\cdot\chi_{k[n+1]}\cdot\chi_{c[n]}$$

3. For $n-m<-1$

$$\chi_{x[n]}\cdot\chi_{y[m]} = q^{(-1)^{n-m}.\langle x,y\rangle}\cdot\chi_{y[m]}\cdot\chi_{x[n]}.$$

The relation (1) follows from the heart property Prop. 6.1. For relation (3), the only extension of $y[m]$ by $x[n]$ is the trivial one, as $[y[m],x[n+1]]=0$ because $n-m<-1$. In the same way, we have $[x[n],y[m+1]]=0$, because $\mathcal{C}$ is of cohomological dimension ≤ 1. Therefore, in order to check the relation (3) we only need to check that

$$g^{x[n]\oplus y[m]}_{x[n],y[m]} = q^{(-1)^{n-m}.\langle x,y\rangle} \qquad g^{x[n]\oplus y[m]}_{y[m],x[n]} = 1,$$

which is a direct consequence of the formula given in Prop. 5.1.

It remains to prove relation (2), which can be easily be rewritten as

$$\chi_{x[n]}\cdot\chi_{y[n+1]} = \sum_{c,k\in\pi_0(\mathcal{C})} \gamma^{k,c}_{x,y}\cdot q^{-\langle c,k\rangle}\cdot\chi_{k[n+1]\oplus c[n]}.$$

By shifting if necessary it is in fact enough to prove that

$$\chi_{x}\cdot\chi_{y[1]} = \sum_{c,k\in\pi_0(\mathcal{C})} \gamma^{k,c}_{x,y}\cdot q^{-\langle c,k\rangle}\cdot\chi_{k[1]\oplus c}.$$

An extension of $y[1]$ by x is classified by a morphism $y[1]\to x[1]$, and therefore by a morphism $u: y\to x$. For such a morphism u, we let c and k be defined by the long exact sequence in $\mathcal{C}$

$$0 \longrightarrow k \longrightarrow y \xrightarrow{u} x \longrightarrow c \longrightarrow 0.$$

Clearly, for a morphism $u: y\to x$, and the corresponding triangle $x \longrightarrow z \longrightarrow y[1] \xrightarrow{u[1]} x[1]$, the object z is isomorphic in $Ho(P(T))$ to $k[1]\oplus c$. Therefore, we can write

$$\chi_{x}\cdot\chi_{y[1]} = \sum_{c,k\in\pi_0(\mathcal{C})} g^{k[1]\oplus c}_{x,y[1]}\cdot\chi_{k[1]\oplus c},$$

and it only remains to show that

$$g^{k[1]\oplus c}_{x,y[1]} = \gamma^{k,c}_{x,y}\cdot q^{-\langle c,k\rangle}.$$

Applying Prop. 5.1 we have

$$g^{k[1]\oplus c}_{x,y[1]} = \frac{|[x,k[1]\oplus c]_y|}{|Aut(x)|.|[x,k]|}.$$

The set $[x, k[1] \oplus c]_y$ can be described as the set of ordered pairs (α, β), where $\alpha : x \to c$ is an epimorphism in $\mathcal{C}$, $\beta \in [x, k[1]]$ is an extension $0 \longrightarrow k \longrightarrow y' \longrightarrow x \longrightarrow 0$, such that the object y is isomorphic to $y' \times_x ker(\alpha)$. For each epimorphism $\alpha : x \to c$, we form the short exact sequence

$$0 \longrightarrow E_\alpha \longrightarrow [x, k[1]] \longrightarrow [ker(\alpha), k[1]] \longrightarrow 0.$$

We see that the set $[x, k[1] \oplus c]_y$ is in bijection with the set of triples (α, β', a), where $\alpha : x \to c$ is an epimorphism, β' is an extension class $0 \longrightarrow k \longrightarrow y' \longrightarrow ker(\alpha) \longrightarrow 0$ such that y' is isomorphic to y, and a is an element in E_α.

Let us now fix an epimorphism $\alpha : x \to c$, and let $V(k, y, \ker(\alpha))$ be the set of exact sequences

$$0 \longrightarrow k \longrightarrow y \longrightarrow ker(\alpha) \longrightarrow 0.$$

The number of possible β' is then equal to

$$\frac{|V(k, y, \ker(\alpha))|}{|Aut(y)|}.|[ker(\alpha), k]|,$$

as $[ker(\alpha), k]$ is (non-canonically) the stabilizer of any point in $V(k, y, \ker(\alpha))$ for the action of $Aut(y)$. Furthermore, considering the long exact sequence

$$0 \longrightarrow [c, k] \longrightarrow [x, k] \longrightarrow [ker(\alpha), k] \longrightarrow [c, k[1]] \longrightarrow [x, k[1]] \longrightarrow [ker(\alpha), k[1]] \longrightarrow 0$$

we see that the number of points in the set E_α is equal to

$$|E_\alpha| = |[c, k[1]]|.|[ker(\alpha), k]|^{-1}.|[x, k]|.|[c, k]|^{-1}.$$

All together, this implies that

$$g_{x,y[1]}^{k[1]\oplus c} = \frac{|[x, k[1] \oplus c]_y|}{|Aut(x)|.|[x, k]|} = \sum_{\alpha : x \twoheadrightarrow c} \frac{|V(k, y, \ker(\alpha))|}{|Aut(y)|.|Aut(x)|}.|[ker(\alpha), k]|.|[x, k]|^{-1}.|E_\alpha|$$

$$= \left(\sum_{\alpha : x \twoheadrightarrow c} \frac{|V(k, y, \ker(\alpha))|}{|Aut(y)|.|Aut(x)|} \right).|[c, k[1]]|.|[c, k]|^{-1} = \gamma_{x,y}^{k,c}.q^{-\langle c,k \rangle}.$$

This finishes the proof of the fact that Θ is a morphism of algebras.

To prove that Θ is bijective we consider the following diagram

$$\begin{tikzcd} \bigotimes_{\mathbb{Z}} \mathcal{H}(\mathcal{C}) \arrow[r, "u"] \arrow[d, "v"'] & B \arrow[dl, "\Theta"] \\ \mathcal{DH}(T), & \end{tikzcd}$$

where the infinite tensor product is the restricted one (i.e. generated by words of finite lengh). The morphisms u and v are defined by

$$u(\otimes_n \chi_{x_n}) := \prod_n^{\to} Z_{x_n}^{[n]}$$

$$v(\otimes_n \chi_{x_n}) := \chi_{\oplus_n x_n[n]},$$

where $\prod_n^{\rightarrow} Z_{x_n}^{[n]}$ means the ordered product of the elements $Z_{x_n}^{[n]}$

$$\prod_n^{\rightarrow} Z_{x_n}^{[n]} = \ldots Z_{x_i}^{[i]}.Z_{x_{i-1}}^{[i-1]}.\ldots.$$

For any two objects x and y in $Ho(P(T))$ with $x \in Ho(P(T))^{\leq 0}$ and $y \in Ho(P(T))^{>0}$ we have

$$g_{x,y}^{x\oplus y} = 1 \qquad g_{x,y}^{z} = 0 \quad if\ \chi_z \neq \chi_{x\oplus y}.$$

In other words $\chi_x \cdot \chi_y = \chi_{x\oplus y}$. This implies on the one hand that the diagram above commutes, and on the other hand that the morphism v is an isomorphism. It remains to show that the morphism u is also an isomorphism, but surjectivity is clear and injectivity follows from injectivity of v. □

It seems to us interesting to note the analogies between the description of $\mathcal{DH}(T)$ given by Prop. 7.1 and the Lattice algebra $L(T)$ of Kapranov introduced in [Ka]. The formal structures of the two algebras $\mathcal{DH}(T)$ and $L(T)$ seem very close, thought $L(T)$ contains an extra factor corresponding the *Cartan subalgebra* (i.e. a copy of the group algebra over the Grothendieck group of T). Therefore, it seems likely that there exists an embedding $j : \mathcal{DH}(T) \hookrightarrow L(T)$. However, I do not see any obvious way to construct such a embedding, and I prefer to leave the question of comparing $\mathcal{DH}(T)$ and $L(T)$ for further research.

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