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# Algebraic and topological aspects of the schematization functor

L. Katzarkov, T. Pantev, B. Toën

**Abstract**

We study some basic properties of schematic homotopy types and the schematization functor. We describe two different algebraic models for schematic homotopy types: cosimplicial Hopf alegbras and equivariant cosimplicial algebras, and provide explicit constructions of the schematization functor for each of these models. We also investigate some standard properties of the schematization functor helpful for the description of the schematization of smooth projective complex varieties. In a companion paper these results are used in the construction of a non-abelian Hodge structure on the schematic homotopy type of a smooth projective variety.

Key words: Schematic homotopy types, homotopy theory, non-abelian Hodge theory.

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# 1 Introduction

The schematization functor is a device which converts a pair $(X, k)$, consisting of topological space $X$ and field $k$, into an algebraic-geometric object $(X \otimes k)^{\mathrm{sch}}$. The characteristic property of $(X \otimes k)^{\mathrm{sch}}$ is that it encodes the $k$-linear part of the homotopy type of $X$. Namely $(X \otimes k)^{\mathrm{sch}}$ captures all the information about local systems of $k$-vector spaces on $X$ and their cohomology. For a simply connected $X$ and $k = \mathbb{Q}$ (respectively $k = \mathbb{F}_p$) the object $(X \otimes k)^{\mathrm{sch}}$ is a model for the rational (repectively $p$-adic) homotopy type of $X$. An important advantage of $(X \otimes k)^{\mathrm{sch}}$ is that it makes sense for non-simply connected $X$ and that it detects non-nilpotent information.

The object $(X \otimes k)^{\mathrm{sch}}$ belongs to a special class of algebraic $\infty$-stacks over $k$, called schematic homotopy types [To1]. The existence and functoriality of the schematization $(X \otimes k)^{\mathrm{sch}}$ are proven in [To1] but the construction is somewhat abstract and unwieldy. In this paper we supplement [To1] by describing explicit algebraic models for $(X \otimes k)^{\mathrm{sch}}$. We also study in detail some of the basic properties of the schematization - the Van Kampen theorem, the schematization of homotopy fibers, de Rham models, etc. These results are used in an essential way in [Ka-Pa-To] in which we construct mixed Hodge structures on schematizations of smooth complex projective varieties.

The paper is organized in three parts. In section 2 we briefly review the definition of schematic homotopy types and the existence results for the schematization functor from [To1]. In section 3 we present two different algebraic models for the schematization of a space, namely equivariant cosimplicial $k$-algebras and cosimplicial Hopf $k$-algebras. These generalize two well known ways for modelling rational homotopy types - via dg algebras and via nilpotent dg Hopf algebras [Ta]. Each of the two models utilizes a different facet of the homotopy theory of a space $X$. The equivariant cosimplicial $k$-algebras codify the cohomology of $X$ with $k$-local system coefficients together with their cup-product structure, whereas the cosimplicial Hopf $k$-algebra is the algebra of representative functions on the simplicial loop group associated with $X$ via Kan's construction [G-J, Section V.5]. The two models have different ranges of aplicability. For instance the cosimplicial Hopf algebra model is needed to construct the weight tower for the mixed Hodge structure (MHS) on the schematization of a smooth projective variety [Ka-Pa-To]. On the other hand the Hodge decomposition on the schematic homotopy type of a smooth projective variety is defined in terms of the equivariant cosimplicial model.

Section 4 gathers some useful facts about the behavior of the schematization functor. As an application of the equivariant cosimplicial algebra model, we describe the schematization of differentiable manifolds in terms of de Rham complexes of flat connections. This description generalizes a theorem of Sullivan's [Su] expressing the real homotopy theory of a manifold in terms of its de Rham complex. To facilitate computations we prove a schematic analogue of the van Kampen theorem which allows us to build schematizations by gluing schematizations of local pieces. Since in the algebraic-geometric setting we can not use contractible neighborhoods as the building blocks we are forced to study the schematizations of Artin neighborhoods and more generally of $K(\pi, 1)$'s. This leads to the notion of $k$-algebraically good groups which are precisely the groups $\Gamma$ with property that the schematization of $K(\Gamma, 1)$ has no higher homotopy groups. We give various examples of such groups and prove that the fundamental groups of Artin neighborhoods are algebraically good. Note that the analogous statement in rational homotopy theory is unknown and probably false. Finally we prove two exactness properties of the schematization functor. First we establish a Lefschetz type right exactness property of schematizations, useful for understanding homotopy types of hyperplane sections. We also give sufficient conditions under which the schematization

commutes with taking homotopy fibers. This criterion is used in the construction of new examples [Ka-Pa-To] of non-Kähler homotopy types.

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*Conventions:* We will fix two universes $\mathbb{U}$ and $\mathbb{V}$, with $\mathbb{U} \in \mathbb{V}$, and we assume $\mathbb{N} \in \mathbb{U}$.

We denote by $k$ a base field in $\mathbb{U}$. We consider $\mathsf{Aff}$, the category of affine schemes over Spec $k$ belonging to $\mathbb{U}$. The category $\mathsf{Aff}$ is a $\mathbb{V}$-small category. We endow it with the faithfully flat and quasi-compact topology, and consider the model category $\mathsf{SPr}(k)$ of presheaves of $\mathbb{V}$-simplicial sets on the site $(\mathsf{Aff}, \mathrm{fpqc})$. We will use the local projective model structure on simplicial presheaves described in [Bl, To1] (note that as the site $\mathsf{Aff}$ is $\mathbb{V}$-small, the model category $\mathsf{SPr}(k)$ exists). We denote by $\mathsf{SPr}(k)_*$ the model category of pointed objects in $\mathsf{SPr}(k)$. The expression *stacks* will always refer to objects in $\mathrm{Ho}(\mathsf{SPr}(k))$. In the same way, *morphisms of stacks* refers to morphisms in $\mathrm{Ho}(\mathsf{SPr}(k))$.

## 2 Review of the schematization functor

In this section, we review the theory of affine stacks and schematic homotopy types introduced in [To1]. The main goal is to recall the theory and fix the notations and the terminology.

We will denote by $\mathsf{Alg}^{\Delta}$ the category of cosimplicial (commutative) $k$-algebras that belong to the universe $\mathbb{V}$. The category $\mathsf{Alg}^{\Delta}$ is endowed with a simplicial closed model category structure for which the fibrations are the epimorphisms and the equivalences are the quasi-isomorphisms. This model category is known to be cofibrantly generated, and even finitely generated [To1, Theorem 2.1.2].

There is a natural spectrum functor

$$\mathrm{Spec} : (\mathsf{Alg}^{\Delta})^{\mathrm{op}} \longrightarrow \mathsf{SPr}(k),$$

defined by the formula

$$\begin{aligned} \mathrm{Spec}\, A \ : \quad & \mathsf{Aff}^{\mathrm{op}} \longrightarrow \mathsf{SSet} \\ & \mathrm{Spec}\, B \mapsto \underline{\mathrm{Hom}}(A, B), \end{aligned}$$

As usual $\underline{\mathrm{Hom}}(A, B)$ denotes the simplicial set of morphisms from the cosimplicial algebra $A$ to the algebra $B$. Explicitly, if $A$ is given by a cosimplicial object $[n] \mapsto A_n$, then the presheaf of $n$-simplices of $\mathrm{Spec}\, A$ is given by $(\mathrm{Spec}\, B) \mapsto \mathrm{Hom}(A_n, B)$.

The functor Spec is a right Quillen functor, and its right derived functor is denoted by

$$\mathbb{R}\,\mathrm{Spec} : \mathrm{Ho}(\mathsf{Alg}^{\Delta})^{\mathrm{op}} \longrightarrow \mathrm{Ho}(\mathsf{SPr}(k)).$$

The restriction of $\mathbb{R}\,\mathrm{Spec}$ to the full sub-category of $\mathrm{Ho}(\mathsf{Alg}^{\Delta})$ consisting of objects isomorphic to a cosimplicial algebra in $\mathbb{U}$ is fully faithful (see [To1, Corollary 2.2.3]). By defintion, an affine stack is an object $F \in \mathrm{Ho}(\mathsf{SPr}(k))$ isomorphic to an object of the form $\mathbb{R}\,\mathrm{Spec}\,A$, for some cosimplicial algebra $A$ in $\mathbb{U}$. Moreover, by [To1, Theorem 2.4.1,2.4.5] the following conditions are equivalent for a given pointed stack $F$:

1. The pointed stack $F$ is affine and connected.
2. The pointed stack $F$ is connected and for all $i > 0$ the sheaf $\pi_i(F, *)$ is represented by an affine unipotent group scheme.
3. There exist a cohomologically connected cosimplicial algebra $A$ (i.e. $H^0(A) \simeq k$), which belongs to $\mathbb{U}$, and such that $F \simeq \mathbb{R}\,\mathit{Spec}\,A$.

Recall next that for a pointed simplicial presheaf $F$, one can define its simplicial presheaf of loops $\Omega_* F$. The functor $\Omega_* : \mathsf{SPr}_*(k) \longrightarrow \mathsf{SPr}(k)$ is right Quillen, and can be derived to a functor defined on the level of homotopy categories

$$\mathbb{R}\Omega_* F : \mathrm{Ho}(\mathsf{SPr}_*(k)) \longrightarrow \mathrm{Ho}(\mathsf{SPr}(k)).$$

A pointed and connected stack $F \in \mathrm{Ho}(\mathsf{SPr}_*(k))$ is called a *pointed affine $\infty$-gerbe* if the loop stack $\mathbb{R}\Omega_* F \in \mathrm{Ho}(\mathsf{SPr}(k))$ is affine. A pointed schematic homotopy type is a pointed affine $\infty$-gerbe which in addition satisfies a cohomological condition (see [To1, Def. 3.1.2] for details).

The main result on affine stacks that we need is the existence theorem of [To1]. Embed the category $\mathsf{SSet}$ into the category $\mathsf{SPr}(k)$ by viewing a simplicial set $X$ as a constant simplicial presheaf on $(\mathsf{Aff}, \mathrm{ffqc})$. With this convention we have the following important definition:

**Definition 2.1** *([To1, Definition* 3.3.1*]) Let $X$ be a pointed and connected simplicial set in $\mathbb{U}$. The schematization of $X$ over $k$ is a pointed schematic homotopy type $(X \otimes k)^{\mathrm{sch}}$, together with a morphism*

$$u : X \longrightarrow (X \otimes k)^{\mathrm{sch}}$$

*in* $\mathrm{Ho}(\mathsf{SPr}_*(k))$ *which is a universal for morphisms from $X$ to pointed schematic homotopy types (in the category* $\mathrm{Ho}(\mathsf{SPr}_*(k))$*).*

We have stated the above definition only for simplicial sets in order to simplify the exposition. However, by using the *singular functor* Sing, attaching to each topological space $T$ the simplicial set of singular chains in $T$ (see e.g. [Ho] for details), one can define the schematization of a pointed connected topological space. In what follows we will always assume implicitly that the functor Sing has been applied when necessary and we will generally not distinguish between topological spaces and simplicial sets when considering the schematization functor.

Finally, recall the main existence theorem.

**Theorem 2.2** *([To1, Theorem 3.3.4]) Any pointed and connected simplicial set $(X, x)$ in $\mathbb{U}$ possesses a schematization over $k$. Furthermore, for any $i > 0$ the sheaf $\pi_i((X \otimes k)^{\mathrm{sch}}, x)$ is represented by an affine group scheme, which is commutative and unipotent for $i > 1$.*

Let $(X, x)$ be a pointed connected simplicial set in $\mathbb{U}$, and let $(X \otimes k)^{\mathrm{sch}}$ be its schematization. Then, one has:

1. The affine group scheme $\pi_1((X \otimes k)^{\mathrm{sch}}, x)$ is naturally isomorphic to the pro-algebraic completion of the discrete group $\pi_1(X, x)$ over $k$.

2. Let $V$ be a local system of finite dimensional $k$-vector spaces on $X$. In particular $V$ corresponds to a linear representation of $\pi_1((X \otimes k)^{\mathrm{sch}}, x)$ and gives rise to a local system $\mathcal{V}$ on $(X \otimes k)^{\mathrm{sch}}$. Then there is a natural isomorphism
$$H^\bullet(X, V) \simeq H^\bullet((X \otimes k)^{\mathrm{sch}}, \mathcal{V}),$$

3. If $X$ is simply connected and of finite type (i.e. the homotopy type of a simply connected finite $CW$ complex), then for any $i > 1$, the group scheme $\pi_i((X \otimes k)^{\mathrm{sch}}, x)$ is naturally isomorphic to the pro-unipotent completion of the discrete groups $\pi_i(X, x)$. In other words, for any $i > 1$
$$\begin{aligned} \pi_i((X \otimes k)^{\mathrm{sch}}, x) &\simeq \pi_i(X, x) \otimes_{\mathbb{Z}} \mathbb{G}_a && \text{if } \mathrm{char}(k) = 0, \\ \pi_i((X \otimes k)^{\mathrm{sch}}, x) &\simeq \pi_i(X, x) \otimes_{\mathbb{Z}} \mathbb{Z}_p && \text{if } \mathrm{char}(k) = p > 0. \end{aligned}$$
Here the groups $\pi_i(X, x)$ appearing in the right hand side are thought of as constant group schemes over $k$.

# 3 Algebraic models

In this section we discuss two different algebraic models for pointed schematic homotopy types: cosimplicial commutative Hopf algebras and equivariant cosimplicial commutative algebras. These two models give rise to two different explicit formulas for the schematization of a space, having their own advantages and disavantages according to the situation. Our first model will allow us to complete the proof of [To1, Theorem 3.2.9].

We will start by introducing an intermediate model category structure on the category of simplicial affine $k$-group schemes. This model category structure will then be localized in order to get the right homotopy category of of cosimplicial Hopf algebras suited for the setting of schematic homotopy type. We will present this intermediate model category in a first separate section as we think that it might have some independant interests.

## 3.1 Simplicial affine group schemes

By a *Hopf algebra* we will mean a *unital and co-unital commutative Hopf $k$-algebra.* The category of Hopf algebras will be denoted by $\mathsf{Hopf}$, which is therefore equivalent to the opposite of the category $\mathsf{GAff}$ of affine $k$-group schemes. Recall that every Hopf algebra is equal to the colimit of

its Hopf subalgebras of finite type (see [De-Ga, III, §3, No. 7]). In particular the category $\mathsf{Hopf}$ is the category of ind-objects in the category of of Hopf algebras of finite type and thus is complete and co-complete [SGA4.1, Exposé 1,Proposition 8.9.1(b)]. We consider the category of cosimplicial Hopf algebras $\mathsf{Hopf}^{\Delta}$, dual to the category of simplicial affine group schemes $\mathsf{sGAff}$. When we will need to specify universes we will write $\mathsf{Hopf}_{\mathbb{U}}$, $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$, $\mathsf{GAff}_{\mathbb{U}}$, $\mathsf{sGAff}_{\mathbb{U}}$ ....

The category $\mathsf{GAff}_{\mathbb{U}}$ has all $\mathbb{U}$-small limits and colimits. In particular, the category of simplicial objects $\mathsf{sGAff}_{\mathbb{U}}$ is naturally endowed with tensor and co-tensor structures over the category $\mathsf{SSet}_{\mathbb{U}}$ of $\mathbb{U}$-small simplicial sets. By duality, the category $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ has also a natural tensor and co-tensor structure over $\mathsf{SSet}_{\mathbb{U}}$. For $X \in \mathsf{SSet}_{\mathbb{U}}$ and $B_* \in \mathsf{Hopf}^{\Delta}$, $G_* = \mathrm{Spec}\, B_*$, we will use the standard notations [G-J, Theorem 2.5]:

$$X \otimes B_*, \qquad B_*^X, \qquad X \otimes G_*, \qquad G_*^X.$$

Explicitly for a simplicial affine group scheme $G_*$ and a simplicial set $X$ we have $(X \otimes G)_n = \coprod_{X_n} G_n$, where the coproduct is taken in the category of affine group schemes. To describe $G^X$ we first define $(G^X)_0$ as the equalizer

$$(G^X)_0 \longrightarrow \prod_n G_n^{X_n} \underset{b}{\overset{a}{\rightrightarrows}} \prod_{p \xrightarrow{u} q} G_q^{X_p}.$$

Here $a$ is the composition $\prod_n G_n^{X_n} \twoheadrightarrow G_p^{X_p} \to G_q^{X_p}$ where the second map is induced from $u$. Similarly $b$ is the composition $\prod_n G_n^{X_n} \twoheadrightarrow G_q^{X_q} \to G_q^{X_p}$ where the second map is induced from $u$. With this definition we can now set $(G^X)_n = (G^{X \times \Delta_n})_0$. The definitions of $X \otimes B_*$ and $B_*^X$ are analogous. Note that

$$\mathrm{Spec}\,(X \otimes B_*) \simeq G_*^X \qquad \text{and} \qquad \mathrm{Spec}\,(B_*^X) \simeq X \otimes B_*.$$

For any simplicial set $K$ and any simplicial affine group scheme $G_*$, we will use the following notation

$$Map(K, G_*) := (G_*^K)_0.$$

Note that $Map(\Delta^n, G_*) \simeq G_n$.

Let $n \geq 0$. We consider the simplicial sphere $S^n := \partial\Delta^{n+1}$, pointed by the vertex $0 \in \Delta^{n+1}$. There is a natural morphism of affine group schemes

$$Map(S^n, G_*) \longrightarrow Map(*, G_*) \simeq G_0.$$

The kernel of this morphism will be denoted by $Map_*(S^n, G_*)$. In the same way, the kernel of the morphism

$$Map(\Delta^{n+1}, G_*) \longrightarrow Map(*, G_*) = G_0$$

will be denoted by $Map_*(\Delta^{n+1}, G_*)$. The inclusion $S^n \subset \Delta^{n+1}$ induces a morphism

$$Map_*(\Delta^{n+1}, G_*) \longrightarrow Map_*(S^n, G_*).$$

The cokernel of this morphism, taken in $\mathsf{sGAff}$, will be denoted by $\Pi_n(G_*)$.

For any $n$ and any $0 \leq k \leq n$, we denote by $\Lambda^{n,k}$ the $k$-th horn of $\Delta^n$, which by definition obtained from $\partial\Delta^n$ by removing its $k$-th face. For a morphism $G_* \longrightarrow H_*$ in $\mathsf{sGAff}$ we then have a morphism of affine group schemes

$$Map(\Delta^n, G_*) \longrightarrow Map(\Lambda^{n,k}, G_*) \times_{Map(\Lambda^{n,k}, G_*)} Map(\Delta^n, H_*),$$

or equivalently

$$G_n \longrightarrow Map(\Lambda^{n,k}, G_*) \times_{Map(\Lambda^{n,k}, G_*)} H_n.$$

**Definition 3.1** *Let $f : G_* \longrightarrow H_*$ be a morphism in $\mathsf{sGAff}$.*

1. *The morphism $f$ is an* equivalence *if for all $n$ the induced morphism*

$$\Pi_n(G_*) \longrightarrow \Pi_n(H_*)$$

   *is an isomorphism.*

2. *The morphism $f$ is a* fibration *if for all $n$ and all $0 \leq k \leq n$ the induced morphism*

$$G_n \longrightarrow Map(\Lambda^{n,k}, G_*) \times_{Map(\Lambda^{n,k}, G_*)} H_n$$

   *is a faithfully flat morphism of schemes.*

3. *The morphism $f$ is a* cofibration *if it has the left lifting property with respect to all morphisms which are fibrations and equivalences.*

It is useful to notice here that a morphism of affine group schemes $G \longrightarrow H$ is faithfully flat if and only if the induced morphism of Hopf algebras $\mathcal{O}(H) \longrightarrow \mathcal{O}(G)$ is injective (see [De-Ga, III, §3, No. 7]).

**Theorem 3.2** *The above definition makes $\mathsf{sGAff}_{\mathbb{U}}$ into a model category.*

*Proof:* We are going to apply the theorem [Ho, Thm. 2.1.19] to the opposite category $\mathsf{sGAff}_{\mathbb{U}}^{op} = \mathsf{Hopf}_{\mathbb{U}}^{\Delta}$. For this, we let $I$ be a set of representative of all morphisms in $\mathsf{sGAff}_{\mathbb{U}}$ which are fibrations $G_* \longrightarrow H_*$ such that for any $n \geq 0$ the group schemes $G_n$ and $H_n$ are of finite type (as schemes over $Spec\, k$). In the same way, we let $J \subset$ be the subset corresponding to morphisms in $I$ which are also equivalences. We need to prove that the category $\mathsf{sGAff}_{\mathbb{U}}^{op} = \mathsf{Hopf}_{\mathbb{U}}^{\Delta}$, the class $W$ of equivalences, and the sets $I$ and $J$ satisfy the following seven conditions.

1. The subcategory $W$ has the two out of the three property and is closed under retracts.
2. The domains and codomains of $I$ are small (relative to $I$-cell).
3. The domains and codomains of $J$ are small (relative to $J$-cell).
4. $J - cell \subset W \cap I - cof$.
5. $I - inj \subset W \cap J - inj$.
6. $W \cap I - cell \subset J - cof$.

These properties will prove the existence of a model category structure on $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ whose equivalences are the one defined in 3.1, and whose cofibrations are generated by the set $I$. Before going futher in the proof of the properties above we check that the cofibrations generated by the set $I$ are precisely the one given in definition 3.1 (note that by duality fibrations in $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ correspond to cofibrations in $\mathsf{sGAff}_{\mathbb{U}}$ and conversly).

**Lemma 3.3** *1. A morphism $G_* \longrightarrow H_*$ in $\mathsf{sGAff}_{\mathbb{U}}$ such that for each $n$, the morphism $G_n \longrightarrow H_n$ is faithfully flat is a fibration.*

*2. A morphism in $\mathsf{sGAff}$ has the left lifting property with respect to $I$ if and only if it has the left lifting property with respect every fibration.*

*Proof of lemma 3.3:* (1) Let $K_*$ be the kernel of the morphism $G_* \longrightarrow H_*$. Let $0 \leq k \leq n$ and set

$$L_n := H_n \times_{Map(\Lambda^{n,k}, H_*)} Map(\Lambda^{n,k}, G_*).$$

We have a commutative diagram of affine group schemes

$$\begin{tikzcd} K_n \arrow[r] \arrow[d] & G_n \arrow[r] \arrow[d] & H_n \arrow[d, "id"] \\ Map(\Lambda^{n,k}, K_*) \arrow[r] & L_n \arrow[r] & H_n. \end{tikzcd}$$

Using a version of the five lemma we see that it is enough to prove that $H_n \longrightarrow Map(\Lambda^{n,k}, K_*)$ is faithfully flat. In other words, we can assume that $H_* = \{e\}$.

The simplicial presheaf $h_{G_*}$ represented by $G_*$ is a presheaf in simplicial groups on the site of all affine schemes. It is a globally fibrant simplicial presheaf (see [Ma, Thm. 17.1]). As the functor $G_* \mapsto h_{G_*}$ commutes with exponential by simplicial sets (because it commutes with arbitrary limits) we see that this implies that for $0 \leq n \leq n$ the morphism

$$G_n \longrightarrow Map(\Lambda^{n,k}, G_*)$$

induces a surjective morphism on the associated presheaves. As $Map(\Lambda^{n,k}, G_*)$ is an affine scheme this implies that this morphism has in fact a section, and thus is faithfully flat by [De-Ga, III. §3, No. 7].

(2) Let $G_* \longrightarrow H_*$ be a morphism having the left lifting property with respect to $I$, and let $K_* \longrightarrow L_*$ be a fibration together with a commutative diagram

$$\begin{tikzcd} G_* \arrow[r] \arrow[d] & K_* \arrow[d] \\ H_* \arrow[r] & L_*. \end{tikzcd}$$

We consider the factorisation $K_* \longrightarrow K'_* \longrightarrow L_*$ into an faithfully flat followed by an injective morphism. The morphism $K'_* \longrightarrow L_*$ stays a fibration, as shown by the following commutative

square with faithfully flat rows

$$\begin{array}{ccc} K_n & \longrightarrow & K'_n \\ \downarrow & & \downarrow \\ Map(\Lambda^{n,k}, K_*) \times_{Map(\Lambda^{n,k}, L_*)} L_n & \longrightarrow & Map(\Lambda^{n,k}, K'_*) \times_{Map(\Lambda^{n,k}, L_*)} L_n. \end{array}$$

Moreover, the morphism $K_* \longrightarrow K'_*$ also is a fibration because of part (1) of the lemma 3.3. This implies that we are reduced to treat two cases, either $K_* \longrightarrow L_*$ is faithfully flat, or it is injective.

We start to treat the case where $K_* \longrightarrow L_*$ is injective. In this case, the induced morphism of simplicial sheaves

$$h_{K_*} \longrightarrow h_{L_*}$$

is a monomorphism and a local fibration. The local lifting property with respect to the inclusion $* \hookrightarrow \Delta^1$ and using that the morphism is mono implies that the induced morphism $\pi_0(h_{K_*}) \longrightarrow \pi_0(h_{L_*})$ is a monomorphism of sheaves. As monomorphisms and local fibrations are stable by exponentiation by a finite simplicial set we also see that the induced morphism $\pi_i(h_{K_*}) \longrightarrow \pi_i(h_{L_*})$ is a monomorphism of sheaves for all $i \geq 0$. Finally, the local lifting property for the inclusion $* \hookrightarrow \Delta^n$ and the fact the morphism is mono also implies that the induced morphism $\pi_n(h_{K_*}) \longrightarrow \pi_n(h_{L_*})$ is surjective for all $n > 0$. In other words, the following square

$$\begin{array}{ccc} h_{K_*} & \longrightarrow & h_{L_*} \\ \downarrow & & \downarrow \\ \pi_0(h_{K_*}) & \longrightarrow & \pi_0(h_{L_*}) \end{array}$$

is cartesian. Equivalently, the following square

$$\begin{array}{ccc} K_* & \longrightarrow & L_* \\ \downarrow & & \downarrow \\ \Pi_0(K_*) & \longrightarrow & \Pi_0(L_*) \end{array}$$

is also cartesian. Therefore, in order to prove the existence of a lifting $H_* \longrightarrow K_*$ we can replace $K_*$ by $\Pi_0(K_*)$ and $L_*$ by $\Pi_0(L_*)$. We are therefore reduced to prove the following fact about affine group schemes: if $p : G \longrightarrow H$ is a morphism of affine group schemes having the left lifting property with respect to every morphism between affine group schemes of finite type, then $p$ is an isomorphism. This last assertion follows easily from the fact that the category of affine group schemes is the category of pro-objects in the category of affine group schemes of finite type.

We now assume that $K_* \longrightarrow L_*$ is faithfully flat. We consider a certain set $X$. Its elements are cosimplicial Hopf sub-algebras $A_* \subset \mathcal{O}(K_*)$ such that:

- the morphism $\mathcal{O}(L_*) \longrightarrow \mathcal{O}(K_*)$ factors throught $A_*$,
- there exists a morphism $H_* \longrightarrow Spec\, A_*$ making the diagram

$$\begin{array}{ccc} G_* & \longrightarrow & Spec\, A_* \\ \downarrow & \nearrow & \downarrow \\ H_* & \longrightarrow & L_* \end{array}$$

commutative.

This set is non-empty as the image of $\mathcal{O}(L_*) \hookrightarrow \mathcal{O}(K_*)$ is an element of $X$. We next order $X$ by the order induced by inclusion of cosimplicial Hopf sub-algebras. The ordered set $X$ is inductive, and we let $A_* \subset \mathcal{O}(K_*)$ be a maximal element. Assume that $A_* \neq \mathcal{O}(K_*)$. We chose a lift $H_* \longrightarrow Spec\, A_*$, and we consider the following commutative diagram

$$\begin{tikzcd} G_* \arrow[r] \arrow[d] & K_* \arrow[d] \\ H_* \arrow[r] & Spec\, A_*. \end{tikzcd}$$

As $A_* \neq \mathcal{O}(K_*)$, there exist a cosimplicial Hopf subalgebra $D_* \subset \mathcal{O}(K_*)$, such that for any $n$ the algebra $D_n$ is of finite type, and with $A_* \not\subseteq D_*$. We let $D'_* := A_* \cap D_*$. Finally, we let $B_*$ be the cosimplicial Hopf sub-algebra of $\mathcal{O}(K_*)$ which is generated by $D_*$ and $A_*$. There exists a commutative diagram

$$\begin{tikzcd} G_* \arrow[r] \arrow[d] & K_* \arrow[r] \arrow[d] & Spec\, B_* \arrow[r] \arrow[dl] & Spec\, D_* \arrow[d] \\ H_* \arrow[r] & Spec\, A_* \arrow[rr] & & Spec\, D'_*, \end{tikzcd}$$

and the square on the right hand side is furthermore cartesian. Finally, as $D'_* \longrightarrow D_*$ is injective, $Spec\, D_* \longrightarrow Spec\, D'_*$ is a fibration (because of part (1) of lemma 3.3) between simplicial affine group schemes of finite type. By assumption a lift $H_* \longrightarrow Spec\, B_*$ exists. But this contredicts the maximality of $A_*$. Therefore, $A_* = \mathcal{O}(K_*)$ and a lift $H_* \longrightarrow K_*$ exists. $\square$

Let us now check that the properties (1) to (5) above are satisfied. The property (1) is true as equivalences are the morphisms inducing isomorphisms on $\Pi_n$ (and isomorphisms of group schemes do have the two out of the three property and are closed under retracts). The points (2) and (3) are true as any object in the category of Hopf algebras is small (with respect to some cardinal depending of this object) with respect to the whole category, therefore the domains and codomains of $I$ and $J$ are also small with respect to the whole category $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$.

Let us now prove (4). We have $J \subset I \subset I - cof$, and therefore $J - cell \subset I - cof$ because $I - cof$ is stable by push-outs and transfinite compositions. In order to prove that $J - cell \subset W$ it is enough to prove the following two properties:

(a) The trivial fibrations in $\mathsf{sGAff}_{\mathbb{U}}$ are stable by base-change.

(b) The trivial fibrations in $\mathsf{sGAff}_{\mathbb{U}}$ are stable by ($\mathbb{U}$-small) filtered limits.

Let

$$\begin{tikzcd} G'_* \arrow[r] \arrow[d, "f'"'] & G_* \arrow[d, "f"] \\ H'_* \arrow[r] & H_* \end{tikzcd}$$

be a cartesian diagram in $\mathsf{sGAff}_{\mathbb{U}}$. For any simplicial set $K$, the diagram

$$\begin{array}{ccc} Map(K, G'_*) & \longrightarrow & Map(K, G_*) \\ \downarrow & & \downarrow f \\ Map(K, H'_*) & \longrightarrow & Map(K, H_*) \end{array}$$

is a cartesian diagram of affine group schemes. Therefore, for any $n$ and $0 \leq k \leq n$ we have a cartesian diagram of affine group schemes

$$\begin{array}{ccc} G'_n & \longrightarrow & G_n \\ \downarrow & & \downarrow \\ Map(\Lambda^{n,k}, G'_*) \times_{Map(\Lambda^{n,k}, G'_*)} H'_n & \longrightarrow & Map(\Lambda^{n,k}, G_*) \times_{Map(\Lambda^{n,k}, G_*)} H_n. \end{array}$$

As the faithfully flat morphisms are stable by base change this implies that $f'$ is a fibration if $f$ is so.

We consider the functor $h : \mathsf{sGAff}_{\mathbb{U}} \longrightarrow \mathsf{SPr}_*(k)$, sending a simplicial affine group scheme $G_*$ to the simplicial presheaf $X \mapsto Hom(X, G_*)$, pointed at the unit of $G_0$. As the functor sending an affine group scheme to the sheaf of groups it represents commutes with limits and quotients (see [De-Ga, III, §3, No. 7]), we see that the homotopy sheaves of $\pi_n(h_{G_*})$ are representable by the group schemes $\Pi_n(G_*)$. Moreover, a morphism $f : G_* \longrightarrow H_*$ is a fibration our sense if and only if the induced morphism $h_{G_*} \longrightarrow h_{H_*}$ is a local fibration in the sense of [J, §1] (i.e. satisfies the local right lifting property with respcect to $\Lambda^{n,k} \subset \Delta^n$). In particular, this morphism of simplicial presheaves provides a long exact sequence on homotopy sheaves when $f$ is a fibration (see [J, Lemma 1.15]). We deduce from this that if $K_*$ denotes the kernel of the morphism $f$, and if $f$ is a fibration, then there exists a long exact sequence of affine group schemes

$$\ldots \Pi_{n+1}(H_*) \longrightarrow \Pi_n(K_*) \longrightarrow \Pi_n(G_*) \longrightarrow \Pi_n(H_*) \longrightarrow \Pi_{n-1}(K_*) \longrightarrow \cdots .$$

Using this fact we deduce that a fibration in $\mathsf{sGAff}_{\mathbb{U}}$ is an equivalence if and only if its kernel $K_*$ is acyclic (i.e; $\Pi_n(K_*) = 0$). As kernels are stable by base change, we see that this implies that trivial fibrations are also stable by base change. This proves the property $(a)$ above.

In order to prove the property $(b)$, let $G_* = lim_\alpha G_*^{(\alpha)}$ be a filtered limit of objects in $\mathsf{sGAff}_{\mathbb{U}}$.

**Lemma 3.4** *For any $n \geq 0$, the natural morphism*

$$\Pi_n(G_*) \longrightarrow lim_\alpha \Pi_n(G_*^\alpha)$$

*is an isomorphism.*

*Proof of the lemma 3.4:* For any $n$, we have a cocartesian square of affine group schemes

$$\begin{array}{ccc} Map_*(\partial\Delta^{n+1}, G_*) & \longrightarrow & Map_*(\Delta^{n+1}, G_*) \\ \downarrow & & \downarrow \\ \{e\} & \longrightarrow & \Pi_n(G_*). \end{array}$$

As the functor $Map(K, G_*)$ commutes with limits, we are reduced to show that filtered limits of affine group schemes preserves cocartesian squares. Let

$$\begin{array}{ccc} E^{(\alpha)} & \longrightarrow & F^{(\alpha)} \\ \downarrow & & \downarrow \\ G^{(\alpha)} & \longrightarrow & H^{(\alpha)} \end{array}$$

be a filtered diagram of cocartesian squares in $\mathsf{GAff}$, et let

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array}$$

be the limit diagram. We need to prove that for any $G_0 \in \mathsf{sGAff}$ the induced diagram of sets

$$\begin{array}{ccc} Hom(H, G_0) & \longrightarrow & Hom(F, G_0) \\ \downarrow & & \downarrow \\ Hom(G, G_0) & \longrightarrow & Hom(E, G_0) \end{array}$$

is cartesian. As any affine group scheme $G_0$ is the projective limit of its quotients of finite type we can assume that $G_0$ is of finite type. But then, $Hom(-, G_0)$ sends filtered limits to filtered colimits, as the Hopf algebra corresponding to $G_0$ is of finite type as an algebra and thus is a compact object in the category of Hopf algebras (i.e. $Hom(\mathcal{O}(G_0), -)$ commutes with filtered colimits). Therefore, the above diagram of sets is isomorphic to

$$\begin{array}{ccc} colim_\alpha Hom(H^{(\alpha)}, G_0) & \longrightarrow & colim_\alpha Hom(F^{(\alpha)}, G_0) \\ \downarrow & & \downarrow \\ colim_\alpha Hom(G^{(\alpha)}, G_0) & \longrightarrow & colim_\alpha Hom(E^{(\alpha)}, G_0). \end{array}$$

This last diagram is cartesian because filtered colimits in sets preserve finite limits. □

The previous lemma implies that equivalences in $\mathsf{sGAff}_{\mathbb{U}}$ are stable by filtered limits, and in particular by transfinite composition on the left. Moreover, faithfully flat morphisms of affine group schemes are also stable by filtered limits, because injective morphisms of Hopf algebras are stable by filtered colimits. By definition of fibrations this implies that the fibrations are also stable by filtered limits. This finishes the proof of the point $(b)$ above. The property (4) is thus proven.

Before proving the last two properties (5) and (6) we need some additional lemmas.

**Lemma 3.5** *A morphism $G_* \longrightarrow H_*$ in $\mathsf{sGAff}_{\mathbb{U}}$ is a fibration if and only if for any trivial cofibration $A \subset B$ of $\mathbb{U}$-small simplicial sets the natural morphism*

$$Map(B, G_*) \longrightarrow Map(A, G_*) \times_{Map(A,H_*)} Map(B, H_*)$$

*is faithfully flat.*

*Proof of the lemma 3.5:* This follows from the definition, from the fact that the morphisms $\Lambda^{n,k} \subset \Delta^n$ generate the trivial cofibrations of simplicial sets (see [Ho, Definition 3.2.1]), and from the fact that faithfully flat morphisms of affine group schemes are stable by pull-backs and filtered compositions (because injective morphisms of Hopf algebras are stable by filtered colimits). $\square$

**Lemma 3.6** *Let $G_* \in \mathsf{sGAff}_{\mathbb{U}}$ and $A \subset B$ be a cofibration of $\mathbb{U}$-small simplicial sets. Then morphism $G_*^B \longrightarrow G_*^A$ is a fibration.*

*Proof of the lemma 3.6:* This follows from the formula

$$Map(K, G_*^A) \simeq Map(A \times K, G_*),$$

from the lemma 3.5, and from the fact that for any $n$ and any $k$ the natural morphism

$$(A \times \Delta^n) \coprod_{A \times \Lambda^{n,k}} B \times \Lambda^{n,k} \longrightarrow B \times \Delta^n$$

is a trivial cofibration of simplicial sets. $\square$

We have $J \subset I$ and thus $I - inj \subset J - inj$. In order to prove (5) we need to prove that $I - inj \subset W$. Let $i : G_* \longrightarrow H_*$ be a morphism in $\mathsf{sGAff}_{\mathbb{U}}$ having the left lifting property with respect to $I$. By the lemma 3.3 we know that it also has the left lifting property with respect to all fibrations. Moreover, lemma 3.3 (1) implies that $G_*$ itself is fibrant, and the lifting property for the diagram

$$\begin{array}{ccc} G_* & \longrightarrow & G_* \\ \downarrow & & \downarrow \\ H_* & \longrightarrow & \{e\} \end{array}$$

implies the existence of $r : H_* \longrightarrow G_*$ such that $r \circ i = id$. The lemma 3.6 implies that $H_*^{\Delta^1} \longrightarrow H_*^{\partial\Delta^1} = H^{\Delta^1} \times H^{\Delta^1}$ is fibration. Therefore, the lifting property for the diagram

$$\begin{array}{ccc} G_* & \xrightarrow{i} & H_*^{\Delta^1} \\ \downarrow & & \downarrow \\ H_* & \xrightarrow[id, i\circ r]{} & H_* \times H_*, \end{array}$$

implies the existence of a morphism

$$h : H_* \longrightarrow H_*^{\Delta^1}$$

which is a homotopy between the identity and $i \circ r$. This implies that $i : h_{G_*} \longrightarrow h_{H_*}$ is a homotopy equivalence of simplicial presheaves, and thus induces isomorphisms on homotopy sheaves. As we have already seen the homotopy sheaves of $h_{G_*}$ and $h_{H_*}$ are representable by the homotopy group schemes of $G_*$ and $H_*$. Therefore, $i$ itself is an equivalence in $\mathsf{sGAff}_{\mathbb{U}}$.

It only remains to prove that the property (5) is satisfied. But as we have already seen during the proof of point (4) the morphism in $I - cell$ are fibrations in $\mathsf{sGAff}$ (because fibrations are stable

by base change and filtered limits). In particular any morphism in $I-cell \cap W$ is a trivial fibration. Because of lemma 3.3 this implies that $I - cell \cap W \subset J - cof$. This finishes the proof of the theorem 3.2. $\square$

Before continuing further we need to make an important remark concerning the dependance of the universe $\mathbb{U}$ for the model category $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$. If $\mathbb{U} \in \mathbb{V}$ are two universes, one gets a natural inclusion functor $\mathsf{Hopf}^{\Delta}_{\mathbb{U}} \longrightarrow \mathsf{Hopf}^{\Delta}_{\mathbb{V}}$. An important fact about this inclusion is that the sets $I$ and $J$ defined in the proof of the previous proposition are independant of the choice of the universe. Therefore, the sets $I$ and $J$ are generating set of cofibrations and trivial cofibrations for both model categories $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ and $\mathsf{Hopf}^{\Delta}_{\mathbb{V}}$. As we know that a morphism is a cofibration if and only if it is a retract of a relative $I$-cell complex (see [Ho, Def. 2.1.9] and [Ho, Prop. 2.1.18 (b)]), we see that a morphism in $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ is a cofibration if and only if it is a cofibration in $\mathsf{Hopf}^{\Delta}_{\mathbb{V}}$. In the same way, fibrations are the morphisms having the right lifting property with respect to $J$, and thus a morphism in $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ is a fibration if and only if it is a fibration in $\mathsf{Hopf}^{\Delta}_{\mathbb{V}}$. Moreover, using the functorial factorization of cosimplicial algebras via the small object argument with respect to $I$ and $J$ (as described in the proof above) one sees that the full sub-category $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ is stable by the functorial factorizations in $\mathsf{Hopf}^{\Delta}_{\mathbb{V}}$. In particular, a path object (see [Ho, Def. 1.2.4 (2)]) in $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ is also a path object in $\mathsf{Hopf}^{\Delta}_{\mathbb{V}}$. As a consequence, we see that two morphisms in $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ between fibrant and cofibrant objects are homotopic (see [Ho, Def. 1.2.4 (5)]) if and only if they are homotopic in $\mathsf{Hopf}^{\Delta}_{\mathbb{V}}$.

A consequence of these remarks and of the fact that the homotopy category of a model category is equivalent to the category of fibrant and cofibrant objects and homotopy classes of objects between them (see [Ho, Thm. 1.2.10]) is that the induced functor

$$\mathrm{Ho}(\mathsf{Hopf}^{\Delta}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(\mathsf{Hopf}^{\Delta}_{\mathbb{V}})$$

is fully faithful. The essential image of this functor consists of all cosimplicial Hopf algebras in $\mathbb{V}$ which are equivalent to a cosimplicial Hopf algebra in $\mathbb{U}$.

The conclusion of this short discussion is that increasing the size of the ambient universe does not destroy the homotopy theory of cosimplicial Hopf algebras, and its only effect is to add more objects to the corresponding homotopy category.

An important additional property of the model category $\mathsf{sGAff}_{\mathbb{U}}$ is its right properness.

**Corollary 3.7** *The model category* $\mathsf{sGAff}_{\mathbb{U}}$ *is right proper (i.e. equivalences are stable by pull-backs along fibrations).*

*Proof:* This follows from lemma 3.3 (1), as it implies in particular that any object in $\mathsf{sGAff}_{\mathbb{U}}$ is fibrant (see [Hir, Corollary 13.1.3]). This could also have been checked directly using the long exact sequence in homotopy for fibrations (see the proof of point (4) of the theorem 3.2). $\square$

Another interesting property is the compatibility between the model structure and the simplicial enrichement.

**Corollary 3.8** *Together with its natural simplicial enrichement, the model category* $\mathsf{sGAff}_{\mathbb{U}}$ *is a simplicial model category.*

*Proof:* This is a consequence of lemma 3.5. Indeed, we need to verify the axiom M7 of [Hir, Definition 9.1.6]. For this, let $i : A \subset B$ be a cofibration of $\mathbb{U}$-small simplicial sets, and $f : G_* \longrightarrow H_*$ a fibration in $\mathsf{sGAff}_{\mathbb{U}}$. The fact that

$$G_*^B \longrightarrow G_*^A \times_{H_*^A} H_*^B$$

is a again fibration follows from the definition, from lemma 3.5 and from the fact that the morphism

$$B \times \Lambda^{n,k} \coprod_{A \times \Lambda^{n,k}} B \times \Delta^n \longrightarrow B \times \Delta^n$$

is a trivial cofibration.

If moreover, $i$ or $f$ is an equivalence then, because of [J, Corollary 1.5] and [J, Lemma 1.1.5], the morphism

$$h_{G_*} \longrightarrow h_{G_*^A \times_{H_*^A} H_*^B} \simeq h_{G_*}^A \times_{h_{H_*}^A} h_{H_*}^B$$

is a local equivalence. This implies that

$$G_*^B \longrightarrow G_*^A \times_{H_*^A} H_*^B$$

is an equivalence. □

The simplicial Homs of the simplicial category $\mathsf{SGAff}_{\mathbb{U}}$ will be denoted by $\underline{Hom}$. By proposition 3.8 these simplicial Homs possess derived version (see [Ho, §4.3])

$$\mathbb{R}\underline{Hom}(-,-) : Ho(\mathsf{sGAff}_{\mathbb{U}})^{op} \times Ho(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow Ho(\mathsf{SSet}_{\mathbb{U}}).$$

## 3.2 The P-localization

Let $K$ be an affine group scheme of finite type over $k$, and $V$ be a finite dimensional linear representation of $K$. We consider $V$ as an affine group scheme (its group law being the addition), as well as the simplicial affine group scheme $K(V,n)$ for $n \geq 0$. By definition $K(V,n)$ is the classifying space of $K(V,n-1)$, as defined for instance in [To1, §1.3], and we set $K(V,0) = V$. The group scheme $K$ acts on $V$ and therefore acts on $K(V,n)$. We will denote by $K(K,V,n)$ the simplicial group scheme which is the semi-direct product of $K$ (considered as a constant simplicial object) by $K(V,n)$. We therefore have a split exact sequence of simplicial affine group schemes

$$1 \longrightarrow K(V,n) \longrightarrow K(K,V,n) \longrightarrow K \longrightarrow 1.$$

Ideally, we would like now to construct the left Bousfield localization of $\mathsf{sGAff}_{\mathbb{U}}$ with respect to the set of objects $K(K,V,n)$, for all $K$, $V$ and $n$. Dually, this would correspond to perform a right Bousfield localization of $\mathsf{Hopf}_{\mathbb{U}}^{\Delta}$ with respect to the corresponding set of objects. The only general result insuring the existence of a right Bousfield localization we are aware about is the theorem [Hir, Theorem 5.1.1] which requires the model category to be cellular. Unfortunately the model category $\mathsf{Hopf}_{\mathbb{U}}^{\Delta}$ is not cellular, as cofibrations are simply not monomorphisms. It is therefore unclear that the localized model structure exists (we think it does). In this section we will show that the existence of a localization functor

$$(-)^P : Ho(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow Ho^P(\mathsf{sGAff}_{\mathbb{U}}),$$

which will be enough to prove the equivalence between pointed schematic homotopy types and cosimplicial Hopf algebras up to P-equivalences.

We start by defining our new equivalences in $\mathsf{sGAff}_{\mathbb{U}}$. We will see later that they are precisely the quasi-isomorphisms of cosimplicial Hopf algebras (see Corollary 3.17).

**Definition 3.9** *A morphism $f : G_* \longrightarrow H_*$ is a* P-equivalence *if for any affine group scheme of finite type $K$, any linear representation of finite dimensional $V$ of $K$, and any $n \geq 1$ the induced morphism*

$$f^* : \mathbb{R}\underline{Hom}(H_*, K(K,V,n)) \longrightarrow \mathbb{R}\underline{Hom}(G_*, K(K,V,n))$$

*is an isomorphism in $Ho(\mathsf{SSet}_{\mathbb{U}})$.*

The definition above gives the new class of equivalences on $\mathsf{sGAff}_{\mathbb{U}}$. The localization of the category $\mathsf{sGAff}_{\mathbb{U}}$ with respect to $P$-equivalences will be denoted by

$$Ho^P(\mathsf{sGAff}_{\mathbb{U}}).$$

As an equivalence is a P-equivalence we have a natural functor

$$l : Ho(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow Ho^P(\mathsf{sGAff}_{\mathbb{U}}).$$

**Proposition 3.10** *The above functor*

$$l : Ho(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow Ho^P(\mathsf{sGAff}_{\mathbb{U}})$$

*possesses a right adjoint*

$$j : Ho^P(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow Ho(\mathsf{sGAff}_{\mathbb{U}})$$

*which is fully faithful. The essential image of $j$ is of the smallest full sub-category of $Ho(\mathsf{sGAff}_{\mathbb{U}})$ containing the objects $K(K,V,n)$ and which is stable by homotopy limits.*

*Proof:* This an application of the existence of a cellularization functor applied to the model category $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ and to set of objects $\{K(K,V,n)\}$ (see [Hir, Proposition 5.2.3, 5.2.4]). For the convenience of the reader we reproduce the argument here.

We let $X$ be a set of representative for the morphisms

$$K(K,V,n)^{\Delta^m} \longrightarrow K(K,V,n)^{\partial\Delta^m},$$

for all affine group scheme of finite type $K$, all finite dimensional linear representation $V$ of $K$, and all integers $n \geq 1$ and $m \geq 0$. Because of Corollary 3.8 all the morphisms in $X$ are fibrations. For a given cofibrant object $G_* \in \mathsf{sGAff}_{\mathbb{U}}$ we construct a tower of cofibrant objects in $G_*/\mathsf{sGAff}_{\mathbb{U}}$

$$G_* \longrightarrow \dots G_*^{(i)} \longrightarrow G_*^{(i-1)} \longrightarrow \cdots \longrightarrow G_*^{(0)} = *,$$

defined inductively in the following way. We let $I_i$ be the set of all commutative squares in $\mathsf{sGAff}_{\mathbb{U}}$

$$\begin{array}{ccc} G_* & \longrightarrow & H_* \\ \downarrow & & \downarrow u \\ G_*^{(i-1)} & \longrightarrow & H'_*, \end{array}$$

where $u \in X$. The set $I_i$ is then $\mathbb{U}$-small.

We now define an object $G_*(i)$ by the pull-back square

$$\begin{array}{ccc} G_*(i) & \longrightarrow & \prod_{j\in I_i} H_* \\ \downarrow & & \downarrow \\ G_*^{(i-1)} & \longrightarrow & \prod_{j\in I_i} H'_*. \end{array}$$

We let $G_* \to G_*^{(i)} = Q(G_*(i))$ be the cofibrant replacement of $F \to G_*(i)$ in $G_*/\mathsf{sGAff}_{\mathbb{U}}$. This defines the tower inductively on $i$. Finally, we consider the morphism

$$\alpha : G_* \longrightarrow \widetilde{G_*} := \mathrm{Lim}_i\, G_*^{(i)}.$$

We first claim that the object $\widetilde{G_*}$ is P-local, in the sense that for any P-equivalence $H_* \longrightarrow H'_*$ the induced morphism

$$\mathbb{R}\underline{Hom}(H'_*, G_*) \longrightarrow \mathbb{R}\underline{Hom}(H_*, G_*)$$

is an isomorphism in $Ho(\mathsf{SSet})$. Indeed, by construction it is a $\mathbb{U}$-small homotopy limit of P-local objects.

It then only remains to see that the morphism $\alpha$ is a $P$-equivalence, as this would imply formally that $G_* \longrightarrow \widetilde{G_*}$ is a P-localization (i.e. un universal P-equivalence with a P-local object). This in turn would imply the result as the functor $G_* \mapsto \widetilde{G_*}$ would then identify the localization $Ho^P(\mathsf{sGAff}_{\mathbb{U}})$ with the full sub-category of P-local objects in $Ho(\mathsf{sGAff}_{\mathbb{U}})$. Moreover, the construction of $\widetilde{G_*}$ shows that the P-local objects are obtained by successive homotopy limits of objects of the form $K(K, V, n)$.

Let us then consider $K(K, V, n)$ for a given affine group scheme of finite type $K$, a finite dimensional linear representation $V$ of $K$ and an integer $n \geq 1$. Using that the simplicial affine group scheme $K(K, V, n)$ is levelwise of finite type and is $n$-truncated we see that it is a $\omega$-cosmall object in $\mathsf{sGAff}_{\mathbb{U}}$. Moreover, as $\widetilde{G_*}$, $G_*^{(i)}$ and $G_*$ are all cofibrant the morphism

$$\alpha^* : \mathbb{R}\underline{\mathrm{Hom}}(\widetilde{G_*}, K(K, V, n)) \longrightarrow \mathbb{R}\underline{\mathrm{Hom}}(\widetilde{G_*}, K(K, V, n))$$

is isomorphic in $\mathrm{Ho}(\mathsf{SSet})$ to the natural morphism

$$\alpha^* : \mathrm{Colim}_i\, \underline{\mathrm{Hom}}(G_*^{(i)}, K(K, V, n)) \longrightarrow \underline{\mathrm{Hom}}(G_*, K(K, V, n)).$$

Thus, by the inductive construction of the tower, we deduce from this that for any $m$, the morphism

$$\mathbb{R}\underline{\mathrm{Hom}}(\widetilde{G_*}, K(K, V, n))^{\partial\Delta^m} \longrightarrow \mathbb{R}\underline{\mathrm{Hom}}(\widetilde{G_*}, K(K, V, n))^{\Delta^m} \times^h_{\mathbb{R}\underline{\mathrm{Hom}}(G_*, K(K,V,n))^{\Delta^m}} \mathbb{R}\underline{\mathrm{Hom}}(G_*, K(K, V, n))^{\partial\Delta^m}$$

is surjective on connected components. This implies that $\alpha^*$ is an isomorphism, and therefore that $G_* \longrightarrow \widetilde{G_*}$ is a $P$-equivalence as required. $\square$

Using the previous proposition, we will always implicitely identify the category $Ho^P(\mathsf{sGAff}_{\mathbb{U}})$ with the full sub-category of $Ho(\mathsf{sGAff}_{\mathbb{U}})$ consisting of P-local objects, and also with the smallest full sub-category of $Ho(\mathsf{sGAff}_{\mathbb{U}})$ containing the $K(K, V, n)$ and which is stable by homotopy limits. The left adjoint of the inclusion functor

$$Ho^P(\mathsf{sGAff}_{\mathbb{U}}) \hookrightarrow Ho(\mathsf{sGAff}_{\mathbb{U}})$$

is isomorphic to the localization functor, and can be identified with the construction $G_* \mapsto \widetilde{G_*}$ given during the proof of the theorem. We will more often denote this functor by

$$G_* \mapsto G_*^P.$$

We will give now a more explicit description of the P-equivalences related to Hochschild cohomology of affine group schemes with coefficients in linear representations. For a Hopf algebra $B$ and a $B$-comodule $V$ (most of the time assumed to be of finite dimension but this is not needed), one can consider the cosimplicial $k$-vector space

$$\begin{array}{rccc} C^*(B,V): & \Delta & \longrightarrow & k\text{-}\mathsf{Vect} \\ & [n] & \mapsto & C^n(B,V) := V \otimes B^{\otimes n}, \end{array}$$

where the transitions morphisms $V \otimes B^{\otimes n} \longrightarrow V \otimes B^{\otimes m}$ are given by the co-action and co-unit morphisms. From a dual point of view, $V$ corresponds to a linear representation of the affine group scheme $G = \mathit{Spec}\, B$, and can be considered as a quasi-coherent sheaf $\mathbb{V}$ on the simplicial affine scheme $BG$. The cosimplicial space $C^*(B,V)$ is by definition the cosimplicial space of sections $\Gamma(BG, \mathbb{V})$ of this sheaf on $BG$.

The cosimplicial vector space $C^*(B,V)$ has an associated total complex, whose cohomology groups will be denoted by

$$H^i(B,V) := H^i(\mathrm{Tot}(C^*(B,V))).$$

These are the Hochschild cohomology groups of $B$ with coefficients in $V$. From a dual point of view, the complex $\mathrm{Tot}(C^*(B,V))$ also computes the cohomology of the affine group scheme $G$ with coefficients in the linear representation $V$. We will also use the notations

$$C^*(G,V) := C^*(B,V) \qquad H^*(G,V) := H^*(B,V).$$

Let now $B_* \in \mathsf{Hopf}_{\mathbb{U}}^{\Delta}$ be a cosimplicial Hopf algebra, and let us consider the Hopf algebra $H^0(B_*)$ of 0-th cohomology of $B_*$. We have $\mathit{Spec}\, H^0(B_*) \simeq \Pi_0(G_*)$, where $G_* := \mathrm{Spec}\ B_*$. By construction $H^0(B_*)$ is the limit (in the category of Hopf algebras) of the cosimplicial diagram $[n] \mapsto B_n$, and therefore comes equipped with a natural co-augmentation $H^0(B_*) \to B_*$. In particular, if $V$ is any $H^0(B_*)$-comodule, $V$ can also be considered naturally as comodule over each $B_n$. We get in this way a cosimplicial object in the category of cosimplicial vector spaces (i.e. a bi-cosimplicial vector space)

$$[n] \mapsto C^*(B_n, V).$$

We will denote the diagonal associated to this bi-cosimplicial space by

$$C^*(B_*, V) := \mathrm{Diag}\left([n] \mapsto C^*(B_n, V)\right) := \left([n] \mapsto C^n(B_n, V)\right).$$

If we denote by $G_* = \mathrm{Spec}\ B_*$ the associated simplicial affine group scheme, we will also use the notation

$$C^*(G_*, V) := C^*(B_*, V).$$

The cohomology of the total complex associated to $C^*(B_*, V)$ is called the *Hochschild cohomology* of the cosimplicial algebra $B_*$ (or equivalently of the simplicial affine group scheme $G_* = \mathrm{Spec}\ B_*$) with coefficients in the comodule $V$, and is denoted by

$$H^n(B_*, V) := H^n(\mathrm{Tot}(C^*(B_*, V))) \qquad H^n(G_*, V) := H^n(\mathrm{Tot}(C^*(G_*, V))).$$

We see that $C^*(G_*, V)$ is the cosimplicial space of sections of $V$ considered as a quasi-coherent sheaf on $h_{BG_*}$, represented by the simplicial affine scheme $BG_*$. As the sheaf $V$ is quasi-coherent we have natural isomorphisms

$$H^i(Tot(C^*(G_*, V))) \simeq H^i(h_{BG_*}, V) := \pi_0(Map_{SPr(k)/h_{BG_*}}(h_{BG_*}, F(V, i))),$$

where $F(V, i) \longrightarrow h_{BG_*}$ is the relative Eleinberg-MacLane construction on the sheaf of abelian groups $V$. This can be easily deduced from the special case of a non-simplicial affine group scheme treated in [To1, §1.3, §1.5], simply by noticing that $h_{BG_*}$ is naturally equivalent to the homotopy colimit of the $h_{BG_n}$ when $n$ varies in $\Delta^{op}$.

**Proposition 3.11** *A morphism $f : G_* \longrightarrow H_*$ is a P-equivalence if and only if it satisfies the following two properties.*

1. *For any finite dimensional linear representation $V$ of $\Pi_0(H_*)$ the induced morphism*

$$f^* : H^*(H_*, V) \longrightarrow H^*(G_*, V)$$

*is an isomorphism.*

2. *The induced morphism $\Pi_0(G_*) \longrightarrow \Pi_0(H_*)$ is an isomorphism.*

*Proof:* Let $G_* \in \mathsf{sGAff}_{\mathbb{U}}$, $K$ an affine group scheme of finite type, and $V$ a finite dimensional linear representation of $K$. We assume that $G_*$ is a cofibrant object. Then, there exists a natural morphism of simplicial sets

$$\underline{Hom}(G_*, K(K, V, n)) \longrightarrow \underline{Hom}_*(Bh_{G_*}, Bh_{K(K,V,n)}),$$

where $\underline{Hom}_*$ denotes the simplicial set of morphisms of the category $\mathsf{SPr}_*(k)$. Composing with the fibrant and cofibrant replacement functors in $\mathsf{SPr}_*(k)$ we get this way a natural morphism in $Ho(\mathsf{SSet}_*)$

$$\mathbb{R}\underline{Hom}(G_*, K(K, V, n)) \longrightarrow \mathbb{R}\underline{Hom}_*(Bh_{G_*}, Bh_{K(K,V,n)}).$$

This morphism comes equiped with a natural projection to the set $Hom(\Pi_0(G_*), K)$

$$\begin{tikzcd}
\mathbb{R}\underline{Hom}(G_*, K(K, V, n)) \arrow[rr] \arrow[dr] & & \mathbb{R}\underline{Hom}_*(Bh_{G_*}, Bh_{K(K,V,n)}) \arrow[dl] \\
& Hom(\Pi_0(G_*), K). &
\end{tikzcd}$$

Moreover, the homotopy fiber $F$ of the right hand side morphism is such that

$$\pi_i(F) \simeq H^{n-i}(G_*, V).$$

This shows that in order to prove the lemma it is enough to prove that the horizontal morphism (recall that $G_*$ is cofibrant, and that $K(K, V, n)$ is always fibrant)

$$\underline{Hom}(G_*, K(K, V, n)) \simeq \mathbb{R}\underline{Hom}(G_*, K(K, V, n)) \longrightarrow \mathbb{R}\underline{Hom}_*(Bh_{G_*}, Bh_{K(K,V,n)})$$

is an isomorphism.

For this, we use the fact that $\underline{Hom}(G_*, K(K,V,n))$ is naturally isomorphic to the total space (see [Hir, Definition 18.6.3]) of the cosimplicial space $m \mapsto \underline{Hom}(G_m, K(K,V,n))$

$$\mathbb{R}\underline{Hom}(G_*, K(K,V,n)) \simeq \underline{Hom}(G_*, K(K,V,n)) \simeq Tot(m \mapsto \underline{Hom}(G_m, K(K,V,n))).$$

In the same way, the simplicial presheaf $Bh_{G_*}$ is equivalent to the homotopy colimit of the diagram $n \mapsto Bh_{G_n}$, and we thus have

$$\mathbb{R}\underline{Hom}_*(Bh_{G_*}, Bh_{K(K,V,n)}) \simeq Holim_n \mathbb{R}\underline{Hom}_*(Bh_{G_m}, Bh_{K(K,V,n)}).$$

Therefore, in order to prove the lemma we need to prove the following two separate statements.

(a) The natural morphism (see [Hir, Definition 18.7.3])

$$Tot\,(m \mapsto \underline{Hom}(G_m, K(K,V,n))) \longrightarrow Holim\,(m \mapsto \underline{Hom}(G_m, K(K,V,n)))$$

is an isomorphism in $Ho(\mathsf{SSet})$.

(b) For any $m$, the natural morphism

$$\underline{Hom}(G_m, K(K,V,n)) \longrightarrow \mathbb{R}\underline{Hom}_*(Bh_{G_m}, Bh_{K(K,V,n)})$$

is an isomorphism in $Ho(\mathsf{SSet})$.

For property $(a)$ we use that $K(K,V,n)$ are abelian group objects in the category of simplicial affine group schemes over $K$. This implies that the morphism

$$Tot\,(m \mapsto \underline{Hom}(G_m, K(K,V,n))) \longrightarrow Holim\,(m \mapsto \underline{Hom}(G_m, K(K,V,n)))$$

is a morphism of abelian group objects in the category of simplicial sets over the set $Hom(G_m; K)$, or in other words is a morphism between disjoint union of simplicial abelian groups. The property $(a)$ above then follows from the fact that for any cosimplicial object $X_*$ in the category of simplicial abelian groups, the natural morphism

$$Tot(X_*) \longrightarrow Holim(X_*)$$

is a weak equivalence (see e.g. [Bo-Ka, X §4.9, XI §4.4]).

The property $(b)$ is clear for $n = 0$, as we simply find a bijection of sets

$$Hom(G_m, K \rtimes V) \simeq Hom_*(Bh_{G_m}, Bh_{K \rtimes V}).$$

Let us assume that $n > 0$. Because we have

$$\pi_i(\underline{Hom}(G_m, K(K,V,n))) \simeq \pi_{i-1}(\underline{Hom}(G_m, K(K,V,n-1)))$$

$$\pi_i(\mathbb{R}\underline{Hom}_*(Bh_{G_m}, Bh_{K(K,V,n)})) \simeq \pi_{i-1}(\mathbb{R}\underline{Hom}_*(Bh_{G_m}, Bh_{K(K,V,n-1)}))$$

it is enough by induction to prove that the natural morphism

$$\pi_0(\underline{Hom}(G_m, K(K,V,n))) \longrightarrow \pi_0(\mathbb{R}\underline{Hom}_*(Bh_{G_m}, Bh_{K(K,V,n)}))$$

is bijective. To prove this it is enough to prove that both morphisms

$$\pi_0(\underline{Hom}(G_m, K(K,V,n))) \longrightarrow Hom(G_m, K)$$

$$\pi_0(\mathbb{R}\underline{Hom}_*(Bh_{G_m}, Bh_{K(K,V,n)})) \longrightarrow Hom(G_m, K)$$

are isomorphisms. As the natural projection $K(K,V,n) \longrightarrow K$ posses a section these two morphisms are surjective. It remains to show that these morphisms are injective. This in turns will follow from the following lemma.

**Lemma 3.12** *Let $G_*$ be a cofibrant simplicial affine group scheme. Then, for any $m \geq 0$ and any faithfully flat morphism of affine group schemes $H \longrightarrow H'$ the induced morphism*

$$Hom(G_m, H') \longrightarrow Hom(G_m, H)$$

*is surjective.*

*Proof of the lemma:* We consider the evaluation functor $H_* \mapsto H_m$, and its right adjoint

$$i_m^* : \mathsf{GAff}_{\mathbb{U}} \longrightarrow \mathsf{sGAff}_{\mathbb{U}}.$$

We have

$$i_m^*(H)_p \simeq H^{Hom([m],[p])},$$

showing that $i_m^*$ preserves faithfully flat morphisms. Moreover, we have for any simplicial set $A$

$$Map(A, i_m^*(H)) \simeq H^{A_m},$$

which easily implies that $\Pi_i(i_m^*(H)) = 0$ for any $i$. In particular, if $H \longrightarrow H'$ is a faithfully flat morphism we see from lemma 3.3 (12) that the induced morphism

$$i_m^*(H) \longrightarrow i_m^*(H')$$

is a trivial fibration. The right lifting property for $G_*$ with respect to this last morphism then precisely says that $Hom(G_m, H') \longrightarrow Hom(G_m, H)$ is surjective. □

The previous lemma implies that for any linear representation $V$ of $G_m$ we have $H^2(G_m, V) = 0$, as this group classifies extensions

$$1 \longrightarrow V \longrightarrow J \longrightarrow G_m \longrightarrow 1.$$

This in turns implies that $H^i(G_m, V)$ for any $i > 1$, and thus that the natural morphism

$$\pi_0(\mathbb{R}\underline{Hom}_*(Bh_{G_m}, Bh_{K(K,V,n)})) \longrightarrow Hom(G_m, K)$$

is injective, as its fibers are in bijections with $H^n(G_m, V)$. For the other morphism, we consider the morphism of affine group schemes

$$Map(\Delta^1, K(K,V,n)) \longrightarrow Map(\Delta^1, K) \times_{Map(\partial\Delta^1, K)} Map(\partial\Delta^1, K(K,V,n)).$$

As the morphism $K(K,V,n) \longrightarrow K$ is a fibration (because of lemma 3.3 (1)) and is relatively (1)-connected (i.e. its fibers are 1-connected), the above morphism is faithfully flat. The right lifting property of $G_m$ with respect to this morphism (which is insured by the sub-lemma 3.12) implies that the morphism

$$\pi_0(\underline{Hom}(G_m, K(K,V,n))) \longrightarrow Hom(G_m, K)$$

is injective. This finishes the proof of the proposition 3.11. □

From the proof of the proposition 3.11 we also extract the following important corollary that will be used in the sequel.

**Corollary 3.13** *Let $G_* \in \mathsf{sGAff}_{\mathbb{U}}$, $K$ be a affine group scheme of finite type, $V$ a finite dimensional linear representation of $K$, and $n \geq 1$. Then the natural morphism*

$$\mathbb{R}\underline{Hom}(G_*, K(K,V,n)) \longrightarrow \mathbb{R}\underline{Hom}_{\mathsf{SGp(k)}}(h_{G_*}, h_{K(K,V,n)}) \simeq \mathbb{R}\underline{Hom}_{\mathsf{SPr_*(k)}}(Bh_{G_*}, Bh_{K(K,V,n)})$$

*is an isomorphism in $Ho(\mathsf{SSet})$.*

### 3.3 Cosimplicial Hopf algebras and schematic homotopy types

We are now ready to explain how cosimplicial Hopf algebras are models for schematic homotopy types.

We now consider $\mathsf{SGp}(k)$, the category of presheaves of $\mathbb{V}$-simplicial groups on $(\mathsf{Aff}, \mathrm{fpqc})$. It will be endowed with the model category structure for which equivalences and fibrations are defined via the forgetful functor $\mathsf{SGp}(k) \longrightarrow \mathsf{SPr}(k)$: a morphism in $\mathsf{SGp}(k)$ is a fibration and/or an equivalence if and only if it is so as a morphism in $\mathsf{SPr}(k)$ (by forgetting the group structure). We consider the Yoneda functor

$$h_{-} : \mathsf{sGAff}_{\mathbb{U}} \longrightarrow \mathsf{SGp}(k).$$

The functor $h$ sends equivalences to local equivalences of simplicial presheaves and thus defines above possesses a functor

$$h : Ho(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow Ho((SGp)(k)).$$

Consider the classifying space functor

$$\boldsymbol{B} : \mathsf{SGp}(k) \longrightarrow \mathsf{SPr}(k)_*,$$

from the category of presheaves of simplicial groups to the category of pointed simplicial presheaves. It is well known (see for example [To1, Theorem 1.4.3] and [To2, Proposition 1.5]) that this functor preserves equivalences and induces a fully faithful functor on the homotopy categories

$$\boldsymbol{B} : \mathrm{Ho}(\mathsf{SGp}(k)) \longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(k)).$$

Composing with the functor

$$h : \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(\mathsf{SGp}(k))$$

one gets a functor

$$\boldsymbol{B}h : \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(k)).$$

With this notations we now have:

**Theorem 3.14** *The functor*

$$\boldsymbol{B}h : \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(k))$$

*is fully faithful when restricted to the full sub-category* $\mathrm{Ho}^P(\mathsf{sGAff}_{\mathbb{U}}) \subset \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}})$ *consisting of P-local objects. Its essential image consists precisely of all psht.*

**Proof:** We first analyze the full faithfulness properties of $\boldsymbol{B}h$. Let $G_*$ and $G'_*$ be two P-local and cofibrant simplicial affine group schemes. We need to prove that the natural morphism of simplicial sets

$$\underline{\mathrm{Hom}}(G_*, G'_*) \to \underline{\mathrm{Hom}}_{\mathsf{SPr}_*(k)}(\boldsymbol{B}h_{G'}, \boldsymbol{B}h_{G'_*}) \to \mathbb{R}\underline{\mathrm{Hom}}_{\mathsf{SPr}_*(k)}(\boldsymbol{B}h_{G'}, \boldsymbol{B}h_{G'_*})$$

is a weak equivalence. As the functor $B$ is fully faithful, it is enough to show that the induced morphism

$$\underline{\mathrm{Hom}}(G_*, G'_*) \to \mathbb{R}\underline{\mathrm{Hom}}_{\mathsf{SGp}(k)}(h_{G'}, h_{G'_*})$$

is an equivalence. As $G'_*$ is P-local, it can be written as a transfinite composition of homotopy pull-backs of objects of the form $K(K, V, n)$ (see [Hir, Theorem 5.1.5], and also the proof of proposition 3.10), for some affine group scheme of finite type $K$, some finite dimensional linear representation $V$ of $K$ and some integer $n \geq 1$.

**Lemma 3.15** *The functor*

$$h : \mathsf{sGAff}_{\mathbb{U}} \longrightarrow \mathsf{SGp}(k)$$

*commutes with homotopy limits of P-local objects.*

*Proof of the lemma 3.15:* It is enough to show that $h$ preserves homotopy pull-backs and homotopy products (possibly infinite) of P-local objects. The case of homotopy pull-backs follows from [J, Lemma 1.15] as we have already seen that the functor $h$ sends fibrations to local fibrations of simplicial presheaves. The case of infinite homotopy product of P-local objects would follow from the fact that $h$ commutes with products and sends P-local objects to fibrant objects in $\mathsf{SGp}(k)$. As the P-local objects are obtained by transfinite composition of homotopy pull-backs of objects of the form $K(K, V, n)$ it is enough to check that $h_{K(K,V,n)}$ is a fibrant simplicial presheaf. But, as a simplicial presheaf it is isomorphic to $h_K \times K(\mathbb{G}_a, n)^d$, where $d$ is the dimension of $V$, which is known to be fibrant as both $h_K$ and $K(\mathbb{G}_a, n)$ are fibrant (see e.g. [To1, Lemma 1.1.2]). □

Using the lemma 3.15 we are reduced to prove that the natural morphism

$$\underline{\mathrm{Hom}}(G_*, K(K, V, n)) \longrightarrow \mathbb{R}\underline{\mathrm{Hom}}_{\mathsf{SPr}_*(k)}(\boldsymbol{B}h_{G'}, \boldsymbol{B}h_{K(K,V,n)})$$

is an equivalence. But this is something we have already seen during the proof of proposition 3.11 (see corollary 3.13).

It remains now to prove that the image of the functor $\boldsymbol{B}h$ consists precisely of all psht. The fact that the image of $\boldsymbol{B}h$ is contained in the category of psht follows from [To1, Corollary 3.2.7] and the fact that the P-local affine group schemes are generated by homotopy limits by the objects $K(K, V, n)$ (note that $\boldsymbol{B}h_{K(K,V,n)}$ is a psht with $\pi_1 = K$, $\pi_n = V$ and $\pi_i = 0$ for $i \neq n$). Now suppose that $F$ is a psht. We will show that $F$ is $\boldsymbol{P}$-equivalent to an object of the form $Bh\,G_*$ for some P-local object $G_* \in \mathsf{sGAff}_{\mathbb{U}}$ (see [To1, Definition 3.1.1] for the notion of $\boldsymbol{P}$-local equivalences).

Since by definition a psht is a $\boldsymbol{P}$-local object, this implies that $F$ is isomorphic to $Bh\, G_*$ as required. For this, we assume that $F$ is cofibrant as an object in $\mathsf{SPr}_*(k)$, and we construct a $\boldsymbol{P}$-local model of $F$ by the small object argument. Let $\mathscr{K}$ be a $\mathbb{U}$-small set of representatives of all psht of the form $\boldsymbol{B}h_{K(K,V,n)}$ for some affine group scheme $K$ which is of finite type as a $k$-algebra, some finite dimensional linear representation $V$ of $K$, and for some $n \geq 1$. We consider the $\mathbb{U}$-small set $K$ of all morphisms of the form

$$G^{\Delta^m} \longrightarrow G^{\partial\Delta^m},$$

for $G \in \mathscr{K}$ and $m \geq 0$. Finally, we let $\Lambda(K)$ be the $\mathbb{U}$-small set of morphisms in $\mathsf{SPr}_*(k)$ which are fibrant approximations of the morphisms of $K$. We construct a tower of cofibrant objects in $F/\mathsf{SPr}_*(k)$

$$F \longrightarrow \dots F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_0 = *,$$

defined inductively in the following way. We let $I_i$ be the set of all commutative squares in $\mathsf{SPr}_*(k)$

$$\begin{array}{ccc} F & \longrightarrow & G_1 \\ \downarrow & & \downarrow u \\ F_{i-1} & \longrightarrow & G_2, \end{array}$$

where $u \in \Lambda(K)$. Let $J_i$ be a sub-set in $I_i$ of representative for the isomorphism classes of objects in the homotopy category of commutative squares in $\mathsf{SPr}_*(k)$ (i.e. $J_i$ is a subset of representative for the equivalent classes of squares in $I_i$). Note that the set $J_i$ is $\mathbb{U}$-small since all stacks in the previous diagrams are affine $\infty$-gerbes.

We now define an object $F_i'$ as the pull-back square

$$\begin{array}{ccc} F_i' & \longrightarrow & \prod_{j\in J_i} G_1 \\ \downarrow & & \downarrow \\ F_{i-1} & \longrightarrow & \prod_{j\in J_i} G_2, \end{array}$$

and finally $F \to F_i = Q(F_i')$ as a cofibrant replacement of $F \to F_i$. This defines the $i$-step of the tower from its $(i-1)$-step. Finally, we consider the morphism

$$\alpha : F \longrightarrow \widetilde{F} := \mathrm{Lim}_i\, F_i.$$

We first claim that the object $\widetilde{F} \in \mathrm{Ho}(\mathsf{SPr}_*(k))$ lies in the essential image of the functor $\boldsymbol{B}\,\mathrm{Spec}$. Indeed, it is a $\mathbb{U}$-small homotopy limit of objects belonging to this essential image, and as the functor $\boldsymbol{B}h$ is fully faithful and commutes with homotopy limits of P-local objects (see 3.15) we see that $\widetilde{F}$ stays in the essential image of $\boldsymbol{B}h$. It remains to see that the morphism $\alpha$ is a $\boldsymbol{P}$-equivalence. For this, let $F_0$ be a psht of the form $\boldsymbol{B}h_{K(K,V,n)}$. Using that $\widetilde{F}$ and $F_0$ are both in the image of $\boldsymbol{B}h$, we see that the morphism

$$\alpha^* : \mathbb{R}\underline{\mathrm{Hom}}(\widetilde{F}, F_0) \longrightarrow \mathbb{R}\underline{\mathrm{Hom}}(F, F_0)$$

is isomorphic in $\mathrm{Ho}(\mathsf{SSet})$ to the natural morphism

$$\alpha^* : \mathrm{Colim}_i\, \mathbb{R}\underline{\mathrm{Hom}}(F_i, F_0) \longrightarrow \mathbb{R}\underline{\mathrm{Hom}}(F, F_0).$$

Thus, by inductive construction of the tower, it is then clear that for any $m$, the morphism

$$\mathbb{R}\underline{\mathrm{Hom}}(\widetilde{F}, F_0)^{\partial\Delta^m} \longrightarrow \mathbb{R}\underline{\mathrm{Hom}}(\widetilde{F}, F_0)^{\Delta^m} \times^h_{\mathbb{R}\underline{\mathrm{Hom}}(F,F_0)^{\Delta^m}} \mathbb{R}\underline{\mathrm{Hom}}(F, F_0)^{\partial\Delta^m}$$

is surjective on connected components. This implies that $\alpha^*$ is an isomorphism, and therefore that $F \longrightarrow \widetilde{F}$ is a $\boldsymbol{P}$-equivalence. □

The following corollary completes the characterization of psht given in [To1, Theorem 3.2.4,Proposition 3.2.9].

**Corollary 3.16** *An object $F \in \mathrm{Ho}(\mathsf{SPr}_*(k))$ is a psht if and only if $\pi_1(F,*)$ is an affine group scheme and $\pi_i(F,*)$ is an affine unipotent group scheme.*

**Proof:** This follows from [To1, Cor. 3.2.7] and the essential surjectivity part of theorem 3.14. □

We are now ready to explain the relations between cosimplicial Hopf algebras and schematic homotopy types. The first step is to identify the P-equivalences in $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$.

**Corollary 3.17** *A morphism of cosimplicial Hopf algebras*

$$f : B_* \longrightarrow B'_*$$

*is a P-equivalence if and only if the induced morphism on the total complexes*

$$Tot(B_*) \longrightarrow Tot(B'_*)$$

*is a quasi-isomorphism.*

*Proof:* We let $G_* \in \mathsf{sGAff}_{\mathbb{U}}$ and $G_* \longrightarrow G_*^P$ be a P-localization. Using corollary 3.13 we see that the induced morphism

$$\boldsymbol{B}h_{G_*} \longrightarrow \boldsymbol{B}h_{G_*^P}$$

is a P-equivalence of simplicial presheaves. In the same way, it has been proved in [To1, Corollary 3.2.7] that the natural morphism

$$\boldsymbol{B}h_{G_*} \longrightarrow \boldsymbol{B}\mathbb{R}Spec\,\mathcal{O}(G_*)$$

is a P-equivalence of simplicial presheaves. As both $\boldsymbol{B}h_{G_*^P}$ and $\boldsymbol{B}\mathbb{R}Spec\,\mathcal{O}(G_*)$ are P-local simplicial presheaves, we deduce that they are naturally equivalent as pointed simplicial presheaves.

From this and theorem 3.14, we deduce that a morphism $G_* \longrightarrow G'_*$ of simplicial affine group schemes is a P-equivalence if and only if the induced morphism of simplicial presheaves

$$\boldsymbol{B}\mathbb{R}Spec\,\mathcal{O}(G_*) \longrightarrow \boldsymbol{B}\mathbb{R}Spec\,\mathcal{O}(G'_*)$$

is an equivalence. This is also equivalent to say that the induced morphism

$$\mathbb{R}Spec\,\mathcal{O}(G_*) \longrightarrow \mathbb{R}Spec\,\mathcal{O}(G'_*)$$

is an equivalence. By [To1, Corollary 2.2.3] this is equivalent to say that

$$\mathcal{O}(G'_*) \longrightarrow \mathcal{O}(G_*)$$

is an equivalence of cosimplicial algebras, or in other words that

$$Tot(\mathcal{O}(G'_*)) \longrightarrow Tot(\mathcal{O}(G_*))$$

is a quasi-isomorphism. □

If we put 3.17 and 3.14 together we find the following result, explaining in which sense cosimplicial Hopf algebras are algebraic models for pointed schematic homotopy types.

**Corollary 3.18** *Let $B_* \mapsto B_*^P$ be a P-localization in $\mathsf{Hopf}_{\mathbb{U}}^{\Delta}$. Then, the composite functor*

$$B_* \mapsto B_*^P \mapsto \boldsymbol{B}h_{Spec\, B_*^P}$$

*induces an equivalence between the $Ho^P(\mathsf{Hopf}_{\mathbb{U}}^{\Delta})^{op}$, the localized category of cosimplicial Hopf algebras along quasi-isomorphisms, and the homotopy category of pointed schematic homotopy types.*

Using cosimplicial Hopf algebras as models for psht we can describe the schematization functor $X \mapsto (X \otimes k)^{\mathrm{sch}}$ of [To1] in the following explicit way.

Let $X$ be a pointed and connected simplicial set in $\mathbb{U}$. As $X$ is connected, one can choose an equivalent $X$ which is a reduced simplicial set (i.e. a simplicial set with when $X_0 = *$). We apply Kan's loop group construction associating to $X$ a simplicial group $GX$. Kan's loop group functor is left adjoint to the classifying space functor associating to each simplicial group its classical classifying space. Explicitly (see [G-J, §V]) $GX$ is the simplicial group whose group of $n$-simplices is the free group generated by the set $X_{n+1} - s_0(X_n)$. Applying the pro-algebraic completion functor lewelwise we turn $GX$ into a simplicial affine group scheme $GX^{\mathrm{alg}}$ in $\mathbb{U}$. The object $GX^{\mathrm{alg}}$ can be thought of as an *algebraic loop space for $X$*. The corresponding cosimplicial algebra $B_* := \mathcal{O}(GX^{\mathrm{alg}})$ of $k$-valued regular functions on $GX^{\mathrm{alg}}$ is such that the level $n$ component $B_n$ of $B_*$ is the sub-Hopf algebra of $k^{GX_n} = \mathrm{Hom}(GX_n, k)$ of functions spanning a finite dimensional sub-$GX_n$-module.

**Corollary 3.19** *With the above notations, there exists a natural isomorphism in* $\mathrm{Ho}(\mathsf{SPr}_*(k))$

$$(X \otimes k)^{\mathrm{sch}} \simeq \boldsymbol{B}\,\mathrm{Spec}\ B_*^P,$$

*where $B_*^P$ is a P-local model for $B_*$.*

*Proof:* As shown in the proof of [To1, Theorem 3.3.4], the schematization $(X \otimes k)^{\mathrm{sch}}$ is given by $\boldsymbol{B}\mathbb{R}Spec\, B_*$. An we have already seen during the proof of corollary 3.17 that there exists a natural equivalence

$$\boldsymbol{B}\mathbb{R}Spec\, B_* \simeq \boldsymbol{B}\,\mathrm{Spec}\ B_*^P.$$

□

### 3.4 Equivariant cosimplicial algebras

In this section we provide another algebraic model for psht. This model is based on equivariant cosimplicial algebras and is quite similar to the approach of [Br-Sz]. The main difference between the two approaches is that we will work equivariantly over a affine group scheme, and not over a discrete group as this is done in [Br-Sz].

*Equivariant stacks*

We will start by some reminders on the notion of equivariant stacks.

For the duration of this section we fix a presheaf of groups $G$ on $(\mathsf{Aff})_{ffqc}$, which will be considered as a group object in $\mathsf{SPr}(k)$. We will make the assumption that $G$ is cofibrant as an object of $\mathsf{SPr}(k)$. For example, $G$ could be representable (i.e. an affine group scheme), or a constant presheaf associated to a group in $\mathbb{U}$.

Because the direct product makes the category $\mathsf{SPr}(k)$ into a cofibrantly generated symmetric monoidal closed model category, for which the monoid axiom of [S-S, Section 3] is satisfied, the category $G\text{-}\mathsf{SPr}(k)$ of simplicial presheaves equipped with a left action of $G$ is again a closed model category [S-S]. Recall that the fibrations (respectively equivalences) in $G\text{-}\mathsf{SPr}(k)$ are defined to be the morphisms inducing fibrations (respectively equivalences) between the underlying simplicial presheaves. The model category $G\text{-}\mathsf{SPr}(k)$ will be called the *model category of $G$-equivariant simplicial presheaves*, and the objects in $\mathrm{Ho}(G\text{-}\mathsf{SPr}(k))$ will be called *$G$-equivariant stacks*. For any $G$-equivariant stacks $F$ and $F'$ we will denote by $\underline{\mathrm{Hom}}_G(F, F')$ the simplicial set of morphisms in $G\text{-}\mathrm{SPr}(k)$, and by $\mathbb{R}\underline{\mathrm{Hom}}_G(F, F')$ its derived version.

On the other hand, to any group $G$ one can associate its classifying simplicial presheaf $BG \in \mathsf{SPr}_*(k)$ (see [To1, §1.3]). The object $BG \in \mathrm{Ho}(\mathsf{SPr}_*(k))$ is uniquely characterized up to a unique isomorphism by the properties

$$\pi_0(BG) \simeq * \qquad \pi_1(BG, *) \simeq G \qquad \pi_i(BG, *) = 0 \; for \; i > 1.$$

Consider the comma category $\mathsf{SPr}(k)/\,BG$, of objects over the classifying simplicial presheaf $BG$, endowed with its natural simplicial closed model structure (see [Ho]). Note that since the model structure on $\mathsf{SPr}(k)$ is proper, the model category $\mathsf{SPr}(k)/\,BG$ is an invariant, up to a Quillen equivalence, of the isomorphism class of $BG$ in $\mathrm{Ho}(\mathsf{SPr}_*(k))$. This implies in particular that one is free to choose any model for $BG \in \mathrm{Ho}(\mathsf{SPr}(k))$ when dealing with the homotopy category $\mathrm{Ho}(\mathsf{SPr}(k)/\,BG)$. We will set $BG := EG\,/G$, where $EG$ is a cofibrant model of $*$ in $G\text{-}\mathsf{SPr}(k)$ which is fixed once for all.

We now define a pair of adjoint functors

$$G\text{-}\mathsf{SPr}(k) \underset{Mo}{\overset{De}{\rightleftarrows}} \mathsf{SPr}(k)/\,BG,$$

where $De$ stands for *descent* and $Mo$ for *monodromy*. If $F$ is a $G$-equivariant simplicial presheaf, then $De(F)$ is defined to be $(EG \times F)/G$, where $G$ acts diagonally on $EG \times F$. Note that there is a natural projection $De(F) \longrightarrow EG\,/G = BG$, and so $De(F)$ is naturally an object in $\mathsf{SPr}(k)/\,BG$. This adjunction is easily seen to be a Quillen adjunction. Furthermore, using the reasoning of [To2], one can also show that this Quillen adjunction is actually a Quillen equivalence. For future reference we state this as a lemma:

**Lemma 3.20** *The Quillen adjunction $(De, Mo)$ is a Quillen equivalence.*

**Proof:** The proof is essentially the same as the proof of [To2, 2.22] and is left to the reader. $\square$

The previous lemma implies that the derived Quillen adjunction induces an equivalence of categories

$$\mathrm{Ho}(G\text{-}\,\mathsf{SPr}(k)) \simeq \mathrm{Ho}(\mathsf{SPr}(k)/\,BG).$$

For any $G$-equivariant stack $F \in \mathrm{Ho}(G\text{-}\,\mathsf{SPr}(\mathbb{C}))$, we define the *quotient stack* $[F/G]$ of $F$ by $G$ as the object $\mathbb{L}\,De(F) \in \mathrm{Ho}(\mathsf{SPr}(k)/\,BG)$ corresponding to $F \in \mathrm{Ho}(G\text{-}\,\mathsf{SPr}(k))$. By construction, the homotopy fiber (taken at the distinguished point of $BG$) of the natural projection

$$p : [F/G] \longrightarrow BG$$

is canonically isomorphic to the underlying stack of the $G$-equivariant stack $F$.

*Equivariant cosimplicial algebras and equivariant affine stacks*

Suppose that $G$ is an affine group scheme and consider the category of $k$-linear representations of $G$. This category will be denoted by $\mathsf{Rep}(G)$. Note that it is an abelian $k$-linear tensor category, which admits all $\mathbb{V}$-limits and $\mathbb{V}$-colimits. The category of cosimplicial $G$-modules is defined to be the category $\mathsf{Rep}(G)^{\Delta}$, of cosimplicial objects in $\mathsf{Rep}(G)$. For any object $E \in \mathsf{Rep}(G)^{\Delta}$ one can construct the normalized cochain complex $\boldsymbol{N}(E)$ associated to $E$, which is a cochain complex in $\mathsf{Rep}(G)$. Its cohomology representations $H^i(\boldsymbol{N}(E)) \in \mathsf{Rep}(G)$ will simply be denoted by $H^i(E)$. This construction is obviously functorial and gives rise to various cohomology functors $H^i : \mathsf{Rep}(G)^{\Delta} \longrightarrow \mathsf{Rep}(G)$. As the category $\mathsf{Rep}(G)$ is $\mathbb{V}$-complete and $\mathbb{V}$-co-complete, its category of cosimplicial objects $\mathsf{Rep}(G)^{\Delta}$ has a natural structure of a simplicial category ([G-J, II,Example 2.8]).

Following the argument of [Q2, II.4], it can be seen that there exists a simplicial finitely generated closed model structure on the category $\mathsf{Rep}(G)^{\Delta}$ with the properties:

- A morphism $f : E \longrightarrow E'$ is an equivalence if and only if for any $i$, the induced morphism $H^i(f) : H^i(E) \longrightarrow H^i(E')$ is an isomorphism.
- A morphism $f : E \longrightarrow E'$ is a cofibration if and only if for any $n > 0$, the induced morphism $f_n : E_n \longrightarrow E'_n$ is a monomorphism.
- A morphism $f : E \longrightarrow E'$ is a fibration if and only if it is an epimorphism whose kernel $K$ is such that for any $n \geq 0$, $K_n$ is an injective object in $\mathsf{Rep}(G)$.

The category $\mathsf{Rep}(G)^{\Delta}$ is endowed with a symmetric monoidal structure, given by the tensor product of cosimplicial $G$-modules (defined levelwise). In particular we can consider the category $G\text{-}\,\mathsf{Alg}^{\Delta}$ of commutative unital monoids in $\mathsf{Rep}(G)^{\Delta}$. It is reasonable to view the objects in $G\text{-}\,\mathsf{Alg}^{\Delta}$ as cosimplicial algebras equipped with an action of the group scheme $G$. Motivated by this remark we will refer to the category $G\text{-}\,\mathsf{Alg}^{\Delta}$ as the *category of $G$-equivariant cosimplicial algebras.* From

another point of view, the category $G\text{-}\mathsf{Alg}^{\Delta}$ is also the category of simplicial affine schemes in $\mathbb{V}$ equipped with an action of $G$.

Every $G$-equivariant cosimplicial algebra $A$ has an underlying cosimplicial $G$-module again denoted by $A \in \mathsf{Rep}(G)^{\Delta}$. This defines a forgetful functor

$$G\text{-}\mathsf{Alg}^{\Delta} \longrightarrow \mathsf{Rep}(G)^{\Delta}$$

which has a left adjoint $L$, given by the free commutative monoid construction.

**Proposition 3.21** *There exists a simplicial cofibrantly generated closed model structure on the category $G\text{-}\mathsf{Alg}^{\Delta}$ satisfying*

- *A morphism $f : A \longrightarrow A'$ is an equivalence if and only if the induced morphism in $\mathsf{Rep}(G)^{\Delta}$ is an equivalence.*
- *A morphism $f : A \longrightarrow A'$ is a fibration if and only if the induced morphism in $\mathsf{Rep}(G)^{\Delta}$ is a fibration.*

**Proof:** This again an application of the small object argument and more precisely of theorem [Ho, Theorem 2.1.19].

Let $I$ and $J$ be sets of generating cofibrations and trivial cofibrations in $\mathsf{Rep}(G)^{\Delta}$. That is, $I$ is the set of monomorphisms between finite dimensional cosimplicial $G$-modules, and $J$ the set of trivial cofibrations between finite dimensional $G$-modules. Consider the forgetful functor $G\text{-}\mathsf{Alg}^{\Delta} \longrightarrow \mathsf{Rep}(G)^{\Delta}$, and its left adjoint $L : \mathsf{Rep}(G)^{\Delta} \longrightarrow G\text{-}\mathsf{Alg}^{\Delta}$. The functor $L$ sends a cosimplicial representation $V$ of $G$ to the free commutative $G$-equivariant cosimplicial algebra generated by $V$. We will apply the small object argument to the sets $L(I)$ and $L(J)$.

By construction, the morphisms in $L(I) - inj$ are precisely the morphisms inducing surjective quasi-isomorphisms on the associated normalized cochain complexes. In the same way, the morphisms in $L(J) - inj$ are precisely the morphisms inducing surjections on the associated normalized cochain complexes. From this, it is easy to see that amoung the conditions (1) to (6) only the inclusion $J - cell \subset W$ requires a proof as well as (6). However, (6) can also be replaced by $W \cap I - inj \subset I - inj$ which is easily seen by the previous descriptions. Therefore, it only remains to check that for any morphism $A \longrightarrow B$ in $J$, and any morphism $L(A) \longrightarrow C$, the induced morphism $C \longrightarrow C \coprod_{L(A)} L(B)$ is an equivalence. But since the forgetful functor $G\text{-}\mathsf{Alg}^{\Delta} \longrightarrow Alg^{\Delta}$ is a left adjoint, it commutes with colimits and so $C \longrightarrow C \coprod_{L(A)} L(B)$ will be an equivalence due to the fact that $Alg^{\Delta}$ is endowed with a model category structure for which $L(A) \longrightarrow L(B)$ is a trivial cofibration (see [To1, Theorem 2.1.2]). $\square$

We will also need the following result:

**Lemma 3.22** *The forgetful functor $G\text{-}\mathsf{Alg}^{\Delta} \longrightarrow \mathsf{Alg}^{\Delta}$ is a left Quillen functor.*

**Proof:** The forgetful functor has a right adjoint

$$F : \mathsf{Alg}^{\Delta} \longrightarrow G\text{-}\mathsf{Alg}^{\Delta},$$

which to a cosimplicial algebra $A \in \mathsf{Alg}^{\Delta}$ assigns the $G$-equivariant algebra $F(A) := \mathcal{O}(G) \otimes A$, where $G$ acts on $\mathcal{O}(G)$ by left translation.

By adjunction, it is enough to show that $F$ is right Quillen. By definition, the functor $F$ preserves equivalences. Moreover, if $A \longrightarrow B$ is a fibration of cosimplicial algebras (i.e. an epimorphism), then the map $\mathcal{O}(G) \otimes A \longrightarrow \mathcal{O}(G) \otimes B$ is again an epimorphism. The kernel of the latter morphism is then isomorphic to $\mathcal{O}(G) \otimes K$, where $K$ is the kernel of $A \longrightarrow B$. But, for any vector space $V$, $\mathcal{O}(G) \otimes V$ is always an injective object in $\mathsf{Rep}(G)$. This implies that $\mathcal{O}(G) \otimes K_n$ is an injective object in $\mathsf{Rep}(G)^{\Delta}$, and therefore $F(A) \longrightarrow F(B)$ is a fibration in $G\text{-}\mathsf{Alg}^{\Delta}$ (see Proposition 3.21). $\square$

For any $G$-equivariant cosimplicial algebra $A$, one can define its (geometric) spectrum $\mathrm{Spec}_G\, A \in G\text{-}\mathsf{SPr}(k)$, by taking the usual spectrum of its underlying cosimplicial algebra and keeping track of the $G$-action. Explicitly, if $A$ is given by a morphism of cosimplicial algebras $A \longrightarrow A \otimes \mathcal{O}(G)$, one finds a morphism of simplicial schemes

$$G \times \mathrm{Spec}\ A \simeq \mathrm{Spec}\,(A \otimes \mathcal{O}(G)) \longrightarrow \mathrm{Spec}\ A,$$

which induces a well defined $G$-action on the simplicial scheme Spec $A$. Hence, by passing to the simplicial presheaves represented by $G$ and Spec $A$, one gets the $G$-equivariant simplicial presheaf $\mathrm{Spec}_G(A)$. This procedure defines a functor

$$\mathrm{Spec}_G : (G\text{-}\mathsf{Alg}^{\Delta})^{\mathrm{op}} \longrightarrow G\text{-}\mathsf{SPr}(k),$$

and we have the following

**Corollary 3.23** *The functor* $\mathrm{Spec}_G$ *is right Quillen.*

**Proof:** Clearly $\mathrm{Spec}_G$ commutes with $\mathbb{V}$-limits. Furthermore, the category $G\text{-}\mathsf{Alg}^{\Delta}$ possesses a small set of small generators in $\mathbb{V}$. For example, one can take a set of representatives of $G$-equivariant cosimplicial algebras $A$ with $A_n$ of finite type for any $n \geq 0$. This implies that $(G\text{-}\mathsf{Alg}^{\Delta})^{\mathrm{op}}$ possesses a small set of small co-generators in $\mathbb{V}$. The existence of the left adjoint to $\mathrm{Spec}_G$ then follows from the special adjoint theorem of [Mac, §V.8].

To prove that $\mathrm{Spec}_G$ is right Quillen it remains to prove that it preserves fibrations and trivial fibrations. In other words, one needs to show that if $A \longrightarrow A'$ is a (trivial) cofibration of $G$-equivariant cosimplicial algebras, then $\mathrm{Spec}_G(A') \longrightarrow \mathrm{Spec}_G(A)$ is a (trivial) fibration in $G\text{-}\mathsf{SPr}(k)$. As fibrations and equivalences in $G\text{-}\mathsf{SPr}(k)$ are defined on the underlying object, this follows immediately from the fact that the non-equivariant Spec is right Quillen and from Lemma 3.22. $\square$

The left adjoint of $\mathrm{Spec}_G$ will be denoted by $\mathcal{O}_G : G\text{-}\mathsf{SPr}(\mathbb{C}) \longrightarrow G\text{-}\mathsf{Alg}^{\Delta}$.

The previous corollary allows one to form the right derived functor of $\mathrm{Spec}_G$:

$$\mathbb{R}\,\mathrm{Spec}_G : \mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta})^{\mathrm{op}} \longrightarrow \mathrm{Ho}(G\text{-}\mathsf{SPr}(k)),$$

which possesses a left adjoint $\mathbb{L}\mathcal{O}_G$. One can then compose this functor with the quotient stack functor $[-/G]$, and obtain a functor

$$[\mathbb{R}\,\mathrm{Spec}_G(-)/G] : \mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta})^{\mathrm{op}} \longrightarrow \mathrm{Ho}(\mathsf{SPr}(k)/\,BG),$$

which still possesses a left adjoint due to the fact that $[-/G]$ is an equivalence of categories. We will denote this left adjoint again by

$$\mathbb{L}\mathcal{O}_G : \mathrm{Ho}(\mathsf{SPr}(k)/\, BG) \longrightarrow \mathrm{Ho}(G\text{-}\,\mathsf{Alg}^{\Delta})^{\mathrm{op}}.$$

**Proposition 3.24** *If $A \in \mathrm{Ho}(G\text{-}\,\mathsf{Alg}^{\Delta})^{\mathrm{op}}$ is isomorphic to some $G$-equivariant cosimplicial algebra in $\mathbb{U}$, then the adjunction morphism*

$$A \longrightarrow \mathbb{L}\mathcal{O}_G(\mathbb{R}\,\mathrm{Spec}_G\, A)$$

*is an isomorphism.*

*In particular, the functors $\mathbb{R}\,\mathrm{Spec}_G$ and $[\mathbb{R}\,\mathrm{Spec}_G(-)/G]$ become fully faithful when restricted to the full sub-category of $\mathrm{Ho}(G\text{-}\,\mathsf{Alg}^{\Delta})$ consisting of $G$-equivariant cosimplicial algebras isomorphic to some object in $\mathbb{U}$.*

**Proof:** Let $A$ be a cofibrant $G$-equivariant cosimplicial algebra in $\mathbb{U}$, and let $F := \mathrm{Spec}_G\, A$. Since $F$ and $\mathrm{hocolim}_{n\in\Delta^{\mathrm{op}}}\, \mathrm{Spec}_G\, A_n$ are weakly equivalent in $G\text{-}\,\mathsf{SPr}(k)$, and since $\mathcal{O}_G$ is a left Quillen functor we get

$$\mathbb{L}\mathcal{O}_G(F) \simeq \mathrm{holim}_{n\in\Delta}\, \mathbb{L}\mathcal{O}_G(\mathrm{Spec}_G\, A_n) \simeq \mathrm{holim}_{n\in\Delta}\, A_n \simeq A.$$

The proposition is proven. □

**Definition 3.25** *An equivariant stack $F \in \mathrm{Ho}(G\text{-}\,\mathsf{SPr}(k))$ is a $G$-equivariant affine stack if it is isomorphic to some $\mathbb{R}\,\mathrm{Spec}_G(A)$, with $A$ a $G$-equivariant cosimplicial algebra in $\mathbb{U}$.*

We conclude this section with a proposition showing that stacks of the form $[\mathbb{R}\,\mathrm{Spec}_G\,(A)/G]$ are often pointed schematic homotopy types.

**Proposition 3.26** *Let $A \in G\text{-}\,\mathsf{Alg}^{\Delta}$ be a $G$-equivariant cosimpicial algebra in $\mathbb{U}$, such that the underlying algebra of $A$ has an augmentation $x : A \longrightarrow k$. Assume also that $A$ is cohomologically connected, i.e. that $H^0(A) \simeq k$. Then, the quotient stack $[\mathbb{R}\,\mathrm{Spec}_G\,(A)/G]$ is a pointed schematic homotopy type.*

**Proof:** The augmentation map $x : A \longrightarrow k$ induces a morphism $* \longrightarrow \mathbb{R}\,\mathrm{Spec}\, A$, and therefore gives rise to a well defined point

$$* \longrightarrow \mathbb{R}\,\mathrm{Spec}\, A \longrightarrow [\mathbb{R}\,\mathrm{Spec}_G\,(A)/G].$$

Consider the natural morphism in $\mathrm{Ho}(\mathsf{SPr}_*(k))$

$$[\mathbb{R}\,\mathrm{Spec}_G\,(A)/G] \longrightarrow BG\,.$$

Its homotopy fiber is isomorphic to $\mathbb{R}\,\mathit{Spec}\, A$. Using the general theorems [To1, Corollary 2.4.10, Theorem 3.2.4] and the long exact sequence on homotopy sheaves, it is then enough to show that $\mathbb{R}\,\mathit{Spec}\, A$ is a connected affine stack. Equivalently, we need to show that $A$ is cohomologically connected, which is true by hypothesis. The proposition is proven. □

We now define a category $\mathrm{Ho}(\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}})$, of *equivariant augmented and 1-connected cosimplicial algebras* in the following way. The objects of $\mathrm{Ho}(\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}})$ are triplets $(G, A, u)$, consisting of

- an affine group scheme $G$ in $\mathbb{U}$
- a $G$-equivariant cosimplicial algebra $A$ in $\mathbb{U}$ such that $H^0(A) \simeq k$ and $H^1(A) = 0$
- a morphism of $G$-equivariant cosimplicial algebras $u : A \to \mathcal{O}(G)$ (where $\mathcal{O}(G)$ is the algebra of functions on $G$ considered as a $G$-equivariant cosimplicial algebra via the regular representation of $G$).

A morphism $f : (G, A, u) \longrightarrow (H, B, v)$ in $\mathrm{Ho}(\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}})$ is a pair $(\phi, a)$, consisting of

- a morphism of affine group schemes $\phi : G \longrightarrow H$
- a morphism in $\mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta}/\mathcal{O}(G))$ (the homotopy category of the model category of objects over $\mathcal{O}(G)$) $a : B \longrightarrow A$. Here, $B$ is considered as an object in $G\text{-}\mathsf{Alg}^{\Delta}/\mathcal{O}(G)$ by composing the action and the augmentation with the morphisms $G \to H$ and $\mathcal{O}(H) \to \mathcal{O}(G)$.

This defines the category $\mathrm{Ho}(\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}})$. Note however that this category is not (a priori) the homotopy category of a model category $\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}}$ (at least we did not define any such model category), and is defined in an ad-hoc way.

Let $(G, A, u) \in \mathrm{Ho}(\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}})$. We consider the associated $G$-equivariant stack $\mathbb{R}\,\mathrm{Spec}_G\, A$. The morphism $u : A \longrightarrow \mathcal{O}(G)$ induces a natural morphism

$$\mathbb{R}\,\mathrm{Spec}_G\, \mathcal{O}(G) \simeq G \longrightarrow \mathbb{R}\,\mathrm{Spec}_G\, A,$$

where $G$ acts on itself by left translations. Therefore, $\mathbb{R}\,\mathrm{Spec}_G\, A$ can be considered in a natural way as an object in $\mathrm{Ho}(G/\, G\text{-}\mathsf{SPr}(k))$, the homotopy category of $G$-equivariant stacks under $G$. Applying the quotient stack functor of [Ka-Pa-To, Definition 1.2.2] we get a well defined object

$$([G/G] \simeq * \longrightarrow [\mathbb{R}\,\mathrm{Spec}_G\, A/G]) \in \mathrm{Ho}([G/G]/\mathsf{SPr}(k)) \simeq \mathrm{Ho}(\mathsf{SPr}_*(k)).$$

Tracing carefully through the definitions, it is straightforward to check that the previous construction defines a functor

$$\begin{aligned} \mathrm{Ho}(\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}})^{\mathrm{op}} &\longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(k)) \\ (G, A, u) &\longmapsto [\mathbb{R}\,\mathrm{Spec}_G\, A/G]. \end{aligned} \tag{1}$$

**Proposition 3.27** *The functor* (1) *is fully faithful and its essential image consists exactly of psht.*

**Proof:** The full faithfulness follows easily from Proposition 3.24. Also, Proposition 3.26 implies that this functor takes values in the sub-category of psht. It remains to prove that any psht is in the essential image of (1).

Let $F$ be a psht. By Theorem 3.14 we can write $F$ as $\boldsymbol{B}G_*$ where $G_*$ is a fibrant object in $\mathsf{sGAff}_{\mathbb{U}}$. We consider the projection $F \longrightarrow BG$, where $G := \pi_0(G_*) \simeq \pi_1(F)$, as well as the cartesian square

$$\begin{tikzcd} F \arrow[r] & BG \\ F^0 \arrow[r] \arrow[u] & EG, \arrow[u] \end{tikzcd}$$

where as usual $EG$ is the simplicial presheaf having $EG_m := G^{m+1}$ and the face and degeneracy maps are given by the projections and diagonal embeddings (see [To1, §1.3] for details). The simplicial presheaf $F^0$ is a pointed affine scheme in $\mathbb{U}$ equipped with a natural action of $G$. Taking its cosimplicial algebra of functions one gets a $G$-equivariant cosimplicial algebra $A := \mathcal{O}(F^0)$. As the fiber of $F^0 \longrightarrow F$ is naturally isomorphic to $G$ one gets furthermore a $G$-equivariant morphism $G \longrightarrow F^0$, giving rise to an $G$-equivariant morphism $u : A \longrightarrow \mathcal{O}(G)$. One can then check that $F^0$ is naturally isomorphic in $\mathrm{Ho}([G/G] - \mathsf{SPr}(k))$ to the image of the object $(G, A, u) \in \mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta}/\mathcal{O}(G))^{\mathrm{op}}$ by the functor $\mathbb{R}\,\mathrm{Spec}_G$ (this follows from the fact that $F^0$ is an affine stack). Finally, one has isomorphisms in $\mathrm{Ho}(\mathsf{SPr}_*(k))$

$$(* \to F) \simeq \big([G/G] \to [F^0/G]\big) \simeq ([\mathbb{R}\,\mathrm{Spec}_G\ \mathcal{O}(G)/G] \to [\mathbb{R}\,\mathrm{Spec}_G\ A/G])\,,$$

showing that $F$ belongs to the essential image of the required functor. □

**Corollary 3.28** *The categories* $\mathrm{Ho}(\mathsf{Hopf}^{\Delta}_{\mathbb{U}})$ *and* $\mathrm{Ho}(\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}})$ *are equivalent.*

**Proof:** Indeed by Theorem 3.14 and Proposition 3.27 they are both equivalent to the full subcategory of $\mathrm{Ho}(\mathsf{SPr}_*(k))$ consisting of psht. □

**Remark 3.29** *The previous corollary is a generalization of the equivalence between reduced nilpotent Hopf dg-algebras and* 1*-connected reduced dga, given by the bar and cobar constructions (see [Ta, §0]). It could be interesting to produce explicit functors between* $\mathrm{Ho}(\mathsf{Hopf}^{\Delta}_{\mathbb{U}})$ *and* $\mathrm{Ho}(\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}})$ *without passing through the category of psht.*

*An explicit model for* $(X \otimes k)^{\mathrm{sch}}$

Let $X$ be a pointed and connected simplicial set in $\mathbb{U}$. In this paragraph we give an explicit model for $(X \otimes k)^{\mathrm{sch}}$ which is based on the notion of equivariant affine stacks we just introduced.

The main idea of the construction is the following observation. Let $G := \pi_1(X, x)^{\mathrm{alg}}$ be the pro-algebraic completion of $\pi_1(X, x)$ over $k$. By Lemma 3.20 and Corollary 3.16, the natural morphism $(X \otimes k)^{\mathrm{sch}} \longrightarrow BG$ corresponds to a $G$-equivariant affine stack. Furthermore, the universal property of the schematization suggests that the corresponding $G$-equivariant cosimplicial algebra is the cosimplicial algebra of cochains of $X$ with coefficients in the local system $\mathcal{O}(G)$. We will show that this guess is actually correct.

Let $\pi_1(X, x) \longrightarrow G$ be the universal morphism, and let $X \longrightarrow B(G(k))$ be the corresponding morphism of simplicial sets. This latter morphism is well defined up to homotopy, and we choose once and for all a representative for it. Let $p : P \longrightarrow X$ be the corresponding $G$-torsor in $\mathsf{SPr}(k)$. More precisely, $P$ is the simplicial presheaf sending an affine scheme $\mathrm{Spec}\, A \in \mathsf{Aff}$ to the simplicial set $P(A) := (EG(A) \times_{BG(A)} X)$, where $EG(A) \longrightarrow BG(A)$ is the natural projection. The morphism $p : P \longrightarrow X$ is then a well defined morphism in $\mathrm{Ho}(G\text{-}\mathsf{SPr}(k))$. Here the group $G$ is acting on $P = (EG \times_{BG} X)$ by its action on $EG$, and trivially on $X$. Alternatively we can describe $P$ by the formula

$$P \simeq (\widetilde{X} \times G)/\pi_1(X, x),$$

where $\widetilde{X}$ is the universal covering of $X$, and $\pi_1(X, x)$ acts on $\widetilde{X} \times G$ by the diagonal action (our convention here is that $\pi_1(X, x)$ acts on $G$ by left translation). We assume at this point that $\widetilde{X}$ is

chosen to be cofibrant in the model category of $\pi_1(X,x)$-equivariant simplicial sets. For example, we may assume that $\widetilde{X}$ is a $\pi_1(X,x)$-equivariant cell complex.

We consider now the $G$-equivariant affine stack $\mathbb{R}\operatorname{Spec}_G \mathcal{O}_G(P) \in \operatorname{Ho}(G\text{-}\mathsf{SPr}(k))$, which comes naturally equipped with its adjunction morphism $P \longrightarrow \mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)$. This induces a well defined morphism in $\operatorname{Ho}(\mathsf{SPr}(k))$:

$$X \simeq [P/G] \longrightarrow [\mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)/G].$$

Furthermore, since $X$ is pointed, this morphism induces a natural morphism in $\operatorname{Ho}(\mathsf{SPr}_*(k))$

$$f : X \longrightarrow [\mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)/G].$$

With this notation we now have the following important

**Theorem 3.30** *The natural morphism $f : X \longrightarrow [\mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)/G]$ is a model for the schematization of $X$.*

**Proof:** Since $P \simeq (\widetilde{X} \times G)/\pi_1(X,x)$, the algebra $\mathbb{L}\mathcal{O}_G(P)$ can be identified with the cosimplicial algebra of cochains on $X$ with coefficients in the local system of algebras $\mathcal{O}(G)$. More precisely, $P$ is equivalent to the homotopy colimit of $\widetilde{X} \times G$ viewed as a $\pi_1(X,x)$-diagram in $G\text{-}\mathsf{SPr}(k)$. As $\mathcal{O}_G$ is left Quillen, one has equivalences

$$\mathbb{L}\mathcal{O}_G(P) \simeq \operatorname{holim}_{\pi_1(X,x)} \mathbb{L}\mathcal{O}_G(\widetilde{X} \times G) \simeq \operatorname{holim}_{\pi_1(X,x)} \mathbb{L}\mathcal{O}_G(G)^{\widetilde{X}},$$

where $(-)^{\widetilde{X}}$ is the exponential functor (which is part of the simplicial structure on $G\text{-}\mathsf{Alg}^{\Delta}$). In particular we have an isomorphism

$$\mathbb{L}\mathcal{O}_G(P) \simeq (\mathcal{O}(G)^{\widetilde{X}})^{\pi_1(X,x)},$$

of $\mathbb{L}\mathcal{O}_G(P)$ with the $\pi_1(X,x)$-invariant $G$-equivariant cosimplicial algebra of $\mathcal{O}(G)^{\widetilde{X}}$. Note that the identification

$$\operatorname{holim}_{\pi_1(X,x)}(\mathcal{O}(G)^{\widetilde{X}}) \simeq \lim_{\pi_1(X,x)}(\mathcal{O}(G)^{\widetilde{X}}) = (\mathcal{O}(G)^{\widetilde{X}})^{\pi_1(X,x)}$$

uses the fact that $\widetilde{X}$ is cofibrant as a $\pi_1(X,x)$-simplicial set. The underlying cosimplicial algebra of $\mathbb{L}\mathcal{O}_G(P)$ is therefore augmented, cohomologically connected and belongs to $\mathbb{U}$. Therefore, by Proposition 3.26 we conclude that $[\mathbb{R}\,\mathit{Spec}_G\, \mathbb{L}\mathcal{O}_G(P)/G]$ is a pointed schematic homotopy type.

In order to finish the proof of the theorem it remains to show that the morphism $f$ is a $\boldsymbol{P}$-equivalence. For this we start with an algebraic group $H$ and a finite dimensional linear representation $V$ of $H$. Let $F$ denote the pointed schematic homotopy type $K(H,V,n)$, and let $F \longrightarrow BH$ be the natural projection. It is instructive to observe that $F$ is naturally isomorphic to $[\mathbb{R}\,\mathit{Spec}_H\, B(V,n)/H]$, where $B(V,n)$ is the cosimplicial cochain algebra of $K(V,n)$ taken together with the natural action of $H$.

There is a commutative diagram

$$\begin{array}{ccc}
\mathbb{R}\underline{\operatorname{Hom}}_*([\mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)/G], F) & \xrightarrow{f^*} & \mathbb{R}\underline{\operatorname{Hom}}_*(X,F) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
\operatorname{Hom}_{\mathsf{GAff}}(G,H) & \longrightarrow & \operatorname{Hom}_{\mathsf{Gp}}(\pi_1(X,x),H),
\end{array}$$

in which the horizontal morphism at the bottom is an isomorphism due to the universal property of the map $\pi_1(X,x) \longrightarrow G$. Therefore it suffices to check that $f^*$ induces an equivalence on the homotopy fibres of the two projections $p$ and $q$. Let $\rho : G \longrightarrow H$ be a morphism of groups, and consider the homotopy fiber of $F \longrightarrow BH$, together with the $G$-action induced from $\rho$. This is a $G$-equivariant affine stack that will be denoted by $F_G$, and whose underlying stack is isomorphic to $K(V,n)$.

The homotopy fibers of $p$ and $q$ at $\rho$ are isomorphic to $\mathbb{R}\underline{\mathrm{Hom}}_G(\mathbb{R}\,\mathrm{Spec}_G\, \mathbb{L}\mathcal{O}_G(P), F_G)$ and $\mathbb{R}\underline{\mathrm{Hom}}_G(P, F_G)$ respectively. But since $F_G$ is a $G$-equivariant affine stack Prop. 3.24 implies that the natural morphism

$$\mathbb{R}\underline{\mathrm{Hom}}_G(\mathbb{R}\,\mathrm{Spec}_G\, \mathbb{L}\mathcal{O}_G(P), F_G) \longrightarrow \mathbb{R}\underline{\mathrm{Hom}}_G(P, F_G)$$

is an equivalence. □

Consider now $\mathcal{O}(G)$ as a locally constant sheaf of algebras on $X$ via the natural action of $\pi_1(X,x)$, and let

$$C^\bullet(X, \mathcal{O}(G)) := (\mathcal{O}(G)^{\widetilde{X}})^{\pi_1(X,x)}$$

be the cosimplicial algebra of cochains on $X$ with coefficients in $\mathcal{O}(G)$. This cosimplicial algebra is equipped with a natural $G$-action, induced by the regular representation of $G$. One can thus consider $C^\bullet(X, \mathcal{O}(G))$ as a object in $G\text{-}\mathsf{Alg}^\Delta$. Thus the previous theorem, and the natural identification $\mathbb{L}\mathcal{O}_G(P) \simeq C^\bullet(X, \mathcal{O}(G))$ as objects in $\mathrm{Ho}(G\text{-}\mathsf{Alg}^\Delta)$ immediately yield the following:

**Corollary 3.31** *With the previous notations, one has*

$$(X \otimes k)^{\mathrm{sch}} \simeq [\mathbb{R}\,\mathrm{Spec}_G\, C^\bullet(X, \mathcal{O}(G))/G].$$

*The case of characteristic zero*

Suppose that $k$ is of characteristic zero. In this case Corollary 3.31 can be reformulated in terms of the pro-reductive completion of the fundamental group.

Let $x \in X$ be a pointed $\mathbb{U}$-small simplicial set, and $G^{\mathrm{red}}$ be the pro-reductive completion of $\pi_1(X,x)$ over $k$. By definition, $G^{\mathrm{red}}$ is the universal reductive affine group scheme over $k$ equipped with a morphism from $\pi_1(X,x)$ with Zariski dense image. We consider $\mathcal{O}(G^{\mathrm{red}})$ as a local system of $k$-algebras on $X$, and consider the cosimplicial algebra $C^*(X, \mathcal{O}(G^{\mathrm{red}}))$ as an object in $G^{\mathrm{red}} - \mathsf{Alg}^\Delta$. The proof of Theorem 3.30 can also be adapted to obtain the following result.

**Corollary 3.32** *There exists a natural isomorphism in* $\mathrm{Ho}(\mathsf{SPr}_*(k))$

$$(X \otimes k)^{\mathrm{sch}} \simeq [\mathbb{R}\,\mathrm{Spec}_{G^{\mathrm{red}}}\, C^\bullet(X, \mathcal{O}(G^{\mathrm{red}}))/G^{\mathrm{red}}].$$

**Proof:** The natural morphism

$$f : X \longrightarrow [\mathbb{R}\,\mathrm{Spec}_{G^{\mathrm{red}}}\, C^\bullet(X, \mathcal{O}(G^{\mathrm{red}}))/G^{\mathrm{red}}] =: \mathcal{X}$$

is defined the same way as in the proof of theorem 3.30. Let $H$ be a reductive linear algebraic group $H$, and $V$ be a linear representation $V$ of $H$. We set $F := K(H, V, n)$, and we let $\mathrm{Hom}^{\mathrm{zd}}_{\mathsf{Gp}}(\pi_1(X, x), H)$ be the subset of morphisms $\pi_1(X, x) \longrightarrow H$ with a Zariski dense image. We also let $\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{zd}}(X, F)$ be defined by the homotopy pull-back diagram

$$\begin{tikzcd} \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{zd}}(X, F) \arrow[r] \arrow[d] & \mathbb{R}\underline{\mathrm{Hom}}(X, F) \arrow[d] \\ \mathrm{Hom}^{\mathrm{zd}}_{\mathsf{Gp}}(\pi_1(X, x), H) \arrow[r] & \mathrm{Hom}_{\mathsf{Gp}}(\pi_1(X, x), H). \end{tikzcd}$$

In the same way, we define $\mathrm{Hom}^{\mathrm{zd}}_{\mathsf{Gp}}(\pi_1(\mathcal{X}, x), H)$ to be the subset of morphisms $\pi_1(\mathcal{X}, x) \longrightarrow H$ with a Zariski dense image. Notice that these are precisely the morphisms $\pi_1(\mathcal{X}, x) \longrightarrow H$ that factor through the natural quotient $\pi_1(\mathcal{X}, x) \longrightarrow G^{\mathrm{red}}$, that is

$$\mathrm{Hom}^{\mathrm{zd}}_{\mathsf{Gp}}(\pi_1(\mathcal{X}, x), H) = \mathrm{Hom}_{\mathsf{Gp}}(G^{\mathrm{red}}, H).$$

Finally, we define $\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{red}}(\mathcal{X}, F)$ by the homotopy pull-back diagram

$$\begin{tikzcd} \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{red}}_*(\mathcal{X}, F) \arrow[r] \arrow[d] & \mathbb{R}\underline{\mathrm{Hom}}_*(\mathcal{X}, F) \arrow[d] \\ \mathrm{Hom}_{\mathsf{Gp}}(G^{\mathrm{red}}, H) \arrow[r] & \mathrm{Hom}_{\mathsf{Gp}}(\pi_1(\mathcal{X}, x), H). \end{tikzcd}$$

We have a commutative diagram

$$\begin{tikzcd} \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{red}}_*(\mathcal{X}, F) \arrow[r, "f^*"] \arrow[d, "p"'] & \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{zd}}_*(X, F) \arrow[d, "q"] \\ \mathrm{Hom}_{\mathsf{GAff}}(G^{\mathrm{red}}, H) \arrow[r] & \mathrm{Hom}^{\mathrm{zd}}_{\mathsf{Gp}}(\pi_1(X, x), H). \end{tikzcd}$$

As in the proof of theorem 3.30 the induced morphisms on the homotopy fibers of the vertical morphisms $p$ and $q$ are all weak equivalences. Since the bottom horizontal morphism is bijective, this shows that the top horizontal morphism

$$f^* : \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{red}}_*(\mathcal{X}, F) \longrightarrow \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{zd}}_*(X, F)$$

is a weak equivalence.

As this is true for any $H$, $V$ and $n$, we find that the natural morphism

$$\pi_1(X, x) \longrightarrow \pi_1(\mathcal{X}, x)$$

induces an equivalence of the categories of reductive linear representations, and that for any such reductive representation $V$, the induced morphism

$$f^* : H^*(\mathcal{X}, V) \longrightarrow H^*(X, V)$$

is an isomorphism. This in turn implies that the morphism $f$ is a $\boldsymbol{P}$-equivalence. □

# 4 Some properties of the schematization functor

In this section we have collected some basic properties of the schematization functor. All these are purely topological but are geared toward the schematization of algebraic varieties.

For a topological space $X$, we will write $(X \otimes k)^{\mathrm{sch}}$ instead of $(S(X) \otimes k)^{sch}$, where $S(X)$ is the singular simplicial set of $X$.

## 4.1 Schematization of smooth manifolds

The purpose of this section is to give a formula for the schematization of a smooth manifold in terms of differential forms.

Let $X$ be a topological space, which is assumed to be locally contractible and for which any open subset is paracompact. For example, $X$ could be any $CW$-complex, including that way any covering of a compact and smooth manifold. Let $G$ be an affine group scheme, and let us consider $\mathsf{Rep}(G)(X)$, the abelian category of sheaves on $X$ with values in the category $\mathsf{Rep}(G)$ of linear representations of $G$. The category $\mathsf{Rep}(G)(X)$ is in particular the category of sheaves on a (small) site with values in a Grothendieck abelian category. Thus, $\mathsf{Rep}(G)(X)$ has enough injectives. This implies that the opposite category possesses enough projectives, and therefore [Q2, II.4.11,Remark 5] can be applied to endow the category $C^+(\mathsf{Rep}(G)(X))$ of positively graded complexes in $\mathsf{Rep}(G)(X)$ with a model structure. Recall that the equivalences in $C^+(\mathsf{Rep}(G)(X))$ are morphisms inducing isomorphisms on cohomology sheaves (i.e. quasi-isomorphisms of complexes of sheaves), and the fibrations are epimorphisms whose kernel $K$ is such that each $K_n$ is an injective object in $\mathsf{Rep}(G)(X)$. One checks immediately that $C^+(\mathsf{Rep}(G)(X))$ is a cofibrantly generated model category.

The Dold-Kan correspondence yields an equivalence of categories

$$D : C^+(\mathsf{Rep}(G)(X)) \longrightarrow \mathsf{Rep}(G)(X)^{\Delta} \qquad C^+(\mathsf{Rep}(G)(X)) \longleftarrow \mathsf{Rep}(G)(X)^{\Delta} : N,$$

where $N$ is the normalization functor and $D$ the denormalization functor. Through this equivalence we can transplant the model structure of $C^+(\mathsf{Rep}(G)(X))$ to a model structure on $\mathsf{Rep}(G)(X)^{\Delta}$. As explained in [G-J, II,Example 2.8], the category $\mathsf{Rep}(G)(X)^{\Delta}$ possesses a natural simplicial structure, and it is easy to check that this simplicial structure is compatible with the model structure. Therefore, $\mathsf{Rep}(G)(X)^{\Delta}$ is a simplicial and cofibrantly generated model category.

The two categories $\mathsf{Rep}(G)(X)^{\Delta}$ and $C^+(\mathsf{Rep}(G)(X))$ have natural symmetric monoidal structures induced by the usual tensor product on $\mathsf{Rep}(G)$. The monoidal structures turn these categories into symmetric monoidal model categories. However, the functors $D$ and $N$ are not monoidal functors, but are related to the monoidal structure via the usual Alexander-Whitney and shuffle products

$$\mathrm{aw}_{X,Y} : N(X) \otimes N(Y) \longrightarrow N(X \otimes Y) \qquad \mathrm{sp}_{X,Y} : D(X) \otimes D(Y) \longrightarrow D(X \otimes Y).$$

The morphisms $\mathrm{aw}_{X,Y}$ are unital and associative, and $\mathrm{sp}_{X,Y}$ are unital, associative and commutative.

We will denote the categories of commutative monoids in the categories $\mathsf{Rep}(G)(X)^{\Delta}$ and $C^+(\mathsf{Rep}(G)(X))$ respectively by

$$G\text{-}\mathsf{Alg}^{\Delta}(X), \text{ and } G\text{-}\mathsf{CDGA}(X),$$

and call them the categories of $G$-equivariant cosimplicial algebras on $X$, and of $G$-equivariant commutative differential algebras on $X$.

The following proposition is standard

**Proposition 4.1** *1. There exist a unique simplicial model structure on $G\text{-}\mathsf{Alg}^{\Delta}$ such that a morphism is an equivalence (respectively a fibration) if and only if the induced morphism in $\mathsf{Rep}(G)(X)^{\Delta}$ is an equivalence, i.e. induces a quasi-isomorphism on the normalized complexes of sheaves (respectively a fibration, i.e. induces an epimorphism on the normalized complexes of sheaves whose kernel is levelwise injective).*

*2. There exist a unique simplicial model structure on $G\text{-}\mathsf{CDGA}(X)$ such that a morphism is an equivalence (respectively a fibration) if and only if the induced morphism in $C^{+}(\mathsf{Rep}(G)(X))$ is an equivalence, i.e. a quasi-isomorphism of complexes of sheaves (respectively a fibration, i.e. a epimorphism of complexes of sheaves whose kernel is levelwise injective).*

By naturality, the functor Th of Thom-Sullivan cochains (see [Hi-Sc, 4.1]) extends to a functor

$$\mathrm{Th} : \mathrm{Ho}(G\text{-}\mathsf{CDGA}(X)) \longrightarrow \mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta}(X)).$$

This functor is an equivalence, and its inverse is the denormalization functor

$$D : \mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta}(X)) \longrightarrow \mathrm{Ho}(G\text{-}\mathsf{CDGA}(X)).$$

Let $f : X \longrightarrow Y$ be a continuous map of topological spaces (again paracompact and loaclly contractible). The inverse and direct image functors of sheaves induce Quillen adjunctions

$$G\text{-}\mathsf{CDGA}(Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} G\text{-}\mathsf{CDGA}(X) \quad \text{and} \quad G\text{-}\mathsf{Alg}^{\Delta}(Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} G\text{-}\mathsf{Alg}^{\Delta}(X).$$

Furthermore, one checks immediately that the following diagram

$$\begin{array}{ccc} \mathrm{Ho}(G\text{-}\mathsf{CDGA}(X)) & \xrightarrow{D} & \mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta}(X)) \\ {\scriptstyle \mathbb{R}f_*}\downarrow & & \downarrow{\scriptstyle \mathbb{R}f_*} \\ \mathrm{Ho}(G\text{-}\mathsf{CDGA}(Y)) & \xrightarrow[D]{} & \mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta}(Y)) \end{array}$$

commutes. As usual, in the case when $Y = *$, the functor $f_*$ will be denoted simply by $\Gamma(X, -)$. The previous diagram should be understood as a functorial isomorphism $D\mathbb{R}\Gamma(X, A) \simeq \mathbb{R}\Gamma(X, D(A))$, for any $A \in \mathrm{Ho}(G\text{-}\mathsf{CDGA}(X))$.

The category $\mathsf{Rep}(G)(X)^{\Delta}$ is naturally enriched over the category of sheaves of simplicial sets on $X$. Indeed, for $F \in \mathsf{SSh}(X)$ a simplicial sheaf, and $E \in \mathsf{Rep}(G)(X)^{\Delta}$, one can define $F \otimes E$ to be the sheaf associated to the presheaf defined by the following formula

$$\begin{aligned} \mathsf{Open}(X)^{\mathrm{op}} &\longrightarrow \mathsf{Rep}(G)^{\Delta} \\ U &\longmapsto F(U) \otimes E(U), \end{aligned}$$

where $\mathsf{Open}(X)$ denotes the category of open sets in $X$ and $F(U) \otimes E(U)$ is viewed as an object in $\mathsf{Rep}(G)^{\Delta}$ via the natural simplicial structure on the model category $\mathsf{Rep}(G)^{\Delta}$. It is straightforward to check that this structure makes the model category $\mathsf{Rep}(G)(X)^{\Delta}$ into a $\mathsf{SSh}(X)$-model category in the sense of [Ho, §4.2], where $\mathsf{SSh}(X)$ is taken with the injective model structure defined in [J]. In particular, one can define functors

$$\begin{aligned} \mathsf{SSh}(X)^{\mathrm{op}} \otimes \mathsf{Rep}(G)(X)^{\Delta} &\to \mathsf{Rep}(G)(X)^{\Delta} \\ (F, E) &\longmapsto E^F, \end{aligned}$$

and

$$\begin{aligned} (\mathsf{Rep}(G)(X)^{\Delta})^{\mathrm{op}} \otimes \mathsf{Rep}(G)(X)^{\Delta} &\longrightarrow \mathsf{SSh}(X) \\ (E, E') &\longmapsto \underline{\mathrm{Hom}}_{\mathsf{SSh}(X)}(E, E'), \end{aligned}$$

which are related by the usual adjunction isomorphisms

$$\underline{\mathrm{Hom}}_{\mathsf{SSh}(X)}(F \otimes E, E') \simeq \underline{\mathrm{Hom}}_{\mathsf{SSh}(X)}(E, (E')^F) \simeq \underline{\mathrm{Hom}}_{\mathsf{SSh}(X)}(F, \underline{\mathrm{Hom}}_{\mathsf{SSh}(X)}(E, E')),$$

where $\underline{\mathrm{Hom}}_{\mathsf{SSh}(X)}$ denotes the internal Hom in $\mathsf{SSh}(X)$.

**Lemma 4.2** *Let $A \in \mathsf{Rep}(G)(X)$ be an injective object. Then, the presheaf of abelian groups on $X$ underlying $A$ is acyclic.*

**Proof:** As the space $X$ is paracompact its sheaf cohomology coincides with its Čech cohomology. Thus it is enough to show that for any open covering $\{U_i\}_{i \in I}$, the Čech complex

$$A(X) \longrightarrow \prod_{i \in I} A(U_i) \longrightarrow \ldots \longrightarrow \prod_{(i_0, \ldots, i_p) \in I^{p+1}} A(U_{i_0} \cap \ldots \cap U_{i_p}) \longrightarrow \ldots$$

is exact.

Since $A$ is an injective object, it is fibrant as a constant cosimplicial object $A \in \mathsf{Rep}(G)(X)^{\Delta}$. Let $\{U_i\}_{i \in I}$ be an open covering of $X$, and let $N(U/X)$ be its nerve. This is the simplicial sheaf on $X$ whose sheaf of $n$-simplices is defined by the following formula

$$\begin{aligned} N(U/X)_n \ : \quad \mathsf{Open}(X)^{\mathrm{op}} &\longrightarrow \mathsf{SSet} \\ V &\longmapsto \textstyle\prod_{(i_0, \ldots, i_n) \in I^{n+1}} \mathrm{Hom}_X(V, U_{i_0} \cap \ldots \cap U_{i_n}). \end{aligned}$$

The simplicial sheaf $N(U/X)$ is contractible (i.e. equivalent to $*$ in $\mathsf{SSh}(X)$) as can be easily seen on stalks. Furthermore, since every object in $\mathsf{SSh}(X)$ is cofibrant, and $A \in \mathsf{Rep}(G)(X)^{\Delta}$ is fibrant, the natural morphism

$$A^{\bullet} \simeq A \longrightarrow A^{N(U/X)}$$

is an equivalence of fibrant objects in $\mathsf{Rep}(G)(X)^{\Delta}$ (here $A^{N(U/X)}$ is part of the $\mathsf{SSh}(X)$-module structure on $\mathsf{Rep}(G)(X)^{\Delta}$)). Therefore, since the global sections functor

$$\Gamma(X, -) : \mathsf{Rep}(G)(X)^{\Delta} \longrightarrow \mathsf{Rep}(G)^{\Delta}$$

is right Quillen, we conclude that the induced morphism $\Gamma(X, A) \longrightarrow \Gamma(X, A^{N(U/X)})$ is a quasi-isomorphism. Since $\Gamma(X, A^{N(U/X)})$ is just the Čech complex of $A$ for the covering $\{U_i\}_{i \in I}$, this implies that $A$ is acyclic on $X$. $\square$

**Lemma 4.3** *Let $A \in G\text{-}\mathsf{CDGA}(X)$ be a $G$-equivariant commutative differential graded algebra over $X$ such that for all $n \geq 0$, the sheaf of abelian groups $A_n$ is an acyclic sheaf on $X$. Then, the natural morphism in* $\mathrm{Ho}(G\text{-}\mathsf{CDGA})$

$$\Gamma(X, A) \longrightarrow \mathbb{R}\Gamma(X, A)$$

*is an isomorphism.*

**Proof:** Let $A \longrightarrow RA$ be a fibrant model of $A$ in $G\text{-}\mathsf{CDGA}(X)$. Then, by the definition of a fibration in $G\text{-}\mathsf{CDAG}(X)$, each $A_n$ is an injective object in $\mathsf{Rep}(G)(X)$, and is therefore acyclic by lemma 4.2. The morphism $A \longrightarrow RA$ is thus a quasi-isomorphism of complexes of acyclic sheaves of abelian groups on $X$, which implies that the induced morphism on global sections

$$\Gamma(X, A) \longrightarrow \Gamma(X, RA)$$

is a quasi-isomorphism of complexes. By the definition of an equivalence in $G\text{-}\mathsf{CDGA}$ this implies that the morphism

$$\Gamma(X, A) \longrightarrow \Gamma(X, RA) =: \mathbb{R}\Gamma(X, A)$$

is actually an isomorphism in $\mathrm{Ho}(G\text{-}\mathsf{CDGA})$. □

Now, let $(X, x)$ be a pointed connected compact smooth manifold and let $X^{\mathrm{top}}$ denote the underlying topological space of $X$. Let $L_B(X)$ be the category of semi-simple local systems of finite dimensional $\mathbb{C}$-vector spaces on $X^{\mathrm{top}}$. It is a rigid $\mathbb{C}$-linear tensor category which is naturally equivalent to the category of finite dimensional semi-simple representations of the fundamental group $\pi_1(X^{\mathrm{top}}, x)$.

The category of semi-simple $C^\infty$ complex vector bundles with flat connections on $X$ will be denoted as before by $L_{DR}(X)$. Recall that the category $L_{DR}(X)$ is a rigid $\mathbb{C}$-linear tensor category, and the functor which maps a flat bundle to its monodromy representations at $x$, induces an equivalence of tensor categories $L_B(X) \simeq L_{DR}(X)$ (this is again the Riemann-Hilbert correspondence).

Let $G_X := \pi_1(X^{\mathrm{top}}, x)^{\mathrm{red}}$ be the pro-reductive completion of the group $\pi_1(X^{\mathrm{top}}, x)$. Note that it is the Tannaka dual of the category $L_B(X)$. The algebra $\mathcal{O}(G_X)$ can be viewed as the left regular representation of $G_X$. Through the universal morphism $\pi_1(X^{\mathrm{top}}, x) \longrightarrow G_X$, we can also consider $\mathcal{O}(G_X)$ as a linear representation of $\pi_1(X^{\mathrm{top}}, x)$. This linear representation is not finite dimensional, but it is admissible in the sense that it equals the union of its finite dimensional sub-representations. Therefore, the algebra $\mathcal{O}(G_X)$ corresponds to an object in the $\mathbb{C}$-linear tensor category $T_B(X)$, of Ind-local systems on $X^{\mathrm{top}}$. By convention all of our Ind-objects are labelled by $\mathbb{U}$-small index categories.

Furthermore, the algebra structure on $\mathcal{O}(G_X)$, gives rise to a morphism

$$\mu : \mathcal{O}(G_X) \otimes \mathcal{O}(G_X) \longrightarrow \mathcal{O}(G_X),$$

which is easily checked to be a morphism in $T_B(X)$. This means that if $\mathcal{O}(G_X)$ is written as the colimit of local systems $\{V_i\}_{i \in I}$, then the product $\mu$ is given by a compatible system of morphisms in $L_B(X)$

$$\mu_{i,k} : V_i \otimes V_i \longrightarrow V_k,$$

for some index $k \in I$ with $i \leq k \in I$. The morphism $\mu = \{\mu_{i,k}\}_{i,k \in I}$, endows the object $\mathcal{O}(G_X) \in T_B(X)$ with a structure of a commutative unital monoid. Through the Riemann-Hilbert

correspondence $T_B(X) \simeq T_{DR}(X)$, the algebra $\mathcal{O}(G_X)$ can also be considered as a commutative monoid in the tensor category $T_{DR}(X)$ of Ind-objects in $L_{DR}(X)$.

Let $\{(V_i, \nabla_i)\}_{i\in I} \in T_{DR}(X)$ be the object corresponding to $\mathcal{O}(G_X)$. For any $i \in I$, one can form the de Rham complex of $C^\infty$-differential forms

$$(A^\bullet_{DR}(V_i), \nabla_i) := A^0(V_i) \xrightarrow{\nabla_i} A^1(V_i) \xrightarrow{\nabla_i} \dots \xrightarrow{\nabla_i} A^n(V_i) \xrightarrow{D_i} \dots.$$

In this way we obtain an inductive system of complexes $\{(A^\bullet_{DR}(V_i), D_i)\}_{i\in I}$ whose colimit complex is defined to be the de Rham complex of the local system $\mathcal{O}(G_X)$ on $X$:

$$(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla) := \mathrm{colim}_{i\in I}(A^\bullet_{DR}(V_i), \nabla_i).$$

The complex $(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla)$ has a natural structure of a commutative differential graded algebra, coming from the commutative monoid structure on $\{(V_i, \nabla_i)\}_{i\in I} \in T_{DR}(X)$. Using wedge products of differential forms, and the monoidal structure, we obtain in the usual fashion morphisms of complexes

$$(A^\bullet_{DR}(V_i), \nabla_i) \otimes (A^\bullet_{DR}(V_j), \nabla_j) \longrightarrow (A^\bullet_{DR}(V_k), \nabla_k)$$

which, after passing to the colimit along $I$, turn $(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla)$ into a commutative differential graded algebra.

The affine group scheme $G_X$ acts via the right regular representation on the Ind-local system $\mathcal{O}(G_X)$, and this action is compatible with the algebra structure. By functoriality, this action gives rise to an action of $G_X$ on the corresponding objects in $T_{DR}(X)$. Furthermore, if $G_X$ acts on an inductive system of flat bundles $(V_i, \nabla_i)$, then it acts naturally on its de Rham complex $\mathrm{colim}_{i\in I}(A^\bullet_{DR}(V_i), \nabla_i)$, by acting on the spaces of differential forms with coefficients in the various $V_i$. Indeed, if the action of $G_X$ is given by a co-module structure

$$\{(V_i, \nabla_i)\}_{i\in I} \longrightarrow \{\mathcal{O}(G_X) \otimes (V_i, \nabla_i)\}_{i\in I},$$

then one obtains a morphism of Ind-$C^\infty$-bundles by tensoring with the sheaf $A^n$ of differential forms on $X$

$$\{V_i \otimes A^n\}_{i\in I} \longrightarrow \{\mathcal{O}(G_X) \otimes (V_i \otimes A^n)\}_{i\in I}.$$

Taking global sections on $X$, one has a morphism

$$\mathrm{colim}_{i\in I}\, A^n(V_i) \longrightarrow \mathrm{colim}_{i\in I}\, A^n(V_i) \otimes \mathcal{O}(G_X),$$

which defines an action of $G_X$ on the space of differential forms with values in the Ind-$C^\infty$-bundle $\{V_i\}_{i\in I}$. Since this action is compatible with the differentials $\nabla_i$, one obtains an action of $G_X$ on the de Rham complex $\mathrm{colim}_{i\in I}(A^\bullet_{DR}(V_i), \nabla_i)$. Furthermore since the action is compatible with the algebra structure on $\mathcal{O}(G_X)$ it follows that $G_X$ acts on $\mathrm{colim}_{i\in I}(A^\bullet_{DR}(V_i), \nabla_i)$ by algebra automorphisms. Thus, the group scheme $G_X$ acts in a natural way on the complex $(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla)$, turning it into a well defined $G_X$-equivariant commutative differential graded algebra.

Applying the denormalization functor $D : \mathrm{Ho}(G_X\text{-}\mathsf{CDGA}) \longrightarrow \mathrm{Ho}(G_X\text{-}\mathsf{Alg}^\Delta)$, we obtain a well defined $G_X$-equivariant cosimplicial algebra denoted by

$$C^\bullet_{DR}(X, \mathcal{O}(G_X)) := D(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla) \in \mathrm{Ho}(G_X\text{-}\mathsf{Alg}^\Delta).$$

To summarize: for any $(X,x)$ - a pointed connected smooth manifold we let $G_X := \pi_1(X,x)^{\mathrm{red}}$ to be the pro-reductive completion of its fundamental group. The $G_X$-equivariant commutative differential graded algebra of de Rham cohomology of $X$ with coefficients in $\mathcal{O}(G_X)$ is denoted by

$$(A^{\bullet}_{DR}(\mathcal{O}(G_X)), \nabla) \in \mathrm{Ho}(G_X\text{-}\,\mathsf{CDGA}).$$

Its denormalization is denoted by

$$C^{\bullet}_{DR}(X, \mathcal{O}(G_X)) := D(A^{\bullet}_{DR}(\mathcal{O}(G_X)), \nabla) \in \mathrm{Ho}(G_X\text{-}\,Alg^{\Delta}).$$

Any smooth map $f : (Y,y) \longrightarrow (X,x)$ of pointed connected smooth manifolds induces a morphism $G_Y := \pi_1(Y,y)^{\mathrm{red}} \longrightarrow G_X := \pi_1(X,x)^{\mathrm{red}}$, and therefore a well defined functor

$$f^* : \mathrm{Ho}(G_X\text{-}\,\mathsf{Alg}^{\Delta}) \longrightarrow \mathrm{Ho}(G_Y\text{-}\,\mathsf{Alg}^{\Delta}).$$

It is not difficult to check that the pull-back of differential forms via $f$ induces a well defined morphism in $\mathrm{Ho}(G_Y\text{-}\,\mathsf{Alg}^{\Delta})$

$$f^* : f^*C^{\bullet}_{DR}(X, \mathcal{O}(G_X)) \longrightarrow C^{\bullet}_{DR}(Y, \mathcal{O}(G_Y)).$$

Furthermore, this morphism depends functorially (in an obvious fashion) on the morphism $f$.

The last discussion, allows us to construct a functor

$$(X,x) \mapsto [\mathbb{R}\,\mathrm{Spec}_{G_X}\ C^{\bullet}_{DR}(X, \mathcal{O}(G_X))/G_X],$$

from the category of pointed connected smooth manifolds to the category of pointed schematic homotopy types. To simplify notation, we will denote this functor by $(X,x) \mapsto (X \otimes \mathbb{C})^{\mathrm{diff}}$.

The following corollary is a generalization of Poincaré lemma.

**Corollary 4.4** *Let $X$ be a pointed connected compact and smooth manifold. Then, there exist an isomorphism in* $\mathrm{Ho}(G\text{-}\,\mathsf{CDGA})$

$$(A_{DR}(\mathcal{O}(G_X)), D) \simeq \mathbb{R}\Gamma(X, \mathcal{O}(G_X)),$$

*which is functorial in $X$.*

**Proof:** This follows from lemma 4.3, and the fact that the sheaves $A^n(\mathcal{O}(G_X))$ of differential forms with coefficients in the Ind-flat bundle associated to $\mathcal{O}(G_X)$, are filtered colimits of soft sheaves on $X$. In particular they are acyclic since $X$ is compact.

The functoriality is straighforward. □

As a consequence of the Corollary 4.4 we obtain the following:

**Corollary 4.5** *Let $X$ be a pointed connected compact and smooth manifold. Then there exist an isomorphism in* $\mathrm{Ho}(G\text{-}\,\mathsf{Alg}^{\Delta})$

$$D(A_{DR}(\mathcal{O}(G_X)), D) \simeq \mathbb{R}\Gamma(X, \mathcal{O}(G_X)),$$

*where $D$ is the denormalization functor, and $\mathcal{O}(G_X)$ is considered as an object in $G\text{-}\,\mathsf{Alg}^{\Delta}(X)$. This isomorphism is furthermore functorial in $X$.*

Let $(S,s)$ be a pointed connected simplicial set, let $\pi := \pi_1(S,s)$ be its fundamental group, and let $G_S$ be the pro-reductive completion of $\pi$. The conjugation action of $\pi$ on $G_S$ is an action by group scheme automorphism, and therefore gives rise to a natural action on the model category $G_S\text{-}\mathsf{Alg}^{\Delta}$. More precisely, if $\gamma \in \pi$ and if $A \in G_S\text{-}\mathsf{Alg}^{\Delta}$ corresponds to a cosimplicial $\mathcal{O}(G_S)$-co-module $E$, one defines $\gamma \cdot E$ to be the co-module structure

$$E \longrightarrow E \otimes \mathcal{O}(G_S) \xrightarrow{Id \otimes \gamma} E \otimes \mathcal{O}(G_S)\ .$$

We will be concerned with the fixed-point model category of $G_S\text{-}\mathsf{Alg}^{\Delta}$ under the group $\pi$, denoted by $(G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$, and described in [Ka-Pa-To].

The category $(G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$ is naturally enriched over the category $\pi\text{-}\mathsf{SSet}$, of $\pi$-equivariant simplicial sets. Indeed, for $X \in \pi\text{-}\mathsf{SSet}$ and $A \in (G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$ one can define $X \otimes A$ whose underlying $G_S$-equivariant cosimplicial algebra is $X^{for} \otimes A^{\text{for}}$ (where $X^{\text{for}}$ and $A^{\text{for}}$ are the underlying simplicial set and $G_S$-equivariant cosimplicial algebra of $X$ and $A$ respectively, and $X^{\text{for}} \otimes A^{\text{for}}$ uses the simplicial structure of $G_S\text{-}\mathsf{Alg}^{\Delta}$). The action of $\pi$ on $X^{\text{for}} \otimes A^{\text{for}}$ is then defined diagonally. In particular, we can use the exponential product

$$A^X \in (G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}, \text{ where } X \in \pi\text{-}\mathsf{SSet} \text{ and } A \in (G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}.$$

The functor

$$\begin{aligned} (\pi\text{-}\mathsf{SSet})^{\text{op}} \times (G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi} &\to (G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi} \\ (X,A) &\longmapsto A^X \end{aligned}$$

is a bi-Quillen functor (see [Ho, §4]), and can be derived into a functor

$$\begin{aligned} \text{Ho}(\pi\text{-}\mathsf{SSet})^{\text{op}} \times \text{Ho}((G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}) &\to \text{Ho}((G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}) \\ (X,A) &\longmapsto A^{\mathbb{R}X}. \end{aligned}$$

Recall that by definition

$$A^{\mathbb{R}X} := (RA)^{QX},$$

where $RA$ is a fibrant model for $A$ in $(G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$, and $QX$ is a cofibrant model for $X$ in $\pi\text{-}\mathsf{SSet}$.

In the following defintion, $\mathcal{O}(G_S)$ is considered together with its $\pi$ and $G_S$ actions, i.e. is viewed as an object in $(G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$.

**Definition 4.6** *The $G_S$-equivariant cosimplicial algebra of cochains of $S$ with coefficients in the local system $\mathcal{O}(G_S)$ is defined to be*

$$C^{\bullet}(S, \mathcal{O}(G_S)) := \mathcal{O}(G_S)^{\mathbb{R}\widetilde{S}} \in \text{Ho}(G_S\text{-}\mathsf{Alg}^{\Delta}),$$

*where $\widetilde{S} \in \text{Ho}(\pi\text{-}\mathsf{SSet})$ is the universal covering of $S$.*

Since $\mathcal{O}(G_S)$ is always a fibrant object in $(G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$, one has

$$C^{\bullet}(S, \mathcal{O}(G_S)) \simeq \mathcal{O}(G_S)^{\widetilde{S}} \in \text{Ho}(G_S\text{-}Alg^{\Delta}),$$

as soon as $\widetilde{S}$ is chosen to be cofibrant in $\pi\text{-}\mathsf{SSet}$ (e.g. is chosen to be a $\pi$-equivariant cell complex).

Going back to our space $X$, which is assumed to be pointed and connected, let $S := S(X)$ be its singular simplicial set, naturally pointed by the image $s \in S$ of $x \in X$. Observe that in this case one has a natural isomorphism $G_S \simeq G_X$, induced by the natural isomorphism $\pi_1(X,x) \simeq \pi_1(S,s)$.

**Proposition 4.7** *There exist an isomorphism in* $\mathrm{Ho}(G_S\text{-}\mathsf{Alg}^{\Delta})$

$$C^{\bullet}(S, \mathcal{O}(G_S)) \simeq \mathbb{R}\Gamma(X, \mathcal{O}(G_S)),$$

*where* $\mathcal{O}(G_S)$ *is considered as an object in* $G_S\text{-}\mathsf{Alg}^{\Delta}(X)$*. Furthermore, this isomorphism is functorial in* $X$*.*

**Proof:** We will prove the existence of the isomorphism. The functoriality statement is straightforward and is left to the reader.

Let $\pi = \pi_1(X, x) \simeq \pi_1(S, s)$ Let $\widetilde{X} \longrightarrow X$ be the universal cover of $X$. Let $\widetilde{S} \longrightarrow S(\widetilde{X})$ be a cofibrant replacement of $S(\widetilde{X}) \in \pi\text{-}\mathsf{SSet}$ and let $p : \widetilde{S} \longrightarrow S(X)$ be the natural $\pi$-equivariant projection. For any open subset $U \subset X$, we will denote by $\widetilde{S}_U$ the fiber product

$$\widetilde{S}_U := \widetilde{S} \times_{S(X)} S(U) \in \pi\text{-}\mathsf{SSet}.$$

Consider the presheaf $C^{\bullet}(-, \mathcal{O}(G_S))$ of $G_S$-equivariant cosimplicial algebras on $X$, defined by

$$\begin{aligned} C^{\bullet}(-, \mathcal{O}(G_S)) \ : \quad \mathsf{Open}(X)^{\mathrm{op}} &\longrightarrow G_S\text{-}\mathsf{Alg}^{\Delta} \\ U &\longmapsto C^{\bullet}(U, \mathcal{O}(G_S)) := \mathcal{O}(G_S)^{\widetilde{S}_U}, \end{aligned}$$

where $\mathcal{O}(G_S)^{\widetilde{S}_U}$ is the exponentiation of $\mathcal{O}(G_S) \in (G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$ by $\widetilde{S}_U \in \pi\text{-}SSet$. We denote by $aC^{\bullet}(-, \mathcal{O}(G_S)) \in G_S\text{-}\mathsf{Alg}^{\Delta}(X)$ the associated sheaf.

We have a natural morphism in $G_S\text{-}\mathsf{Alg}^{\Delta}$

$$C^{\bullet}(X, \mathcal{O}(G_S)) \longrightarrow \Gamma(X, aC^{\bullet}(-, \mathcal{O}(G_S))) \longrightarrow \Gamma(X, RaC^{\bullet}(-, \mathcal{O}(G_S))) \longleftarrow \Gamma(X, R\mathcal{O}(G)_S),$$

where $RaC^{\bullet}(-, \mathcal{O}(G_S)))$ is a fibrant replacement of $aC^{\bullet}(-, \mathcal{O}(G_S)))$, and $R\mathcal{O}(G)_S$ is a fibrant replacement of $\mathcal{O}(G)_S$. Since $X$ is assumed to be locally contractible, the natural morphism in $G_S\text{-}\mathsf{Alg}^{\Delta}(X)$

$$\mathcal{O}(G_S) \longrightarrow C^{\bullet}(-, \mathcal{O}(G_S)),$$

induced over every open subset $U \subset X$ by the projection $\widetilde{S}_U \longrightarrow *$, is an equivalence. Therefore, it only remains to show that the morphism

$$C^{\bullet}(X, \mathcal{O}(G_S)) \longrightarrow \Gamma(X, RaC^{\bullet}(-, \mathcal{O}(G_S))$$

is an equivalence in $G_S\text{-}Alg^{\Delta}$.

For this, let $U_{\bullet} \longrightarrow X$ be an open hyper-cover of $X$ such that each $U_n$ is the disjoint union of contractible open subset of $X$ (such a hyper-cover exists again due to local contractibility of $X$). The simplicial sheaf represented by $U_{\bullet}$ is equivalent to $*$ in $\mathsf{SSh}(X)$. As $RaC^{\bullet}(-, \mathcal{O}(G_S))$ is fibrant we obtain a natural equivalence in $G_S\text{-}\mathsf{Alg}^{\Delta}$

$$\Gamma(X, RaC^{\bullet}(-, \mathcal{O}(G_S)) \simeq \Gamma(X, RaC^{\bullet}(-, \mathcal{O}(G_S))^{U_{\bullet}}) \simeq \mathrm{holim}_{[n]\in\Delta}\, \Gamma(U_n, RaC^{\bullet}(-, \mathcal{O}(G_S))).$$

Furthermore, it is shown is [To2, Lemma 2.10] that the natural morphism

$$\mathrm{hocolim}_{[n]\in\Delta^{\mathrm{op}}}\, U_{\bullet} \longrightarrow X$$

is a weak equivalence, and therefore

$$\mathrm{hocolim}_{[n]\in\Delta^{\mathrm{op}}}\widetilde{S}_{U_\bullet}\longrightarrow\widetilde{S}_X=\widetilde{S}$$

is an equivalence in $\pi$- $\mathsf{SSet}$. This implies that the natural morphism in $G_S$- $\mathsf{Alg}^\Delta$

$$C^\bullet(X,\mathcal{O}(G_S))\longrightarrow\mathrm{holim}_{[n]\in\Delta}C^\bullet(U_n,\mathcal{O}(G_S))$$

is an equivalence. Moreover, there exist a commutative diagram

$$\begin{array}{ccc} C^\bullet(X,\mathcal{O}(G_S)) & \longrightarrow & holim_{[n]\in\Delta}C^\bullet(U_n,\mathcal{O}(G_S)) \\ \downarrow & & \downarrow \\ \Gamma(X,RaC^\bullet(-,\mathcal{O}(G_S)) & \longrightarrow & \mathrm{holim}_{[n]\in\Delta}\Gamma(U_n,RaC^\bullet(-,\mathcal{O}(G_S))), \end{array}$$

i.e. without a loss of generality we may assume that $X$ is contractible. Under this assumption it only remains to check that the natural morphism

$$\mathcal{O}(G_S)\longrightarrow\mathbb{R}\Gamma(X,\mathcal{O}(G_S))$$

is an isomorphism in $\mathrm{Ho}(G_S\text{-}\mathsf{Alg}^\Delta)$. But, by lemma 4.2 and the description of fibrant objects in $G_S$- $\mathsf{Alg}^\Delta(X)$, one has

$$H^n(\mathbb{R}\Gamma(X,\mathcal{O}(G_S)))\simeq H^n(X,\mathcal{O}(G_S)).$$

Since the space $X$ is paracompact and contractible, its sheaf cohomology with coefficients in the constant local system $\mathcal{O}(G_S)$ is trivial, which completes the proof of the proposition. □

As an immediate corollary of Corollary4.5 and Proposition 4.7 one obtains the following description of the schematization of a smooth manifold in terms of its differential forms.

**Proposition 4.8** *Let $X$ be a pointed connected compact and smooth manifold. Then, there exist an isomorphism in* $\mathrm{Ho}(G\text{-}\mathsf{Alg}^\Delta)$

$$D(A_{DR}(\mathcal{O}(G_X)),D)\simeq C^\bullet(S(X),\mathcal{O}(G_X)),$$

*where $D$ is the denormalization functor, and $\mathcal{O}(G_X)$ is considered as an object in $G$- $\mathsf{Alg}^\Delta(X)$. In other words, one has a natural isomorphism*

$$(X\otimes k)^{\mathrm{sch}}\simeq[\mathbb{R}\,\mathrm{Spec}_{G_X}D(A_{DR}(\mathcal{O}(G_X)),D)/G_X].$$

## 4.2 A schematic Van Kampen theorem

Let $X$ be a pointed and connected ($\mathbb{U}$-small) topological space, and $\{U_i\}_{i\in I}$ be a finite open covering. We will assume that each $U_i$ contains the base point, and that each of the $p+1$-uple intersection $U_{i_0,\dots,i_{p+1}}:=U_{i_0}\cap\cdots\cap U_{i_p}$ is connected. Here we have in mind the example of a smooth projective complex variety $X$ covered by Zariski open subsets containing the base point. We form the poset

$N(U)$ (the nerve of $\{U_i\}_{i\in I}$), whose objects are strings of indices $(i_0,\dots,i_p)\in I^{p+1}$ (for various lenghts $p\geq 0$), and where $(i_0,\dots,i_p)\leq(j_0,\dots,j_p)$ if and only if $U_{i_0,\dots,i_{p+1}}\subset U_{j_0,\dots,j_{p+1}}$. There exists a natural functor

$$\begin{aligned} N(U) &\longrightarrow \mathsf{Top}_*^{\mathrm{con}} \\ (i_0,\dots,i_p) &\longmapsto U_{i_0,\dots,i_p}, \end{aligned}$$

where $\mathsf{Top}_*^{\mathrm{con}}$ is the category of pointed and connected $\mathbb{U}$-topological spaces. This diagram is furthermore augmented to the constant diagram $X$. Applying the schematization functor, we get a diagram of pointed simplicial presheaves

$$\begin{aligned} N(U) &\longrightarrow \mathsf{SPr}_*(k) \\ (i_0,\dots,i_p) &\longmapsto (U_{i_0,\dots,i_p}\otimes k)^{\mathrm{sch}} \end{aligned}$$

which is naturally augmented towards $(X\otimes k)^{\mathrm{sch}}$ (here we use a version of the schematization functor which is defined on the level of simplicial sets and not only on its homotopy category. For example, one can use the functor $Z\mapsto \boldsymbol{B}\operatorname{Spec}\mathcal{O}(GZ^{\mathrm{alg}})^P$ where $(-)^P$ is a cofibrant replacement functor for the P-local model structure on $\mathsf{Hopf}^{\Delta}$, see Corollary 3.19).

**Proposition 4.9** *For any psht $F$, the natural morphism*

$$\mathbb{R}\underline{\mathrm{Hom}}((X\otimes k)^{\mathrm{sch}},F)\longrightarrow \operatorname{Holim}_{\alpha\in N(U)}\mathbb{R}\underline{\mathrm{Hom}}((U_\alpha\otimes k)^{\mathrm{sch}},F),$$

*induced by the augmentation, is an equivalence of simplicial sets.*

**Proof:** Using the universal property of the schematization functor it is enough to prove that the natural morphism

$$\operatorname{Hocolim}_{\alpha\in N(U)}U_\alpha\longrightarrow X$$

is a weak equivalence of topological spaces. When $X$ is a CW complex this is something well known (see for example [To2, Lemma 2.10]). In the general case it is enough to consider the commutative square

$$\begin{tikzcd} \operatorname{Hocolim}_{\alpha\in N(U)}|S(U_\alpha)| \arrow[r] \arrow[d] & |S(X)| \arrow[d] \\ \operatorname{Hocolim}_{\alpha\in N(U)}U_\alpha \arrow[r] & X, \end{tikzcd}$$

where $|S(-)|$ is the geometric realization of the singular functor. Since the functors $|-|$ and $S(-)$ form a Quillen equivalence, the vertical morphisms are both weak equivalences. The top horizontal morphism being an equivalence this finishes the proof of the proposition. $\square$

**Remark 4.10** Another interpretation of Proposition 4.9 is to say that $(X\otimes k)^{\mathrm{sch}}$ is the homotopy colimit, *in the category of psht*, of the diagram $\alpha\mapsto U_\alpha$. This point of view however is somewhat trickier since, when apropriately defined, the homotopy colimits of psht are not the same as homotopy colimits of pointed simplicial presheaves. These subtleties go beyond the scope of the present paper and we will not discuss them here.

Proposition 4.9 is a Van Kampen type result, as it states that the schematization of a space $X$ is uniquely determined by the diagram $\alpha \mapsto (U_\alpha \otimes k)^{\mathrm{sch}}$. This property will be useful only if the space $X$ behaves well locally with respect to the schematization functor. This is for example the case when $X$ is a smooth projective complex variety. Indeed, locally for the Zariski topology $X$ is a $K(\pi, 1)$, where $\pi$ is a successive extension of free groups of finite type. In Theorem 4.16 below, we will show that such groups $\pi$ are *algebraically good*, and therefore $(K(\pi, 1) \otimes k)^{\mathrm{sch}}$ is itself 1-truncated. So in a way the schematization of a smooth projective complex variety is well understood locally for the Zariski topology, and proposition 4.9 tells us that globally the schematization is obtained by the homotopy colimits of the schematization of a covering.

## 4.3 Good groups

Recall that a discrete group $\Gamma$ in $\mathbb{U}$ is algebraically good (relative to the field $k$) if the natural morphism

$$(K(\Gamma, 1) \otimes k)^{\mathrm{sch}} \longrightarrow K(\Gamma^{\mathrm{alg}}, 1)$$

is an isomorphism. The terminology *algebraically good* mimicks the corresponding pro-finite notion introduced by J.P.Serre in [Se]. It is justified by the following lemma. Let $H^\bullet_H(\Gamma^{\mathrm{alg}}, V)$ denote the Hochschild cohomology of the affine group scheme $\Gamma^{\mathrm{alg}}$ with coefficients in a linear representation $V$ (as defined in [SGA3, I]).

**Lemma 4.11** *Let $\Gamma$ be a group in $\mathbb{U}$ and $\Gamma^{\mathrm{alg}}$ be its pro-algebraic completion. Then, $\Gamma$ is an algebraically good group if and only if for every finite dimensional linear representation $V$ of $\Gamma^{\mathrm{alg}}$, the natural morphism*

$$H^\bullet_H(\Gamma^{\mathrm{alg}}, V) \longrightarrow H^\bullet(\Gamma, V)$$

*is an isomorphism.*

**Proof:** Using [To1, §1.5] and [To1, Corollary 3.3.3], the fact that $H^\bullet_H(\Gamma^{\mathrm{alg}}, V) \cong H^\bullet(\Gamma, V)$ for all $V$ is just a reformulation of the fact that $K(\Gamma, 1) \longrightarrow K(\Gamma^{\mathrm{alg}}, 1)$ is a $\boldsymbol{P}$-equivalence. Since $K(\Gamma^{\mathrm{alg}}, 1)$ is a pointed schematic homotopy type, this implies the lemma. $\square$

We will also use the following very general fact.

**Proposition 4.12** *Let $\Gamma$ be a group in $\mathbb{U}$, $\Gamma^{\mathrm{alg}}$ be its pro-algebraic completion, and $n > 1$ an integer. The following are equivalent.*

*(1) For any linear representation $V$ of $\Gamma^{\mathrm{alg}}$ the induced morphism*

$$H^i_H(\Gamma^{\mathrm{alg}}, V) \longrightarrow H^i(\Gamma, V)$$

*is an isomorphism for $i < n$ and injective for $i = n$.*

*(2) For any linear representation $V$ of $\Gamma^{\mathrm{alg}}$ the induced morphism*

$$H^i_H(\Gamma^{\mathrm{alg}}, V) \longrightarrow H^i(\Gamma, V)$$

*is surjective for $i < n$.*

**Proof:** Let $\mathsf{Rep}(\Gamma)$ be the category of linear complex representations of $\Gamma$ (possibly infinite dimensional), and $\mathsf{Rep}(\Gamma^{\mathrm{alg}})$ the category of linear representation of the affine group scheme $\Gamma^{\mathrm{alg}}$ (again maybe of infinite dimension). The proposition follows from [An, Lemma 11] applied to the inclusion functor $\mathsf{Rep}(\Gamma^{\mathrm{alg}}) \longrightarrow \mathsf{Rep}(\Gamma)$. $\square$

As an immediate corollary we obtain that finite groups are algebraically good over $k$. Furthermore for any finitely generated group $\Gamma$ the universal property of the pro-algebraic completion implies that the natural map $\Gamma \to \Gamma^{\mathrm{alg}}$ induces an isomorphism on cohomology in degrees 0 and 1. Therefore the previous proposition implies that the natural map $H^2_H(\Gamma^{\mathrm{alg}}, V) \longrightarrow H^2(\Gamma, V)$ is always injective. In particular any free group of finite type will be algebraically good over $k$. In the same vein we have:

**Proposition 4.13** *The following groups are algebraically good groups.*

*(1) Abelian groups of finite type.*

*(2) Fundamental groups of a compact Riemann surface when $k = \mathbb{C}$.*

**Proof:** (1) For any abelian group of finite type $M$, there exist a morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M_0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & H \arrow[r] & M^{\mathrm{alg}} \arrow[r] & M_0^{\mathrm{alg}} \arrow[r] & 0,
\end{tikzcd}$$

where $H$ is a finite group, and $M_0$ is torsion free. The comparison of the associated Leray spectral sequences shows that without a loss of generality one may assume $M$ to be torsion free. Then, again comparing the two Leray spectral sequences for the rows of the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z}^{n-1} \arrow[r] \arrow[d] & \mathbb{Z}^n \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & (\mathbb{Z}^{n-1})^{\mathrm{alg}} \arrow[r] & (\mathbb{Z}^n)^{\mathrm{alg}} \arrow[r] & \mathbb{Z}^{\mathrm{alg}} \arrow[r] & 0,
\end{tikzcd}$$

and proceeding by induction on the rank of $M$, one reduces the proof to the case $M = \mathbb{Z}$. But, as $\mathbb{Z}$ is a free group it is algebraically good.

(2) Note that by (1) one can suppose that $g > 2$. As $\Gamma_g$ is the fundamental group of a compact Riemann surface of genus $g$, one has $H^i(\Gamma_g, V) = 0$ for any $i > 2$ and any linear representation $V$. Applying 4.12 it is enough to prove that for any finite dimensional linear representation $V$ of $\Gamma$, the natural morphism $H^2_H(\Gamma_g^{\mathrm{alg}}, V) \longrightarrow H^2(\Gamma_g, V)$ is an isomorphism.

**Lemma 4.14** *Let $V$ be any linear representation of finite dimension of $\Gamma_g$, and $x \in H^2(\Gamma_g, V)$. Then, there exist a linear representation of finite dimension $V'$, and an injective morphism $j : V \hookrightarrow V'$, such that $j(x) = 0 \in H^2(\Gamma_g, V')$.*

**Proof:** Using Poincare duality the assertion of the lemma can be restated as follows. Given any finite dimensional linear representation $E$ of $\Gamma_g$ and any invariant vector $e \in H^0(\Gamma_g, E) = E^{\Gamma_g}$ show that there exists a finite dimensional linear representation $E'$ and a $\Gamma_g$-equivariant surjection $q : E' \to E$ so that $e \notin \mathrm{im}[H^0(\Gamma_g, E') \to H^0(\Gamma_g, E)]$. Let $\mathbb{I}$ denote the one dimensional trivial representation of $\Gamma_g$. Then the element $e \in H^0(\Gamma_g, E)$ can be viewed as an injective homomorphism $e : \mathbb{I} \to E$ of finite dimensional $\Gamma_g$-modules. Let $F := E/\mathbb{I}$ be the corresponding quotient module and let $\Sigma$ be an irreducible $r$-dimensional representation of $\Gamma_g$ for some $r \geq 1$. The short exact sequence of $\Gamma_g$-modules

$$0 \to \mathbb{I} \xrightarrow{e} E \to F \to 0$$

induces a long exact sequence of Ext's in the category of $\Gamma_g$-modules:

$$\ldots \longrightarrow \mathrm{Ext}^1(F, \Sigma) \longrightarrow \mathrm{Ext}^1(E, \Sigma) \xrightarrow{ev_e} \mathrm{Ext}^1(\mathbb{I}, \Sigma) \longrightarrow \mathrm{Ext}^2(F, \Sigma) \longrightarrow \ldots,$$

Now observe that $\dim \mathrm{Ext}^1(\mathbb{I}, \Sigma) = \dim H^1(\Gamma_g, \Sigma) \geq \chi(\Gamma_g, \Sigma) = 2r(g-1)$. Furthermore by Poincare duality we have $\mathrm{Ext}^2(F, \Sigma) \cong \mathrm{Hom}(\Sigma, F)^\vee$ and since $\Sigma$ contains only finitely many irreducible subrepresentations of $\Gamma_g$ it follows that for the generic choice of $\Sigma$, the natural map

$$\mathrm{Ext}^1(E, \Sigma) \xrightarrow{ev_e} \mathrm{Ext}^1(\mathbb{I}, \Sigma)$$

will be surjective. Choose such a $\Sigma$ and let $\epsilon \in \mathrm{Ext}^1(E, \Sigma)$ be an element such that $ev_e(\epsilon) \neq 0 \in \mathrm{Ext}^1(\mathbb{I}, \Sigma)$. Let now

$$0 \to \Sigma \to E' \xrightarrow{q} E \to 0$$

be the extension of $\Gamma_g$ modules corresponding to $\epsilon$. Then the image of $e \in H^0(\Gamma_g, E)$ under the first edge homomorphism $H^0(\Gamma_g, E) \to H^1(\Gamma_g, \Sigma) \cong \mathrm{Ext}^1(\mathbb{I}, \Sigma)$ is precisely $ev_e(\epsilon) \neq 0$. The lemma is proven. □

Now, let $V$ by any linear representation of finite dimension of $\Gamma_g$, $x \in H^2(\Gamma_g, V)$, and let $V \hookrightarrow V'$ as in the previous lemma. Then, the morphism of long exact sequences

$$\begin{array}{ccccccc}
H^1_H(\Gamma_g^{\mathrm{alg}}, V'/V) & \longrightarrow & H^2_H(\Gamma_g^{\mathrm{alg}}, V) & \longrightarrow & H^2_H(\Gamma_g^{\mathrm{alg}}, V') & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(\Gamma_g, V'/V) & \longrightarrow & H^2(\Gamma_g, V) & \longrightarrow & H^2(\Gamma_g, V') & \longrightarrow & \cdots
\end{array}$$

shows that $x$ can be lifted to $H^2(\Gamma_g^{\mathrm{alg}}, V)$. This implies that $H^2_H(\Gamma_g^{\mathrm{alg}}, V) \longrightarrow H^2(\Gamma_g, V)$ is surjective, and finishes the proof of the proposition. □

Before we construct more examples of algebraically good groups we describe a convenient criterion (which was already used implicitly in the proof of proposition 4.13) for checking if a given group is good. For a finitely generated group $\Gamma$ we denote by $\mathsf{Rep}(\Gamma)$ the category of all representations of $\Gamma$ in $k$-vector spaces (possibly infinite dimensional) and by $\mathsf{Rep}(\Gamma^{\mathrm{alg}})$ the category of all algebraic representations of the affine pro-algebraic group $\Gamma^{\mathrm{alg}}$. Recall also that a group $\Gamma$ is of type $(\boldsymbol{F})$ (over $k$) if: (a) for every $n$ and every finite dimensional complex representation $V$ of $\Gamma$ the group $H^n(\Gamma, V)$ is finite dimensional, and (b) $H^\bullet(\Gamma, -)$ commutes with inductive limits of finite dimensional complex representations of $\Gamma$. With this notation we now have:

**Lemma 4.15** *Let $\Gamma$ be a finitely generated group of type $(\boldsymbol{F})$. The following properties of $\Gamma$ are equivalent:*

**(a)** *$\Gamma$ is algebraically good over $k$.*

**(b)** *For every positive integer $n$, every finite dimensional representation $L \in \mathsf{Rep}(\Gamma)$ and every class $\alpha \in H^n(\Gamma, L)$, there exists an injection $\imath : L \hookrightarrow W_\alpha$ in some finite dimensional $\Gamma$-module $W_\alpha \in \mathsf{Rep}(\Gamma)$ so that the induced map $\imath^* : H^n(\Gamma, L) \to H^n(\Gamma, W_\alpha)$, annihilates $\alpha$, i.e. $\imath^*(\alpha) = 0$.*

**(c)** *For every positive integer $n$, every finite dimensional representation $L \in \mathsf{Rep}(\Gamma)$, there exists an injection $\imath : L \hookrightarrow W_\alpha$ in some finite dimensional $\Gamma$-module $W_\alpha \in \mathsf{Rep}(\Gamma)$ so that the induced map $\imath^* : H^n(\Gamma, L) \to H^n(\Gamma, W_\alpha)$, is identically zero, i.e. $\imath \equiv 0$.*

**Proof:** First we show that **(a)** $\Rightarrow$ **(b)**. Suppose that $\Gamma$ is good over $k$. Fix an integer $n > 0$, a representation $L \in \mathsf{Rep}(\Gamma)$ with $\dim(L) < +\infty$ and some class $\alpha \in H^n(\Gamma, L)$.

Consider the regular representation of $\Gamma^{\mathrm{alg}}$ on the algebra $\mathcal{O}(\Gamma^{\mathrm{alg}})$ of $k$-valued regular functions on the affine group $\Gamma^{\mathrm{alg}}$. The natural map $s : \Gamma \to \Gamma^{\mathrm{alg}}$ allows us to view $\mathcal{O}(\Gamma^{\mathrm{alg}})$ as a $\Gamma$-module and so the tensor product

$$L' := L \otimes_{\mathbb{C}} \mathcal{O}(\Gamma^{\mathrm{alg}})$$

can be interpreted both as a $\Gamma$-module and as a $\Gamma^{\mathrm{alg}}$-module.

Note that when viewed as an object in $\mathsf{Rep}(\Gamma^{\mathrm{alg}})$, the module $L'$ is injective. In particular

$$H^i_H(\Gamma^{\mathrm{alg}}, L') = 0, \qquad \text{for all } i > 0.$$

Now write $L'$ as an inductive limit $L' = \operatorname{colim} L'_i$, with finite dimensional $\Gamma$-modules $L'_i$. We consider the natural inclcusion $L \hookrightarrow L'$, which induces inclusions $L' \hookrightarrow L'_i$ for all $i \gg 0$. Since $\Gamma$ is algebraically good we have

$$H^n(\Gamma, L) \cong H^n_H(\Gamma^{\mathrm{alg}}, L),$$

and so we may view $\alpha$ as an element in $H^n_H(\Gamma^{\mathrm{alg}}, L)$. Furthermore since $\Gamma$ is of type $(\boldsymbol{F})$ we get

$$H^n_H(\Gamma^{\mathrm{alg}}, L') = H^n_H(\Gamma^{\mathrm{alg}}, \operatorname{colim}_{\to} L'_i) = \operatorname{colim}_{\to} H^n_H(\Gamma^{\mathrm{alg}}, L'_i),$$

and since $L'$ was chosen so that $H^n_H(\Gamma^{\mathrm{alg}}, L') = 0$, it follows that

$$\alpha \to 0 \in H^n_H(\Gamma^{\mathrm{alg}}, L'_i)$$

for all sufficiently big $i$. Combined with the fact that $L \hookrightarrow L'_i$ for all $i \gg 0$ this yields the implication **(a)** $\Rightarrow$ **(b)**.

We will prove the implication **(b)** $\Rightarrow$ **(a)** by induction on $n$. More precisely, for every integer $n > 0$ consider the condition

$(*_n)$ for every finite dimensional represenation $V \in \mathsf{Rep}(\Gamma)$ the natural map $s : \Gamma \to \Gamma^{\mathrm{alg}}$ induces an isomorphism

$$s^* : H^k_H(\Gamma^{\mathrm{alg}}, V) \xrightarrow{\cong} H^k(\Gamma, V)$$

for all $k < n$.

We need to show that $(*_n)$ holds for all integers $n > 0$. By the universal property of pro-algebraic completions we know that $(*_1)$ holds. This provides the base of the induction. Assume next that $(*_n)$ holds. This automatically implies that

$$s^* : H^n_H(\Gamma^{\mathrm{alg}}, L) \to H^n(\Gamma, L)$$

is injective, and so we only have to show that $s^* : H^n_H(\Gamma^{\mathrm{alg}}, L) \to H^n(\Gamma, L)$ is also surjective.

Fix $\alpha \in H^n(\Gamma, L)$ and let $W \in \mathsf{Rep}(\Gamma)$ be a finite dimensional representation for which we can find an injection $\imath : L \hookrightarrow W$ so that $\imath(\alpha) = 0$. In particular we can find a class $\beta \in H^{n-1}(\Gamma, W/L)$ which is mapped to $\alpha$ by the edge homomorphism of the long exact sequence in cohomology associated to the sequence of $\Gamma$-modules

$$0 \to L \xrightarrow{\imath} W \to W/L \to 0.$$

However, by the inductive hypothesis $(*_n)$ we have an isomorphism

$$H^{n-1}(\Gamma, W/L) \cong H^{n-1}_H(\Gamma^{\mathrm{alg}}, W/L),$$

and so from the commutative diagram

$$\begin{tikzcd}
H^n_H(\Gamma^{\mathrm{alg}}, L) \arrow[r, "s^*"] & H^n(\Gamma, L) \\
H^{n-1}_H(\Gamma^{\mathrm{alg}}, W/L) \arrow[u] \arrow[r, "\cong", "s^*"'] & H^{n-1}(\Gamma, W/L) \arrow[u]
\end{tikzcd}$$

it follows that $\alpha$ comes from $H^n_H(\Gamma^{\mathrm{alg}}, L)$.

The implication **(c)** $\Rightarrow$ **(b)** is obvious. For the implication **(b)** $\Rightarrow$ **(c)** we need to construct a finite dimensional representation $W \in \mathsf{Rep}(L)$ and a monomorphism $\imath : L \to W$ so that $\imath(H^n(\Gamma, L)) = \{0\} \subset H^n(\Gamma, W)$. Choose a basis $e_1, e_2, \ldots, e_m$ of $H^n(\Gamma, L)$. By **(b)** we can find monomorphisms $\imath_1 : L \hookrightarrow W_1$, $\imath_2 : L \hookrightarrow W_2$, $\ldots$, $\imath_m : L \hookrightarrow W_m$, so that $\imath_i(e_i) = 0 \in H^n(\Gamma, W_i)$ for $i = 1, \ldots, m$. Consider now the sequence of finite dimensional representations $V_k \in \mathsf{Rep}(\Gamma)$ and monomorphisms $\jmath_k : L \hookrightarrow V_k$ constructed inductively as follows:

- $V_1 := W_1$, $\jmath_1 := \imath_1$.
- Assuming that $\jmath_{k-1} : L \hookrightarrow V_{k-1}$ has already been constructed, define $V_k$ as the pushout:

$$\begin{tikzcd}
L \arrow[r, hook, "-\imath_k"] \arrow[d, hook, "\jmath_{k-1}"'] & W_k \arrow[d] \\
V_{k-1} \arrow[r] & V_k
\end{tikzcd}$$

  in $\mathsf{Rep}(\Gamma)$. Explicitly $V_k = (V_{k-1} \oplus W_k)/\operatorname{im}[L \xrightarrow{\jmath_{k-1} \times (-\imath_k)} V_{k-1} \oplus W_k]$. Moreover the natural map $\jmath_{k-1} \times \imath_k : L \to V_{k-1} \oplus W_k$ induces a monomorphism $\jmath_k : L \hookrightarrow V_k$ which completes the step of the induction.

Let now $W := V_m$ and $\imath := \jmath_m$. By construction we have inclusions $W_k \subset W$ for all $k = 1, \ldots, m$ and for each $k$ the map $\jmath$ factors as

$$\begin{tikzcd} L \arrow[rr, hook, "\imath"] \arrow[dr, hook, "\imath_k"'] & & W \\ & W_k \arrow[ur, hook] & \end{tikzcd}$$

In particular $\imath(L) \subset \cap_{k=1}^m W_k \subset W$ and so $\imath(H^n(\Gamma, L)) = 0$. The lemma is proven. □

Using the basic good groups (e.g. free, finite, abelian, surface groups) as building blocks and the criterion from Lemma 4.15 we can construct more good groups as follows:

**Theorem 4.16** *Suppose that*

$$1 \to F \to \Gamma \to \Pi \to 1 \tag{2}$$

*is a short exact sequence of finitely generated groups of type* $(\boldsymbol{F})$*, such that* $\Pi$ *is algebraically good over* $k$ *and* $F$ *is free.* $\Gamma$ *is algebraically good over* $k$*.*

**Proof:** The proof is essentially contained in an argument of Beilinson which appears in [Be, Lemma 2.2.1 and 2.2.2] in a slightly different guise. Since the statement of the theorem is an important ingredient in the localization technique for computing schematic homotopy types, we decided to write up the proof in detail in our context.

Let $p : X \to S$ be the Serre fibration corresponding to the short exact sequence (2) and let $Y$ denote the homotopy fiber of $p$. Note that $X = K(\Gamma, 1)$, $S = K(\Pi, 1)$ and $Y = K(F, 1)$. Given a representation $L \in \mathsf{Rep}(\Gamma)$ (respectively in $\mathsf{Rep}(\Pi)$ or $\mathsf{Rep}(F)$) we write $\mathbb{L}$ for the corresponding local system of $k$-vector spaces on $X$ (respectively $S$ or $Y$).

To prove that $\Gamma$ is good it suffices (see Lemma 4.15) to show that for every positive integer $n$ and any finite dimensional representation $L \in \mathsf{Rep}(\Gamma)$ we can find an injection $\imath : L \to W$ into a finite dimensional $W \in \mathsf{Rep}(\Gamma)$ so that the induced map $\imath : H^n(X, \mathbb{L}) \to H^n(X, \mathbb{W})$ is identically zero.

The representation $W$ will be constructed in three steps:

**Step 1.** Fix $n$ and $L$ as above. Then we can find a finite dimensional $W \in \mathsf{Rep}(\Gamma)$ and an injection $\imath : L \hookrightarrow W$ such that the induced map

$$H^n(S, p_*\mathbb{L}) \to H^n(S, p_*\mathbb{W})$$

is identically zero.

Indeed, since $\Pi$ is assumed to be algebraically good and since $p_*\mathbb{L}$ is a finite dimensional local system on $S$ we can find (see Lemma 4.15) a finite dimensional local system $\mathbb{V}$ on $S$ and an injection $p_*\mathbb{L} \hookrightarrow \mathbb{V}$ which induces the zero map

$$H^n(S, p_*\mathbb{L}) \xrightarrow{0} H^n(S, \mathbb{V})$$

on cohomology.

Now define a local system $\mathbb{W} \to X$ as the pushout

$$\begin{tikzcd} p^*p_*\mathbb{L} \arrow[r, hook] \arrow[d] & p^*\mathbb{V} \arrow[d] \\ \mathbb{L} \arrow[r, hook] & \mathbb{W} \end{tikzcd}$$

i.e. $\mathbb{W} := (p^*\mathbb{V} \oplus \mathbb{L})/p^*p_*\mathbb{L}$. By pushing this cocartesian square down to $S$ and using the projection formula $p_*p^*p_*\mathbb{L} \cong p_*\mathbb{L}$, we get a commutative diagram of local systems on $S$:

$$\begin{tikzcd} & p_*p^*\mathbb{V} = \mathbb{V} \arrow[dd] \\ p_*\mathbb{L} \arrow[ur] \arrow[dr] & \\ & p_*\mathbb{W}. \end{tikzcd}$$

In particular the natural map

$$H^n(S, p_*\mathbb{L}) \to H^n(S, p_*\mathbb{W})$$

factors through

$$H^n(S, p_*\mathbb{L}) \xrightarrow{0} H^n(S, \mathbb{V})$$

and so is identically zero. This finishes the proof of Step 1. In fact, the same reasoning can be used to prove the following enhancement of Step 1.

**Step 2.** Let $n$ be a positive integer and let $L, P \in \mathsf{Rep}(\Gamma)$ be some finite dimensional representations. Then we can find a finite dimensional representation $W \in \mathsf{Rep}(\Gamma)$ and an injection of $\Gamma$-modules $L \hookrightarrow W$ so that the induced map $H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})) \to H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{W}))$ vanishes identically.

Indeed, the goodness of $\Pi$ together with the fact that $p_*(\mathbb{P}^\vee \otimes \mathbb{L})$ is finite dimensional again implies the existence of an injection $p_*(\mathbb{P}^\vee \otimes \mathbb{L}) \hookrightarrow \mathbb{M}$ into a finite dimensional local system $\mathbb{M}$ on $S$, so that the induced map

$$H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})) \to H^2(S, p_*\mathbb{M})$$

is identically zero. Now define a local system $\mathbb{W} \to X$ as the pushout

$$\begin{tikzcd} p^*p_*(\mathbb{P}^\vee \otimes \mathbb{L}) \otimes \mathbb{P} \arrow[r, hook] \arrow[d] & p^*\mathbb{M} \otimes \mathbb{P} \arrow[d] \\ \mathbb{L} \arrow[r, hook] & \mathbb{W} \end{tikzcd}$$

where $p^*p_*(\mathbb{P}^\vee \otimes \mathbb{L}) \otimes \mathbb{P} \to \mathbb{L}$ is the natural morphism corresponding to $\mathrm{id}_{p_*(\mathbb{P}^\vee \otimes \mathbb{L})}$ under the identifications

$$Hom(p_*(\mathbb{P}^\vee \otimes \mathbb{L}), p_*(\mathbb{P}^\vee \otimes \mathbb{L})) = Hom(p^*p_*(\mathbb{P}^\vee \otimes \mathbb{L}), \mathbb{P}^\vee \otimes \mathbb{L}) = Hom(p^*p_*(\mathbb{P}^\vee \otimes \mathbb{L}) \otimes \mathbb{P}, \mathbb{L}).$$

Note that the definition of $\mathbb{W}$ implies that the natural map $p^*p_*(\mathbb{P}^\vee \otimes \mathbb{L}) \hookrightarrow \mathbb{P}^\vee \otimes \mathbb{L}$ factors through $p^*\mathbb{M}$ and so by pushing forward to $S$ and using the projection formula we get a commutative diagram of finite dimensional local systems on $S$:

$$\begin{tikzcd} p_*(\mathbb{P}^\vee \otimes \mathbb{L}) \arrow[r, hook] \arrow[dr] & \mathbb{M} \arrow[d] \\ & p_*(\mathbb{P}^\vee \otimes \mathbb{W}). \end{tikzcd}$$

Thus the inclusion $\mathbb{L} \hookrightarrow \mathbb{W}$ will induce the zero map

$$H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})) \xrightarrow{0} H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{W}))$$

as claimed. Step 2 is proven.

**Step 3.** Let $L \in \mathsf{Rep}(\Gamma)$ be a finite dimensional representation and let

$$\begin{tikzcd} Y \arrow[r, hook] & X \arrow[d, "p"] \\ & S \end{tikzcd}$$

be a fibration corresponding to the sequence (2). Then there exists an injection $L \hookrightarrow W$ into a finite dimensional $W \in \mathsf{Rep}(\Gamma)$ inducing the zero map

$$H^1(Y, \mathbb{L}_{|Y}) \xrightarrow{0} H^1(Y, \mathbb{W}_{|Y})$$

on the fiberwise cohomology.

In order to construct $\mathbb{W}$ we begin by choosing an injection $\imath_Y : \mathbb{L}_{|Y} \hookrightarrow \mathbb{E}_Y$ into a suitable local system $\mathbb{E}_Y \to Y$, so that the natural map

$$H^1(Y, \mathbb{L}_{|Y}) \to H^1(Y, \mathbb{E}_Y),$$

that $\imath_Y$ induces on cohomology is identically zero.

We will choose $\mathbb{E}_Y$ as follows. Start with the trivial rank one local system $\mathbb{I}_Y \to Y$ and consider the trivial local system

$$\mathbb{P}_Y := H^1(Y, \mathbb{L}_{|Y}) \otimes_{\mathbb{C}} \mathbb{I}_Y$$

on $Y$. Let

$$(e)\ 0 \longrightarrow \mathbb{L}_{|Y} \xrightarrow{\imath_Y} \mathbb{E}_Y \longrightarrow \mathbb{P}_Y \longrightarrow 0$$

be the associated tautological extension of local systems on $Y$ (the universal extension of $\mathbb{P}_Y$ by $\mathbb{L}_{|Y}$). The definition of $\mathbb{E}_Y$ ensures that the pushforward of any extension $0 \to \mathbb{L}_{|Y} \to \mathbb{L}' \to \mathbb{I}_Y \to 0$ via the map $\imath_Y : \mathbb{L}_{|Y} \to \mathbb{E}_Y$ will be split. In particular the map $H^1(Y, \mathbb{L}_{|Y}) \to H^1(Y, \mathbb{E}_Y)$ induced by $\imath_Y$ is identically zero.

Observe also that we can view $\mathbb{P}_Y$ as the restriction $\mathbb{P}_Y = \mathbb{P}_{|Y}$ of the global local system $\mathbb{P} := p^*(R^1 p_* \mathbb{L})$. If it happens that the extension $(e)$ is also a restriction from a global extension of local systems on $X$, we can take as $W$ the monodromy representation of this global local system and this will complete the proof of Step 2.

By universality we know that $e \in Ext^1_Y(\mathbb{P}_Y, \mathbb{L}_{|Y})$ will be fixed by the monodromy action of $\Pi$ and so

$$e \in Ext^1_Y(\mathbb{P}_{|Y}, \mathbb{L}_{|Y})^{\Pi} = H^0(S, R^1 p_*(\mathbb{P}^\vee \otimes \mathbb{L})).$$

The group $H^0(S, R^1 p_*(\mathbb{P}^\vee \otimes \mathbb{L}))$ fits in the following chunk of the Leray spectral sequence for the map $p : X \to S$:

$$0 \to H^1(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})) \to H^1(X, \mathbb{P}^\vee \otimes \mathbb{L}) \to H^0(S, R^1 p_*(\mathbb{P}^\vee \otimes \mathbb{L})) \xrightarrow{\delta} H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})).$$

The obstruction for lifting $\mathbb{E}_Y$ to a global local system on $X$ is precisely the class

$$\delta(e) \in H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})).$$

We can kill this obstruction by using Step 2. Indeed, by the Step 2 we can find an injection $L \hookrightarrow V$ of finite dimensional $\Gamma$ modules inducing the zero map

$$H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})) \xrightarrow{0} H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{V})).$$

Since the morphism $\mathbb{L} \hookrightarrow \mathbb{V}$ induces a morphism of long exact sequences

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^1(X, \mathbb{P}^\vee \otimes \mathbb{L}) & \longrightarrow & H^0(S, R^1 p_*(\mathbb{P}^\vee \otimes \mathbb{L})) & \xrightarrow{\delta} & H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & \\
\cdots \longrightarrow & H^1(X, \mathbb{P}^\vee \otimes \mathbb{V}) & \longrightarrow & H^0(S, R^1 p_*(\mathbb{P}^\vee \otimes \mathbb{V})) & \xrightarrow{\delta} & H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{V})) & \longrightarrow \cdots
\end{array}$$

we conclude that the push-forward

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{L}_{|Y} & \longrightarrow & \mathbb{E}_Y & \longrightarrow & \mathbb{P}_{|Y} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{V}_{|Y} & \longrightarrow & \mathbb{W}_Y & \longrightarrow & \mathbb{P}_{|Y} & \longrightarrow & 0
\end{array}$$

of the extension $(e)$ via the map $\mathbb{L}_{|Y} \hookrightarrow \mathbb{V}_{|Y}$ is the restriction to $Y$ of some global extension

$$0 \to \mathbb{V} \to \mathbb{W} \to \mathbb{P} \to 0$$

of finite dimensional local systems on $X$. Now by constrcution the natural map $H^1(Y, \mathbb{L}_{|Y}) \to H^1(Y, \mathbb{W}_{|Y})$, induced by the injection $\mathbb{L} \hookrightarrow \mathbb{W}$, will vanish identically, which completes the proof of Step 3.

By putting together the existence results established in Steps 1-3 we can now finish the proof of the theorem. Fix a positive integer $n$ and let $L \in \mathsf{Rep}(\Gamma)$ be a finite dimensional representation of $\Gamma$. By Step 3 we can find an injection $L \hookrightarrow W$ in a finite dimensional representation $W \in \mathsf{Rep}(\Gamma)$, so that the induced map $R^1 p_* \mathbb{L} \to R^1 p_* \mathbb{W}$ vanishes identically. The Leray spectral sequence for $p$ yields a commutative diagram with exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(S, p_*\mathbb{L}) \arrow[r] \arrow[d] & H^n(X, \mathbb{L}) \arrow[r] \arrow[d] & H^{n-1}(S, R^1 p_*\mathbb{L}) \arrow[r] \arrow[d, "0"'] & \cdots \\
\cdots \arrow[r] & H^n(S, p_*\mathbb{W}) \arrow[r] & H^n(X, \mathbb{W}) \arrow[r] & H^{n-1}(S, R^1 p_*\mathbb{W}) \arrow[r] & \cdots
\end{tikzcd}$$

and so

$$H^n(X, \mathbb{L}) \to \mathrm{im}[H^n(S, p_*\mathbb{W}) \to H^n(X, \mathbb{W})] \subset H^n(X, \mathbb{W}).$$

Furthermore, by Step 1 we can find a finite dimensional local system $\mathbb{W}'$ and an injection $\mathbb{W} \hookrightarrow \mathbb{W}'$ inducing

$$H^n(S, p_*\mathbb{W}) \xrightarrow{0} H^n(S, p_*\mathbb{W}').$$

By functoriality of the Leray spectral sequence this implies that the natural map

$$H^n(X, \mathbb{L}) \to H^n(X, \mathbb{W}),$$

induced by the injection $\mathbb{L} \hookrightarrow \mathbb{W} \subset \mathbb{W}'$, is identically zero. The theorem is proven. $\square$

**Remark 4.17 (i)** The only property of free groups that was used in the proof of the previous theorem is the fact that free groups have cohomological dimension $\leq 1$. Therefore over $\mathbb{C}$ we can use the same reasoning (in fact one only needs Step 1) as in the proof of the theorem to argue that the extension of a good group by a finite group will also be good.

**(ii)** By a classical result of M.Artin every point $x$ on a complex smooth variety $X$ admits a Zariski neighborhood $x \in U \subset X$ which is a tower of smooth affine morphisms of relative dimension one. In particular the underlying topological space of $U$ (for the classical topology) is a $K(\pi, 1)$ with $\pi$ being a successive extension of free groups of finite type. Since $\pi$ is manifestly of type $(\boldsymbol{F})$ the previous theorem implies that $\pi$ is algebraically good over $\mathbb{C}$. In fact an easy application of the Leray spectral sequence shows that any successive extension of free groups of finite type will be of type $(\boldsymbol{F})$ and so the previous theorem again implies that such successive extensions are algebraically good.

## 4.4 Lefschetz exactness

Let $f : X \longrightarrow Y$ be a morphism of connected topological spaces, and $n \geq 1$ be an integer. We will say that $f$ is an $n$-epimorphism if induced morphism $\pi_i(X) \longrightarrow \pi_i(Y)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$. In the same way, one defines the notion of an $n$-epimorphism of psht. A morphism $f : F \longrightarrow F'$ of psht is an $n$-epimorphism if the induced morphism of sheaves $\pi_i(F) \longrightarrow \pi_i(F')$ is an isomorphism for $i < n$ and an epimorpism for $i = n$.

**Proposition 4.18** *Let $f : X \longrightarrow Y$ be an $n$-epimorphism of pointed and connected topological spaces (in $\mathbb{U}$), then the induced morphism $f : (X \otimes k)^{\mathrm{sch}} \longrightarrow (Y \otimes k)^{\mathrm{sch}}$ is an $n$-epimorphism of psht.*

**Proof:** For $n = 1$ the proposition is clear, as $\pi_1((X \otimes k)^{\mathrm{sch}}) \simeq \pi_1(X)^{\mathrm{alg}}$ and as the functor $G \mapsto G^{\mathrm{alg}}$ preserves epimorphisms. So we will assume $n > 1$.

One first observes that $f$ is an $n$-epimorphism if and only if it satisfies the following two conditions:

- The induced morphism $\pi_1(X) \longrightarrow \pi_1(Y)$ is an isomorphism.
- For any local system of abelian groups $M$ on $Y$, the induced morphism $H^i(Y, M) \longrightarrow H^i(X, f^*M)$ is an isomorphism for $i < n$ and a monomorphism for $i = n$.

In the same way, $f : (X \otimes k)^{\mathrm{sch}} \longrightarrow (Y \otimes k)^{\mathrm{sch}}$ is an $n$-epimorphism if and only if the two following conditions are satisfied.

- The induced morphism $\pi_1((X \otimes k)^{\mathrm{sch}}) \longrightarrow \pi_1((Y \otimes k)^{\mathrm{sch}})$ is an isomorphism.
- For any local system of $k$-vector spaces of finite dimension $L$ on $Y$, the induced morphism $H^i(Y, L) \longrightarrow H^i(X, f^*L)$ is an isomorphism for $i < n$ and a monomorphism for $i = n$.

The proposition now follows immediately from the fundamental property of the schematization [To1, Definition 3.3.1]. □

**Corollary 4.19** *Let $Y$ be a smooth projective (connected) complex variety of dimension $n+1$, and $X \hookrightarrow Y$ a smooth hyperplane section. Then, for any base point $x \in X$, the induced morphism*

$$\pi_i((X \otimes k)^{\mathrm{sch}}, x) \longrightarrow \pi_i((Y \otimes k)^{\mathrm{sch}}, f(x))$$

*is an isomorphism for $i < n$ and an epimorphism for $i = n$.*

**Proof:** This follows from Proposition 4.18 and the Lefschetz theorem on hyperplane sections. □

## 4.5 Homotopy fibers of schematizations

Let $X \longrightarrow B$ be a morphism of pointed and connected $\mathbb{U}$-simplicial sets, and suppose that its homotopy fiber $Z$ is also connected. We want to relate the schematization of $Z$ with the homotopy fiber $F$ of $(X \otimes k)^{\mathrm{sch}} \longrightarrow (Y \otimes k)^{\mathrm{sch}}$. The universal property of the schematization induces a natural morphism of psht

$$(Z \otimes k)^{\mathrm{sch}} \longrightarrow F$$

which in general is far from being an isomorphism. The following proposition gives a sufficient condition in order for this morphism to be an isomorphism.

**Proposition 4.20** *Assume that $Z$ is 1-connected and of $\mathbb{Q}$-finite type. Then the natural morphism $(Z \otimes k)^{\mathrm{sch}} \longrightarrow F$ is an equivalence.*

**Proof:** Note first that $(Z \otimes k)^{\mathrm{sch}} \simeq (Z \otimes k)^{\mathrm{uni}}$, as $Z$ is simply connected (see [To1, Corollary 3.3.8]).

The fibration sequence $Z \to X \to B$ is classified by a morphism (well defined in the homotopy category) $B \longrightarrow B\mathbb{R}\underline{\mathrm{Aut}}(Z)$, where $\mathbb{R}\underline{\mathrm{Aut}}(Z)$ is the space of auto-equivalences of $Z$. We consider the morphism $B\mathbb{R}\underline{\mathrm{Aut}}(Z) \longrightarrow B\mathbb{R}\underline{\mathrm{Aut}}((Z\otimes k)^{\mathrm{uni}})$, and we observe that $B\mathbb{R}\underline{\mathrm{Aut}}((Z\otimes k)^{\mathrm{uni}})$ is the pointed simplicial set of global sections of the pointed stack $B\mathbb{R}\mathcal{AUT}((Z\otimes k)^{\mathrm{uni}})$. The finitness assumtion on $Z$ implies that $B\mathbb{R}\mathcal{AUT}((Z\otimes k)^{\mathrm{uni}})$ is a psht, and therefore the morphism $B \longrightarrow B\mathbb{R}\underline{\mathrm{Aut}}((Z\otimes k)^{\mathrm{uni}})$ can be inserted in a (homotopy) commutative square of pointed simplicial sets

$$\begin{array}{ccc} B & \longrightarrow & B\mathbb{R}\underline{\mathrm{Aut}}((Z\otimes k)^{\mathrm{uni}}) \\ \downarrow & & \downarrow \\ (B\otimes k)^{\mathrm{sch}} & \longrightarrow & B\mathbb{R}\mathcal{AUT}((Z\otimes k)^{\mathrm{uni}}). \end{array}$$

By passing to the associated fibration sequences, one gets a morphism of fibration sequences of pointed simplicial presheaves

$$\begin{array}{ccccc} Z & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ (Z\otimes k)^{\mathrm{sch}} & \longrightarrow & \widetilde{F} & \longrightarrow & (B\otimes k)^{\mathrm{sch}}. \end{array}$$

We deduce from this diagram a morphism between the Leray spectral sequences, for any local system $L$ on $B$ of finite dimensional $k$-vector spaces

$$\begin{array}{ccc} H^p(B, H^q(Z,L)) & \Longrightarrow & H^{p+q}(X,L) \\ \downarrow & & \downarrow \\ H^p((B\otimes k)^{sch}, H^q((Z\otimes k)^{sch}, L)) & \Longrightarrow & H^{p+q}(\widetilde{F}, L). \end{array}$$

The finiteness condition on $Z$ ensures that this morphism is an isomorphisms on the $E_2$-term, showing that $X \longrightarrow \widetilde{F}$ is a $\boldsymbol{P}$-equivalence. Therefore, the induced morphism $(X\otimes k)^{\mathrm{sch}} \longrightarrow \widetilde{F}$ is an equivalence of psht, which induces the proposition. □

The important consequence of Proposition 4.20 is that under these conditions the schematization of the base $(B\otimes k)^{\mathrm{sch}}$ *acts* on the schematization of the fiber $(Z\otimes k)^{\mathrm{sch}}$. By this we mean that the fibration sequence $(Z\otimes k)^{\mathrm{sch}} \longrightarrow (Z\otimes k)^{\mathrm{sch}} \longrightarrow (B\otimes k)^{\mathrm{sch}}$ gives rise to a classification morphism

$$(B\otimes k)^{\mathrm{sch}} \longrightarrow B\mathbb{R}\mathcal{AUT}((Z\otimes k)^{\mathrm{sch}}),$$

where $\mathbb{R}\mathcal{AUT}$ denotes the stack of aut-equivalences. Using [To1, Theorem 1.4.3] this morphism can also be considered as a morphism of $H_\infty$-stacks

$$\Omega_*(B\otimes k)^{sch} \longrightarrow \mathbb{R}\mathcal{AUT}((Z\otimes k)^{\mathrm{sch}}),$$

or in other words as an action loop stack $\Omega_*(B\otimes k)^{\mathrm{sch}}$ on $(Z\otimes k)^{\mathrm{sch}}$. This action contains of course the monodromy action of $\pi_1(B)$ on the homotopy groups of $(Z\otimes k)^{\mathrm{sch}}$, but also higher homotopical invariants, as for examples higher monodromy maps

$$\pi_i((B\otimes k)^{\mathrm{sch}}) \otimes \pi_n((Z\otimes k)^{\mathrm{sch}}) \longrightarrow \pi_{n+i-1}((Z\otimes k)^{\mathrm{sch}}).$$

A typical situation of applications of Proposition 4.20 is the case where $X \longrightarrow B$ is a smooth projective familly of simply connected complex varieties over a smooth and projective base. In this situation, the Hodge decomposition constructed in [Ka-Pa-To] is compatible with the monodromy action of $(B \otimes \mathbb{C})^{\mathrm{sch}}$ on the schematization of the fiber. In other words, if $Z$ is the homotopy type of the fiber, then $(Z \otimes \mathbb{C})^{\mathrm{sch}}$ has an action of $\Omega_*(B \otimes \mathbb{C})^{\mathrm{sch}}$ which is $\mathbb{C}^*$-equivariant with respect to the Hodge decomposition. This can also be interpreted by saying that $(Z \otimes \mathbb{C})^{\mathrm{sch}}$ form a *variation of non-abelian schematic Hodge structures on $B$*. Of course, this variation contains more information than the variations of Hodge structures on the rational homotopy groups of the fiber, since it captures higher homotopical data. An example of such non-trivial higher invariants is given in [Si].

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