Simplicial presheaves and derived algebraic geometry.

Bertrand Toen

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Bertrand Toën
Laboratoire Emile Picard
Université Paul Sabatier, Bat 1R2
31062 Toulouse Cedex , France
email: bertrand.toen@math.univ-toulouse.fr

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Introduction

Chapter 1

Motivations and objectives

The purpose of this first lecture is to present some motivations for derived algebraic geometry, and to present the objectives of the series of lectures. I will start by a brief review of the notion of moduli problems and moduli spaces. In a second part I will present the particular example of the moduli problem of linear representations of a discrete group. The study of this example will show the importance of two constructions useful to produce and understand moduli spaces: intersections (or more generally fiber products), and group quotients (or more generally quotient by groupoids). As many algebraic constructions these are not *exact* in some sense and possess derived versions. This will provides motivations for derived algebraic geometry, which is a geometrico-algebraic setting in which these derived versions exist and are well behaved.

We warn the reader that this section is highly informal and that several notions and ideas will be explained more formally later during the lectures.

1.1 The notion of moduli spaces

The main object studied in algebraic geometry are schemes (or more generally algebraic spaces, these notions will be redefined later). They often appear as solutions to *moduli functors* (or equivalently *moduli problems*), which intuitively means that their points classify certain geometico-algebraic objects (e.g. a scheme whose points are in one-to-one correspondence with algebraic subvarieties of the projective space $\mathbb{P}^n$). More precisely, we often are given a *moduli functor*

$$F : Comm \longrightarrow Set,$$

from the category of commutative rings to the category of sets. The set $F(A)$ has to be thought as the set of families of objects parametrized by the scheme $Spec\,A$ (we will see later many examples). When it exists, a scheme X is then called a *moduli space for* F (or a *solution to the moduli problem* F, we also say that *the scheme* X *represents* F or that F *is representable by* X) if there are functorial bijections

$$F(A) \simeq Hom(Spec\, A, X).$$

An important special case is when X is an affine scheme, say $Spec\, B$. Then, X represents F if and only if there are functorial isomorphisms

$$F(A) \simeq Hom(B, A),$$

where $Hom(B, A)$ is the set of ring morphisms from B to A.

We mention here a very basic, but foundamental, example of a moduli space, namely the projective space $\mathbf{P}^n$. Of crouse, more elaborate examples will be given later in these notes. We define a functor $P_n : Comm \longrightarrow Set$ as follows. For $A \in Comm$ we set $P_n(A)$ to be the set of sub-A-modules $M \subset A^{n+1}$ such that the quotient A^{n+1}/M is projective of rank 1 over A (i.e. is an invertible A-module). For $A \longrightarrow B$, a morphism in $Comm$, the application $F(A) \longrightarrow F(B)$ sends $M \subset A^{n+1}$ to $M \otimes_A B \subset B^{n+1}$. Note here that as A^{n+1}/M is projective, M is a direct factor in A^{n+1}, and thus $M \otimes_A B$ is a sub-A-module of B^{n+1}. This defines the functor P_n. It is well known that this functor is representable by a scheme, denoted by $\mathbb{P}_n$, and called the projective space of dimension n (see for instance ??).

The notion of moduli space is extremelly important for at least two reasons:

1. A good geometric understanding of the moduli space of a given moduli problem can be considered as a step towards a solution to the corresponding classification problem. For instance, a good enough understanding of the moduli space of algebraic curves could be understood as a solution to the problem of classifiying algebraic curves.

2. The notion of moduli problems is a rich source to construct new and interesting schemes. Indeed, the fact that a given scheme X is the solution to a moduli problem often makes its geometry rather rich. Typically the scheme will have interesting subschemes corresponding to objects satisfying certain additional properties.

1.2 Construction of moduli spaces: one example

For a given moduli problem $F : Comm \longrightarrow Set$ the question of the existence of a moduli space is never an easy question. There are two general strategies to prove the existence of such a moduli space, either by applying the so-called *Artin's representability theorem*, or by a more direct approach consisting of constructing the moduli space explicitly. The first approach is the most powerful to prove the existence, but the second one is often needed to have a better understanding of the moduli space itself (e.g. to prove that it satisfies some further properties). In this paragraph we will study the particular example of the moduli problem of linear representations of a discrete group, and will try to construct the corresponding moduli space by a direct approach. This example is chosen so that the moduli space does not exist, which is most often the case, but still the approach to the construction we present is right, at leats when it will be done in the context of derived algebraic geometry (we will of course come back to this fundamental example later on when the techniques of derived stacks will be at our disposal).

So let Γ be a group that will be assumed to be finitely presented. We want to study finite dimensional linear representations of Γ and for this we are looking for a moduli space of those. We start to define a moduli functor

$$R(\Gamma) : Comm \longrightarrow Set,$$

sending a commutative ring A to the set of isomorphism classes of $A[\Gamma]$-modules whose underlying A-module is projective and of finite type over A. As projective A-modules of finite type correspond to vector bundles on the scheme $Spec\, A$, $R(\Gamma)(A)$ can also be identified with the set of isomorphism classes of vector bundles on $Spec\, A$ endowed with an action of Γ. For a morphism of commutative rings $A \to A'$, we have a base change functor $- \otimes_A A'$ from $A[\Gamma]$-modules to $A'[\Gamma]$-modules, which induces a morphism

$$R(\Gamma)(A) \longrightarrow R(\Gamma)(A').$$

This defines the moduli functor $R(\Gamma)$.

The strategy to try to construct a solution to this moduli problem is to start to study a *framed* (or *rigidified*) version of it. We introduce for any integer n an auxiliary moduli problem $R'_n(\Gamma)$, whose values at a commutative ring A is the set of group morphisms $\Gamma \longrightarrow Gl_n(A)$. We set $R'(\Gamma) := \coprod_n R'_n(\Gamma)$, and we define a morphism (i.e. a natural transformation of functors)

$$\pi : R'(\Gamma) \longrightarrow R(\Gamma),$$

sending a morphism $\rho : \Gamma \longrightarrow Gl_n(A)$ to the A-module A^n together with the action of Γ defined by ρ. At this point we would like to argue on two steps:

1. The moduli functor $R'(\Gamma)$ is representable.
2. The moduli functor $R(\Gamma)$ is the disjoint union (for all n) of the quotients of the schemes $R'_n(\Gamma)$ by the group schemes Gl_n.

For the point (1), we write a presentation of Γ by generators and relations

$$\Gamma \simeq < g_1, \ldots, g_m > / < r_1, \ldots, r_p > .$$

From this presentation, we deduce the existence of a cartesian square of moduli functors

$$\begin{array}{ccc} R'_n(\Gamma) & \longrightarrow & Gl_n^m \\ \downarrow & & \downarrow \\ \{1\} & \longrightarrow & Gl_n^p, \end{array}$$

where Gl_n is the functor $A \mapsto Gl_n(A)$. The functor Gl_n is representable by an affine scheme. Indeed, if we set

$$C_n : \mathbb{Z}[T_{i,j}][Det(T_{i,j})^{-1}],$$

where $T_{i,j}$ are formal variables with $1 \leq i, j \leq n$, then the affine scheme $Spec\, C_n$ represents the functor Gl_n. This implies that Gl_n^r is also representable by an affine scheme for any integer r, precisely $Spec\,(C_n^{\otimes r})$. And finally, we see that $R'_n(\Gamma)$ is representable by the affine scheme

$$Spec\,(C_n^{\otimes m} \otimes_{C_r^{\otimes p}} \mathbb{Z}).$$

This sounds good, but an important observation here is that in general $C_n^{\otimes m}$ is not a flat $C_r^{\otimes p}$-algebra, and thus that the tensor product $C_n^{\otimes m} \otimes_{C_r^{\otimes p}} \mathbb{Z}$ is not well behaved from the point of view of homological algebra. Geometrically this is related to the fact that $R'_n(\Gamma)$ is the intersection of two subschemes in $Gl_n^m \times Gl_n^p$, namely the graph of the morphism $Gl_n^m \longrightarrow Gl_n^p$ and $Gl_n^m \times \{1\}$, and that these two subschemes are not in general position. Direct consequences of this is the fact that the scheme $R'_n(\Gamma)$ can be badly singular at certain points, precisely the points for which the above intersection is not transversal. Another bad consequence is that the tangent complex (a derived version of the tangent space, we will review this notion later in the course) is not easy to compute for the scheme $R'_n(\Gamma)$. The main philosophy of derived algebraic geometry is that the tensor product $C_n^{\otimes m} \otimes_{C_r^{\otimes p}} \mathbb{Z}$ should be replaced by its derived version $C_n^{\otimes m} \otimes^{\mathbb{L}}_{C_r^{\otimes p}} \mathbb{Z}$ which also encodes the higher Tor's $Tor_*^{C_r^{\otimes p}}(C_n^{\otimes m}, \mathbb{Z})$, for instance by considering simplicial commutative rings. Of course $C_n^{\otimes m} \otimes^{\mathbb{L}}_{C_r^{\otimes p}} \mathbb{Z}$ is no longer a commutative ring and thus the notion of schemes should be extended in order to be able to consider object of the form $''Spec\, A''$, where A is now a simplicial commutative ring.

Exercise 1.2.1. *Suppose that* $\Gamma = \mathbb{Z}^2$*, presented by the standard presentation* $\Gamma =< g_1, g_2 > / < [g_1, g_2] >$*. Show that the morphism* $C_n \longrightarrow C_n^{\otimes 2}$ *is indeed a non-flat morphism.*

We now consider the point (2). The functor $Gl_n : Comm \longrightarrow Set$ sending A to $Gl_n(A)$ is a group object (in the category of functors) and it acts naturally on $R'_n(\Gamma)$. For a given $A \in Comm$, the action of $Gl_n(A)$ on $R'_n(\Gamma)(A) = Hom(\Gamma, Gl_n(A))$ is the one induced by the conjugaison action of $Gl_n(A)$ on itself.

The morphism $R'_n(\Gamma) \longrightarrow R(\Gamma)$ is equivariant for this action and thus factorizes as a morphism

$$R'_n(\Gamma)/Gl_n \longrightarrow R(\Gamma).$$

We thus obtain a morphism of functors

$$\coprod_n R'_n(\Gamma)/Gl_n \longrightarrow R(\Gamma).$$

Intuitively this morphism should be an isomorphism and in fact it is close to be. It is a monomorphism but it is not an epimorphism because not every projective A-module of finite type is free. However, up to a localization for the Zariski topology on $Spec\, A$ this is the case, and therefore we see that the above morphism is an epimorphism in the sense of sheaf theory. In other words, this morphism is an isomorphism if the left hand side is understood as the *quotient sheaf* with respect to the Zariski topology on the category $Comm^{op}$. This sounds like a good situation as both functors $R'_n(\Gamma)$ and Gl_n are representable by affine schemes. However, the quotient sheaf of an affine scheme by the action of an affine group scheme is in general not a scheme when the action has fixed points. It is for instance not so hard to see that the quotient sheaf $\mathbb{A}^1/(\mathbb{Z}/2)$ is not representable by a scheme (here $\mathbb{A}^1 = Spec\, \mathbb{Z}[T]$ is the affine line and the action is induced by the involution $T \mapsto -T$, see Ex. 1.2.2). In our situation the action of Gl_n on $R'_n(\Gamma)$ has many fixed points, as for a given $A \in Comm$, the stabilizer of a given morphism $\Gamma \longrightarrow Gl_n(A)$ is precisely the group of automorphisms of the corresponding $A[\Gamma]$-module. We see here that the reason for the non-representability of the quotients $R'_n(\Gamma)/Gl_n$ is the existence of non trivial automorphism groups. Here the philosophy is the same of for the previous point, the quotient construction is not exact in some sense and should be derived. The derived quotient of a group G acting on a set X is the groupoid $[X/G]$, whose objects are the points of x and whose morphisms from x to y are the elements $g \in G$ such that $g.x = y$. The set of isomorphism classes of objects in $[X/G]$ is the usual quotient X/G, but the derived quotient $[X/G]$ also remembers the stabilizers of the action in the automorphism groups of $[X/G]$. This suggests that the right think to do is to replace $\coprod_n R'_n(\Gamma)/Gl_n$ by the more evolved construction $\coprod_n [R'_n(\Gamma)/Gl_n]$, which is now a functor from $Comm$ to the catgeory of groupoids rather than the category of sets. In the same way, this suggests that the functor $R(\Gamma)$ should rather be replaced by $\underline{R}(\Gamma)$, sending a commutative ring A to the whole groupoid of $A[\Gamma]$-modules whose underlying A-module is projective and of finite type. We see here that we again need to extend the notion of schemes in order to be able to find a geometric object representing $\underline{R}(\Gamma)$, as the functor represented by a scheme is always set valued by definition.

Exercise 1.2.2. *Let $\mathbb{Z}/2$ acts on the scheme $\mathbb{A}^1 = Spec\, \mathbb{Z}[T]$, by $T \mapsto -T$. We note $F : Comm \longrightarrow Set$ the functor represented by $\mathbb{A}^1$.*

1. *Show that the quotient of $\mathbb{A}^1$ by $\mathbb{Z}/2$ exists in the category of affine schemes, and is isomorphic to $Spec\, (\mathbb{Z}[T]^{\mathbb{Z}/2}) \simeq \mathbb{A}^1$.*

2. *Suppose now that we are given a Grothendieck topology τ on $Comm^{op} = Aff$, the category of affine schemes. Let $F_0 = F/\mathbb{Z}/2$ be the quotient sheaf for this topology. Prove that the natural morphism*

$$F \times \mathbb{Z}/2 \longrightarrow F \times_{F_0} F$$

is an epimorphism of sheaves.

3. *Show that if the topology τ is sub-canonical then F_0 is represented by an affine scheme if and only if it is represented by $\mathbb{A}^1$.*

4. *Assume now that τ is the ffp (flat and finitely presented) topology. Use (2) and (3) to show that F_0 can not be represented by an affine scheme.*

1.3 Conclusions

We arrive at the conclusions of this first lecture. The fundamental objects of algebraic geometry are functors

$$Comm \longrightarrow Set.$$

However we have seen that certain constructions on rings (tensor products), or on sets (quotients), are not exact and should rather be derived in order to be better behaved. Deriving the tensor product for commutative rings forces us to introduce simplicial commutative rings, an deriving quotients forces us to introduce groupoids (when it is a quotient by a group) and more generally simplicial sets (when it is a more complicate quotient). The starting point of derived algebraic geometry is that its fundamental objects are functors

$$sComm \longrightarrow SSet,$$

from the category of simplicial commutative rings to the category of simplicial sets. The main objective of the series of lectures is to explain how the basic notions of algebraic geometry (schemes, algebraic spaces, flat, smooth and étale morphisms ...) can be extended to this derived setting, and how this is useful for the study of moduli problems.

We will proceed in two steps. We will first explain how to do half of the job and to allow derived quotients but not derived tensor products (i.e. considering functors $Comm \longrightarrow SSet$). In other words we will start to explain formally how the quotient problem (point (2) of the discussion of the last paragraph) can be solved. This will be done by introducing the notions of *stacks* and *algebraic stacks*, which is based on the well known homotopy theory of simplicial presheaves of Joyal-Jardine. Later on we will explain how to incorporate derived tensor products and simplicial commutative rings to the picture.

Chapter 2

Simplicial presheaves as stacks

The purpose of this second lecture is to present the homotopy theory of simplicial presheaves on a Grothendieck site, and explain how these are models for stacks. In the next lecture, simplicial presheaves will be used to produce models for (higher) stacks in the context of algebraic geometry and will allow us to define the notion of algebraic n-stacks, a far reaching generalization of the notion of schemes for which all quotients by reasonable equivalence relations exists.

2.1 Review of the model category of simplicial presheaves

We let (C, τ) be a Grothendieck site. Recall that this means that we are given a category C, together a Grothendieck topology τ on C. The Grothendieck topology τ is the data for any object $X \in C$ of a family $cov(X)$ of sieves over X (i.e. sub-functors of the representable functor $h_X := Hom(-, X)$) satisfying the following three conditions.

1. For any $X \in C$, we have $h_X \in cov(x)$.
2. For any morphism $f : Y \to X$ in C, and any $u \in cov(X)$, we have $f^*(u) := u \times_{h_X} h_Y \in cov(Y)$.
3. Let $X \in C$, $u \in cov(X)$, and v be any sieve on X. If for all $Y \in C$ and any $f \in u(Y) \subset Hom(Y, X)$ we have $f^*(v) \in cov(Y)$, then $v \in cov(X)$.

Recall that for such a Grothendieck site we have its associated category of presheaves $Pr(C)$, which by definition is the category of all functors from C^{op} to the category of sets. The full sub-category of sheaves $Sh(C)$ is defined to be the sub-category of presheaves $F : C^{op} \longrightarrow Set$ such that for any $X \in C$ and any $u \in cov(X)$, the natural morphism

$$F(X) \simeq Hom_{Pr(C)}(h_X, F) \longrightarrow Hom_{Pr(C)}(u, F)$$

is bijective.

A standard result from sheaf theory states that the inclusion functor

$$i : Sh(C) \hookrightarrow Pr(C)$$

has an exact (i.e. commutes with finite limits) left adjoint

$$a : Pr(C) \longrightarrow Sh(C)$$

called the associated sheaf functor.

We now let $SPr(C)$ be the category of simplicial objects in $Pr(C)$. We start to endow the category $SPr(C)$ with a levelwise model category structure defined as follows.

Definition 2.1.1. *Let $f : F \longrightarrow F$ be a morphism in $Pr(C)$.*

1. *The morphism f is a* global fibration *if for any $X \in C$ the induced morphism*

$$F(X) \longrightarrow F'(X)$$

   is a fibration of simplicial sets (for the standard model category structure, i.e. is a Kan fibration).

2. *The morphism f is an* global equivalence *if for any $X \in C$ the induced morphism*

$$F(X) \longrightarrow F'(X)$$

   is an equivalence[1] of simplicial sets (again for the standard model category structure on simplicial sets).

3. *The morphism f is a* global cofibration *if it has the right lifting property with respect to every fibration which is also an equivalence.*

It is well known that the above definitions endow the category $SPr(C)$ with a cofibrantly generated model category structure. This model category is moreover proper and cellular (in the sense of [Hi]). This model structure will be refered to the *global model structure*. There is a small set theory problem here when the category C is not *small*. This problem can be easily solved by fixing universes and will be simply neglected in the sequel.

We now take into account the Grothendieck topology τ on C in order to rafine the global model structure. This is an important step as when the quotient of group action on a scheme exists, the presheaf represented y the quotient scheme is certainly not the quotient presheaf. However, for free actions the sheaf represented y the quotient scheme is the quotient sheaf.

[1] In these notes the expression *equivalence* always refers to *weak equivalence*.

We start by introducing the so-called homotopy sheaves of a simplicial presheaf $F : C^{op} \longrightarrow SSet$. We define a presheaf

$$\pi_0^{pr}(F) : C^{op} \longrightarrow Set$$

simply by sending $X \in C$ to $\pi_0(F(X))$. In the same way, for any $X \in C$ and any 0-simplex $s \in F(X)_0$ we define presheaves of groups on C/X

$$\pi_i^{pr}(F, s) : (C/X)^{op} \longrightarrow Gp$$

sending $f : Y \to X$ to $\pi_i(F(Y), f^*(s))$. Here, $F(Y)$ is the simplicial set of values of F over Y, $f^*(s) \in F(Y)_0$ is the inverse image of the base point s, and finally $\pi_i(F(Y), f^*(s))$ denotes the *correct* homotopy groups of the simplicial set $F(Y)$ based at $f^*(s)$. By *correct* we mean either the simplicial (or combinatorial) homotopy groups of a fibrant model for $F(Y)$, or more easily the topological homotopy groups of the the geometric realization $|F(Y)|$.

The associated sheaves to the presheaves $\pi_0^{pr}(F)$ and $\pi_i^{pr}(F, s)$ will be denoted by $\pi_0(F)$ and $\pi_i(F, s)$. These are called the *homotopy sheaves of* F. They are functorial in F.

Definition 2.1.2. *Let $f : F \longrightarrow F'$ be a morphism of simplicial presheaves.*

1. *The morphism f is a* local equivalence *if it satisfies the following two conditions*
   (a) *The induced morphism $\pi_0(F) \longrightarrow \pi_0(F')$ is an isomorphism of sheaves.*
   (b) *For any $X \in C$, any $s \in F(X)_0$ and any $i > 0$ the induced morphism $\pi_i(F, s) \longrightarrow \pi_i(F', f(s))$ is an isomorphism of sheaves on C/X.*
2. *The morphism f is a* local cofibration *if it is a global cofibration in the sense of definition 2.1.1.*
3. *The morphism f is a* local fibration *if it has the left lifting property with respect to every local cofibration which is also a local equivalence.*

For simplicity, we will use the expressions equivalence, fibrations *and* cofibration *in order to refer to* local equivalence, local fibration *and* local cofibration.

It is also well known that the above definition endows the category $SPr(C)$ with a model category structure, but this is a much harder result than the existence of the global model structure. This result, as well as several small modifications, is due to Joyal (for simplicial sheaves) and Jardine (for simplicial presheaves), and we refer to [Bl, DHI] for recent references. Unless the contrary is specified we will always assume that the category $SPr(C)$ is endowed with this model category structure, which will be called the *local model structure*.

A nice result proved in [DHI] is the following characterization of fibrant objects in $SPr(C)$ (for the local model structure). Recall first that a hypercovering of an object $X \in C$ is the data of a simplicial presheaf F together with a morphism $H \longrightarrow X$ and satisfying the following two conditions.

1. For any integer n the presheaf H_n is a disjoint union of representable presheaves.
2. For any $n \geq 0$ the morphism of presheaves (of sets)

$$H_n \simeq Hom(\Delta^n, H) \longrightarrow Hom(\partial\Delta^n, H) \times_{Hom(\partial\Delta^n, X)} Hom(\Delta^n, X)$$

   induces an epimorphism on the associated sheaves.

Here Δ^n denotes the simplicial simplex of dimension n as well as the corresponding constant simplicial presheaf. In the same way $\partial\Delta^n$ is the $(n-1)$-skeleton of Δ^n and is considered here as a constant simplicial presheaf. Finally, Hom denote here the presheaves of morphisms between two simplicial presheaves. This second condition can equivalently be stated by saying that for any $Y \in C$, and any commutative square of simplicial sets

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & H(Y) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & X(Y) = Hom(Y, X), \end{array}$$

there exists a covering sieve $u \in cov(Y)$ such that for any $f : U \to Y$ in the sieve u there exists a morphism $\Delta^n \longrightarrow H(U)$ making the following square commutative

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & H(U) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & X(U) = Hom(U, X). \end{array}$$

This property is also called the *local lifting property*, and is the local analog of the lifting property characterizing trivial fibrations of simplicial sets. In particular it implies that the homotopy sheaves of H and of X coincide, and thus that $H \longrightarrow X$ is a local equivalence. In low dimensions, this local lifting condition says the following:

- (n=0) The morphism $H_0 \longrightarrow X$ induces an epimorphism on the associated sheaves.
- (n=1) The morphism $H_1 \longrightarrow H_0 \times_X H_0$ induces an epimorphism on the associated sheaves.
- (n=2) The morphism $H_2 \longrightarrow H_1 \times_{H_0 \times H_0} (H_1 \times_{H_0} H_1)$ induces an epimorphism on the associated sheaves.

Note that for $n > 1$ the simplicial set $\partial\Delta^n$ is connected, and thus the morphism $X^{\Delta^n} \longrightarrow X^{\partial\Delta^n}$ is, in this case, an isomorphism. This implies that for $n > 1$ the condition 2. of being an hypercovering is equivalent to the simpler condition that

$$H_n \longrightarrow H^{\partial\Delta^n}$$

induces an epimorphism on the associated sheaves. Therefore, this condition only depends on H_* for $n > 1$, and not upon X or upon the morphism $H_* \longrightarrow X$.

For $F \in SPr(C)$, any $X \in C$ and any hypercovering H of X, we can define an augmented cosimplicial diagram of simplicial sets

$$F(X) \longrightarrow ([n] \mapsto F(H_n)) .$$

Here, each H_n is a coproduct of representables, say $H_n = \coprod_i H_{n,i}$, and by definition we set $F(H_n) = \prod_i F(H_{n,i})$ (when this product is infinite it should be taken with some care, by first replacing each $F(H_{n,i})$ by their fibrant models).

With these notions and notations it is possible to prove (see [DHI]) that an object $F \in SPr(C)$ is fibrant (for the local model structure) if and only if it satisfies the following two conditions.

1. For any $X \in C$ the simplicial set $F(X)$ is fibrant.
2. For any $X \in C$ and any hypercovering $H \longrightarrow X$ the natural morphism

$$F(X) \longrightarrow Holim_{[n]\in\Delta} F(H_n)$$

is an equivalence of simplicial sets.

The above first condition is rather anodyne and the second condition is of course the important one. It is a homotopy analog of the sheaf conditions, in the sense that when F is a presheaf of sets, considered as a simplicial presheaf constant in the simplicial direction, then this second condition for F is equivalent to the fact that F is a sheaf (this is because the homotopy limits is then simply a usual limits in the category of sets and the condition becomes the usual descent conditions for sheaves).

Definition 2.1.3. *1. An object $F \in SPr(C)$ is called a* stack *if for any $X \in C$ and any hypercovering $H \longrightarrow X$ the natural morphism*

$$F(X) \longrightarrow Holim_{[n]\in\Delta} F(H_n)$$

is an equivalence of simplicial sets.

2. The homotopy category $Ho(SPr(C))$ will be called the homotopy category of stacks on the site (C, τ) *(or simply the* category of stacks*). Most often objects* *in $Ho(SPr(C))$ will simply be called* stacks. *The expressions* morphism of stacks *and* isomorphism of stacks*, will refer to morphisms and isomorphisms in $Ho(SPr(C))$. The set of morphisms of stacks from F to F' will be denoted by $[F, F']$.*

The following exercise is to understand that, for a given simplicial presheaf, being a stack and being a sheaf of simplicial sets are two different notions having nothing in common.

Exercise 2.1.4. *1. Show that a presheaf of sets $F : C^{op} \longrightarrow Set$, considered as a simplicial presheaf, is a stack if and only if it is a sheaf of sets.*

2. Let G be sheaf of groups on C and consider the simplicial presheaf

$$\begin{array}{rccc} BG: & C^{op} & \longrightarrow & SSet \\ & X & \mapsto & B(G(X)). \end{array}$$

Here, if H is any discrete group BH is its simplicial classifying space, whose set of n-simplices is G^n, and whose face maps are given by the group structure together with the various projection, and whose degeneracies are given by the generalized diagonal maps. Prove that BG is a sheaf in simplicial sets, but that it is not a stack as soon as there exists an object $X \in C$ with $H^1(X, G) \neq *$.

3. For any $X \in C$ we let $\mathcal{F}(X)$ be the nerve of the groupoid of sheaves of sets over X (its objects are sheaves of sets on the site C/X, and its morphisms are isomorphisms between such sheaves). Show how to make $X \mapsto \mathcal{F}(X)$ into a simplicial presheaf on C. Show that $\mathcal{F}$ is a stack which is not a sheaf of simplicial sets in general (e.g. show that the set valued presheaf of 0-simplices $\mathcal{F}_0$ is not a sheaf on C).

In the sequel, we will often use the following terminology and notations.

- For a diagram of stacks $F \longrightarrow H \longleftarrow G$, we denote by $F \times^h_H G$ the corresponding homotopy fiber product of simplicial presheaves (note that this constuction is not functorially defined on $Ho(SPr(C))$ and requires some lift of the diagrams to $SPr(C)$).
- A morphism of stacks $F \longrightarrow Ho(SPr(C))$ is an *epimorphism* if the induced morphism $\pi_0(F) \longrightarrow \pi_0(F')$ is a sheaf epimorphism.
- A morphism of stacks $F \longrightarrow F'$ is a *monomorphism* if the diagonal morphism $F \longrightarrow F \times^h_{F'} F$ is an isomorphism.

Exercise 2.1.5. *1. Show that a morphism of stacks $F \longrightarrow F'$ is a monomorphism if and only if it satisfies the following two conditions*

(a) The induced morphism $\pi_0(F) \longrightarrow \pi_0(F')$ is a monomorphism.

(b) *For all* $X \in C$ *and all* $s \in F(X)$ *the induced morphisms* $\pi_i(F, s) \longrightarrow \pi_i(F', f(s))$ *are isomorphisms for all* $i > 0$.

2. *Show that a morphism of stacks* $F \longrightarrow F'$ *is an epimorphism (resp. a monomorphism), if and only if for any* $X \ni C$ *and any morphism* $X \longrightarrow F'$ *in* $Ho(SPr(C))$, *the induced projection* $F \times^h_{F'} X \longrightarrow X$ *is an epimorphism (resp. a monomorphism).*

2.2 Basic examples

The most foundamental example of a stack is the *stack of stacks*, whose existence express that *stacks can be defined locally and glued.* This example of a stack is important for conceptual reasons, but also because it can be used to construct many examples of other stacks. Its precise definition goes as follows. For $X \in C$, we consider $SPr^W(C/X)$, the category whose objects are simplicial presheaves on C/X and whose morphisms are the local equivalences. For a morphism $f : Y \to X$ in C we have a base change functor $SPr^W(C/X) \longrightarrow SPr^W(C/Y)$, making $X \mapsto SPr^W(C/X)$ into a presheaf of categories. Taking the nerve of all the categories $SPr^W(C/X)$ we obtain a simplicial presheaf

$$\begin{array}{cccc} \mathcal{S}: & C^{op} & \longrightarrow & SSEt \\ & X & \mapsto & N(SPr^W(C/X)). \end{array}$$

Theorem 2.2.1. *The simplicial presheaf* $\mathcal{S}$ *defined above is a stack. It is called the* stack of stacks.

Sketch of a proof: There exist several different ways to prove the above theorem, unfortunately none of them being really easy. We sketche here the main steps for one of them, but a complete and detailed proof would be much too long for these notes, as well as not very instructive.

Step 1: We first extend the functor $\mathbb{S} : C^{op} \longrightarrow SSet$, to a functor

$$\mathbb{S}' : SPr(C)^{op} \longrightarrow SSet.$$

This will cause some set theoretic troubles because $SPr(C)$ is a *non-small* category. This issue can be solved in at least two different ways, either by using universes, or by choosing a large enough bound on the cardinalities of the presheaves we want to consider. Now, for $F \in SPr(C)^{op}$, we consider Fib^W/F, the category whose objects are fibrations $F' \longrightarrow F$ in $SPr(C)$, and whose morphismes are local equivalences in $SPr(C)/F$. When $F' \longrightarrow F$ we have a base change functor

$$F' \times_F - : Fib^W/F \longrightarrow Fib^W/F'.$$

This does not define a presheaf of categories on $SPr(C)$, but only a lax functor. However, any lax functor is equivalent to a strict functor by a natural construction called rectification (see for example [Hol, §3.3]). We will therefore do as if

$F \mapsto Fib^W/F$ were a genuine presheaf of categories. Taking the nerves of all the categories Fib^W/F provides a simplicial presheaf

$$\begin{array}{cccc} \mathcal{S}' : & SPr(C)^{op} & \longrightarrow & SSEt \\ & F & \mapsto & N(Fib^W/F). \end{array}$$

This simplicial presheaf, restricted to $C \hookrightarrow SPr(C)$ is naturally equivalent to $\mathbb{S}$, as it values on $X \in C$ is the nerve of the category of equivalences between fibrant objects in $SPr(C)/X \simeq SPr(C/X)$. This nerve is itself equivalent to the nerve of local equivalences between all simplicial objects in $SPr(C/X)$, as the fibrant replacement functor gives an inverse up to homotopy of the natural inclusion.

The conclusion of this first step is that $\mathcal{S}$ possesses, up to a natural equivalence, an extension $\mathcal{S}'$ has a presheaf on $SPr(C)$.

Step 2: The functor $\mathcal{S}' : SPr(C)^{op} \longrightarrow SSet$ sends local equivalences to equivalences, and homotopy colimits ni $SPr(C)$ to homotopy limits in $SSet$. Indeed, the fact that $\mathcal{S}'$ preserves equivalences follows formally from the fact that the model category $SPr(C)$ is right proper. That it sends homotopy colimits to homotopy limits is more subtle. First of all, any homotopy colimit can be obtained by a succession of homotopy push-outs and homotopy disjoint unions. Therefore, in order to prove that $\mathcal{S}'$ sends homotopy colimits to homotopy limits it is enough to prove the following two statements.

1. For any family of objects $\{F_i\}_{i \in I}$ in $SPr(C)$, the natural morphism

$$\mathcal{S}'(\coprod_i F_i) \longrightarrow \prod_i \mathcal{S}'(F_i)$$

is a weak equivalence.

2. For any homotopy push-out diagram in $SPr(C)$

$$\begin{array}{ccc} F_0 & \longrightarrow & F_1 \\ \downarrow & & \downarrow \\ F_2 & \longrightarrow & F, \end{array}$$

the induced diagram

$$\begin{array}{ccc} \mathcal{S}'(F) & \longrightarrow & \mathcal{S}'(F_2) \\ \downarrow & & \downarrow \\ \mathcal{S}'(F_1) & \longrightarrow & \mathcal{S}'(F_0) \end{array}$$

is homotopy cartesian in $SSet$.

The statement (1) above follows from the fact that the model category $SPr(C)/(\coprod_i F_i)$ is the product of the model categories $SPr(C)/F_i$ (there is a small issue with infinite products that we do not mention here). The statement (2) is the key of the proof of the theorem and is the hardest point. It can be deduced from [Re, Thm. 1.4] as follows. We assume that we have a diagram $F_1 \xleftarrow{i} F_0 \xrightarrow{j} F_2$, with i and j cofibrations between cofibrant objects (requiring one of the two morphisms to be a cofibration would be enough here). We let F be the push-out of this diagram in $SPr(C)$, which is therefore also a homotopy push-out. We have a diagram of Quillen adjunctions obtained by base change (we write here the right adjoints)

$$\begin{array}{ccc} SPr(C)/F & \longrightarrow & SPr(C)/F_1 \\ \downarrow & & \downarrow \\ SPr(C)/F_2 & \longrightarrow & SPr(C)/F_0. \end{array}$$

Because of [Re, Thm. 1.4] this diagram of model categories satisfies the (opposite) conditions of [To2, Lem. 4.2], which insures that the corresponding diagram of simplicial sets obtained by taking the nerve of the categories of equivalences between fibrant objects

$$\begin{array}{ccc} Fib^W/F & \longrightarrow & Fib^W/F_1 \\ \downarrow & & \downarrow \\ Fib^W/F_2 & \longrightarrow & Fib^W/F_0 \end{array}$$

is homotopy cartesian.

Step 3: We can now conclude from steps 1 and 2 that $\mathcal{S}$ is a stack. Indeed, let $H \longrightarrow X$ be a hypercovering. The morphism

$$\mathcal{S}(X) \longrightarrow Holim_n \mathcal{S}(H_n)$$

is equivalent to

$$\mathcal{S}'(X) \longrightarrow Holim_n \mathcal{S}'(H_n).$$

But, as $\mathcal{S}'$ converts homotopy colimits to homotopy limits we see that this last morphism is an equivalence because the morphism

$$H \simeq Hocolim H_* \longrightarrow X$$

is a local equivalence. □

To finish this section we present some basic and general examples of stacks. These are very general examples and we will see more specific examples in the context of algebraic geometry in the next lectures.

Sheaves: We start by noticing that there is a full embedding

$$Sh(C) \longrightarrow Ho(SPr(C))$$

from the category of sheaves (of sets) to the homotopy category of stacks, simply by considering a sheaf of sets as a simplicial presheaf (constant in the simplicial direction). This inclusion functor has a left adjoint, which sends a simplicial presheaf F to the sheaf $\pi_0(F)$. This will allow us to consider any sheaf as a stack, and in the sequel we will do this implicitely. In this way, the category of stacks $Ho(SPr(C))$ is an extension of the category of sheaves. Moreover, any objects in $Ho(SPr(C))$ is isomorphic to a homotopy colimit of sheaves (this is because any simplicial set X is naturally equivalent to the homotopy colimit of the diagram $[n] \mapsto X_n$), which shows that stacks are obtained from sheaves by taking derived quotients.

Classifying stacks: Let G be a group object in $SPr(C)$, that is a presheaf of simplicial groups. From it we construct a simplicial presheaf BG by applying levelwise the classifying space construction. More explicitely BG is the simplicial presheaf whose presheaf of n-simplices is G_n^n, and whose face and degeneracies are defined using the composition and units in G as well as the face and degeneracies of the underlying simplicial set of G. The simplicial presheaf BG has a natural global point $*$, and by construction we have

$$\pi_i(BG, *) \simeq \pi_{i-1}(G, e).$$

When G is abelian, the simplicial presheaf BG is again an abelian group object in $SPr(C)$, and the construction can then be iterated.

When A is a sheaf of abelian groups on C we let

$$K(A, n) := \underbrace{B \dots B}_{n\ times}(A).$$

By construction $K(A, n)$ is a pointed simplicial presheaf such that

$$\pi_i(K(A, n), *) \simeq 0\ if\ i \neq n \qquad \pi_n(K(A, n), *) \simeq A,$$

and this property characterizes $K(A, n)$ uniquely up to an isomorphism in $Ho(SPr(C))$.

Exercise 2.2.2. *1. Let $X \in C$ and $H_* \longrightarrow X$ by a hypercovering. Show that there exists natural isomorphisms*

$$\pi_i(Holim_{n \in \Delta} K(A, n)(H_n)) \simeq \check{H}^{n-i}(H_*/X, A),$$

where the right hand side is Chech cohomology of X with coefficient in A and with respect to the hypercovering H_* (see [Ar-Ma]).

2. *Deduce from (1) that the simplicial presheaf $K(A,n)$ is a stack if and only if A is a locally acyclic sheaf (i.e. for any $X \in C$, we have $H^i(X,A) = 0$ for $i > 0$).*

3. *Use (2) and induction on n to prove that for any $X \in C$ there exist natural isomorphisms*
$$[X, K(A,n)] \simeq H^n(X,A),$$
where the left hand side is the set of morphisms in $Ho(SPr(C))$ and the right hand side denotes sheaf cohomology.

Truncations and n-stacks: A stack $F \in Ho(SPr(C))$ is *n-truncated*, or en *n-stack*, if for any $X \in C$ and any $s \in F(X)_0$ we have $\pi_i(F,s) = 0$ for all $i > n$. The full sub-category of n-stacks will be denoted by $Ho(SPr_{\leq n}(C))$. We note that $Ho(SPr_{\leq 0}(C))$ is the essential image of the inclusion morphism $Sh(C) \longrightarrow Ho(SPr(C))$, and thus that there exists an equivalence of categories $Sh(C) \simeq Ho(SPr_{\leq 0}(C))$.

The inclusion functor $Ho(SPr_{\leq n}(C)) \hookrightarrow Ho(SPr(C))$ admits a left adjoint
$$t_{\leq n} : Ho(SPr(C)) \longrightarrow Ho(SPr_{\leq n}(C))$$
called the *truncation functors*. We have $t_{\leq 0} \simeq \pi_0$, and in general $t_{\leq n}$ is obtained by applying levelwise the usual truncation functor for simplicial sets. Another possible understanding of this situation is by intoducing the left Bousfield localization (in the sense of [Hi]) of the model category $SPr(C)$, by inverting all the morphisms $\partial\Delta^{n+2} \times X \longrightarrow X$, for all $X \in C$. The fibrant objects for this localized model structure are precisely the n-truncated fibrant simplicial presheaves, and its homotopy category can be naturally identified with $Ho(SPr_{\leq n}(C))$. The functor $t_{\leq n}$ is then the localization functor for this left Bousfield localization.

For any stack F, there exists a tower of stacks
$$F \longrightarrow \cdots \longrightarrow t_{\leq n}(F) \longrightarrow t_{\leq n-1}(F) \longrightarrow \cdots \longrightarrow t_{\leq 0}(F) = \pi_0(F),$$
called the Postnikov tower for F. A new feature here is that this tower does not converge in general, or in other words the natural morphism
$$F \longrightarrow Holim_n t_{\leq n}(F)$$
is not an equivalence in general. It is the case under some rather strong boundness conditions on the cohomological dimension of the sheaves of groups $\pi_i(F)$.

Exercise 2.2.3. *Suppose that for there exists an integer d such that for any $X \in C$ and any sheaf of abelian groups A on C/X we have $H^i(X,A) = 0$ for all $i > d$. Prove that for any stack F the natural morphism*
$$F \longrightarrow Holim_n t_{\leq n}(F)$$
is an isomorphism in $Ho(SPr(C))$.

Internal Hom: An important property of the category $Ho(SPr(C))$ is that it admits internal Homs (i.e. is cartesian closed). One way to see this is to use the injective model structure on $SPr(C)$, originally introduced in [Ja]. In order to distinguish this model structure from the projective model structure we are using in these notes, we denote by $SPr_{inj}(C)$ the category of simplicial presheaves endowed with the injective model structure. Its equivalences are the local equivalences of 2.1.2, and its cofibrations are the monomorphisms of simplicial presheaves. The nice property of the model category $SPr_{inj}(C)$ is that it becomes a monoidal model category in the sense of [Ho], when endowed with the monoidal structure given by the direct product. A formal consequence of this is that $Ho(SPr_{inj}(C)) = Ho(SPr(C))$ is cartesian closed (see for instance [Ho, Thm. 4.3.2]).

Explicitely, if F and F' are two stacks, we define a simplicial presheaf

$$\mathbb{R}\underline{\mathcal{H}om}(F, F') : C^{op} \longrightarrow SSet,$$

by

$$\mathbb{R}\underline{\mathcal{H}om}(F, F')(X) := \underline{Hom}(X \times F, R(F')),$$

where $\underline{Hom}$ denotes the natural simplicial enrichement of the category $SPr(C)$and $R(F')$ is a fibrant model for F' as an object in $SPr_{inj}(C)$. When the object $\mathbb{R}\underline{\mathcal{H}om}(F, F')$ is considered in $Ho(SPr(C))$ it is possible to show that we have functorial isomorphisms

$$[F'', \mathbb{R}\underline{\mathcal{H}om}(F, F')] \simeq [F'' \times F, F']$$

for any $F'' \in Ho(SPr(C))$. The stack $\mathbb{R}\underline{\mathcal{H}om}(F, F')$ is called the stack of morphisms from F to F'.

Exercise 2.2.4. *Assume that C possesses finite products. Prove that the model category $SPr(C)$, as usual with its projective local model structure, is a monoidal model category for the monoidal structure given by the direct product (for this use e.g. [Ho, Cor. 4.2.5] and the explicit generating cofibrations $A \times X \to B \times X$, where $A \to B$ is a cofibration in $SSet$).*

Substacks defined by local conditions: Let F be a stack on C, and F_0 its presheaf of 0-dimensional simplices. A *condition on objects on F* is, by definition, a sub-presheaf $G_0 \subset F_0$. Such a condition is *saturated* if there exists a pull-back square

$$\begin{array}{ccc} G_0 & \longrightarrow & F_0 \\ \downarrow & & \downarrow \\ E & \longrightarrow & \pi_0^{pr}(F). \end{array}$$

Equivalently, the condition is saturated if for any $X \in C$, the subset $G_0(X) \subset F_0(X)$ is a union of connected components. Finally, we say that a saturated condition on F is *local*, if for any $X \in C$, and any $x \in F_0(X)$ we have

$$(x \in G_0(X)) \Longleftrightarrow (\exists\, u \in cov(X)\ s.t.\ f^*(x) \in G_0(U)\ \forall\, f : U \to X\ in\ u)\,.$$

Let $G_0 \subset F_0$ be a saturated local condition on F. We define a sub-simplicial presheaf $G \subset F$ as follows: for $[n] \in \Delta$, $G(X)_n$ is the the subset of $F(X)_n$ consisting of all n-dimensional simplices α such that all 0-dimensional faces of α belong to $G_0(X)$. As the condition is saturated $G(X)$ is a union of connected components of $F(X)$. Moreover, as the condition is local and as F is a stack, it is easily seen that G is itself a stack. The stack G is called the *sub-stack of F defined by the condition G_0*.

Twisted forms: We let $F \in Ho(SPr(C))$ be a stack. We consider the following condition G_F on the stack of stacks $\mathcal{S}$. For $X \in C$, the set $\mathcal{S}(X)_0$ is by definition the set of simplicial presheaves on C/X. We let $G_F(X) \subset \mathcal{S}(X)_0$ be the subset corresponding of all simplicial presheaves F' such that there exists a covering sieve $u \in cov(x)$, such that for all $U \to X$ in u, the restrictions of F and F' are isomorphic in $Ho(SPr(C/U))$. This condition is a saturated and local condition on $\mathcal{S}$, and therefore defines, as explained in our previous example, a sub-stack $\mathcal{S}_F \subset \mathcal{S}$. The stack $\mathcal{S}_F$ is called the *stack of twisted forms of F*.

Exercise 2.2.5. *Let F and G be two stacks. Prove that there exists a natural bijection between $[G, \mathcal{S}_F]$ and the subset of isomorphism classes of objects in $Ho(SPr(C)/G)$ consisting of all objects $G' \longrightarrow G$ satisfying the following condition:*

there is a family of objects $\{X_i\}$ of C and an epimorphism $\coprod_i X_i \longrightarrow G$, such that for any i, the stack $G' \times^h_G X_i$ is isomorphic, in $Ho(SPr(C/X_i))$ to the restriction of F.

More about twisted forms: Let F be a given stack, for which we want to better understand the stack of twisted forms $\mathcal{S}_F$. We consider the following presheaf of simplicial monoids

$$\mathbb{R}\underline{\mathcal{E}nd}(F) : X \mapsto \underline{Hom}(X \times R(F), R(F)),$$

where $R(F)$ is an injective fibrant model for F. The monoid structure on this presheaf is simply induced by composing endomorphisms. We define another presheaf of simplicial monoids by the following homotopy pull-back square

$$\begin{array}{ccc} \mathbb{R}\underline{\mathcal{A}ut}(F) & \longrightarrow & \mathbb{R}\underline{\mathcal{E}nd}(F) \\ \downarrow & & \downarrow \\ \pi_0(\mathbb{R}\underline{\mathcal{E}nd}(F))^{inv} & \longrightarrow & \pi_0(\mathbb{R}\underline{\mathcal{E}nd}(F)), \end{array}$$

where $\pi_0(\mathbb{R}\underline{\mathcal{E}nd}(F))^{inv}$ denotes the subsheaf of invertible elements in the sheaf of monoids $\pi_0(\mathbb{R}\underline{\mathcal{E}nd}(F))$.

The stack $\mathbb{R}\underline{\mathcal{A}ut}(F)$ is called the stack of auto-equivalence of F. It is a represented by a presheaf in simplicial monoids for which all elements are invertible up to homotopy. Even if this is not strictly speaking a presheaf of simplicial groups we can apply the classifying space construction to get a new stack $B\mathbb{R}\underline{\mathcal{A}ut}(F)$. There exists a natural morphism of stacks

$$B\mathbb{R}\underline{\mathcal{A}ut}(F) \longrightarrow \mathcal{S}$$

constructed as follows. The simplicial monoid $\underline{\mathcal{A}ut}(F)$ acts on the simplicial presheaf RF in an abvious way. We form the Borel construction for this action to get a new simplicial presheaf $[F/\underline{\mathcal{A}ut}(F)]$, which is, by definition, the homotopy colimit of the standard simplicial object

$$([n] \mapsto \underline{\mathcal{A}ut}(F)^n \times F)\,.$$

There exists a natural projection

$$[F/\underline{\mathcal{A}ut}(F)] \longrightarrow [*/\underline{\mathcal{A}ut}(F)] = B/\underline{\mathcal{A}ut}(F),$$

giving an object in $Ho(SPr(C)/B/\underline{\mathcal{A}ut}(F))$. By exercise 2.2.5 this object corresponds to a well defined morphism in $Ho(SPr(C))$

$$B\mathbb{R}\underline{\mathcal{A}ut}(F) \longrightarrow \mathcal{S}_F \subset \mathcal{S}.$$

Moreover, [HAGII, Prop. A.0.6] implies that the morphism

$$B\mathbb{R}\underline{\mathcal{A}ut}(F) \longrightarrow \mathcal{S}_F$$

is in fact an isomorphism.

An important example is when $F = K(A, n)$, for A a sheaf of abelian groups, as twisted forms of F are sometimes refered to *n-gerbes with coefficients in A*. It can be shown that the monoid $\mathbb{R}\underline{\mathcal{A}ut}(F)$ is the semi-direct product of $K(A, n)$ by the sheaf of group $aut(A)$. In other words, we have

$$\mathcal{S}_{K(A,n)} \simeq [K(A + 1, n)/aut(A)],$$

or, inother words we have a split fibration sequence

$$K(A, n + 1) \longrightarrow \mathcal{S}_{K(A,n)} \longrightarrow Baut(A).$$

As a consequence we see that the set of equivalence classes of n-gerbes on X with coefficients in A is in bijection with the set of pairs (ρ, α), where $\rho \in H^1(X, aut(A))$, and $\alpha \in H^{n+1}(X, A_\rho)$, where A_ρ is the twisted form of the sheaf A determined by ρ.

Another important application is the description of the inductive construction of the Postnikov tower of a stack F. We first assume that F is simply connected and connected (i.e. that the sheaves π_0 and $\pi_1(F)$ are trivial on C). This implies that we have globally defined sheaves $\pi_n(F)$ on C. The natural projection in the Postnikov tower

$$t_{\leq n+1}(F) \longrightarrow t_{\leq n}(F)$$

produces an object in $Ho(SPr(C))/t_{\leq n}(F)$, which by exercise 2.2.5 produces a morphism of stacks

$$t_{\leq n}(F) \longrightarrow \mathcal{S}_{K(\pi_n,n)},$$

which is the n-th Postnikov invariant of F. In determines, in particular, a class $k_n \in H^{n+1}(t_{\leq n}(F), \pi_n)$ which completely determines the object $t_{\leq n+1}(F)$. For a general F, possibly non-connected and non-simply connected, it is possible, by changing the base site C, to reduce to the connected and simply connected case.

Exercise 2.2.6. *Let F be any stack and $\Pi_1(F)$ be its associate presheaf of fundamental groupoids, sending X to $\Pi_1(F(X))$. We let $p : D := \int_C \Pi_1(F) \longrightarrow C$ be the Grothendieck construction of the functor $\Pi_1(F)$, endowed with the topology induced from the one on C. Recall that objects in D are pairs (X, x), consisting of $X \in C$ and $x \in \Pi_1(F(X))$, and morphisms $(X, x) \to (Y, y)$ consist of pairs (u, f), with $u : X \to Y$ is in C and $f : u^*(x) \to y$ is in $\Pi_1(F(Y))$.*

1. *Show that the functor*

   $$D \longrightarrow Ho(SPr(C)/t_{\leq 1}(F)),$$

   sending (X, x) to $x : X \longrightarrow F$, extends to an equivalence

   $$Ho(SPr(D)) \simeq Ho(SPr(C)/t_{\leq 1}(F)).$$

2. *Show that under this equivalence the image of $F \longrightarrow t_{\leq 1}(F)$ is a connected and simply connected object in $Ho(SPr(D))$.*

3. *Deduce the existence of Postnikov invariants $k_n \in H^{n+1}(t_{\leq n}(F), \pi_n(F))$ for F, where the cohomology group now means cohomology with coefficients in a sheaf $\pi_n(F)$ leaving on $t_{\leq 1}(F)$.*

Chapter 3

Algebraic stacks

In the previous lecture we have introduced the notion of stacks over some site. We will now consider the more specific case of stacks over the étale site of affine schemes and introduce an important class of stacks called *algebraic stacks*. These are generalizations of schemes and algebraic spaces for which quotients by smooth actions always exist.

All along this lecture we will consider $Comm$ the category of commutative rings and set $Aff := Comm^{op}$. For $A \in Comm$ we denote by $Spec\,A$ the corresponding object in Aff (therefore "*Spec*" stands for a formal notation here). We endow Aff with the étale topology defined as follows. Recall that a morphism of commutative rings $A \longrightarrow B$ is *étale* if it satisfies the following three conditions:

1. B is flat as an A-module.
2. B is finitely presented as a commutative A-algebra (i.e. of the form $A[T_1, \dots, T_n]/(P_1, \dots, P_r)$).
3. B is flat as a $B \otimes_A B$-module.

There exists several equivalent characterizations of étale morphisms (see e.g. [SGA1]) , for instance the third condition can be equivalently replaced by the condition $\Omega^1_{B/A} = 0$, where $\Omega^1_{B/A}$ is the B-module of relative Kahler derivations (corepresenting the functor sending a B-module M to the set of A-linear derivations on B with coefficients in M). Etale morphisms are stable by base change and composition in Aff, i.e. by cobase change and composition in $Comm$. Geometrically an étale morphism $A \longrightarrow B$ should be thought as a "local isomorphism" of schemes $Spec\,B \longrightarrow Spec\,A$, though here *local* should not be understood in the sense of the Zariski topology.

Now, a family of morphisms $\{A \longrightarrow A_i\}_{i \in I}$ is an étale covering if each morphism $A \longrightarrow A_i$ is étale and if the family of base change functors

$$- \otimes_A A_i : A - Mod \longrightarrow A_i - Mod$$

is conservative. This defines a topology on Aff by defining that a sieve on $Spec\, A$ is a covering sieve if it is generated by an étale covering family.

Finally, a morphism $Spec\, B \longrightarrow Spec\, A$ is a Zariski open immersion if it is étale and a monomorphism (this is equivalent to say that the natural morphism $B \otimes_A B \longrightarrow B$ is an isomorphism, or equivalently that the forgetful functor $B - Mod \longrightarrow A - Mod$ is fully faithful).

3.1 Schemes and algebraic n-stacks

We start by the definition of schemes and then define algebraic n-stacks as certain succesive quotients of schemes.

For $Spec\, A \in Aff$ we can consider the presheaf represented by $Spec\, A$

$$Spec\, A : Aff^{op} = Comm \longrightarrow Set,$$

by setting $(Spec\, A)(B) = Hom(A, B)$. A standard result of commutative algebra (faithfully flat descent) states that the presheaf $Spec\, A$ is always a sheaf. We thus consider $Spec\, A$ has a stack and as an object in $Ho(SPr(Aff))$. This defines a fully faithful functor

$$Aff \longrightarrow Ho(SPr(Aff)).$$

Any objects in $Ho(SPr(Aff))$ isomorphic to a sheaf of the form $Spec\, A$ will be called an *affine scheme*. The full sub-category of $Ho(SPr(Aff))$ consisting of affine schemes is equivalent to $Aff = Comm^{op}$, and these two categories will be implicitly identified.

Definition 3.1.1. *1. Let $Spec\, A$ be an affine scheme, F a stack and $i : F \longrightarrow Spec\, A$ a morphism. We say that i is a* Zariski open immersion *(or simply an* open immersion*) if it satisfies the following two conditions.*

(a) The stack F is a sheaf (i.e. 0-truncated) and the morphism i is a monomorphism of sheaves.

(b) There exists a family of Zariski open immersions $\{A \longrightarrow A_i\}_i$ such that F is the image of the morphism of sheaves

$$\coprod_i Spec\, A_i \longrightarrow Spec\, A.$$

2. A morphism of stacks $F \longrightarrow F'$ is a Zariski open immersion *(or simply an* open immersion*) if for any affine scheme $Spec\, A$ and any morphism $Spec\, A \longrightarrow F'$, the induced morphism*

$$F \times^h_{F'} Spec\, A \longrightarrow Spec\, A$$

is a Zariski open immersion in the sense above.

3. *A stack F is a* scheme *if there exists a family of affine schemes $\{Spec\, A_i\}_i$ and Zariski open immersions $Spec\, A_i \longrightarrow F$, such that the induced morphism of sheaves*
$$\coprod_i Spec\, A_i \longrightarrow F$$
is an epimorphism. Such a family of morphisms $\{Spec\, A_i \longrightarrow F\}$ will be called a Zariski altas for F.

Exercise 3.1.2. 1. *Show that any Zariski open immersion $F \longrightarrow F'$ is a monomorphism of stacks.*

2. *Deduce from this that a scheme F is always 0-truncated, and thus equivalent to a sheaf.*

We now pass to the definition of algebraic stacks. These are stacks obtained by gluing schemes along smooth quotient, and we first need to recall the notion of smooth morphisms of schemes.

Recall that a morphism of commutative rings $A \longrightarrow B$ is *smooth*, if it is flat of finite presentation and if moreover B is of finite Tor dimension as a $B \otimes_A B$-module. Smooth morphisms are the algebraic analog of submersions, and there exists equivalent definitions making this analogy more clear (see [SGA1]). Smooth morphisms are stable by compositions and base change in Aff. The notion of smooth morphisms can be extended to a notion for all schemes by the following way. We say that a morphism of schemes $X \longrightarrow Y$ is smooth if there exists Zariski atlas $\{Spec\, A_i \longrightarrow X\}$ and $\{Spec\, A_j \longrightarrow Y\}$ together with commutative squares
$$\begin{array}{ccc} X & \longrightarrow & Y \\ \uparrow & & \uparrow \\ Spec\, A_i & \longrightarrow & Spec\, A_j, \end{array}$$
with $Spec\, A_i \longrightarrow Spec\, A_j$ a smooth morphism (here j depends on i). Again, smooth morphisms of schemes are stable by composition and base change.

We are now ready to define the notion of algebraic stack. The definition is by induction on an algebraicity index n representing the number of successive smooth quotients we take. This index will be forgetten after the definition is acheived.

Definition 3.1.3. 1. *A stack F is* 0-algebraic *if it is a scheme.*

2. *A morphism of stacks $F \longrightarrow F'$ is* 0-algebraic *(or* 0-representable*) if for any scheme X and any morphism $X \longrightarrow F'$ the stack $F \times^h_{F'} X$ is 0-algebraic (i.e. a scheme).*

3. *A 0-algebraic morphism of stacks $F \longrightarrow F'$ is smooth if for any scheme X and any morphism $X \longrightarrow F'$ the morphism of schemes $F \times^h_{F'} X \longrightarrow X$ is smooth.*

4. *We now let $n > 0$, and we assume that the notions of $(n-1)$-algebraic stack, $(n-1)$-algebraic morphisms and smooth $(n-1)$-algebraic morphisms have been defined.*

   (a) *A stack F is* n-algebraic *if there exists a scheme X together with smooth $(n-1)$-algebraic morphisms $X \longrightarrow F$ which is an epimorphism. Such a morphism $X \longrightarrow F$ is called a* smooth n-atlas for F.

   (b) *A morphism of stacks $F \longrightarrow F'$ is* n-algebraic *(or* n-representable*) if for any scheme X and any morphism $X \longrightarrow F'$ the stack $F \times_{F'} X$ is n-algebraic.*

   (c) *An n-algebraic morphism of stacks $F \longrightarrow F'$ is smooth if for any scheme X and any morphism $X \longrightarrow F'$ there exists a smooth n-atlas $Y \longrightarrow F \times^h_{F'} X$ such that each morphism $Y \longrightarrow X$ is a smooth morphism of schemes.*

5. *An* algebraic stack *is a stack which is n-algebraic for some integer n. An* algebraic n-stack *is an algebraic stack which is also an n-stack. An* algebraic space *is an algebraic* 0*-stack.*

6. *A morphism of stacks $F \longrightarrow F'$ is* algebraic *(or* representable*) if it is n-algebraic for some n.*

7. *A morphism of stacks $F \longrightarrow F'$ is* smooth *if it is n-algebraic and smooth for some integer n.*

Long, but formal arguments show that algebraic stacks satisfy the following properties.

- Algebraic stacks are stable by finite homotopy limits (i.e. by homotopy pullbacks).
- Algebraic stacks are stable by disjoint union.
- Algebraic morphisms of stacks are stable by composition and base change.
- Algebraic stacks are stable by smooth quotients. To be more precise, if $F \longrightarrow F'$ is a smooth epimorphism of stacks, then F' is algebraic if and only if F is so.

Exercise 3.1.4. *Let F be an algebraic n-stack, $X \in Aff$, and $x : X \longrightarrow F$ a morphism of stacks. Show that the sheaf $\pi_n(F, x)$ is reprentable by an algebraic space, locally of finite type over X.*

The standard finiteness properties of schemes can be extended to algebraic stacks in the following way.

- An algebraic morphism $F \longrightarrow F'$ is *locally of finite presentation* if for any scheme X and any morphism $X \longrightarrow F'$, there exists a smooth atlas $Y \to F \times^h_{F'} X$ such that the induced morphism $Y \longrightarrow X$ is locally of finite presentation.

- An algebraic morphism $F \longrightarrow F'$ is *quasi-compact*, if for any affine scheme X and any morphism $X \longrightarrow F'$, there exists a smooth atlas $Y \longrightarrow F \times^h_{F'} X$ with Y an affine scheme.

- An algebraic stack F is *strongly quasi-compact* if for all integer n the induced morphism
$$F \longrightarrow \mathbb{R}\underline{\mathcal{H}om}(\partial\Delta^n, F)$$
is quasi-compact.

- An algebraic stack morphism $F \longrightarrow F'$ is *strongly of finite presentation* if for any affine scheme X and any morphism $X \longrightarrow F'$ the stack $F \times^h_{F'} X$ is locally of finite presentation and strongly quasi-compact.

Note that when $n = 0$ we have $\mathbb{R}\underline{\mathcal{H}om}(\partial\Delta^n, F) \simeq F \times F$, and the condition of strongly quasi-compactness implies in particular that the diagonal morphism $F \longrightarrow F \times F$ is quasi-compact. In general, being strongly quasi-compact involves quasi-compactness conditions for all the "higher diagonals".

Exercise 3.1.5. *Let X be an affine scheme and G be a sheaf of groups on Aff/X. We form the classifying stack $K(G,1) \in Ho(SPr(Aff)/X)$, and consider it in $Ho(SPr(Aff))$.*

1. *Show that if $K(G,1)$ is an algebraic stack then G is represented by an algebraic space locally of finite type.*

2. *Conversely, if G is representable by an algebraic space, which is smooth over X, then $K(G,1)$ is an algebraic stack.*

3. *Assume that $K(G,1)$ is algebraic. Show that $K(G,1)$ is quasi-compact. Show that $K(G,1)$ is strongly quasi-compact if and only if G is quasi-compact.*

3.2 Some examples

Classifying stacks: Suppose that G is a sheaf of groups over some affine scheme X, and assume that G is an algebraic space, flat and of finite presentation over X. We can form $K(G,1) \in Ho(SPr(Aff))$, the classifying stack of the group G, as explained in §2.2. The stack $K(G,1)$ is however not exactly the right object to consider, at least when G is not smooth over X. Indeed, for Y and affine scheme over X, $[Y, K(G,1)]$ classifies G-torsors over Y which are locally trivial for the étale topology on Y. This is a rather unnatural condition as there exists G-torsors, locally trivial for the flat topology on Y, which are not étale locally trivial (for instance, when $X = Spec\, k$ is a perfect field of caracteristic p, the Frobenius map $Fr : \mathbb{G}_m \longrightarrow \mathbb{G}_m$ is a μ_p-torsor over $\mathbb{G}_m$ which is not étale locally trivial). To remedy this we introduce a slight modification of the classifying stack $K(G, 1)$ by changing the topology in the following way. We consider the simplicial presheaf $BG : X \mapsto B(G(X))$, consider as an object in $SPr_{ffqc}(Aff)$, the model category of simplicial presheaves on the site of affine schemes endowed with the faithfully flat and quasi-compact topology (ffqc for short). Note that étale coverings are ffqc coverings, and therefore we have a natural full embedding

$$Ho(SPr_{ffqc}(SPr(Aff)) \subset Ho(SPr(Aff)),$$

objects in $Ho(SPr_{ffqc}(SPr(Aff))$ being the stacks satisfying the more restrictive descent condition for ffqc hypercoverings. We consider the simplicial presheaf $BG \in SPr(Aff)$, and denote $K_{fl}(G, 1) \in Ho(SPr_{ffqc}(Aff)) \subset Ho(SPr(Aff))$ a fibrant replacement of BG in the model category of stacks for the ffqc topology. It is a non-trivial statement that $K_{fl}(G, 1)$ is an algebraic stack (see for instance [La-Mo] ??). Moreover, the natural morphism $K_{fl}(G, 1) \longrightarrow X$ is smooth. Indeed, we chose a smooth and surjective morphism $Y \longrightarrow K_{fl}(G, 1)$, with Y an affine scheme. The composition $Y \longrightarrow X$ is clearly a flat surjective morphism of finite presentation. We let $X' := Y \times^h_{K_{fl}(G,1)} X$, and consider the diagram of stacks

$$\begin{array}{ccccc} X' & \xrightarrow{v} & Y & \longrightarrow & K_{fl}(G,1) \\ & {\scriptstyle u}\searrow & \downarrow {\scriptstyle q} & \swarrow & \\ & & X. & & \end{array}$$

In this diagram, v is a flat surjective morphism of finite presentation, because it is the base change of the trivial section $X \longrightarrow K_{fl}(G, 1)$ which is flat, surjective and of finite presentation. Moreover, u is a smooth morphism, because it is the base change of the smooth atlas $Y \longrightarrow K_{fl}(G, 1)$. We conclude that the morphism q is also smooth by []??.

Higher classifying stacks: Assume now that A is a sheaf of abelian groups over an affine scheme X, which is an algebraic space, flat and of finite presentation over X. We form the simplicial prehseaf $B^n(A) = B(B^{n-1}(A))$, by iterating the classifying space construction. We denote by $K_{fl}(A, n) \in Ho(SPr_{ffqc}(Aff)) \subset Ho(SPr(Aff))$ a fibrant model for $B^n(A)$ with respect to the ffqc topology. It is again true that $K_{fl}(A, n)$ is an algebraic n-stack when $n > 1$. Indeed, $K(A, n)$ is the quotient of X by the trivial action by the group stack $K(A, n - 1)$. As this group stack is algebraic and smooth for $n > 1$, the quotient stack is again an algebraic stack.

Groupoid quotients: We start here by the standard way to construct algebraic stacks using quotients by smooth groupoid actions. We start by a simplicial objects in $SPr(C)$

$$F_* : \Delta^{op} \longrightarrow SPr(Aff).$$

We will say that F_* is a *Segal groupoid* if it satisfies the following two conditions

1. For any $n > 1$, the natural morphism

$$F_n \longrightarrow F_1 \times^h_{F_0} F_1 \cdots \times^h_{F_0} F_1,$$

   induced by the morphism $[1] \to [n]$ sending 0 to i and 1 to $i+1$ (for $0 \leq i < n$), is an isomorphism of stacks.

2. The natural morphism

$$F_2 \longrightarrow F_1 \times^h_{F_0} F_1$$

   induced by the morphism $[1] \to [2]$ sending 0 to 0 and 1 to 1 or 2, is an isomorphism of stacks.

Exercise 3.2.1. *Let F_* be a Segal monoid object in $SPr(Aff)$, and suppose that $F_n(X)$ is a set for all n and all X. Show that F_* is the nerve of a presheaf of groupoids on Aff.*

We now assume that F_* is a Segal groupoid and moreover that all the face morphisms $F_1 \longrightarrow F_0$ are smooth morphism between algebraic stacks. We consider the homotopy colimit of the diagram $[n] \mapsto F_n$, and denote it by $|F_*| \in Ho(SPr(Aff))$. The stack $|F_*|$ is called the *quotient stack of the Segal groupoid* F_*. It can been proved that $|F_*|$ is again an algebraic stack. Moreover if each F_i is an algebraic n-stack then $|F_*|$ is an algebraic $(n+1)$-stack. This is a formal way to produce higher algebraic stacks starting say from schemes, but this is often not the way stacks arise in practice.

An important very special case of the quotient stack constuction is the case of a smooth group scheme G acting on a scheme X. In this case we form the groupoid object $B(X,G)$ whose values in degre n is $X \times G^n$, and whose transition morphisms are given by the action of G on X. This is a groupoid object in schemes and thus can be considered as a groupoid objects in sheaves and therefore as a very special kind of Segal groupoid. The quotient stack of this Segal groupoid is denoted by $[X/G]$ and is called the quotient stack of X by G. It is an algebraic 1-stack for which a natural smooth atlas is the natural projection $X \longrightarrow [X/G]$. It can be characterize by a universal property: morphisms of stacks $[X/G] \longrightarrow F$ are in one-to-one correspondence with morphisms of G-equivariant stacks $X \longrightarrow F$ (here we need to use a model category $G - SPr(Aff)$ of G-equivariant simplicial presheaves in order to have the correct homotopy category of G-equivariant stacks).

Simplicial presentation: Algebraic stacks can also be characterized as the simplicial presheaves having being represented by certain kind of simplicial schemes. For this we let X_* be a simplicial object in the category of schemes. For any finite simplicial set K (finite here means generated by a finite number of cells) we can form X_*^K, which is the scheme of morphism from K to X_*. It is, by definition the equalizer of the two natural morphisms

$$\prod_n X_n^{K_n} \rightrightarrows \prod_{[p]\rightarrow[q]} X_p^{K_q}.$$

This equalizer exists as a scheme when K is finite (because it then only invovles finite limits).

A simplicial scheme X_* is then called a *weak smooth groupoid* if for any $0 \leq k \leq n$, the natural morphism

$$X_n = X_*^{\Delta^n} \longrightarrow X_*^{\Lambda^{n,k}}$$

is a smooth and surjective morphism of schemes (surjective here has to be understood pointwise, but as the morphism is smooth this is equivalent to say that it induces an epimorphism on the corresponding sheaves). A weak smooth groupoid X_* is moreover *n-truncated* if for any $k > n+1$ the natural morphism

$$X_k = X^{\Delta^k} \longrightarrow X_*^{\partial\Delta^k}$$

is an isomorphism.

It is then possible to prove that a stack F is an algebraic n-stack if there exists an n-truncated weak smooth groupoid X_* and an isomorphism in $Ho(SPr(Aff))$ $F \simeq X_*$. We refer to [Pr] for details.

Some famous algebraic 1-stacks: We review here two famous examples of algebraic 1-stacks, the stack of smooth and proper curves and the stack of vector bundles on curve. We refer to [La-Mo] for more details.

For $X \in Aff$ an affine scheme we let $\mathcal{M}_g(X)$ be the full sub-groupoid of sheaves F on Aff/X such that the corresponding morphism of sheaves $F \longrightarrow X$ is representable by a smooth and proper curve of genus g over X (i.e. F is itself a scheme, the morphism $F \longrightarrow X$ is smooth, proper, with geometric fibers being connected curves of genus g). For $Y \longrightarrow X$ in Aff, we have a restriction functor from sheaves on Aff/X to sheaves on Aff/Y, and this defines a natural functor of groupoids

$$\mathcal{M}_g(X) \longrightarrow \mathcal{M}_g(Y).$$

This defines a presheaf in groupoids on Aff, and taking the nerve of these groupoids gives a simplicial presheaf denoted by $\mathcal{M}_g$. The stack $\mathcal{M}_g$ is called the *stack of smooth curves of genus g*. It is such that for $X \in Aff$, $\mathcal{M}_g(X)$ is a 1-truncated simplicial set whose π_0 is the set of isomorphism classes of smooth proper curves of genus g over X, and whose π_1 at a given curve is its automorphism group. It is a well known theorem that $\mathcal{M}_g$ is an algebraic 1-stack which is smooth and of finite presentation over $Spec\,\mathbb{Z}$. This stack is even Deligne-Mumford, that is the diagonal morphism $\mathcal{M}_g \longrightarrow \mathcal{M}_g \times \mathcal{M}_g$ is unramified (i.e. locally for the étale topology an closed immersion). Equivalently this means that there exists an atlas $X \longrightarrow \mathcal{M}_g$ which is étale rather than only smooth.

Another very important and famous example of an algebraic 1-stack is the stack of G-bundles on some smooth projective curve C (say over some base field k). Let G be a smooth affine algebraic group over k. We start by consider the stack BG, which is a stack over $Spec\,k$. It is the quotient stack $[Spec\,k/G]$ for the trivial action of G on $Spec\,k$. As G is a smooth algebraic group this stack is an algebraic 1-stack. When C is a smooth and proper curve over $Spec\,k$ we can consider the stack of morphisms (of stacks over $Spec\,k$)

$$Bun_G(C) := \mathbb{R}\underline{\mathcal{H}om}_{Aff/Spec\,k}(C, BG),$$

which by definition is the stack of principal G-bundles on C. By definition, for $X \in Aff$, $Bun_G(C)(X)$ is a 1-truncated simplicial set whose π_0 is the set of isomorphism classes of principal G-bundles on C and whose π_1 at a given bundle is its automorphism group. It is also a well known theorem that the stack $Bun_G(C)$ is an algebraic 1-stack, which is smooth and locally of finite presentation over $Spec\,k$. However, this stack is not quasi-compact and is only a countable union of quasi-compact open substack.

Higher linear stacks: Let $X = Spec\,A$ be an affine scheme and E be positively graded cochain complex of A-modules. We assume that E is perfect, i.e. it is quasi-isomorphic to a bounded complex of projective A-modules of finite type. We define a stack $\mathbb{V}(E)$ over X in the following way. To any commutative A-algebra B we set

$$\mathbb{V}(E)(B) := Map(E, B),$$

where Map denotes the mapping spaces of the model category of complexes of A-modules. More explicitly, $\mathbb{V}(E)(B)$ is the simplicial set whose set of n-simplicies is the set $Hom(Q(E) \otimes_A C_*(\Delta^n, A), B)$. Here $Q(E)$ is a cofibrant resolution of E in the model category of complexes A-modules (for the projective model structure for which equivalences are quasi-isomorphisms and fibrations are epimorphisms), $C_*(\Delta^n, A)$ is the homology complexe of the simplicial set Δ^n with coefficients in A, and the Hom is taken in the category of complexes of A-modules. In other words, $\mathbb{V}(E)(B)$ is the simplicial set obtained from the complex $\underline{Hom}^*(Q(E), B)$ by the Dold-Kan correspondence. When B varies in the category of commutative A-algebras this defines a simplicial presheaf $\mathbb{V}(E)$ together with a morphism $\mathbb{V}(E) \longrightarrow X = Spec\,A$. For any commutative A-algebras we have

$$\pi_i(\mathbb{V}(E)(B)) \simeq Ext^{-i}(E, B).$$

It can be shown that the stack $\mathbb{V}(E)$ is an algebraic n-stack strongly of finite presentation over X, where n is such that $H^i(E) = 0$ for all $i > n$, and that $\mathbb{V}(E)$ is smooth if and only if the Tor amplitude of E is non negative (i.e. E is quasi-isomorphic to a complex of projective A-modules of finite type which is moreover concentrated in non negative degrees). For this, we can first assume that E is a bounded complex of projective modules of finite type. We then set $K = E^{\leq 0}$ the part of E which is concentrated in non positive degrees, and we have a natural morphism of complexes $E \longrightarrow K$. This morphism induces a morpihsm of stacks

$$\mathbb{V}(K) \longrightarrow \mathbb{V}(E).$$

Now, by definition $\mathbb{V}(K)$ is naturally equivalent to the affine scheme $Spec\, A[H^0(K)]$, with $A[H^0(K)]$ the free commutative A-algebra generated by the A-module $H^0(K)$. It is well known that $\mathbb{V}(H^0(k))$ is a smooth over $Spec\, A$ if and only if $H^0(K)$ is projective and of finite type. This is equivalent to say that E has non negative Tor amplitude. The only thing to check is then that the natural morphism

$$\mathbb{V}(K) \longrightarrow \mathbb{V}(E),$$

is $(n-1)$-algebraic and smooth. But this follows by induction on n as this morphism is locally on $\mathbb{V}(E)$ of the form $Y \times \mathbb{V}(L) \longrightarrow Y$, for L the homotopy cofiber (i.e. the cone) of the morphism $E \longrightarrow K$. This homotopy cofiber is itself quasi-isomorphic to $E^{>0}[1]$, and thus is a perfect complex of non negative Tor amplitude with $H^i(L) = 0$ for $i > n-1$.

Exercise 3.2.2. *Let $X = \mathbb{A}^1 = Spec\, \mathbb{Z}[T]$, and E be the complex of $\mathbb{Z}[T]$-modules given by*

$$0 \longrightarrow \mathbb{Z}[T] \xrightarrow{\times T} \mathbb{Z}[T] \longrightarrow 0,$$

concentrated in degrees 1 and 2. Show that $\mathbb{V}(E)$ is an algebraic 2-stack which is such that the sheaf $\pi_1(\mathbb{V}(E))$ is not representable by an affine scheme (it is in fact not representable by an algebraic space).

The algebraic 2-stack of abelian categories: This is a non trivial example of an algebraic 2-stack. The material is taken from [An]. For a commutative ring A we consider $Ab(A)$ the following category. Its objects are abelian A-linear categories which are equivalent to $R-Mod$, the category of left R-modules for some associative A-algebra R which is projective and of finite type as an A-module. The morphism in $Ab(A)$ are the A-linear equivalences of categories. For a morphism of commutative rings $A \longrightarrow B$ we have a functor

$$Ab(A) \longrightarrow Ab(B)$$

sending an abelian category $\mathcal{C}$ to $\mathcal{C}^{B/A}$, the category of B-modules in $\mathcal{C}$. Precisely $\mathcal{C}^{B/A}$ can be taken to be the category of all A-linear functors from BB, the A-linear category with a unique object and B as its A-algebra of endomorphisms, to $\mathcal{C}$. This defines a presheaf of categories $A \mapsto Ab(A)$ on Aff. Taking the nerves of these categories we obtain a simplicial presheaf $\mathbf{Ab} \in SPr(Aff)$. The simplicial presheaf $\mathbf{Ab}$ is not a stack, but we still consider it as an object in $Ho(SPr(Aff))$. Of the main result of [An] states that $\mathbf{Ab}$ is an algebraic 2-stack which is locally of finite presentation.

The algebraic n-stack of $[n, 0]$-perfect complexes: For an commutative ring A we consider a category $P(A)$ defined as follows. Its objetcs are the cofibrant complexes of A-modules (for the projective model structures) which are perfect (i.e. quasi-isomorphic to a bounded complex of projective modules of finite type). The morphisms in $P(A)$ are the quasi-isomorphisms of complexes of A-modules. For a morphism of commutative ring $A \longrightarrow B$ we have a base change functor

$$- \otimes_A B : P(A) \longrightarrow P(B).$$

This does not however define stricly speaking a presheaf of categories, as the base change functors are only compatible with composition up to a natural isomorphism. In other words, $A \mapsto P(A)$ is only a weak functor from $Comm$ to the 2-category of categories. Hopfully there exists a standard procedure to replace any weak functor by an equivalent strict functor: it consists of replacing P by the presheaf of cartesian sections of the Grothendieck construction $\int P \longrightarrow Comm$ (see [SGA1]). In few words, we define a new category $P'(A)$ whose objects consist of the following data.

1. For any morphism of commutative A-algebra B an object $E_B \in P(B)$.
2. For any commutative A-algebra B and any commutative B-algebra C, an isomorphism in $P'(C)$

$$\phi_{B,C} : E_B \otimes_B C \simeq E_C.$$

We require moreover that for any commutative A-algebra B, any commutative B-algebra C and any commutative C-algebra D, the two possible isomorphisms

$$\phi_{C,D} \circ (\phi_{B,C} \otimes_C D) : (E_B \otimes_B C) \otimes_C D) \simeq E_B \otimes_B D \longrightarrow E_D$$

$$\phi_{B,D} : E_B \otimes_B D \longrightarrow E_D$$

are equal. The morphisms in $P'(A)$ are simply taken to be families of morphisms $E_B \longrightarrow E'_B$ which commute with the $\phi_{B,C}$'s and the $\phi'_{B,C}$.

With these definitions $A \mapsto P'(A)$ is a functor $Comm \longrightarrow Cat$, and there is moreover an equivalence of lax funcors $P' \longrightarrow P$. We compose the functor P' with the nerve construction and we get a simplicial presheaf **Perf** on Aff. It can be proved that the simplicial presheaf **Perf** is a stack in the sense of definition 2.1.3 (1). This is not an abvious result (see for instance [H-S] for a proof), and can be reduced to a the well known flat cohomological descent for quasi-coherent complexes. It can also be proved that for $X = Spec\, A \in Aff$, the simplicial set **Perf**(X) satisfies the following properties.

1. The set $\pi_0(\mathbf{Perf}(X))$ is in a natural bijection with the set of quasi-isomorphism classes of perfect complexes of A-modules.

2. For $x \in \mathbf{Perf}(X)$ corresponding to a perfect complex E, we have

$$\pi_1(\mathbf{Perf}(X), x) \simeq Aut(E),$$

where the automorphism group is taken in the derived category $D(A)$ of the ring A.

3. For $x \in \mathbf{Perf}(X)$ corresponding to a perfect complex E, we have

$$\pi_i(\mathbf{Perf}(X), x) \simeq Ext^{1-i}(E, E)$$

for any $i > 1$. Again, these ext groups are computed in the triangle category $D(A)$.

For any $n \geq 0$ and $a \leq b$ with $b - a = n$ we can define a subsimplicial presheaf $\mathbf{Perf}^{[a,b]} \subset \mathbf{Perf}$ which consists of all perfect complexes of Tor amplitude contained in the interval $[a, b]$ (i.e. complexes quasi-isomorphic to a complex of projective modules of finite type concentrated in degrees $[a, b]$). It can be proved that the substacks $\mathbf{Perf}^{[a,b]}$ form an open covering of $\mathbf{Perf}$. Moreover, $\mathbf{Perf}^{[a,b]}$ is an algebraic $(n+1)$-stack which is locally of finite presentation. This way, even thought $\mathbf{Perf}$ is not stricly speaking an algebraic stack (because it is not an n-stack for any n), it is an increasing union of open algebraic substacks. We say that $\mathbf{Perf}$ is *locally algebraic*. The fact that $\mathbf{Perf}^{[a,b]}$ is an algebraic $(n+1)$-stack is also not easy. We refer to [To-Va] for a complete proof.

Exercise 3.2.3. *1. Show how to define a stack* $\mathbf{MPerf}$ *of morphisms between perfect complexes, whose values at* $X \in Aff$ *is equivalent to the nerve of the category of quasi-isomorphisms in the category of morphisms between perfect complexes over* X.

2. Show that the morphism source *and* target*, define an algebraic morphism of stacks*

$$\pi : \mathbf{MPerf} \longrightarrow \mathbf{Perf} \times \mathbf{Perf}$$

(Here, you will need the following result of homotopical algebra: if M *is a model category,* $Mor(M)$ *the model category of morphisms, then the homotopy fiber of the source and target map* $N(wMor(M)) \longrightarrow N(wM) \times N(wM)$*, taken at a point* (x, y)*, is naturally equivalent to the mapping space* $Map(x, y)$*).*

3. Show that the morphism π *is smooth locally near any point corresponding to a morphism* $E \to E'$ *of perfect complexes such that* $Ext^i(E, E') = 0$ *for all* $i > 0$.

3.3 Coarse moduli spaces and homotopy sheaves

The purpose of this part is to show that algebraic n-stacks strongly of finite presentation can be approximated by schemes by mean of some *dévissage*. The existence of this approximation has several important consequences about the behaviour of algebraic n-stacks, such as the existence of virtual coarse moduli space or homotopy group schemes. Conceptually, the results of this part show that algebraic n-stacks are not that far from being schemes or algebraic spaces, and that for many purposes they behave like *convergent series of schemes.*

Convention: All along this part all algebraic n-stacks will be strongly of finite presentation over some base affine scheme $Spec\, k$ (for k some commutative ring).

The key notion is that of *total gerbe*, whose precise definition is as follows.

Definition 3.3.1. *Let F be an algebraic n-stack. We say that F is a* total (n-)gerbe *for all $i > 0$ the natural projection*

$$I_F^{(i)} := \mathbb{R}\underline{Hom}(S^i, F) \longrightarrow F$$

is a flat morphism.

In the previous definition, $I_F^{(i)}$ is called the *i-th inertia stack of F*. Note that when F is a 1-stack and that $I_F^{(1)}$ is equivalent to the inertia stack of F in the sense of [La-Mo] ??. In particular, for an an algebraic 1-stack, being a total gerbe un the sense of definition 3.3.1 is equivalent to the fact that the projection morphism

$$F \times^h_{F\times F} F \longrightarrow F$$

is flat, and thus equivalent to the usual notion of gerbes for algebraic 1-stacks (see [La-Mo] ??).

Proposition 3.3.2. *Let F be an algebraic n-stack which is a total gerbe. Then the following conditions are satisfied.*

1. *We let $M(F)$ be the sheaf associated to $\pi_0(F)$ for the flat (ffqc) topology. Then $M(F)$ is represented by an algebraic space and the morphism $F \longrightarrow M(F)$ is flat and of finite presentation.*

2. *For any $X \in Aff$, and any morphism $x : X \longrightarrow F$, $\pi_i^{fl}(F, x)$, the sheaf on X associated to $\pi_i(F, x)$ with respect to the ffqc topology, is an algebraic space, flat, and of finite presentation over X.*

Proof: The condition (1) follows from a well known theorem of Artin, insuring the representability by algebraic spaces of quotients of schemes by flat equivalence relations. The argument goes as follows. We chose a smooth atlas $X \longrightarrow F$ with

X an affine scheme, and we let $X_1 := X \times^h_F X$. We define $R \subset X \times X$, the sub-ffqc-scheaf image of $X_1 \longrightarrow X \times X$, which defines an equivalence relation on X. Clearly, $M(F)$ is isomorphic to the quotient ffqc-sheaf $(X/R)^{fl}$. We now prove the following two properties.

1. The sheaf R is an algebraic space.
2. The two projections $R \longrightarrow X$ are smooth.

To prove (1) we consider the natural projection $X_1 \longrightarrow R$. Let $x, y : Y \longrightarrow R \subset X \times X$ a morphism with Y affine, then $X_1 \times^h_R Y$ is equivalent to $\Omega_{x,y}F \simeq Y \times^h_F Y$, the stack of paths from x to y. As the objects x and y are locally, for the flat topology, equivalent on Y (because $(x, y) \in R$), the stack $\Omega_{x,y}F \simeq Y \times^h_F Y$ is algebraic and locally, for the flat topology on Y, equivalent to the loop stack $\Omega_x F$, defined by the homotopy cartesian square

$$\begin{array}{ccc} \Omega^F_x & \longrightarrow & I^{(1)}_F \\ \downarrow & & \downarrow \\ Y & \longrightarrow & F. \end{array}$$

By hypothesis on F we deduce that $X_1 \times^h_R Y \longrightarrow Y$ is flat, surjective and of finite presentation. As this is true for any $Y \to R$ we have that the morphism of stacks $X_1 \longrightarrow R$ is surjective, flat and finitely presented morphism. If $U \to X_1$ is a smooth atlas, we have that the sheaf R is isomorphic to the quotient ffqc-sheaf

$$R \simeq Colim\,(U \times_{X \times X} U \rightrightarrows U)\,,$$

and by what we have just seen the projections $U \times_{X \times X} U \to U$ are flat and finitely presentated morphisms of affine schemes. By [La-Mo] ?? we have that R is an algebraic space.

We now consider the property (2). For this we consider the diagram

$$U \longrightarrow R \longrightarrow X.$$

The first morphism is a flat and finitely presented cover, and the second morphism is the composition $U \to X_1 \to X$, and is thus smooth. We thus see that $R \longrightarrow X$ is locally for the flat finitely presented topology on R a smooth morphism, and therefore is smooth by []??. This finishes the proof of the first part of the proposition, as $X \longrightarrow M(F)$ is now a smooth atlas, showing that $M(F)$ is an algebraic space.

To prove the second statement of the proposition we will use (1) applied to certain stacks of iterated loops. We let $x : X \longrightarrow F$, and consider the loop stack $\Omega_x F$ of F at x, defined by

$$\Omega_x F := X \times^h_F X.$$

In the same way we have the iterated loop stacks

$$\Omega_x^{(i)}F := \Omega_x(\Omega_x^{(i-1)}F).$$

Note that we have homotopy cartesian squares

$$\begin{array}{ccc} \Omega_x^{(i)}F & \longrightarrow & I_F^{(i)} \\ \downarrow & & \downarrow \\ X & \longrightarrow & F, \end{array}$$

showing that $\Omega_x^{(i)}F \longrightarrow X$ is flat for any i. Moreoever, for any $Y \in Aff$, and any $s : Y \longrightarrow \Omega_x^{(i)}F$ we have a homotopy cartesian square

$$\begin{array}{ccc} \Omega_s^{(j)}\Omega_x^{(i)}F & \longrightarrow & I^{(j)}_{\Omega_x^{(i)}F} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \Omega_x^{(i)}F. \end{array}$$

Now, as $\Omega_x^{(i)}F$ is a group object over X, we have isomorphisms of stacks over Y

$$\Omega_s^{(j)}\Omega_x^{(i)}F \simeq \Omega_x^{(j+i)}F \times_X Y$$

obtained by translating along the section s. Therefore, we have that

$$I^{(j)}_{\Omega_x^{(i)}F} \longrightarrow \Omega_x^{(i)}F$$

is flat for any i and any j. We can therefore apply (1) to the stacks $\Omega_x^{(i)}F$. As we have

$$M(\Omega_x^{(i)}F) \simeq \pi_i^{fl}(F, x),$$

this gives that the sheaves $\pi_i^{fl}(F, x)$ are algebraic spaces. Moreover, the morphism $\Omega_x^{(i)}F \longrightarrow \pi_i^{fl}(F, x)$ is flat, surjective and of finite presentation, showing that so is $\pi_i^{fl}(F, x)$ as an algebraic space over X. □

Exercise 3.3.3. *1. Let $f : F \longrightarrow F'$ be a morphism of finite presentation between algebraic stacks strongly of finite presentation over some affine scheme. Assume that F' is reduced. Show that there exists a non-empty open sub-stack $U \subset F'$ such that the base change morphism $F \times^h_{F'} U \longrightarrow U$ is flat (use smooth altases and the generic flatness theorem statement that the result is true when F and F' are affine schemes).*

2. *Deduce from* (1) *that if F is a reduced algebraic stack strongly of finite presentation over some affine scheme then F has a non-empty open sub-stack $U \subset F$ which is a total gerbe in the sense of definition 3.3.1.*

The previous exercise, together with the proposition 3.3.2, has the following important consequence.

Corollary 3.3.4. *Let F be an algebraic stack strongly of finite presentation over some affine scheme X. There exists a finite sequence of closed sub-stacks*

$$\emptyset \subset F_r \subset F_{r-1} \subset \cdots \subset F_1 \subset F_0 = F$$

such that eah $F_i - F_{i+1}$ is a total gerbe. We can moreover chose the F_i with the following properties:

1. *For all i, the ffqc-sheaf $M(F_i - F_{i+1})$ is a scheme of finite type over X.*

2. *For all i, all affine scheme Y, all morphism $y : Y \longrightarrow (F_i - F_{i+1})$, and all $j > 0$ the ffqc-sheaf $\pi_j(F_i - F_{i+1}, y)$ is a flat algebraic space of finite presentation over Y.*

In other words, any aglgebraic stack F, strongly finitely presented over some affine scheme, give rise to several schemes $M(F_i - F_{i+1})$, which are stratified pieces of the non-existing coarse moduli space for F. Over each of these schemes, locally for the étale topology, we have the flat groups $\pi_j(F_i - F_{i+1})$. Therefore, up to a stratification, the stack F behave very much like a homotopy type for which its homotopy groups would be represented by schemes (or algebraic spaces).

Exercise 3.3.5. *Recall that an algebraic stack is* Deligne-Mumford *if it possesses an étale atlas (rather than simply smooth).*

1. *Let F be an algebraic stack which is étale over an affine scheme X. Prove that F is Deligne-Mumford and that F is a total gerbe. Show also that the projection $F \longrightarrow M(F)$ is an étale morphism.*

2. *Let F be a Deligne-Mumford stack, and let $p : F \longrightarrow t_{\leq 1}(F)$ be its 1-truncation. Show that $t_{\leq 1}(F)$ is itself a Deligne-Mumford* 1*-stack and that p is etale.*

Chapter 4

Simplicial commutative algebras

In this lecture we review the homotopy theory of simplicial commutative rings. It will be used all along the next lectures in order to define and study the notion of derived schemes and derived stacks.

4.1 Quick review of the model category of commutative simplicial algebras and modules

We let $sComm$ be the category of simplicial objects in $Comm$, that is of simplicial commutative algebras. For $A \in sComm$ a simplicial commutative algebra, we let $sA - Mod$ be the category of simplicial A-modules. Recall that an object in $sA - Mod$ is the data of a simplicial abelian group M_n, together with A_n-module structures on M_n in a way that the transition morphisms $M_n \longrightarrow M_m$ are morphisms of A_n-modules (for the A_n-module structure on M_m induced by the morphism $A_n \longrightarrow A_m$). We will say that a morphism in $sComm$ or in $sA - Mod$ is an equivalence (resp. a fibration), if it the morphism induced on the underlying simplicial sets is so. It is well known that this defines model category structures on $sComm$ and $sA - Mod$. These model category are cofibrantly generated, proper and cellular.

To any simplicial commutative algebras A let $\pi_*(A) := \oplus_n \pi_n(A)$. We do not specify base points as the underlying simplicial sets of a simplicial algebra is a simplicial abelian group, and thus its homotopy groups do not depend on the base point (by convention we will take 0 as base point). The graded abelian group $\pi_*(A)$ has a natural structure of a graded commutative (in the graded sense) algebra. The multiplication of two elements $a \in \pi_n(A)$ and $b \in \pi_m(A)$ is defined as follows. We represent a and b by morphisms of pointed simplicial sets

$$a : S^n := (S^1)^{\wedge n} \longrightarrow A \qquad b : S^m := (S^1)^{\wedge n} \longrightarrow A,$$

where S^1 is a model for the pointed simplicial circle. We then consider the induced morphism

$$a \otimes b : S^n \times S^m \longrightarrow A \times A \longrightarrow A \otimes A.$$

Composing with the multiplication in A we get a morphism of simplicial set

$$a.b : S^n \times S^m \longrightarrow A.$$

This last morphism sends $S^n \times *$ and $* \times S^m$ to the base point $0 \in A$. Therefore, it factorizes as a morphism

$$S^n \wedge S^m \simeq S^{n+m} \longrightarrow A.$$

As the left hand side has the homotopy type of S^{n+m} we obtain a morphism

$$S^{n+m} \longrightarrow A$$

which gives an element $ab \in \pi_{n+m}(A)$. This multiplication is associative, unital and graded commutative. In the same way, if A is a simplicial commutative algebra and M a simplicial A-module, $\pi_*(M) = \oplus_n \pi_n(M)$ has a natural structure of a graded $\pi_*(A)$-module.

For a morphism of simplicial commutative rings $f : A \longrightarrow B$ we have an adjunction

$$- \otimes_A B : sA - Mod \longrightarrow sB - Mod \qquad sA - Mod \longleftarrow sB - Mod : f^*,$$

where the right adjoint f^* is the forgetful functor. This adjunction is a Quillen adjunction which is moreover a Quillen equivalence if f is an equivalence of simplicial algebras. The left derived functor of $- \otimes_A B$ will be denoted by

$$- \otimes^{\mathbb{L}}_A B : Ho(sA - Mod) \longrightarrow Ho(sB - Mod).$$

Finally, a (non simplicial) commutative ring will always be considered as a constant simplicial commutative ring and thus as an object in $sComm$. This induces a fully faithful functor $Comm \longrightarrow sComm$ which induces a fully faithful embedding on the level of the homotopy category

$$Comm \longrightarrow Ho(sComm).$$

This last functor possesses a left adjoint

$$\pi_0 : Ho(sComm) \longrightarrow Comm.$$

In the same manner, if $A \in sComm$ any (non simplicial) $\pi_0(A)$-module can be considered as a constant simplicial A-module, and thus as an object in $sA - Mod$. This also defines a full embedding

$$\pi_0(A) - Mod \longrightarrow Ho(sA - Mod)$$

which admits a left adjoint

$$\pi_0 : Ho(sA - Mod) \longrightarrow \pi_0(A) - Mod.$$

4.2 Cotangent complexes

We start to recall tha notion of trivial square zero extension of a commutative ring by a module. For any commutative ring A and any A-module M we define another commutative ring $A \oplus M$. The underlying abelian group of $A \oplus M$ is the direct sum of A and M, and the multiplication is defined by the following formula

$$(a, m).(a', m') := (aa', am' + a'm).$$

The commutative ring $A \oplus M$ is called the *trivial square zero extension of A by M*. It is an augmented A-algebra by the natural morphisms

$$A \longrightarrow A \oplus M \longrightarrow A,$$

send respectively a to $(a, 0)$ and (a, m) to a. The main property of the ring $A \oplus M$ is that the set of sections (as morphisms or rings) of the projection $A \oplus M \longrightarrow A$ is in natural bijection with the set $Der(A, M)$ of derivations from A to M (this can be taken as a definition of $Der(A, M)$). A standard result states that the functor

$$A - Mod \longrightarrow Set$$

sending M to $Der(A, M)$ is corepresented by an A-module Ω^1_A, the A-module of Kahler differentials on A.

Exercise 4.2.1. *Let A be a commutative ring and consider $Comm/A$, the category of commutative rings augmented over A. Show that $M \mapsto A \oplus M$ defines an equivalence of categories, between $A - Mod$, the category of A-modules, and $Ab(Comm/A)$, the category of abelian group objects in $Comm/A$. Under this equivalence, show that Ω^1_A is the free abelian group object in $Comm/A$ over a single generator.*

The generalization of the above notion to the context of simplicial commutative rings leads to the notion of cotangent complexes and André-Quillen homology (and cohomology). We let A be a simplicial commutative ring and $M \in sA - Mod$ be a simplicial A-module. By applying the construction of the trivial square zero extension levelwise for each A_n and each M_n we obtain a new simplicial commutative ring $A \oplus M$, together with two morphisms

$$A \longrightarrow A \oplus M \longrightarrow A.$$

The model category $sComm$ is a simplicial model category and we will denote by $\underline{Hom}$ its simplicial Hom sets, and by $\mathbb{R}\underline{Hom}$ its derived version (i.e. $\mathbb{R}\underline{Hom}(A, B) := \underline{Hom}(Q(A), B)$ where $Q(A)$ is a cofibrant model for A). The simplicial set $\mathbb{R}Der(A, M)$, of derived derivation from A to M is by definition the homotopy fiber of the natural morphism

$$\mathbb{R}\underline{Hom}(A, A \oplus M) \longrightarrow \mathbb{R}\underline{Hom}(A, A)$$

taken at the identity. In another way we have

$$\mathbb{R}Der(A, M) \simeq \mathbb{R}\underline{Hom}_{/A}(A, A \oplus M),$$

where now $\underline{Hom}_{/A}$ denotes the simplicial Hom sets of the model category $sComm/A$ of commutative simplicial rings over A. It is a well known result that the functor

$$\mathbb{R}Der(A, -) : Ho(sA - Mod) \longrightarrow Ho(SSet)$$

is corepresented by a simplicial A-module $\mathbb{L}_A$ called the *cotangent complex of* A (see for example [Q, Go-Ho]). One possible construction of $\mathbb{L}_A$ is has follows. We start by considering a cofibrant replacement $Q(A) \longrightarrow A$ for A. We then apply the construction of Kahler differentials levelwise for $Q(A)$ to get a simplicial $Q(A)$-module $\Omega^1_{Q(A)}$. We then set

$$\mathbb{L}_A := \Omega^1_{Q(A)} \otimes^{\mathbb{L}}_{Q(A)} A \in Ho(sA - Mod).$$

In this way, $A \mapsto \mathbb{L}_A$ is the left derived functor of $A \mapsto \Omega^1_A$. We note that by adjunction we always have

$$\pi_0(\mathbb{L}_A) \simeq \Omega^1_{\pi_0(A)}.$$

The cotangent complex is functorial in A. Therefore for any morphism of simplicial commutative rings $A \longrightarrow B$ we have a morphism $\mathbb{L}_A \longrightarrow \mathbb{L}_B$ in $Ho(sA-Mod)$. By adjunction this morphism can also be considered as a morphism in $Ho(sB - Mod)$

$$\mathbb{L}_A \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{L}_B.$$

The homotopy cofiber of this morphism will be denoted by $\mathbb{L}_{B/A}$, is called the relative cotangent complex of B over A.

An important fact concerning cotangent complexes is that it can be used in order to describe Postnikov invariants of commutative rings as follows. A simplicial commutative ring A is said to be n-truncated if $\pi_i(A) = 0$ for all $i > n$. The inclusion functor of the full subcategory $Ho(sComm_{\leq n})$ of n-truncated simplicial commutative rings has a left adjoint

$$\tau_{\leq n} : Ho(sComm) \longrightarrow Ho(sComm_{\leq n}),$$

called the n-th truncation functor. These functors can be easily obtained by applying the general mahcinery of left Bousfield localizations to $sComm$. They are the localization functors associated to the left Bousfield localizations of $sComm$ with respect to the morphism $S^{n+1} \otimes \mathbb{Z}[T] \longrightarrow \mathbb{Z}[T]$. For any $A \in Ho(sComm)$ we then have a Postnikov tower

$$A \longrightarrow \cdots \qquad \tau_{\leq n}(A) \longrightarrow \tau_{n-1}(A) \longrightarrow \cdots \longrightarrow \tau_{\leq 0}(A) = \pi_0(A).$$

It can be proved that for any $n > 0$ there is a homotopy pull-back square

$$\begin{array}{ccc} \tau_{\leq n}(A) & \longrightarrow & \tau_{\leq n-1}(A) \\ \downarrow & & \downarrow{\scriptstyle 0} \\ \tau_{\leq n-1}(A) & \xrightarrow[k_n]{} & \tau_{\leq n-1}(A) \oplus \pi_n(A)[n+1], \end{array}$$

where $\pi_n(A)[n+1]$ is the simplicial A-module $S^{n+1} \otimes \pi_n(A)$ (i.e. the $(n+1)$-suspension of $\pi_n(A)$), 0 stands for the trivial derivation and k_n is a certain derived derivation with values in $\pi_n(A)[n+1]$. This derivation is an element in $[\mathbb{L}_{\tau_{\leq n-1}(A)}, \pi_n(A)[n+1]]$ which is by definition the n-Postnikov invariant of A. This element completely determines the simplicial commutative ring $\tau_{\leq n}(A)$ from $\tau_{\leq n-1}(A)$ and the $\pi_0(A)$-module $\pi_n(A)$. It is non-zero precisely when the projection $\tau_{\leq n}(A) \longrightarrow \tau_{\leq n-1}(A)$ has no sections (in $Ho(sComm)$). It is zero precisely when $\tau_{\leq n}(A)$ is equivalent (as an object over $\tau_{\leq n-1}(A)$) to $\tau_{\leq n-1}(A) \oplus \pi_n(A)[n]$.

Exercise 4.2.2. *1. Let A be a simplicial commutative ring with $\pi_0(A)$ being isomorphic either to $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{Z}/p$. Show that the natural projection $A \longrightarrow \pi_0(A)$ has a section in $Ho(sComm)$. Show moreover that this section is unique when $\pi_0(A)$ is either $\mathbb{Z}$ or $\mathbb{Q}$.*

2. Give an example of a simplicial commutative ring A such that the natural projection $A \longrightarrow \pi_0(A)$ has no sections in $Ho(sComm)$.

4.3 Flat, smooth and étale morphisms

We arrive at the three fundamental notions of flat, smooth and étale morphisms of commutative simplicial rings. The material of this paragraph is less standard that the one of the previous paragraph and thus refer to [HAGII, §2.2.2] for the details.

Definition 4.3.1. *Let $f : A \longrightarrow B$ be a morphism of simplicial commutative rings.*

1. *The morphism f is* homotopically of finite presentation *if for any filtered system of commutative simplicial A-algebras $\{C_\alpha\}$ the natural morphism*

$$Colim_\alpha \mathbb{R}\underline{Hom}_{A/sComm}(C_\alpha, B) \longrightarrow \mathbb{R}\underline{Hom}_{A/sComm}(Colim_\alpha C_\alpha, B)$$

is an equivalence.

2. *The morphism f* flat *is the base change functor*

$$- \otimes^{\mathbb{L}}_A B : Ho(sA - Mod) \longrightarrow Ho(sB - Mod)$$

commutes with homotopy pull-backs.

3. *The morphism f is* formally étale *if* $\mathbb{L}_{B/A} \simeq 0$.

4. *The morphism f is* formally smooth *if for any simplicial B-module M with $\pi_0(M) = 0$ we have $[\mathbb{L}_{B/A}, M] = 0$.*

5. *The morphism f is* smooth *if it is formally smooth and homotopically of finite presentation.*

6. *The morphism f is* étale *if it is formally étale and homotopically of finite presentation.*

7. *The morphism f is a* Zariski open immersion *if it is flat, homotopically of finite presentation and if moreover the natural morphism $B \otimes^{\mathbb{L}}_A B \longrightarrow B$ is an equivalence.*

All the above notions of morphisms are stable by composition in $Ho(sComm)$. They are also stable by homotopy cobase change in the sense that if a morphism $f : A \longrightarrow B$ is homotopically of finite presentation (resp. flat, resp. formally étale ...) , then for any $A \longrightarrow A'$ the induced morphism $A' \longrightarrow A' \otimes^{\mathbb{L}}_A B$ is again homotopically of finite presentation (resp. flat, resp. formally étale ...).

Here follows a sample of standard results concerning the above notions.

- A Zariski open immersion is étale, an étale morphism smooth and a smooth morphism is flat.

- A morphism $A \longrightarrow B$ is flat (resp. étale, resp. smooth, resp. a Zariski open immersion) if and only if it satisfies the two following properties.

  1. The induced morphism of rings $\pi_0(A) \longrightarrow \pi_0(B)$ is flat (resp. étale, resp. smooth, resp. a Zariski open immersion) in the usual sense.
  2. For all $i > 0$ the induced morphism
  $$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow \pi_i(B)$$
  is bijective.

- An important direct consequence of the last point is that a morphism of (non simplicial) commutative rings is flat (resp. étale, resp. smooth, resp. a Zariski open immersion) in the usual sense if and only if it is so in the sense of definition 4.3.1.

- A morphism $A \longrightarrow B$ is homotopically of finite presentation if and only if B is equivalent to a retract of a finite cellular A-algebra. Recall here that a finite cellular A-algebra is a commutative A-algebra B' such that there exists a finite sequence
$$A = B'_0 \longrightarrow B'_1 \dots \longrightarrow B'_n = B',$$

such that for any i there exists a cocartesian square of commutativr simplicial rings

$$\begin{array}{ccc} B'_i & \longrightarrow & B'_{i+1} \\ \uparrow & & \uparrow \\ \partial\Delta^{n_i} \otimes A & \longrightarrow & \Delta^{n_i} \otimes A. \end{array}$$

In particular, if $A \longrightarrow B$ is a morphism of commutative rings, being homotopically finitely presented is a much stronger condition than being finitely presented in the usual sense. For instance, if B is a a commutative k-algebra of finite type (for k a field) which admits a singularity which is not a local complete intersection, then B is not homotopically finitely presented over k. Intuitively, B is homotopically finitely presented over A when it admits (up to a retract) a finite resolution by free A-algebras of finite type.

- For a given $A \in sComm$, there exists a functor

$$\pi_0 : Ho(A/sComm) \longrightarrow \pi_0(A)/Comm$$

from the homotopy category of commutative simplicial A-algebras to the category of commutative $\pi_0(A)$-algebras. This functor induces an equivalence on the full subcategories of étale morphisms Zariski open immersions. The corresponding fact for smooth and flat morphisms is not true.

Exercise 4.3.2. *Let $A \longrightarrow B$ be a morphism in sComm. Show that $A \longrightarrow B$ is formally étale if and only if $B \otimes^{\mathbb{L}}_A B \longrightarrow B$ is a formally smooth.*

Exercise 4.3.3. *1. Let $A \longrightarrow B$ be a morphsm in sComm which is formally étale and such that $\pi_0(A) \longrightarrow \pi_0(B)$ is an isomorphism. Show that f is an isomorphism in $Ho(sComm)$.*

2. Deduce from (1) that if $A \longrightarrow B$ is formally étale the natural morphism

$$B \longrightarrow B \otimes^{\mathbb{L}}_{B \otimes^{\mathbb{L}}_A B} B$$

is an isomorphism in $Ho(sComm)$.

3. Let $A \longrightarrow B$ be a morphism in $Ho(sComm)$ which is formally étale. Show that for any $C \in A/sComm$ the mapping space $Map_{A/sComm}(B, C)$ is homotopically discrete (i.e. equivalent to a set).

Exercise 4.3.4. *Let $f : A \longrightarrow B$ a morphism of non-simplicial commutative rings. Show that f is hpf in sComm if and only if it is finitely presented as a morphism of rings and $\mathbb{L}_{B/A}$ is a perfect simplicial B-module (i.e. correspond to a perfect complex of B-modules by the Dold-Kan correspondence).*

Chapter 5

Derived stacks and derived algebraic stacks

We arrive at the notion of derived stacks and derived algebraic stacks. We first present the homotopy theory of derived stacks, which is very similar to the homotopy theory of simplicial presheaves presented in §2, but where the category $Comm$ must be replaced by the more complicated model category $sComm$. The new feature here is to take into account correctly the model category structure of $sComm$ which makes the definitions a bit more technical.

5.1 Derived stacks

We set $dAff := sComm^{op}$, which by definition is the category of derived affine schemes. It is endowed with the dual model category structure of $sComm$. An object in $dAff$ corresponding to $A \in sComm$ will be denoted formally by $Spec\, A$. This $Spec$ has only a formal meaning, and we will define another, more interesting, Spec functor that will be denoted by $\underline{\mathbb{R}Spec}$.

We consider $SPr(dAff)$, the category of simplicial presheaves on $dAff$. We will define three different model category structures on $SPr(dAff)$, each one being a left Bousfield localization of the previous one. In order to avoid confusion we will use different notations for these three model categories (contrary to what we have done in the sections §2 and §3), even thought the underlying categories are identical. They will be denoted by $SPr(dAff)$, $dAff^{\wedge}$ and $dAff^{\sim}$.

The first model structure is the projective levelwise model category structure on $SPr(dAff)$, for which equivalences and fibrations are defined levelwise. We do not give any specific name to this model category. We consider the Yoneda embedding

$$\begin{array}{rccc} h: & dAff & \longrightarrow & SPr(dAff) \\ & X & \mapsto & h_X = Hom(-,X). \end{array}$$

Here h_X is a presheaf of sets and is consider as a simplicial presheaf constant in the simplicial direction. For any equivalence $X \longrightarrow Y$ in $dAff$ we deduce a morphism $h_X \longrightarrow h_Y$ in $SPr(dAff)$. By definition the model category $dAff^{\wedge}$ is the left Bousfield localization of the model category $SPr(dAff)$ with respect to the set of all morphisms $h_X \longrightarrow h_Y$ obtained from equivalences $X \longrightarrow Y$ in $dAff$. The model category $dAff^{\wedge}$ is called the *model category of prestacks over* $dAff$.

By definition, the fibrant objects in $dAff^{\wedge}$ are the simplicial presheaves $F : dAff^{op} \longrightarrow SSet$ satisfying the following two conditions.

1. For any $X \in dAff$ the simplicial set $F(X)$ is fibrant.
2. For any equivalence $X \longrightarrow Y$ in $dAff$, the induced morphism $F(Y) \longrightarrow F(X)$ is an equivalence of simplicial sets.

The first condition above is anodyne, but the second one is not. This second condition is called the *prestack* condition. This is the essential new features of derived stack theory compare with stack theory for which this condition did not appear (simply because there is no notion of equivalence conisdered in $Comm$ except the trivial one: the notion of isomorphism). The standard results about left Bousfield localizations imply that $Ho(dAff^{\wedge})$ is naturally equivalent to the full subcategory of $Ho(SPr(dAff))$ consisting of all simplicial presheaves satisfying condition (2) above. We will implicitely identify these two categories. Moreover, the inclusion functor

$$Ho(dAff^{\wedge}) \longrightarrow Ho(SPr(dAff))$$

has a left adjoint which simply consists of sending a simplicial presheaf F to its fibrant model.

Exercise 5.1.1. *The natural projection* $dAff \longrightarrow Ho(dAff)$ *induces a functor*

$$Ho(SPr(Ho(dAff))) \longrightarrow Ho(dAff^{\vee}).$$

Show that this functor is not an equivalence of categories.

We come back to the Yoneda functor

$$h : dAff \longrightarrow SPr(dAff) = dAff^{\wedge}.$$

We compose it with the natural functor $dAff^{\wedge} \longrightarrow Ho(dAff^{\wedge})$ and we obtain a functor

$$h : dAff \longrightarrow Ho(dAff^{\wedge}).$$

By construction this functor sends equivalences in $dAff$ to isomorphisms in $Ho(dAff^{\wedge})$. Therefore it induces a well defined functor

$$Ho(h) : Ho(dAff) \longrightarrow Ho(dAff^{\wedge}).$$

A general result, called the Yoneda lemma for model categories (see [HAGI]), states two properties concerning $Ho(h)$.

1. The functor $Ho(h)$ is fully faithful. This is the model category version of the Yoneda lemma for categories.

2. For $X \in dAff$, the object $Ho(h)(X) \in Ho(SPr(dAff))$ can be described as follows. We take RX a fibrant model for X in $dAff$ (i.e. if $X = Spec\,A$, $RX = Spec\,, Q(A)$ for $Q(A)$ a cofibrant model for A in $sComm$). We consider the simplicial presheaf $\underline{h}_{RX}\ Y \mapsto \underline{Hom}(Y, RX)$ where $\underline{Hom}$ denotes the simplicial Hom sets of $dAff$. When $X = Spec\,A$, this simplicial presheaf is also $Spec\,B \mapsto \underline{Hom}(Q(A), B)$. Then, the simplicial presheaf $Ho(h)(X)$ is equivalent to $\underline{h}_{RX}$. When $X = Spec\,A$ we will also use the following notation
$$\mathbb{R}\underline{Spec}\,A := Ho(h)(X) \simeq \underline{h}_{RX} \simeq \underline{Hom}(Q(A), -).$$

In an equivalent way the Yoneda lemma in this setting states that the functor $A \mapsto \underline{Hom}(Q(A), -)$ induces a fully faithful functor
$$Ho(Comm)^{op} \longrightarrow Ho(dAff^{\wedge}) \subset Ho(SPr(dAff)).$$

We now introduce the notion of local equivalences for morphisms in $dAff^{\wedge}$ which will be our equivalences for the final model structure. For this we endow the category $Ho(dAff)$ with a Grothendieck topology as follows. We say that a family of morphisms $\{A \longrightarrow A_i\}_i$ is an *étale covering* if each of the morphisms $A \longrightarrow A_i$ is étale in the sense of definition 4.3.1 and if the family of functors
$$\{- \otimes^{\mathbb{L}}_A A_i : Ho(sA - Mod) \longrightarrow Ho(sA_i - Mod)\}_i$$
is conservative. By definition, the étale topology on $Ho(dAff)$ is the topology for which covering sieves are the one generated by étale covering families. In the same way, for any fixed $X \in dAff$ we define an étale topology on $Ho(dAff/X)$.

The étale topology on $Ho(dAff)$ can be used in order to define homotopy sheaves for objects $F \in Ho(dAff^{\wedge})$. We start to define the homotopy presheaves as follows. Let $F : dAff^{op} \longrightarrow SSet$ be an object in $Ho(dAff^{\wedge})$, so in particular we assume that F sends equivalences in $dAff$ to equivalences in $SSet$. We consider the presheaf of sets $X \mapsto \pi_0(F(X))$. This presheaf sends equivalences in $dAff$ to isomorphisms in Set and thus factorizes as a functor $\pi_0^{pr}(F) : Ho(dAff)^{op} \longrightarrow Set$. In the same way, for $X \in dAff$ et $s \in F(X)$, we define a presheaf of groups on $dAff/X$ wich sends $f : Y \longrightarrow X$ to $pi_i(F(Y), f^*(s))$. Again this presheaf sends equivalences to isomorphisms and thus induces a functor $pi_i^{pr}(F, s) : Ho(dAff/X)^{op} \longrightarrow Set$. With these notations, the associated sheaves (for the étale topology defined above) to $\pi_0^{pr}(F)$ and $\pi_i^{pr}(F, s)$ are denoted by $\pi_0(F)$ and $\pi_i(F, s)$ and are called the *homotopy sheaves of* F. These are defined for $F : dAff^{op} \longrightarrow SSet$ sending equivalences to equivalences. Now, for a general simplicial presheaf we set
$$\pi_0(F) := \pi_0(F^{\wedge}) \qquad \pi_i(F, s) := \pi_i(F^{\wedge}, s)$$
where $F^{\wedge}$ is a fibrant model for F in $dAff^{\wedge}$.

Definition 5.1.2. *Let $f : F \longrightarrow F'$ be a morphism of simplicial presheaves on $dAff$.*

1. *The morphism f is a* local equivalence *if it satisfies the following two conditions*
   (a) *The induced morphism $\pi_0(F) \longrightarrow \pi_0(F')$ is an isomorphism of sheaves on $Ho(dAff)$.*
   (b) *For any $X \in dAff$, any $s \in F(X)_0$ and any $i > 0$ the induced morphism $\pi_i(F, s) \longrightarrow \pi_i(F', f(s))$ is an isomorphism of sheaves on $dAff/X$.*
2. *The morphism f is a* local cofibration *if it is a cofibration in $dAff^\wedge$ (or equivalently in $SPr(dAff)$).*
3. *The morphism f is a* local fibration *if it has the left lifting property with respect to every local cofibration which is also a local equivalence.*

For simplicity, we will use the expressions equivalence, fibrations *and* cofibration *in order to refer to* local equivalence, local fibration *and* local cofibration.

It can be proved (see [HAGI]) that these notions of equivalences, fibrations and cofibrations define a model category structure on $SPr(dAff)$. This model category will be denoted by $dAff^\sim$. As for the case of simplicial presheaves it is possible to characterize fibrant objects in $dAff^\sim$ as functors $F : dAff^{op} \longrightarrow SSet$ satisfying the following three conditions (we de not precise the definition of étale hypercovering in this context, it is very similar to the one we gave for simplicial presheaves in §2.1).

1. For any $X \in dAff$ the simplicial set $F(X)$ is fibrant.
2. For any equivalence $X \longrightarrow Y$ in $dAff$ the induced morphism $F(Y) \longrightarrow F(X)$ is an equivalence of simplicial sets.
3. For any $X \in dAff$ and any étale hypercovering $H \longrightarrow X$ the natural morphism
$$F(X) \longrightarrow Holim_{[n]\in\Delta} F(H_n)$$
is an equivalence of simplicial sets.

Definition 5.1.3.
1. *An object $F \in SPr(dAff)$ is called a* derived stack *if it satisfies the conditions* (2) *and* (3) *above.*
2. *The homotopy category $Ho(dAff^\sim)$ will be called the* homotopy category of derived stacks*. Most often objects in $Ho(dAff^\sim)$ will simply be called* derived stacks*. The expressions* morphism of derived stacks *and* isomorphism of derived stacks*, will refer to morphisms and isomorphisms in $Ho(dAff^\sim)$. The set of morphisms of derived stacks from F to F' will be denoted by* $[F, F']$.

To finish this first paragraph we mention how stacks and derived stacks are compared. For this we consider the functor $Comm \longrightarrow sComm$ which consists of considering a commutative ring as a constant simplicial commutative ring. This induces a functor $i : Aff \longrightarrow dAff$. Pulling back along this functor induces a functor

$$i^* : dAff^{\sim} \longrightarrow SPr(Aff).$$

This functor is seen to be right Quillen whose left adjoint is denoted by

$$i_! : SPr(Aff) \longrightarrow dAff^{\sim}.$$

The derived Quillen adjunction is denoted by

$$j : Ho(SPr(Aff)) \longrightarrow Ho(dAff^{\sim}) \qquad Ho(SPr(Aff)) \longleftarrow Ho(dAff^{\sim}) : h^0.$$

The functor j is fully faithful, as this follows from the fact that the functor $Comm \longrightarrow Ho(sComm)$ is fully faithful and compatible with the étale topologies on both sides. Therefore, any stack can be considered as a derived stacks. The functor h^0 is called the *classical part functor*, and remembers only the part related to non simplicial commutative rings of a given derived stack. Using the functor j we will see any stack as a derived stack.

Definition 5.1.4. *Given a stack $F \in Ho(SPr(Aff))$, a* derived extension of F *is the data of a derived stack $\widetilde{F} \in Ho(dAff^{\sim})$ together with an isomorphism of stacks $F \simeq h^0(\widetilde{F})$.*

The existence of the full embedding j implies that any stack admits a derived extension $j(F)$, but this extension is somehow the trivial one. The striking fact about derived algebraic geometry is that most (if not all) of the moduli problems admits natural derived extensions, and these are not the trivial one in general. We will see many such examples in the next lecture.

5.2 Algebraic derived n-stacks

We now mimick the definitions of schemes and algebraic stacks given in §3 for our new context of derived stacks.

We start by considering the Yoneda embedding

$$Ho(dAff) \longrightarrow Ho(dAff^{\wedge}).$$

The fathfully flat descent stays true in the derived setting and this embedding induces a fully faithful functor

$$Ho(dAff) \longrightarrow Ho(dAff^{\sim}).$$

Equivalently, this means that for any $A \in sComm$, the prestack $\mathbb{R}\underline{Spec}\,A$, sending B to $\underline{Hom}(Q(A), B)$, satisties the descent condition for étale hypercoverings (i.e. is a derived stack). Objects in the essential image of this functor will be called *derived affine schemes*, and the full subcategory of $Ho(dAff^{\sim})$ consisting of derived affine schemes will be implicitly identified with $Ho(dAff)$.

One of the major difference between stacks and derived stacks is that derived affine schemes are not 0-truncated. The definition of Zariski open immersion given in definition 3.1.1 has therefore to be slightly modified.

Definition 5.2.1. *1. A morphism of derived stacks $F \longrightarrow F'$ is a* monomorphism *if the induced morphism $F \longrightarrow F \times^h_{F'} F$ is an equivalence.*

2. A morphism of derived stacks $F \longrightarrow F'$ is an epimorphism *if the induced morphism $\pi_0(F) \longrightarrow \pi_0(F')$ is an epimorphism of sheaves.*

3. Let $X = \mathbb{R}\underline{Spec}\,A$ be a derived affine scheme, F a derived stack and $i : F \longrightarrow \mathbb{R}\underline{Spec}\,A$ a morphism. We say that i is a Zariski open immersion *(or simply an* open immersion*) if it satisfies the following two conditions.*

(a) The morphism i is a monomorphism.

(b) There exists a family of Zariski open immersions $\{A \longrightarrow A_i\}_i$ such that the morphism $\mathbb{R}\underline{Spec}\,A_i \longrightarrow \mathbb{R}\underline{Spec}\,A$ all factor throught F in a way that the resulting morphism

$$\coprod_i \mathbb{R}\underline{Spec}\,A_i \longrightarrow \mathbb{R}\underline{Spec}\,A$$

is an epimorphism.

4. A morphism of derived stacks $F \longrightarrow F'$ is a Zariski open immersion *(or simply an* open immersion*) if for any derived affine scheme X and any morphism $X \longrightarrow F'$, the induced morphism*

$$F \times^h_{F'} X \longrightarrow X$$

is a Zariski open immersion in the sense above.

5. A derived stack F is a derived scheme *if there exists a family of derived affine schemes $\{\mathbb{R}\underline{Spec}\,A_i\}_i$ and Zariski open immersions $\mathbb{R}\underline{Spec}\,A_i \longrightarrow F$, such that the induced morphism of sheaves*

$$\coprod_i \mathbb{R}\underline{Spec}\,A_i \longrightarrow F$$

is an epimorphism. Such a family of morphisms $\{\mathbb{R}\underline{Spec}\,A_i \longrightarrow F\}$ will be called a Zariski altas for *F.*

We say that a morphism of derived schemes $X \longrightarrow Y$ is smooth (resp. flat, resp. étale) if there exists Zariski atlas $\{\mathbb{R}\underline{Spec}\, A_i \longrightarrow X\}$ and $\{\mathbb{R}\underline{Spec}\, A_j \longrightarrow Y\}$ together with commutative squares (in $Ho(dAff^{\sim})$)

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \uparrow & & \uparrow \\ \mathbb{R}\underline{Spec}\, A_i & \longrightarrow & \mathbb{R}\underline{Spec}\, A_j, \end{array}$$

with $\mathbb{R}\underline{Spec}\, A_i \longrightarrow \mathbb{R}\underline{Spec}\, A_j$ a smooth (resp. flat, resp. étale) morphism (here j depends on i). Smooth morphisms of derived schemes are stable by compositions and homotopy base change.

The following is the main definition of this series of lectures.

Definition 5.2.2. *1. A derived stack F is* 0-algebraic *if it is a derived scheme.*

2. *A morphism of derived stacks $F \longrightarrow F'$ is* 0-algebraic *(or* 0-representable*) if for any derived scheme X and any morphism $X \longrightarrow F'$ the derived stack $F \times^h_{F'} X$ is 0-algebraic (i.e. a derived scheme).*

3. *A 0-algebraic morphism of derived stacks $F \longrightarrow F'$ is smooth if for any derived scheme X and any morphism $X \longrightarrow F'$ the morphism of derived schemes $F \times^h_{F'} X \longrightarrow X$ is smooth.*

4. *We now let $n > 0$, and we assume that the notions of $(n-1)$-algebraic derived stack, $(n-1)$-algebraic morphisms and smooth $(n-1)$-algebraic morphisms have been defined.*

   (a) *A derived stack F is* n-algebraic *if there exists a derived scheme X together with smooth $(n-1)$-algebraic morphisms $X \longrightarrow F$ which is an epimorphism. Such a morphism $X \longrightarrow F$ is called a* smooth n-atlas for F.

   (b) *A morphism of derived stacks $F \longrightarrow F'$ is* n-algebraic *(or* n-representable*) if for any derived scheme X and any morphism $X \longrightarrow F'$ the derived stack $F \times_{F'} X$ is n-algebraic.*

   (c) *An n-algebraic morphism of derived stacks $F \longrightarrow F'$ is* smooth *(resp.* flat*, resp.* étale*) if for any derived scheme X and any morphism $X \longrightarrow F'$ there exists a smooth n-atlas $Y \longrightarrow F \times^h_{F'} X$ such that each morphism $Y \longrightarrow X$ is a smooth (resp. flat, resp. étale) morphism of derived schemes.*

5. *An* derived algebraic stack *is a derived stack which is n-algebraic for some integer n.*

6. *A morphism of derived stacks $F \longrightarrow F'$ is* algebraic *(or* representable*) if it is n-algebraic for some n.*

7. *A morphism of derived stacks $F \longrightarrow F'$ is* smooth *(resp.* flat*, resp.* étale*) if it is n-algebraic and smooth (resp. flat, resp. étale) for some integer n.*

We finish this part by some basic properties of derived algebraic stacks, and in particular with a comparison between the notions of algebraic stacks and derived algebraic stacks.

- Derived algebraic stacks are stable by finite homotopy limits (i.e. homotopy pull-backs).

- Derived algebraic stacks are stable by disjoint union.

- Algebraic morphisms of derived stacks are stable by composition and homotopy base change.

- Derived algebraic stacks are stable by smooth quotients. To be more precise, if $F \longrightarrow F'$ is a smooth epimorphism of derived stacks, then F' is algebraic if and only if F is so.

- A (non derived) stack F is algebraic if and only if the derived stack $j(F)$ is algebraic.

- If F is an algebraic derived stack then the stack $h^0(F)$ is an algebraic stack. When $h^0(F)$ is an algebraic n-stack we say that F is a derived algebraic n-stack (thought it is not n-truncated as a simplicial presheaf on $dAff$).

- A derived algebraic space is a derived algebraic stack F such that $h^0(F)$ is an algebraic space. In other words a derived algebraic space is a derived algebraic 0-stack

- If F is an algebraic derived n-stack, and A is an m-truncated commutative simplicial ring then $F(A)$ is an $(n+m)$-truncated simplicial set.

- If $f : F \longrightarrow F'$ is a flat morphism of derived algebraic stack, and if F' is an algebraic stack (i.e. of the form $j(F'')$ for an algebraic stack F''), then F is itself an algebraic stack.

We see that the formal properties of derived algebraic stacks are the same as the formal properties of non derived algebraic stacks. However, we would like to make the important comment here that the inclusion functor $j : Ho(SPr(Aff)) \hookrightarrow Ho(dAff^{\sim})$ from stacks to derived stacks does not commute with homotopy pull-backs. In other words, if $F \longleftarrow H \longrightarrow G$ is a diagram of stacks then the natural morphism

$$j(F \times^h_H G) \longrightarrow j(F) \times^h_{j(H)} j(G)$$

is not an isomorphism in general. As this morphism induces an isomorphism on h^0, this is an example of a non trivial derived extension of a stack as a derived stack. Each time a stack is presented as a certain finite homotopy limit of other stacks it has a natural, and in general non trivial, derived extension by considering the same homotopy limit in the bigger category of derived stacks.

5.3 Cotangent complexes

To finish this lecture we now explain the notion of cotangent complexes of a derived stack at given point. We let F be an algebraic derived stack and $X = Spec\,A$ be a (non derived) affine scheme. We fix a point (i.e. a morphism of stacks)

$$x : X \longrightarrow F.$$

We let $D^{\leq 0}(A)$ be the non positive derived category of cochain complexes of A-modules. By the Dold-Kan correspondence we will also identify $D^{\leq 0}(A)$ with $Ho(sA-Mod)$ the homotopy category of simplicial A-modules. We define a functor

$$\mathbb{D}er_x(F, -) : D^{\leq 0}(A) \longrightarrow Ho(SSet)$$

by the following way. For $M \in D^{\leq 0}(A)$ we form $A \oplus M$, which is now a commutative simplicial ring (here we consider M as a simplicial A-module), and we set $X[M] := \mathbb{R}\underline{Spec}\,A \oplus M$. The natural projection $A \oplus M$ induces a morphism of derived schemes $X \longrightarrow X[M]$. By definition, the simplicial set $\mathbb{D}er_x(F, M)$ is the homotopy fiber of the natural morphism

$$F(X[M]) \longrightarrow F(X)$$

taken at the point x (here we use the Yoneda lemma stating that $\pi_0(F(X)) \simeq [X, F]$). The simplicial set $\mathbb{D}er_x(F, M)$ is called the simplicial set of derivations of F at the point x with coefficients in M. This is functorial in M and thus defines a functor

$$\mathbb{D}er_x(F, -) : D^{\leq 0}(A) \longrightarrow Ho(SSet).$$

It can be proved that this functor is corepresentable by a complex of A-modules. More precisely, there exists a complex of A-modules (a priori not concentrated in non positive degrees anymore) $\mathbb{L}_{F,x}$, called the cotangent complex of F at x, and such that there exist natural isomorphisms in $Ho(SSet)$

$$\mathbb{D}er_x(F, M) \simeq Map(\mathbb{L}_{F,x}, M),$$

where Map are the mapping spaces of the model category of (unbounded) complexes of A-modules. When the derived stack F is affine this is a reformulation of the existence of a cotangent complex as recalled in §4. In general one reduces the statement to the affine case by a long a tedious induction (on n proving the result for algebraic derived n-stacks). Finally, with a bit of care we can show that $\mathbb{L}_{F,x}$ is unique and functorial (but this requires to state a rafined universal property, see [HAGII]).

Definition 5.3.1. *With the notation above the complex* $\mathbb{L}_{F,x}$ *is called the* cotangent complex of F at the point x. *Its dual* $\mathbb{T}_{F,x} := \mathbb{R}\underline{Hom}(\mathbb{L}_{F,x}, A)$ *is called the* tangent complex of F at x. *The cohomology groups*

$$T^i_{F,x} := H^i(\mathbb{T}_{F,x})$$

are called the higher tangent spaces of F at x.

For a derived algebraic n-stack F and $x : X = Spec\, A \longrightarrow F$ a point, the cotangent complex $\mathbb{L}_{F,x}$ belongs to $D^{\leq n}(A)$ the derived category of complexes concentrated in degrees $]-\infty, n]$. The part of $\mathbb{L}_{F,x}$ concentrated in negative degree is the one related to the *derived part of* F (i.e. the one making the difference between commutative rings and commutative simplicial rings), and the non negative part is related to the *stacky part* of F (i.e. the part related to the higher homotopy sheaves of $h^0(F)$). For instance, when F is a derived scheme its stacky part is trivial and thus $\mathbb{L}_{F,x}$ belongs to $D^{\leq 0}(A)$. On the other hand, when F is a smooth algebraic n-stack (say over $Spec\, \mathbb{Z}$ to simplify), then $\mathbb{L}_{F,x}$ is concentrated in degrees $[0, n]$ (even more is true, it is of Tor-amplitude concentrated in degrees $[0, n]$).

We will in the next lecture several examples of derived stacks which will show that the tangent complexes contain interesting cohomological information. Also, tangent complexes are very useful to provide smoothness and étaleness criterion, which are in general easy to check in practice. For this reason, proving smoothness is in general much more easy in the context of derived algebraic geometry than in the usual context of algebraic geometry. Here is for instance a smoothness criterion (see [HAGII] for details).

Let $f : F \longrightarrow F'$ be a morphism of algebraic derived stack. For any affine scheme $X := Spec\, A$ and any point $x : X \longrightarrow F$ we consider the homotopy fiber of the natural morphism

$$\mathbb{L}_{F,x} \longrightarrow \mathbb{L}_{F',x},$$

which is called the relative cotangent complex of f at x and is denoted by $\mathbb{L}_{f,x}$. We assume that f is locally homotopically of finite presentation (i.e. F and F' admits atlases compatible with f so that the induced morphism on the atlases is homotopically of finite presentation). Then f is smooth if and only if for any affine scheme $X = Spec\, A$ and any $x : X \longrightarrow F$, the complex $\mathbb{L}_{f,x}$ is of non negative Tor amplitude (i.e. for all $M \in D^{\leq -1}(A)$ we have $[\mathbb{L}_{f,x}, M] = 0$).

Chapter 6

Some examples of derived algebraic stacks

In this last lecture we present examples of derived algebraic stacks.

6.1 The derived moduli space of local systems

We come back to the example we presented in the first lecture, the moduli problem of linear representations of a discrete group. We will now reconsider it from the point of view of derived algebraic geometry. We will try to treat this example with some details as we think it is a rather simple, but interesting, example of a derived algebraic stack.

A linear representation of group G can also be interpreted as a local system on the space BG. We will therefore study the moduli problem from this topological point of view. We fix a finite CW complex X and we are going to define a derived stack $\mathbb{R}Loc(X)$, classifying local systems on X. We will see that this stack is an algebraic derived 1-stack and we will describe its higher tangent spaces in termes of cohomology groups of X. When $X = BG$ for a discrete group G the derived algebraic stack $\mathbb{R}Loc(X)$ is the *correct moduli space* of linear representations of G.

We start to consider the non derived algebraic 1-stack **Vect** classifying projective modules of finite type. By definition, **Vect** sends a commutative ring A to the nerve of the groupoid of projective A-modules of finite type. The stack **Vect** is a 1-stack. It is easy to see that **Vect** is an algebraic 1-stack. Indeed, we have a decomposition

$$\mathbf{Vect} \simeq \coprod_n \mathbf{Vect}_n,$$

where $\mathbf{Vect}_n \subset \mathbf{Vect}$ is the substack of projective modules of rank n (recall that a projective A-module of finite type M is of rank n is for any field K and any morphism $A \longrightarrow K$ the K-vector space $M \otimes_A K$ is of dimension n). It is therefore enough to prove that $\mathbf{Vect}_n$ is an algebraic 1-stack. This last statement will itsefl follow from the identification

$$\mathbf{Vect}_n \simeq [*/Gl_n] = BGl_n,$$

where Gl_n is the affine group scheme sending A to $Gl_n(A)$. In order to prove that $\mathbf{Vect}_n \simeq BGl_n$ we construct a morphism of simplicial presheaves

$$BGl_n \longrightarrow \mathbf{Vect}_n$$

by sending the base point of BGl_n to the trivial projective module of rank n: for a given commutative ring A the morphism

$$BGl_n(A) \longrightarrow \mathbf{Vect}_n(A)$$

sends the base point to A^n and identifies $Gl_n(A)$ with the automorphism group of A^n. The claim is that the morphism $BGl_n \longrightarrow \mathbf{Vect}_n$ is a local equivalence of simplicial presheaves. As by construction this morphism induces isomorphisms on all higher homotopy sheaves it only remains to show that it induces an isomorphism on the sheaves π_0. But this in turn follows from the fact that $\pi_0(\mathbf{Vect}_n) \simeq *$, because any projective A-module of finite type is locally free for the Zariski topology on $Spec\, A$.

The algebraic stack **Vect** is now considered as an algebraic derived stack using the inclusion functor $j : Ho(SPr(Aff)) \longrightarrow Ho(dAff^{\sim})$. We consider $F \in dAff^{\sim}$, a fibrant model for $j(\mathbf{Vect})$, and we define a new simplicial presheaf

$$\mathbb{R}Loc(X) : dAff^{op} \longrightarrow SSet$$

which sends $A \in sComm$ to $Map(X, |F(A)|)$ the simplicial set of continuous maps from X to $|F(A)|$.

Definition 6.1.1. *The derived stack* $\mathbb{R}Loc(X)$ *defined above is called the* derived moduli stack of local systems on X.

We will now describe some basic properties of the derived stack $\mathbb{R}Loc(X)$. We start by a description of its classical part $h^0(\mathbb{R}Loc(X))$, which will show that it does classify local systems on X. We will then show that $\mathbb{R}Loc(X)$ is an algebraic derived stack locally of finite presentation over $Spec\, \mathbb{Z}$, and that it can be written as

$$\mathbb{R}Loc(X) \simeq \coprod_n \mathbb{R}Loc_n(X)$$

where $\mathbb{R}Loc_n(X)$ is the part classifiying local systems of rank n and is itself strongly of finite type. Finally, we will compute its tangent spaces in terms of the cohomology of X.

For $A \in Comm$, $h^0(\mathbb{R}Loc(X))(A)$ is by definition the simplicial set $Map(X, |F(A)|)$. Now, $F(A)$ is a fibrant model for $j(\mathbf{Vect})(A) \simeq \mathbf{Vect}(A)$, so is equivalent to the nerve of the groupoid of projective A-modules of finite rank. The simplicial set $Map(X, |F(A)|)$ is then naturally equivalent to the nerve of the groupoid of functors $Fun(\Pi_1(X), F(A))$, from the fundamental groupoid of X to $F(A)$. This last groupoid is in turn equivalent to the groupoid of local systems of projective A-modules of finite type on the space X. Thus, we see that $h^0(\mathbb{R}Loc(X))(A)$ is naturally equivalent to the nerve of the groupoid of local systems of projective A-modules of finite type on the space X. We thus have the following properties.

1. The set $\pi_0(h^0(\mathbb{R}Loc(X))(A))$ is functorially in bijection with the set of isomorphism classes of local systems of projective A-modules of finite type on X. In particular, when A is a field this is also the set of local systems of finite dimensional vector spaces over X.

2. For a local system $E \in \pi_0(h^0(\mathbb{R}Loc(X))(A))$ we have

$$\pi_1(h^0(\mathbb{R}Loc(X))(A), E) = Aut(E),$$

the automorphism group of E as a sheaf of A-modules on X.

3. For all $i > 1$ and all $E \in \pi_0(h^0(\mathbb{R}Loc(X))(A))$ we have $\pi_i(h^0(\mathbb{R}Loc(X))(A), E) = 0$.

Let us explain now why the derived stack $\mathbb{R}Loc(X)$ is algebraic. We start by the trivial case where X is a contractible space. Then by definition we have $\mathbb{R}Loc(X) \simeq \mathbb{R}Loc(*) \simeq j(\mathbf{Vect})$. As we already know that $j(\mathbf{Vect})$ is an algebraic stack this implies that $\mathbb{R}Loc(X)$ is an algebraic derived stack when X is contractible.

The next step is to prove that $\mathbb{R}Loc(S^n)$ is algebraic for any $n \geq 0$. This can be seen by induction on n. The case $n = 0$ is obvious. Moreover, for any $n > 0$ we have a homotopy push-out of topological spaces

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & S^n, \end{array}$$

where D^n is the n-dimensional ball. This implies the existence of a homotopy pull-back diagram of derived stack

$$\begin{array}{ccc} \mathbb{R}Loc(S^n) & \longrightarrow & \mathbb{R}Loc(D^n) \\ \downarrow & & \downarrow \\ \mathbb{R}Loc(D^n) & \longrightarrow & \mathbb{R}Loc(S^{n-1}). \end{array}$$

By induction on n and by what have just seen before the derived stacks $\mathbb{R}Loc(D^n)$ and $\mathbb{R}Loc(S^{n-1})$ are algebraic. By the stability of algebraic derived stacks by homotopy pull-backs we deduce that $\mathbb{R}Loc(S^n)$ is an algebraic derived stack.

We are now ready to show that $\mathbb{R}Loc(X)$ is algebraic. We write X_k the k-th skeleton of X. As X is a finite CW complex there is an n such that $X = X_n$. Moreover, for any k there exists a homotopy push-out diagram of topological spaces

$$\begin{array}{ccc} X_{k-1} & \longrightarrow & X_k \\ \downarrow & & \downarrow \\ \coprod S^{k-1} & \longrightarrow & \coprod D^k, \end{array}$$

where the disjoint unions are finite. This implies that we have a homotopy pull-back square of derived stacks

$$\begin{array}{ccc} \mathbb{R}Loc(X_k) & \longrightarrow & \mathbb{R}Loc(X_{k-1}) \\ \downarrow & & \downarrow \\ \prod^h \mathbb{R}Loc(S^{k-1}) & \longrightarrow & \prod^h \mathbb{R}Loc(S^k). \end{array}$$

By the stability of algebraic derived stacks by finite homotopy limits we deduce that $\mathbb{R}Loc(X_k)$ is algebraic by induction on k (the case $k = 0$ being clear as $\mathbb{R}Loc(X_0)$ is a finite product of $\mathbb{R}Loc(*)$).

To finish the study of this example we will compute the higher tangent spaces of the derived stack $\mathbb{R}Loc(X)$. We let A be a commutative algebra and we consider the natural morphism

$$\mathbb{R}Loc(*)(A \oplus A[i]) \longrightarrow \mathbb{R}Loc(*)(A).$$

This morphism This morphism has a natural section and its homotopy fiber at an A-module E is equivalent to $K(End(E), i+1)$. It is therefore naturally equivalent to

$$[K(End(-), i+1)/Vect(A)] \longrightarrow N(Vect(A)),$$

where $Vect(A)$ is the groupoid of projective A-modules of finite type, $N(Vect(A))$ is its nerve, and $[K(End(-), i+1)/Vect(A)]$ is the homotopy colimit of the simplicial presheaf $Vect(A) \longrightarrow SSet$ sending E to $K(End(E), i+1)$ (this is a general fact, for any simplicial presheaf $F : I \longrightarrow SSet$ we have a natural morphism $Hocolim_I F \longrightarrow N(I) \simeq Hocolim_I *$). We consider the geometric realization of this morphism to get a morphism of topological spaces

$$|[K(End(-), i+1)/Vect(A)]| \longrightarrow |N(Vect(A))|,$$

which is equivalent to the geometric realization of

$$\mathbb{R}Loc(*)(A \oplus A[i]) \longrightarrow \mathbb{R}Loc(*)(A).$$

We take the image of this morphism by $Map(X,-)$ to get

$$\mathbb{R}Loc(X)(A\oplus A[i]) \simeq Map(X,\mathbb{R}Loc(*)(A\oplus A[i])) \longrightarrow \mathbb{R}Loc(X)(A) \simeq Map(X,\mathbb{R}Loc(*)(A)).$$

This implies that the morphism

$$\mathbb{R}Loc(*)(A \oplus A[i]) \longrightarrow \mathbb{R}Loc(*)(A)$$

is equivalent to the morphism

$$Map(X, |[K(End(-), i+1)/Vect(A)]|) \longrightarrow Map(X, |N(Vect(A))|).$$

A morphism $X \longrightarrow |N(Vect(A))|$ correspond to a local system E of projective A-module of finite type on X. The homotopy fiber of the above morphism at E is then equivalent to the simplicial set of homotopy lifts of $X \longrightarrow |N(Vect(A))|$ to a morphism $X \longrightarrow |[K(End(-), i+1)/Vect(A)]|$. This simplicial set is in turn naturally equivalent to $DK(C^*(X, End(E))[i+1])$, the simplicial set obtained from the complex $C^*(X, End(E))[i+1]$ by the Dold-Kan construction. Here $C^*(X, End(E))$ denotes the complex of cohomology of X with coefficients in the local system $End(E)$. We therefore have the following formula for the higher tangent complexes

$$T^i_E\mathbb{R}Loc(X) \simeq H^0(C^*(X, End(E))[i+1]) \simeq H^{i+1}(X, End(E)).$$

More generally, it is possible to prove that there is an isomorphism in $D(A)$

$$\mathbb{T}_E\mathbb{R}Loc(X) \simeq C^*(X, End(E))[1].$$

6.2 The derived moduli of maps

As for non derived stacks, the homotopy category of derived stacks $Ho(dAff^{\sim})$ is cartesian closed. The corresponding internal Homs will be denoted by $\mathbb{R}\underline{\mathcal{H}om}$. Note that even thought we use the same notations for the internal Homs of stacks and derived stacks the inclusion functor

$$j : Ho(SPr(Aff)) \longrightarrow Ho(dAff^{\sim})$$

does not commute avec them. However, we always have

$$h^0(\mathbb{R}\underline{\mathcal{H}om}(F, F')) \simeq \mathbb{R}\underline{\mathcal{H}om}(h^0(F), h^0(F'))$$

for any derived stacks F and F'. The situation is therefore very similar to the case of homotopy pull-backs.

We have just seen an example of a derived stack constructed as an internal Hom between two stacks. Indeed, if we use again the notations of the last example we have

$$\mathbb{R}Loc(X) \simeq \mathbb{R}\underline{\mathcal{H}om}(K, \mathbf{Vect}),$$

where $K := S_*(X)$ is the singular simplicial set of X.

We now consider another example. Let X and Y be two schemes, and assume that X is flat and proper (say over $Spec\, k$ for some base ring k), and that Y is smooth over k. It is possible to prove that the derived stack $\mathbb{R}\underline{\mathcal{H}om}_{dAff/Spec\, k}(X, Y)$ is a derived scheme which is homotopically finitely presented over $Spec\, k$. We will not sketch the argument here which is out of the scope of these lectures, and we refer to [HAGII] for more details. The derived scheme $\mathbb{R}\underline{\mathcal{H}om}(X, Y)$ is called the derived moduli space of maps from X to Y. Its classical part $h^0(\mathbb{R}\underline{\mathcal{H}om}(X, Y))$ is the usual moduli scheme of maps from X to Y, and for such a map we have

$$\mathbb{T}_f \mathbb{R}\underline{\mathcal{H}om}_{dAff/Spec\, k}(X, Y) \simeq C^*(X, f^*(\mathbb{T}_Y),$$

(where all these tangent complexes are relative to $Spec\, k$).

We mention here that these derived mapping space of maps can also be used in order to construct the so-called derived moduli of stable maps to an algebraic variety, by letting X varies in the moduli space of stable curves. We refer to [To1] for more details about this construction, and for some explanations of how Gromov-Witten theory can be extracted from this derived stack of stable maps.

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