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A Primal Condition for Approachability with Partial Monitoring

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Abstract

In approachability with full monitoring there are two types of conditions that are known to be equivalent for convex sets: a primal and a dual condition. The primal one is of the form: a set \mathcal{C} is approachable if and only all containing half-spaces are approachable in the one-shot game; while the dual one is of the form: a convex set C is approachable if and only if it intersects all payoff sets of a certain form. We consider approachability in games with partial monitoring. In previous works (Perchet, 2011a, Mannor et al., 2011) we provided a dual characterization of approachable convex sets; we also exhibited efficient strategies in the case where \mathcal{C} is a polytope. In this paper we provide primal conditions on a convex set to be approachable with partial monitoring. They depend on a modified reward function and lead to approachability strategies, based on modified payoff functions, that proceed by projections similarly to Blackwell's (1956) strategy; this is in contrast with previously studied strategies in this context that relied mostly on the signaling structure and aimed at estimating well the distributions of the signals received. Our results generalize classical results by Kohlberg (1975) (see also Mertens et al. 1994) and apply to games with arbitrary signaling structure as well as to arbitrary convex sets.

1. Introduction

Approachability theory dates back to the seminal paper of Blackwell (1956). In this paper Blackwell presented conditions under which a player can guarantee that the long-term average vector-valued reward is asymptotically inside some target set regardless of the opponent actions. If this guarantee can be made, we say that the set is approachable. In

*This work began after an interesting objection of and further discussions with Jean-François Mertens during the presentation of our earlier results (Mannor et al., 2011) on the dual characterization of approachability with partial monitoring at the conference *Games Toulouse 2011*. The material presented in this article was developed further with Sylvain Sorin back in Paris, whom we thank deeply for his advice and encouragements. Sadly, Jean-François Mertens, a close collaborator of Sylvain Sorin, passed away in the mean time. This contribution is in honor of both of these important contributors to the theory of repeated games.

the full monitoring case studied in Blackwell (1956) there are two equivalent conditions for a convex set to be approachable. The first one, known as a primal condition (or later termed the “B” condition in honor for Blackwell), states that every half-space that contains the target set is also approachable. It turns out the approachability of a half-space is equivalent to solving the value of a standard (one-dimensional valued) game above or below zero. The second characterization, known as the dual condition, states that for every mixed action of the opponent, the player can guarantee that the one-shot vector-valued reward is inside the target set.

Approachability theory has found many applications in game theory, online learning, and related fields. Among the key advantages of using it is first, the simplicity of checking if the dual condition holds (this is usually quite simple for convex sets); and second, the derivation of a concrete approaching strategy from the primal one simply needs to solve a simple zero-sum game at every stage of the repeated vector-valued game.

Approachability theory was applied to repeated zero-sum games where there is imperfect knowledge about the game or partial monitoring. The work of Kohlberg (1975) (see also Mertens et al. 1994) uses approachability to derive strategies for games with incomplete information. The general case of repeated vector-valued games with partial monitoring was studied only recently. A dual characterization of approachable convex sets with partial monitoring was presented by Perchet (2011a). However, it is not useful in deriving concrete approaching strategies since it would essentially require to run a calibration algorithm, which is known to be computationally hard. In a recent work (Mannor et al. 2011) we derived efficient strategies for approachability in games with partial monitoring in some cases, e.g., when the convex set to be approached is a polytope. However, these strategies are based on the dual condition, and not on any primal one; they thus do not shed light on the structure of the game.

In this paper we provide a primal condition for approachability in games with partial monitoring. It can be stated, as in Blackwell (1956), as a requirement for every half-space containing the target set to be one-shot approachable. However, the reward function has to be modified in some cases for the condition to be sufficient. We also show how it leads to an efficient approachability strategy, at least in the case of approachable polytopes.

Outline. The paper is organized as follows. In Section 2 we define the partial monitoring model and recall some of the basic results from approachability (both in terms of primal and dual characterizations of approachability). In Section 3 we explain the current state-of-the-art, recall the dual condition for approachability with partial monitoring, and outline what we set out to do. In Section 4 we provide results for approaching half-spaces as they have the simplest characterization of approachability ; we show that the signaling structure has no impact on approachability, only the payoff structure does. This is not the case anymore when more complicated convex sets are to be approached, which is the focus of the subsequent sections. In Section 5 we discuss the case of a target set that is an orthant and some additional properties on the payoff structure are satisfied; we show that a natural primal condition holds. As an intermezzo, we link our results to Kohlberg (1975) in Section 6 and show that repeated games with imperfect

information can be analyzed using our approach for games with imperfect monitoring. In Section 7 we then analyze the case of a general signaling structure for the approachability of orthants and provide an efficient approaching strategy based on the exhibited primal conditions. Finally, we relax the shape of the target set from an orthant to a polyhedron in Section 8, and then to a general convex set in Section 9. Our generalization shows that the *same* primal condition holds in all cases.

2. Model and preliminaries

We now define the model of interest and then recall some basic results from approachability theory for repeated vector-valued games (with full monitoring).

Model, notation

We consider a vector-valued game between two players, a decision maker (or player) and Nature, with respective finite action sets \mathcal{I} and \mathcal{J} , whose cardinalities are referred to as I and J . We denote by d the dimension of the reward vector and equip \mathbb{R}^d with the ℓ^2 -norm $\|\cdot\|_2$. The payoff function of the player is given by a mapping $r : \mathcal{I} \times \mathcal{J} \rightarrow \mathbb{R}^d$, which is multi-linearly extended to $\Delta(\mathcal{I}) \times \Delta(\mathcal{J})$, the set of product-distributions over $\mathcal{I} \times \mathcal{J}$.

At each round, the player and Nature simultaneously choose their actions $i_n \in \mathcal{I}$ and $j_n \in \mathcal{J}$, at random according to probability distributions denoted by $x_n \in \Delta(\mathcal{I})$ and $y_n \in \Delta(\mathcal{J})$. At the end of a round, the player does not observe Nature's action j_n nor even the payoff $r_n := r(i_n, j_n)$ he obtains; he only gets to see some signal. More precisely, there is a finite set \mathcal{H} of possible signals; the signal $s_n \in \mathcal{H}$ that is shown to the player is drawn at random according to the distribution $H(i_n, j_n)$, where the mapping $H : \mathcal{I} \times \mathcal{J} \rightarrow \Delta(\mathcal{H})$ is known by both players.

The player is said to have full monitoring if $H(i, j) = \text{Full}(i, j) := \delta_j$, i.e., if the action of Nature is observed. We speak of a game in the dark when the signaling structure H is not informative at all, i.e., when \mathcal{H} is reduced to a single signal referred to as \emptyset ; when denote this situation by $H = \text{Dark}$.

Of major interest will be *maximal informative mapping* $\mathbf{H} : \Delta(\mathcal{J}) \rightarrow \Delta(\mathcal{J})^{\mathcal{I}}$, which is defined as follows. The image of each $j \in \mathcal{J}$ is the vector $\mathbf{H}(j) = (H(i, j))_{i \in \mathcal{I}}$; and this definition is extended linearly onto $\Delta(\mathcal{J})$. An element of the image $\mathcal{F} = \mathbf{H}(\Delta(\mathcal{J}))$ of \mathbf{H} is referred as a *flag*. For every $x \in \Delta(\mathcal{I})$ and $\mathbf{h} \in \mathcal{F}$ the set of payoffs compatible with \mathbf{h} is

$$\overline{m}(x, \mathbf{h}) = \{r(x, y) : y \in \Delta(\mathcal{J}) \text{ such that } \mathbf{H}(y) = \mathbf{h}\}. \quad (1)$$

The set $\overline{m}(x, \mathbf{h})$ represents all the rewards that are statistically compatible with the flag \mathbf{h} ; this set is essentially the set of all possible rewards we cannot distinguish from. Note that with full monitoring, \mathcal{H} reduces to $\Delta(\mathcal{J})$ and one has $\overline{m}(x, y) = \{r(x, y)\}$ for all $y \in \Delta(\mathcal{J})$.

Finally, we denote by M a uniform ℓ_2 bound on r , that is,

$$M = \max_{i,j} \|r(i, j)\|.$$

Also, for every $n \in \mathbb{N}$ and sequence $(a_m)_{m \in \mathbb{N}}$, the average of the first n elements is referred to as $\bar{a}_n = (1/n) \sum_{m=1}^n a_m$; and $d_{\mathcal{C}}$ is the distance to a set \mathcal{C} .

A behavioral strategy σ of the player is a mapping from the set of his finite histories $\cup_{n \in \mathbb{N}} (\mathcal{I} \times \mathcal{H})^n$ into $\Delta(\mathcal{I})$; similarly, a strategy τ of nature is a mapping from $\cup_{n \in \mathbb{N}} (\mathcal{I} \times \mathcal{H} \times \mathcal{J})^n$ into $\Delta(\mathcal{J})$. As usual, we denote by $\mathbb{P}_{\sigma, \tau}$ the probability induced by the pair (σ, τ) onto $(\mathcal{I} \times \mathcal{H} \times \mathcal{J})^{\mathbb{N}}$.

Definition and some properties of approachability

A set $\mathcal{C} \subseteq \mathbb{R}^d$ is r -approachable for the signaling structure H , or, in short, is (r, H) -approachable, if, for all $\varepsilon > 0$, there exists a strategy σ_ε of the player and a natural number $N \in \mathbb{N}$ such that, for all strategies τ of Nature,

$$\mathbb{P}_{\sigma_\varepsilon, \tau} \{ \exists n \geq N \text{ s.t. } d_{\mathcal{C}}(\bar{r}_n) \geq \varepsilon \} \leq \varepsilon.$$

We refer to the strategy σ_ε as an ε -approachability strategy of \mathcal{C} . It is easy to show that the approachability of \mathcal{C} implies the existence of a strategy ensuring that the sequence of the average vector-valued payoffs converges to the set \mathcal{C} almost surely, uniformly with respect to the strategies of Nature. By analogy such a strategy is called a 0-approachability strategy of \mathcal{C} .

Conversely, a set \mathcal{C} is r -excludable for the signaling structure H if, for some $\delta > 0$, the complement of the δ -neighborhood of \mathcal{C} is r -approachable by Nature for the signaling structure H .

Primal characterization. We now discuss characterizations of approachability in the case of full monitoring; we will need the stronger notion of one-shot approachability. (The notion of one-shot excludability is stated only for later purposes.)

Definition 1. *A set \mathcal{C} is one-shot r -approachable if there exists $x \in \Delta(\mathcal{I})$ such that for all $y \in \Delta(\mathcal{J})$, one has $r(x, y) \in \mathcal{C}$. A set \mathcal{C} is one-shot r -excludable if for some $\delta > 0$, the complement of the δ -neighborhood of \mathcal{C} is one-shot r -approachable by Nature.*

Blackwell (1956) (see also Mertens et al. 1994) provided the following *primal characterization* of approachable convex¹ set; a set that satisfies it is called a B -set.

Theorem 1. *A convex set \mathcal{C} is (r, Full) -approachable if and only if any containing half-space $\mathcal{C}_{\text{hs}} \supseteq \mathcal{C}$ is one-shot r -approachable.*

¹This primal characterization was actually stated by Blackwell (1956) in a more general way for all, non-necessarily convex, sets.

This characterization also leads to an approachability strategy, which we describe with a slight modification with respect to its most classical statement. We denote by $r'_t = r(x_t, j_t)$ an auxiliary payoff obtained at round t ; this quantity is observed. At stage n , if $\bar{r}'_n \notin \mathcal{C}$, let $\pi_{\mathcal{C}}(\bar{r}'_n)$ denote the projection of \bar{r}'_n onto \mathcal{C} and consider the containing half-space $\mathcal{C}_{\text{hs}, n+1}$ whose defining hyperplane is tangent to \mathcal{C} at $\pi_{\mathcal{C}}(\bar{r}'_n)$. The strategy then consists of choosing the mixed action $x_{n+1} \in \Delta(\mathcal{I})$ associated with the one-shot approachability of $\mathcal{C}_{\text{hs}, n+1}$. If $\bar{r}'_n \in \mathcal{C}$, any choice $x_{n+1} \in \Delta(\mathcal{I})$ is suitable. The above strategy ensures that for all $y \in \Delta(\mathcal{J})$,

$$\langle r(x_{n+1}, y) - \pi_{\mathcal{C}}(\bar{r}'_n), \bar{r}'_n - \pi_{\mathcal{C}}(\bar{r}'_n) \rangle \leq 0; \quad (2)$$

which in turn ensures the convergence to \mathcal{C} of the mixed payoffs at a rate independent of d , namely

$$d_{\mathcal{C}}\left(\frac{1}{n} \sum_{t=1}^n r(x_t, j_t)\right) \leq \frac{2M}{\sqrt{n}}. \quad (3)$$

The uniform convergence of \bar{r}_n to \mathcal{C} is deduced by martingale convergence theorems (e.g., the Hoeffding–Azuma inequality) from the above uniform convergence of \bar{r}'_n to \mathcal{C} .

Dual characterization. In the specific case of closed convex sets, using von Neumann’s min-max theorem, the primal characterization stated above can be transformed into the following *dual characterization*:

$$\mathcal{C} \subseteq \mathbb{R}^d \text{ is } (r, \text{Full})\text{-approachable} \iff \forall y \in \Delta(\mathcal{Y}), \exists x \in \Delta(\mathcal{I}), r(x, y) \in \mathcal{C}. \quad (4)$$

This characterization might be simpler to formulate and to check, yet it does not provide an explicit approachability strategy.

3. Related literature and the objective of this paper

In this section we first recall the existing results on approachability with partial monitoring and then explain technically what the paper sets out to do.

Results on approachability with partial monitoring

Concerning the primal characterization. Kohlberg (1975)—see also Mertens et al. 1994—studied specific frameworks (induced by games with incomplete information; see Section 6) in which approachability depends mildly on the signaling structure. A so-called product property holds between the payoff function r and the signaling structure H , based on which it is rather straightforward to show that the primal characterization for the (r, H) -approachability of orthants stated in Theorem 1 still holds. Section 6 provides more details on this matter.

Concerning the dual characterization. Perchet (2011a) provided the following dual characterization of approachable closed convex sets under partial monitoring:

$$\mathcal{C} \subseteq \mathbb{R}^d \text{ is } (r, H)\text{-approachable} \iff \forall \mathbf{h} \in \mathcal{F}, \exists x \in \Delta(\mathcal{I}), \overline{m}(x, \mathbf{h}) \subseteq \mathcal{C}. \quad (5)$$

It indeed generalizes Blackwell’s dual characterization (4) with full monitoring, as \mathcal{F} can be identified with $\Delta(\mathcal{J})$ in this case.

Based on (5), Perchet constructed the first (r, H) -approachability strategy of any closed convex set \mathcal{C} ; it was based on calibrated forecasts of the vectors of \mathcal{F} . Because of this, the per-stage computational complexity of this strategy increases indefinitely and rates of convergence cannot be inferred. Moreover, the construction of this strategy is unhelpful to infer a generic primal characterization.

Mannor et al. (2011) tackled the issue of complexity and devised an efficient (r, H) -approachability strategy for the case when the target set is some polytope. This strategy has a fixed and bounded per-stage computational complexity. Moreover, its rates of convergence are independent of d : they are of the order of $n^{-1/5}$, where n is the number of stages.

On the other hand, Perchet and Quincampoix (2011) unified the setups of approachability with full or partial monitoring and characterized approachable closed (convex) sets using some lifting to the Wasserstein space of probability measures on $\Delta(\mathcal{I}) \times \Delta(\mathcal{J})$.

Objectives and technical content of the paper

This paper focuses on the primal characterization of approachable closed convex sets with partial monitoring, which was unclear at the time of the writing of this paper. First, note that if a closed convex set is (r, H) -approachable, then it is also (r, Full) -approachable, and therefore, by (4), any containing half-space is necessarily one-shot r -approachable. The question is when the latter statement is a sufficient condition for (r, H) -approachability. The difficulty, as noted already by Perchet (2011a) and recalled at the beginning of Section 5, is that since the notions of approachability with full or partial monitoring do not coincide, it can be that a closed convex sets is not (r, H) -approachable while every containing half-space is one-shot r -approachable.

One class of exceptions when the usual dual characterization is indeed sufficient is formed first, by the cases when half-spaces are to be approached (with no condition on the game), and second, by the cases of games whose structure (r, H) has the product property and when orthants are to be approached. This first series of results is detailed in Sections 4 and 5. Some light is then shed in Section 6 on the construction of Kohlberg (1975) for the case of games with incomplete information.

The rest of the paper (Sections 7, 8, and 9), in the lack of an assumption on the structure (r, H) of the game, discusses a primal condition based on one-shot approachability of half-spaces with respect to a modified payoff function \tilde{r}_H that encompasses the links between the signaling structure H and the original payoff function r . Depending on the geometry of the closed convex set to be approached, this primal condition is stated in the original payoff space (for orthants, Section 7) or in some lifted space (for polytopes

or general convex sets, see Section 8 and 9). We explain in Example 1 why such a lifting seems inevitable.

We also illustrate how the exhibited primal condition leads to a new (and efficient) strategy for (r, H) -approachability in the case of target sets given by polytopes (Section 7 does it for orthants and the result extends to polytopes via Lemma 2). This new strategy is based on sequences of (modified) payoffs, as in Kohlberg (1975), and is not only based on sequences of signals, as in Perchet (2011a), Mannor et al. (2011). The construction of this strategy also entails some non-linear approachability results (both in full and partial monitoring).

4. Primal approachability of half-spaces

We first focus on half-spaces, not only because they are the simplest convex sets, but because they are the cornerstones of the primal characterization of Blackwell (1956). The following proposition ties one-shot r -approachability with (r, H) -approachability of half-spaces.

Proposition 1. *For all half-spaces \mathcal{C}_{hs} , for all signaling structures H ,*

$$\mathcal{C}_{\text{hs}} \text{ is } (r, H)\text{-approachable} \iff \mathcal{C}_{\text{hs}} \text{ is one-shot } r\text{-approachable}.$$

This result is a mere interpolation of two well-known results for the extremal cases when $H = \text{Full}$ and $H = \text{Dark}$. The former case corresponds to Blackwell's primal characterization. In the latter case, since Nature could always play the same y at all rounds, the player, to ensure (r, Dark) -approachability, indeed needs an action x such that $r(x, y) \in \mathcal{C}_{\text{hs}}$ for all y , which is the action he should play at all stages.

Stated differently, the above proposition indicates that as far as half-spaces are concerned, the approachability is independent of the signaling structure.

Proof. Only the direct implication is to be proven; the converse implication is immediate by the above discussion about games in the dark. We thus assume that \mathcal{C}_{hs} is (r, H) -approachable. Using the characterization (5) of (r, H) -approachable sets, one then has

$$\begin{aligned} & \forall \mathbf{h} \in \mathcal{F}, \exists x \in \Delta(\mathcal{I}), \overline{m}(x, \mathbf{h}) \subset \mathcal{C}_{\text{hs}}, \\ \text{which implies that } & \forall y \in \Delta(\mathcal{J}), \exists x \in \Delta(\mathcal{I}), r(x, y) \in \mathcal{C}_{\text{hs}}. \end{aligned}$$

The implication holds because $r(x, y) \in \overline{m}(x, \mathbf{h})$ when $\mathbf{h} = \mathbf{H}(y)$. Now, let $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that $\mathcal{C}_{\text{hs}} = \{\omega \in \mathbb{R}^d : \langle \omega, a \rangle \leq b\}$. The last implication of obtained can be restated as

$$\begin{aligned} & \forall y \in \Delta(\mathcal{J}), \exists x \in \Delta(\mathcal{I}), \langle r(x, y), a \rangle \leq b, \\ \text{or equivalently, } & \max_{y \in \Delta(\mathcal{J})} \min_{x \in \Delta(\mathcal{I})} \langle r(x, y), a \rangle \leq b. \end{aligned}$$

By von Neumann's min-max theorem, we then have that

$$\min_{x \in \Delta(\mathcal{I})} \max_{y \in \Delta(\mathcal{J})} \langle r(x, y), a \rangle \leq b, \quad \text{that is, } \exists x_0 \in \Delta(\mathcal{I}), \forall y \in \Delta(\mathcal{J}), \langle r(x_0, y), a \rangle \leq b.$$

This is exactly the one-shot approachability of \mathcal{C}_{hs} . □

Since the complement of any δ -neighborhood of a half-space is also a half-space, we get the following additional equivalence, in view of the respective definitions of excludability and one-shot excludability.

Corollary 1. *For all half-spaces \mathcal{C}_{hs} , for all signaling structures H ,*

$$\mathcal{C}_{\text{hs}} \text{ is } (r, H)\text{-excludable} \iff \mathcal{C}_{\text{hs}} \text{ is one-shot } r\text{-excludable}.$$

5. Primal approachability of orthants under a product property

This section is devoted to stating a primal characterization of (r, H) -approachable orthants, i.e., sets in the form

$$\mathcal{C}_{\text{orth}}(a) = \{a - \omega : \omega \in (\mathbb{R}_+)^d\}$$

for some $a \in \mathbb{R}^d$. Orthants are the key for extension to polyhedra, because, as we will discuss later, up to some lifting in higher dimensions, every polyhedral set can be seen as an orthant.

We start by indicating that the primal characterization stated in the previous section in terms of the original payoff function r does not extend directly to general convex sets, not even to orthants—at least under no additional assumption. In this section we however state such a sufficient assumption for its extension. We study the most general primal characterization in Section 7; it will involve a modified payoff function for the one-shot approachability of half-spaces.

Counter-example. This example is extracted from Perchet (2011a). We first show that the equivalence of Proposition 1 does not hold in general if the convex set \mathcal{C} at hand is not a half-space; we do so by exhibiting a game and a set \mathcal{C} which is (r, Full) -approachable but not (r, Dark) -approachable. The game is a straightforward modification of an example studied in Perchet (2011a). We set $\mathcal{I} = \{T, B\}$ and $\mathcal{J} = \{L, R\}$, and the payoff function r is given by the matrix

	L	R
T	$(0, 0)$	$(1, -1)$
B	$(-1, 1)$	$(0, 0)$

We consider the set $\mathcal{C}_{\text{orth}}((0, 0)) = (\mathbb{R}_-)^d$. This set is (r, Full) -approachable as indicated by the dual characterization (4): to each mixed action y_α of Nature, indexed by $\alpha \in [0, 1]$, we associate the mixed action x_α of the player indexed by the same α ,

$$r(\alpha T + (1 - \alpha)B, \alpha L + (1 - \alpha)R) = (0, 0) \in \mathcal{C}_{\text{orth}}((0, 0)).$$

On the other hand, under the signaling structure $H = \text{Dark}$, for which the only flag is \emptyset , we have, for all actions of the player, that is, for all $\alpha \in [0, 1]$,

$$\begin{aligned} \overline{m}(\alpha T + (1 - \alpha)B, \emptyset) &= \left\{ r(\alpha T + (1 - \alpha)B, y) : y \in \Delta(\mathcal{J}) \right\} \\ &= \{(\lambda, -\lambda) ; \lambda \in [-\alpha, 1 - \alpha]\} \not\subseteq \mathcal{C}_{\text{orth}}((0, 0)) ; \end{aligned}$$

therefore, the characterization of r -approachable closed convex sets (5) does not hold when playing in the dark.

Product property. We define the upper-right corner function $R : \Delta(\mathcal{I}) \times \mathcal{F} \rightarrow \mathbb{R}^d$ of the compatible payoff function \overline{m} in a component-wise manner. We write the coordinates of R as $R = (R_1, \dots, R_d)$ and set, for all $k \in \{1, \dots, d\}$,

$$\forall x \in \Delta(\mathcal{I}), \forall \mathbf{h} \in \mathcal{F}, \quad R_k(x, \mathbf{h}) = \max \left\{ \omega^k : \omega = (\omega^1, \dots, \omega^d) \in \overline{m}(x, \mathbf{h}) \right\}.$$

Note that the ℓ_2 -norm of R is bounded by, e.g., $M\sqrt{d}$.

The name of an upper-right corner stems from the fact that $R(x, \mathbf{h})$ is the (component-wise) smaller a such that $\overline{m}(x, \mathbf{h}) \subseteq \mathcal{C}_{\text{orth}}(R(x, \mathbf{h}))$. This point $R(x, \mathbf{h})$ is in some sense the worst-case payoff vector associated with $\overline{m}(x, \mathbf{h})$; controlling its distance to the orthant to approach entails a control of the distance of the whole set $\overline{m}(x, \mathbf{h})$ to it.

Of course, $R(x, \mathbf{h})$ is in general not a feasible payoff vector, i.e., $R(x, \mathbf{h}) \notin \overline{m}(x, \mathbf{h})$. We are interested in this section in the case when the upper-right corners are indeed feasible payoffs—an assumption that we call the product property in reference to the simplest case of sets whose upper-right corners are elements of the set: rectangles (which are given by Cartesian products of intervals).

Definition 2. *The game (r, H) with partial monitoring has the product property if*

$$\forall x \in \Delta(\mathcal{I}), \forall \mathbf{h} \in \mathcal{F}, \quad R(x, \mathbf{h}) \in \overline{m}(x, \mathbf{h}).$$

Of course, games with full monitoring have the product property, as for them \mathcal{F} can be identified with $\Delta(\mathcal{J})$ and \overline{m} can be identified with the function $\{r\}$ with values in the set of all singleton subsets of \mathbb{R}^d .

Primal characterization under the product property. The following proposition was implicitly used by Kohlberg (1975).

Proposition 2. *For all games (r, H) with partial monitoring that have the product property, for all orthants $\mathcal{C}_{\text{orth}}(a)$, where $a \in \mathbb{R}^d$,*

$$\begin{aligned} &\mathcal{C}_{\text{orth}}(a) \text{ is } (r, H)\text{-approachable} \\ \iff &\text{every half-space } \mathcal{C}_{\text{hs}} \supset \mathcal{C}_{\text{orth}}(a) \text{ is one-shot } r\text{-approachable.} \end{aligned}$$

Stated differently, using Blackwell's primal characterization of approachability (Theorem 1), an orthant $\mathcal{C}_{\text{orth}}(a)$ is (r, H) -approachable in a game (r, H) satisfying the product property if and only if $\mathcal{C}_{\text{orth}}(a)$ is r -approachable with full monitoring.

Proof. The direct implication is by applying Proposition 1 to any half-space $\mathcal{C}_{\text{hs}} \supset \mathcal{C}_{\text{orth}}(a)$, which in particular is (r, H) -approachable as soon as $\mathcal{C}_{\text{orth}}(a)$ is. The interesting implication is thus the converse one.

So, we assume that every half-space $\mathcal{C}_{\text{hs}} \supset \mathcal{C}_{\text{orth}}(a)$ is one-shot r -approachable and, following Kohlberg's original proof and inspired by Blackwell's strategy in the case of full monitoring, we construct an (r, H) -approachability strategy of $\mathcal{C}_{\text{orth}}(a)$.

Flags observed, mixed payoffs obtained. For simplicity, assume first that after stage n , the observation made by the player is not just the random signal s_n but the entire vector of probability distributions over the signals $\mathbf{h}_n = \mathbf{H}(y_n)$. (We will recall below why this is not a restrictive assumption.) We consider the surrogate payoff vector $R_n = R(x_n, \mathbf{h}_n)$, which is a quantity thus observed by the player. When \bar{R}_n does not already belong to $\mathcal{C}_{\text{orth}}(a)$, since the latter set is convex, the half-space $\mathcal{C}_{\text{hs}, n}$ defined by

$$\mathcal{C}_{\text{hs}, n} = \left\{ \omega \in \mathbb{R}^d : \langle \omega - \pi_{\mathcal{C}_{\text{orth}}(a)}(\bar{R}_n), \bar{R}_n - \pi_{\mathcal{C}_{\text{orth}}(a)}(\bar{R}_n) \rangle \leq 0 \right\}$$

contains $\mathcal{C}_{\text{orth}}(a)$; therefore, by assumption, $\mathcal{C}_{\text{hs}, n}$ is one-shot r -approachable. That is, there exists $x_{n+1} \in \Delta(\mathcal{I})$ such that

$$\forall y \in \Delta(\mathcal{J}), \quad \langle r(x_{n+1}, y) - \pi_{\mathcal{C}_{\text{orth}}(a)}(\bar{R}_n), \bar{R}_n - \pi_{\mathcal{C}_{\text{orth}}(a)}(\bar{R}_n) \rangle \leq 0.$$

By the product property, $R_{n+1} \in \bar{m}(x_{n+1}, \mathbf{h}_{n+1})$, which entails that there exists $y'_{n+1} \in \Delta(\mathcal{J})$ such that $\mathbf{H}(y'_{n+1}) = \mathbf{h}_{n+1}$ and $R_{n+1} = r(x_{n+1}, y'_{n+1})$. As a consequence, R_{n+1} belongs to $\mathcal{C}_{\text{hs}, n}$ and the sequence (R_n) satisfies the following condition, usually referred to as *Blackwell's condition*:

$$\langle R_{n+1} - \pi_{\mathcal{C}_{\text{orth}}(a)}(\bar{R}_n), \bar{R}_n - \pi_{\mathcal{C}_{\text{orth}}(a)}(\bar{R}_n) \rangle \leq 0.$$

This condition is trivially satisfied when \bar{R}_n already belongs to $\mathcal{C}_{\text{orth}}(a)$. Now, just as (2) leads to (3), this condition implies that $d_{\mathcal{C}_{\text{orth}}(a)}(\bar{R}_n) \leq 2M\sqrt{d}/\sqrt{n}$. (Recall that R is bounded by $M\sqrt{d}$ in ℓ_2 -norm.) Now, $(1/n) \sum_{t=1}^n r(x_t, y_t) \in (1/n) \sum_{t=1}^n \bar{m}(x_t, \mathbf{h}_t)$ and, by definition of R as the upper-right corner function, $(1/n) \sum_{t=1}^n \bar{m}(x_t, \mathbf{h}_t) \subseteq \mathcal{C}_{\text{orth}}(\bar{R}_n)$. That is, $(1/n) \sum_{t=1}^n r(x_t, y_t)$ is component-wise smaller than \bar{R}_n . Since the distance to the orthant $\mathcal{C}_{\text{orth}}(a)$ equals, for all $\omega \in \mathbb{R}^d$,

$$d_{\mathcal{C}_{\text{orth}}(a)}(\omega) = \sqrt{\sum_{k=1}^d \max\{\omega_k - a_k, 0\}^2}, \quad (6)$$

we get that

$$d_{\mathcal{C}_{\text{orth}}(a)}\left(\frac{1}{n} \sum_{t=1}^n r(x_t, y_t)\right) \leq d_{\mathcal{C}_{\text{orth}}(a)}(\bar{R}_n) \leq \frac{2M\sqrt{d}}{\sqrt{n}}.$$

Finally, by martingale convergence theorems (e.g., the Hoeffding–Azuma inequality), the sequence of the distances $d_{\mathcal{C}_{\text{orth}}(a)}(\bar{r}_n)$ is also uniformly controlled against all strategies of Nature.

Flags not observed, only random signals observed, pure payoffs. It remains to relax the assumption that the flags \mathbf{h}_n are observed, while only the signals s_n drawn at random according to $H(i_n, j_n)$ are. A standard trick in the literature of partial monitoring (see Kohlberg 1975, Mertens et al. 1994, Lugosi et al. 2008) solves the issue, together with martingale convergence theorems and the fact that the upper-right corner R is a Lipschitz function for a well-chosen metric over sets (see Lemma 1 below). We briefly describe this trick without working out the lengthy details. Time is divided into blocks of time with (large) length L , indexed by $n = 1, 2, \dots$, in which the same mixed distribution $x^{(n)}$ is played at all stages by the player; that is, $x_{nL+t} = x^{(n)}$ for all $1 \leq t \leq L$. Furthermore, this distribution $x^{(n)}$ puts a positive probability mass of at least some $\gamma > 0$ on all actions. Doing so, arbitrarily precise estimators of the average flags on each block can be constructed (at arbitrarily small costs); and, informally, each block plays now the role of a stage in the setting above when flags were observed. Also, similar martingale convergence arguments show that measuring payoffs in terms of the mixed actions x_t or pure actions i_t does not matter. As indicated, we omit the technical proof of these facts (it already appeared in all the given references) but notice, however, that rates of convergence are affected by this trick. \square

Remark 1. The construction in the proof above shows that under the product property, it is necessary and sufficient to control the behavior of the upper-right corners R_n .

This property was used only to show that R_{n+1} is equal to some $r(x_{n+1}, y'_{n+1})$ and thus that the sequence (R_n) satisfies Blackwell's condition. When the property is not satisfied anymore, the sequence of the upper-right corners may fail this condition. For instance, in the counter-example at the beginning of the present section, the upper-right corners equal

$$\forall \alpha \in [0, 1], \quad R(\alpha T + (1 - \alpha)B, \emptyset) = (\alpha, 1 - \alpha),$$

so that, for all strategies of the player, $\bar{R}_n = (\lambda, 1 - \lambda)$ for some $\lambda \in [0, 1]$. Thus, the distance of \bar{R}_n to $\mathcal{C}_{\text{orth}}((0, 0))$ is always larger than $1/\sqrt{2}$.

6. Intermezzo:

Kohlberg's games with incomplete information

We consider in this section a different, yet related framework, which is the main focus of Kohlberg (1975). We first describe a setting where d games with partial monitoring are to be played simultaneously, and then establish the formal connection with Kohlberg's results.

Simultaneous games with partial monitoring. We consider d such games, with common action sets \mathcal{I} and \mathcal{J} for the player and Nature and common set \mathcal{H} of signals, but with possibly different payoff functions and signaling structures. We index these games by g . For each game $g \in \{1, \dots, d\}$, the player's payoff function is denoted by $r^{(g)} : \mathcal{I} \times \mathcal{J} \rightarrow \mathbb{R}$ and the signaling structure is given by $H^{(g)} : \mathcal{I} \times \mathcal{J} \rightarrow \Delta(\mathcal{H})$, with associated maximal informative mapping $\mathbf{H}^{(g)} : \Delta(\mathcal{J}) \rightarrow \Delta(\mathcal{H})^{\mathcal{I}}$.

We put some restrictions on the strategies of the player and of Nature. The player may only choose one action $x_t \in \Delta(\mathcal{I})$ at each stage t , the same for all games g . On the other hand, Nature can choose different mixed actions $y_t^{(g)}$ in each game g , but these need to be *non-revealing*, that is, they need to induce the same flags; i.e., they need to be picked in the following set, which we assume is non-empty:

$$\text{NR} = \left\{ (y^{(1)}, \dots, y^{(d)}) \in \Delta(\mathcal{J})^d : \mathbf{H}^{(1)}(y^{(1)}) = \dots = \mathbf{H}^{(d)}(y^{(d)}) \right\}.$$

The above framework of simultaneous games can be embedded into some equivalent game that fits the model studied in the previous sections. Indeed, by linearity of each $\mathbf{H}^{(g)}$, the set NR of non-revealing actions is a polytope, thus it is the convex hull of its finite set of extremal points, whose cardinality we denote by K and whose elements we write as

$$\mathcal{K} = \left\{ (y_1^{(g)})_{1 \leq g \leq d}, \dots, (y_K^{(g)})_{1 \leq g \leq d} \right\}.$$

Each $(y^{(g)})_{1 \leq g \leq d} \in \text{NR}$ can then be represented by an element of $\Delta(\mathcal{K})$; conversely, each $z = (z_k)_{k \leq K} \in \Delta(\mathcal{K})$ induces the following element of NR:

$$Y(z) = (Y^{(g)}(z))_{1 \leq g \leq d} = \sum_{k=1}^K z_k (y_k^{(g)})_{1 \leq g \leq d}.$$

So, with no loss of generality, we can assume that \mathcal{K} is the finite set of actions of Nature and that, given $z \in \Delta(\mathcal{K})$ and $x \in \Delta(\mathcal{I})$, the payoff in the game g is $r^{(g)}(x, Y^{(g)}(z))$.

This defines naturally an auxiliary game with linear vector-valued payoff function $r : \Delta(\mathcal{I}) \times \Delta(\mathcal{K}) \rightarrow \mathbb{R}^d$ and signaling structure $\mathbf{H} : \Delta(\mathcal{K}) \rightarrow \Delta(\mathcal{H})^{\mathcal{I}}$ defined by

$$r(x, z) = \left(r^{(g)}(x, Y^{(g)}(z)) \right)_{1 \leq g \leq d} \quad \text{and} \quad \mathbf{H}(z) = \mathbf{H}^{(g)}(Y^{(g)}(z)) \quad \text{for all } g;$$

the definition of \mathbf{H} is independent of g by construction, as we restricted Nature to use non-revealing strategies. This maximal informative mapping \mathbf{H} corresponds to a signaling structure which we denote by $H : \Delta(\mathcal{I}) \times \Delta(\mathcal{K}) \rightarrow \Delta(\mathcal{H})$.

The game (r, H) constructed above satisfies the product property. Indeed, for all $\mathbf{h} \in \mathbf{H}(\Delta(\mathcal{K}))$ and all $x \in \Delta(\mathcal{I})$,

$$\begin{aligned} \bar{m}(x, \mathbf{h}) &= \left\{ \left(r^{(1)}(x, Y^{(1)}(z)), \dots, r^{(d)}(x, Y^{(d)}(z)) \right) : z \in \Delta(\mathcal{K}) \text{ s.t. } \mathbf{H}(z) = \mathbf{h} \right\} \\ &= \left\{ \left(r^{(1)}(x, y^{(1)}), \dots, r^{(d)}(x, y^{(d)}) \right) : \right. \\ &\quad \left. (y^{(1)}, \dots, y^{(d)}) \in \Delta(\mathcal{J})^d \text{ s.t. } \mathbf{H}^{(1)}(y^{(1)}) = \dots = \mathbf{H}^{(d)}(y^{(d)}) = \mathbf{h} \right\}. \end{aligned}$$

Because of the equality constraint to a fixed \mathbf{h} , the set

$$\left\{ (y^{(1)}, \dots, y^{(d)}) \in \Delta(\mathcal{J})^d : \mathbf{H}^{(1)}(y^{(1)}) = \dots = \mathbf{H}^{(d)}(y^{(d)}) = \mathbf{h} \right\}$$

is a Cartesian product of subsets of $\Delta(\mathcal{J})$ and thus, its image $\overline{m}(x, \mathbf{h})$ by the mapping

$$(y^{(1)}, \dots, y^{(d)}) \in \Delta(\mathcal{J})^d \mapsto (r^{(1)}(x, y^{(1)}), \dots, r^{(d)}(x, y^{(d)}))$$

is also a Cartesian product of closed intervals of \mathbb{R} . In particular, the latter set contains its upper-right corner, that is, $R(x, \mathbf{h}) \in \overline{m}(x, \mathbf{h})$, as claimed.

We assume, with no loss of generality, that in these simultaneous games, Nature maximizes the payoffs and the player minimizes them. A question that naturally arises—and whose answer will be needed below—is to determine for which vectors $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ the player can simultaneously guarantee that his average payoff will be smaller than a_g in the limit in each game $g \in \{1, \dots, d\}$; that is, to determine which orthants $\mathcal{C}_{\text{orth}}(a)$ are (r, H) -approachable. By the exhibited product property, Proposition 2 shows that a necessary and sufficient condition for this is that all containing half-spaces of $\mathcal{C}_{\text{orth}}(a)$ be one-shot r -approachable. These half-spaces are parameterized by the convex distributions $q \in \Delta(\{1, \dots, d\})$ and are denoted by

$$\mathcal{C}_{\text{hs}}^{(q)} = \{\omega \in \mathbb{R}^d : \langle q, \omega \rangle \leq \langle a, q \rangle\}. \quad (7)$$

Stated equivalently, the orthant $\mathcal{C}_{\text{orth}}(a)$ is (r, H) -approachable if and only if the value of the zero-sum game with payoff function $(x, z) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{K}) \mapsto \langle q, r(x, z) \rangle$ is smaller than $\langle a, q \rangle$ for all $q \in \Delta(\{1, \dots, d\})$.

Kohlberg's model of games with incomplete information. The setting of games with incomplete information, introduced by Aumann and Maschler (1955), is the same as above, except for two modifications. First, Nature is in principle not restricted to use non-revealing strategies. Second, a probability distribution $p \in \Delta(\{1, \dots, d\})$ is considered by Nature; it is known by both the player (who gets informed of it) and Nature. The elements in $\{1, \dots, d\}$ are called states. With no loss of generality, we may assume that p has full support. A state G is chosen at random according to p by Nature, but the player is not informed of the choice. A repeated game with partial monitoring then takes place between the player and Nature in the G -th game only. However, payoffs are evaluated in expectation with respect to the random choice of G ; furthermore, for simplicity, we assume that all mappings $\mathbf{H}^{(g)}$ have the same range². Because of these two properties, the considered setting of games with incomplete information is, for the player, the same described above as far as the signaling structures are concerned, while the payoffs, for a given choice of p , are formed by the inner products $(x, z) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{K}) \mapsto \langle p, r(x, z) \rangle$ of r with p . We recall that the player is informed of p .

In this game also, Nature maximizes the payoff and the player minimizes it. For each $q \in \Delta(\{1, \dots, d\})$, denote by $u(q)$ the value of the zero-sum game $\Gamma(q)$ with payoffs

²In full generality, when this is not the case, Nature may resort to strategies that reveal that the true state G belongs to some strict subset of $\{1, \dots, d\}$, and the player must adapt his strategy in correspondence with this knowledge, see Kohlberg (1975). But our assumption already enables to capture the basic idea of the use of approachability in this framework and the alluded technical adaptations are beyond the scope of this paper.

$(x, z) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{K}) \mapsto \langle q, r(x, z) \rangle$. We now consider the meta-game in which Nature chooses the distribution p and then the game $\Gamma(p)$ is played. We show that, as proved in the mentioned references, the value U of this game, as a function of the choice p , may be larger than $u(p)$ and is given by $\text{cav}[u](p)$, where $\text{cav}[u]$ is the smallest concave function above u . First, the so-called splitting lemma (see Aumann and Maschler 1955, see also Mertens et al. 1994, Section V.1), which corresponds to a two-stage random draw of the state G according to any convex decomposition of p in terms of other distributions over the stages, shows that U is concave. Therefore, we have $U \geq \text{cav}[u]$. The interesting and non-trivial inequality is the converse one. (Kohlberg, 1975, Corollary 2.4) states that for all $p \in \Delta(\{1, \dots, d\})$, there exists some $a_p \in \mathbb{R}^d$ such that

$$\text{cav}[u](p) = \langle a_p, p \rangle \quad \text{and} \quad \forall q \in \Delta(\{1, \dots, d\}), \quad \text{cav}[u](q) \leq \langle a_p, q \rangle;$$

in particular, $u(q) \leq \langle a_p, q \rangle$ for all $q \in \Delta(\{1, \dots, d\})$. Now, the equivalence stated after (7) exactly indicates that thus, $\mathcal{C}_{\text{orth}}(a_p)$ is (r, H) -approachable. This result in the simultaneous model entails in particular, in the model with incomplete information and for the choice p as a distribution by Nature, that

$$\{\langle p, \omega \rangle : \omega \in \mathcal{C}_{\text{orth}}(a_p)\} = (-\infty, \langle p, a_p \rangle) = (-\infty, \text{cav}[u](p))$$

is approachable. This shows that $U(p) \leq \text{cav}[u](p)$.

In conclusion, Kohlberg (1975) implicitly used the properties detailed above when constructing an optimal strategy for the uninformed player. A close inspection reveals that Lemma 5.4 therein does not hold anymore in the more general framework without the product property (in particular, one might want to read it again with Remark 1 in mind).

7. Primal approachability of orthants in the general case

We noted that the primal characterization in terms of one-shot r -approachability of containing half-spaces stated in Proposition 2 did not extend to games (r, H) without the product property. We show in this section that it holds true in the general case when one-shot approachability is with respect to the modified payoff function $\tilde{r}_H : \Delta(\mathcal{I}) \times \Delta(\mathcal{J}) \rightarrow \mathbb{R}^d$ defined as follows:

$$\forall x \in \Delta(\mathcal{I}), \forall y \in \Delta(\mathcal{J}), \quad \tilde{r}_H(x, y) = R(x, \mathbf{H}(y)).$$

The change of payoff function can be intuitively explained as follows. As noted in Section 5, since orthants are to be approached (and only because of this), the behavior of (averages of) sets of compatible payoffs is dictated by their upper-right corners. Now, the product property indicated that even when measuring payoffs with r , the worst-case payoffs were given by the upper-right corners and that it was thus necessary and sufficient to consider the latter. If this property does not hold, then evaluating actions with \tilde{r}_H enables and forces the consideration of these corners.

Of course, in the case of full monitoring, as follows from the comments after Definition 2, no modification takes place in the payoff function, that is, $\tilde{r}_{\text{Full}} = r$.

The main result of this section is the following primal characterization; subsequent subsections of this section show how it leads to a new approachability strategy under partial monitoring, based on surrogate payoffs (upper-right corner payoffs) and not, as usually done, on signals only (on estimated flags).

Theorem 2. *For all games (r, H) with partial monitoring, for all orthants $\mathcal{C}_{\text{orth}}(a)$, where $a \in \mathbb{R}^d$,*

$$\begin{aligned} & \mathcal{C}_{\text{orth}}(a) \text{ is } (r, H)\text{-approachable} \\ \iff & \text{ every half-space } \mathcal{C}_{\text{hs}} \supset \mathcal{C}_{\text{orth}}(a) \text{ is one-shot } \tilde{r}_H\text{-approachable.} \end{aligned}$$

The proof of this theorem is as follows. The dual characterization (5) indicates that a necessary and sufficient condition of (r, H) -approachability for $\mathcal{C}_{\text{orth}}(a)$ is that for all $y \in \Delta(\mathcal{J})$, there exists $x \in \Delta(\mathcal{I})$ such that $\overline{m}(x, \mathbf{H}(y)) \subseteq \mathcal{C}_{\text{orth}}(a)$. Since, by construction of R , the smallest orthant (in the sense of inclusion) in which $\overline{m}(x, \mathbf{H}(y))$ is contained is precisely $\mathcal{C}_{\text{orth}}(\tilde{r}_H(x, y))$, the necessary and sufficient condition can be restated as the requirement that for all $y \in \Delta(\mathcal{J})$, there exists $x \in \Delta(\mathcal{I})$ such that $\tilde{r}_H(x, y) \in \mathcal{C}_{\text{orth}}(a)$. Now, this reformulated dual characterization of approachability in the context of orthants is seen to be equivalent to the following primal characterization, which concludes the proof of the theorem.

Proposition 3. *For all games (r, H) with partial monitoring, for all orthants $\mathcal{C}_{\text{orth}}(a)$, where $a \in \mathbb{R}^d$,*

$$\begin{aligned} & \forall y \in \Delta(\mathcal{J}), \exists x \in \Delta(\mathcal{I}), \quad \tilde{r}_H(x, y) \in \mathcal{C}_{\text{orth}}(a) \\ \iff & \text{ every half-space } \mathcal{C}_{\text{hs}} \supset \mathcal{C}_{\text{orth}}(a) \text{ is one-shot } \tilde{r}_H\text{-approachable.} \end{aligned}$$

Before proving this proposition, we need to state some properties of the function \tilde{r}_H . Given two points $a, a' \in \mathbb{R}^d$, the notation $a \preceq a'$ means that a is component-wise smaller than a' —or equivalently, that a belongs to the orthant $\mathcal{C}_{\text{orth}}(a')$.

Lemma 1. *The function \tilde{r}_H is Lipschitz continuous; it is also convex in its first argument and concave in its second argument, in the sense that, for all $x, x' \in \Delta(\mathcal{I})$, all $y, y' \in \Delta(\mathcal{J})$, and all $\lambda \in [0, 1]$,*

$$\begin{aligned} & \tilde{r}_H(\lambda x + (1 - \lambda)x', y) \preceq \lambda \tilde{r}_H(x, y) + (1 - \lambda) \tilde{r}_H(x', y) \\ \text{and} & \quad \lambda \tilde{r}_H(x, y) + (1 - \lambda) \tilde{r}_H(x, y') \preceq \tilde{r}_H(x, \lambda y + (1 - \lambda)y'). \end{aligned}$$

Proof. The convexity and the concavity stem from the concavity and the convexity of \overline{m} for inclusion; formally, it follows from the very definition (1) of \overline{m} and from the linearity of r and \mathbf{H} that, for all $x, x' \in \Delta(\mathcal{I})$, all $\mathbf{h}, \mathbf{h}' \in \mathcal{F}$, and $\lambda \in [0, 1]$,

$$\begin{aligned} & \overline{m}(\lambda x + (1 - \lambda)x', \mathbf{h}) \subseteq \lambda \overline{m}(x, \mathbf{h}) + (1 - \lambda) \overline{m}(x', \mathbf{h}) \\ \text{and} & \quad \lambda \overline{m}(x, \mathbf{h}) + (1 - \lambda) \overline{m}(x, \mathbf{h}') \subseteq \overline{m}(x, \lambda \mathbf{h} + (1 - \lambda)\mathbf{h}'). \end{aligned}$$

The second part of the lemma follows by taking upper-right corners, which is a linear and non-decreasing operation (for the respective partial orders \subseteq and \preceq).

As for the Lipschitzness of \tilde{r}_H , it follows from a rewriting of $\overline{m}(x, \mathbf{H}(y))$ as

$$\overline{m}(x, \mathbf{H}(y)) = \sum_{b \in \mathcal{B}} \phi_b(\mathbf{H}(y)) r(x, \mathbf{H}^{-1}(b)),$$

where \mathcal{B} is a finite subset of \mathcal{F} , the ϕ_b are Lipschitz functions $\mathcal{F} \rightarrow [0, 1]$, and \mathbf{H}^{-1} is the pre-image function of \mathbf{H} , which takes values in the set of compact subsets of $\Delta(\mathcal{J})$. This rewriting was proved in Mannor et al. (2011, Lemma 6.1 and Remark 6.1). We equip the set of compact subsets of the Euclidian ball with center $(0, \dots, 0)$ and radius M , in which \overline{m} takes its values, with the Hausdorff distance. For this distance, $x \mapsto r(x, \mathbf{H}^{-1}(b))$ is M -Lipschitz for each $b \in \mathcal{B}$. All in all, given the boundedness of the ϕ_b and of r , the mapping $(x, y) \mapsto \overline{m}(x, \mathbf{H}(y))$ is also Lipschitz continuous. Since taking the upper-right corner is a Lipschitz mapping as well (for the Hausdorff distance), we get, by composition, the desired Lipschitzness of \tilde{r}_H . \square

We are now ready to prove Proposition 3. (Note that it indeed needs a proof and it is not implied by the various results discussed in Section 5; indeed, (\tilde{r}_H, H) is an auxiliary game which, by construction, has the product property, but \tilde{r}_H is not linear, while linearity of the payoff function was a crucial feature of the setting studied therein.)

Proof of Proposition 3. We start with the direct implication and consider some containing half-space \mathcal{C}_{hs} ; the latter is parameterized by $\alpha \in \mathbb{R}^d$ and $\beta \in \mathbb{R}$, and equals

$$\mathcal{C}_{\text{hs}} = \{\omega \in \mathbb{R}^d : \langle \alpha, \omega \rangle \leq \beta\};$$

since \mathcal{C}_{hs} contains the orthant $\mathcal{C}_{\text{orth}}(a)$, there are sequences of ω in it with components tending to $-\infty$; therefore, we necessarily have that $\alpha \succcurlyeq 0$. The convexity/concavity of \tilde{r}_H in the sense of \preceq thus entails that the function $G_{\alpha, \beta} : (x, y) \mapsto \langle \alpha, \tilde{r}_H(x, y) \rangle - \beta$ is also convex/concave. The continuity of $G_{\alpha, \beta}$ stems from the one of \tilde{r}_H . The Sion–Fan lemma applies and guarantees that

$$\min_{x \in \Delta(\mathcal{I})} \max_{y \in \Delta(\mathcal{J})} G_{\alpha, \beta}(x, y) = \max_{y \in \Delta(\mathcal{J})} \min_{x \in \Delta(\mathcal{I})} G_{\alpha, \beta}(x, y),$$

(the suprema and infima are all attained and are denoted by maxima and minima). Now, by assumption, for all $y \in \Delta(\mathcal{J})$, there exists $x \in \Delta(\mathcal{I})$ such that $\tilde{r}_H(x, y) \in \mathcal{C}_{\text{orth}}(a)$; this means that the above $\max \min G_{\alpha, \beta}$ is non-positive. Putting all things together, we have proved that

$$\min_{x \in \Delta(\mathcal{I})} \max_{y \in \Delta(\mathcal{J})} G_{\alpha, \beta}(x, y) \leq 0.$$

That is, there exists $x_0 \in \Delta(\mathcal{I})$, e.g., the element attaining the above maximum, such that for all $y \in \Delta(\mathcal{J})$, one has $G_{\alpha, \beta}(x_0, y) \leq 0$, or, equivalently, $\tilde{r}_H(x_0, y) \in \mathcal{C}_{\text{orth}}(a)$. This property is exactly the stated one-shot \tilde{r}_H -approachability of $\mathcal{C}_{\text{orth}}(a)$.

Conversely, assume that there exists some $y_0 \in \Delta(\mathcal{J})$ such that, for all $x \in \Delta(\mathcal{I})$, one has $\tilde{r}_H(x, y_0) \notin \mathcal{C}_0$. By continuity of \tilde{r}_H and closedness of $\mathcal{C}_{\text{orth}}(a)$, there exists some

$\delta > 0$ such that $d_{\mathcal{C}_{\text{orth}}(a)}(\tilde{r}_H(x, y_0)) \geq \delta$ for all $x \in \Delta(\mathcal{I})$. Now, as indicated around (6), the distance to $\mathcal{C}_{\text{orth}}(a)$ is non-decreasing for \preceq , which, in view of the convexity of \tilde{r}_H in its first argument, shows that we also have $d_{\mathcal{C}_{\text{orth}}(a)}(z) \geq \delta$ for all elements z of the convex hull $\mathcal{C}_{\tilde{r}_H, y_0}$ of the set $\{\tilde{r}_H(x, y_0) : x \in \Delta(\mathcal{I})\}$. That is, the closed convex sets $\mathcal{C}_{\tilde{r}_H, y_0}$, which is compact, and $\mathcal{C}_{\text{orth}}(a)$ are disjoint and thus, by the Hahn–Banach theorem, are strictly separated by some hyperplane. One of the two half-spaces thus defined, namely, the one not containing $\mathcal{C}_{\tilde{r}_H, y_0}$, is not one-shot \tilde{r}_H -approachable. \square

A new approachability strategy of an orthant under partial monitoring

In the case its characterization is satisfied, Theorem 2 suggests an approachability strategy based on surrogate payoffs and not only the information gained, i.e., based on the mapping \tilde{r}_H and not only on the signaling structure H (and the estimated flags). The first approach was already considered by Kohlberg (1975) while other works, like Perchet (2011a) and Mannor et al. (2011), resorted to the second one. The considered strategy is an adaptation of Blackwell’s strategy (which was recalled after the statement of Theorem 1); such an adaptation is possible as the latter strategy only relies on the one-shot approachability of half-spaces, which is satisfied here with the surrogate payoffs \tilde{r}_H .

Description and convergence analysis of the strategy. As in the proof of Proposition 2, we assume in a first time that flags $\mathbf{h}_t = \mathbf{H}(y_t)$ are observed at the end of each round t and that mixed payoffs are to be controlled. The player then knows his mixed payoffs $\tilde{r}_{H,t} := \tilde{r}_H(x_t, y_t) = R(x_t, \mathbf{h}_t)$ and aims at controlling his average payoffs, which we recall are denoted by $\bar{\tilde{r}}_{H,n}$. Similarly to what was done in the proof of Proposition 2, the one-shot \tilde{r}_H -approachability of the containing half-spaces of $\mathcal{C}_{\text{orth}}(a)$ entails that for each round n , there exists $x_{n+1} \in \Delta(\mathcal{I})$ such that

$$\forall y \in \Delta(\mathcal{J}), \quad \left\langle \tilde{r}_H(x_{n+1}, y) - \pi_{\mathcal{C}_{\text{orth}}(a)}(\bar{\tilde{r}}_{H,n}), \bar{\tilde{r}}_{H,n} - \pi_{\mathcal{C}_{\text{orth}}(a)}(\bar{\tilde{r}}_{H,n}) \right\rangle \leq 0.$$

The sequence $(\bar{\tilde{r}}_{H,n})$ thus satisfies Blackwell’s condition and as a result (see again the proof of Proposition 2), we get

$$d_{\mathcal{C}_{\text{orth}}(a)}\left(\frac{1}{n} \sum_{t=1}^n \tilde{r}_H(x_t, y_t)\right) \leq \frac{2M\sqrt{d}}{\sqrt{n}}.$$

Since $r(x_t, y_t) \in \bar{m}(x_t, \mathbf{H}(y_t)) \subseteq \mathcal{C}_{\text{orth}}(\tilde{r}_H(x_t, y_t))$, we get $r(x_t, y_t) \preceq \tilde{r}_H(x_t, y_t)$ and, in view again of (6),

$$d_{\mathcal{C}_{\text{orth}}(a)}\left(\frac{1}{n} \sum_{t=1}^n r(x_t, y_t)\right) \leq d_{\mathcal{C}_{\text{orth}}(a)}\left(\frac{1}{n} \sum_{t=1}^n \tilde{r}_H(x_t, y_t)\right) \leq \frac{2M\sqrt{d}}{\sqrt{n}}.$$

The same trick of playing i.i.d. in blocks as in the second part of the proof of Proposition 2, together with martingale convergence arguments, relaxes the assumptions of flags being observed and payoffs being evaluated with mixed actions, leading to the desired

(r, H) -approachability strategy. (This is where we need the Lipschitzness properties stated in Lemma 1 and its proof.) A more careful study, which we omit here for the sake of concision, shows that (r, H) -approachability takes place at a $n^{-1/5}$ -rate.

What we proved in passing. We proved in a constructive way that when an orthant is (r, H) -approachable, it is also $(\tilde{r}_H, \text{Full})$ -approachable.

Conversely, assume that the equivalent conditions in Theorem 2 are not satisfied, i.e., that the orthant at hand, $\mathcal{C}_{\text{orth}}(a)$, is not (r, H) -approachable. Then, (the proof of) Proposition 3 indicates that there exists some $y_0 \in \Delta(\mathcal{J})$ such that the set $\mathcal{C}_{\text{orth}}(a)$ and the convex hull of $\{\tilde{r}_H(x, y_0), x \in \Delta(\mathcal{I})\}$ are strictly separated. This implies in particular that $\mathcal{C}_{\text{orth}}(a)$ is $(\tilde{r}_H, \text{Full})$ -excludable, and thus is not $(\tilde{r}_H, \text{Full})$ -approachable.

Putting all things together, we have proved the following equivalence:

$$\begin{aligned} & \mathcal{C}_{\text{orth}}(a) \text{ is } (r, H)\text{-approachable} \\ \iff & \mathcal{C}_{\text{orth}}(a) \text{ is } (\tilde{r}_H, \text{Full})\text{-approachable} \\ \iff & \mathcal{C}_{\text{orth}}(a) \text{ is not } (\tilde{r}_H, \text{Full})\text{-excludable.} \end{aligned}$$

Note that the $(\tilde{r}_H, \text{Full})$ -approachability is a form of non-linear approachability, by which we mean that the function \tilde{r}_H is not linear and yet, approachability is possible. This result could be generalized (but we omit the description of the extension for the sake of concision).

On the computational complexity of the above-described strategy. The strategy we have exhibited reduces to solving, at each stage, a program of the form

$$\min_{x_{n+1} \in \Delta(\mathcal{I})} \left\{ \max_{y \in \Delta(\mathcal{J})} \langle \tilde{r}_H(x_{n+1}, y) - \beta, \alpha \rangle \right\}$$

for some vectors $\alpha, \beta \in \mathbb{R}^d$. At first sight, it cannot be written as a finite linear program as \tilde{r}_H is not a linear function of its arguments. However, as proved in Mannor et al. (2011, Section 7.1), the function \tilde{r}_H is actually piecewise linear; that is, there exist some finite liftings of $\Delta(\mathcal{I})$ and $\Delta(\mathcal{J})$ with respect to which \tilde{r}_H is linear. (These liftings only need to be computed once, before the game starts.) Moreover, the per-step computational complexity of our strategy is constant (in fact, it is polynomial in the sizes of these liftings; see Mannor et al. 2011 for more details).

8. Primal approachability of polyhedra

Recall that a convex set $\mathcal{C}_{\text{polyh}}$ is a polyhedron if it is the intersection of a finite number of half-spaces $\{\omega \in \mathbb{R}^d : \langle \omega, a_\ell \rangle \leq b_\ell\}$, for $a_\ell, b_\ell \in \mathbb{R}^d$ and ℓ ranging in some finite set \mathcal{L} . That is,

$$\mathcal{C}_{\text{polyh}} = \bigcap_{\ell \in \mathcal{L}} \left\{ \omega \in \mathbb{R}^d : \langle \omega, a_\ell \rangle \leq b_\ell \right\} = \left\{ \omega \in \mathbb{R}^d : \max_{\ell \in \mathcal{L}} \langle \omega, a_\ell \rangle - b_\ell \leq 0 \right\}. \quad (8)$$

The following lemma (which is a mere exercise of rewriting) states that an approachability problem of a polyhedron can be transformed into an approachability problem of some negative orthant; we denote by $(0)_{\mathcal{L}} = (0, \dots, 0)$ the null vector of $\mathbb{R}^{\mathcal{L}}$. The negative orthant of $\mathbb{R}^{\mathcal{L}}$ is then denoted by $\mathcal{C}_{\text{orth}}((0)_{\mathcal{L}})$.

Lemma 2. *The convex polyhedron $\mathcal{C}_{\text{polyh}}$ defined in (8) is (r, H) -approachable if and only if the negative orthant $\mathcal{C}_{\text{orth}}((0)_{\mathcal{L}})$ is (s, H) -approachable, where the vector-valued payoff function $s : \Delta(\mathcal{I}) \times \Delta(\mathcal{J}) \rightarrow \mathbb{R}^{\mathcal{L}}$ is defined as*

$$s(x, y) = \left[\langle r(x, y), a_{\ell} \rangle - b_{\ell} \right]_{\ell \in \mathcal{L}}. \quad (9)$$

Proof. The result follows from the equivalence (see, e.g., property 3 in Appendix A.1 of Perchet 2011b) of the distances to $\mathcal{C}_{\text{polyh}}$ given by

$$d_{\mathcal{C}_{\text{polyh}}} \quad \text{and} \quad d_{\mathcal{C}_{\text{orth}}((0)_{\mathcal{L}})}(T(\cdot)),$$

where $T : \mathbb{R}^d \rightarrow \mathbb{R}^{\mathcal{L}}$ the linear transformation $\omega \mapsto [\langle \omega, a_{\ell} \rangle - b_{\ell}]_{\ell \in \mathcal{L}}$. □

Theorem 2 can then be rewritten, using Lemma 2 above, to provide the desired primal characterization of polyhedra.

Corollary 2. *Consider the convex polyhedron $\mathcal{C}_{\text{polyh}}$ given by (8), together with the payoff function s defined in (9). Then,*

$$\begin{aligned} & \mathcal{C}_{\text{polyh}} \text{ is } (r, H)\text{-approachable} & (10) \\ \iff & \text{ every containing half-space of } \mathcal{C}_{\text{orth}}((0)_{\mathcal{L}}) \text{ is one-shot } \tilde{s}_H\text{-approachable.} \end{aligned}$$

When $\mathcal{C}_{\text{polyh}}$ is indeed (r, H) -approachable, the results of the previous section provide an approachability strategy of $\mathcal{C}_{\text{orth}}((0)_{\mathcal{L}})$ based on the transformed payoffs \tilde{s}_H . This strategy also approaches $\mathcal{C}_{\text{polyh}}$ in view of Lemma 2, however it might not be representable in the original space \mathbb{R}^d , as demonstrated in the following (counter-)example.

Example 1. Consider on the one hand the polytope $\mathcal{C}_{\text{polyh}} = \{\omega \in \mathbb{R} : \omega \in [-1, 1]\}$ and the associated linear transformation T defined by $T(\omega) = (\omega - 1, -\omega - 1) \in \mathbb{R}^2$ for all $\omega \in \mathbb{R}$. Consider on the other hand the following game. The sets of pure actions are $\mathcal{I} = \{T, B\}$ and $\mathcal{J} = \{L, R\}$, the signaling structure is $H = \text{Dark}$ (with single signal denoted by \emptyset), and the payoff function r is given by the matrix

	L	R
T	-1	2
B	-2	1

We identify $\Delta(\mathcal{I})$ and $\Delta(\mathcal{J})$ with $[0, 1]$.

We first discuss the dual condition (5) for (r, Dark) -approachability. For all $x \in [0, 1]$, we have $\overline{m}(x, \emptyset) = [-2 + x, 1 + x]$. Thus, no mixed action x of the player is such that $\overline{m}(x, \emptyset)$ is included in $\mathcal{C}_{\text{polyh}}$, which is therefore not (r, Dark) -approachable.

We now turn to the primal condition as stated by Corollary 2. We denote by $T = (T_1, T_2)$ the components of the linear transformation T . From the linearity of T , we deduce from the above-stated expression of \overline{m} (based on r) that the sets of compatible payoffs in terms of $s = T(r)$ are of the form $T(\overline{m}(x, \emptyset))$. Taking the maxima, we thus get, for all mixed actions $x \in [0, 1]$ (and all $y \in [0, 1]$ as the game is played in the dark),

$$\tilde{s}_{\text{Dark}}(x, y) = \left(\max T_1([-2 + x, 1 + x]), \max T_2([-2 + x, 1 + x]) \right) = (x, 1 - x).$$

Again, the necessary and sufficient condition for (r, H) -approachability of $\mathcal{C}_{\text{polyh}}$ fails, as no containing half-space of the negative orthant but two of them is one-shot \tilde{s}_{Dark} -approachable. More precisely, these half-spaces are parameterized by $(p, 1 - p)$ where $p \in [0, 1]$ and correspond to the points $\{(t_1, t_2) \in \mathbb{R}^2 : p_1 t_1 + p_2 t_2 \leq 0\}$. Except for the case when $p = 0$ or $p = 1$, these half-spaces are strictly separated from the convex set $\{(x, 1 - x) : x \in [0, 1]\}$.

The question is now whether we could have determined this by satisfying some primal condition in the original space \mathbb{R} . First, consider some containing half-space of $\mathcal{C}_{\text{polyh}}$, typically, either $(-\infty, 1]$ or $[-1, +\infty)$. Their transformations by T into subsets of \mathbb{R}^2 are included respectively in $(-\infty, 0] \times \mathbb{R}$ or $\mathbb{R} \times (-\infty, 0]$. These are precisely the only two half-spaces that were one-shot \tilde{s}_{Dark} -approachable (by resorting to one of the pure actions). Now, and more importantly, consider the containing half-space of the negative orthant in \mathbb{R}^2 parameterized by $p = 1/2$, that is, $\mathcal{C}_{\text{hs}, 1/2} = \{(t_1, t_2) \in \mathbb{R}^2 : t_1 + t_2 \leq 0\}$; as indicated above, it is not one-shot \tilde{s}_{Dark} -approachable. However, this half-space contains all the original space, in the sense that $T(\mathbb{R}) \subset \mathcal{C}_{\text{hs}, 1/2}$, as follows from a simple calculus: $(\omega - 1) + (-\omega - 1) = -2$. Therefore, there is no hope to prove, based even on general subsets of the original game with payoffs in \mathbb{R} , that the necessary and sufficient condition on the half-space $\mathcal{C}_{\text{hs}, 1/2}$ in the transformed space \mathbb{R}^2 fails.

The fundamental reason why the primal characterization in the transformed space cannot be checked based on considerations in the original space is because, in the absence of a product property, the range of \tilde{s}_{Dark} is outside the range of s ; and we can only access to the latter based on the original space. The moral of example is that we have to consider some *hidden* containing half-spaces of the polytope $\mathcal{C}_{\text{polyh}}$ in order to establish some primal characterization; this is precisely what Condition (10) does.

9. Primal approachability of general convex sets

We consider in this section the primal approachability of general closed convex sets \mathcal{C} . In the case of polytopes, Lemma 2 was essentially indicating that only finitely many directions in \mathbb{R}^d (the ones given by the a_ℓ) had to be considered. In the case of general convex sets, all directions are to be studied; we do so by resorting to support functions, which we define based on the unit Euclidean sphere $\mathcal{S} = \{\omega \in \mathbb{R}^d : \|\omega\| = 1\}$. More formally, the support function $\phi_{\mathcal{C}} : \mathcal{S} \rightarrow \mathbb{R} \cup \{+\infty\}$ of a set $\mathcal{C} \subseteq \mathbb{R}^d$ is defined by

$$\forall s \in \mathcal{S}, \quad \phi_{\mathcal{C}}(s) = \sup\{\langle c, s \rangle : c \in \mathcal{C}\}.$$

We now construct a lifted setting in which one-shot approaching the containing half-spaces for some payoff function will be equivalent to (r, H) -approaching the original closed convex set \mathcal{C} . This setting is given by some set of integrable functions on \mathcal{S} . We equip the latter with the (induced) Lebesgue measure, for which \mathcal{S} has a finite measure and whose integration against will be denoted by ds . That is, we consider the set $\mathbb{L}^2(\mathcal{S})$ of Lebesgue square integrable functions $\mathcal{S} \rightarrow \mathbb{R}$, equipped with the inner product

$$(f, g) \in \mathbb{L}^2(\mathcal{S}) \times \mathbb{L}^2(\mathcal{S}) \mapsto \int_{\mathcal{S}} f(s) g(s) ds.$$

The orthant in $\mathbb{L}^2(\mathcal{S})$ corresponding to $\mathcal{C} \subseteq \mathbb{R}^d$ is

$$\mathcal{C}_{\text{orth}}(\phi_{\mathcal{C}}) = \{f \in \mathbb{L}^2(\mathcal{S}) : f \leq \phi_{\mathcal{C}}\}.$$

The description of the lifted setting is concluded by stating the considered payoff function $\Phi : \Delta(\mathcal{I}) \times \Delta(\mathcal{J}) \rightarrow \mathbb{L}^2(\mathcal{S})$; it indicates, as in the previous sections, how to transform payoffs given the signaling structure H . Formally,

$$\forall x \in \Delta(\mathcal{I}), \forall y \in \Delta(\mathcal{J}), \quad \Phi(x, y) = \phi_{\overline{m}(x, \mathbf{H}(y))};$$

the square integrability of $\Phi(x, y)$ follows its boundedness, which stems from the boundedness of $\overline{m}(x, \mathbf{H}(y))$; see Lemma 3 in appendix, property 1, for a reminder of this well-known result (and others) on support functions.

We are now ready to state the primal characterization of approachability with partial monitoring in the general case.

Theorem 3. *For all games (r, H) , for all closed convex sets $\mathcal{C} \subset \mathbb{R}^d$,*

$$\begin{aligned} & \mathcal{C} \text{ is } (r, H)\text{-approachable} \\ \iff & \text{ every half-space } \mathcal{C}_{\text{hs}} \supset \mathcal{C}_{\text{orth}}(\phi_{\mathcal{C}}) \text{ is one-shot } \Phi\text{-approachable.} \end{aligned}$$

Proof. We first note that we can assume with no loss of generality that \mathcal{C} is bounded thus compact; indeed, \mathcal{C} is (r, H) -approachable if and only if its intersection $\mathcal{C} \cap r(\Delta(\mathcal{I} \times \mathcal{J}))$ with the bounded convex set of feasible payoffs is approachable. This entails that $\phi_{\mathcal{C}} \in \mathbb{L}^2(\mathcal{S})$.

Now, the proof follows the lines of the proof of Theorem 2. In particular, we exploit the dual characterization (5), that indicates that for all $y \in \Delta(\mathcal{J})$, there exists $x \in \Delta(\mathcal{I})$ such that $\overline{m}(x, \mathbf{H}(y)) \subseteq \mathcal{C}$. It can be restated equivalently (see Lemma 3 in appendix, property 3) as stating that for all $y \in \Delta(\mathcal{J})$, there exists $x \in \Delta(\mathcal{I})$ such that $\Phi(x, y) \leq \phi_{\mathcal{C}}$. We thus only need to show that the stated primal characterization is equivalent to the latter condition.

We start with the direct implication (from the dual condition to the primal condition). As recalled in the proof of Lemma 1, the function \overline{m} is concave/convex, which, together with properties 3 and 4 of Lemma 3, shows that Φ is also convex/concave. Moreover, as proved at the end of the proof of Lemma 1, the function $(x, y) \mapsto \overline{m}(x, \mathbf{H}(y))$ is a Lipschitz function, with Lipschitz constant denoted by $L_{\overline{m}}$, when the set of compact

subsets of the Euclidean ball of \mathbb{R}^d with center $(0, \dots, 0)$ and radius M is equipped with the Hausdorff distance. This entails that Φ is also a Lipschitz function, with constant $L_{\overline{m}}V$, where V is the volume of \mathcal{S} for the induced Lebesgue measure. This is because the Hausdorff distance δ between two sets \mathcal{D}_1 and \mathcal{D}_2 translates to a $V\delta$ -Euclidean distance between $\phi_{\mathcal{D}_1}$ and $\phi_{\mathcal{D}_2}$. Indeed, we have, by definition of the Hausdorff distance, $\mathcal{D}_1 \subseteq \mathcal{D}_2 + B_\delta$ and $\mathcal{D}_2 \subseteq \mathcal{D}_1 + B_\delta$, where B_δ is the Euclidian ball of \mathbb{R}^d with center $(0, \dots, 0)$ and radius δ . Properties 4 and 1 of Lemma 3 respectively yield the inequalities

$$|\phi_{\mathcal{D}_1} - \phi_{\mathcal{D}_2}| = \max\{\phi_{\mathcal{D}_1} - \phi_{\mathcal{D}_2}, \phi_{\mathcal{D}_2} - \phi_{\mathcal{D}_1}\} \leq \phi_{B_\delta} \leq \delta,$$

with, by integration, $\|\phi_{\mathcal{D}_1} - \phi_{\mathcal{D}_2}\| \leq V\delta$.

The above-stated properties of Φ imply that for all $\psi \in \mathbb{L}^2(\mathcal{S})$ with $\psi \geq 0$, the game $(x, y) \mapsto \langle \psi, \Phi(x, y) \rangle$ has a value $v(\psi)$, and that this value is achieved: there exists some $x_\psi \in \Delta(\mathcal{I})$ such that

$$\max_{y \in \Delta(\mathcal{J})} \langle \psi, \Phi(x_\psi, y) \rangle = v(\psi) = \max_{y \in \Delta(\mathcal{J})} \min_{x \in \Delta(\mathcal{I})} \langle \psi, \Phi(x, y) \rangle.$$

Now, consider some half-space \mathcal{C}_{hs} containing $\mathcal{C}_{\text{orth}}(\phi_{\mathcal{C}})$; it is of the form

$$\mathcal{C}_{\text{hs}} = \{f \in \mathbb{L}^2(\mathcal{S}) : \langle \psi, f \rangle \leq \beta\},$$

where necessarily, as can be shown by contradiction, $\psi \geq 0$. The dual condition is satisfied by assumption, that is, for all $y \in \Delta(\mathcal{J})$, there exists $x \in \Delta(\mathcal{I})$ such that $\Phi(x, y) \in \mathcal{C}_{\text{orth}}(\phi_{\mathcal{C}})$, and therefore, $\Phi(x, y) \in \mathcal{C}_{\text{hs}}$. Thus, $v(\psi) \leq \beta$, as can be seen with its expression as a max/min. The mixed action x_ψ thus satisfies that $\langle \psi, \Phi(x_\psi, y) \rangle \leq \beta$ for all $y \in \Delta(\mathcal{J})$, which is exactly saying that $\Phi(x_\psi, y) \in \mathcal{C}_{\text{hs}}$ for all $y \in \Delta(\mathcal{J})$. We therefore proved the desired one-shot Φ -approachability of \mathcal{C}_{hs} .

Conversely, we assume that the dual condition is not satisfied, i.e., that there exists some $y_0 \in \Delta(\mathcal{J})$ such that for no $x \in \Delta(\mathcal{I})$ one has $\Phi(x, y_0) \in \mathcal{C}_{\text{orth}}(\phi_{\mathcal{C}})$. We consider the continuous thus compact image $\Phi(\Delta(\mathcal{I}), y_0)$ of $\Delta(\mathcal{I})$ by $\Phi(\cdot, y_0)$. Its Euclidean distance to the closed set $\mathcal{C}_{\text{orth}}(\phi_{\mathcal{C}})$ is thus positive, we denote it by $\delta > 0$. Now, the distance of an element $f \in \mathbb{L}^2(\mathcal{S})$ to $\mathcal{C}_{\text{orth}}(\phi_{\mathcal{C}})$ is given by

$$d_{\mathcal{C}_{\text{orth}}(\phi_{\mathcal{C}})}(f) = \int_{\mathcal{S}} (f(s) - \phi_{\mathcal{C}}(s))_+ \, ds.$$

Since in addition, Φ is convex in its first argument (as shown in the first part of this proof), we have that $d_{\mathcal{C}_{\text{orth}}(\phi_{\mathcal{C}})}(f) \geq \delta$ not only for all $f \in \Phi(\Delta(\mathcal{I}), y_0)$ but also for all f in the convex hull of $\Phi(\Delta(\mathcal{I}), y_0)$. The latter set is pointwise bounded (by M , as follows from property 1 of Lemma 3) and is formed by equicontinuous functions (they all are M -Lipschitz continuous, as follows from property 2 of the same lemma); the Arzela-Ascoli theorem thus ensures that the closure of this set is compact for the supremum norm $\|\cdot\|_\infty$ over \mathcal{S} . As by integration $\|\cdot\|_\infty \geq \|\cdot\|/V$, the closure of the convex hull of $\Phi(\Delta(\mathcal{I}), y_0)$ and the set $\mathcal{C}_{\text{orth}}(\phi_{\mathcal{C}})$ are still δ/V -separated in $\|\cdot\|_\infty$ -norm, thus are disjoint. Since the former set is a convex and compact set, and the latter is a closed convex set, the Hahn-Banach theorem entails that they are strictly separated by some hyperplane. In particular, one of the two thus-defined half-spaces is not one-shot Φ -approachable. \square

The above result is a generalization of the polytopial case

In Section 8 we showed that when approaching a polyhedra, there are only finitely many directions (i.e., finitely many elements of the sphere \mathcal{S}) of interest, namely, the directions corresponding to the defining hyperplanes. On the contrary, with general convex sets, every direction might be relevant, as a general convex set is defined as the intersection of infinitely many half-spaces, one per element of \mathcal{S} . This is the reason why, for general convex sets, we introduced a lifting into the space of real-valued mappings on the whole set \mathcal{S} . We also resorted the uniform Lebesgue measure since all directions are equally important.

On the other hand, as we recalled above, only a few directions matter in the polytopial framework of Section 8. The results we obtained therein can in fact be obtained as a corollary of Theorem 3 when the latter is stated (and proved) with a different measure instead of the Lebesgue measure, given by the sums of the Dirac masses on the directions of the defining hyperplanes.

References

- R.J. Aumann and M.B. Maschler. *Repeated Games with Incomplete Information*. MIT Press, 1955.
- D. Blackwell. An analog of the minimax theorem for vector payoffs. *Pacific Journal of Mathematics*, 6:1–8, 1956.
- E. Kohlberg. Optimal strategies in repeated games with incomplete information. *International Journal of Game Theory*, 4:7–243, 1975.
- G. Lugosi, S. Mannor, and G. Stoltz. Strategies for prediction under imperfect monitoring. *Mathematics of Operations Research*, 33:513–528, 2008.
- S. Mannor, V. Perchet, and G. Stoltz. Robust approachability and regret minimization in games with partial monitoring. An extended abstract was published in the *Proceedings of COLT'11*, 2011. URL <http://hal.archives-ouvertes.fr/hal-00595695>.
- J.-F. Mertens, S. Sorin, and S. Zamir. Repeated games. Technical Report no. 9420, 9421, 9422, Université de Louvain-la-Neuve, 1994.
- V. Perchet. Approachability of convex sets in games with partial monitoring. *Journal of Optimization Theory and Applications*, 149:665–677, 2011a.
- V. Perchet. Internal regret with partial monitoring: Calibration-based optimal algorithms. *Journal of Machine Learning Research*, 12:1893–1921, 2011b.
- V. Perchet and M. Quincampoix. On an unified framework for approachability in games with or without signals. 2011. URL <https://sites.google.com/site/vianneyperchet/cache>.

Appendix

A brief survey of some well-known properties of support functions

For the sake of self-completeness only we summarize in the lemma below some simple and well-known properties of support functions.

Lemma 3. *We consider a set $\mathcal{C} \subseteq \mathbb{R}^d$.*

1. *If \mathcal{C} is bounded in Euclidian norm by C , then $\phi_{\mathcal{C}}$ is bounded in supremum norm by C and in Euclidean norm by VC , where V is the volume of \mathcal{S} under the (induced) Lebesgue measure.*
2. *If \mathcal{C} is bounded in Euclidian norm by C , then $\phi_{\mathcal{C}}$ is a Lipschitz function, with Lipschitz constant C .*
3. *For all $\mathcal{C}' \subseteq \mathbb{R}^d$, if $\mathcal{C} \subseteq \mathcal{C}'$, then $\phi_{\mathcal{C}} \leq \phi_{\mathcal{C}'}$. The converse implication holds if in addition \mathcal{C}' is a closed convex set.*
4. *The function ϕ is linear, in the sense that for all $\gamma \geq 0$ and all $\mathcal{C}' \subseteq \mathbb{R}^d$, one has $\phi_{\gamma\mathcal{C} + \mathcal{C}'} = \gamma\phi_{\mathcal{C}} + \phi_{\mathcal{C}'}$.*

Proof. Property 1 follows from the Cauchy–Schwarz inequality: for all $s \in \mathcal{S}$,

$$|\phi_{\mathcal{C}}(s)| \leq \sup_{c \in \mathcal{C}} |\langle c, s \rangle| \leq \sup_{c \in \mathcal{C}} \|c\| \|s\| = \sup_{c \in \mathcal{C}} \|c\|,$$

as the elements $s \in \mathcal{S}$ have unit norm. The bound in Euclidean norm follows by integration over \mathcal{S} .

For property 2, we note that $s \in \mathcal{S} \mapsto \langle c, s \rangle$ is a $\|c\|$ -Lipschitz function (again, by the Cauchy–Schwarz inequality). Therefore $\phi_{\mathcal{C}}$ is the supremum of C -Lipschitz functions and as such is also a C -Lipschitz function.

The first part of property 3 is by definition of a supremum. To prove the converse implication, we use an argument by contradiction. We consider two sets \mathcal{C} and \mathcal{C}' , where \mathcal{C}' is closed and convex. We assume that \mathcal{C} is not included in \mathcal{C}' and show that the existence of a $s \in \mathcal{S}$ such that $\phi_{\mathcal{C}}(s) > \phi_{\mathcal{C}'}(s)$. The set $\mathcal{C} \setminus \mathcal{C}'$ is not empty, let x be one of its elements. The convex sets $\{x\}$, which is compact, and \mathcal{C}' , which is closed, are disjoint sets. The Hahn–Banach theorem ensures the existence of a strictly separating hyperplane between these convex sets, which we can write in the form $\{\omega \in \mathbb{R}^d : \langle s, \omega \rangle = \beta\}$ for some $s \in \mathcal{S}$ and $\beta \in \mathbb{R}$ such that

$$\phi_{\{x\}}(s) = \langle s, x \rangle > \beta \quad \text{and} \quad \forall c' \in \mathcal{C}', \quad \langle s, c' \rangle < \beta.$$

This entails that $\phi_{\mathcal{C}'}(s) \leq \beta < \phi_{\{x\}}(s) \leq \phi_{\mathcal{C}}(s)$.

Finally, the last property is because by definition

$$\gamma\mathcal{C} + \mathcal{C}' = \{\gamma c + c' : c \in \mathcal{C}, c' \in \mathcal{C}'\}$$

and thus, for all $s \in \mathcal{S}$,

$$\sup_{c'' \in \gamma \mathcal{C} + \mathcal{C}'} \langle c'', s \rangle = \sup_{c \in \mathcal{C}, c' \in \mathcal{C}'} \gamma \langle c, s \rangle + \langle c', s \rangle = \gamma \sup_{c \in \mathcal{C}} \langle c, s \rangle + \sup_{c' \in \mathcal{C}'} \langle c', s \rangle,$$

where we used the fact that $\gamma > 0$ in the last equality. □