Cohomological Dimension of
Laumon 1-motives up to Isogenies

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Abstract
We prove that the category of Laumon 1-motives up isogenies over a field of characteristic zero is of cohomological dimension $\leq 1$. As a consequence this implies the same result for the category of formal Hodge structures of level $\leq 1$ (over $\mathbb{Q}$).

MSC: 14C99, 14L15.

Contents

1 Laumon 1-motives
1.1 fppf sheaves .............................................. 2

2 Extensions
2.1 The group of $n$-extensions .............................. 6
2.1.1 A lemma on 2-fold extensions ....................... 6
2.2 Ext of 1-motives up to isogenies ...................... 7

References .............................................. 10

Introduction

In [6] Deligne defined 1-motive over a field $k$ as $\text{Gal}(k_{\text{sep}}|k)$-equivariant morphism $[u : X \to G(k_{\text{sep}})]$ where $X$ is a free $\text{Gal}(k_{\text{sep}}|k)$-module and $G$ is
a semi-abelian algebraic group over \( k \). They form a category that we shall denote by \( \mathcal{M}_{1,k} \) or \( \mathcal{M}_{1} \).

Deligne’s definition was motivated by Hodge theory. In fact the category of 1-motives over the complex numbers is equivalent, via the so called Hodge realization functor, to the category \( \text{MHS}_1 \) of mixed Hodge structures of level \( \leq 1 \). It is known the the category \( \text{MHS}_1 \) is of cohomological dimension 1 (see [3]) and the same holds for \( \mathcal{M}_{1,C} \).

F. Orgogozo proved more generally that for any field \( k \), the category \( \mathcal{M}_{1,k} \otimes \mathbb{Q} \) is of cohomological dimension \( \leq 1 \) (see [14] Prop. 3.2.4).

Over a field of characteristic 0 it is possible to define the category \( \mathcal{M}_{1,k}^o \) of Laumon 1-motives generalizing that of Deligne 1-motives (See [11]). In [3] L. Barbieri-Viale generalized the Hodge realization functor to Laumon 1-motives. He defined the category \( \text{FHS}_1 \) of formal Hodge structures of level \( \leq 1 \) containing \( \text{MHS}_1 \) and proved that \( \text{FHS}_1 \) is equivalent to the category of Laumon 1-motives over \( \mathbb{C} \) (compatibly with the Hodge realization).

In this paper we prove that the category of Laumon 1-motives up to isogenies is of cohomological dimension 1.

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## 1 Laumon 1-motives

In this paper \( k \) is a field of characteristic 0 and \( \bar{k} \) is its algebraic closure. As explained in the [11] we can assume that the categories of formal and algebraic groups are full sub-category of \( \text{Ab}_k \), i.e. the category of abelian sheaves on the category of affine \( k \)-schemes w.r.t. the fpff topology.

**Definition 1.1.** A Laumon 1-motive over \( k \) (or an effective free 1-motive over \( k \), cf. [2] 1.4.1]) is the data of

i) A (commutative) formal group \( F \) over \( k \), such that \( \text{Lie} F \) is a finitely generated and \( F(\bar{k}) = \lim_{k'} F(k') \) is finitely generated and torsion-free \( \text{Gal}(\bar{k}/k) \)-module.

ii) A connected commutative algebraic group scheme \( G \) over \( k \).

iii) A morphism \( u : F \to G \) in the category \( \text{Ab}_k \).

Note that we can consider a Laumon 1-motive \( M = [u : F \to G] \) as a complex of sheaves in \( \text{Ab}_k \) concentrated in degree 0, 1.

It is known that any formal \( k \)-group \( F \) splits canonically as product \( F^0 \times F_{\text{et}} \) where \( F^0 \) is the identity component of \( F \) and is a connected formal \( k \)-group, and \( F_{\text{et}} = F/F^0 \) is étale. Moreover, \( F_{\text{et}} \) admits a maximal sub-group
scheme $F_{tor}$, étale and finite, such that the quotient $F_{et}/F_{tor} = F_{fr}$ is constant of the type $\mathbb{Z}'$ over $k$. One says that $F$ is torsion-free if $F_{tor} = 0$.

By a theorem of Chevalley any connected algebraic group scheme $G$ is extension of an abelian variety $A$ by a linear $k$-group scheme $L$ that is product of its maximal sub-torus $T$ with a vector $k$-group scheme $V$. (See [7] for more details on algebraic and formal groups)

**Definition 1.2.** A *morphism* of Laumon 1-motives is a commutative square in the category $\text{Ab}_k$. We denote by $\mathcal{M}_1 = \mathcal{M}_{1,k}$ the category of Laumon $k$-1-motives, i.e. the full sub-category of $C^h(\text{Ab}_k)$ whose objects are Laumon 1-motives.

**Remark 1.3.** The category of Deligne 1-motives (over $k$) is the full sub-category $\mathcal{M}_1$ of $\mathcal{M}_1^a$ whose objects are $M = [u : F \to G]$ such that $F^o = 0$ and $G$ is semi-abelian (cf. [6] §10.1.2]).

**Proposition 1.4.** The category $\mathcal{M}_1^a$ of Laumon 1-motives (over $k$) is an additive category with kernels and co-kernels.

**Proof.** See [11] Prop. 5.1.3. \qed

**Remark 1.5.** i) Let

\[
\begin{array}{ccc}
F & \xrightarrow{f} & F' \\
\downarrow{u} & & \downarrow{u'} \\
G & \xrightarrow{g} & G'
\end{array}
\]

be a morphism from $M = [u : F \to G]$ to $M' = [u' : F' \to G']$. Then from previous proof we get

\[
\text{Ker}(f, g) = [u^*\text{Ker}(g)^o \to \text{Ker}(g)^o]
\]

(1)

where $u : \text{Ker}(f) \to \text{Ker}(g)$, and

\[
\text{Coker}(f, g) = [\text{Coker}(f)^{fr} \to \text{Coker}(g)]
\]

(2)

ii) The category of Laumon 1-motives is not abelian. In fact consider a surjective morphism of connected algebraic groups $g : G \to G'$. Then $\text{Ker}(g)$ is not necessarily connected. Hence in the category of connected algebraic groups the canonical map

\[
\text{Coim}(g) = G/\text{Ker}(g)^o \to \text{Im}(g) = G'
\]

is not an isomorphism in general.

Note that the category of connected algebraic groups is fully embedded in $\mathcal{M}_1^a$. 

3
According to [13] we define the category $\mathcal{M}_1^a \otimes \mathbb{Q}$ of Laumon 1-motives up to isogenies: the objects are the same of $\mathcal{M}_1^a$; the Hom groups are $\text{Hom}_{\mathcal{M}_1^a}(M, M') \otimes \mathbb{Z} \otimes \mathbb{Q}$.

**Remark 1.6.** Note that a morphism $(f, g) : M \to M'$ is an isogeny (i.e. an isomorphism in $\mathcal{M}_1^a \otimes \mathbb{Q}$) if and only if $f$ is injective with finite co-kernel and $g$ is surjective with finite kernel.

**Proposition 1.7.** The category of Laumon 1-motives up to isogenies is abelian.

**Proof.** By construction $\mathcal{M}_1^a \otimes \mathbb{Q}$ is an additive category. Let $(f, g) : M \to M'$ be a morphism of Laumon 1-motives. We know that the group $\pi_0(\text{Ker}(g)) = \text{Ker}(g)/\text{Ker}(g)^0$ is a finite group scheme, hence there exists an integer $n$ such that the following diagram commutes in $\text{Ab}_k$

$$
\begin{array}{c}
\text{Ker}(f) \\
\downarrow \quad n^\ast \\
\text{Ker}(g)^0 \quad \longrightarrow \quad \text{Ker}(g) \quad \longrightarrow \quad \pi_0(\text{Ker}(g))
\end{array}
$$

Hence $n^\ast$ factors through $\text{Ker}(g)^0$. Then it is easy to check that $\text{Ker}((f, g)) = [(u^\ast \text{Ker}(g)^0) \to \text{Ker}(g)^0]$ is isogenous to $[\text{Ker}(f) \to \text{Ker}(g)^0]$.

It follows that $\text{Coi}(f, g)$ is isogenous to $[(\text{F} / \text{Ker}(f))_0 \to \text{G} / \text{Ker}(g)]$. As $\text{G} / \text{Ker}(g)^0 \to \text{G} / \text{Ker}(g)$ is an isogeny we get that the canonical map $\text{Coi}(f, g) \to \text{Im}(f, g)$ an isogeny too.

This is enough to prove that the category $\mathcal{M}_1^a \otimes \mathbb{Q}$ is abelian. \[\square\]

**Remark 1.8.** We can define the category $\mathcal{M}_1^a$ of 1-motives with torsion (over $k$) as the full sub-category of $D^b(\text{Ab}_k)$ with objects complexes $[u : F \to G]$ concentrated in degree 0, 1 such that

i) A (commutative) formal group $F$ over $k$, such that Lie $F$ is a finitely generated and $F(\bar{k}) = \lim_{\substack{\text{fin. gen.} \quad F(k') \text{ is finitely gen.} \quad \text{(non necessarily torsion-free!)}}} \text{Gal}(\bar{k}/k)$-module.

ii) A connected commutative algebraic group scheme $G$ over $k$.

iii) A morphism $u : F \to G$ in the category $\text{Ab}_k$.

Also we denote by $\mathcal{M}_1 \subset \mathcal{M}_1^a$ the full sub-category whose object are of the form $[u : F \to G]$ with $F^n = 0$ and $G$ semi-abelian.

In [13, C 7.3] is proven that the canonical functor $\mathcal{M}_1 \to \mathcal{M}_1$ induces an equivalence of the same categories up to isogeny, i.e. $\mathcal{M}_1 \otimes \mathbb{Q} \cong \mathcal{M}_1 \otimes \mathbb{Q}$. The same result holds for Laumon 1-motives, in fact all the arguments given in [13] work also in this setting. Hence there is an equivalence of categories

$$
\mathcal{M}_1^a \otimes \mathbb{Q} \cong \mathcal{M}_1^a \otimes \mathbb{Q}.
$$
A Deligne 1-motive is endowed with an increasing filtration (of sub-1-motives) called the weight filtration ([6] §10.1.4) defined as follows

$$W_i = W_i M := \begin{cases} [u : X \to G] & i \geq 0 \\ [0 \to G] & i = -1 \\ [0 \to T] & i = -2 \\ [0 \to 0] & i \leq -3 \end{cases}$$

hence we get

$$\text{gr}^W_i M = \begin{cases} [X \to 0] & i = 0 \\ [0 \to A] & i = -1 \\ [0 \to T] & i = -2 \\ [0 \to 0] & \text{otherwise} \end{cases}$$

According to [4] C.11.1 we extend the weight filtration to Laumon 1-motives. Let $M = [u : F \to G]$ be an Laumon 1-motive, then

$$W_{-3} = 0 \subset W_{-2} = [0 \to L] \subset W_{-1} = [0 \to G] \subset W_0 = M.$$ 

### 1.1 fppf sheaves

Let $\text{Sch}_k$ be the category of schemes over $k$ and $\text{Aff}_k$ be the full sub-category of affine schemes. According to [1] Exp. IV §6.3 the fppf topology on $\text{Sch}_k$ is the one generated by: the families of jointly surjective open immersions in $\text{Sch}_k$; the finite families of jointly surjective, flat, of finite presentation and quasi-finite morphisms in $\text{Aff}_k$.

Recall that $\text{Ab}_k$ is the category of abelian sheaves on $\text{Aff}_k$ w.r.t. the fppf topology.

**Proposition 1.9.** i) The category of commutative group schemes over $k$ is a full sub-category of $\text{Ab}_k$ via the functor of points $G \mapsto h_G := \text{Hom}_{\text{sch}_k}(-, G)$.

ii) Let $\text{char}(k) = 0$. The category of formal group schemes is a full sub-category of $\text{Ab}_k$ via the functor of points $F = \text{Spf}(A) \mapsto h_F := \text{Hom}^\text{cont}_{\text{alg}}(A, -)$ (where $\text{Hom}^\text{cont}_{\text{alg}}(-, -)$ denote the set of continuous homomorphisms of $k$-algebras).

**Proof.** By a result of Grothendieck ([8] Part I, §2.3.6)) every scheme (over $k$) is a sheaf (on sets) w.r.t. the fppf topology on $\text{Sch}_k$. Hence it is also a fppf-sheaf on the sub-category $\text{Aff}_k \subset \text{Sch}_k$. From this fact (i) and (ii) follow for étale formal groups.

By the decomposition theorem for formal groups over a perfect field [7] it remains to prove that any connected (or local) formal group is a sheaf. It is sufficient to note that

$$\widehat{\text{Gal}} \cong \text{colim} \text{Spec}(k[t]/(t^{n+1}))$$
is a direct limit of affine schemes, hence a direct limit of sheaves of sets w.r.t. the fppf topology.

2 Extensions

2.1 The group of $n$-extensions

Let $A$ be any abelian category (we don’t suppose it has enough injective objects), then we can define its derived category $D(A)$ and the group of $n$-fold extension classes

$$\text{Ext}^n_A(A, B) := \text{Hom}_{D(A)}(A, B[n]) \quad A, B \in A.$$ 

As usual we identify this group with the group of classes of Yoneda extensions, i.e. the set of exact sequences

$$0 \to B \to E_1 \to \cdots \to E_n \to A \to 0$$

modulo congruences (See [10] or [9]).

2.1.1 A lemma on 2-fold extensions

Consider a 2-fold extension $\gamma \in \text{Ext}^2_A(M, M')$. It is represented by an exact sequence

$$0 \to M' \to E_1 \to E_2 \to M \to 0. \quad (3)$$

This can be written as the product of two 1-fold extensions as follows. Let $E := \text{Ker}(E_2 \to M) = \text{Coker}(M' \to E_1)$, then let $\gamma_1 \in \text{Ext}^1_A(E, M')$, $\gamma_2 \in \text{Ext}^1_A(M, E)$ be the classes represented by

$$0 \to M' \to E_1 \to E \to 0, \quad 0 \to E \to E_2 \to M \to 0 \quad (4)$$

respectively. Then $\gamma = \gamma_1 \cdot \gamma_2$.

As a particular case, consider $W_{-2} \subset W_{-1} \subset W_0$ a sequence of objects of $A$. We have the following exact sequences

$$\gamma : \quad 0 \to W_{-2} \to W_{-1} \to W_0/W_{-2} \to W_0/W_{-1} \to 0$$

$$\gamma_1 : \quad 0 \to W_{-2} \to W_{-1} \to W_{-1}/W_{-2} \to 0$$

$$\gamma_2 : \quad 0 \to W_{-1}/W_{-2} \to W_0/W_{-2} \to W_0/W_{-1} \to 0$$

and $\gamma = \gamma_1 \cdot \gamma_2 \in \text{Ext}^2_A(W_0/W_{-1}, W_{-2})$. In this particular case we get

Lemma 2.1. $\gamma = 0$ in $\text{Ext}^2_A(W_0/W_{-1}, W_{-2})$.


□
2.2 Ext of 1-motives up to isogenies

From now on we call 1-motive a Laumon 1-motive (over \( k \)) and \( \text{Ext}^1_Q(M, M') \) is the group of classes of 1-fold extensions in \( M^a_1 \otimes Q \).

We are going to prove that \( M^a_1 \otimes Q \) is of cohomological dimension \( \leq 1 \). We start with the following result.

**Lemma 2.2.** Let \( \text{gr}^W_i M^a_1 \) be the full sub-category of \( M^a_1 \) with objects 1-motives pure of weight \( i \), \( i = 0, -1, -2 \). Then \( \text{gr}^W_i M^a_1 \otimes Q \) is an abelian thick sub-category of \( M^a_1 \otimes Q \) and it is of cohomological dimension 0.

**Proof.** First we consider the case \( M = F[1], M' = F'[1] \) pure of weight 0 (i.e. formal groups). Let \( 0 \to F'[1] \to E \to F[1] \to 0 \) an exact sequence of 1-motives modulo isogenies. Then \( E \) is also of weight 0 (this follows directly from the definitions). Hence \( \text{Ext}^1_Q(F[1], F'[1]) \) is isomorphic to the group of classes of extensions in the category of formal groups over \( k \) modulo isogenies. We know that of \( k \)-vector spaces \( \text{Mod}_k \) is semi-simple, and so is the \( Q \)-linearized category of free \( \text{Gal}(\bar{k}/k) \)-modules, \( \text{Mod}^{\text{free}}_\text{Gal}(\bar{k}/k) \otimes Q \), by the lemma of Maschke (See [13], p. 47), for the representations of finite groups; the case of pro-finite is a direct consequence. Hence the category of formal groups up to isogeny, equivalent to \( \text{Mod}_k \times \text{Mod}^{\text{free}}_\text{Gal}(\bar{k}/k) \otimes Q \), is of cohomological dimension 0.

The second case is that of abelian varieties (weight \(-1\)). Again using the definitions we get that \( \text{Ext}^1_Q(A', A) \) correspond to the group of extensions in the category of abelian varieties modulo isogenies. This group is zero (See [13], p. 173).

The third case is that of linear groups (weight \(-2\)). This can be reduced to the first case by Cartier duality (See [11], §5) or proved explicitly. \( \square \)

**Lemma 2.3.** Let \( M, M' \) be pure Laumon 1-motives with weights \( w < w' \). Then \( \text{Ext}^1_Q(M, M') = 0. \)

**Proof.** Fix a 2-fold extension \( \gamma \in \text{Ext}^1_Q(M, M') \) represented by

\[
0 \to M' \to E_1 \to E_2 \to M \to 0
\]

and take \( \gamma_2 \in \text{Ext}^1_Q(M, E), \; \gamma_1 \in \text{Ext}^1_Q(E, M') \) (as in \( \Theta \), \( \Delta \)) such that \( \gamma = \gamma_1 \cdot \gamma_2 \).

We have an exact sequence

\[
\text{Ext}^1_Q(M, W_{-1}E) \to \text{Ext}^1_Q(M, E) \to \text{Ext}^1_Q(M, \text{gr}_0 E)
\]

By assumption \(-2 \leq w < w' \leq 0\), then \( M \) is pure of weight \(-1\) or \(-2\). In this case we get easily \( \text{Ext}^1_Q(M, \text{gr}_0 E) = 0 \) and we can lift \( \gamma_2 \) to \( \gamma_2' \in \text{Ext}^1_Q(M, W_{-1}E) \). Then let \( \gamma_1' \) be the image of \( \gamma_1 \) via \( \text{Ext}^1_Q(E, M') \to \text{Ext}^1_Q(W_{-1}E, M') \). Now using

\[
\text{Ext}^1_Q(\text{gr}_{-1} E, M') \to \text{Ext}^1_Q(W_{-1} E, M') \to \text{Ext}^1_Q(W_{-2} E, M')
\]

7
we can reduce to consider $E$ pure of weight $-1$, in fact $\text{Ext}^1_Q(W_{-2}E, M') = 0$ because $w' > -2$. From this follows that $\gamma_1 = \gamma_2 = 0$. \hfill \square

**Lemma 2.4.** Let $M, M'$ be pure Laumon 1-motives with weights $w > w'$. Then $\text{Ext}^2_Q(M, M') = 0$.

**Proof.** As in the previous proof fix a 2-fold extension $\gamma \in \text{Ext}^2_Q(M, M')$ represented by

$$0 \to M' \to E_1 \to E_2 \to M \to 0$$

and take $\gamma_2 \in \text{Ext}^1_Q(M, E), \gamma_1 \in \text{Ext}^1_Q(E, M')$ (as in [3]) such that $\gamma = \gamma_1 \cdot \gamma_2$.

We have to consider three cases: (a) $M = F[1], M' = A[0]$; (b) $M = F[1], M' = L[0]$; (c) $M = A[0], M' = L[0]$. Where $F$ is a formal group, $A$ an abelian variety, $L$ a linear group.

Case (a): now $\gamma_1 \in \text{Ext}^1_Q(E, A), \gamma_2 \in \text{Ext}^1_Q(F[1], E)$. Then $E = [F' \to A]$ is such that $W_{-2}E = 0$. Consider the exact sequence

$$0 \to \text{gr}_{-1} E \to E \to \text{gr}_0 E \to 0$$

applying $\text{Hom}_Q(F[1], -)$ to it we get

$$\text{Ext}^1_Q(F[1], \text{gr}_{-1} E) \to \text{Ext}^1_Q(F[1], E) \to \text{Ext}^1_Q(F[1], \text{gr}_0 E)$$

We proved that $\text{Ext}^1_Q(F[1], \text{gr}_0 E) = 0$ so we can lift $\gamma_2$ to a class $\gamma_2' \in \text{Ext}^1_Q(F[1], \text{gr}_{-1} E)$ (This lifting is not canonical). Similarly using $\text{Hom}_Q(-, A)$ to it we get an exact sequence

$$\text{Ext}^1_Q(\text{gr}_0 E, A) \to \text{Ext}^1_Q(E, A) \to \text{Ext}^1_Q(\text{gr}_{-1} E, A)$$

and we can map $\gamma_1 \mapsto \gamma_1' \in \text{Ext}^1_Q(\text{gr}_{-1} E, A)$. By standard arguments it holds $\gamma_1' \cdot \gamma_2' = \gamma_1 \cdot \gamma_2 = \gamma$. Recalling that $\text{Ext}^1_Q(\text{gr}_{-1} E, A) = 0$ we get the result.

Case (c): is similar to case (a).

Case (b): now $\gamma \in \text{Ext}^1_Q(F[1], L)$. We want to reduce to the hypothesis of lemma [2.1]. Thus we have to show: we can take $E$ pure of weight 1 (i.e. an abelian variety); there exists a 1-motive $N$ such that $\gamma_1 \in \text{Ext}^1_Q(E, L)$ is represented by $0 \to W_{-2}N \to W_{-1}N \to \text{gr}_{-1}N \to 0$; $\gamma_2 \in \text{Ext}^1_Q(F[1], E)$ is represented by $0 \to \text{gr}_{-1} N \to W_0N/W_{-2} \to \text{gr}_0 N \to 0$.

We know that $\text{Ext}^1_Q(F[1], \text{gr}_0 E) = 0$, so like in case (a) we can lift $\gamma_2$ to a class $\gamma_2' \in \text{Ext}^1_Q(F[1], W_{-1}E)$. Let $\gamma_1'$ the image of $\gamma_1$ via $\text{Ext}^1_Q(E, L) \to \text{Ext}^1_Q(W_{-1}E, L)$. Hence $\gamma_1' \cdot \gamma_2' = \gamma_1 \cdot \gamma_2$.

Now we can suppose $E$ of weight $\leq -1$. Using the same argument we can lift $\gamma_1'$ to $\gamma_1'' \in \text{Ext}^1_Q(\text{gr}_{-1} E, L)$ (because $\text{Ext}^1_Q(\text{gr}_{-2} E, L) = 0$) and send $\gamma_2' \mapsto \gamma_2'' \in \text{Ext}^1_Q(F[1], \text{gr}_{-1} E)$. We proved that there exists an abelian variety $A, \gamma_1 \in \text{Ext}^1_Q(A, A), \gamma_2 \in \text{Ext}^1_Q(F[1], A)$, such that $\gamma_1 \cdot \gamma_2 = \gamma$. We
claim that $\gamma_1, \gamma_2$ can be represented by extensions in the category Laumon-1-motives. In fact let

$$\gamma_1 : \quad 0 \to L \xrightarrow{f \otimes n^{-1}} G \xrightarrow{g \otimes m^{-1}} A \to 0$$

be an extension in the category of 1-motives modulo isogenies: $f, g$ are morphism of algebraic groups, $n, m \in \mathbb{Z}$. Then consider the push-forward by $n^{-1}$ and the pull-back by $m^{-1}$, we get the following commutative diagram with exact rows in $\mathcal{M}_1^{a,fr} \otimes \mathbb{Q}$.

\[
\begin{array}{cccccc}
0 & \to & L & \xrightarrow{f/n} & G & \xrightarrow{g/m} & A & \to & 0 \\
& & \downarrow{id} & & \downarrow{id} & & \\
0 & \to & L & \xrightarrow{f} & G & \xrightarrow{g} & A & \to & 0 \\
& & \downarrow{id} & & \downarrow{id} & & \\
0 & \to & L & \xrightarrow{f} & G & \xrightarrow{g} & A & \to & 0
\end{array}
\]

The exactness of the last row is equivalent to the following: Ker $f$ is finite; let $(\text{Ker } g)^0$ be the connected component of Ker $g$, then $\text{Im } f \to (\text{Ker } g)^0$ is surjective with finite kernel $K$; $g$ is surjective. So after replacing $L, A$ with isogenous groups we have an exact sequence in $\mathcal{M}_1^{a,fr}$

$$0 \to L \to G \to A \to 0$$

Explicitly

\[
\begin{array}{cccccc}
0 & \to & L & \xrightarrow{f} & G & \xrightarrow{g} & A & \to & 0 \\
& & \downarrow{id} & & \downarrow{id} & & \\
0 & \to & L/\text{Ker } f & \xrightarrow{g} & A & \to & 0 \\
& & \downarrow{id} & & \downarrow{id} & & \\
0 & \to & \text{Im } f/K & \xrightarrow{g} & A & \to & 0 \\
& & \downarrow{id} & & \downarrow{id} & & \\
0 & \to & \text{Im } f/K & \xrightarrow{g'} & G/(\text{Ker } g)^0 & \to & 0
\end{array}
\]

With similar arguments we can prove that $\gamma_2$ is represented by an extension in the category $\mathcal{M}_1^{a,fr}$

$$0 \to A \to N \to F[1] \to 0$$

with $N = [u : F \to A]$.

To apply lemma [21] we need to prove that there is lifting $u' : F \to G$. First suppose $F = F_{et}$: consider the long exact sequence

$$\text{Hom}_{Ab}(F, G) \to \text{Hom}_{Ab}(F, A) \to \text{Ext}^1_{Ab}(F, L)$$
We know ([12]) that $\text{Ext}^1_{\text{Ab}_k}(F, L)$ is a torsion group. So modulo replacing $F$ with an isogenous lattice we get $\partial u = 0$ and the lift exists. In case $F = F^o$ is a connected formal group we have a commutative diagram in $\text{Ab}_k$

\[
\begin{array}{ccc}
\hat{G} & \xrightarrow{\hat{\pi}} & \hat{A} \\
\downarrow & & \downarrow \pi \\
G & \xrightarrow{\pi} & A
\end{array}
\]

where $\hat{?}$ is the formal completion at the origin of $? = G, A$. The formal completion is an exact functor so $\hat{\pi}$ is an epimorphism. The category of formal groups is of cohomological dimension 0, then we can choose a section of $\hat{\pi}$ and lift $u$. \hfill \Box

**Theorem 2.5.** The category $\mathcal{M}_1^\pi \otimes \mathbb{Q}$ is of cohomological dimension 1.

**Proof.** First note that we can restrict to consider pure motives $M, M'$ (a 1-motive is pure if it is isomorphic to one of its graded pieces w.r.t. the weight filtration). In fact given $M, M'$ 1-motives, not necessarily pure, we have the canonical exact sequences given by the weight filtration

\[
0 \to W_{-1}M' \to M' \to \text{gr}_W^0 M' \to 0
\]
\[
0 \to W_{-2}M' \to W_{-1}M' \to \text{gr}_W^{-1} M' \to 0
\]

Hence applying $\text{Hom}_\mathbb{Q}(M, -)$ we get two long exact sequences

\[
\cdots \text{Ext}_2^\mathbb{Q}(M, W_{-1}M') \to \text{Ext}_2^\mathbb{Q}(M, M') \to \text{Ext}_2^\mathbb{Q}(M, \text{gr}_W^0 M') \cdots
\]
\[
\cdots \text{Ext}_2^\mathbb{Q}(M, W_{-2}M') \to \text{Ext}_2^\mathbb{Q}(M, W_{-1}M') \to \text{Ext}_2^\mathbb{Q}(M, \text{gr}_W^{-1} M') \cdots
\]

from this follows that we can reduce to prove $\text{Ext}_{\mathbb{Q}}^2(M, M') = 0$ for $M'$ pure. In the same way we reduce to consider $M$ pure.

Then using the previous three lemmas we conclude. \hfill \Box

**References**


