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THE COAGULATION - FRAGMENTATION EQUATION AND ITS STOCHASTIC COUNTERPART

EDUARDO CEPEDA

ABSTRACT. We consider a coagulation multiple-fragmentation equation, which describes the concentration $c_t(x)$ of particles of mass $x \in (0, \infty)$ at the instant $t \geq 0$ in a model where fragmentation and coalescence phenomena occur. We study the existence and uniqueness of measured-valued solutions to this equation for homogeneous-like kernels of homogeneity parameter $\lambda \in (0, 1]$ and bounded fragmentation kernels, although a possibly infinite number of fragments is considered. We also study a stochastic counterpart of this equation where a similar result is shown. We ask to the initial state to have a finite λ -moment.

This work relies on the use of a Wasserstein-type distance, which has shown to be particularly well-adapted to coalescence phenomena. It was introduced in previous works on coagulation and coalescence.

Mathematics Subject Classification (2000): 45K05, 60K35.

Keywords: Coagulation Multi-Fragmentation equation, Coalescence - Fragmentation process, Interacting stochastic particle systems.

1. INTRODUCTION

The coagulation-fragmentation equation is a deterministic equation that models the evolution in time of a system of a very big number of particles (mean-field description) undergoing coalescences and fragmentations. The particles in the system grow and decrease due to successive mergers and dislocations, each particle is fully identified by its mass $x \in (0, \infty)$, we do not consider its position in space, its shape nor other geometrical properties. Examples of applications of these models arise in polymers, aerosols and astronomy.

The first works (see [1, 8, 6]) were concentrated on the binary fragmentation where the particles dislocate only into two particles :

Binary Model .- Denoting $c_t(x)$ the concentration of particles of mass $x \in (0, \infty)$ at time t , the dynamics of c is given by

$$\begin{aligned} \partial_t c_t(x) &= \frac{1}{2} \int_0^x K(y, x-y) c_t(y) c_t(x-y) dy - c_t(x) \int_0^\infty K(x, y) c_t(y) dy \\ &\quad + \int_x^\infty F(x, y-x) c_t(y) dy - \frac{1}{2} c_t(x) \int_0^x F(y, x-y) dy, \end{aligned}$$

for $(t, x) \in (0, \infty)^2$. The coagulation kernel $K(x, y) = K(y, x) \geq 0$ models the likelihood that two particles with respective masses x and y merge into a single one with mass $x + y$. On the other hand, the fragmentation kernel F is also a symmetric function and $F(x, y)$ is the rate of fragmentation of particles of mass $x + y$ into particles of masses x and y .

The coagulation-only ($F \equiv 0$) equation is known as Smoluchowski's equation and it has been studied by several authors, Norris in [14] gives the first general well-posedness result

and Fournier and Laurençot [9] give a result of existence and uniqueness of a measured-valued solution for a class of homogeneous-like kernels. The fragmentation-only ($K \equiv 0$) equation has been studied in [3, 13]. In particular, Bertoin characterized the self-similar fragmentations using a fragmentation kernel of the type $F(x) = x^\alpha$ for $\alpha \in \mathbb{R}$ and where the particles may undergo multi-fragmentations.

We are interested in the version of the equation which takes into account a mechanism of dislocation with a possibly infinite number of fragments:

Multifragmentation Model .- Denoting as before $c_t(x)$ the concentration of particles of mass $x \in (0, \infty)$ at time t , the dynamics of c is given by

$$\begin{aligned} \partial_t c_t(x) &= \frac{1}{2} \int_0^x K(y, x-y) c_t(y) c_t(x-y) dy - c_t(x) \int_0^\infty K(x, y) c_t(y) dy \\ &+ \int_\Theta \left[\sum_{i=1}^\infty \frac{1}{\theta_i} F\left(\frac{x}{\theta_i}\right) c_t\left(\frac{x}{\theta_i}\right) - F(x) c_t(x) \right] \beta(d\theta). \end{aligned} \quad (1.1)$$

This equation describes two phenomena. On the one hand, the coalescence of two particles of mass x and y giving birth a new one of mass $x + y$, $\{x, y\} \rightarrow x + y$ with a rate proportional to the coagulation kernel $K(x, y)$. On the other hand, the fragmentation of a particle of mass x giving birth a new set of smaller particles $x \rightarrow \{\theta_1 x, \theta_2 x, \dots\}$, where $\theta_i x$ represents the fragments of x , with a rate proportional to $F(x)\beta(\theta)$ and where $F : (0, \infty) \rightarrow (0, \infty)$ and β is a positive measure on the set $\Theta = \left\{ \theta = (\theta_i)_{i \geq 1}, \text{ with } \sum_{i \geq 1} \theta_i \leq 1 \right\}$.

Note that we can obtain the continuous coagulation binary-fragmentation equation, for example, by considering β with support in $\{\theta : \theta_1 + \theta_2 = 1\}$ and $\beta(d\theta) = b(\theta_1) d\theta_1 \delta_{\{\theta_2=1-\theta_1\}}$, and setting $F(x, y) = F(x+y)b\left(\frac{x}{x+y}\right)$ where $b(\cdot)$ is a continuous function on $[0, 1]$ and symmetric at $1/2$.

The study of the coagulation-fragmentation is more recent, for example in [15, 16, 6] the authors give a result of existence and uniqueness to the binary fragmentation model. In [12, 11] a well-posedness result is given for a multi-fragmentation model, where the existence holds in the functional set $X = \{f \in L^1(0, \infty) : \int_0^\infty (1+x)|f(x)|dx < \infty\}$. The authors used a compactness method.

In this paper we extend the method in [9] concerning only coagulation, and we show existence and uniqueness to (1.1) for a class of homogeneous-like coagulation kernels and bounded fragmentation kernels, in the class of measures having a finite moment of order the degree of homogeneity of the coagulation kernel. Unfortunately this method does not extend to unbounded fragmentation kernels. Our assumptions on F are not very restrictive for small masses, since we do not ask to F to be zero on a neighbourhood of 0. On the other hand, we control the big masses imposing to the fragmentation kernel to be bounded near infinity.

We also study the existence and uniqueness of a stochastic process of coalescence - fragmentation. We follow the same ideas in [10], we construct a stochastic particle system. We point out that the mass-conservation property allows us to consider in particular self-similar fragmentation kernels as defined in [3] and, more generally, unbounded fragmentation kernels.

The paper is organized as follows: the deterministic equation (1.1) is studied in Sections 2 and 3. A stochastic counterpart is studied in Sections 4, 5, 6 and 7 and in Appendix A we give some technical details which are useful in this case.

2. THE COAGULATION MULTI-FRAGMENTATION EQUATION.- NOTATION AND DEFINITIONS

We first give some notation and definitions. We consider the set of non-negative Radon measures \mathcal{M}^+ and for $\lambda \in \mathbb{R}$ and $c \in \mathcal{M}^+$, we set

$$M_\lambda(c) := \int_0^\infty x^\lambda c(dx), \quad \mathcal{M}_\lambda^+ = \{c \in \mathcal{M}^+, M_\lambda(c) < \infty\}.$$

Next, for $\lambda \in (0, 1]$ we introduce the space \mathcal{H}_λ of test functions,

$$\mathcal{H}_\lambda = \left\{ \phi \in \mathcal{C}([0, \infty)) \text{ such that } \phi(0) = 0 \text{ and } \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^\lambda} < \infty \right\}.$$

Note that $\mathcal{C}_c^1((0, \infty)) \subset \mathcal{H}_\lambda$.

Here and below, we use the notation $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$ for $(x, y) \in (0, \infty)^2$.

Hypothesis 2.1 (Coagulation and Fragmentation Kernels). *Consider $\lambda \in (0, 1]$ and a symmetric coagulation kernel $K : (0, \infty) \times (0, \infty) \rightarrow [0, \infty)$ i.e., $K(x, y) = K(y, x)$. Assume that K belongs to $W^{1, \infty}((\varepsilon, 1/\varepsilon)^2)$ for every $\varepsilon > 0$ and that it satisfies*

$$K(x, y) \leq \kappa_0(x + y)^\lambda, \quad (2.1)$$

$$(x^\lambda \wedge y^\lambda) |\partial_x K(x, y)| \leq \kappa_1 x^{\lambda-1} y^\lambda, \quad (2.2)$$

for all $(x, y) \in (0, \infty)^2$ and for some positive constants κ_0 and κ_1 . Consider also a fragmentation kernel $F : (0, \infty) \rightarrow [0, \infty)$ and assume that F belongs to $W^{1, \infty}((\varepsilon, 1/\varepsilon))$ for every $\varepsilon > 0$ and that it satisfies

$$F(x) \leq \kappa_2, \quad (2.3)$$

$$|F'(x)| \leq \kappa_3 x^{-1}, \quad (2.4)$$

for $x \in (0, \infty)$ and some positive constants κ_2 and κ_3 .

For example, the coagulation kernels listed below, taken from the mathematical and physical literature, satisfy Hypothesis 2.1.

$$\begin{aligned} K(x, y) &= (x^\alpha + y^\alpha)^\beta && \text{with } \alpha \in (0, \infty), \beta \in (0, \infty) \text{ and } \lambda = \alpha\beta \in (0, 1], \\ K(x, y) &= x^\alpha y^\beta + x^\beta y^\alpha && \text{with } 0 \leq \alpha \leq \beta \leq 1 \text{ and } \lambda = \alpha + \beta \in (0, 1], \\ K(x, y) &= (xy)^{\alpha/2} (x + y)^{-\beta} && \text{with } \alpha \in (0, 1], \beta \in [0, \infty) \text{ and } \lambda = \alpha - \beta \in (0, 1], \\ K(x, y) &= (x^\alpha + y^\alpha)^\beta |x^\gamma - y^\gamma| && \text{with } \alpha \in (0, \infty), \beta \in (0, \infty), \gamma \in (0, 1] \text{ and } \lambda = \alpha\beta + \gamma \in (0, 1], \\ K(x, y) &= (x + y)^\lambda e^{-\beta(x+y)^{-\alpha}} && \text{with } \alpha \in (0, \infty), \beta \in (0, \infty), \text{ and } \lambda \in (0, 1]. \end{aligned}$$

On the other hand, the following fragmentation kernels satisfy Hypothesis 2.1.

$$\begin{aligned} F(x) &\equiv 1, \\ &\text{all non-negative function } F \in C^2(0, \infty), \text{ bounded, convex and non-decreasing,} \\ &\text{all non-negative function } F \in C^2(0, \infty), \text{ bounded, concave and non-increasing.} \end{aligned}$$

We define the set of ratios by

$$\Theta = \{\theta = (\theta_k)_{k \geq 1} : 1 > \theta_1 \geq \theta_2 \geq \dots \geq 0\}.$$

Hypothesis 2.2 (The β measure.-). *We consider on Θ a measure $\beta(\cdot)$ and assume that it satisfies*

$$\beta \left(\sum_{k \geq 1} \theta_k > 1 \right) = 0, \quad (2.5)$$

$$C_\beta^\lambda := \int_\Theta \sum_{k \geq 2} \theta_k^\lambda \beta(d\theta) < \infty \quad \text{for some } \lambda \in (0, 1]. \quad (2.6)$$

Remark 2.3. *i) The property (2.5) means that there is no gain of mass due to the dislocation of a particle. Nevertheless, it does not exclude a loss of mass due to the dislocation of the particles.*

ii) Note that under (2.5) we have $\sum_{k \geq 1} \theta_k - 1 \leq 0$ β -a.e., and since $\theta_k \in [0, 1)$ for all $k \geq 1$, $\theta_k \leq \theta_k^\lambda$, we have

$$\begin{cases} 1 - \theta_1^\lambda \leq 1 - \theta_1 \leq (1 - \theta_1)^\lambda \leq \left(\sum_{k \geq 2} \theta_k \right)^\lambda \leq \sum_{k \geq 2} \theta_k^\lambda & \beta - \text{a.e.}, \\ \sum_{k \geq 1} \theta_k^\lambda - 1 = \sum_{k \geq 2} \theta_k^\lambda - (1 - \theta_1^\lambda) \leq \sum_{k \geq 2} \theta_k^\lambda, \end{cases} \quad (2.7)$$

implying the following bounds:

$$\begin{cases} \int_\Theta (1 - \theta_1) \beta(d\theta) \leq C_\beta^\lambda, & \int_\Theta (1 - \theta_1^\lambda) \beta(d\theta) \leq C_\beta^\lambda, \\ \int_\Theta (1 - \theta_1)^\lambda \beta(d\theta) \leq C_\beta^\lambda, & \int_\Theta \left(\sum_{k \geq 1} \theta_k^\lambda - 1 \right) \beta(d\theta) \leq C_\beta^\lambda. \end{cases} \quad (2.8)$$

Definition 2.4 (Weak solution to (1.1)). *Let $c^{in} \in M_\lambda^+$. A family $(c_t)_{t \geq 0} \subset M^+$ is a $(c^{in}, K, F, \beta, \lambda)$ -weak solution to (1.1) if $c_0 = c^{in}$,*

$$t \mapsto \int_0^\infty \phi(x) c_t(dx) \text{ is differentiable on } [0, \infty)$$

for each $\phi \in \mathcal{H}_\lambda$, and for every $t \in [0, \infty)$,

$$\sup_{s \in [0, t]} M_\lambda(c_s) < \infty, \quad (2.9)$$

and for all $\phi \in \mathcal{H}_\lambda$

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \phi(x) c_t(dx) &= \frac{1}{2} \int_0^\infty \int_0^\infty K(x, y) (A\phi)(x, y) c_t(dx) c_t(dy) \\ &+ \int_0^\infty F(x) \int_\Theta (B\phi)(\theta, x) \beta(d\theta) c_t(dx), \end{aligned} \quad (2.10)$$

where the functions $(A\phi) : (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ and $(B\phi) : \Theta \times (0, \infty) \rightarrow \mathbb{R}$ are defined by

$$(A\phi)(x, y) = \phi(x + y) - \phi(x) - \phi(y), \quad (2.11)$$

$$(B\phi)(\theta, x) = \sum_{i=1}^{\infty} \phi(\theta_i x) - \phi(x). \quad (2.12)$$

This equation can be split into two parts, the first integral explains the evolution in time of the system under coagulation and the second integral explains the behaviour of the system when undergoing fragmentation and it corresponds to a growth in the number of particles of

masses $\theta_1 x, \theta_2 x, \dots$, and to a decrease in the number of particles of mass x as a consequence of their fragmentation.

According to (2.1), (2.3), Lemma 3.1. below, (2.9) and (2.6), the integrals in (2.10) are absolutely convergent and bounded with respect to $t \in [0, s]$ for every $s \geq 0$.

The main result reads as follows.

Theorem 2.5. *Consider $\lambda \in (0, 1]$ and $c^{in} \in \mathcal{M}_\lambda^+$. Assume that the coagulation kernel K , the fragmentation kernel F and the measure β satisfy Hypotheses 2.1. and 2.2 with the same λ .*

Then, there exists a unique $(c^{in}, K, F, \beta, \lambda)$ -weak solution to (1.1).

It is important to note that the main interest of this result is that only one moment is asked to the initial condition c^{in} . The assumptions on the coagulation kernel K and the measure β are reasonable. Whereas the main limitation is that we need to assume that the fragmentation kernel is bounded. It is also worth to point out that we have chosen to study this version of the equation because of its easy physical intuition.

For other result on well-posedness of the coagulation multi-fragmentation equation we refer to [11, 12]. Roughly, the solution is given in a functional space (the solutions are not measures) and it is assumed for the initial condition that $M_0(c^{in}) + M_1(c^{in}) < \infty$. The coagulation kernel is assumed to satisfy $K(x, y) \leq C(1+x)^\mu(1+y)^\mu$ with $\mu \in [0, 1)$, the number of fragments on each dislocation is assumed to be bounded by N and the measure β is supposed to be integrable. However, F (or its equivalent) is not assumed to be bounded.

3. PROOFS

We begin giving some properties of the operators $(A\phi)$ and $(B\phi)$ for $\phi \in \mathcal{H}_\lambda$ which allow us to justify the weak formulation (2.10).

Lemma 3.1. *Consider $\lambda \in (0, 1]$, $\phi \in \mathcal{H}_\lambda$. Then there exists C_ϕ depending on ϕ , θ and λ such that*

$$\begin{aligned} (x+y)^\lambda |(A\phi)(x, y)| &\leq C_\phi (xy)^\lambda, \\ |(B\phi)(\theta, x)| &\leq C_\phi x^\lambda \sum_{i \geq 2} \theta_i^\lambda, \end{aligned}$$

for all $(x, y) \in (0, \infty)^2$ and for all $\theta \in \Theta$.

Prof of Lemma 3.1. For $(A\phi)$ we recall [9, Lemma 3.1]. Next, consider $\lambda \in (0, 1]$ and $\phi \in \mathcal{H}_\lambda$ then, since $\phi(0) = 0$,

$$\begin{aligned} |(B\phi)(\theta, x)| &\leq |\phi(\theta_1 x) - \phi(x)| + \sum_{i \geq 2} |\phi(\theta_i x) - \phi(0)| \\ &\leq C_\phi x^\lambda (1 - \theta_1)^\lambda + C_\phi x^\lambda \sum_{i \geq 2} \theta_i^\lambda \leq C_\phi x^\lambda \sum_{i \geq 2} \theta_i^\lambda. \end{aligned}$$

We used (2.7). □

We are going to work with a distance between solutions depending on λ . This distance involves the primitives of the solution of (1.1), thus we recall [9, Lemma 3.2].

Lemma 3.2. For $c \in \mathcal{M}^+$ and $x \in (0, \infty)$, we put

$$F^c(x) := \int_0^\infty \mathbb{1}_{(x, \infty)}(y) c(dy),$$

If $c \in \mathcal{M}_\lambda^+$ for some $\lambda \in (0, 1]$, then

$$\int_0^\infty x^{\lambda-1} F^c(x) dx = M_\lambda(c)/\lambda, \quad \lim_{x \rightarrow 0} x^\lambda F^c(x) = \lim_{x \rightarrow \infty} x^\lambda F^c(x) = 0,$$

and $F^c \in L^\infty(\varepsilon, \infty)$ for each $\varepsilon > 0$.

We give now a very important inequality on which the existence and uniqueness proof relies.

Proposition 3.3. Consider $\lambda \in (0, 1]$, a coagulation kernel K , a fragmentation kernel F and a measure β on Θ satisfying Hypotheses 2.1. and 2.2. with the same λ . Let c^{in} and $d^{in} \in \mathcal{M}_\lambda^+$ and denote by $(c_t)_{t \in [0, \infty)}$ a $(c^{in}, K, F, \beta, \lambda)$ -weak solution to (2.10) and by $(d_t)_{t \in [0, \infty)}$ a $(d^{in}, K, F, \beta, \lambda)$ -weak solution to (2.10). In addition, we put $E(t, x) = F^{c_t}(x) - F^{d_t}(x)$, $\rho(x) = x^{\lambda-1}$ and

$$R(t, x) = \int_0^x \rho(z) \text{sign}(E(t, z)) dz \text{ for } (t, x) \in [0, \infty) \times (0, \infty).$$

Then, for each $t \in [0, \infty)$, $R(t, \cdot) \in \mathcal{H}_\lambda$ and

$$\begin{aligned} \frac{d}{dt} \int_0^\infty x^{\lambda-1} |E(t, x)| dx &\leq \frac{1}{2} \int_0^\infty \int_0^\infty K(x, y) [\rho(x+y) - \rho(x)] (c_t + d_t)(dy) |E(t, x)| dx \\ &\quad + \frac{1}{2} \int_0^\infty \int_0^\infty \partial_x K(x, y) (AR(t))(x, y) (c_t + d_t)(dy) E(t, x) dx \\ &\quad + \int_0^\infty F'(x) \int_\Theta (BR(t))(\theta, x) \beta(d\theta) E(t, x) dx \\ &\quad + \int_0^\infty F(x) x^{\lambda-1} |E(t, x)| \int_\Theta \left(\sum_{i \geq 1} \theta_i^\lambda - 1 \right) \beta(d\theta) dx. \end{aligned} \quad (3.1)$$

Before to give the proof of Proposition 3.3., we state two auxiliary results. In Lemma 3.4. are given some inequalities which are useful to verify that the integrals on the right-hand side of (3.1) are convergent, and in Lemma 3.5. we study the time differentiability of E .

Lemma 3.4. Under the notation and assumptions of Proposition 3.3., there exists a positive constant C such that for $(t, x, y) \in [0, \infty) \times (0, \infty)^2$,

$$\begin{aligned} K(x, y) |\rho(x+y) - \rho(x)| &\leq C x^{\lambda-1} y^\lambda, \\ K(x, y) |(AR(t))(x, y)| &\leq C x^\lambda y^\lambda, \\ |\partial_x K(x, y) (AR(t))(x, y)| &\leq C x^{\lambda-1} y^\lambda, \\ \int_\Theta |(BR(t))(\theta, x)| \beta(d\theta) &\leq C C_\beta^\lambda x^\lambda. \end{aligned} \quad (3.2)$$

Proof. The first three inequalities were proved in [9, Lemma 3.4]. In particular, recall that

$$|(AR(t))(x, y)| \leq \frac{2}{\lambda} (x \wedge y)^\lambda, \quad (3.3)$$

for $(t, x, y) \in [0, \infty) \times (0, \infty)^2$. Next, using (2.8) and (2.6) we deduce

$$\begin{aligned}
\int_{\Theta} |(BR(t))(\theta, x)| \beta(d\theta) &= \left| \int_{\Theta} \left[\sum_{i \geq 1} R(t, \theta_i x) - R(t, x) \right] \beta(d\theta) \right| \\
&= \int_{\Theta} \left| \sum_{i \geq 2} \int_0^{\theta_i x} \partial_x R(t, z) dz - \int_{\theta_1 x}^x \partial_x R(t, z) dz \right| \beta(d\theta) \\
&\leq \int_{\Theta} \left(\sum_{i \geq 2} \int_0^{\theta_i x} z^{\lambda-1} dz + \int_{\theta_1 x}^x z^{\lambda-1} dz \right) \beta(d\theta) \\
&\leq \frac{2}{\lambda} C_{\beta}^{\lambda} x^{\lambda}.
\end{aligned}$$

□

Lemma 3.5. Consider $\lambda \in (0, 1]$, a coagulation kernel K , a fragmentation kernel F and a measure β on Θ satisfying the Hypotheses 2.1. with the same λ . Let $c^{in} \in \mathcal{M}_{\lambda}^{+}$ and denote by $(c_t)_{t \in [0, \infty)}$ a $(c^{in}, K, F, \beta, \lambda)$ -weak solution to (2.10). Then

$(x, t) \mapsto \partial_t F^{c_t}(x)$ belongs to $L^{\infty}(0, s; L^1(0, \infty; x^{\lambda-1} dx))$, for each $s \in [0, \infty)$.

Proof. Following the same ideas as in [9], we consider $\vartheta \in \mathcal{C}([0, \infty))$ with compact support in $(0, \infty)$, we put

$$\phi(x) = \int_0^x \vartheta(y) dy, \quad \text{for } x \in (0, \infty),$$

this function belongs to \mathcal{H}_{λ} . First, performing an integration by parts and using Lemma 3.2. we obtain

$$\int_0^{\infty} \vartheta(x) F^{c_t}(x) dx = \int_0^{\infty} \phi(x) c_t(dx).$$

Next, on the one hand recall that in [9, eq. (3.7)] was proved that

$$\begin{aligned}
\int_0^{\infty} \int_0^{\infty} K(x, y) (A\phi)(x, y) c_t(dy) c_t(dx) dz \\
= \int_0^{\infty} \vartheta(z) \int_0^z \int_0^z \mathbb{1}_{[z, \infty)}(x+y) K(x, y) c_t(dy) c_t(dx) dz \\
- \int_0^{\infty} \vartheta(z) \int_z^{\infty} \int_z^{\infty} K(x, y) c_t(dy) c_t(dx) dz.
\end{aligned}$$

On the other hand, using the Fubini Theorem, we have

$$\begin{aligned}
\int_0^{\infty} F(x) \int_{\Theta} (B\phi)(\theta, x) \beta(d\theta) c_t(dx) \\
= \int_0^{\infty} F(x) \int_{\Theta} \left[\sum_{i \geq 1} \int_0^{\theta_i x} \vartheta(z) dz - \int_0^x \vartheta(z) dz \right] \beta(d\theta) c_t(dx) \\
= \int_0^{\infty} \vartheta(z) \int_{\Theta} \left[\sum_{i \geq 1} \int_{z/\theta_i}^{\infty} F(x) c_t(dx) - \int_z^{\infty} F(x) c_t(dx) \right] \beta(d\theta) dz.
\end{aligned}$$

Thus, from (2.10) we infer that

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \vartheta(x) F^{c_t}(x) dx &= \frac{1}{2} \int_0^\infty \vartheta(z) \int_0^z \int_0^z \mathbb{1}_{[z, \infty)}(x+y) K(x, y) c_t(dy) c_t(dx) dz \\ &\quad - \frac{1}{2} \int_0^\infty \vartheta(z) \int_z^\infty \int_z^\infty K(x, y) c_t(dy) c_t(dx) dz \\ &\quad + \int_0^\infty \vartheta(z) \int_\Theta \left[\sum_{i \geq 1} \int_{z/\theta_i}^\infty F(x) c_t(dx) - \int_z^\infty F(x) c_t(dx) \right] \beta(d\theta) dz, \end{aligned}$$

whence

$$\begin{aligned} \partial_t F^{c_t}(z) &= \frac{1}{2} \int_0^z \int_0^z \mathbb{1}_{[z, \infty)}(x+y) K(x, y) c_t(dy) c_t(dx) - \frac{1}{2} \int_z^\infty \int_z^\infty K(x, y) c_t(dy) c_t(dx) \\ &\quad + \int_\Theta \left[\sum_{i \geq 1} \int_{z/\theta_i}^\infty F(x) c_t(dx) \beta(d\theta) - \int_z^\infty F(x) c_t(dx) \right] \beta(d\theta), \end{aligned} \quad (3.4)$$

for $(t, z) \in [0, \infty) \times (0, \infty)$. First, in [9, Lemma 3.5] it was shown that,

$$\begin{aligned} \int_0^\infty z^{\lambda-1} \left| \frac{1}{2} \int_0^z \int_0^z \mathbb{1}_{[z, \infty)}(x+y) K(x, y) c_t(dy) c_t(dx) - \frac{1}{2} \int_z^\infty \int_z^\infty K(x, y) c_t(dy) c_t(dx) \right| dz \\ \leq \frac{2\kappa_0}{\lambda} M_\lambda(c_t)^2. \end{aligned}$$

Thus, from (2.3) and the Fubini Theorem follows that, for each $t \in [0, \infty)$,

$$\begin{aligned} \int_0^\infty z^{\lambda-1} |\partial F^{c_t}(z)| dz &\leq \frac{2\kappa_0}{\lambda} M_\lambda(c_t)^2 \\ &\quad + \int_0^\infty z^{\lambda-1} \left| \int_\Theta \left(\sum_{i \geq 2} \int_{z/\theta_i}^\infty F(x) c_t(dx) - \int_z^{z/\theta_1} F(x) c_t(dx) \right) \right| \beta(d\theta) dz \\ &\leq \frac{2\kappa_0}{\lambda} M_\lambda(c_t)^2 + \kappa_2 \int_\Theta \int_0^\infty \left(\sum_{i \geq 2} \int_0^{\theta_i x} z^{\lambda-1} dz + \int_{\theta_1 x}^x z^{\lambda-1} dz \right) c_t(dx) \beta(d\theta) \\ &\leq \frac{2\kappa_0}{\lambda} M_\lambda(c_t)^2 + \frac{\kappa_2}{\lambda} M_\lambda(c_t) \left[\int_\Theta \left(\sum_{i \geq 2} \theta_i^\lambda + (1 - \theta_1^\lambda) \right) \beta(d\theta) \right] \\ &\leq \frac{2\kappa_0}{\lambda} M_\lambda(c_t)^2 + \frac{2C_\beta^\lambda \kappa_2}{\lambda} M_\lambda(c_t), \end{aligned}$$

where we have used (2.8). Finally, since the right-hand side of the above inequality is bounded on $[0, t]$ for all $t > 0$ by (2.9), we obtain the expected result. \square

Proof of Proposition 3.3. Let $t \in [0, \infty)$. We first note that, since $s \mapsto M_\lambda(c_s)$ and $s \mapsto M_\lambda(d_s)$ are in $L^\infty(0, t)$ by (2.9), it follows from Lemmas 3.2. and 3.4. that the integrals in (3.1) are

absolutely convergent. Furthermore, for $t \geq 0$ and $x > y$, we have

$$\begin{aligned} |R(t, x) - R(t, y)| &= \left| \int_y^x z^{\lambda-1} \text{sign}(E(t, z)) dz \right| \\ &\leq \frac{1}{\lambda} (x^\lambda - y^\lambda) = \frac{1}{\lambda} ((x - y + y)^\lambda - y^\lambda) \\ &\leq \frac{1}{\lambda} (x - y)^\lambda, \end{aligned}$$

since $\lambda \in (0, 1]$. Thus $R(t, \cdot) \in \mathcal{H}_\lambda$ for each $t \in [0, \infty)$.

Next, by Lemmas 3.2 and 3.5, $E \in W^{1, \infty}(0, s; L^1(0, \infty; x^{\lambda-1} dx))$ for every $s \in (0, T)$, so that

$$\begin{aligned} \frac{d}{dt} \int_0^\infty x^{\lambda-1} |E(t, x)| dx &= \int_0^\infty x^{\lambda-1} \text{sign}(E(t, x)) \partial_t E(t, x) dx \\ &= \int_0^\infty \partial_x R(t, x) (\partial_t F^{c_t}(x) - \partial_t F^{d_t}(x)) dx. \end{aligned}$$

We use (3.4) to obtain

$$\begin{aligned} &\frac{d}{dt} \int_0^\infty x^{\lambda-1} |E(t, x)| dx \\ &= \frac{1}{2} \int_0^\infty \partial_x R(t, z) \int_0^z \int_0^z \mathbf{1}_{[z, \infty)}(x+y) K(x, y) (c_t(dy) c_t(dx) - d_t(dy) d_t(dx)) dz \\ &\quad - \frac{1}{2} \int_0^\infty \partial_x R(t, z) \int_z^\infty \int_z^\infty K(x, y) (c_t(dy) c_t(dx) - d_t(dy) d_t(dx)) dz \\ &\quad + \int_0^\infty \partial_x R(t, z) \int_\Theta \left[\sum_{i \geq 1} \int_{z/\theta_i}^\infty F(x) (c_t - d_t)(dx) - \int_z^\infty F(x) (c_t - d_t)(dx) \right] \beta(d\theta) \end{aligned} \quad (3.5)$$

Recalling [9, eq. (3.8)] and using the Fubini Theorem we obtain

$$\frac{d}{dt} \int_0^\infty x^{\lambda-1} |E(t, x)| dx = \frac{1}{2} \int_0^\infty I^c(t, x) (c_t - d_t)(dx) + \int_0^\infty I^f(t, x) (c_t - d_t)(dx), \quad (3.6)$$

where

$$\begin{aligned} I^c(t, x) &= \int_0^\infty K(x, y) (AR(t))(x, y) (c_t + d_t)(dy), & x \in (0, \infty) \\ I^f(t, x) &= F(x) \int_\Theta (BR(t))(\theta, x) \beta(d\theta), & x \in (0, \infty). \end{aligned}$$

It follows from (3.2) with (2.3) that

$$|I^f(t, x)| \leq C x^\lambda, \quad x \in (0, \infty), \quad t \in [0, \infty). \quad (3.7)$$

We would like to be able to perform an integration by parts in the second integral of the right hand of (3.6). However, I^f is not necessarily differentiable with respect to x . We thus fix $\varepsilon \in (0, 1)$ and put

$$I_\varepsilon^f(t, x) = F(x) \int_\Theta (BR(t))(\theta, x) \beta_\varepsilon(d\theta), \quad x \in (0, \infty),$$

where β_ε is the finite measure $\beta|_{\Theta_\varepsilon}$ with $\Theta_\varepsilon = \{\theta \in \Theta : \theta_1 \leq 1 - \varepsilon\}$ and note that

$$\beta_\varepsilon(\Theta) = \int_{\Theta} \mathbb{1}_{\{1-\theta_1 \geq \varepsilon\}} \beta(d\theta) \leq \frac{1}{\varepsilon} \int_{\Theta} (1 - \theta_1) \beta(d\theta) \leq \frac{1}{\varepsilon} C_\beta^\lambda < \infty. \quad (3.8)$$

Since F belongs to $W^{1,\infty}(\alpha, 1/\alpha)$ for $\alpha \in (0, 1)$ and $|R(t, x)| \leq x^\lambda/\lambda$ and $|\partial_x R(t, x)| \leq x^{\lambda-1}$ we deduce that $I_\varepsilon^f \in W^{1,\infty}(\alpha, 1/\alpha)$ for $\alpha \in (0, 1)$ with

$$\partial_x I_\varepsilon^f(t, x) = F'(x) \int_{\Theta} (BR(t))(\theta, x) \beta_\varepsilon(d\theta) + F(x) \int_{\Theta} \left[\sum_{i \geq 1} \theta_i \partial_x R(t, \theta_i x) - \partial_x R(t, x) \right] \beta_\varepsilon(d\theta). \quad (3.9)$$

We now perform an integration by parts to obtain

$$\begin{aligned} \int_0^\infty I^f(t, x) (c_t - d_t) (dx) &= \int_0^\infty (I^f - I_\varepsilon^f)(t, x) (c_t - d_t) (dx) - [I_\varepsilon^f(t, x) E(t, x)]_{x=0}^{x=\infty} \\ &\quad + \int_0^\infty \partial_x I_\varepsilon^f(t, x) E(t, x) dx. \end{aligned} \quad (3.10)$$

First, using (2.7) we have

$$\begin{aligned} \left| \int_0^\infty (I^f - I_\varepsilon^f)(t, x) (c_t - d_t) (dx) \right| &\leq \int_0^\infty |(I^f - I_\varepsilon^f)(t, x)| (c_t + d_t) (dx) \\ &\leq \kappa_2 \int_0^\infty \int_{\Theta} |(BR(t))(\theta, x)| (\beta - \beta_\varepsilon)(d\theta) (c_t + d_t) (dx) \\ &\leq \kappa_2 \int_0^\infty \int_{\Theta} \left(\sum_{i \geq 2} \int_0^{\theta_i x} z^{\lambda-1} dz + \int_{\theta_1 x}^x z^{\lambda-1} dz \right) \mathbb{1}_{\{1-\theta_1 < \varepsilon\}} \beta(d\theta) (c_t + d_t) (dx) \\ &\leq \frac{2\kappa_2}{\lambda} \int_0^\infty x^\lambda \int_{\Theta} \sum_{i \geq 2} \theta_i^\lambda \mathbb{1}_{\{1-\theta_1 < \varepsilon\}} \beta(d\theta) (c_t + d_t) (dx) \\ &= \frac{2\kappa_2}{\lambda} M_\lambda (c_t + d_t) \int_{\Theta} \sum_{i \geq 2} \theta_i^\lambda \mathbb{1}_{\{1-\theta_1 < \varepsilon\}} \beta(d\theta), \end{aligned}$$

whence,

$$\lim_{\varepsilon \rightarrow 0} \int_0^\infty (I^f - I_\varepsilon^f)(t, x) (c_t - d_t) (dx) = 0. \quad (3.11)$$

Next, it follows from (3.7) that

$$|I_\varepsilon^f(t, x) E(t, x)| \leq C x^\lambda (F^{c_t}(x) + F^{d_t}(x)), \quad x \in (0, \infty), \quad t \in [0, \infty),$$

we can thus easily conclude by Lemma 3.2. that

$$\lim_{x \rightarrow 0} I_\varepsilon^f(t, x) E(t, x) = \lim_{x \rightarrow \infty} I_\varepsilon^f(t, x) E(t, x) = 0. \quad (3.12)$$

Finally, (2.4), Lemma 3.2. and (3.2) imply that

$$\lim_{\varepsilon \rightarrow 0} \int_0^\infty F'(x) \int_{\Theta} (BR(t))(\theta, x) \beta_\varepsilon(d\theta) E(t, x) dx = \int_0^\infty F'(x) \int_{\Theta} (BR(t))(\theta, x) \beta(d\theta) E(t, x) dx, \quad (3.13)$$

while

$$\begin{aligned}
& \limsup_{\varepsilon \rightarrow 0} \int_0^\infty F(x) \int_{\Theta} \left[\sum_{i \geq 1} \theta_i \partial_x R(t, \theta_i x) - \partial_x R(t, x) \right] \beta_\varepsilon(d\theta) E(t, x) dx \\
&= \limsup_{\varepsilon \rightarrow 0} \int_0^\infty F(x) \int_{\Theta} \left(\sum_{i \geq 1} \theta_i^\lambda x^{\lambda-1} \text{sign}(E(t, \theta_i x)) - x^{\lambda-1} \text{sign}(E(t, x)) \right) \beta_\varepsilon(d\theta) E(t, x) dx \\
&= \limsup_{\varepsilon \rightarrow 0} \int_0^\infty F(x) x^{\lambda-1} \text{sign}(E(t, x)) E(t, x) \int_{\Theta} \left(\sum_{i \geq 1} \theta_i^\lambda \text{sign}(E(t, \theta_i x) E(t, x)) - 1 \right) \beta_\varepsilon(d\theta) dx \\
&\leq \limsup_{\varepsilon \rightarrow 0} \int_0^\infty F(x) x^{\lambda-1} |E(t, x)| \int_{\Theta} \left(\sum_{i \geq 1} \theta_i^\lambda - 1 \right) \beta_\varepsilon(d\theta) dx \\
&= \int_0^\infty F(x) x^{\lambda-1} |E(t, x)| \int_{\Theta} \left(\sum_{i \geq 1} \theta_i^\lambda - 1 \right) \beta(d\theta) dx.
\end{aligned} \tag{3.14}$$

We have used (2.3). Recall (3.6), the term involving I^c was treated in [9, Proposition 3.3], while from (3.10) with (3.11), (3.12), (3.13) and (3.14) we deduce the inequality (3.1), which completes the proof of Proposition 3.3. \square

Note that it is straightforward that under the notation and assumptions of Proposition 3.3., as in [9, Corollary 3.6], from (2.3), (2.4), (2.8) and using Lemma 3.4., there exists a positive constant C_1 depending on λ , κ_0 and κ_1 and a positive constant C_2 depending on κ_2 , κ_3 and C_β^λ such that for each $t \in [0, \infty)$,

$$\frac{d}{dt} \int_0^\infty x^{\lambda-1} |E(t, x)| dx \leq C_1 M_\lambda(c_t + d_t) \int_0^\infty x^{\lambda-1} |E(t, x)| dx + C_2 \int_0^\infty x^{\lambda-1} |E(t, x)| dx. \tag{3.15}$$

3.1. Proof of Theorem 2.5.

Uniqueness. Owing to (2.9) and (3.15), the uniqueness assertion of Theorem 2.5. readily follows from the Gronwall Lemma. \square

Existence. The proof of the existence assertion of Theorem 2.5. is split into three steps. The first step consists in finding an approximation to the coagulation-fragmentation equation by a version of (2.10) with finite operators: we will show existence in the set of positive measures with finite total variation, i.e. \mathcal{M}_0^+ , using the Picard method.

Next, we will show existence of a weak solution to (1.1) with an initial condition c^{in} in $\mathcal{M}_\lambda^+ \cap \mathcal{M}_2^+$, the final step consists in extending this result to the case where c^{in} belongs only to \mathcal{M}_λ^+ .

Bounded Case : existence and uniqueness in \mathcal{M}_0^+ .

We consider a bounded coagulation kernel and a fragmentation mechanism which gives only a finite number of fragments. This is

$$\begin{cases} K(x, y) \leq \overline{K}, & \text{for some } \overline{K} \in \mathbb{R}^+ \\ F(x) \leq \overline{F}, & \text{for some } \overline{F} \in \mathbb{R}^+ \\ \beta(\Theta) < \infty, \\ \beta(\Theta \setminus \Theta_k) = 0, & \text{for some } k \in \mathbb{N}, \end{cases} \quad (3.16)$$

where

$$\Theta_k = \{\theta = (\theta_n)_{n \geq 1} \in \Theta : \theta_{k+1} = \theta_{k+2} = \dots = 0\}.$$

We will show in this paragraph that under this assumptions there exists a global weak-solution to (1.1). We will use the notation $\|\cdot\|_\infty$ for the sup norm on $L^\infty[0, \infty)$ and $\|\cdot\|_{VT}$ for the total variation norm on measures. The result reads as follows.

Proposition 3.6. *Consider $\mu^{in} \in \mathcal{M}_0^+$. Assume that the coagulation and fragmentation kernels K and F and the measure β satisfy the assumptions (3.16). Then, there exists a unique non-negative weak-solution $(\mu_t)_{t \geq 0}$ starting at $\mu_0 = \mu^{in}$ to (1.1). Furthermore, it satisfies for all $t \geq 0$,*

$$\sup_{[0, t]} \|\mu_s\|_{VT} \leq C_t \|\mu^{in}\|_{VT}, \quad (3.17)$$

where C_t is a positive constant depending on $t, \overline{K}, \overline{F}$ and β .

To prove this proposition we need to replace the operator A in (2.10) by an equivalent one, this new operator will be easier to manipulate. We consider, for ϕ a bounded function, the following operators

$$(\tilde{A}\phi)(x, y) = K(x, y) \left[\frac{1}{2} \phi(x+y) - \phi(x) \right], \quad (3.18)$$

$$(L\phi)(x) = F(x) \int_{\Theta} \left(\sum_{i \geq 1} \phi(\theta_i x) - \phi(x) \right) \beta(d\theta). \quad (3.19)$$

Thus, (2.10) can be rewritten as

$$\frac{d}{dt} \int_0^\infty \phi(x) c_t(dx) = \int_0^\infty \left[\int_0^\infty (\tilde{A}\phi)(x, y) c_t(dy) + (L\phi)(x) \right] c_t(dx). \quad (3.20)$$

The Proposition will be proved using an implicit scheme for equation (3.20). First, we need to provide a unique and non-negative solution to this scheme.

Lemma 3.7. *Consider $\mu^{in} \in \mathcal{M}_0^+$ and let $(\nu_t)_{t \geq 0}$ be a family of measures in \mathcal{M}_0^+ such that $\sup_{[0, t]} \|\nu_s\|_{VT} < \infty$ for all $t \geq 0$. Then, under the assumptions (3.16), there exists a unique non-negative solution $(\mu_t)_{t \geq 0}$ starting at $\mu_0 = \mu^{in}$ to*

$$\int_0^\infty \phi(x) \mu_t(dx) = \int_0^\infty \phi(x) \mu_0(dx) + \int_0^t \int_0^\infty \left[\int_0^\infty (\tilde{A}\phi)(x, y) \nu_s(dy) + (L\phi)(x) \right] \mu_s(dx) ds \quad (3.21)$$

for all $\phi \in L^\infty(\mathbb{R}^+)$. Furthermore, the solution satisfies for all $t \geq 0$,

$$\sup_{[0, t]} \|\mu_s\|_{VT} \leq C_t \|\mu^{in}\|_{VT}, \quad (3.22)$$

where C_t is a positive constant depending on $t, \overline{K}, \overline{F}$ and β .

The constant C_t does not depend on $\sup_{[0, t]} \|\nu_s\|_{VT}$.

We will prove this lemma in two steps. First, we show that (3.21) is equivalent to another equation. This new equation is constructed in such a way that the negative terms of equation (3.21) are eliminated. Next, we prove existence and uniqueness for this new equation. This solution will be proved to be non-negative and it will imply existence, uniqueness and non-negativity of a solution to (3.21).

Proof. Step 1.- First, we give now an auxiliary result which allows to differentiate equation (3.24) when the test function depends on t .

Lemma 3.8. *Let $(t, x) \mapsto \phi_t(x) : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ be a bounded measurable function, having a bounded partial derivative $\partial\phi/\partial t$ and consider $(\mu_t)_{t \geq 0}$ a weak-solution to (3.21). Then, for all $t \geq 0$,*

$$\frac{d}{dt} \int_0^\infty \phi_t(x) \mu_t(dx) = \int_0^\infty \frac{\partial}{\partial t} \phi_t(x) \mu_t(dx) + \int_0^\infty \int_0^\infty (\tilde{A}\phi_t)(x, y) \mu_t(dx) \nu_t(dy) + \int_0^\infty (L\phi_t)(x) \mu_t(dx).$$

Proof. First, note that for $0 \leq t_1 \leq t_2$ we have,

$$\begin{aligned} & \int_0^\infty \phi_{t_2}(x) \mu_{t_2}(dx) - \int_0^\infty \phi_{t_1}(x) \mu_{t_1}(dx) \\ &= \int_0^\infty (\phi_{t_2}(x) - \phi_{t_1}(x)) \mu_{t_2}(dx) + \int_0^\infty \phi_{t_1}(x) (\mu_{t_2} - \mu_{t_1})(dx) \\ &= \int_{t_1}^{t_2} \int_0^\infty \frac{\partial}{\partial t} \phi_s(x) \mu_{t_2}(dx) ds + \int_{t_1}^{t_2} \frac{d}{dt} \int_0^\infty \phi_{t_1}(x) \mu_t(dx) dt \\ &= \int_{t_1}^{t_2} \int_0^\infty \frac{\partial}{\partial t} \phi_s(x) \mu_{t_2}(dx) ds \\ & \quad + \int_{t_1}^{t_2} \left[\int_0^\infty \int_0^\infty (\tilde{A}\phi_{t_1})(x, y) \mu_s(dx) \nu_s(dy) + \int_0^\infty (L\phi_{t_1})(x) \mu_s(dx) \right] ds. \end{aligned}$$

Thus, fix $t > 0$ and set for $n \in \mathbb{N}$, $t_k = t \frac{k}{n}$ with $k = 0, 1, \dots, n$, we get

$$\begin{aligned} \int_0^\infty \phi_t(x) \mu_t(dx) &= \int_0^\infty \phi_0(x) \mu_0(dx) + \sum_{k=1}^n \left[\int_0^\infty \phi_{t_k}(x) \mu_{t_k}(dx) - \int_0^\infty \phi_{t_{k-1}}(x) \mu_{t_{k-1}}(dx) \right] \\ &= \int_0^\infty \phi_0(x) \mu_0(dx) + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_0^\infty \frac{\partial}{\partial t} \phi_s(x) \mu_{t_k}(dx) ds \\ & \quad + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left[\int_0^\infty \int_0^\infty (\tilde{A}\phi_{t_{k-1}})(x, y) \mu_s(dx) \nu_s(dy) + \int_0^\infty (L\phi_{t_{k-1}})(x) \mu_s(dx) \right] ds. \end{aligned}$$

Next, for $s \in [t_{k-1}, t_k]$ we set $k = \lfloor \frac{ns}{t} \rfloor$ and use the notation $\bar{s}_n := t_k = \frac{t}{n} \lfloor \frac{ns}{t} \rfloor$ and $\underline{s}_n := t_{k-1}$.

Thus, the equation above can be rewritten as

$$\begin{aligned} \int_0^\infty \phi_t(x) \mu_t(dx) &= \int_0^\infty \phi_0(x) \mu_0(dx) + \int_0^t \int_0^\infty \frac{\partial}{\partial t} \phi_s(x) \mu_{\bar{s}_n}(dx) ds \\ & \quad + \int_0^t \int_0^\infty \int_0^\infty (\tilde{A}\phi_{\underline{s}_n})(x, y) \mu_s(dx) \nu_s(dy) ds + \int_0^t \int_0^\infty (L\phi_{\underline{s}_n})(x) \mu_s(dx) ds, \end{aligned}$$

and the lemma follows from letting $n \rightarrow \infty$ since $\bar{s}_n \rightarrow s$. \square

Next, we introduce a new equation. We put for $t \geq 0$,

$$\gamma_t(x) = \exp \left[\int_0^t \left(\int_0^\infty K(x, y) \nu_s(dy) - F(x) \right) ds \right], \quad (3.23)$$

and we consider the equation

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \phi(x) \tilde{\mu}_t(dx) &= \int_0^\infty \left[\int_0^\infty \frac{1}{2} K(x, y) (\phi \gamma_t)(x + y) \nu_t(dy) \right. \\ &\quad \left. + F(x) \int_\Theta \sum_{i \geq 1} (\phi \gamma_t)(\theta_i x) \beta(d\theta) \right] \gamma_t^{-1}(x) \tilde{\mu}_t(dx) \end{aligned} \quad (3.24)$$

Now, we give a result that relates (3.21) to (3.24).

Lemma 3.9. *Consider $\mu^{in} \in \mathcal{M}_0^+$ and recall (3.23). Then, $(\mu_t)_{t \geq 0}$ with $\mu_0 = \mu^{in}$ is a weak-solution to (3.21) if and only if $(\tilde{\mu}_t)_{t \geq 0}$ with $\tilde{\mu}_0 = \mu^{in}$ is a weak-solution to (3.24), where $\tilde{\mu}_t = \gamma_t \mu_t$ for all $t \geq 0$.*

Proof. First, assume that $(\mu_t)_{t \geq 0}$ is a weak-solution to (3.21).

We have $\frac{\partial}{\partial t} \gamma_t(x) = \gamma_t(x) \left[\int_0^\infty K(x, y) \nu_t(dy) - F(x) \right]$. Note that γ_t , γ_t^{-1} and $\frac{\partial}{\partial t} \gamma_t$ are bounded on $[0, t]$ for all $t \geq 0$, by (3.16) and since $\sup_{[0, t]} \|\nu_s\|_{VT} < \infty$.

Set $\tilde{\mu}_t = \gamma_t \mu_t$, recall (3.18) and (3.19), by Lemma 3.8., for all bounded measurable functions ϕ , we have

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \phi(x) \tilde{\mu}_t(dx) &= \int_0^\infty \phi(x) \gamma_t(x) \left[\int_0^\infty K(x, y) \nu_t(dy) - F(x) \right] \mu_t(dx) \\ &\quad + \int_0^\infty \int_0^\infty \left[\frac{1}{2} (\phi \gamma_t)(x + y) - (\phi \gamma_t)(x) \right] K(x, y) \nu_t(dy) \mu_t(dx) \\ &\quad + \int_0^\infty F(x) \int_\Theta \left(\sum_{i \geq 1} (\phi \gamma_t)(\theta_i x) - (\phi \gamma_t)(x) \right) \beta(d\theta) \mu_t(dx) \\ &= \int_0^\infty \int_0^\infty \frac{1}{2} K(x, y) (\phi \gamma_t)(x + y) \nu_t(dy) \mu_t(dx) \\ &\quad + \int_0^\infty F(x) \int_\Theta \sum_{i \geq 1} (\phi \gamma_t)(\theta_i x) \beta(d\theta) \mu_t(dx) \\ &= \int_0^\infty \left[\int_0^\infty \frac{1}{2} K(x, y) (\phi \gamma_t)(x + y) \nu_t(dy) \right. \\ &\quad \left. + F(x) \int_\Theta \sum_{i \geq 1} (\phi \gamma_t)(\theta_i x) \beta(d\theta) \right] \gamma_t^{-1}(x) \tilde{\mu}_t(dx), \end{aligned}$$

and the result follows.

For the reciprocal assertion, we assume that $(\tilde{\mu}_t)_{t \geq 0}$ is a weak-solution to (3.24), set $\mu_t = \gamma_t^{-1} \tilde{\mu}_t$ and we show in the same way that $(\mu_t)_{t \geq 0}$ is a weak-solution to (3.21). \square

We note that, since all the terms between the brackets are non-negative, the right-hand side of equation (3.24) is non-negative whenever $\tilde{\mu}_t \geq 0$. Thus, γ_t is an integrating factor that removes the negative terms of equation (3.21).

Step 2.- We define the following explicit scheme for (3.24): we set $\tilde{\mu}_t^0 = \mu^{in}$ for all $t \geq 0$ and for $n \geq 0$

$$\begin{cases} \frac{d}{dt} \int_0^\infty \phi(x) \tilde{\mu}_t^{n+1}(dx) &= \int_0^\infty \left[\int_0^\infty \frac{1}{2} K(x, y) (\phi \gamma_t)(x + y) \nu_t(dy) \right. \\ &\quad \left. + F(x) \int_\Theta \sum_{i \geq 1} (\phi \gamma_t)(\theta_i x) \beta(d\theta) \right] \gamma_t^{-1}(x) \tilde{\mu}_t^n(dx) \\ \tilde{\mu}_0^{n+1} &= \mu^{in}. \end{cases} \quad (3.25)$$

Recall (3.16), note that the following operators are bounded:

$$\left\| \gamma_t^{-1}(\cdot) \int_0^\infty \frac{1}{2} K(\cdot, y) (\phi \gamma_t)(\cdot + y) \nu_t(dy) \right\|_\infty \leq C_t \|\phi\|_\infty, \quad (3.26)$$

$$\left\| \gamma_t^{-1}(\cdot) F(\cdot) \int_\Theta \sum_{i \geq 1} (\phi \gamma_t)(\theta_i \cdot) \beta(d\theta) \right\|_\infty \leq C_t \|\phi\|_\infty, \quad (3.27)$$

where C_t is a positive constant depending on \bar{K} , \bar{F} , β and $\sup_{[0,t]} \|\nu_s\|_{VT}$.

Thus, we consider ϕ bounded, integrate in time (3.25), use (3.26) and (3.27) to obtain

$$\begin{aligned} \int_0^\infty \phi(x) (\tilde{\mu}_t^{n+1}(dx) - \tilde{\mu}_t^n(dx)) &\leq C_{1,t} \|\phi\|_\infty \int_0^t \|\tilde{\mu}_s^n - \tilde{\mu}_s^{n-1}\|_{VT} ds \\ &\quad + C_{2,t} \|\phi\|_\infty \int_0^t \|\tilde{\mu}_s^n - \tilde{\mu}_s^{n-1}\|_{VT} ds, \end{aligned}$$

note that the the difference of the initial conditions vanishes since they are the same. We take the sup over $\|\phi\|_\infty \leq 1$ and use $\sup_{[0,t]} \|\nu_s\|_{VT} < \infty$ to deduce

$$\|\tilde{\mu}_t^{n+1} - \tilde{\mu}_t^n\|_{VT} \leq C_t \int_0^t \|\tilde{\mu}_s^n - \tilde{\mu}_s^{n-1}\|_{VT} ds,$$

where C_t is a positive constant depending on \bar{K} , \bar{F} , β , $\sup_{[0,t]} \|\nu_s\|_{VT}$ and $\|\phi\|_\infty$. Hence, by classical arguments, $(\tilde{\mu}_t^n)_{t \geq 0}$ converges in \mathcal{M}_0^+ uniformly in time to $(\tilde{\mu}_t)_{t \geq 0}$ solution to (3.24), and since $\tilde{\mu}_t^n \geq 0$ for all n , we deduce $\tilde{\mu}_t \geq 0$ for all $t \geq 0$. The uniqueness for (3.24) follows from similar computations.

Thus, by Lemma 3.9. we deduce existence and uniqueness of $(\mu_t)_{t \geq 0}$ solution to (3.21), and since $\tilde{\mu}_t \geq 0$ we have $\mu_t \geq 0$ for all $t \geq 0$.

Finally, it remains to prove (3.22). For this, we apply (3.21) with $\phi(x) \equiv 1$, remark that $(\tilde{A}1)(x, y) \leq 0$ and that $(L1)(x) \leq \bar{F}(k-1)\beta(\Theta)$. Since $\mu_t \geq 0$ for all $t \geq 0$, this implies

$$\|\mu_t\|_{VT} = \int_0^\infty \mu_t(dx) \leq \|\mu_0\|_{VT} + \bar{F}(k-1)\beta(d\Theta) \int_0^t \|\mu_s\|_{VT} ds.$$

Using the Gronwall Lemma, we conclude

$$\sup_{[0,t]} \|\mu_s\|_{VT} \leq \|\mu^{in}\|_{VT} e^{Ct} \quad \text{for all } t \geq 0,$$

where C is a positive constant depending only on \bar{K} , \bar{F} and β . We point out that the term $\sup_{[0,t]} \|\nu_s\|_{VT}$ is not involved since it is relied to the coagulation part of the equation, which is negative and bounded by 0. This ends the proof of Lemma 3.7. \square

Proof of Proposition 3.6. We define the following implicit scheme for (3.20): $\mu_t^0 = \mu^{in}$ for all $t \geq 0$ and for $n \geq 0$,

$$\begin{cases} \frac{d}{dt} \int_0^\infty \phi(x) \mu_t^{n+1}(dx) &= \int_0^\infty \int_0^\infty (\tilde{A}\phi)(x, y) \mu_t^{n+1}(dx) \mu_t^n(dy) + \int_0^\infty (L\phi)(x) \mu_t^{n+1}(dx) \\ \mu_0^{n+1} &= \mu^{in}. \end{cases} \quad (3.28)$$

First, from Lemma 3.7. for $n \geq 0$ we have existence of $(\mu_t^{n+1})_{t \geq 0}$ unique and non-negative solution to (3.28) whenever $(\mu_t^n)_{t \geq 0}$ is non-negative and $\sup_{[0,t]} \|\mu_s^n\|_{VT} < \infty$ for all $t \geq 0$. Hence, since $\mu^{in} \in \mathcal{M}_0^+$, by recurrence we deduce existence, uniqueness and non-negativity of $(\mu_t^{n+1})_{t \geq 0}$ for all $n \geq 0$ solution to (3.28).

Moreover, from (3.22), this solution is bounded uniformly in n on $[0, t]$ for all $t \geq 0$ since this bound does not depend on μ_t^n , i.e.,

$$\sup_{n \geq 1} \sup_{[0,t]} \|\mu_s^{n+1}\|_{VT} \leq C_t \|\mu^{in}\|_{VT}. \quad (3.29)$$

Next, note that the operators \tilde{A} and L are bounded:

$$\|L\phi\|_\infty \leq \bar{F}(k+1)\beta(\Theta)\|\phi\|_\infty, \quad (3.30)$$

$$\left\| \int_0^\infty (\tilde{A}\phi)(\cdot, y) \mu(dy) \right\|_\infty \leq \frac{3}{2} \bar{K} \|\phi\|_\infty \|\mu\|_{VT}. \quad (3.31)$$

From (3.31) and (3.30),

$$\begin{aligned} & \frac{d}{dt} \int_0^\infty \phi(x) (\mu_t^{n+1}(dx) - \mu_t^n(dx)) \\ &= \int_0^\infty \int_0^\infty (\tilde{A}\phi)(x, y) (\mu_t^{n+1}(dx) \mu_t^n(dy) - \mu_t^n(dx) \mu_t^{n-1}(dy)) \\ & \quad + \int_0^\infty (L\phi)(x) (\mu_t^{n+1} - \mu_t^n)(dx) \\ &= \int_0^\infty \int_0^\infty (\tilde{A}\phi)(x, y) [(\mu_t^{n+1} - \mu_t^n)(dx) \mu_t^n(dy) + \mu_t^n(dx) (\mu_t^n - \mu_t^{n-1})(dy)] \\ & \quad + \int_0^\infty (L\phi)(x) (\mu_t^{n+1} - \mu_t^n)(dx) \\ &\leq \frac{3}{2} \bar{K} \|\phi\|_\infty \|\mu_t^n\|_{VT} \left[\int_0^\infty |\mu_t^{n+1} - \mu_t^n|(dx) + \int_0^\infty |\mu_t^n - \mu_t^{n-1}|(dy) \right] \\ & \quad + \bar{F}(k+1)\beta(\Theta)\|\phi\|_\infty \|\mu_t^{n+1} - \mu_t^n\|_{VT}, \end{aligned}$$

implying,

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \phi(x) (\mu_t^{n+1}(dx) - \mu_t^n(dx)) &\leq \|\phi\|_\infty \left(\frac{3}{2} \bar{K} \|\mu_t^n\|_{VT} + \bar{F}(k+1)\beta(\Theta) \right) \|\mu_t^{n+1} - \mu_t^n\|_{VT} \\ &\quad + \frac{3}{2} \bar{K} \|\phi\|_\infty \|\mu_t^n\|_{VT} \|\mu_t^n - \mu_t^{n-1}\|_{VT}. \end{aligned}$$

We integrate on t , take the sup over $\|\phi\|_\infty \leq 1$, and use (3.29), to deduce that there exist two constants $C_{1,t}$ and $C_{2,t}$ depending on t but not on n such that

$$\|\mu_t^{n+1} - \mu_t^n\|_{VT} \leq C_{1,t} \int_0^t \|\mu_s^{n+1} - \mu_s^n\|_{VT} ds + C_{2,t} \int_0^t \|\mu_s^n - \mu_s^{n-1}\|_{VT} ds.$$

Note that the difference of initial conditions vanishes since they are the same. We obtain using the Gronwall Lemma.

$$\|\mu_t^{n+1} - \mu_t^n\|_{VT} \leq C_{2,t} e^{t C_{1,t}} \int_0^t \|\mu_s^n - \mu_s^{n-1}\|_{VT} ds.$$

Hence, by usual arguments, $(\mu_t^n)_{t \geq 0}$ converges in \mathcal{M}_0^+ uniformly in time to the desired solution, which is also unique. Moreover, for some finite constant C depending on t, \bar{K}, \bar{F} and β , this solution satisfies (3.17) by (3.29).

This concludes the proof of Proposition 3.6. \square

Existence and uniqueness for $c^{in} \in \mathcal{M}_\lambda^+ \cap M_2^+$.

We are no longer under (3.16), more generally we assume Hypotheses 2.1. and 2.2. This paragraph is devoted to show existence in the case where the initial condition satisfies:

$$c^{in} \in \mathcal{M}_\lambda^+ \cap M_2^+.$$

First, for $n \geq 1$, we consider $c^{in,n}(dx) = \mathbf{1}_{[1/n,n]} c^{in}(dx)$, this measure belongs to \mathcal{M}_0^+ and satisfies

$$\sup_{n \geq 1} M_\lambda(c^{in,n}) \leq M_\lambda(c^{in}). \quad (3.32)$$

We also note that $(F^{c^{in,n}})$ converges towards $F^{c^{in}}$ in $L^1(0, \infty; x^{\lambda-1} dx)$ as $n \rightarrow \infty$. Define K_n by $K_n(x, y) = K(x, y) \wedge n$ for $(x, y) \in (0, \infty)^2$. Notice that (2.1) and (2.2) warrant that

$$\begin{aligned} K_n(x, y) &\leq \kappa_0(x+y)^\lambda, \\ (x^\lambda \wedge y^\lambda) |\partial_x K_n(x, y)| &\leq \kappa_1 x^{\lambda-1} y^\lambda. \end{aligned} \quad (3.33)$$

Furthermore, we consider the set $\Theta(n)$ defined by $\Theta(n) = \{\theta \in \Theta : \theta_1 \leq 1 - \frac{1}{n}\}$, we consider also the projector

$$\begin{aligned} \psi_n : \Theta &\rightarrow \Theta_n \\ \theta &\mapsto \psi_n(\theta) = (\theta_1, \dots, \theta_n, 0, \dots), \end{aligned} \quad (3.34)$$

and we put

$$\beta_n = \mathbf{1}_{\theta \in \Theta(n)} \beta \circ \psi_n^{-1}. \quad (3.35)$$

The measure β_n can be seen as the restriction of β to the projection of $\Theta(n)$ onto Θ_n . Note that $\Theta(n) \subset \Theta(n+1)$ and that since we have excluded the degenerated cases $\theta_1 = 1$ we have $\bigcup_n \Theta(n) = \Theta$.

Then, K_n, F and β_n satisfy (3.16) (use (3.8)) and since $c^{in,n} \in \mathcal{M}_0^+$, we have from Proposition 3.6. that for each $n \geq 1$, there exists a $(c^{in,n}, K_n, F, \beta_n, \lambda)$ -weak solution $(c_t^n)_{t \geq 0}$ to (2.10).

Note that since we have fragmentation it is not evident that $M_\lambda(c_t)$ remains finite in time. We need to control $M_\lambda(c_t)$ to verify (2.9). For this, we set $\phi(x) = x^\lambda$, from (2.10) and since $(A\phi)(x, y) \leq 0$ we have

$$\begin{aligned} \frac{d}{dt} \int_0^\infty x^\lambda c_t^n(dx) &= \frac{1}{2} \int_0^\infty \int_0^\infty K_n(x, y) (A\phi)(x, y) c_t^n(dx) c_t^n(dy) \\ &\quad + \int_\Theta \int_0^\infty F(x) \left(\sum_{i \geq 1} \theta_i^\lambda - 1 \right) x^\lambda c_t^n(dx) \beta_n(d\theta) \\ &\leq \kappa_2 C_\beta^\lambda M_\lambda(c_t^n), \end{aligned}$$

where we used that clearly, $C_{\beta_n}^\lambda \leq C_\beta^\lambda$ for all $n \geq 1$ (recall (2.6)).

Using the Gronwall Lemma and (3.32) we deduce, for all $t \geq 0$

$$\sup_{n \geq 1} \sup_{[0, t]} M_\lambda(c_s^n) \leq C_t, \quad (3.36)$$

where C_t is a positive constant. Next, apply (2.10) with $\phi(x) = x^2$ and since $\sum_{i \geq 1} \theta_i^2 - 1 \leq 0$ the fragmentation part is negative. In [5, Lemma A.3. (ii)] was shown that there exists a constant C depending only on λ and κ_0 such that $K_n(x, y) |(A\phi)(x, y)| \leq K(x, y) |(A\phi)(x, y)| \leq C(x^2 y^\lambda + x^\lambda y^2)$. Thus,

$$\begin{aligned} \frac{d}{dt} \int_0^\infty x^2 c_t^n(dx) &\leq \frac{C}{2} \int_0^\infty \int_0^\infty (x^2 y^\lambda + x^\lambda y^2) c_t^n(dx) c_t^n(dy) \\ &= C M_\lambda(c_t^n) M_2(c_t^n). \end{aligned}$$

Using the Gronwall Lemma, we obtain

$$M_2(c_t^n) \leq M_2(c_0^n) e^{C \int_0^t M_\lambda(c_s^n) ds},$$

for $t \geq 0$ and for each $n \geq 1$. Hence, using (3.36) we get

$$\sup_{n \geq 1} \sup_{[0, t]} M_2(c_s^n) \leq C_t, \quad (3.37)$$

where C_t is a positive constant.

We set $E_n(t, x) = F^{c_t^{n+1}}(x) - F^{c_t^n}(x)$ and define $R_n(t, x) = \int_0^x z^{\lambda-1} \text{sign}(E_n(t, x)) dz$. Recall (3.4) and (3.5),

$$\begin{aligned}
& \frac{d}{dt} \int_0^\infty x^{\lambda-1} |E_n(t, x)| dx \\
&= \frac{1}{2} \int_0^\infty \partial_x R_n(t, z) \int_0^z \int_0^z \mathbb{1}_{[z, \infty)}(x+y) K_{n+1}(x, y) (c_t^{n+1}(dy) c_t^{n+1}(dx) - c_t^n(dy) c_t^n(dx)) dz \\
&\quad - \frac{1}{2} \int_0^\infty \partial_x R_n(t, z) \int_z^\infty \int_z^\infty K_{n+1}(x, y) (c_t^{n+1}(dy) c_t^{n+1}(dx) - c_t^n(dy) c_t^n(dx)) dz \\
&\quad + \int_0^\infty \partial_x R_n(t, z) \int_\Theta \sum_{i \geq 1} \int_{z/\theta_i}^\infty F(x) (c_t^{n+1} - c_t^n)(dx) \beta_{n+1}(d\theta) dz \\
&\quad - \int_0^\infty \partial_x R_n(t, z) \int_\Theta \int_z^\infty F(x) (c_t^{n+1} - c_t^n)(dx) \beta_{n+1}(d\theta) dz \\
&\quad + \frac{1}{2} \int_0^\infty \partial_x R_n(t, z) \int_0^z \int_0^z \mathbb{1}_{[z, \infty)}(x+y) (K_{n+1}(x, y) - K_n(x, y)) c_t^n(dy) c_t^n(dx) dz \\
&\quad - \frac{1}{2} \int_0^\infty \partial_x R_n(t, z) \int_z^\infty \int_z^\infty (K_{n+1}(x, y) - K_n(x, y)) c_t^n(dy) c_t^n(dx) dz \\
&\quad + \int_0^\infty \partial_x R_n(t, z) \int_\Theta \sum_{i \geq 1} \int_{z/\theta_i}^\infty F(x) c_t^n(dx) (\beta_{n+1} - \beta_n)(d\theta) dz \\
&\quad - \int_0^\infty \partial_x R_n(t, z) \int_\Theta \int_z^\infty F(x) c_t^n(dx) (\beta_{n+1} - \beta_n)(d\theta) dz.
\end{aligned}$$

Thus after some computations, we obtain

$$\frac{d}{dt} \int_0^\infty x^{\lambda-1} |E_n(t, x)| dx = I_1^n(t, x) + I_2^n(t, x) + I_3^n(t, x) + I_4^n(t, x), \quad (3.38)$$

where $I_1^n(t, x)$ and $I_2^n(t, x)$ are respectively the equivalent terms to the coagulation and fragmentation parts in (3.6) and

$$\begin{aligned}
I_3^n(t, x) &= \frac{1}{2} \int_0^\infty \int_0^\infty (K_{n+1}(x, y) - K_n(x, y)) (AR_n(t))(x, y) c_t^n(dy) c_t^n(dx) \\
I_4^n(t, x) &= \int_0^\infty F(x) \int_\Theta (BR_n(t))(\theta, x) (\beta_{n+1} - \beta_n)(d\theta) c_t^n(dx),
\end{aligned}$$

which are the terms resulting of the approximation.

Exactly as in (3.15), since the bounds in (3.33) do not depend on n and that β_n satisfies (2.6) uniformly in n , we get

$$I_1^n(t, x) + I_2^n(t, x) \leq C_1 M_\lambda (c_t^n + c_t^{n+1}) \int_0^\infty x^{\lambda-1} |E_n(t, x)| dx + C_2 \int_0^\infty x^{\lambda-1} |E_n(t, x)| dx. \quad (3.39)$$

Next, since

$$K_{n+1}(x, y) - K_n(x, y) = \mathbb{1}_{\{K(x, y) > n+1\}} + (K(x, y) - n) \mathbb{1}_{\{n < K(x, y) \leq n+1\}} \leq \mathbb{1}_{\{K(x, y) > n\}} \leq \frac{K(x, y)^2}{n^2}$$

and using (3.3), we have

$$\begin{aligned}
|I_3^n(t, x)| &= \frac{1}{2} \left| \int_0^\infty \int_0^\infty (K_{n+1}(x, y) - K_n(x, y)) (AR_n(t))(x, y) c_t^n(dy) c_t^n(dx) \right| \\
&\leq \frac{1}{2} \int_0^\infty \int_0^\infty \frac{K(x, y)^2}{n^2} |(AR_n(t))(x, y)| c_t^n(dy) c_t^n(dx) \\
&\leq \frac{2^{2\lambda+1} \kappa_0^2}{2\lambda n^2} \int_0^\infty \int_0^\infty (x \vee y)^{2\lambda} (x \wedge y)^\lambda c_t^n(dy) c_t^n(dx) \\
&\leq \frac{C}{n^2} M_{2\lambda}(c_t^n) M_\lambda(c_t^n) \\
&\leq \frac{1}{n^2} C_t,
\end{aligned} \tag{3.40}$$

we have used $M_{2\lambda}(c_t) \leq M_\lambda(c_t) + M_2(c_t)$ together with (3.36) and (3.37).

Finally, since $\int_\Theta (BR_n(t))(\theta, x) \beta_n(d\theta) = \int_\Theta (BR_n(t))(\psi_n(\theta), x) \mathbf{1}_{\{\theta \in \Theta(n)\}} \beta(d\theta)$, we have

$$\begin{aligned}
|I_4^n(t, x)| &= \left| \int_0^\infty F(x) \int_\Theta \left\{ [(BR_n(t))(\psi_{n+1}(\theta), x) - (BR_n(t))(\psi_n(\theta), x)] \mathbf{1}_{\Theta(n) \cap \Theta(n+1)} \right. \right. \\
&\quad \left. \left. + (BR_n(t))(\psi_{n+1}(\theta), x) \mathbf{1}_{\Theta(n+1) \setminus \Theta(n)} \right\} \beta(d\theta) c_t^n(dx) \right| \\
&\leq \int_0^\infty F(x) \int_\Theta |R_n(t, \theta_{n+1}x)| \mathbf{1}_{\Theta(n+1) \cap \Theta(n)} \beta(d\theta) c_t^n(dx) \\
&\quad + \int_0^\infty F(x) \int_\Theta \left| \sum_{i=1}^{n+1} R_n(t, \theta_i x) - R_n(t, x) \right| \mathbf{1}_{\Theta(n+1) \setminus \Theta(n)} \beta(d\theta) c_t^n(dx) \\
&\leq C \int_0^\infty x^\lambda c_t^n(dx) \int_\Theta \theta_{n+1}^\lambda \mathbf{1}_{\{\Theta(n+1) \cap \Theta(n)\}} \beta(d\theta) \\
&\quad + C \int_0^\infty x^\lambda c_t^n(dx) \int_\Theta \sum_{i \geq 2} \theta_i^\lambda \mathbf{1}_{\{\Theta(n+1) \setminus \Theta(n)\}} \beta(d\theta) \\
&\leq C_t \int_\Theta \theta_{n+1}^\lambda \beta(d\theta) + C_t \int_\Theta \sum_{i \geq 2} \theta_i^\lambda \mathbf{1}_{\{\Theta(n+1) \setminus \Theta(n)\}} \beta(d\theta),
\end{aligned} \tag{3.41}$$

we used (2.8) and (3.36). Gathering (3.39), (3.40) and (3.41) in (3.38), we obtain

$$\begin{aligned}
\frac{d}{dt} \int_0^\infty x^{\lambda-1} |E_n(t, x)| dx &\leq C_t M_\lambda(c^{in}) \int_0^\infty x^{\lambda-1} |E_n(t, x)| dx + \frac{1}{n^2} C_t \\
&\quad + C_t \int_\Theta \theta_{n+1}^\lambda \beta(d\theta) + C_t \int_\Theta \sum_{i \geq 2} \theta_i^\lambda \mathbf{1}_{\{\Theta(n+1) \setminus \Theta(n)\}} \beta(d\theta).
\end{aligned}$$

Thus by the Gronwall Lemma we obtain

$$\begin{aligned} \int_0^\infty x^{\lambda-1} \left| F^{c_t^{n+1}}(x) - F^{c_t^n}(x) \right| dx &\leq C_t \left(\int_0^\infty x^{\lambda-1} \left| F^{c^{in,n+1}}(x) - F^{c^{in,n}}(x) \right| dx + \frac{1}{n^2} \right. \\ &\quad \left. + \int_\Theta \theta_{n+1}^\lambda \beta(d\theta) + \int_\Theta \sum_{i \geq 2} \theta_i^\lambda \mathbb{1}_{\{\Theta(n+1) \setminus \Theta(n)\}} \beta(d\theta) \right), \end{aligned}$$

for $t \geq 0$ and $n \geq 1$ and where C_t is a positive constant depending on $\lambda, \kappa_0, \kappa_1, \kappa_2, \kappa_3, C_\beta^\lambda, t$ and c^{in} . Recalling that

$$t \mapsto F^{c_t^n} \text{ belongs to } \mathcal{C}([0, \infty); L^1(0, \infty; x^{\lambda-1} dx)),$$

for each $n \geq 1$ by Lemma 3.2. and Lemma 3.5, and since the last three terms in the right-hand side of the inequality above are the terms of convergent series, we conclude that $(t \mapsto F^{c_t^n})_{n \geq 1}$ is a Cauchy sequence in $\mathcal{C}([0, \infty); L^1(0, \infty; x^{\lambda-1} dx))$ and there is

$$f \in \mathcal{C}([0, \infty); L^1(0, \infty; x^{\lambda-1} dx))$$

such that

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, t]} \int_0^\infty x^{\lambda-1} \left| F^{c_s^{n+1}}(x) - f(s, x) \right| dx = 0 \quad \text{for each } t \in [0, \infty). \quad (3.42)$$

As a first consequence of (3.42), we obtain that $x \mapsto f(t, x)$ is a non-decreasing and non-negative function for each $t \in [0, \infty)$. Furthermore,

$$\lim_{\varepsilon \rightarrow 0} \sup_{s \in [0, t]} \left[\int_0^\varepsilon x^{\lambda-1} f(s, x) dx + \int_{1/\varepsilon}^\infty x^{\lambda-1} f(s, x) dx \right] = 0 \quad (3.43)$$

for each $t \in (0, \infty)$ since $f \in \mathcal{C}([0, \infty); L^1(0, \infty; x^{\lambda-1} dx))$.

We will show that this convergence implies tightness of $(c_t^n)_{n \geq 1}$ in \mathcal{M}_λ^+ , uniformly with respect to $s \in [0, t]$. We consider $\varepsilon \in (0, 1/4)$, and since $x \mapsto F^{c_s^n}(x)$ is non-decreasing and $\lambda \in (0, 1]$, it follows from Lemma 3.2.:

$$\int_0^\varepsilon x^\lambda c_t^n(dx) + \int_{1/\varepsilon}^\infty x^\lambda c_t^n(dx) \leq \int_0^\varepsilon x^{\lambda-1} F^{c_t^n}(x) dx + \int_{1/(2\varepsilon)}^\infty x^{\lambda-1} F^{c_t^n}(x) dx.$$

The Lebesgue dominated convergence Theorem, (3.42) and (3.43) give

$$\lim_{\varepsilon \rightarrow 0} \sup_{n \geq 1} \sup_{s \in [0, t]} \left[\int_0^\varepsilon x^\lambda c_t^n(dx) + \int_{1/\varepsilon}^\infty x^\lambda c_t^n(dx) \right] = 0, \quad (3.44)$$

for every $t \in [0, \infty)$. Denoting by $c_t(dx) := -\partial_x f(t, x)$ the derivative with respect to x of f in the sense of distributions for $t \in (0, \infty)$, we deduce from (3.36), (3.42) and (3.44) that $c_t(dx) \in \mathcal{M}_\lambda^+$ with $M_\lambda(c_t) \leq e^{\kappa_2 C_\beta^\lambda t} M_\lambda(c^{in})$.

Consider now $\phi \in \mathcal{C}_c^1((0, \infty))$ and recall that $|\phi'(x)| \leq Cx^{\lambda-1}$ for some positive constant C . On the one hand, the time continuity of f implies that

$$t \mapsto \int_0^\infty \phi(x) c_t(dx) = \int_0^\infty \phi'(x) f(t, x) dx$$

is continuous on $[0, \infty)$. On the other hand, the convergence (3.42) entails

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{s \in [0, t]} \left| \int_0^\infty \phi(x)(c_s^n - c_s)(dx) \right| &= \lim_{n \rightarrow \infty} \sup_{s \in [0, t]} \left| \int_0^\infty \phi'(x) \left(F^{c_s^n}(x) - F^{c_s}(x) \right) dx \right| \\ &\leq \lim_{n \rightarrow \infty} \sup_{s \in [0, t]} \left| C \int_0^\infty x^{\lambda-1} \left(F^{c_s^n}(x) - F^{c_s}(x) \right) dx \right| \\ &= 0, \end{aligned} \tag{3.45}$$

for every $t \geq 0$. We then infer from (3.44), (3.45), Lemma 3.1., (3.2) and a density argument that for every $\phi \in \mathcal{H}_\lambda$, the map $t \mapsto \int_0^\infty \phi(x)c_t(dx)$ is continuous and

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{s \in [0, t]} \left| \int_0^\infty \phi(x)(c_s^n - c_s)(dx) \right| &= 0, \\ \lim_{n \rightarrow \infty} \sup_{s \in [0, t]} \left| \frac{1}{2} \int_0^\infty \int_0^\infty (A\phi)(x, y) K(x, y) (c_s^n(dx)c_s^n(dy) - c_s(dx)c_s(dy)) \right. \\ &\quad \left. + \int_0^\infty F(x) \int_\Theta (B\phi)(\theta, x) \beta(d\theta) (c_s^n - c_s)(dx) \right| = 0. \end{aligned}$$

We may thus pass to the limit as $n \rightarrow \infty$ in the integrated form of (2.10) for $(c_t^n)_{t \geq 0}$ and deduce that for all $t \geq 0$ and $\phi \in \mathcal{H}_\lambda$, we have

$$\begin{aligned} \int_0^\infty \phi(x)c_t(dx) &= \int_0^\infty \phi(x)c^{in}(x) dx \\ &\quad + \frac{1}{2} \int_0^\infty \int_0^\infty [\phi(x+y) - \phi(x) - \phi(y)] K(x, y) c_t(dx) c_t(dy) \\ &\quad + \int_0^\infty \int_\Theta \left[\sum_{i=1}^\infty \phi(\theta_i x) - \phi(x) \right] F(x) \beta(d\theta) c_t(dx). \end{aligned} \tag{3.46}$$

Classical arguments then allows us to differentiate (3.46) with respect to time and conclude that $(c_t^n)_{t \geq 0}$ is a $(c^{in}, K, F, \beta, \lambda)$ -weak solution to (1.1).

Existence and uniqueness for $c^{in} \in \mathcal{M}_\lambda^+$.

We have shown existence for $c^{in} \in \mathcal{M}_\lambda^+ \cap M_2^+$. Now we are going to extend the previous result to an initial condition only in \mathcal{M}_λ^+ . For this, we consider $(a_n)_{n \geq 1}$ and $(A_n)_{n \geq 1}$ two sequences in \mathbb{R}^+ such that a_n is non-increasing and converging to 0 and A_n non-decreasing and tending to $+\infty$ with $0 < a_0 \leq A_0$. We set $B_n = [a_n, A_n]$ and define

$$c^{in, n}(dx) := c^{in}|_{B_n}(dx),$$

note that trivially we have $M_2(c^{in, n}) < \infty$. Next, we call $(\tilde{c}_t^n)_{t \geq 1}$ the $(c^{in, n}, K, F, \beta, \lambda)$ -weak solution to (1.1) constructed in the previous section.

Owing to Proposition 3.3. and (3.15), we have for $t \geq 0$ and $n \geq 1$

$$\int_0^\infty x^{\lambda-1} \left| F^{\tilde{c}_t^{n+1}}(x) - F^{\tilde{c}_t^n}(x) \right| dx \leq e^{Ct} \int_0^\infty x^{\lambda-1} \left| F^{c^{in, n+1}}(x) - F^{c^{in, n}}(x) \right| dx,$$

Next, we have

$$\begin{aligned} & \int_0^\infty x^{\lambda-1} \left| F^{c^{in,n+1}}(x) - F^{c^{in,n}}(x) \right| dx \\ &= \int_0^{+\infty} x^{\lambda-1} \left| \int_0^{+\infty} \mathbf{1}_{[x,+\infty)}(y) (c^{in}|_{B_n} - c^{in}|_{B_{n+1}})(dy) \right| dx \\ &= \int_0^{+\infty} x^{\lambda-1} \int_0^{+\infty} \mathbf{1}_{[x,+\infty)}(y) (\mathbf{1}_{[a_{n+1},a_n)}(y) + \mathbf{1}_{[A_n,A_{n+1})}(y)) c^{in}(dy) dx, \end{aligned}$$

note that since $\sum_{n \geq 0} [\mathbf{1}_{[a_{n+1},a_n)}(y) + \mathbf{1}_{[A_n,A_{n+1})}(y)] \leq \mathbf{1}_{\mathbb{R}^+}(y)$ the term in the right-hand of the last inequality is summable. We conclude that $(t \mapsto F^{\tilde{c}_t^n})_{n \geq 1}$ is a Cauchy sequence in $\mathcal{C}([0, \infty); L^1(0, \infty; x^{\lambda-1} dx))$ and there is

$$f \in \mathcal{C}([0, \infty); L^1(0, \infty; x^{\lambda-1} dx)),$$

such that

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, t]} \int_0^\infty x^{\lambda-1} \left| F^{\tilde{c}_s^{n+1}}(x) - f(s, x) \right| dx = 0 \quad \text{for each } t \in [0, \infty).$$

and we conclude using the same arguments as in the previous case, setting $c_t := -\partial_x f(t, x)$ in the sense of distributions, that $(c_t)_{t \geq 0}$ is a $(c^{in}, K, F, \beta, \lambda)$ -weak solution to (1.1) in the sense of Definition 2.4.

This completes the proof of Theorem 2.5. \square

4. STOCHASTIC COALESCENCE-FRAGMENTATION PROCESSES

Let \mathcal{S}^\downarrow the set of non-increasing sequences $m = (m_n)_{n \geq 1}$ with values in $[0, +\infty)$. A state m in \mathcal{S}^\downarrow represents the sequence of the ordered masses of the particles in a particle system. Next, for $\lambda \in (0, 1]$, consider

$$\ell_\lambda = \left\{ m = (m_k)_{k \geq 1} \in \mathcal{S}^\downarrow, \|m\|_\lambda := \sum_{k=1}^\infty m_k^\lambda < \infty \right\}.$$

Consider also the sets of finite particle systems, completed for convenience with infinitely many 0-s.

$$\ell_{0+} = \{ m = (m_k)_{k \geq 1} \in \mathcal{S}^\downarrow, \inf\{k \geq 1, m_k = 0\} < \infty \}.$$

Remark 4.1. Note that for all $0 < \lambda_1 < \lambda_2$, $\ell_{0+} \subset \ell_{\lambda_1} \subset \ell_{\lambda_2}$. Note also that, since $\|m\|_1 \leq \|m\|_\lambda^{\frac{1}{\lambda}}$ the total mass of $m \in \ell_\lambda$ is always finite.

Hypothesis 4.2. We consider a coagulation kernel K bounded on every compact set in $[0, \infty)^2$. There exists $\lambda \in (0, 1]$ such that for all $a > 0$ there exists a constant $\kappa_a > 0$ such that for all $x, y, \tilde{x}, \tilde{y} \in (0, a]$,

$$|K(x, y) - K(\tilde{x}, \tilde{y})| \leq \kappa_a [|x^\lambda - \tilde{x}^\lambda| + |y^\lambda - \tilde{y}^\lambda|], \quad (4.1)$$

We consider also a fragmentation kernel $F : (0, \infty) \mapsto [0, \infty)$, bounded on every compact set in $[0, \infty)$. There exists $\alpha \in [0, \infty)$ such that for all $a > 0$ there exists a constant $\mu_a > 0$ such that for all $x, \tilde{x} \in (0, a]$,

$$|F(x) - F(\tilde{x})| \leq \mu_a |x^\alpha - \tilde{x}^\alpha|. \quad (4.2)$$

Finally, we consider a measure β on Θ satisfying (2.5), (2.6).

We will use the following conventions

$$\begin{aligned} K(x, 0) &= 0 \quad \text{for all } x \in [0, \infty), \\ F(0) &= 0. \end{aligned}$$

Remark that this convention is also valid, for example, for $K = 1$. Actually, 0 is a symbol used to refer to a particle that does not exist. For $\theta \in \Theta$ and $x \in (0, \infty)$ we will write $\theta \cdot x$ to say that the particle of mass x of the system splits into $\theta_1 x, \theta_2 x, \dots$.

Consider $m \in \ell_\lambda$, the dynamics of the process is as follows. A pair of particles m_i and m_j coalesce with rate given by $K(m_i, m_j)$ and is described by the map $c_{ij} : \ell_\lambda \mapsto \ell_\lambda$ (see below). A particle m_i fragmentates following the dislocation configuration $\theta \in \Theta$ with rate given by $F(m_i)\beta(d\theta)$ and is described by the map $f_{i\theta} : \ell_\lambda \mapsto \ell_\lambda$, with

$$\begin{aligned} c_{ij}(m) &= \text{reorder}(m_1, \dots, m_{i-1}, m_i + m_j, m_{i+1}, \dots, m_{j-1}, m_{j+1}, \dots), \\ f_{i\theta}(m) &= \text{reorder}(m_1, \dots, m_{i-1}, \theta \cdot m_i, m_{i+1}, \dots), \end{aligned} \quad (4.3)$$

the reordering being in the decreasing order.

Distances on S^\downarrow

We endow S^\downarrow with the pointwise convergence topology, which can be metrized by the distance

$$d(m, \tilde{m}) = \sum_{k \geq 1} 2^{-k} |m_k - \tilde{m}_k|. \quad (4.4)$$

Also, for $\lambda \in (0, 1]$ and $m, \tilde{m} \in \ell_\lambda$, we set

$$d_\lambda(m, \tilde{m}) = \sum_{k \geq 1} |m_k^\lambda - \tilde{m}_k^\lambda| \quad (4.5)$$

Infinitesimal generator $\mathcal{L}_{K,F}^\beta$

Consider some coagulation and fragmentation kernels K and F and a measure β . We define the infinitesimal generator $\mathcal{L}_{K,F}^\beta$ for any $\Phi : \ell_\lambda \rightarrow \mathbb{R}$ sufficiently regular and for any $m \in \ell_\lambda$ by

$$\mathcal{L}_{K,F}^\beta \Phi(m) = \sum_{1 \leq i < j < \infty} K(m_i, m_j) [\Phi(c_{ij}(m)) - \Phi(m)] + \sum_{i \geq 1} F(m_i) \int_{\Theta} [\Phi(f_{i\theta}(m)) - \Phi(m)] \beta(d\theta). \quad (4.6)$$

5. RESULTS

We define first the finite coalescence - fragmentation process. In order to prove the existence of this process we need to add two properties to the measure β . Namely, the measure of Θ must be finite and the number of fragments at each fragmentation must be bounded:

$$\begin{cases} \beta(\Theta) < \infty, \\ \beta(\Theta \setminus \Theta_k) = 0 \quad \text{for some } k \in \mathbb{N}, \end{cases} \quad (5.1)$$

where

$$\Theta_k = \{\theta = (\theta_n)_{n \geq 1} \in \Theta : \theta_{k+1} = \theta_{k+2} = \dots = 0\}.$$

Proposition 5.1 (Finite Coalescence - Fragmentation processes). *Consider $\lambda \in (0, 1]$ and $m \in \ell_{0+}$. Assume that the coagulation kernel K , the fragmentation kernel F and a measure β satisfy Hypotheses 4.2. Furthermore, suppose that β satisfies (5.1).*

Then, there exists a unique (in law) strong Markov process $(M(m, t))_{t \geq 0}$ starting at $M(m, 0) = m$ and with infinitesimal generator $\mathcal{L}_{K, F}^\beta$.

We wish to extend this process to the case where the initial condition consists of infinitely many particles and for more general fragmentation measures β . For this, we will build a particular sequence of finite coalescence - fragmentation processes, the result will be obtained by passing to the limit.

Lemma 5.2 (Definition.- The finite process $M^n(m, t)$). *Consider $\lambda \in (0, 1]$, $\alpha \geq 0$ and $m \in \ell_{0+}$. Assume that the coagulation kernel K , the fragmentation kernel F and the measure β satisfy Hypotheses 4.2. Furthermore, recall β_n as defined by (3.35).*

Then, there exists a unique (in law) strong Markov process $(M^n(m, t))_{t \geq 0}$ starting at m and with infinitesimal generator $\mathcal{L}_{K, F}^{\beta_n}$.

This lemma is straightforward, it suffices to note that β_n satisfies (5.1) and to use Proposition 5.1. Indeed, recall (2.8), for $n \geq 1$

$$\beta_n(\Theta) = \int_{\Theta} \mathbb{1}_{\{1 - [\psi_n(\theta)]_1 \geq \frac{1}{n}\}} \beta(d\theta) \leq n \int_{\Theta} (1 - \theta_1) \beta(d\theta) \leq n C_\beta^\lambda < \infty.$$

Our main result concerning stochastic Coalescence-Fragmentation processes is the following.

Theorem 5.3. *Consider $\lambda \in (0, 1]$, $\alpha \geq 0$. Assume that the coagulation K and the fragmentation F kernels and that a measure β satisfy Hypotheses 4.2. Endow ℓ_λ with the distance d_λ .*

i) *For any $m \in \ell_\lambda$, there exists a (necessarily unique in law) strong Markov process $(M(m, t))_{t \geq 0} \in \mathbb{D}([0, \infty), \ell_\lambda)$ satisfying the following property.*

For any sequence $m^n \in \ell_{0+}$ such that $\lim_{n \rightarrow \infty} d_\lambda(m^n, m) = 0$, the sequence $(M^n(m^n, t))_{t \geq 0}$ defined in Lemma 5.2, converges in law, in $\mathbb{D}([0, \infty), \ell_\lambda)$, to $(M(m, t))_{t \geq 0}$.

ii) *The obtained process is Feller in the sense that for all $t \geq 0$, the map $m \mapsto \text{Law}(M(m, t))$ is continuous from ℓ_λ into $\mathcal{P}(\ell_\lambda)$ (endowed with the distance d_λ).*

iii) *For all bounded $\Phi : \ell_\lambda \rightarrow \mathbb{R}$ satisfying $|\Phi(m) - \Phi(\tilde{m})| \leq a d(m, \tilde{m})$ for some $a > 0$, the process*

$$\Phi(M(m, t)) - \Phi(m) - \int_0^t \mathcal{L}_{K, F}^\beta(M(m, s)) ds$$

is a local martingale.

We have chosen an explicit sequence of measure $(\beta_n)_{n \geq 1}$ because it will be easier to manipulate when coupling two coalescence-fragmentation processes. Nevertheless, more generally, taking any sequence of measures β_n satisfying (5.1) and converging towards β in a suitable sense as n tends to infinity should provide the same result.

This result extends those of Fournier [7] concerning only coalescence and Bertoin [3, 2] concerning only fragmentation. We point out that in [3] is not assumed $C_\beta^\lambda < \infty$ but only $C_\beta^1 < \infty$. However, we believe that in presence of coalescence our hypotheses on β are optimal. We refer to [4] for an extensive study of coagulation and fragmentation systems.

5.1. A Poisson-driven S.D.E.. We now introduce a representation of the stochastic processes of coagulation - fragmentation in terms of Poisson measures, in order to couple two of these processes with different initial data.

Definition 5.4. Assume that a coagulation kernel K , a fragmentation kernel F and a measure β satisfy Hypotheses 4.2.

- a) For the coagulation, we consider a Poisson measure $N(dt, d(i, j), dz)$ on $[0, \infty) \times \{(i, j) \in \mathbb{N}^2, i < j\} \times [0, \infty)$ with intensity measure $dt [\sum_{k < l} \delta_{(k, l)}(d(i, j))] dz$, and denote by $(\mathcal{F}_t)_{t \geq 0}$ the associated canonical filtration.
- b) For the fragmentation, we consider $M(dt, di, d\theta, dz)$ a Poisson measure on $[0, \infty) \times \mathbb{N} \times \Theta \times [0, \infty)$ with intensity measure $dt (\sum_{k \geq 1} \delta_k(di)) \beta(d\theta) dz$, and denote by $(\mathcal{G}_t)_{t \geq 0}$ the associated canonical filtration. M is independent of N .

Finally, we consider $m \in \ell_\lambda$. A càdlàg $(\mathcal{H}_t)_{t \geq 0} = (\sigma(\mathcal{F}_t, \mathcal{G}_t))_{t \geq 0}$ -adapted process $(M(m, t))_{t \geq 0}$ is said to be a solution to $SDE(K, F, m, N, M)$ if it belongs a.s. to $\mathbb{D}([0, \infty), \ell_\lambda)$ and if for all $t \geq 0$, a.s.

$$\begin{aligned} M(m, t) = & m + \int_0^t \int_{i < j} \int_0^\infty [c_{ij}(M(m, s-)) - M(m, s-)] \mathbb{1}_{\{z \leq K(M_i(m, s-), M_j(m, s-))\}} \\ & N(dt, d(i, j), dz) \\ & + \int_0^t \int_i \int_\Theta \int_0^\infty [f_{i\theta}(M(m, s-)) - M(m, s-)] \mathbb{1}_{\{z \leq F(M_i(m, s-))\}} M(dt, di, d\theta, dz) \end{aligned} \quad (5.2)$$

Remark that due to the independence of the Poisson measures only a coagulation or a fragmentation mechanism occurs at each instant t .

Proposition 5.5. Let $m \in \ell_{0+}$. Consider the coagulation kernel K , the fragmentation kernel F , the measure β and the Poisson measures N and M as in Definition 5.4, we furthermore suppose that β satisfies (5.1).

Then there exists a unique process $(M(m, t))_{t \geq 0}$ which solves $SDE(K, F, m, N, M)$. This process is a finite Coalescence-Fragmentation process in the sense of Proposition 5.1.

This proposition will be proved using an a priori estimate, we will show that in such a system the number of particles remains finite, we will then use that the total rate of jumps of the system is bounded by the number of particles to conclude. We begin the proof by checking that the integrals in (5.2) always make sense.

Lemma 5.6. Let $\lambda \in (0, 1]$ and $\alpha \geq 0$, consider K , F , β and the Poisson measures N and M as in Definition 5.4. For any $(\mathcal{H}_t)_{t \geq 0}$ -adapted process $(M(t))_{t \geq 0}$ belonging a.s. to $\mathbb{D}([0, \infty), \ell_\lambda)$, a.s.

$$\begin{aligned} I_1 &= \int_0^t \int_{i < j} \int_0^\infty [c_{ij}(M(s-)) - M(s-)] \mathbb{1}_{\{z \leq K(M_i(s-), M_j(s-))\}} N(dt, d(i, j), dz), \\ I_2 &= \int_0^t \int_i \int_\Theta \int_0^\infty [f_{i\theta}(M(s-)) - M(s-)] \mathbb{1}_{\{z \leq F(M_i(s-))\}} M(dt, di, d\theta, dz), \end{aligned}$$

are well-defined and finite for all $t \geq 0$.

5.2. A Gronwall type inequality. We will also check a fundamental inequality, which shows that the distance between two coagulation-fragmentation processes cannot increase excessively while their moments of order λ remain finite.

Proposition 5.7. *Let $\lambda \in (0, 1]$, $\alpha \geq 0$ and $m, \tilde{m} \in \ell_\lambda$. Consider K, F, β and the Poisson measures N and M as in Definition 5.4. Assume that there exist solutions $M(m, t)$ and $M(\tilde{m}, t)$ to $SDE(K, F, m, N, M)$ and $SDE(K, F, \tilde{m}, N, M)$.*

i) *The map $t \mapsto \|M(m, t)\|_1$ is a.s. non-increasing. Furthermore, for all $t \geq 0$*

$$\mathbb{E} \left[\sup_{s \in [0, t]} \|M(m, s)\|_\lambda \right] \leq \|m\|_\lambda e^{\bar{F}_m C_\beta^\lambda t},$$

where $\bar{F}_m = \sup_{[0, \|m\|_1]} F(x)$.

ii) *We define, for all $x > 0$, the stopping time $\tau(m, x) = \inf\{t \geq 0, \|M(m, t)\|_\lambda \geq x\}$. Then for all $t \geq 0$ and all $x > 0$,*

$$\mathbb{E} \left[\sup_{s \in [0, t \wedge \tau(m, x) \wedge \tau(\tilde{m}, x)]} d_\lambda(M(m, s), M(\tilde{m}, s)) \right] \leq d_\lambda(m, \tilde{m}) e^{C(x+1)t}.$$

where C is a positive constant depending on $K, F, C_\beta^\lambda, \|m\|_1$ and $\|\tilde{m}\|_1$.

6. PROOFS

In this section we give the proves to the results in Section 5.

6.1. Proof of Lemma 5.6. The processes in the integral being càdlàg and adapted, it suffices to check the compensators are a.s. finite. We have to show that a.s., for all $k \geq 1$, all $t \geq 0$,

$$\begin{aligned} C_k(t) &= \int_0^t ds \sum_{i < j} K(M_i(s), M_j(s)) |[c_{ij}(M(s))]_k - M_k(s)| \\ &\quad + \int_0^t ds \int_{\Theta} \beta(d\theta) \sum_{i \geq 1} F(M_i(s)) |[f_{i\theta}(M(s))]_k - M_k(s)| < \infty \end{aligned}$$

Note first that for all $s \in [0, t]$, $\sup_i M_i(s) \leq \sup_{[0, t]} \|M(s)\|_1 \leq \sup_{[0, t]} \|M(s)\|_\lambda^{1/\lambda} =: a_t < \infty$ a.s. since M belongs a.s. to $\mathbb{D}([0, \infty), \ell_\lambda)$. Next, let

$$\bar{K}_t = \sup_{(x, y) \in [0, a_t]^2} K(x, y) \quad \text{and} \quad \bar{F}_t = \sup_{x \in [0, a_t]} F(x) \quad (6.1)$$

which are a.s. finite since K and F are bounded on every compact in $[0, \infty)^2$ and $[0, \infty)$ respectively. Then using (A.15) and (A.17) with (2.7) and (2.8), we write:

$$\begin{aligned} \sum_{k \geq 1} 2^{-k} C_k(t) &= \int_0^t ds \sum_{i < j} K(M_i(s), M_j(s)) d(c_{ij}(M(s)), M(s)) \\ &\quad + \int_0^t ds \int_{\Theta} \beta(d\theta) \sum_{i \geq 1} F(M_i(s)) d(f_{i\theta}(M(s)), M(s)) \\ &\leq \bar{K}_t \int_0^t ds \sum_{i < j} \frac{3}{2} 2^{-i} M_j(s) + C_\beta^\lambda \bar{F}_t \int_0^t ds \sum_{i \geq 1} 2^{-i} M_i(s) \\ &\leq \left(\frac{3}{2} \bar{K}_t + C_\beta^\lambda \bar{F}_t \right) \int_0^t \|M(s)\|_1 ds \leq t \left(\frac{3}{2} \bar{K}_t + C_\beta^\lambda \bar{F}_t \right) \sup_{[0, t]} \|M(s)\|_\lambda^{1/\lambda} < \infty. \end{aligned}$$

6.2. Proof of Proposition 5.7. Let $\lambda \in (0, 1]$, $\alpha \geq 0$ and $m \in \ell_\lambda$, and consider $(M(m, t))_{t \geq 0}$ the solution to $SDE(K, F, m, N, M)$. We begin studying the behavior of the moments of this solution.

First, we will see that under our assumptions the total mass $\|\cdot\|_1$ does *a.s.* not increase in time. This property is fundamental in this approach since that we will use the bound $\sup_{[0, \|M(m, 0)\|_1]} F(x)$, which is finite whenever $\|M(m, 0)\|_\lambda$ is. This will allow us to bound lower moments of $M(m, t)$ for $t \geq 0$.

Next, we will prove that the λ -moment remains finite in time. Finally, we will show that the distance d_λ between two solutions to (5.2) are bounded in time while their λ -moments remain finite.

Moments Estimates.- The aim of this paragraph is to prove *i*).

The solution to $SDE(K, F, m, N, M)$ will be written $M(t) := M(m, t)$ for simplicity. First, from (5.2) we have for $k \geq 1$,

$$\begin{aligned} M_k(t) &= M_k(0) + \int_0^t \int_{i < j} \int_0^\infty [[c_{ij}(M(s-))]_k - M_k(s-)] \mathbf{1}_{\{z \leq K(M_i(s-), M_j(s-))\}} \\ &\quad N(dt, d(i, j), dz) \\ &\quad + \int_0^t \int_i \int_\Theta \int_0^\infty [[f_{i\theta}(M(s-))]_k - M_k(s-)] \mathbf{1}_{\{z \leq F(M_i(s-))\}} M(dt, di, \beta(d\theta), dz), \end{aligned}$$

and summing on k , we deduce

$$\begin{aligned} \|M(t)\|_1 &= \|m\|_1 + \int_0^t \int_{i < j} \int_0^\infty [\|c_{ij}(M(s-))\|_1 - \|M(s-)\|_1] \mathbf{1}_{\{z \leq K(M_i(s-), M_j(s-))\}} \\ &\quad N(dt, d(i, j), dz) \\ &\quad + \int_0^t \int_i \int_\Theta \int_0^\infty [\|f_{i\theta}(M(s-))\|_1 - \|M(s-)\|_1] \mathbf{1}_{\{z \leq F(M_i(s-))\}} M(dt, di, \beta(d\theta), dz) \end{aligned} \quad (6.2)$$

Note that, clearly $\|c_{ij}(m)\|_1 = \|m\|_1$ and $\|f_{i\theta}(m)\|_1 = \|m\|_1 + m_i \left(\sum_{k \geq 0} \theta_k - 1 \right) \leq \|m\|_1$ for all $m \in \ell_\lambda$, since $\sum_{k \geq 0} \theta_k \leq 1$ β -a.e. Then,

$$\sup_{[0, t]} \|M(s)\|_1 \leq \|m\|_1, \text{ a.s. } \forall t \geq 0.$$

This implies for all $s \in [0, t]$, $\sup_i M_i(s) \leq \sup_{[0, t]} \|M(s)\|_1 \leq \|m\|_1$ *a.s.* We set

$$\overline{K}_m = \sup_{(x, y) \in [0, \|m\|_1]^2} K(x, y) \quad \text{and} \quad \overline{F}_m = \sup_{x \in [0, \|m\|_1]} F(x) \quad (6.3)$$

which are finite since K and F are bounded on every compact in $[0, \infty)^2$ and $[0, \infty)$ respectively.

In the same way, from (5.2) for $\lambda \in (0, 1)$ we have for $k \geq 1$,

$$\begin{aligned} [M_k(t)]^\lambda &= [M_k(0)]^\lambda + \int_0^t \int_{i < j} \int_0^\infty [[c_{ij}(M(s-))]_k^\lambda - [M_k(s-)]^\lambda] \mathbf{1}_{\{z \leq K(M_i(s-), M_j(s-))\}} \\ &\quad N(dt, d(i, j), dz) \\ &\quad + \int_0^t \int_i \int_\Theta \int_0^\infty [[f_{i\theta}(M(s-))]_k^\lambda - [M_k(s-)]^\lambda] \mathbf{1}_{\{z \leq F(M_i(s-))\}} M(dt, di, \beta(d\theta), dz), \end{aligned}$$

and summing on k , we deduce

$$\begin{aligned} \|M(t)\|_\lambda &= \|m\|_\lambda + \int_0^t \int_{i < j} \int_0^\infty [\|c_{ij}(M(s-))\|_\lambda - \|M(s-)\|_\lambda] \mathbf{1}_{\{z \leq K(M_i(s-), M_j(s-))\}} \\ &\quad N(dt, d(i, j), dz) \\ &\quad + \int_0^t \int_i \int_\Theta \int_0^\infty [\|f_{i\theta}(M(s-))\|_\lambda - \|M(s-)\|_\lambda] \mathbf{1}_{\{z \leq F(M_i(s-))\}} M(dt, di, \beta(d\theta), dz). \end{aligned}$$

We take the expectation, use (A.4) and (A.5) with (2.8) and (6.3), to obtain

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [0, t]} \|M(s)\|_\lambda \right] &\leq \|m\|_\lambda + C_\beta^\lambda \int_0^t \mathbb{E} \left[\sum_{i \geq 1} F(M_i(s)) M_i^\lambda(s) \right] ds \\ &\leq \|m\|_\lambda + \bar{F}_m C_\beta^\lambda \int_0^t \mathbb{E} [\|M(s)\|_\lambda] ds. \end{aligned}$$

We conclude using the Gronwall Lemma.

Bound for d_λ . The aim of this paragraph is to prove *ii*). For this, we consider for $m, \tilde{m} \in \ell_\lambda$ some solutions to $SDE(K, F, m, N, M)$ and $SDE(K, F, \tilde{m}, N, M)$ which will be written $M(t) := M(m, t)$ and $\tilde{M}(t) := M(\tilde{m}, t)$ for simplicity. Since M and \tilde{M} solve (5.2) with the same Poisson measures N and M , we have

$$d_\lambda(M(t), \tilde{M}(t)) = d_\lambda(m, \tilde{m}) + A_t^c + B_t^c + C_t^c + A_t^f + B_t^f + C_t^f, \quad (6.4)$$

where

$$\begin{aligned} A_t^c &= \int_0^t \int_{i < j} \int_0^\infty \left\{ d_\lambda \left(c_{ij}(M(s-)), c_{ij}(\tilde{M}(s-)) \right) - d_\lambda \left(M(s-), \tilde{M}(s-) \right) \right\} \\ &\quad \mathbf{1}_{\{z \leq K(M_i(s-), M_j(s-)) \wedge K(\tilde{M}_i(s-), \tilde{M}_j(s-))\}} N(ds, d(i, j), dz), \end{aligned}$$

$$\begin{aligned} B_t^c &= \int_0^t \int_{i < j} \int_0^\infty \left\{ d_\lambda \left(c_{ij}(M(s-)), \tilde{M}(s-) \right) - d_\lambda \left(M(s-), \tilde{M}(s-) \right) \right\} \\ &\quad \mathbf{1}_{\{K(\tilde{M}_i(s-), \tilde{M}_j(s-)) \leq z \leq K(M_i(s-), M_j(s-))\}} N(ds, d(i, j), dz), \end{aligned}$$

$$\begin{aligned} C_t^c &= \int_0^t \int_{i < j} \int_0^\infty \left\{ d_\lambda \left(M(s-), c_{ij}(\tilde{M}(s-)) \right) - d_\lambda \left(M(s-), \tilde{M}(s-) \right) \right\} \\ &\quad \mathbf{1}_{\{K(M_i(s-), M_j(s-)) \leq z \leq K(\tilde{M}_i(s-), \tilde{M}_j(s-))\}} N(ds, d(i, j), dz), \end{aligned}$$

$$\begin{aligned} A_t^f &= \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ d_\lambda \left(f_{i\theta}(M(s-)), f_{i\theta}(\tilde{M}(s-)) \right) - d_\lambda \left(M(s-), \tilde{M}(s-) \right) \right\} \\ &\quad \mathbf{1}_{\{z \leq F(M_i(s-)) \wedge F(\tilde{M}_i(s-))\}} M(ds, di, d\theta, dz), \end{aligned}$$

$$\begin{aligned} B_t^f &= \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ d_\lambda \left(f_{i\theta}(M(s-)), \tilde{M}(s-) \right) - d_\lambda \left(M(s-), \tilde{M}(s-) \right) \right\} \\ &\quad \mathbf{1}_{\{F(\tilde{M}_i(s-)) \leq z \leq F(M_i(s-))\}} M(ds, di, d\theta, dz), \end{aligned}$$

$$C_t^f = \int_0^t \int_i \int_{\Theta} \int_0^\infty \left\{ d_\lambda \left(M(s-), f_{i\theta}(\tilde{M}(s-)) \right) - d_\lambda \left(M(s-), \tilde{M}(s-) \right) \right\} \mathbb{1}_{\{F(M_i(s-)) \leq z \leq F(\tilde{M}_i(s-))\}} M(ds, di, d\theta, dz).$$

Note also that

$$\left| d_\lambda \left(c_{ij}(M(s-)), \tilde{M}(s-) \right) - d_\lambda \left(M(s-), \tilde{M}(s-) \right) \right| \leq d_\lambda \left(c_{ij}(M(s-)), M(s-) \right) \quad (6.5)$$

$$\left| d_\lambda \left(f_{i\theta}(M(s-)), \tilde{M}(s-) \right) - d_\lambda \left(M(s-), \tilde{M}(s-) \right) \right| \leq d_\lambda \left(f_{i\theta}(M(s-)), M(s-) \right) \quad (6.6)$$

We now search for an upper bound to the expression in (6.4). We define, for all $x > 0$, the stopping time $\tau(m, x) := \inf\{t \geq 0; \|M(m, t)\|_\lambda \geq x\}$. We set $\tau_x = \tau(m, x) \wedge \tau(\tilde{m}, x)$.

Furthermore, since for all $s \in [0, t]$, $\sup_i M_i(s) \leq \sup_{[0, t]} \|M(s)\|_1 \leq \|m\|_1 := a_m$ a.s, equivalently for \tilde{M} , we put $a_{\tilde{m}} = \|\tilde{m}\|_1$. For $a := a_m \vee a_{\tilde{m}}$ we set κ_a and μ_a the constants for which the kernels K and F satisfy (4.1) and (4.2). Finally, we set \bar{F}_m as in (6.3).

Term A_t^c : using (A.8) we deduce that this term is non-positive, we bound it by 0.

Term B_t^c : we take the expectation, use (6.5), (A.6) and (4.1), to obtain

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [0, t \wedge \tau_x]} B_s^c \right] &\leq \mathbb{E} \left[\int_0^{t \wedge \tau_x} \sum_{i < j} 2M_j^\lambda(s) \left| K(M_i(s), M_j(s)) - K(\tilde{M}_i(s), \tilde{M}_j(s)) \right| ds \right] \\ &\leq 2\kappa_a \mathbb{E} \left[\int_0^{t \wedge \tau_x} \sum_{i < j} M_j^\lambda(s) \left(\left| M_i^\lambda(s) - \tilde{M}_i^\lambda(s) \right| + \left| M_j^\lambda(s) - \tilde{M}_j^\lambda(s) \right| \right) ds \right] \\ &\leq 2\kappa_a \mathbb{E} \left[\int_0^{t \wedge \tau_x} \sum_{i \geq 1} \left| M_i^\lambda(s) - \tilde{M}_i^\lambda(s) \right| \sum_{j \geq i+1} M_j^\lambda(s) ds \right] \\ &\quad + 2\kappa_a \mathbb{E} \left[\int_0^{t \wedge \tau_x} \sum_{j \geq 2} \left| M_j^\lambda(s) - \tilde{M}_j^\lambda(s) \right| \sum_{i=1}^{j-1} M_i^\lambda(s) ds \right] \\ &\leq 4\kappa_a \mathbb{E} \left[\int_0^{t \wedge \tau_x} \|M(s)\|_\lambda d_\lambda \left(M(s), \tilde{M}(s) \right) ds \right] \\ &\leq 4\kappa_a x \int_0^t \mathbb{E} \left[\sup_{u \in [0, s \wedge \tau_x]} d_\lambda \left(M(u), \tilde{M}(u) \right) \right] ds, \end{aligned} \quad (6.7)$$

we used that for $m \in \ell_\lambda$, $\sum_{i=1}^{j-1} m_i^\lambda \leq \sum_{i=1}^{j-1} m_i^\lambda \leq \|m\|_\lambda$.

Term C_t^c : it is treated exactly as B_t^c .

Term A_t^f : We take the expectation, and use (A.9) together with (2.7) and (2.8), to obtain

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \in [0, t \wedge \tau_x]} A_s^f \right] &\leq C_\beta^\lambda \mathbb{E} \left[\int_0^{t \wedge \tau_x} \sum_{i \geq 1} \left(F(M_i(s)) \wedge F(\tilde{M}_i(s)) \right) \left| M_i^\lambda(s) - \tilde{M}_i^\lambda(s) \right| ds \right] \\
&\leq \bar{F}_m C_\beta^\lambda \mathbb{E} \left[\int_0^{t \wedge \tau_x} \sum_{i \geq 1} \left| M_i^\lambda(s) - \tilde{M}_i^\lambda(s) \right| ds \right] \\
&\leq \bar{F}_m C_\beta^\lambda \int_0^t \mathbb{E} \left[\sup_{u \in [0, s \wedge \tau_x]} d_\lambda \left(M(u), \tilde{M}(u) \right) \right] ds. \tag{6.8}
\end{aligned}$$

Term B_t^f : we take the expectation and use (4.2) (recall $a := a_m \vee a_{\tilde{m}}$), (6.6), (A.7) together with (2.7) and (2.8), (A.3) and finally Proposition 5.7. *ii*), to obtain

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \in [0, t \wedge \tau_x]} B_s^f \right] &\leq 2C_\beta^\lambda \mathbb{E} \left[\int_0^{t \wedge \tau_x} \sum_{i \geq 1} \left| F(M_i(s)) - F(\tilde{M}_i(s)) \right| M_i^\lambda(s) ds \right] \\
&\leq 2\mu_a C_\beta^\lambda \mathbb{E} \left[\int_0^{t \wedge \tau_x} \sum_{i \geq 1} \left| M_i(s)^\alpha - \tilde{M}_i(s)^\alpha \right| \left(M_i^\lambda(s) + \tilde{M}_i^\lambda(s) \right) ds \right] \\
&\leq 2\mu_a C_\beta^\lambda C \mathbb{E} \left[\int_0^{t \wedge \tau_x} \left(\|M(s)\|_1^\alpha + \|\tilde{M}(s)\|_1^\alpha \right) \sum_{i \geq 1} \left| M_i^\lambda(s) - \tilde{M}_i^\lambda(s) \right| ds \right] \\
&\leq 4\mu_a C_\beta^\lambda C \left(\|m\|_1^\alpha \vee \|\tilde{m}\|_1^\alpha \right) \times \int_0^t \mathbb{E} \left[\sup_{u \in [0, s \wedge \tau_x]} d_\lambda \left(M(u), \tilde{M}(u) \right) \right] ds. \tag{6.9}
\end{aligned}$$

Term C_t^f : it is treated exactly as B_t^f .

Conclusion. - we take the expectation on (6.4) and gather (6.7), (6.8) and (6.9) to obtain

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \in [0, t \wedge \tau_x]} d_\lambda \left(M(s), \tilde{M}(s) \right) \right] &\leq d_\lambda(m, \tilde{m}) + [8\kappa_a x + 8\mu_a C_\beta^\lambda C \left(\|m\|_1^\alpha \vee \|\tilde{m}\|_1^\alpha \right) + \bar{F}_m C_\beta^\lambda] \\
&\quad \times \int_0^t \mathbb{E} \left[\sup_{u \in [0, s \wedge \tau_x]} d_\lambda \left(M(u), \tilde{M}(u) \right) \right] ds. \tag{6.10}
\end{aligned}$$

We conclude using the Gronwall Lemma:

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \in [0, t \wedge \tau_x]} d_\lambda \left(M(s), \tilde{M}(s) \right) \right] &\leq d_\lambda(m, \tilde{m}) \times e^{C(x \vee 1 \vee \|m\|_1^\alpha \vee \|\tilde{m}\|_1^\alpha) t} \\
&\leq d_\lambda(m, \tilde{m}) e^{C(x+1)t}.
\end{aligned}$$

Where C is a positive constant depending on $\lambda, \alpha, \kappa_a, \mu_a, K, F, C_\beta^\lambda, \|m\|_1$ and $\|\tilde{m}\|_1$.

This ends the proof of Proposition 5.7.

6.3. Existence and uniqueness, finite case. The aim of this section is to prove Proposition 5.1 which is a consequence of Proposition 5.5. We will prove existence and uniqueness of the Finite Coalescence - Fragmentation processes showing an a priori estimate of such a process.

Lemma 6.1. *Let $m \in \ell_{0+}$, consider K, F, β and the Poisson measures N and M as in Definition 5.4. and assume that β satisfies (5.1). Assume that there exists $(M(m, t))_{t \geq 0}$ solution to SDE(K, F, m, N, M).*

i) The number of particles in the system remains a.s. bounded,

$$\sup_{[0, t]} N_s < \infty, \text{ a.s. for all } t \geq 0,$$

where $N_t = \text{card}\{M_i(m, t) : M_i(m, t) > 0\} = \sum_{i \geq 1} \mathbb{1}_{\{M_i(m, t) > 0\}}$.

ii) The coalescence and fragmentation jump rates of the process $(M(m, t))_{t \geq 0}$ are a.s. bounded,

$$\sup_{[0, t]} \rho_c(s) < \infty, \text{ a.s. for all } t \geq 0,$$

$$\sup_{[0, t]} \rho_f(s) < \infty, \text{ a.s. for all } t \geq 0,$$

where $\rho_c(t) := \sum_{i < j} K(M_i(m, t), M_j(m, t))$ and $\rho_f(t) := \beta(\Theta) \sum_{i \geq 1} F(M_i(m, t))$.

Proof. First, recall that the measure β satisfies (5.1). We put $M(t) = M(m, t)$ for simplicity and define $\Phi(m) = \sum_{n \geq 1} \mathbb{1}_{\{m_n > 0\}}$.

Recall also (4.6) and use $\Phi(c_{ij}(m)) - \Phi(m) \leq 0$ and $\bar{F}_m := \sup_{[0, \|m\|_1]} F(x)$, to obtain

$$\begin{aligned} \mathcal{L}_{K, F}^\beta \Phi(m) &= \sum_{1 \leq i < j < \infty} K(m_i, m_j) [\Phi(c_{ij}(m)) - \Phi(m)] \\ &\quad + \sum_{i \geq 1} \int_{\Theta} F(m_i) [\Phi(f_{i\theta}(m)) - \Phi(m)] \beta(d\theta) \\ &\leq \bar{F}_m \sum_{i \geq 1} \int_{\Theta} \left[\sum_{n \geq 1} \mathbb{1}_{\{\theta_n m_i > 0\}} - \mathbb{1}_{\{m_i > 0\}} \right] \beta(d\theta) \\ &\leq (k-1) \bar{F}_m \beta(\Theta) \Phi(m), \end{aligned}$$

we used $\theta_j m_i = 0$ for all $j \geq k+1$.

Next, from Proposition 5.7. *i)*, we have for all $s \in [0, t]$, $\sup_i M_i(s) \leq \sup_{[0, t]} \|M(s)\|_1 \leq \|m\|_1$ a.s, we deduce

$$\mathbb{E} \left[\sup_{[0, t]} N_s \right] \leq N_0 + (k-1) \bar{F}_m \beta(\Theta) \int_0^t \mathbb{E} [N_s] ds.$$

We use the Gronwall Lemma to obtain

$$\mathbb{E} \left[\sup_{[0, t]} N_s \right] \leq N_0 e^{(k-1) \bar{F}_m \beta(\Theta) t},$$

and *i)* follows.

Finally, we set $\bar{K}_m = \sup_{[0, \|m\|_1]^2} K(x, y)$, then we get

$$\sup_{s \in [0, t]} \rho_c(s) \leq \bar{K}_m \sup_{[0, t]} (N_s)^2 \leq \bar{K}_m \left(\sup_{s \in [0, t]} N_s \right)^2 < \infty, \text{ a.s. for all } t \geq 0.$$

On the other hand, since $\beta(\Theta) < \infty$ by (5.1), we get

$$\sup_{s \in [0, t]} \rho_f(t) \leq \bar{F}_m \beta(\Theta) \sup_{s \in [0, t]} N_s < \infty, \text{ a.s. for all } t \geq 0.$$

This ends the proof of Lemma 6.1. \square

Let $\lambda \in (0, 1]$, $\alpha \geq 0$ and $m \in \ell_{0+}$, and consider K, F, β and the Poisson measures N and M as in Proposition 5.5.

From Lemma 6.1. we deduce that the total rate of jumps of the system is uniformly bounded. Thus, pathwise existence and uniqueness holds for $(M(m, t))_{t \geq 0}$ solution to $SDE(K, F, m, N, M)$. This proves, furthermore, that the system $(M(m, t))_{t \geq 0}$ is a strong Markov process in continuous time with infinitesimal generator $\mathcal{L}_{K, F}^\beta$ and Proposition 5.1. follows.

7. EXISTENCE AND UNIQUENESS FOR SDE

We may now prove well-posedness of (SDE) . For the existence we will build a sequence of coupled finite Coalescence-Fragmentation process which will be proved to be a Cauchy sequence in $\mathbb{D}([0, \infty), \ell_\lambda)$.

Theorem 7.1. *Let $\lambda \in (0, 1]$, $\alpha \geq 0$ and $m \in \ell_\lambda$. Consider the coagulation kernel K , the fragmentation kernel F , the measure β and the Poisson measures N and M as in Definition 5.4.*

Then, there exists a unique solution $(M(m, t))_{t \geq 0}$ to $SDE(K, F, m, N, M)$.

First, we need the following lemma.

Lemma 7.2. *For $\lambda \in (0, 1]$ and $\alpha \geq 0$ fixed. Consider the coagulation kernel K , the fragmentation kernel F , the measure β and the Poisson measures N and M as in Definition 5.4. We consider also a subset \mathcal{A} of ℓ_λ such that $\sup_{m \in \mathcal{A}} \|m\|_\lambda < \infty$ and $\lim_{i \rightarrow \infty} \sup_{m \in \mathcal{A}} \sum_{k \geq i} m_k^\lambda = 0$.*

Assume that for each $m \in \mathcal{A}$ there is a $(M(m, t))_{t \geq 0}$ solution to $SDE(K, F, m, N, M)$ and define $\tau(m, x) = \inf\{t \geq 0 : \|M(m, t)\|_\lambda \geq x\}$, then for each $t \geq 0$ we have $\lim_{x \rightarrow \infty} \alpha(t, x) = 0$, where

$$\alpha(t, x) := \sup_{m \in \mathcal{A}} P \left[\sup_{[0, t]} \|M(m, s)\|_\lambda \geq x \right].$$

Proof. It suffices to remark that from Proposition 5.7. *i)*, we have

$$\sup_{m \in \mathcal{A}} P \left[\sup_{[0, t]} \|M(m, s)\|_\lambda \geq x \right] \leq \frac{1}{x} \sup_{m \in \mathcal{A}} \mathbb{E} \left[\sup_{[0, t]} \|M(m, s)\|_\lambda \right] \leq \frac{1}{x} \sup_{m \in \mathcal{A}} \|m\|_\lambda e^{\bar{F}_m C_\beta^\lambda t}.$$

We make x tend to infinity and the lemma follows. \square

Proof of Theorem 7.1. Uniqueness.- Let $m \in \ell_\lambda$ and consider $(M(m, t))_{t \geq 0}$ and $(\tilde{M}(m, t))_{t \geq 0}$ two solutions to $SDE(K, F, m, N, M)$. For $x \geq 0$ we set $\tau_x = \inf\{t \geq 0 : \|M(m, t)\|_\lambda \geq x\}$ and $\tilde{\tau}_x$ the object concerning \tilde{M} , thus from Proposition 5.7. *ii)*, we have

$$\mathbb{E} \left[\sup_{s \in [0, t \wedge \tau_x \wedge \tilde{\tau}_x]} d_\lambda \left(M(m, s), \tilde{M}(m, s) \right) \right] = 0.$$

This implies uniqueness on the interval $[0, t \wedge \tau_x \wedge \tilde{\tau}_x]$. Since *a.s.* $\tau_x \wedge \tilde{\tau}_x \xrightarrow{x \rightarrow \infty} \infty$, the uniqueness assertion on $[0, t]$ for all $t \geq 0$ follows from making x tend to infinity.

Existence. - First, recall ψ_n defined by (3.34) and the measure $\beta_n = \mathbb{1}_{\theta \in \Theta(n)} \beta \circ \psi_n^{-1}$. Consider the Poisson measure $M(dt, di, d\theta, dz)$ associated to the fragmentation, as in Definition 5.4.

We set $M_n = \mathbb{1}_{\Theta(n)} M \circ \psi_n^{-1}$. This means that writing M as $M = \sum_{k \geq 1} \delta_{(T_k, i_k, \theta_k, z_k)}$, we have $M_n = \sum_{k \geq 1} \delta_{(T_k, i_k, \psi_n(\theta_k), z_k)} \mathbb{1}_{\theta \in \Theta(n)}$. Defined in this way, M_n is a Poisson measure on $[0, \infty) \times \mathbb{N} \times \Theta \times [0, \infty)$ with intensity measure $dt \left(\sum_{k \geq 1} \delta_k(di) \right) \beta_n(d\theta) dz$.

We define $m^n \in \ell_{0+}$ by $m^n = (m_1, m_2, \dots, m_n, 0, \dots)$ and denote $M^n(t) := M(m^n, t)$ the unique solution to $SDE(K, F, m^n, N, M_n)$ obtained in Proposition 5.5. Note that $M^n(t)$ satisfies the following equation

$$\begin{aligned} M^n(t) &= m^n + \int_0^t \int_{i < j} \int_0^\infty [c_{ij}(M^n(s-)) - M^n(s-)] \mathbb{1}_{\{z \leq K(M_i^n(s-), M_j^n(s-))\}} N(dt, d(i, j), dz) \\ &\quad + \int_0^t \int_i \int_\Theta \int_0^\infty [f_{i\psi_n(\theta)}(M^n(s-)) - M^n(s-)] \mathbb{1}_{\{z \leq F(M_i^n(s-))\}} \mathbb{1}_{\{\theta \in \Theta(n)\}} \\ &\hspace{15em} M(dt, di, d\theta, dz). \end{aligned} \tag{7.1}$$

This setting allows us to couple the processes since they are driven by the same Poisson measures.

Convergence $M_t^n \rightarrow M_t$. - Consider $p, q \in \mathbb{N}$ with $1 \leq p < q$, from (7.1) we obtain

$$d_\lambda(M^p(t), M^q(t)) \leq d_\lambda(m^p, m^q) + A_c^{p,q}(t) + B_c^{p,q}(t) + C_f^{p,q}(t) + A_f^{p,q}(t) + B_f^{p,q}(t) + C_f^{p,q}(t) + D_c^{p,q}(t). \tag{7.2}$$

We obtain this equality, exactly as in (6.4), by replacing M by M^p and \tilde{M} by M^q . The terms concerning the coalescence are the same. The terms concerning the fragmentation are, equivalently:

$$\begin{aligned} A_f^{p,q}(t) &= \int_0^t \int_i \int_\Theta \int_0^\infty \{d_\lambda(f_{i\psi_p(\theta)}(M^p(s-)), f_{i\psi_p(\theta)}(M^q(s-))) - d_\lambda(M^p(s-), M^q(s-))\} \\ &\quad \mathbb{1}_{\{\theta \in \Theta(p)\}} \mathbb{1}_{\{z \leq F(M_i^p(s-)) \wedge F(M_i^q(s-))\}} M(ds, di, d\theta, dz), \\ B_f^{p,q}(t) &= \int_0^t \int_i \int_\Theta \int_0^\infty \{d_\lambda(f_{i\psi_p(\theta)}(M^p(s-)), M^q(s-)) - d_\lambda(M^p(s-), M^q(s-))\} \\ &\quad \mathbb{1}_{\{\theta \in \Theta(p)\}} \mathbb{1}_{\{F(M_i^q(s-)) \leq z \leq F(M_i^p(s-))\}} M(ds, di, d\theta, dz), \\ C_f^{p,q}(t) &= \int_0^t \int_i \int_\Theta \int_0^\infty \{d_\lambda(f_{i\psi_p(\theta)}(M^q(s-)), M^p(s-)) - d_\lambda(M^p(s-), M^q(s-))\} \\ &\quad \mathbb{1}_{\{\theta \in \Theta(p)\}} \mathbb{1}_{\{F(M_i^p(s-)) \leq z \leq F(M_i^q(s-))\}} M(ds, di, d\theta, dz), \end{aligned}$$

Finally, the term $D_f^{p,q}(t)$ is the term that collects the errors.

$$\begin{aligned} D_f^{p,q}(t) &= \int_0^t \int_i \int_\Theta \int_0^\infty d_\lambda(f_{i\psi_p(\theta)}(M^q(s-)), f_{i\psi_q(\theta)}(M^q(s-))) \mathbb{1}_{\{\theta \in \Theta(p)\}} \\ &\quad \mathbb{1}_{\{z \leq F(M_i^q(s-))\}} M(ds, di, d\theta, dz) \\ &\quad + \int_0^t \int_i \int_\Theta \int_0^\infty \{d_\lambda(f_{i\psi_q(\theta)}(M^q(s-)), M^p(s-)) - d_\lambda(M^p(s-), M^q(s-))\} \\ &\quad \mathbb{1}_{\{z \leq F(M_i^q(s-))\}} \mathbb{1}_{\{\theta \in \Theta(q) \setminus \Theta(p)\}} M(ds, di, d\theta, dz). \end{aligned}$$

The first term of $D_f^{p,q}(t)$ results from the triangulation that gives $A_f^{p,q}(t)$ and $C_f^{p,q}(t)$. The second term is issued from fragmentation of M^q when θ belongs to $\Theta(q) \setminus \Theta(p)$. This induces a fictitious jump to M^p which does not undergo fragmentation.

We proceed to bound each term. We define, for all $x > 0$ and $n \geq 1$, the stopping time $\tau_n^x = \inf\{t \geq 0 : \|M^n(t)\|_\lambda \geq x\}$.

From Proposition 5.7. we have for all $s \in [0, t]$, $\sup_{n \geq 1} \sup_{i \geq 1} M_i^n(s) \leq \sup_{n \geq 1} \sup_{i \geq 1} \sup_{[0, t]} \|M^n(s)\|_1 \leq \|m\|_1 := a_m$ a.s. We set κ_{a_m} and μ_{a_m} the constants for which the kernels K and F satisfy (4.1) and (4.2). Finally, we set $\bar{F}_m = \sup_{[0, a_m]} F(x)$.

The terms concerning coalescence are upper bounded on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ with $t \geq 0$, exactly as in (6.4).

Term $A_f^{p,q}(t)$: we take the sup on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ and then the expectation. We use (A.9) together with (2.7) and (2.8). We thus obtain exactly the same bound as for A_t^f .

Term $B_f^{p,q}(t)$: we take the sup on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ and then the expectation. We use (6.6), (A.7) with (2.7) and (2.8), and (4.2). We thus obtain exactly the same bound as for B_t^f .

Term $C_f^{p,q}(t)$: it is treated exactly as $B_f^{p,q}(t)$.

Term $D_f^{p,q}(t)$: we take the sup on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ and then the expectation. For the first term we use (A.10). For the second term we use (6.6) and (A.7) together with (2.7) and (2.8). Finally, we use Proposition 5.7. *i*). to obtain

$$\begin{aligned}
& \mathbb{E} \left[\sup_{s \in [0, t \wedge \tau_p^x \wedge \tau_q^x]} D_f^{p,q}(t) \right] \\
& \leq \mathbb{E} \left[\int_0^{t \wedge \tau_p^x \wedge \tau_q^x} \sum_{i \geq 1} F(M_i^q(s)) \int_{\Theta} \mathbb{1}_{\{\theta \in \Theta(p)\}} \sum_{k=p+1}^q \theta_k^\lambda [M_i^q(s)]^\lambda \beta(d\theta) ds \right] \\
& \quad + \mathbb{E} \left[\int_0^{t \wedge \tau_p^x \wedge \tau_q^x} \sum_{i \geq 1} F(M_i^q(s)) [M_i^q(s)]^\lambda ds \int_{\Theta} 2 \sum_{k \geq 2} \theta_k^\lambda \mathbb{1}_{\{\theta \in \Theta(q) \setminus \Theta(p)\}} \beta(d\theta) \right] \\
& \leq \bar{F}_m \int_{\Theta} \sum_{k > p} \theta_k^\lambda \beta(d\theta) \int_0^t \mathbb{E} \left[\sup_{u \in [0, t]} \|M^q(u)\|_\lambda \right] ds \\
& \quad + 2\bar{F}_m \int_{\Theta} \sum_{k \geq 2} \theta_k^\lambda \mathbb{1}_{\{\theta \in \Theta \setminus \Theta(p)\}} \beta(d\theta) \int_0^t \mathbb{E} \left[\sup_{u \in [0, t]} \|M^q(u)\|_\lambda \right] ds \\
& \leq \bar{F}_m t \|m\|_\lambda e^{\bar{F}_m C_\beta^\lambda t} (A(p) + 2B(p)),
\end{aligned}$$

where $A(p) := \int_{\Theta} \sum_{k > p} \theta_k^\lambda \beta(d\theta)$ and $B(p) := \int_{\Theta} \sum_{k \geq 2} \theta_k^\lambda \mathbb{1}_{\{\theta \in \Theta \setminus \Theta(p)\}} \beta(d\theta)$. Note that by (2.6) and since $\Theta \setminus \Theta(p)$ tends to the empty set, $A(p)$ and $B(p)$ tend to 0 as p tends to infinity.

Thus, gathering the terms as for the bound (6.10), we get

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [0, t \wedge \tau_q^x \wedge \tau_p^x]} d_\lambda (M^p(s), M^q(s)) \right] &\leq d_\lambda (m^p, m^q) + D_1 t [A(p) + B(p)] + [8\kappa_1 x + CC_\beta^\lambda \|m\|_1^\alpha] \\ &\quad \times \int_0^t \mathbb{E} \left[\sup_{u \in [0, s \wedge \tau_q^x \wedge \tau_p^x]} d_\lambda (M^p(u), M^q(u)) \right] ds, \end{aligned} \quad (7.3)$$

where $D_1 = 2\bar{F}_m \|m\|_\lambda e^{\bar{F}_m C_\beta^\lambda t}$. The Gronwall Lemma allows us to obtain

$$\mathbb{E} \left[\sup_{s \in [0, t \wedge \tau_q^x \wedge \tau_p^x]} d_\lambda (M^p(s), M^q(s)) \right] \leq \{d_\lambda (m^p, m^q) + D_1 [A(p) + B(p)] t\} \times e^{D_2 x t}, \quad (7.4)$$

where D_2 is a positive constants depending on $\lambda, \alpha, \kappa_{a_m}, \mu_{a_m}, K, F, C_\beta^\lambda$ and $\|m\|_1$.

Since $\lim_{n \rightarrow \infty} d_\lambda (m^n, m) = 0$, we deduce from Lemma 7.2. that for all $t \geq 0$,

$$\lim_{x \rightarrow \infty} \alpha(t, x) = 0 \text{ where } \alpha(t, x) := \sup_{n \geq 1} P[\tau(m^n, x) \leq t]. \quad (7.5)$$

This means that the stopping times τ_n^x tend to infinity as $x \rightarrow \infty$, uniformly in n .

Next, from (7.4), (7.5) and since $(m^n)_{n \geq 1}$ is a Cauchy sequence for d_λ and $(A(n))_{n \geq 1}$ and $(B(n))_{n \geq 1}$ converge to 0, we deduce that for all $\varepsilon > 0, T > 0$ we may find $n_\varepsilon > 0$ such that for $p, q \geq n_\varepsilon$ we have

$$P \left[\sup_{[0, T]} d_\lambda (M^p(t), M^q(t)) \geq \varepsilon \right] \leq \varepsilon. \quad (7.6)$$

Indeed, for all $x > 0$,

$$\begin{aligned} P \left[\sup_{[0, T]} d_\lambda (M^p(t), M^q(t)) \geq \varepsilon \right] &\leq P[\tau_p^x \leq T] + P[\tau_q^x \leq T] + \frac{1}{\varepsilon} \mathbb{E} \left[\sup_{[0, T \wedge \tau_p^x \wedge \tau_q^x]} d_\lambda (M^p(t), M^q(t)) \right] \\ &\leq 2\alpha(T, x) + \frac{1}{\varepsilon} [d_\lambda (m^p, m^q) + D_1 T (A(p) + B(p))] \times e^{D_2 x T}. \end{aligned}$$

Choosing x large enough so that $\alpha(T, x) \leq \varepsilon/8$ and n_ε large enough to have both $A(p)$ and $B(p) \leq (\varepsilon^2/4D_1 T)e^{-D_2 x T}$ and in a such a way that for all $p, q \geq n_\varepsilon, d_\lambda (m^p, m^q) \leq (\varepsilon^2/4)e^{-D_2 x T}$, we conclude that (7.6) holds.

We deduce from (7.6) that the sequence of processes $(M_t^n)_{t \geq 0}$ is Cauchy in probability in $\mathbb{D}([0, \infty), \ell_\lambda)$, endowed with the uniform norm in time on compact intervals. We are thus able to find a subsequence (not relabelled) and a (\mathcal{H}_t) -adapted process $(M(t))_{t \geq 0}$ belonging *a.s.* to $\mathbb{D}([0, \infty), \ell_\lambda)$ such that for all $T > 0$,

$$\lim_{n \rightarrow \infty} \sup_{[0, T]} d_\lambda (M^n(t), M(t)) = 0. \text{ a.s.} \quad (7.7)$$

Setting now $\tau^x := \inf\{t \geq 0 : \|M(t)\|_\lambda \geq x\}$, due to Lebesgue Theorem,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{[0, T \wedge \tau_n^x \wedge \tau^x]} d_\lambda (M^n(t), M(t)) \right] = 0. \quad (7.8)$$

We have to show now that the limit process $(M(t))_{t \geq 0}$ defined by (7.7) solves the equation $SDE(K, F, m, N, M)$ defined in (5.2).

We want to pass to the limit in (7.1), it suffices to show that $\lim_{n \rightarrow \infty} \Delta_n(t) = 0$, where

$$\begin{aligned} \Delta_n(t) = & \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} \int_{i < j} \int_0^\infty \sum_{k \geq 1} 2^{-k} |([c_{ij}(M(s-))]_k - M_k(s-)) \mathbb{1}_{\{z \leq K(M_i(s-), M_j(s-))\}} \right. \\ & - ([c_{ij}(M^n(s-))]_k - M_k^n(s-)) \mathbb{1}_{\{z \leq K(M_i^n(s-), M_j^n(s-))\}} | N(dt, d(i, j), dz) \\ & + \int_0^{t \wedge \tau_n^x \wedge \tau^x} \int_i \int_\Theta \int_0^\infty \sum_{k \geq 1} 2^{-k} |([f_{i\theta}(M(s-))]_k - [M(s-)]_k) \mathbb{1}_{\{z \leq F(M_i(s-))\}} \\ & \left. - ([f_{i\psi_n(\theta)}(M^n(s-))]_k - M_k^n(s-)) \mathbb{1}_{\{z \leq F(M_i^n(s-))\}} \mathbb{1}_{\{\theta \in \Theta(n)\}} | M(dt, di, d\theta, dz) \right]. \end{aligned}$$

Indeed, due to (7.7), for all $x > 0$ and for n large enough, *a.s.* $\tau_n^x \geq \tau^{x/2}$. Thus M will solve $SDE(K, F, M(0), N, M)$ on the time interval $[0, \tau^{x/2})$ for all $x > 0$, and thus on $[0, \infty)$ since *a.s.* $\lim_{x \rightarrow \infty} \tau^x = \infty$, because $M \in \mathbb{D}([0, \infty), \ell_\lambda)$.

Note that

$$\begin{aligned} & |([c_{ij}(M(s))]_k - M_k(s)) \mathbb{1}_{\{z \leq K(M_i(s), M_j(s))\}} \\ & \quad - ([c_{ij}(M^n(s))]_k - M_k^n(s)) \mathbb{1}_{\{z \leq K(M_i^n(s), M_j^n(s))\}}| \\ & \leq \left| ([c_{ij}(M(s))]_k - M_k(s)) - ([c_{ij}(M^n(s))]_k - M_k^n(s)) \right| \mathbb{1}_{\{z \leq K(M_i(s), M_j(s))\}} \\ & \quad + \left| [c_{ij}(M^n(s))]_k - M_k^n(s) \right| \left| \mathbb{1}_{\{z \leq K(M_i(s), M_j(s))\}} - \mathbb{1}_{\{z \leq K(M_i^n(s), M_j^n(s))\}} \right| \end{aligned}$$

and

$$\begin{aligned} & |([f_{i\theta}(M(s))]_k - M_k(s)) \mathbb{1}_{\{z \leq F(M_i(s))\}} \\ & \quad - ([f_{i\psi_n(\theta)}(M^n(s))]_k - M_k^n(s)) \mathbb{1}_{\{z \leq F(M_i^n(s))\}} \mathbb{1}_{\{\theta \in \Theta(n)\}}| \\ & \leq \left| ([f_{i\theta}(M(s))]_k - M_k(s)) - ([f_{i\theta}(M^n(s))]_k - M_k^n(s)) \right| \mathbb{1}_{\{z \leq F(M_i(s))\}} \\ & \quad + \left| ([f_{i\theta}(M^n(s))]_k - [f_{i\psi_n(\theta)}(M^n(s))]_k) \right| \mathbb{1}_{\{z \leq F(M_i(s))\}} \\ & \quad + \left| [f_{i\psi_n(\theta)}(M^n(s))]_k - M_k^n(s) \right| \left| \mathbb{1}_{\{z \leq F(M_i(s))\}} - \mathbb{1}_{\{z \leq F(M_i^n(s))\}} \right| \\ & \quad + \left| [f_{i\psi_n(\theta)}(M^n(s))]_k - M_k^n(s) \right| \mathbb{1}_{\{z \leq F(M_i^n(s))\}} \mathbb{1}_{\{\theta \in \Theta(n)^c\}}, \end{aligned}$$

where $\Theta(n)^c = \Theta \setminus \Theta(n)$. We thus obtain the following bound

$$\Delta_n(t) \leq A_n^c(t) + B_n^c(t) + A_n^f(t) + B_n^f(t) + C_n^f(t) + D_n^f(t).$$

First, $A_n^c(t) = \sum_{i < j} A_n^{ij}(t)$ with

$$\begin{aligned} A_n^{ij}(t) = & \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} K(M_i(s), M_j(s)) \sum_{k \geq 1} 2^{-k} \right. \\ & \left. |([c_{ij}(M(s))]_k - M_k(s)) - ([c_{ij}(M^n(s))]_k - M_k^n(s))| ds \right], \end{aligned}$$

and using

$$\begin{aligned} & \left| \mathbb{1}_{\{z \leq K(M_i(s), M_j(s))\}} - \mathbb{1}_{\{z \leq K(M_i^n(s), M_j^n(s))\}} \right| \\ &= \mathbb{1}_{\{K(M_i(s), M_j(s)) \wedge K(M_i^n(s), M_j^n(s)) \leq z \leq K(M_i(s), M_j(s)) \vee K(M_i^n(s), M_j^n(s))\}}, \\ B_n^c(t) &= \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} \sum_{i < j} |K(M_i(s), M_j(s)) - K(M_i^n(s), M_j^n(s))| \right. \\ & \quad \left. \sum_{k \geq 1} 2^{-k} |[c_{ij}(M^n(s))]_k - M_k^n(s)| ds \right]. \end{aligned}$$

For the fragmentation terms we have

$$\begin{aligned} A_n^f(t) &= \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} \int_{\Theta} \sum_{i \geq 1} F(M_i(s)) \right. \\ & \quad \left. \sum_{k \geq 1} 2^{-k} |([f_{i\theta}(M(s))]_k - M_k(s)) - ([f_{i\theta}(M^n(s))]_k - M_k^n(s))| \beta(d\theta) ds \right], \\ B_n^f(t) &= \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} \int_{\Theta} \sum_{i \geq 1} F(M_i(s)) \right. \\ & \quad \left. \sum_{k \geq 1} 2^{-k} |([f_{i\theta}(M^n(s))]_k - [f_{i\psi_n(\theta)}(M^n(s))]_k)| \beta(d\theta) ds \right], \end{aligned}$$

using

$$\begin{aligned} & \left| \mathbb{1}_{\{z \leq F(M_i(s))\}} - \mathbb{1}_{\{z \leq F(M_i^n(s))\}} \right| = \mathbb{1}_{\{F(M_i(s)) \wedge F(M_i^n(s)) \leq z \leq F(M_i(s)) \vee F(M_i^n(s))\}}, \\ C_n^f(t) &= \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} \int_{\Theta} \mathbb{1}_{\{\theta \in \Theta(n)\}} \sum_{i \geq 1} |F(M_i(s)) - F(M_i^n(s))| \right. \\ & \quad \left. \sum_{k \geq 1} 2^{-k} |[f_{i\psi_n(\theta)}(M^n(s))]_k - M_k^n(s)| \beta(d\theta) ds \right], \end{aligned}$$

and finally,

$$\begin{aligned} D_n^f(t) &= \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} \int_{\Theta} \mathbb{1}_{\{\theta \in \Theta(n)^c\}} \sum_{i \geq 1} F(M_i^n(s)) \right. \\ & \quad \left. \sum_{k \geq 1} 2^{-k} |[f_{i\psi_n(\theta)}(M^n(s))]_k - M_k^n(s)| \beta(d\theta) ds \right], \end{aligned}$$

We will show that each term converges to 0 as n tends to infinity.

Fist, from Proposition Proposition 5.7. *i)* we have that for all $s \in [0, t]$, $\sup_i M_i(s) \leq \sup_{[0, t]} \|M(s)\|_1 \leq \|m\|_1 := a_m$ a.s, equivalently for M^n , we have $a_{m^n} = \|m^n\|_1 \leq \|m\|_1$. We set κ_{a_m} and μ_{a_m} the constants for which the kernels K and F satisfy (4.1) and (4.2). Finally, we set $\bar{K}_m = \sup_{[0, a_m]^2} K(x, y)$ and $\bar{F}_m = \sup_{[0, a_m]} F(x)$.

We prove that $A_n^c(t)$ tends to 0 using the Lebesgue dominated convergence Theorem. It suffices to show that:

- a) for each $1 \leq i < j$, $A_n^{ij}(t)$ tends to 0 as n tends to infinity,
b) $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i+j \geq k} A_n^{ij}(t) = 0$.

Now, for $A_n^{ij}(t)$ using (A.16), (A.14) and Proposition 5.7. i), we have

$$\begin{aligned} A_n^{ij}(t) &\leq \bar{K}_m \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} d(c_{ij}(M(s)), c_{ij}(M^n(s))) + d(M(s), M^n(s)) ds \right] \\ &\leq \bar{K}_m \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} (2^i + 2^j + 1) d(M(s), M^n(s)) ds \right] \\ &\leq C \bar{K}_m (2^i + 2^j + 1) \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} (\|M(s)\|_1^{1-\lambda} \vee \|M^n(s)\|_1^{1-\lambda}) d_\lambda(M(s), M^n(s)) ds \right] \\ &\leq C \bar{K}_m (2^i + 2^j + 1) t \|m\|_1^{1-\lambda} \mathbb{E} \left[\sup_{[0, t \wedge \tau_n^x \wedge \tau^x]} d_\lambda(M(s), M^n(s)) \right]. \end{aligned}$$

which tends to 0 as $n \rightarrow \infty$ due to (7.8). On the other hand, using (A.15) we have

$$\begin{aligned} A_n^{ij}(t) &\leq \bar{K}_m \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} d(c_{ij}(M(s)), M(s)) + d(c_{ij}(M^n(s)), M^n(s)) ds \right] \\ &\leq \frac{3\bar{K}_m}{2} 2^{-i} \int_0^t \mathbb{E} [M_j(s) + M_j^n(s)] ds. \end{aligned}$$

Since $\sum_{i \geq 1} 2^{-i} = 1$ and $\sum_{j \geq 1} \int_0^t \mathbb{E} [M_j(s)] ds \leq \|m\|_1 t$, b) reduces to

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{j \geq k} \int_0^t \mathbb{E} [M_j^n(s)] ds = 0.$$

But for each $k \geq 1$, since $M^n(s)$ and $M(s)$ belong to ℓ_1 for all $s \geq 0$ a.s by Proposition 5.7. i), and since the map $m \mapsto \sum_{j=1}^{k-1} m_j$ is continuous for the pointwise convergence topology,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_0^t \mathbb{E} \left[\sum_{j \geq k} M_j^n(s) \right] &= \int_0^t ds \left\{ \lim_{n \rightarrow \infty} \|M^n(s)\|_1 - \lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{j=1}^{k-1} M_j^n(s) \right] \right\} ds \\ &= \int_0^t \left\{ \|M(s)\|_1 - \mathbb{E} \left[\sum_{j=1}^{k-1} M_j(s) \right] \right\} ds \\ &= \int_0^t \mathbb{E} \left[\sum_{j=k}^{\infty} M_j(s) \right] ds. \end{aligned}$$

We easily conclude using that a.s. $\|M(s)\|_1 \leq \|m\|_1$ for all $s \geq 0$.

Using (4.1), (A.15) and Proposition 5.7. *i*), we obtain

$$\begin{aligned}
B_n^c(t) &\leq \kappa_{a_m} \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} \sum_{i < j} [|M_i^n(s)^\lambda - M_i(s)^\lambda| + |M_j^n(s)^\lambda - M_j(s)^\lambda| ds] \right. \\
&\quad \left. \times d(c_{ij}(M^n(s)), M^n(s)) \right] \\
&\leq \frac{3}{2} \kappa_{a_m} \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} \sum_{i < j} [|M_i^n(s)^\lambda - M_i(s)^\lambda| + |M_j^n(s)^\lambda - M_j(s)^\lambda|] 2^{-i} M_j^n(s) ds \right] \\
&\leq 3\kappa_{a_m} t \|m\|_1 \mathbb{E} \left[\sup_{[0, t \wedge \tau_n^x \wedge \tau^x]} d_\lambda(M(s), M^n(s)) \right],
\end{aligned}$$

which tends to 0 as $n \rightarrow \infty$ due to (7.8).

We use (A.18) and (A.17) to obtain

$$\begin{aligned}
A_n^f(t) &\leq \bar{F}_m \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} \sum_{i \geq 1} \int_{\Theta} \left[\left(d(f_{i\theta}(M(s)), f_{i\theta}(M^n(s))) + d(M(s), M^n(s)) \right) \right. \right. \\
&\quad \left. \left. \wedge \left(d(f_{i\theta}(M(s)), M(s)) + d(f_{i\theta}(M^n(s)), M^n(s)) \right) \right] \beta(d\theta) ds \right] \\
&\leq 2\bar{F}_m \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} \int_{\Theta} \sum_{i \geq 1} \left[\left(d(M(s), M^n(s)) \right) \wedge \left(2^{-i}(1 - \theta_1)(M_i(s) + M_i^n(s)) \right) \right] \beta(d\theta) ds \right].
\end{aligned}$$

We split the integral on Θ and the sum on i into two parts. Consider $\Theta_\varepsilon = \{\theta \in \Theta : \theta_1 \leq 1 - \varepsilon\}$ and $N \in \mathbb{N}$. Using (A.14) and Proposition 5.7. *i*), we deduce

$$\begin{aligned}
&\int_{\Theta} \sum_{i \geq 1} \left[\left(d(M(s), M^n(s)) \right) \wedge \left(2^{-i}(1 - \theta_1)(M_i(s) + M_i^n(s)) \right) \right] \beta(d\theta) \\
&\leq \int_{\Theta_\varepsilon} \sum_{i=1}^N d(M(s), M^n(s)) \beta(d\theta) + \int_{\Theta_\varepsilon^c} (1 - \theta_1) \beta(d\theta) \sum_{i \geq 1} (M_i(s) + M_i^n(s)) \\
&\quad + \int_{\Theta} \sum_{i > N} 2^{-i}(1 - \theta_1)(M_i(s) + M_i^n(s)) \beta(d\theta) \\
&\leq C \|m\|_1^{1-\lambda} N \beta(\Theta_\varepsilon) d_\lambda(M(s), M^n(s)) + 2 \|m\|_1 \int_{\Theta_\varepsilon^c} (1 - \theta_1) \beta(d\theta) \\
&\quad + 2 \|m\|_1 \int_{\Theta} (1 - \theta_1) \beta(d\theta) \sum_{i > N} 2^{-i}
\end{aligned}$$

Note that $\beta(\Theta_\varepsilon) = \int_{\Theta} \mathbf{1}_{\{1 - \theta_1 \geq \varepsilon\}} \beta(d\theta) \leq \frac{1}{\varepsilon} \int_{\Theta} (1 - \theta_1) \beta(d\theta) \leq \frac{1}{\varepsilon} C_\beta^\lambda < \infty$. Thus, we get

$$\begin{aligned}
A_n^f(t) &\leq \frac{2t}{\varepsilon} C_\beta^\lambda N \bar{F}_m C \|m\|_1^{1-\lambda} \mathbb{E} \left[\sup_{[0, t \wedge \tau_n^x \wedge \tau^x]} d_\lambda(M(s), M^n(s)) \right] + 4t \bar{F}_m \|m\|_1 \int_{\Theta_\varepsilon^c} (1 - \theta_1) \beta(d\theta) \\
&\quad + 2t \bar{F}_m \|m\|_1 C_\beta^\lambda 2^{-N}.
\end{aligned}$$

Thus, due to (7.8) we have for all $\varepsilon > 0$ and $N \geq 1$,

$$\limsup_{n \rightarrow \infty} A_n^f(t) \leq 4t\bar{F}_m \|m\|_1 \int_{\Theta_\varepsilon^c} (1 - \theta_1) \beta(d\theta) + 2t\bar{F}_m \|m\|_1 C_\beta^\lambda 2^{-N}.$$

Since Θ_ε^c tends to the empty set as $\varepsilon \rightarrow 0$ we conclude using (2.8) with (2.6) and making $\varepsilon \rightarrow 0$ and $N \rightarrow \infty$.

Next, use (A.19) and Proposition 5.7. *i*) to obtain

$$B_n^f(t) \leq t\bar{F}_m \|m\|_1 \int_{\Theta} \sum_{k > n} \theta_k \beta(d\theta).$$

which tends to 0 as $n \rightarrow \infty$ due to (2.5).

Using (4.2), (A.17) with (2.7) and (2.8), (A.3), (A.14) and Proposition 5.7. *i*), we obtain

$$\begin{aligned} C_n^f(t) &\leq 2\mu_{a_m} \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} \int_{\Theta(n)} \sum_{i \geq 1} |[M_i(s)]^\alpha - [M_i^n(s)]^\alpha| 2^{-i} (1 - \theta_1) M_i(s) \beta(d\theta) ds \right] \\ &\leq 2\mu_{a_m} C_\beta^\lambda \mathbb{E} \left[\int_0^{t \wedge \tau_n^x \wedge \tau^x} \sum_{i \geq 1} 2^{-i} |M_i(s) - M_i^n(s)| ([M_i^n(s)]^\alpha + [M_i(s)]^\alpha) ds \right] \\ &\leq 2\mu_{a_m} C C_\beta^\lambda t \|m\|_1^{1-\lambda+\alpha} \mathbb{E} \left[\sup_{[0, t \wedge \tau_n^x \wedge \tau^x]} d_\lambda(M(s), M^n(s)) \right], \end{aligned}$$

which tends to 0 as $n \rightarrow \infty$ due to (7.8).

Finally, we use (A.17) with (2.7) and (2.8) and Proposition 5.7. *i*), to obtain

$$D_n^f(t) \leq 2t\bar{F}_m \|m\|_1 \int_{\Theta} \mathbb{1}_{\{\theta \in \Theta(n)^c\}} (1 - \theta_1) \beta(d\theta)$$

which tends to 0 as n tends to infinity since $\int_{\Theta} (1 - \theta_1) \beta(d\theta) \leq C_\beta^\lambda$ and $\Theta(n)^c$ tends to the empty set.

This ends the proof of Theorem 7.1. \square

Conclusion.- It remains to conclude the proof of Theorem 5.3.

We start with some boundness of the operator $\mathcal{L}_{K,F}^\beta$.

Lemma 7.3. *Let $\lambda \in (0, 1]$, $\alpha \geq 0$, the coagulation kernel K , fragmentation kernel F and the measure β satisfying Hypotheses 4.2. Let $\Phi : \ell_\lambda \rightarrow \mathbb{R}$ satisfy, for all $m, \tilde{m} \in \ell_\lambda$, $|\Phi(m)| \leq a$ and $|\Phi(m) - \Phi(\tilde{m})| \leq ad(m, \tilde{m})$. Recall (4.6). Then $m \mapsto \mathcal{L}_{K,F}^\beta \Phi(m)$ is bounded on $\{m \in \ell_\lambda, \|m\|_\lambda \leq c\}$ for each $c > 0$.*

Proof. This Lemma is a straightforward consequence of the hypotheses on the kernels and Lemma A.3. Let $c > 0$ be fixed, and set $A := c^{1/\lambda}$. Notice that if $\|m\|_\lambda \leq c$, then for all $k \geq 1$ $m_k \leq A$.

Setting $\sup_{[0,A]^2} K(x, y) = \bar{K}$ and $\sup_{[0,A]} F(x) = \bar{F}$. We use (A.15) and (A.17) with (2.7) and (2.8), and deduce that for all $m \in \ell_\lambda$ such that $\|m\|_\lambda \leq c$,

$$\begin{aligned} |\mathcal{L}_{K,F}^\beta \Phi(m)| &\leq \bar{K} \sum_{1 \leq i < j < \infty} [\Phi(c_{ij}(m)) - \Phi(m)] + \bar{F} \sum_{i \geq 1} \int_{\Theta} [\Phi(f_{i\theta}(m)) - \Phi(m)] \beta(d\theta) \\ &\leq a\bar{K} \sum_{1 \leq i < j < \infty} d(c_{ij}(m), m) + a\bar{F} \int_{\Theta} \sum_{i \geq 1} d(f_{i\theta}(m), m) \beta(d\theta) \\ &\leq \frac{3}{2} a\bar{K} \|m\|_1 + 2a\bar{F} C_\beta^\lambda \|m\|_1 \leq \left(\frac{3}{2} \bar{K} + 2\bar{F} C_\beta^\lambda \right) ac^{1/\lambda}. \end{aligned}$$

□

Finally, it remains to conclude the proof of Theorem 5.3.

Proof of Theorem 5.3. We consider the Poisson measures N and M as in Definition 5.4., and we fix $m \in \ell_\lambda$. We consider $M(t) := M(m, t)$ the unique solution to $SDE(K, F, M(0), N, M)$ built in Section 7. M is a strong Markov Process, since it solves a time-homogeneous Poisson-driven S.D.E. for which pathwise uniqueness holds. We now check the points *i*) and *ii*).

Consider any sequence $m^n \in \ell_{0+}$ such that $\lim_{n \rightarrow \infty} d_\lambda(m^n, m) = 0$ and $M^n(t) := M(m^n, t)$ the solution to $SDE(K, F, m^n, N, M_n)$ obtained in Proposition 5.5. Denote by $\tau^x = \inf\{t \geq 0, \|M(m, t)\|_\lambda \geq x\}$ and by τ_n^x the stopping time concerning M^n . We will prove that for all $T \geq 0$ and $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} P \left[\sup_{[0,T]} d_\lambda(M(t), M^n(t)) > \varepsilon \right] = 0. \quad (7.9)$$

For this, consider the sequence $m^{(n)} \in \ell_{0+}$ defined by $m^{(n)} = (m_1, \dots, m_n, 0, \dots)$ and $M^{(n)}(t) := M(m^{(n)}, t)$ the solution to $SDE(K, F, m^{(n)}, N, M_n)$ obtained in Proposition 5.5. and denote by $\tau_{(n)}^x$ the stopping time concerning $M^{(n)}$.

First, note that since $\lim_{n \rightarrow \infty} d_\lambda(m^{(n)}, m) = \lim_{n \rightarrow \infty} d_\lambda(m^n, m) = 0$, we deduce that $\sup_{n \geq 1} \|m^{(n)}\|_\lambda < \infty$ and from Lemma 7.2. that for all $t \geq 0$,

$$\lim_{x \rightarrow \infty} \alpha_1(t, x) = 0 \text{ where } \alpha_1(t, x) := \sup_{n \geq 1} P[\tau_{(n)}^x \leq t], \quad (7.10)$$

$$\lim_{x \rightarrow \infty} \alpha_2(t, x) = 0 \text{ where } \alpha_2(t, x) := \sup_{n \geq 1} P[\tau_n^x \leq t]. \quad (7.11)$$

Thus, using Proposition 5.7. *ii*) we get for all $x > 0$

$$\begin{aligned} &P \left[\sup_{[0,T]} d_\lambda(M(t), M^n(t)) > \varepsilon \right] \\ &\leq P \left[\sup_{[0,T]} d_\lambda(M(t), M^{(n)}(t)) > \varepsilon \right] + P \left[\sup_{[0,T]} d_\lambda(M^{(n)}(t), M^n(t)) > \varepsilon \right] \\ &\leq P[\tau^x \leq T] + \alpha_1(T, x) + \frac{1}{\varepsilon} \mathbb{E} \left[\sup_{[0, T \wedge \tau_{(n)}^x \wedge \tau^x]} d_\lambda(M(t), M^{(n)}(t)) \right] \\ &\quad + \alpha_1(T, x) + \alpha_2(T, x) + \frac{1}{\varepsilon} e^{C(x+1)T} d_\lambda(m^{(n)}, m^n). \end{aligned}$$

We first make n tend to infinity and use (7.8), then x to infinity and use (7.10) and (7.11). We thus conclude that (7.9) holds.

We may prove point *ii*) using a similar computation that for *i*). The proof is easier since we do not need to use a triangulation.

Finally, consider $(M(m, t))_{t \geq 0}$ solution to $SDE(K, F, m, N, M)$ and the sequence of stopping times $(\tau^{x_n})_{n \geq 1}$ where $\tau^{x_n} = \inf\{t \geq 0, \|M(m, t)\|_\lambda \geq x_n\}$, with $x_n = n$. From Lemma 7.2. we have that $(\tau^{x_n})_{n \geq 1}$ is non-decreasing and $\tau^{x_n} \xrightarrow[n \rightarrow \infty]{} \infty$ and from Lemma 7.3. we deduce that $(\mathcal{L}_{K, F}^\beta \Phi(M(m, s)))_{s \in [0, \tau^{x_n}]}$ is uniformly bounded.

We thus apply Itô's Formula to $\Phi(M(m, t))$ on the interval $[0, t \wedge \tau^{x_n}]$ to obtain

$$\begin{aligned} \Phi(M(m, t \wedge \tau^{x_n})) - \Phi(m) &= \\ &\int_0^{t \wedge \tau^{x_n}} \int_{i < j} \int_0^\infty [\Phi(c_{ij}(M(m, s-))) - \Phi(M(m, s-))] \mathbf{1}_{\{z \leq K(M_i(m, s-), M_j(m, s-))\}} \\ &\quad \tilde{N}(dt, d(i, j), dz) \\ &+ \int_0^{t \wedge \tau^{x_n}} \int_i \int_\Theta \int_0^\infty [\Phi(f_{i\theta}(M(m, s-))) - \Phi(M(m, s-))] \mathbf{1}_{\{z \leq F(M_i(m, s-))\}} \tilde{M}(dt, di, d\theta, dz) \\ &+ \int_0^{t \wedge \tau^{x_n}} \mathcal{L}_{K, F}^\beta(M(m, s)) ds, \end{aligned}$$

where \tilde{N} and \tilde{M} are two compensated Poisson measures and point *iii*) follows.

This ends the proof of Theorem 5.3.

APPENDIX A. ESTIMATES CONCERNING c_{ij} , $f_{i\theta}$, d AND d_λ

Here we put all the auxiliary computations needed in Sections 6 and 7.

Lemma A.1. Fix $\lambda \in (0, 1]$. Consider any pair of finite permutations $\sigma, \tilde{\sigma}$ of \mathbb{N} . Then for all m and $\tilde{m} \in \ell_\lambda$,

$$d(m, \tilde{m}) \leq \sum_{k \geq 1} 2^{-k} |m_{\sigma(k)} - \tilde{m}_{\tilde{\sigma}(k)}|, \quad (\text{A.1})$$

$$d_\lambda(m, \tilde{m}) \leq \sum_{k \geq 1} |m_{\sigma(k)}^\lambda - \tilde{m}_{\tilde{\sigma}(k)}^\lambda|. \quad (\text{A.2})$$

This lemma is a consequence of [7, Lemma 3.1].

We also have the following inequality: for all $\alpha, \beta > 0$, there exists a positive constant $C = C_{\alpha, \beta}$ such that for all $x, y \geq 0$,

$$(x^\alpha + y^\alpha)|x^\beta - y^\beta| \leq 2|x^{\alpha+\beta} - y^{\alpha+\beta}| \leq C(x^\alpha + y^\alpha)|x^\beta - y^\beta|. \quad (\text{A.3})$$

We now give the inequalities concerning the action of c_{ij} and $f_{i\theta}$ on d_λ and $\|\cdot\|_\lambda$.

Lemma A.2. *Let $\lambda \in (0, 1]$ and $\theta \in \Theta$. Then for all m and $\tilde{m} \in \ell_\lambda$, all $1 \leq i < j < \infty$,*

$$\|c_{ij}(m)\|_\lambda = \|m\|_\lambda + (m_i^\lambda + m_j^\lambda) - m_i^\lambda - m_j^\lambda \leq \|m\|_\lambda, \quad (\text{A.4})$$

$$\|f_{i\theta}(m)\|_\lambda = \|m\|_\lambda + m_i^\lambda \left(\sum_{k \geq 1} \theta_k^\lambda - 1 \right), \quad (\text{A.5})$$

$$d_\lambda(c_{ij}(m), m) \leq 2m_j^\lambda, \quad (\text{A.6})$$

$$d_\lambda(f_{i\theta}(m), m) \leq m_i^\lambda (1 - \theta_1^\lambda) + m_i^\lambda \sum_{k \geq 2} \theta_k^\lambda, \quad (\text{A.7})$$

$$d_\lambda(c_{ij}(m), c_{ij}(\tilde{m})) \leq d_\lambda(m, \tilde{m}), \quad (\text{A.8})$$

$$d_\lambda(f_{i\theta}(m), f_{i\theta}(\tilde{m})) \leq d_\lambda(m, \tilde{m}) + |m_i^\lambda - \tilde{m}_i^\lambda| \left(\sum_{k \geq 1} \theta_k^\lambda - 1 \right). \quad (\text{A.9})$$

On the other hand, recall (3.34), we have, for $u, v \in \mathbb{N}$ with $1 \leq u < v$,

$$d_\lambda(f_{i\psi_u(\theta)}(m), f_{i\psi_v(\theta)}(m)) \leq \sum_{k=u+1}^v \theta_k^\lambda m_i^\lambda. \quad (\text{A.10})$$

Proof. First (A.4) and (A.5) are evident. Next, (A.6) and (A.8) are proved in [10, Lemma A.2].

To prove (A.7) let $\theta = (\theta_1, \dots) \in \Theta$, $i \geq 1$ and $p \geq 2$ and set $l := l(m) = \min\{k \geq 1 : m_k \leq \theta_p m_i\}$, we consider the largest particle of the original system (before dislocation of m_i) that is smaller than the p -th fragment of m_i , this is m_l . Consider now σ , the finite permutation of \mathbb{N} that achieves:

$$\begin{aligned} (f_k)_{k \geq 1} &:= \left([f_{i\theta}(m)]_{\sigma(k)} \right)_{k \geq 1} \\ &= (m_1, \dots, m_{i-1}, \theta_1 m_i, m_{i+1}, \dots, m_{l-1}, m_l, \theta_2 m_i, \theta_3 m_i, \dots, \theta_p m_i, [f_{i\theta}(m)]_{l+1}) \end{aligned} \quad (\text{A.11})$$

It suffices to compute the d_λ -distance of the sequences $(f_k)_k$ and $(m_k)_k$:

$$\begin{array}{cccccccccccccccc} m_1 & \cdots & m_{i-1} & \theta_1 m_i & m_{i+1} & \cdots & m_{l-1} & m_l & \theta_2 m_i & \theta_3 m_i & \cdots & \theta_p m_i & f_{l+p} & \cdots \\ m_1 & \cdots & m_{i-1} & m_i & m_{i+1} & \cdots & m_{l-1} & m_l & m_{l+1} & m_{l+2} & \cdots & m_{l+p-1} & m_{l+p} & \cdots \end{array} \quad (\text{A.12})$$

Thus, using (A.2), we have

$$\begin{aligned}
d_\lambda(f_{i\theta}(m), m) &\leq \sum_{k \geq 1} |f_k^\lambda - m_k^\lambda| = \left(\sum_{k=1}^l + \sum_{k=l+1}^{l+p-1} + \sum_{k \geq l+p} \right) |f_k^\lambda - m_k^\lambda| \\
&\leq (1 - \theta_1^\lambda) m_i^\lambda + \sum_{k=l+1}^{l+p-1} |\theta_{k-l+1}^\lambda m_i^\lambda - m_k^\lambda| + \sum_{k \geq l+p} |f_k^\lambda - m_k^\lambda| \\
&\leq (1 - \theta_1^\lambda) m_i^\lambda + \left(\sum_{k=2}^p \theta_k^\lambda m_i^\lambda + \sum_{k=l+1}^{l+p-1} m_k^\lambda \right) + \sum_{k \geq l+p} (f_k^\lambda + m_k^\lambda) \\
&= (1 - \theta_1^\lambda) m_i^\lambda + m_i^\lambda \sum_{k=2}^{\infty} \theta_k^\lambda + 2 \sum_{k > l} m_k^\lambda.
\end{aligned}$$

For the last equality it suffices to remark that $\sum_{k \geq l} f_k^\lambda$ contains all the remaining fragments of m_i^λ and all the particles m_k^λ with $k > l$.

Note that if $m \in \ell_{0+}$ the last sum consists of a finite number of terms and it suffices to take p large enough (implying l large) to cancel this term. On the other hand, if $m \in \ell_\lambda \setminus \ell_{0+}$ then the last sum is the tail of a convergent serie and since $l \rightarrow \infty$ whenever $p \rightarrow \infty$, we conclude by making p tend to infinity and (A.7) follows.

To prove (A.9) consider \tilde{m} , $l := l(m) \vee l(\tilde{m})$ and the permutations σ and $\tilde{\sigma}$ associated to this l , exactly as in (A.11). Let f and \tilde{f} be the corresponding objects concerning m and \tilde{m} :

$$\begin{array}{cccccccccccccccc}
m_1 & \cdots & m_{i-1} & \theta_1 m_i & m_{i+1} & \cdots & m_{l-1} & m_l & \theta_2 m_i & \theta_3 m_i & \cdots & \theta_p m_i & f_{l+p} & \cdots \\
\tilde{m}_1 & \cdots & \tilde{m}_{i-1} & \theta_1 \tilde{m}_i & \tilde{m}_{i+1} & \cdots & \tilde{m}_{l-1} & \tilde{m}_l & \theta_2 \tilde{m}_i & \theta_3 \tilde{m}_i & \cdots & \theta_p \tilde{m}_i & \tilde{f}_{l+p} & \cdots
\end{array} \quad (\text{A.13})$$

Using again (A.2) for $(f_k)_k$ and $(\tilde{f}_k)_k$, we have

$$\begin{aligned}
d_\lambda(f_{i\theta}(m), f_{i\theta}(\tilde{m})) &\leq \sum_{k \geq 1} |f_k^\lambda - \tilde{f}_k^\lambda| = \left(\sum_{k=1}^l + \sum_{k=l+1}^{l+p-1} + \sum_{k \geq l+p} \right) |f_k^\lambda - \tilde{f}_k^\lambda| \\
&= \sum_{k=1}^l |m_k^\lambda - \tilde{m}_k^\lambda| - |m_i^\lambda - \tilde{m}_i^\lambda| + \sum_{k=1}^p \theta_k^\lambda |m_i^\lambda - \tilde{m}_i^\lambda| + \sum_{k \geq l+p} (f_k^\lambda + \tilde{f}_k^\lambda) \\
&= \sum_{k=1}^l |m_k^\lambda - \tilde{m}_k^\lambda| - |m_i^\lambda - \tilde{m}_i^\lambda| + \sum_{k=1}^p \theta_k^\lambda |m_i^\lambda - \tilde{m}_i^\lambda| + \sum_{k > p} \theta_k^\lambda (m_i^\lambda + \tilde{m}_i^\lambda) \\
&\quad + \sum_{k > l} (m_k^\lambda + \tilde{m}_k^\lambda) \\
&= \sum_{k=1}^l |m_k^\lambda - \tilde{m}_k^\lambda| + |m_i^\lambda - \tilde{m}_i^\lambda| \left(\sum_{k=1}^p \theta_k^\lambda - 1 \right) + (m_i^\lambda + \tilde{m}_i^\lambda) \sum_{k > p} \theta_k^\lambda \\
&\quad + \sum_{k > l} (m_k^\lambda + \tilde{m}_k^\lambda).
\end{aligned}$$

Notice that the last two sums are the tails of convergent series, note also that $l \rightarrow \infty$ whenever $p \rightarrow \infty$. We thus conclude making p tend to infinity.

Finally, to prove (A.10) we consider the permutation σ as in (A.11) with $p = v$ and $l := l(m)$. Recall (A.13), we have

$$\begin{aligned} d_\lambda(f_{i\psi_u(\theta)}(m), f_{i\psi_v(\theta)}(m)) &= d_\lambda(f_{i\psi_v(\psi_u(\theta))}(m), f_{i\psi_v(\theta)}(m)) \\ &\leq \sum_{k \geq 1} |[f_{i\psi_v(\psi_u(\theta))}(m)]_{\sigma(k)} - [f_{i\psi_v(\theta)}(m)]_{\sigma(k)}| \\ &\leq \sum_{k=u+1}^v \theta_k^\lambda m_i^\lambda + 2 \sum_{k \geq l} m_k^\lambda. \end{aligned}$$

We used that $[\psi_v(\psi_u(\theta))]_k = 0$ for $k = u+1, \dots, v$. Since $m \in \ell_\lambda$, we conclude making l tend to infinity. \square

Lemma A.3. Consider $m, \tilde{m} \in S^\downarrow$ and $1 \leq i < j < \infty$. Recall the definition of d (4.4), d_λ (4.5), $c_{ij}(m)$ and $f_{i\theta}(m)$ (4.3) and $\psi_n(\theta)$ (3.34). For $\lambda \in (0, 1]$ and for all $m, \tilde{m} \in \ell_\lambda$ there exists a positive constant C depending on λ such that

$$d(m, \tilde{m}) \leq C(\|m\|_1^{1-\lambda} \vee \|\tilde{m}\|_1^{1-\lambda}) d_\lambda(m, \tilde{m}). \quad (\text{A.14})$$

Next,

$$d(c_{ij}(m), m) \leq \frac{3}{2} 2^{-i} m_j, \quad \sum_{1 \leq k < l < \infty} d(c_{kl}(m), m) \leq \frac{3}{2} \|m\|_1, \quad (\text{A.15})$$

$$d(c_{ij}(m), c_{ij}(\tilde{m})) \leq (2^i + 2^j) d(m, \tilde{m}). \quad (\text{A.16})$$

$$d(f_{i\theta}(m), m) \leq 2(1 - \theta_1) 2^{-i} m_i, \quad (\text{A.17})$$

$$d(f_{i\theta}(m), f_{i\theta}(\tilde{m})) \leq d(m, \tilde{m}), \quad (\text{A.18})$$

$$d(f_{i\theta}(m), f_{i\psi_n(\theta)}(m)) \leq 2^{-i} m_i \sum_{k > n} \theta_k. \quad (\text{A.19})$$

Proof. The inequality (A.14) comes from (A.3), with $\alpha = 1 - \lambda$ and $\beta = \lambda$. The inequalities (A.15) and (A.16) involving d are proved in [7, Corollary 3.2.].

We prove (A.17) exactly as (A.7). Consider p, l and the permutation σ defined by (A.11), from (A.1) and since $i \leq l+1 \leq l+p$, we obtain

$$\begin{aligned} d(f_{i\theta}(m), m) &\leq \left(\sum_{k=1}^l + \sum_{k=l+1}^{l+p-1} + \sum_{k \geq l+p} \right) 2^{-k} |f_k - m_k| \\ &\leq (1 - \theta_1) 2^{-i} m_i + \sum_{k=l+1}^{l+p-1} 2^{-k} |\theta_{k-l+1} m_i - m_k| + \sum_{k \geq l+p} 2^{-k} |f_k - m_k| \\ &\leq (1 - \theta_1) 2^{-i} m_i + \left(\sum_{k=2}^p 2^{-i} \theta_k m_i + \sum_{k=l+1}^{l+p-1} m_k \right) + \sum_{k \geq l+p} 2^{-i} (f_k + m_k) \\ &\leq (1 - \theta_1) 2^{-i} m_i + 2^{-i} m_i \sum_{k=2}^{\infty} \theta_k + 2 \sum_{k > l} m_k. \end{aligned}$$

Since $m \in \ell_1$, we conclude using (2.5) and making l tend to infinity.

Next, we prove (A.18) exactly as (A.9). Consider p, l and the permutations σ and $\tilde{\sigma}$ defined by (A.11). Recall (A.13), from (A.1) and since, $i \leq l+1 \leq l+p$ we obtain

$$\begin{aligned}
d(f_{i\theta}(m), f_{i\theta}(\tilde{m})) &\leq \left(\sum_{k=1}^l + \sum_{k=l+1}^{l+p-1} + \sum_{k \geq l+p} \right) 2^{-k} |f_k - \tilde{f}_k| \\
&\leq \sum_{k=1}^l 2^{-k} |m_k - \tilde{m}_k| + (\theta_1 - 1) 2^{-i} |m_i - \tilde{m}_i| + \sum_{k=l+1}^{l+p-1} 2^{-k} \theta_{k-l+1} |m_i - \tilde{m}_i| \\
&\quad + \sum_{k \geq l+p} 2^{-k} (f_k + \tilde{f}_k) \\
&\leq \sum_{k=1}^l |m_k - \tilde{m}_k| + 2^{-i} |m_i - \tilde{m}_i| \left(\sum_{k=1}^p \theta_k - 1 \right) + (m_i + \tilde{m}_i) \sum_{k > p} \theta_k \\
&\quad + \sum_{k > l} (m_k + \tilde{m}_k) \\
&\leq \sum_{k=1}^l |m_k - \tilde{m}_k| + (m_i + \tilde{m}_i) \sum_{k > p} \theta_k + \sum_{k > l} (m_k + \tilde{m}_k).
\end{aligned}$$

We used that for $k \geq l+p$, f_k contains all the remaining fragments of m_i and the particles m_j with $j > l$ and (2.5). Since $m, \tilde{m} \in \ell_1$ we conclude making p tend to infinity.

For the inequality (A.19), let $i \geq 1, p \geq 1$ and $l := l_p(m) = \min\{k \geq 1 : m_k \leq (\theta_n/p)m_i\}$ and consider σ , the finite permutation of \mathbb{N} that achieves:

$$\begin{aligned}
(f_k)_{k \geq 1} &:= \left([f_{i\theta}(m)]_{\sigma(k)} \right)_{k \geq 1} \\
&= (m_1, \dots, m_{i-1}, \theta_1 m_i, \dots, \theta_n m_i, m_{i+1}, \dots, m_{l-1}, m_l, [f_{i\theta}(m)]_{l+n}, \dots). \quad (\text{A.20})
\end{aligned}$$

Thus, from (A.1) and since $i \leq l+1 \leq l+n+1$, we deduce

$$\begin{aligned}
d((f_{i\theta}(m), f_{i\psi_n(\theta)}(m))) &\leq \left(\sum_{k=1}^l + \sum_{k=l+1}^{l+n-1} + \sum_{k \geq l+n} \right) 2^{-k} |[f_{i\theta}(m)]_{\sigma(k)} - [f_{i\psi_n(\theta)}(m)]_{\sigma(k)}| \\
&\leq \sum_{k > n} 2^{-i} \theta_k m_i + 2 \sum_{k > l} 2^{-i} m_k.
\end{aligned}$$

The last sum being the tail of a convergent series we conclude making $l \rightarrow \infty$.

This concludes the proof of Lemma A.3. □

REFERENCES

- [1] D.J. Aldous. Deterministic and Stochastic Models for Coalescence (Aggregation, Coagulation): A Review of the Mean-Field Theory of Probabilists. *Bernoulli*, 5:3–48, 1999.
- [2] J. Bertoin. Homogeneous fragmentation processes. *Probab. Theory Relat. Fields*, 121:301–318, 2001.
- [3] J. Bertoin. Self-similar fragmentations. *Ann. I. H. Poincaré*, 38:319–340, 2002.
- [4] J. Bertoin. *Random Fragmentation and Coagulation Processes*. Cambridge University Press, 2006.
- [5] E. Cepeda and N. Fournier. Smoluchowski's equation: rate of convergence of the Marcus-Lushnikov process. *Stochastic Process. Appl.*, 121(6):1411–1444, 2011.
- [6] P. Laurençot. On a class of continuous-fragmentation equations. *J. Differential Equations*, 167:245–274, 2000.
- [7] N. Fournier. On some stochastic coalescents. *Proba. Theory Related Fields*, 136(4):509–523, 2006.

- [8] N. Fournier and J. S. Giet. On small particles in coagulation-fragmentation equations. *J. Stat. Phys.*, 10(2):95–106, 2004.
- [9] N. Fournier and Ph. Laurençot. Well-Possedness of Smoluchowski’s Coagulation Equation for a Class of Homogeneous Kernels. *J. Functional Analysis*, 233(2):351–379, 2006.
- [10] N. Fournier and E. Löcherbach. Stochastic coalescence with homogeneous-like interaction rates. *Stoch. Proc. Appl.*, 119:45–73, 2009.
- [11] A. K. Giri. On the uniqueness for coagulation and multiple fragmentation equation. *arXiv:1208.0413 [math.AP]*, 2012.
- [12] E. K. Giri, J. Kumar, and G. Warnecke. The continuous coagulation equation with multiple fragmentation. *J. Mathematical Analysis and App.*, 374(1):71–87, 2011.
- [13] B. Haas. Loss of mass in deterministic and random fragmentations. *Stochastic Process. Appl.*, 106(2):245–277, 2003.
- [14] J.R. Norris. Smoluchowski’s coagulation equation: uniqueness, non-uniqueness and hydrodynamic limit for the stochastic coalescent. *Ann. Appl. Probab.*, 9(1):78–109, 1999.
- [15] I. W. Stewart. A global existence theorem for the general coagulation fragmentation equation with unbounded kernels. *Math. Methods Appl. Sci.*, 11:627–648, 1989.
- [16] I. W. Stewart. A uniqueness existence theorem for the coagulation fragmentation equation. *Math. Proc. Camb. Phil. Soc.*, 107:573–578, 1990.

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