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# Surface waves supported by thin-film/substrate interactions

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## Abstract

A systematic approximation to the linear equations for small-amplitude surface waves in an elastic half space, interacting with a residually-stressed thin film of different material bonded to its plane boundary, is developed in powers of the film thickness, assuming the latter to be small compared to the wavelength of the disturbance. The theory is illustrated by calculating asymptotic expansions of the wave speeds for Love and Rayleigh waves valid to second order in the dimensionless film thickness for a transversely isotropic film bonded to an isotropic substrate.

## 1 Introduction

We are concerned in this work with the problem of the acoustic interaction between an elastic half-space and a thin film of a different material attached to its surface. In particular, we seek an alternative to the classical approach of solving the exact differential equations in the film and the continuity conditions at the film/substrate interface and instead seek an approximate two-dimensional model for the film that can be used to obtain systematic approximations for small thickness. The same objective motivated the early fundamental works of Achenbach & Keshava (1967) and Tiersten (1969), in which the film is modelled

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as an elastic plate acted upon by a distributed force supplied by the substrate. In these works the model for the film is based on classical ideas in plate theory (e.g., Landau & Lifschitz, 1986) and accordingly suffers from the ambiguities inherent in theories of this kind. Indeed, open questions about the relationship between such theories and three-dimensional elasticity have furnished the impetus for ongoing research (see, for example, Pichugin & Rogerson, 2002; Kaplunov *et al.*, 2006; Paroni, 2006; Steigmann, 2007). Efforts to address these issues, or, more accurately, to side-step them, have been based in the present context on models in which the film is regarded as an elastic boundary with essentially zero thickness, as in the classical theory of capillary surfaces. Murdoch's work (1976), in which the elastic boundary is regarded as a pre-stressed membrane, typifies this approach. That of Ogden & Steigmann (2002) seeks to extend it by accounting for the flexural stiffness of the film in addition to its extensional stiffness.

An alternative approach, discussed recently by Lembo & Podio-Guidugli (2007), is to impose constraints of the Kirchhoff-Love type on the through-thickness variation of the three-dimensional displacement field and to use the associated reactive stresses to enhance the accuracy of the three-dimensional stress field generated by the resulting two-dimensional model. While this approach has met with success, our view is that the thickness-wise variation of the displacement field should emerge as a prediction of a theory for thin bodies rather than being imposed at the outset. Indeed, the present approach yields predictions of precisely this kind (cf. equations (3.17), (5.11), (5.20) and (5.21) below) that are not of the form typically imposed in constrained theories. In this respect our approach is similar to that of Ciarlet (2000). A further drawback of the approach based on constraints is that while it is clearly intended for application to thin bodies, there is no *a priori* estimate of the range of thicknesses, if any, over which the constraints are appropriate.

The approach followed here is based on a systematic small-thickness approximation of

the exact three-dimensional mechanical energy density of the film material in which the thickness arises explicitly. This approximation may be truncated at any desired level. The approximate equations of motion emerge from the stationarity of the associated action integral. This method was introduced by Steigmann (2007) for nonlinear elasticity. Here it is applied in the setting of linear elasticity to derive equations of motion for the film/substrate interface directly. Conventionally, such models furnish equations for the midsurface of the considered thin body. Membrane effects are associated with the order  $h$  problem, where  $h$  is the film thickness, and corrections, associated with bending effects, typically emerge at order  $h^3$  if the midsurface is a plane of symmetry of the material properties. Here we derive equations for the film/substrate interface instead of the film midsurface. Corrections to the membrane approximation now emerge at order  $h^2$  and include bending effects in a non-standard manner. The form of the order  $h^2$  correction is the primary concern of the present work. Higher-order corrections may be derived by following the described procedure.

Vapour deposition of material onto substrates may be expected to yield films that are *transversely isotropic* in their solid phases with the isotropic plane being parallel to the film/substrate interface. In the setting of linear elasticity, this assumption accommodates fiber symmetry or hexagonal crystal symmetry and allows for residual stress in the film. Accordingly, we develop the theory for transversely isotropic films interacting with isotropic substrates. Section 2 contains prerequisite material on transversely isotropic linearly elastic materials *with residual stress*. Fundamental theory for linear elasticity with residual stress is developed in Man & Carlson (1994). In Section 3, we outline the procedure used to generate systematic estimates of the film energy in powers of the thickness. This is used together with Hamilton's principle in Section 4 to derive the approximate equations of motion for the interface between arbitrary linearly elastic films and substrates. The model is specialized to transversely isotropic films and isotropic substrates in Section 5 and used

in Section 6 to analyze the propagation of surface waves.

Standard notation is used, with bold face representing vectors and tensors and subscripts denoting their components with respect to standard orthogonal axes. Latin subscripts take values in  $\{1, 2, 3\}$  while Greek subscripts take values in  $\{1, 2\}$ . The latter are associated with coordinates in the interface and in-plane vector and tensor components. A dot between bold symbols is used to denote the standard inner product for either vectors or second-order tensors. In particular, if  $\mathbf{A}$  and  $\mathbf{B}$  are second-order tensors, then  $\mathbf{A} \cdot \mathbf{B} = \text{tr}(\mathbf{A}\mathbf{B}^T)$ , where  $\text{tr}$  is the trace and the superscript  $T$  is used to denote the transpose. The linear operator  $\text{Sym}$  delivers the symmetric part of its second-order tensor argument. The notation  $\otimes$  identifies the standard tensor product of vectors. If  $\mathcal{C}$  is a fourth-order tensor, then  $\mathcal{C}[\mathbf{A}]$  is the second-order tensor with components  $\mathcal{C}_{ijkl}A_{kl}$ . Finally, we use the symbols  $\text{Div}$  and  $\text{Grad}$  to denote the three-dimensional divergence and gradient operators, while  $\text{div}$  and  $\nabla$  are reserved for their two-dimensional counterparts. Thus, for example,  $\text{Div} \mathbf{A} = A_{ij,j} \mathbf{e}_i$  and  $\text{div} \mathbf{A} = A_{i\alpha,\alpha} \mathbf{e}_i$ , where  $\{\mathbf{e}_i\}$  is an orthonormal basis and subscripts preceded by commas are used to denote partial derivatives with respect to coordinates. Time derivatives are denoted by a subscript  $t$ .

## 2 Three-dimensional strain-energy function for a transversely isotropic solid with residual stress

To apply linear elasticity theory we require an expression for the strain energy of the bulk material of which the film is made, valid to quadratic order in the three-dimensional displacement gradient  $\mathbf{H}$ . Let  $U(\mathbf{H})$  be the required strain-energy function, and let  $\bar{U}(\mathbf{E})$  be the strain-energy function expressed in terms of the Lagrange strain

$$\mathbf{E} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H}). \quad (2.1)$$

Let  $f(\tau) = U(\mathbf{H}(\tau))$ , where  $\mathbf{H}(\tau) = \tau\mathbf{H}_0$ , with  $\mathbf{H}_0$  fixed, is the parametric representation of a straight line in displacement-gradient space. To ensure kinematic admissibility we require that  $\det(\mathbf{I} + \tau\mathbf{H}_0) > 0$ , where  $\mathbf{I}$  is the three-dimensional identity, this being assured for all  $\mathbf{H}_0$  if  $\tau$  is sufficiently small. Further, the expression for  $U(\mathbf{H})$  need only apply in an open ball containing the origin. Since the latter is a convex set, the use of straight-line paths to derive  $U(\mathbf{H})$  entails no loss of generality.

We have  $f(\tau) = \bar{U}(\mathbf{E}(\tau))$ , where  $\mathbf{E}(\tau)$  is the (curved) image of the straight line in strain space. Thus,

$$f(\tau) = \tau\dot{f}(0) + \frac{1}{2}\tau^2\ddot{f}(0) + o(\tau^2), \quad (2.2)$$

apart from an unimportant constant, where a superposed dot represents a derivative with respect to  $\tau$ ,

$$\dot{f}(0) = \mathbf{S}_R \cdot \dot{\mathbf{E}}, \quad \ddot{f}(0) = \dot{\mathbf{E}} \cdot \mathcal{C}[\dot{\mathbf{E}}] + \mathbf{S}_R \cdot \ddot{\mathbf{E}}, \quad (2.3)$$

wherein

$$\dot{\mathbf{E}} = \text{Sym}\mathbf{H}_0, \quad \ddot{\mathbf{E}} = \mathbf{H}_0^T \mathbf{H}_0 \quad (2.4)$$

are the derivatives of  $\mathbf{E}(\tau)$  at  $\tau = 0$ ,  $\mathbf{S}_R$  is the (symmetric) residual second Piola-Kirchhoff stress at zero strain, and  $\mathcal{C}$  is the classical fourth-order tensor of elastic moduli associated with the undeformed material. The latter possesses the minor symmetries

$$\mathbf{A} \cdot \mathcal{C}[\mathbf{B}] = \mathbf{A}^T \cdot \mathcal{C}[\mathbf{B}], \quad \mathbf{A} \cdot \mathcal{C}[\mathbf{B}] = \mathbf{A} \cdot \mathcal{C}[\mathbf{B}^T] \quad (2.5)$$

and the major symmetry

$$\mathbf{A} \cdot \mathcal{C}[\mathbf{B}] = \mathbf{B} \cdot \mathcal{C}[\mathbf{A}] \quad (2.6)$$

for all second-order tensors  $\mathbf{A}, \mathbf{B}$ . Accordingly, (2.2) furnishes the quadratic-order energy

$$U(\mathbf{H}) = \mathbf{S}_R \cdot \mathbf{H} + \frac{1}{2}(\mathbf{H}\mathbf{S}_R \cdot \mathbf{H} + \mathbf{H} \cdot \mathcal{C}[\mathbf{H}]). \quad (2.7)$$

This expression is consistent with that obtained by Hoger (1995), who notes that the linear term may be discarded if the residual stress is required to be self-equilibrating in the

sense that  $\text{Div} \mathbf{S}_R$  vanishes in the interior of the body and  $\mathbf{S}_R \mathbf{n}$  vanishes on its boundary (with exterior unit normal  $\mathbf{n}$ ). That this is so follows easily from the fact that  $\mathbf{H} = \text{D}\mathbf{u}$ , the gradient of the displacement field  $\mathbf{u}$ , yielding  $\mathbf{S}_R \cdot \mathbf{H} = \text{Div}(\mathbf{S}_R \mathbf{u}) - \mathbf{u} \cdot \text{Div} \mathbf{S}_R = \text{Div}(\mathbf{S}_R \mathbf{u})$ ; the volume integral of this term is expressible as the surface integral of  $\mathbf{u} \cdot \mathbf{S}_R \mathbf{n}$  over the boundary, which vanishes by virtue of the restriction imposed thereon. In the present work we do not impose equilibrium of the residual stress in the film in the exact three-dimensional sense and thus retain the linear term in (2.7). Rather, we impose equilibrium in a certain approximate sense to be described. This furnishes restrictions on the residual stress that are compatible with our model.

The model derived in Sections 3 and 4 below requires the derivative  $U_{\mathbf{H}}$ . To obtain it we use (2.7) to write

$$U_{\mathbf{H}} \cdot \dot{\mathbf{H}} = \dot{U} = (\mathbf{S}_R + \mathbf{H} \mathbf{S}_R + \mathcal{C}[\mathbf{H}]) \cdot \dot{\mathbf{H}}, \quad (2.8)$$

where we have invoked the symmetry of  $\mathbf{S}_R$  and the major symmetry of  $\mathcal{C}$ . Thus,

$$U_{\mathbf{H}} = \mathbf{S}_R + \mathbf{H} \mathbf{S}_R + \mathcal{C}[\mathbf{H}]. \quad (2.9)$$

In the classical theory without residual stress the energy reduces to  $\frac{1}{2} \mathbf{H} \cdot \mathcal{C}[\mathbf{H}]$ , and, in that setting, is usually assumed to be a positive definite function of  $\boldsymbol{\varepsilon} = \text{Sym} \mathbf{H}$ ; the minor symmetries of  $\mathcal{C}$  imply that the energy vanishes if  $\mathbf{H}$  is skew. Here we assume that the weaker condition of strong ellipticity is satisfied, i.e.

$$\mathbf{a} \otimes \mathbf{b} \cdot \mathcal{C}[\mathbf{a} \otimes \mathbf{b}] > 0 \quad \text{for all} \quad \mathbf{a} \otimes \mathbf{b} \neq \mathbf{0}. \quad (2.10)$$

It is easy to show that the symmetric part of  $\mathbf{a} \otimes \mathbf{b}$  vanishes only if  $\mathbf{a} \otimes \mathbf{b}$  vanishes, so that (2.10) is meaningful. Then, since  $\mathcal{C}$  and  $\mathbf{S}_R$  are independent, (2.10) also holds in the presence of residual stress.

In the case of transverse isotropy relative to the undeformed state of the body, the

components of  $\mathbf{C}$  relative to an orthonormal basis  $\{\mathbf{e}_i\}$  are (Spencer, 1984)

$$\begin{aligned} \mathcal{C}_{ijkl} = & \lambda\delta_{ij}\delta_{kl} + \mu_T(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \alpha(\delta_{ij}m_k m_l + m_i m_j \delta_{kl}) \\ & + (\mu_L - \mu_T)(m_i m_k \delta_{jl} + m_i m_l \delta_{jk} + m_j m_k \delta_{il} + m_j m_l \delta_{ik}) + \beta m_i m_j m_k m_l, \end{aligned} \quad (2.11)$$

where  $\delta_{ij}$  is the Kronecker delta,  $\alpha, \beta, \lambda, \mu_T$  and  $\mu_L$  are material constants, and the unit vector  $\mathbf{m}$ , with components  $m_i$ , is the axis of transverse isotropy. For example,  $m_i = \delta_{i3}$  if the basis is chosen such that  $\mathbf{e}_3 = \mathbf{m}$ . Spencer (1984) shows that  $\mu_T$  is the shear modulus for shearing in planes transverse to  $\mathbf{m}$ , whereas  $\mu_L$  is the shear modulus for shearing parallel to  $\mathbf{m}$ . The remaining material constants in (2.11) may be interpreted in terms of extensional moduli and Poisson ratios (Spencer, 1984).

The general form of the residual stress may be derived by enumerating the strain invariants for transverse isotropy that are linear in the (infinitesimal) strain. These are  $\mathbf{I} \cdot \boldsymbol{\varepsilon}$  and  $\mathbf{m} \otimes \mathbf{m} \cdot \boldsymbol{\varepsilon}$ , as given in Spencer (1984), and they are unchanged if  $\mathbf{H}$  is substituted for  $\boldsymbol{\varepsilon}$ . Comparison with the linear term in (2.7) then furnishes

$$\mathbf{S}_R = S_T(\mathbf{I} - \mathbf{m} \otimes \mathbf{m}) + S_L \mathbf{m} \otimes \mathbf{m}, \quad (2.12)$$

where  $S_T$  is the residual stress in the isotropic plane and  $S_L$  is the residual uniaxial stress along  $\mathbf{m}$ .

The strong-ellipticity inequality (2.10) is equivalent to the positive definiteness of the acoustic tensor  $\boldsymbol{\Gamma}(\mathbf{b})$  with components

$$\Gamma_{ik} = \mathcal{C}_{ijkl} b_j b_l, \quad (2.13)$$

where  $\mathbf{b}$  is an arbitrary unit vector. For  $b_i = m_i = \delta_{i3}$  we have

$$\Gamma_{ik} = \mathcal{C}_{i3k3} = \mu_L \delta_{ik} + (\varphi - \mu_L) \delta_{i3} \delta_{k3}, \quad (2.14)$$

where

$$\varphi = \lambda + 4\mu_L - 2\mu_T + 2\alpha + \beta. \quad (2.15)$$



Strong ellipticity thus requires that

$$\mu_L > 0, \quad \varphi > 0. \quad (2.16)$$

For  $b_i = \delta_{i1}$  we obtain

$$\Gamma_{ik} = \mathcal{C}_{i1k1} = (\lambda + 2\mu_T)\delta_{i1}\delta_{k1} + \mu_T\delta_{i2}\delta_{k2} + \mu_L\delta_{i3}\delta_{k3}, \quad (2.17)$$

and so it is further necessary that

$$\mu_T > 0, \quad \lambda + 2\mu_T > 0, \quad (2.18)$$

these results also being obtained in the case  $b_i = \delta_{i2}$ . Isotropy is recovered by setting  $\alpha = \beta = 0$  and  $\mu_L = \mu_T$ . Both sets of inequalities then reduce to the classical necessary and sufficient conditions

$$\mu > 0, \quad \lambda + 2\mu > 0. \quad (2.19)$$

Necessary and sufficient conditions for strong ellipticity to hold in the case of transverse isotropy are (2.16) and (2.18) together with

$$|\lambda + \alpha + \mu_L| < \mu_L + \sqrt{\varphi(\lambda + 2\mu_T)}; \quad (2.20)$$

see, for example, Payton (1983) and Merodio & Ogden (2003). For ease of reference we note the connections

$$c_{11} = \lambda + 2\mu_T, \quad c_{33} = \varphi, \quad c_{12} = \lambda, \quad c_{13} = \lambda + \alpha, \quad c_{44} = \mu_L$$

with the standard Voigt notation.

### 3 Small-thickness estimate of the film energy

We require estimates for the strain energy and kinetic energy of the film valid for small thickness. In particular, we seek a model that can be used to derive systematic corrections

to membrane models of thin films (Murdoch, 1976). To this end we regard the film as a material body with a reference configuration generated by the parallel translation of a simply-connected plane region  $\Omega$  (the film/substrate interface) in the direction orthogonal to  $\Omega$ . The film occupies the volume  $\kappa_f = \bar{\Omega} \times [0, h]$ , where  $\bar{\Omega} = \Omega \cup \partial\Omega$ ,  $\partial\Omega$  being the closed curve bounding  $\Omega$ , and  $h$  is the (uniform) thickness. Let  $l$  be another length scale such as the wavelength of a surface wave. We assume that  $\epsilon := h/l \ll 1$ . The material of the film is assumed to have uniform properties. Position in the three-dimensional reference placement of the film is given by

$$\mathbf{x} = \boldsymbol{\xi} + \varsigma \mathbf{m}, \quad (3.1)$$

where  $\boldsymbol{\xi} \in \Omega$ ,  $\mathbf{m}$  is the (fixed) unit normal to  $\Omega$ , and  $\varsigma \in [0, h]$ . We take the origin of the position  $\mathbf{x}$  to be on  $\Omega$ . The projection

$$\mathbf{1} = \mathbf{I} - \mathbf{m} \otimes \mathbf{m} \quad (3.2)$$

is the (two-dimensional) identity on the translation (vector) space  $\Omega'$  of  $\Omega$ , and may be used to expand the displacement gradient  $\tilde{\mathbf{H}} = \tilde{\mathbf{H}}\mathbf{I}$  in the form

$$\tilde{\mathbf{H}} = \mathbf{H} + \mathbf{a} \otimes \mathbf{m}, \quad (3.3)$$

where

$$\mathbf{H} = \tilde{\mathbf{H}}\mathbf{1}, \quad \mathbf{a} = \tilde{\mathbf{H}}\mathbf{m}. \quad (3.4)$$

Here we use the notation  $\tilde{\mathbf{H}}$  to denote the three-dimensional displacement gradient of the previous section and reserve the symbol  $\mathbf{H}$  for its action on  $\Omega'$ . Use of  $d\tilde{\mathbf{u}} = \tilde{\mathbf{H}}d\mathbf{x}$  with  $\hat{\mathbf{u}}(\boldsymbol{\xi}, \varsigma, t) = \tilde{\mathbf{u}}(\boldsymbol{\xi} + \varsigma\mathbf{m}, t)$  and  $d\boldsymbol{\xi} \in \Omega'$ , where  $\tilde{\mathbf{u}}$  is the three-dimensional displacement field, yields the alternative representation

$$(\tilde{\mathbf{H}}\mathbf{1})d\boldsymbol{\xi} + \tilde{\mathbf{H}}\mathbf{m}d\varsigma = d\hat{\mathbf{u}} = (\nabla\hat{\mathbf{u}})d\boldsymbol{\xi} + \hat{\mathbf{u}}'d\varsigma, \quad (3.5)$$

where  $\nabla$  is the (two-dimensional) gradient with respect to  $\boldsymbol{\xi}$  at fixed  $\varsigma$  and the prime is used to denote the partial derivative with respect to  $\varsigma$  at fixed  $\boldsymbol{\xi}$ . It follows that

$$\mathbf{H} = \nabla \hat{\mathbf{u}}, \quad \mathbf{a} = \hat{\mathbf{u}}'. \quad (3.6)$$

The total strain energy of the film in a given motion is

$$\int_{\kappa_f} U(\tilde{\mathbf{H}}(\mathbf{x}, t)) \, dv = \int_{\Omega} \int_0^h U(\hat{\mathbf{H}}(\boldsymbol{\xi}, \varsigma, t)) \, d\varsigma da, \quad (3.7)$$

where  $\hat{\mathbf{H}}(\boldsymbol{\xi}, \varsigma, t) = \tilde{\mathbf{H}}(\boldsymbol{\xi} + \varsigma \mathbf{m}, t)$ . We write the through-thickness integral in the form

$$I(h) = \int_0^h G(\varsigma) d\varsigma, \quad (3.8)$$

where  $G(\cdot) = U(\hat{\mathbf{H}}(\boldsymbol{\xi}, \cdot, t))$ . Then, by the Leibniz rule and Taylor's theorem,

$$I(h) = hG_0 + \frac{1}{2}h^2G'_0 + O(h^3), \quad (3.9)$$

where the zero subscript identifies function values at  $\varsigma = 0$  and where, by the chain rule,

$$G_0 = U(\hat{\mathbf{H}}_0), \quad G'_0 = U_{\hat{\mathbf{H}}}(\hat{\mathbf{H}}_0) \cdot \hat{\mathbf{H}}'_0. \quad (3.10)$$

Now, from (3.3),

$$\hat{\mathbf{H}} = \mathbf{H} + \mathbf{a} \otimes \mathbf{m}, \quad \hat{\mathbf{H}}' = \mathbf{A} + \mathbf{b} \otimes \mathbf{m} \quad (3.11)$$

with

$$\mathbf{A} = \mathbf{H}', \quad \mathbf{b} = \mathbf{a}'. \quad (3.12)$$

Use of these with (3.6) and the symmetry of mixed partial derivatives furnishes

$$\mathbf{A} = (\nabla \hat{\mathbf{u}})' = \nabla \mathbf{a}. \quad (3.13)$$

We thus obtain

$$\mathbf{H}_0 = \nabla(\hat{\mathbf{u}}_0), \quad \mathbf{A}_0 = \nabla(\mathbf{a}_0), \quad \mathbf{a}_0 = \hat{\mathbf{u}}'_0, \quad \mathbf{b}_0 = \mathbf{a}'_0, \quad (3.14)$$

in which  $\hat{\mathbf{u}}_0$ ,  $\mathbf{a}_0$  and  $\mathbf{b}_0$  are *independent* functions of  $\boldsymbol{\xi}$  and  $t$ .

Henceforth we suppress the zero subscript and write the total strain energy as

$$\int_{\kappa_f} U(\tilde{\mathbf{H}})dv = \int_{\Omega} W(\mathbf{a}, \mathbf{b}, \mathbf{H}, \mathbf{A})da + O(h^3), \quad (3.15)$$

where

$$W = hU(\mathbf{H} + \mathbf{a} \otimes \mathbf{m}) + \frac{1}{2}h^2U_{\tilde{\mathbf{H}}}(\mathbf{H} + \mathbf{a} \otimes \mathbf{m}) \cdot (\mathbf{A} + \mathbf{b} \otimes \mathbf{m}) \quad (3.16)$$

is the strain energy through order  $O(h^2)$  per unit area of  $\Omega$ ,  $\mathbf{u}(\boldsymbol{\xi}, t)$  ( $= \hat{\mathbf{u}}_0$ ) is the displacement of the film/substrate interface, and  $\mathbf{a}(\boldsymbol{\xi}, t)$ ,  $\mathbf{b}(\boldsymbol{\xi}, t)$  are *director* fields defined on  $\Omega$ . These are the coefficient vectors in the expansion

$$\tilde{\mathbf{u}}(\boldsymbol{\xi}, \varsigma, t) = \mathbf{u}(\boldsymbol{\xi}, t) + \varsigma\mathbf{a}(\boldsymbol{\xi}, t) + \frac{1}{2}\varsigma^2\mathbf{b}(\boldsymbol{\xi}, t) + O(\varsigma^3). \quad (3.17)$$

In the same way we use a formula like (3.9) to write the total kinetic energy as

$$\int_{\kappa_f} Tdv = \int_{\Omega} Kda + O(h^3), \quad (3.18)$$

where

$$2T = \rho\hat{\mathbf{u}}_t \cdot \hat{\mathbf{u}}_t \quad (3.19)$$

and, in the case of uniform density,

$$2K = \rho(h\mathbf{u}_t \cdot \mathbf{u}_t + h^2\mathbf{u}_t \cdot \mathbf{a}_t). \quad (3.20)$$

The total mechanical energy in a fixed volume  $\kappa_f$  of the film is then given by

$$\int_{\kappa_f} (U + T)dv = \int_{\Omega} (W + K)da + O(h^3). \quad (3.21)$$

The estimates (3.15), (3.18) and (3.21) are meaningful provided that  $h$  is small against any other length scale. In the present work the film/substrate combination is a half-space that supports a propagating surface disturbance whose wavelength  $l$  (the reciprocal of the wavenumber  $k$ ) furnishes the only available length scale to which  $h$  can be compared. Accordingly, the validity of the foregoing estimates is contingent on satisfaction of the strong inequality  $\epsilon \ll 1$ , where  $\epsilon = hk$ .

## 4 Equations of motion and energy flux

To obtain the equations of motion for the film/substrate combination we proceed as in Hilgers (1997) to use Hamilton's principle (Achenbach, 1973) together with the foregoing expressions for the elastic and kinetic energies. The required equations are recovered by rendering the associated action integral stationary. The latter is given by

$$\mathcal{A} = \int_{t_1}^{t_2} dt \left[ \int_{\kappa_s} (T_s - U_s) dv + \int_{\kappa_f} (T - U) dv \right], \quad (4.1)$$

where  $[t_1, t_2]$  is an assigned time interval,  $T_s$  and  $U_s$  are the substrate kinetic and elastic energy densities, and  $\kappa_s$  is the volume occupied by the substrate in its reference configuration. The volume  $\kappa = \kappa_s \cup \kappa_f$  of the film/substrate combination need not be the total volume of the film and substrate, assumed here to be a half-space of unbounded extent with the film situated on top of it. Rather,  $\kappa$  is taken to be the support of the variation of the displacement field, the latter being assumed to vanish identically in time in the region exterior to  $\kappa$ . These variations are further assumed to vanish identically in time on  $\partial\kappa_s \setminus \Omega$  and on  $\partial\Omega \times [0, h]$ , and to be arbitrary in the interior of  $\kappa$  and on the upper surface  $\partial\kappa_f^+ = \Omega \times \{h\}$  of the thin film. Under these restrictions the stationarity of the action integral yields equations of motion for the film/substrate combination but does not supply edge conditions. The latter are not required in this work. Variations are further supposed to vanish at the instants  $t_1$  and  $t_2$ .

Using (3.15) and (3.18) we estimate the action to be

$$\mathcal{A} = A + O(h^3), \quad (4.2)$$

where

$$A = \int_{t_1}^{t_2} dt \left[ \int_{\kappa_s} (T_s - U_s) dv + \int_{\Omega} (K - W) da \right]. \quad (4.3)$$

The stationarity of  $A$  thus furnishes equations of motion modulo terms of order  $h^3$ . Using

superposed dots to denote variations, we have

$$\int_{t_1}^{t_2} dt \left[ \int_{\kappa_s} (\dot{T}_s - \dot{U}_s) dv + \int_{\Omega} (\dot{K} - \dot{W}) da \right] = 0, \quad (4.4)$$

where

$$\dot{T}_s = \rho_s [(\mathbf{w}_t \cdot \dot{\mathbf{w}})_t - \dot{\mathbf{w}} \cdot \mathbf{w}_{tt}], \quad \dot{U}_s = \boldsymbol{\sigma} \cdot \mathbf{D}\dot{\mathbf{w}}, \quad (4.5)$$

$$\dot{K} = \rho \{ [(h\mathbf{u}_t + \frac{1}{2}h^2\mathbf{a}_t) \cdot \dot{\mathbf{u}}]_t - \dot{\mathbf{u}} \cdot (h\mathbf{u}_{tt} + \frac{1}{2}h^2\mathbf{a}_{tt}) + \frac{1}{2}h^2 [(\mathbf{u}_t \cdot \dot{\mathbf{a}})_t - \dot{\mathbf{a}} \cdot \mathbf{u}_{tt}] \}, \quad (4.6)$$

and

$$\dot{W} = W_{\mathbf{H}} \cdot \nabla \dot{\mathbf{u}} + W_{\mathbf{A}} \cdot \nabla \dot{\mathbf{a}} + W_{\mathbf{a}} \cdot \dot{\mathbf{a}} + W_{\mathbf{b}} \cdot \dot{\mathbf{b}}. \quad (4.7)$$

Here  $\mathbf{w}$  is the displacement field in the substrate,  $\mathbf{D}\mathbf{w}$  is the substrate displacement gradient,  $\rho_s$  is the substrate density, and  $\boldsymbol{\sigma}$  is the (symmetric) substrate stress. We assume that the substrate is free of residual stress and is bonded to the film in the sense that  $\mathbf{w}|_{\Omega} = \mathbf{u}$ . Invoking the stated restrictions on the variations, assuming a degree of smoothness sufficient to justify interchanging the order of integration, and using the Green-Stokes and divergence theorems, we reduce (4.4) to

$$\int_{t_1}^{t_2} dt \left\{ \int_{\kappa_s} \dot{\mathbf{w}} \cdot (\text{Div} \boldsymbol{\sigma} - \rho_s \mathbf{w}_{tt}) dv + \int_{\Omega} \dot{\mathbf{u}} \cdot [\text{div} W_{\mathbf{H}} - \boldsymbol{\sigma} \mathbf{m} - \rho (h\mathbf{u}_{tt} + \frac{1}{2}h^2\mathbf{a}_{tt})] da \right\} + \int_{t_1}^{t_2} dt \int_{\Omega} \left[ \dot{\mathbf{a}} \cdot (\text{div} W_{\mathbf{A}} - W_{\mathbf{a}} - \frac{1}{2}\rho h^2 \mathbf{u}_{tt}) - \dot{\mathbf{b}} \cdot W_{\mathbf{b}} \right] da = 0, \quad (4.8)$$

where we have used the fact that  $\mathbf{m}$  is the exterior unit normal to  $\kappa_s$  on  $\Omega$ . On using  $\dot{\mathbf{w}}|_{\Omega} = \dot{\mathbf{u}}$  and the mutual independence of the variations  $\dot{\mathbf{u}}, \dot{\mathbf{a}}, \dot{\mathbf{b}}$ , this yields the equation of motion

$$\text{Div} \boldsymbol{\sigma} = \rho_s \mathbf{w}_{tt} \quad (4.9)$$

in the substrate, together with

$$\text{div} W_{\mathbf{H}} - \boldsymbol{\sigma} \mathbf{m} = \rho (h\mathbf{u}_{tt} + \frac{1}{2}h^2\mathbf{a}_{tt}), \quad \text{div} W_{\mathbf{A}} - W_{\mathbf{a}} = \frac{1}{2}\rho h^2 \mathbf{u}_{tt}, \quad W_{\mathbf{b}} = \mathbf{0} \quad (4.10)$$

on the film/substrate interface. In the course of deriving (4.10) we have also used the fact that  $\dot{\mathbf{u}}$  and  $\dot{\mathbf{a}}$  vanish on  $\partial\Omega$ . This in turn follows from the restriction  $\dot{\mathbf{u}} = \mathbf{0}$ , which holds identically on  $\partial\Omega \times [0, h]$ .

To render these equations explicit we combine (2.9) with (3.16) to write

$$W = hU(\mathbf{H} + \mathbf{a} \otimes \mathbf{m}) + \frac{1}{2}h^2\{\mathbf{S}_R + (\mathbf{H} + \mathbf{a} \otimes \mathbf{m})\mathbf{S}_R + \mathbf{C}[\mathbf{H} + \mathbf{a} \otimes \mathbf{m}]\} \cdot (\mathbf{A} + \mathbf{b} \otimes \mathbf{m}). \quad (4.11)$$

Using the symmetries of  $\mathbf{S}_R$  and  $\mathbf{C}$ , we obtain

$$\begin{aligned} \dot{W} &= \{\mathbf{S}_R + (\mathbf{H} + \mathbf{a} \otimes \mathbf{m})\mathbf{S}_R + \mathbf{C}[\mathbf{H} + \mathbf{a} \otimes \mathbf{m}]\} \cdot [h(\dot{\mathbf{H}} + \dot{\mathbf{a}} \otimes \mathbf{m}) + \frac{1}{2}h^2(\dot{\mathbf{A}} + \dot{\mathbf{b}} \otimes \mathbf{m})] \\ &\quad + \frac{1}{2}h^2\{(\mathbf{A} + \mathbf{b} \otimes \mathbf{m})\mathbf{S}_R + \mathbf{C}[\mathbf{A} + \mathbf{b} \otimes \mathbf{m}]\} \cdot (\dot{\mathbf{H}} + \dot{\mathbf{a}} \otimes \mathbf{m}). \end{aligned} \quad (4.12)$$

Comparison with (4.7) then yields

$$W_{\mathbf{b}} = \frac{1}{2}h^2\{\mathbf{S}_R + (\mathbf{H} + \mathbf{a} \otimes \mathbf{m})\mathbf{S}_R + \mathbf{C}[\mathbf{H} + \mathbf{a} \otimes \mathbf{m}]\}\mathbf{m}, \quad (4.13)$$

$$W_{\mathbf{A}} = \frac{1}{2}h^2\{\mathbf{S}_R + (\mathbf{H} + \mathbf{a} \otimes \mathbf{m})\mathbf{S}_R + \mathbf{C}[\mathbf{H} + \mathbf{a} \otimes \mathbf{m}]\}\mathbf{1}, \quad (4.14)$$

$$W_{\mathbf{a}} = h\{\mathbf{S}_R + (\mathbf{H} + \mathbf{a} \otimes \mathbf{m})\mathbf{S}_R + \mathbf{C}[\mathbf{H} + \mathbf{a} \otimes \mathbf{m}]\}\mathbf{m} + \frac{1}{2}h^2\{(\mathbf{A} + \mathbf{b} \otimes \mathbf{m})\mathbf{S}_R + \mathbf{C}[\mathbf{A} + \mathbf{b} \otimes \mathbf{m}]\}\mathbf{m}, \quad (4.15)$$

and

$$W_{\mathbf{H}} = h\{\mathbf{S}_R + (\mathbf{H} + \mathbf{a} \otimes \mathbf{m})\mathbf{S}_R + \mathbf{C}[\mathbf{H} + \mathbf{a} \otimes \mathbf{m}]\}\mathbf{1} + \frac{1}{2}h^2\{(\mathbf{A} + \mathbf{b} \otimes \mathbf{m})\mathbf{S}_R + \mathbf{C}[\mathbf{A} + \mathbf{b} \otimes \mathbf{m}]\}\mathbf{1}. \quad (4.16)$$

We require the undeformed material to be in equilibrium and thus impose the appropriate specialization of (4.10) in the absence of deformation. This yields the restrictions

$$\mathbf{S}_R\mathbf{m} = \mathbf{0}, \quad \text{div}(\mathbf{S}_R\mathbf{1}) = \mathbf{0} \quad (4.17)$$

on the residual stress in the film at the film/substrate interface. With these satisfied, equations (4.10) reduce to

$$(\mathbf{C}[\mathbf{H} + \mathbf{a} \otimes \mathbf{m}])\mathbf{m} = \mathbf{0}, \quad \text{div}(\{\mathbf{H}\mathbf{S}_R + \mathbf{C}[\mathbf{H} + \mathbf{a} \otimes \mathbf{m}]\}\mathbf{1}) = \{\mathbf{C}[\mathbf{A} + \mathbf{b} \otimes \mathbf{m}]\}\mathbf{m} + \rho\mathbf{u}_{tt} \quad (4.18)$$

and

$$\frac{1}{2}h^2\operatorname{div}(\{\mathbf{A}\mathbf{S}_R+\mathbf{C}[\mathbf{A}+\mathbf{b}\otimes\mathbf{m}]\}\mathbf{1})+h\operatorname{div}(\{\mathbf{H}\mathbf{S}_R+\mathbf{C}[\mathbf{H}+\mathbf{a}\otimes\mathbf{m}]\}\mathbf{1})-\boldsymbol{\sigma}\mathbf{m}=\rho(h\mathbf{u}_{tt}+\frac{1}{2}h^2\mathbf{a}_{tt}). \quad (4.19)$$

In Section 5 we use the equations (4.18) to eliminate  $\mathbf{a}$  and  $\mathbf{b}$  in favour of the derivatives of  $\mathbf{u}$ ; the results are then substituted into (4.19) in order to formulate a problem for determining the displacement of the interface.

**Remark:** Equations (4.17)<sub>1</sub> and (4.18)<sub>1</sub> imply that the film is in a state of plane stress. In classical treatments as summarized in, for example, Landau & Lifschitz (1986), plane stress is imposed *a priori* in the course of constructing the energy functional for a thin plate from three-dimensional elasticity theory. In that setting the condition approximately satisfies the exact zero-traction conditions imposed on the lateral surfaces of the plate. The resulting energy functional purports to be valid for all geometrically possible configurations (subject to the restrictions inherent in the underlying three-dimensional theory), and yields the classical plate-bending equation as a consequence of the stationarity of the energy functional so constructed. Herein lies a logical defect, however. The *a priori* satisfaction of the restriction on lateral traction entails the incorporation of a natural boundary condition into the energy functional, whether or not the latter is stationary. This is not consistent with conventional variational theory, according to which such conditions emerge as consequences of the stationarity of the energy rather than being imposed at the outset. In contrast, the present treatment, which yields plane stress as a necessary condition for the stationarity of the action integral, restores logical consistency with the programme of conventional variational theory.

The energetic interaction between film and substrate may be determined from the equations of motion. To this end we integrate the time derivative  $W_t$  over a simply-connected part  $\Omega$  of the interface and use an expression like (4.7) in which variations are



replaced by time derivatives. The Green-Stokes theorem then yields

$$\begin{aligned} \int_{\Omega} W_t da &= \int_{\partial\Omega} (\mathbf{u}_t \cdot W_{\mathbf{H}}\boldsymbol{\nu} + \mathbf{a}_t \cdot W_{\mathbf{A}}\boldsymbol{\nu}) ds \\ &+ \int_{\Omega} [W_{\mathbf{b}} \cdot \mathbf{b}_t + (W_{\mathbf{a}} - \operatorname{div} W_{\mathbf{A}}) \cdot \mathbf{a}_t - \operatorname{div} W_{\mathbf{H}} \cdot \mathbf{u}_t] da, \end{aligned} \quad (4.20)$$

where  $\boldsymbol{\nu}$  is the outward unit normal to  $\partial\Omega$  in the plane of  $\Omega$ . We use (4.10) to reduce this to

$$\int_{\Omega} (E_t + \operatorname{div} \mathbf{E}) da = \int_{\Omega} \mathbf{E}_s \cdot \mathbf{m} da, \quad (4.21)$$

where  $E = W + K$  is the mechanical energy density of the film,

$$\mathbf{E} = -(W_{\mathbf{H}})^{\mathbf{T}} \mathbf{u}_t - (W_{\mathbf{A}})^{\mathbf{T}} \mathbf{a}_t \quad (4.22)$$

is the energy flux in the film, and

$$\mathbf{E}_s = -\boldsymbol{\sigma} \mathbf{w}_t \quad (4.23)$$

is the energy flux in the substrate. The arbitrariness of  $\Omega$  may be used to extract the local energy equation

$$E_t + \operatorname{div} \mathbf{E} = \mathbf{E}_s \cdot \mathbf{m}. \quad (4.24)$$

Similarly, the substrate energy flux satisfies

$$(E_s)_t + \operatorname{Div} \mathbf{E}_s = 0, \quad (4.25)$$

where  $E_s = U_s + T_s$  is the mechanical energy density of the substrate. From (4.24) it is evident that energy is supplied to the film through the component of substrate energy flux in the direction orthogonal to the interface.

## 5 Transversely isotropic film interacting with an isotropic substrate

We assume the substrate to be uniform, isotropic and free of residual stress, so that

$$\boldsymbol{\sigma} = \lambda_s \operatorname{tr}(\mathbf{D}\mathbf{w}) \mathbf{I} + \mu_s [\mathbf{D}\mathbf{w} + (\mathbf{D}\mathbf{w})^{\mathbf{T}}], \quad (5.1)$$

where  $\lambda_s$  and  $\mu_s$  are the associated Lamé constants. These are assumed to satisfy the ellipticity conditions (2.19), i.e.

$$\mu_s > 0, \quad \lambda_s + 2\mu_s > 0, \quad (5.2)$$

and (4.9) reduces to the well-known Navier equation

$$(\lambda_s + \mu_s)\text{Grad}(\text{Div}\mathbf{w}) + \mu_s\Delta\mathbf{w} = \rho_s\mathbf{w}_{tt}, \quad (5.3)$$

where  $\Delta$  is the Laplacian. The interaction term in (4.19) is given by

$$\boldsymbol{\sigma}\mathbf{m} = [\lambda_s w_{\alpha,\alpha} + (\lambda_s + 2\mu_s)w_{3,3}]\mathbf{m} + \mu_s(w_{\alpha,3} + w_{3,\alpha})\mathbf{e}_\alpha. \quad (5.4)$$

This is to be evaluated on the interfacial plane  $x_3 = 0$ , while equations (5.1)–(5.3) hold in the half-space  $x_3 < 0$ .

For transversely isotropic films we conclude from (4.17) that the residual stress in the film satisfies

$$\mathbf{S}_R = S\mathbf{1}, \quad \nabla S = 0, \quad (5.5)$$

where  $S = S_T$  and  $S_L = 0$ . For an isotropic film we have  $S_L = S_T$  and the residual stress vanishes, while here the residual stress is a uniform two-dimensional isotropic tension or compression in the plane of the interface and is assumed to be specified.

The plane-stress condition (4.18)<sub>1</sub> is equivalent to

$$\mathcal{C}_{i3k\alpha}H_{k\alpha} + \mathcal{C}_{i3k3}a_k = 0. \quad (5.6)$$

The strong-ellipticity conditions (2.16) imply that this yields  $\mathbf{a}$  in terms of the interfacial displacement gradient  $\mathbf{H} = \nabla\mathbf{u}$ . To see this we use (2.11) to deduce that

$$\mathcal{C}_{i3k\alpha} = (\lambda + \alpha)m_i\delta_{k\alpha} + \mu_L\delta_{i\alpha}m_k \quad (5.7)$$

and thus reduce (5.6) to

$$(\lambda + \alpha)(\text{div}\mathbf{v})\mathbf{m} + \mu_L\nabla w = -\mu_L\mathbf{a}_{\parallel} - \varphi\mathbf{a}\mathbf{m}, \quad (5.8)$$

where

$$\mathbf{v} = \mathbf{1}\mathbf{u}, \quad w = \mathbf{u} \cdot \mathbf{m} \quad (5.9)$$

are the tangential and normal interface displacements, and where

$$\mathbf{a}_{\parallel} = \mathbf{1}\mathbf{a}, \quad a = \mathbf{a} \cdot \mathbf{m} \quad (5.10)$$

are the corresponding components of the director  $\mathbf{a}$ . These are easily obtained from (5.8) and furnish

$$\mathbf{a} = -\nabla w - (\bar{\lambda} + \bar{\alpha})(\operatorname{div} \mathbf{v})\mathbf{m}, \quad (5.11)$$

where  $\bar{\lambda} = \lambda/\varphi$ ,  $\bar{\alpha} = \alpha/\varphi$ .

**Remark:** The expansion (3.17) gives

$$\hat{\mathbf{u}} = \mathbf{v} - \varsigma \nabla w + [w - \varsigma(\bar{\lambda} + \bar{\alpha})(\operatorname{div} \mathbf{v})]\mathbf{m} + o(\varsigma). \quad (5.12)$$

This agrees with classical Kirchhoff-Love kinematics (Landau & Lifschitz, 1986) apart from the term involving  $\operatorname{div} \mathbf{v}$ , which represents a transverse thickness strain of the film in response to in-plane areal strain. We assume that

$$\lambda + \alpha > 0 \quad (5.13)$$

so that areal dilation or contraction is accompanied by thinning or thickening of the film, respectively. Note that this inequality does not conflict with (2.20).

Regarding (4.18)<sub>2</sub>, we proceed as in the foregoing to obtain

$$\{\mathcal{C}[\mathbf{A} + \mathbf{b} \otimes \mathbf{m}]\}\mathbf{m} = (\lambda + \alpha)(\operatorname{div} \mathbf{a}_{\parallel})\mathbf{m} + \mu_L \nabla a + \mu_L \mathbf{b}_{\parallel} + \varphi b \mathbf{m}, \quad (5.14)$$

where  $\mathbf{b}_{\parallel}$  and  $b$  are defined as in (5.10) with  $\mathbf{a}$  replaced by the director  $\mathbf{b}$ . We also require  $\{\mathcal{C}[\mathbf{H} + \mathbf{a} \otimes \mathbf{m}]\}\mathbf{1}$ , where

$$\mathbf{H} + \mathbf{a} \otimes \mathbf{m} = \nabla \mathbf{v} - (\bar{\lambda} + \bar{\alpha})(\operatorname{div} \mathbf{v})\mathbf{m} \otimes \mathbf{m} + \mathbf{m} \otimes \nabla w - \nabla w \otimes \mathbf{m}. \quad (5.15)$$

The minor symmetry of  $\mathcal{C}$  yields

$$\{\mathcal{C}[\mathbf{H} + \mathbf{a} \otimes \mathbf{m}]\}\mathbf{1} = [\mathcal{C}_{i\gamma\alpha\beta}v_{\alpha,\beta} - (\bar{\lambda} + \bar{\alpha})(\operatorname{div}\mathbf{v})\mathcal{C}_{i\gamma 33}]\mathbf{e}_i \otimes \mathbf{e}_\gamma, \quad (5.16)$$

where

$$\mathcal{C}_{i\gamma 33} = (\lambda + \alpha)\delta_{i\gamma}, \quad \mathcal{C}_{i\gamma\alpha\beta} = \lambda\delta_{i\gamma}\delta_{\alpha\beta} + \mu_T(\delta_{i\alpha}\delta_{\gamma\beta} + \delta_{i\beta}\delta_{\gamma\alpha}). \quad (5.17)$$

Accordingly, with some effort we derive

$$\operatorname{div}(\{\mathcal{C}[\mathbf{H} + \mathbf{a} \otimes \mathbf{m}]\}\mathbf{1}) = \varphi[\bar{\lambda} + \bar{\mu}_T - (\bar{\lambda} + \bar{\alpha})^2]\nabla(\operatorname{div}\mathbf{v}) + \mu_T\Delta\mathbf{v}, \quad (5.18)$$

where  $\Delta$  is now the two-dimensional Laplacian and  $\bar{\mu}_T = \mu_T/\varphi$ . Further, (3.6) and (5.5) combine to give  $\mathbf{HS}_R\mathbf{1} = S\nabla\mathbf{u}$  and

$$\operatorname{div}(\mathbf{HS}_R\mathbf{1}) = S(\Delta\mathbf{v} + \Delta w\mathbf{m}). \quad (5.19)$$

These results are substituted into (4.18)<sub>2</sub>, which is then solved to obtain

$$\mu_L\mathbf{b}_\parallel = (\mu_T + S)\Delta\mathbf{v} + \varphi[\bar{\lambda} + \bar{\mu}_T - (\bar{\lambda} + \bar{\alpha})(\bar{\lambda} + \bar{\alpha} - \bar{\mu}_L)]\nabla(\operatorname{div}\mathbf{v}) - \rho\mathbf{v}_{tt}, \quad (5.20)$$

where  $\bar{\mu}_L = \mu_L/\varphi$ , and

$$\varphi b = (\lambda + \alpha + S)\Delta w - \rho w_{tt}. \quad (5.21)$$

The strong-ellipticity conditions (2.16) then furnish  $\mathbf{b}$  uniquely.

It remains to substitute (5.11), (5.20) and (5.21) into (4.19) to derive the final equations of motion for the interface displacement field. We accomplish this in several steps. First, we use  $\mathbf{A} = \nabla\mathbf{a}$  and (5.11) to write

$$\mathbf{A} = -\nabla(\nabla w) + \mathbf{m} \otimes \nabla a \quad (5.22)$$

and

$$\mathcal{C}[\mathbf{A} + \mathbf{b} \otimes \mathbf{m}] = -\mathcal{C}[\nabla(\nabla w)] + \mathcal{C}[\mathbf{m} \otimes \nabla a] + \mathcal{C}[\mathbf{b} \otimes \mathbf{m}], \quad (5.23)$$

where

$$\mathbf{C}[\nabla(\nabla w)] = \mathcal{C}_{ij\alpha\beta} w_{,\alpha\beta} \mathbf{e}_i \otimes \mathbf{e}_j, \quad \mathbf{C}[\mathbf{m} \otimes \nabla a] = \mathcal{C}_{ij3\beta} a_{,\beta} \mathbf{e}_i \otimes \mathbf{e}_j, \quad \mathbf{C}[\mathbf{b} \otimes \mathbf{m}] = \mathcal{C}_{ijk3} b_k \mathbf{e}_i \otimes \mathbf{e}_j. \quad (5.24)$$

Thus,

$$\operatorname{div}(\{\mathbf{C}[\mathbf{A} + \mathbf{b} \otimes \mathbf{m}]\}\mathbf{1}) = (-\mathcal{C}_{i\gamma\alpha\beta} w_{,\alpha\beta\gamma} + \mathcal{C}_{i\gamma3\beta} a_{,\beta\gamma} + \mathcal{C}_{i\gamma k3} b_{k,\gamma}) \mathbf{e}_i. \quad (5.25)$$

We use (5.17)<sub>1,2</sub> and

$$\mathcal{C}_{i\gamma3\beta} = \mu_L m_i \delta_{\gamma\beta} \quad (5.26)$$

to reduce this to

$$\operatorname{div}(\{\mathbf{C}[\mathbf{A} + \mathbf{b} \otimes \mathbf{m}]\}\mathbf{1}) = -(\lambda + 2\mu_T) \nabla(\Delta w) + (\lambda + \alpha) \nabla b + \mu_L (\Delta a + \operatorname{div} \mathbf{b}_{\parallel}) \mathbf{m}. \quad (5.27)$$

Further, a calculation like that leading to (5.19) results in

$$\operatorname{div}(\mathbf{A} \mathbf{S}_R \mathbf{1}) = S[(\Delta a) \mathbf{m} - \nabla(\Delta w)]. \quad (5.28)$$

Substitution of (5.18), (5.19), (5.27) and (5.28) into (4.19) and use of  $\operatorname{div}(\Delta \mathbf{v}) = \operatorname{div}[\nabla(\operatorname{div} \mathbf{v})] = \Delta(\operatorname{div} \mathbf{v})$  leads to the final equations of motion for the interface displacement, namely

$$\begin{aligned} & \frac{1}{2} h^2 [(\bar{\lambda} + \bar{\alpha})(\lambda + \alpha + S) - (\lambda + 2\mu_T + S)] \nabla(\Delta w) \\ & + h \{ (S + \mu_T) \Delta \mathbf{v} + \varphi [\bar{\lambda} + \bar{\mu}_T - (\bar{\lambda} + \bar{\alpha})^2] \nabla(\operatorname{div} \mathbf{v}) \} - \mathbf{1}(\boldsymbol{\sigma} \mathbf{m}) \\ & = \frac{1}{2} h^2 \rho (\bar{\lambda} + \bar{\alpha} - 1) \nabla w_{tt} + h \rho \mathbf{v}_{tt} \end{aligned} \quad (5.29)$$

and

$$\begin{aligned} & \frac{1}{2} h^2 [S + \mu_T - (\bar{\lambda} + \bar{\alpha})(\lambda + \alpha + S)] \Delta(\operatorname{div} \mathbf{v}) + h S \Delta w \\ & - \mathbf{m} \cdot \boldsymbol{\sigma} \mathbf{m} = \frac{1}{2} h^2 \rho (1 - \bar{\lambda} - \bar{\alpha}) (\operatorname{div} \mathbf{v}_{tt}) + h \rho w_{tt}. \end{aligned} \quad (5.30)$$

For the case of vanishing residual stress and with the  $O(h^2)$  terms neglected, these equations are consistent with equation (4.6) of Fu (2007), which is valid for materials having general symmetry. Further papers relevant to this subject are those by Cai & Fu (2000) and Every & Shuvalov (2002).

## 6 Surface waves

We are concerned with the acoustic interaction of the film and substrate. Accordingly, we study harmonic surface waves whose amplitudes decay with depth in the substrate. We seek solutions to the associated dispersion relations. Love waves are treated first, followed by consideration of Rayleigh waves. The special forms of the residual stress and material symmetry considered lead to uncoupling of the two kinds of waves. In each case the substrate is taken to occupy the region  $x_3 < 0$  and the waves propagate in the  $x_1$  direction.

### 6.1 Love waves

Love waves have the form

$$w_i = \delta_{i2}F(x_1, x_3, t), \quad F(x_1, x_3, t) = A \exp(\eta k x_3) \exp[ik(x_1 - ct)], \quad (6.1)$$

where  $\eta$  and  $k$  are positive constants,  $c$  is the wavespeed and  $A$  is a constant. The induced deformation of the film/substrate interface is

$$w = 0, \quad v_\alpha = \delta_{\alpha 2}F(\xi_1, 0, t), \quad (6.2)$$

where we have used  $\xi_\alpha = x_\alpha$ . From (2.6) and (6.1) the interaction term is calculated as

$$\boldsymbol{\sigma} \mathbf{m} = \mu_s \eta k F(\xi_1, 0, t) \mathbf{e}_2. \quad (6.3)$$

In the substrate, equation (5.3) is satisfied provided that (Achenbach, 1973; Murdoch, 1976)

$$\eta = \sqrt{1 - s^2}, \quad s \equiv c/c_s < 1, \quad (6.4)$$

where

$$c_s = \sqrt{\mu_s/\rho_s} \quad (6.5)$$

is the transverse wavespeed in the substrate.

According to (6.2)<sub>2</sub>, we have  $\operatorname{div} \mathbf{v} = 0$  identically at the interface. With (6.2), we find that (5.30) is trivially satisfied and that (5.29) reduces to

$$h(S + \mu_T)\Delta \mathbf{v} - \boldsymbol{\sigma} \mathbf{m} = h\rho \mathbf{v}_{tt}, \quad (6.6)$$

while all of the order  $h^2$  terms are zero. This is satisfied if

$$\eta = hk \left( rs^2 - \frac{S + \mu_T}{\mu_s} \right) \quad (6.7)$$

modulo terms of order  $h^3$ , with

$$1 > s > \sqrt{\frac{|S + \mu_T|}{r\mu_s}}, \quad (6.8)$$

where we have set  $\rho/\rho_s = r$ . Eliminating  $\eta$  between (6.4) and (6.7) yields the dispersion relation

$$\sqrt{1 - s^2} = \epsilon \left( rs^2 - \frac{S + \mu_T}{\mu_s} \right), \quad (6.9)$$

where  $\epsilon = hk$ . For the case of zero pre-stress this agrees with the result obtained by Murdoch (1976), who assumed the interface to be an elastic boundary of vanishing thickness. Accordingly he did not use the length scale  $h$ , but correlation with his result is nevertheless achieved by identifying  $h$  with the length scale used in that work. The theory used here to obtain this relation purports to be valid only if  $\epsilon \ll 1$ , so that  $1 - s^2 = O(\epsilon^2)$ . Thus, we assume that  $s^2 = 1 - \epsilon^2 C^2 + o(\epsilon^2)$ , solve for  $C$ , and obtain

$$s \sim 1 - \frac{1}{2}\epsilon^2 \left( r - \frac{S + \mu_T}{\mu_s} \right)^2 \quad (6.10)$$

to leading order.

The exact dispersion relation for an isotropic film of thickness  $h$ , without initial stress, is well known (see Achenbach, 1973, equation (6.51)). For small  $\epsilon$  and  $S = 0$  it agrees precisely with (6.9) at leading order, with deviations from (6.9) occurring at order  $\epsilon^3$ ; this is consistent with the foregoing findings based on the present approximate model, according to which the order  $h^2$  terms vanish identically.

## 6.2 Rayleigh waves

Rayleigh waves have the form

$$w_1 = A_1 \exp(\eta k x_3) \exp[ik(x_1 - ct)], \quad w_2 = 0, \quad w_3 = A_3 \exp(\eta k x_3) \exp[ik(x_1 - ct)], \quad (6.11)$$

where  $k$  is again the (positive) wavenumber and  $c$  the wavespeed,  $A_1$  and  $A_3$  are constants and  $\eta$  is to be determined. Substitution of (6.11) into the equation of motion (5.3) yields a quadratic equation for  $\eta^2$  and a connection between  $A_1$  and  $A_3$ . The relevant (positive) solutions for  $\eta$  are given by

$$\eta = \eta^{(1)} \equiv \sqrt{1 - \zeta}, \quad \eta = \eta^{(2)} \equiv \sqrt{1 - \zeta'}, \quad (6.12)$$

where we have defined

$$\zeta = \rho_s c^2 / \mu_s, \quad \zeta' = \mu_s \zeta / (\lambda_s + 2\mu_s). \quad (6.13)$$

The associated connections

$$i\eta^{(1)} A_3^{(1)} = A_1^{(1)}, \quad iA_3^{(2)} = \eta^{(2)} A_1^{(2)} \quad (6.14)$$

follow, where the superscript labels refer to the two solutions. The resulting general solutions for  $w_1$  and  $w_3$  that have the appropriate decay properties as  $x_3 \rightarrow -\infty$  are

$$w_1 = \{A_1^{(1)} \exp(k\eta^{(1)} x_3) + A_1^{(2)} \exp(k\eta^{(2)} x_3)\} \exp[ik(x_1 - ct)], \quad (6.15)$$

$$w_3 = \{A_3^{(1)} \exp(k\eta^{(1)} x_3) + A_3^{(2)} \exp(k\eta^{(2)} x_3)\} \exp[ik(x_1 - ct)], \quad (6.16)$$

with (6.12)–(6.14).

The interaction terms in (5.4) are then calculated as

$$\mathbf{1}(\boldsymbol{\sigma}\mathbf{m}) = \mu_s k \{[\eta^{(1)} A_1^{(1)} + \eta^{(2)} A_1^{(2)} + i(A_3^{(1)} + A_3^{(2)})]\} \exp[ik(x_1 - ct)] \mathbf{e}_1, \quad (6.17)$$

$$\mathbf{m} \cdot \boldsymbol{\sigma}\mathbf{m} = k [i\lambda_s (A_1^{(1)} + A_1^{(2)}) + (\lambda_s + 2\mu_s)(\eta^{(1)} A_3^{(1)} + \eta^{(2)} A_3^{(2)})] \exp[ik(x_1 - ct)], \quad (6.18)$$



and the induced displacements on the interface are

$$v_1 = w_1|_{x_3=0} = (A_1^{(1)} + A_1^{(2)}) \exp[ik(x_1 - ct)], \quad (6.19)$$

$$w = w_3|_{x_3=0} = (A_3^{(1)} + A_3^{(2)}) \exp[ik(x_1 - ct)]. \quad (6.20)$$

Using (6.12)–(6.14) and (6.17)–(6.20) in the interface equations (5.29) and (5.30), we obtain, after some manipulations, the following pair of equations for  $A_3^{(1)}$  and  $A_3^{(2)}$ , correct to order  $\epsilon^2$ :

$$\begin{aligned} & \{-\mu_s(2 - \zeta) + \epsilon\eta^{(1)}(\rho c^2 - B) + \frac{1}{2}\epsilon^2[\rho c^2(\bar{\lambda} + \bar{\alpha} - 1) - C]\}A_3^{(1)} \\ & + \{-2\mu_s\eta^{(2)} + \epsilon(\rho c^2 - B) + \frac{1}{2}\epsilon^2\eta^{(2)}[\rho c^2(\bar{\lambda} + \bar{\alpha} - 1) - C]\}A_3^{(2)} = 0, \end{aligned} \quad (6.21)$$

$$\begin{aligned} & \{-2\mu_s\eta^{(1)} + \epsilon(\rho c^2 - S) + \frac{1}{2}\epsilon^2\eta^{(1)}[\rho c^2(\bar{\lambda} + \bar{\alpha} - 1) + D]\}A_3^{(1)} \\ & + \{-\mu_s(2 - \zeta) + \epsilon\eta^{(2)}(\rho c^2 - S) + \frac{1}{2}\epsilon^2[\rho c^2(\bar{\lambda} + \bar{\alpha} - 1) + D]\}A_3^{(2)} = 0. \end{aligned} \quad (6.22)$$

To avoid lengthy expressions in (6.21) and (6.22) we have introduced the notations

$$B = S + \mu_T + \varphi[\bar{\lambda} + \bar{\mu}_T - (\bar{\lambda} + \bar{\alpha})^2], \quad (6.23)$$

$$C = (\bar{\lambda} + \bar{\alpha})(\lambda + \alpha + S) - (\lambda + 2\mu_T + S), \quad (6.24)$$

$$D = S + \mu_T - (\bar{\lambda} + \bar{\alpha})(\lambda + \alpha + S). \quad (6.25)$$

The secular equation, i.e. the equation that determines  $c$ , is obtained by setting to zero the determinant of coefficients in equations (6.21) and (6.22). This yields, after some tedious calculations,

$$I + \epsilon J + \epsilon^2 K + o(\epsilon^2) = 0, \quad (6.26)$$

where

$$I = (2 - \zeta)^2 - 4\eta^{(1)}\eta^{(2)}, \quad J = r\zeta[\eta^{(1)}(\zeta - \bar{B}) + \eta^{(2)}(\zeta - \bar{S})] \quad (6.27)$$

and

$$\begin{aligned} K = & r^2(\zeta - \bar{B})(\zeta - \bar{S})[1 + \eta^{(1)}\eta^{(2)}] + r[\zeta(\bar{\lambda} + \bar{\alpha} - 1) - \bar{C}][\eta^{(1)}\eta^{(2)} + \frac{1}{2}(\zeta - 2)] \\ & + r[\zeta(\bar{\lambda} + \bar{\alpha} - 1) + \bar{D}][\eta^{(1)}\eta^{(2)} + \frac{1}{2}(\zeta - 2)], \end{aligned} \quad (6.28)$$

in which

$$r = \rho/\rho_s, \quad \bar{B} = \rho_s B/\rho\mu_s, \quad \bar{C} = \rho_s C/\rho\mu_s, \quad \bar{D} = \rho_s D/\rho\mu_s, \quad \bar{S} = \rho_s S/\rho\mu_s. \quad (6.29)$$

To solve (6.26) we assume that  $\zeta$  admits the asymptotic expansion

$$\zeta = \zeta_0 + \epsilon\zeta_1 + \epsilon^2\zeta_2 + o(\epsilon^2). \quad (6.30)$$

This induces the associated expansions

$$I = I_0 + \epsilon I_1 + \epsilon^2 I_2 + o(\epsilon^2), \quad J = J_0 + \epsilon J_1 + \epsilon^2 J_2 + o(\epsilon^2), \quad K = K_0 + \epsilon K_1 + \epsilon^2 K_2 + o(\epsilon^2), \quad (6.31)$$

which reduce (6.26) to

$$I_0 + \epsilon(I_1 + J_0) + \epsilon^2(I_2 + J_1 + K_0) + o(\epsilon^2) = 0, \quad (6.32)$$

where  $I_0, J_0, K_0$  are given by (6.27) and (6.28) with  $\zeta$  replaced by  $\zeta_0$ ,

$$I_1 = 2(\zeta_0 - 2)\zeta_1 + 2\sqrt{1 - \zeta_0}\sqrt{1 - \zeta_0'} \left( \frac{\zeta_1}{1 - \zeta_0} + \frac{\zeta_1'}{1 - \zeta_0'} \right), \quad (6.33)$$

$$\begin{aligned} J_1 = r\zeta_0 \left\{ \sqrt{1 - \zeta_0} \left( 1 - \frac{1}{2} \frac{\zeta_0 - \bar{B}}{1 - \zeta_0} \right) \zeta_1 + \sqrt{1 - \zeta_0'} \left( \zeta_1 - \frac{1}{2} \frac{\zeta_0 - \bar{S}}{1 - \zeta_0'} \zeta_1' \right) \right\} \\ + r\zeta_1 \left[ \sqrt{1 - \zeta_0}(\zeta_0 - \bar{B}) + \sqrt{1 - \zeta_0'}(\zeta_0 - \bar{S}) \right], \end{aligned} \quad (6.34)$$

and

$$\begin{aligned} I_2 = 2(\zeta_0 - 2)\zeta_2 + 2\sqrt{1 - \zeta_0}\sqrt{1 - \zeta_0'} \left( \frac{\zeta_2}{1 - \zeta_0} + \frac{\zeta_2'}{1 - \zeta_0'} \right) \\ + \zeta_1^2 + \frac{1}{2}\sqrt{1 - \zeta_0}\sqrt{1 - \zeta_0'} \left( \frac{\zeta_1}{1 - \zeta_0} - \frac{\zeta_1'}{1 - \zeta_0'} \right)^2, \end{aligned} \quad (6.35)$$

with

$$\zeta_i' = \mu_s \zeta_i / (\lambda_s + 2\mu_s), \quad i = 0, 1, 2. \quad (6.36)$$

These results follow from the expansions

$$\eta^{(1)} = \sqrt{1 - \zeta_0} \left\{ 1 - \frac{\epsilon}{2} \frac{\zeta_1}{1 - \zeta_0} - \frac{\epsilon^2}{2} \left[ \frac{\zeta_2}{1 - \zeta_0} + \frac{1}{4} \left( \frac{\zeta_1}{1 - \zeta_0} \right)^2 \right] \right\} + o(\epsilon^2) \quad (6.37)$$

and

$$\eta^{(2)} = \sqrt{1 - \zeta'_0} \left\{ 1 - \frac{\epsilon}{2} \frac{\zeta'_1}{1 - \zeta'_0} - \frac{\epsilon^2}{2} \left[ \frac{\zeta'_2}{1 - \zeta'_0} + \frac{1}{4} \left( \frac{\zeta'_1}{1 - \zeta'_0} \right)^2 \right] \right\} + o(\epsilon^2), \quad (6.38)$$

which are derived from (6.12) and (6.30).

For (6.32) to represent an asymptotic expansion of (6.26), it is necessary and sufficient that

$$I_0 = 0, \quad I_1 = -J_0, \quad I_2 = -K_0 - J_1. \quad (6.39)$$

The first of these yields the standard equation for the Rayleigh wave speed (see, for example, Achenbach, 1973), namely

$$(2 - \zeta_0)^2 = 4\sqrt{1 - \zeta_0}\sqrt{1 - \zeta'_0}. \quad (6.40)$$

This implies that  $\zeta_0 \neq 1$  and  $\zeta'_0 \neq 1$ , thus ensuring that equations (6.33)–(6.38) are meaningful. From (6.33) and (6.36) it is evident that the second of equations (6.39) furnishes a linear equation for the first-order correction  $\zeta_1$  in terms of the Rayleigh wave speed. Substituting the result into (6.34) and the third of equations (6.39) and using (6.35), we finally obtain a linear equation for the second-order correction  $\zeta_2$ . The solutions to these equations are entirely straightforward and therefore omitted. The interested reader may compare the results thus obtained with equation (3.20) of Fu (2007) and equation (11) of Shuvalov and Every (2002).

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## REFERENCES

ACHENBACH, J.D. (1973) *Wave Propagation in Elastic Solids*. Amsterdam: North-Holland.

- ACHENBACH, J.D. & KESHAVA, S.P. (1967) Free waves in a plate supported by a semi-infinite continuum. *J. Appl. Mech.*, **34**, 397–404.
- CAI, Z.X. & FU, Y.B. (2000) Exact and asymptotic stability analyses of a coated elastic half space. *Int. J. Solids Structures*, **37**, 3101–3119.
- CIARLET, P.G. (2000) *Mathematical Elasticity, Vol. 3: Theory of Shells*. Amsterdam: North-Holland.
- FU, Y.B. (2007) Linear and nonlinear wave propagation in coated or uncoated elastic half spaces. In: *Waves in Nonlinear Pre-stressed Materials* (CISM Courses and Lecture Notes). (M. Destrade & G. Saccomandi, eds.), in press.
- HILGERS, M.G. (1997) Dynamics of elastic sheets with bending stiffness. *Quart. J. Mech. Appl. Math.*, **50**, 525–543.
- HOGER, A. (1995) Positive definiteness of the elasticity tensor of a residually stressed material. *J. Elasticity*, **36**, 201–226.
- KAPLUNOV, J.D., PICHUGIN, A.V. & ROGERSON, G.A. (2006) On a Lamb-type problem for a bi-axially pre-stressed incompressible elastic plate. *IMA J. Appl. Math.*, **71**, 171–185.
- LANDAU, L.D. & LIFSHITZ, E.M. (1986) *Theory of Elasticity*, 3rd Edn., Vol. 7 of the Course of Theoretical Physics. Oxford: Pergamon.
- LEMBO, M. & PODIO-GUIDUGLI, P. (2007) How to use reactive stresses to improve plate-theory approximations of the stress field in a linearly elastic plate-like body. *Int. J. Solids Structures*, **44**, 1337–1369.
- MAN, C.-S. & CARLSON, D. (1994) On the traction problem of dead loading in linear elasticity with initial stress. *Arch. Ration. Mech. Anal.*, **128**, 223–247.
- MERODIO, J. & OGDEN, R.W. (2003) A note on strong ellipticity for transversely isotropic linearly elastic solids. *Quart. J. Mech. Appl. Math.*, **56**, 589–591.
- MURDOCH, I.A. (1976) The propagation of surface waves in bodies with material boundaries. *J. Mech. Phys. Solids*, **24**, 137–146.
- OGDEN, R.W. & STEIGMANN, D.J. (2002) Plane strain dynamics of elastic solids with intrinsic boundary elasticity, with application to surface wave propagation. *J. Mech. Phys. Solids*, **50**, 1869–1896.

- PARONI, R. (2006) Theory of linearly elastic residually stressed plates. *Math. Mech. Solids*, **11**, 137–159.
- PAYTON, R.G. (1983) *Elastic Wave Propagation in Transversely Isotropic Media*. The Hague: Martinus Nijhoff.
- PICHUGIN, A.V. & ROGERSON, G.A. (2002) An asymptotic membrane-like theory for long-wave motion in a pre-stressed elastic plate. *Proc. R. Soc. Lond. A*, **458**, 1447–1468.
- SHUVALOV, A.L. & EVERY, A.G. (2002) Some properties of surface acoustic waves in anisotropic coated solids, studied by the impedance method. *Wave Motion*, **36**, 257–273.
- SPENCER, A.J.M. (1984) Constitutive theory for strongly anisotropic solids. In: *Continuum Theory of the Mechanics of Fibre-Reinforced Composites* (CISM Courses and Lectures, Vol. 282). (A.J.M. Spencer, ed.). Vienna: Springer, pp. 1–32.
- STEIGMANN, D.J. (2007) Thin-plate theory for large elastic deformations. *Int. J. Non-linear Mech.*, in press.
- TIERSTEN, H.F. (1969) Elastic surface waves guided by thin films. *J. Appl. Phys.*, **40**, 770–789.