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DEGREE AND CLASS OF CAUSTICS BY REFLECTION FOR A GENERIC SOURCE

ALFREDERIC JOSSE AND FRANÇOISE PÈNE

ABSTRACT. Given any irreducible algebraic (mirror) curve $\mathcal{C} \subseteq \mathbb{P}^2 := \mathbb{P}^2(\mathbb{C})$ and any (light position) $S \in \mathbb{P}^2$, the caustic by reflection $\Sigma_S(\mathcal{C})$ of \mathcal{C} from S is the Zariski closure of the envelope of the reflected lines got from the lines coming from S after reflection on \mathcal{C} . In [7, 8], we established formulas for the degree and class (with multiplicity) of $\Sigma_S(\mathcal{C})$ for any \mathcal{C} and any S . In this paper, we prove the birationality of the caustic map for a generic S in \mathbb{P}^2 . Moreover, we give simple formulas for the degree and class (without multiplicity) of $\Sigma_S(\mathcal{C})$ for any \mathcal{C} and for a generic S in \mathbb{P}^2 .

1. INTRODUCTION

We are interested in the study of caustics by reflection in the projective complex plane \mathbb{P}^2 . Given an irreducible algebraic curve $\mathcal{C} = V(F) \subset \mathbb{P}^2$ of degree $d \geq 2$ and given $S = [x_0 : y_0 : z_0] \in \mathbb{P}^2$, the **caustic by reflection** $\Sigma_S(\mathcal{C})$ of \mathcal{C} from S is the Zariski closure of the envelope of the reflected lines on \mathcal{C} of the lines coming from S .

For $m \in \mathcal{C}$, the reflected line $\mathcal{R}_{m,S,\mathcal{C}}$ is defined as the orthogonal symmetric of the (incident) line (mS) with respect to the tangent line to \mathcal{C} at m . In [7, 8], we detail the construction of the reflected lines and we define two rational maps $\rho_{F,S}$ and $\Phi_{F,S}$ from \mathbb{P}^2 into itself satisfying the following property: For a generic m in \mathcal{C} , $\rho_{F,S}(m)$ corresponds to an equation of the reflected line $\mathcal{R}_{m,S,\mathcal{C}}$ and this line is tangent to $\Phi_{F,S}(\mathcal{C})$ at $\Phi_{F,S}(m)$. Hence the caustic $\Sigma_S(\mathcal{C})$ is the Zariski closure of $\Phi_{F,S}(\mathcal{C})$ and $\Phi_{F,S}$ is called the **caustic map** of \mathcal{C} from S . Observe that the Zariski closure of $\rho_{F,S}(\mathcal{C})$ is then the dual curve of the caustic $\Sigma_S(\mathcal{C})$. In [7, 8], we used this approach to establish precise formulas for the degree and class (both with multiplicity) of $\Sigma_S(\mathcal{C})$ for any \mathcal{C} and any S . The degree with multiplicity of $\Sigma_S(\mathcal{C})$ means its degree multiplied by the degree of the rational map $\Phi_{F,S}$ restricted to \mathcal{C} . The class with multiplicity of $\Sigma_S(\mathcal{C})$ means its class multiplied by the degree of the rational map $\rho_{F,S}$ restricted to \mathcal{C} . Our formulas complete the formula obtained by Chasles in [3] for the class of a caustic by reflection (for a generic \mathcal{C} and a generic S). Let us indicate that, in [1], Brocard and Lemoyne gave, without any proof, formulas for the degree and class of caustics by reflection (for a Plücker curve \mathcal{C} and for S not at infinity). It seems that their formulas come from an incorrect composition of formulas

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by Salmon and Cayley [10] for some characteristic invariants of pedal and evolute curves (using the representation of caustics by reflection given by Quetelet and Dandelin). This is discussed in [8]. Let us also mention the work of Catanese and Trifogli on focal loci, which generalize evolutes to higher dimension [11, 2].

The question of the birationality of the rational maps $\rho_{F,S}$ and $\Phi_{F,S}$ on \mathcal{C} is not evident even if S is not at infinity. Indeed, according to results of Quetelet and Dandelin [9, 4], when S is not at infinity, the caustic $\Sigma_S(\mathcal{C})$ is the evolute of the S -centered homothety (with ratio 2) of the pedal of \mathcal{C} from S (i.e. the evolute of the orthotomic of \mathcal{C} with respect to S). But we just know that the evolute map is birational for a generic algebraic curve (see [5] by Fantechi).

In this note, we prove the birationality on \mathcal{C} of the maps $\rho_{F,S}$ and $\Phi_{F,S}$ for any irreducible algebraic curve $\mathcal{C} \subset \mathbb{P}^2$ of degree $d \geq 2$ and for a generic S in \mathbb{P}^2 . This result enables us to establish simple formulas for the degree and class of caustics by reflection valid for any irreducible algebraic curves $\mathcal{C} \subset \mathbb{P}^2$ of degree $d \geq 2$ and for a generic S in \mathbb{P}^2 . In this study, the cyclic points $I = [1 : i : 0]$ and $J = [1 : -i : 0]$ play a particular role. We will also use the canonical projection $\pi : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$.

2. BIRATIONALITY

Theorem 2.1. *Let $\mathcal{C} = V(F) \subset \mathbb{P}^2$ be any irreducible algebraic curve of degree $d \geq 2$. For a generic $S \in \mathbb{P}^2$, the maps $\rho_{F,S}$ and $\Phi_{F,S}$ are birational on \mathcal{C} .*

Before going into the proof of our Theorem, let us introduce some notations and recall some facts (see [7]). For any line $\mathcal{D} = V(ax + by + cz) \in \mathbb{P}^2$ such that $a^2 + b^2 \neq 0$, we define the orthogonal symmetric with respect to \mathcal{D} as the rational map $\sigma_{\mathcal{D}} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ (which is an involution) given by

$$\sigma_{\mathcal{D}}[x : y : z] = \pi((a^2 + b^2) \cdot (x, y, z) + (ax + by + cz) \cdot (a, b, 0)).$$

Let $\mathcal{C} = V(F) \subset \mathbb{P}^2$ be an irreducible algebraic curve of degree $d \geq 2$ and let $S \in \mathbb{P}^2 \setminus \{I, J\}$ be a (light) position. We define $\mathcal{C}_0 := \mathcal{C} \setminus V(F_x^2 + F_y^2)$. Observe that this set corresponds to the complement in \mathcal{C} of the cyclic apparent contour of \mathcal{C} (the cyclic apparent contour of \mathcal{C} from the cyclic points). We recall that the reflected line $\mathcal{R}_{m,S,\mathcal{C}}$ at $m \in \mathcal{C}_0 \setminus \{S\}$ is the line $(m \sigma_{\mathcal{T}_m \mathcal{C}}(S))$, where $\mathcal{T}_m \mathcal{C}$ is the tangent to \mathcal{C} at m . For any $m \in \mathcal{C}_0$, we define the normal line $\mathcal{N}_m \mathcal{C}$ to \mathcal{C} at m as the line containing m and $[F_x(m) : F_y(m) : 0]$.

For any $m \in \mathcal{C}_0$, we consider the set K_m of points $S \in \mathbb{P}^2$ such that there exists $m' \in \mathcal{C}_0 \setminus \{m\}$ satisfying $\rho_{F,S}(m') = \rho_{F,S}(m) \neq 0$. Observe that the set \mathcal{A} of $S \in \mathbb{P}^2$ such that $\rho_{F,S}$ is not birational can be written $\mathcal{A} = \bigcup_{E \subset \mathcal{C}_0 : \#E < \infty} \bigcap_{m \in \mathcal{C}_0 \setminus E} K_m$. To prove that $\rho_{F,S}$ is birational for a generic S in \mathbb{P}^2 , we prove that \mathcal{A} is contained in a subvariety of codimension at least 1 in \mathbb{P}^2 . Our proof is based on the following lemma.

Lemma 2.2. *For any $m \in \mathcal{C}_0$, the set K_m is contained in a (possibly non irreducible) algebraic curve \bar{K}_m of degree at most $2d^2 + 2$.*

Proof. Let us consider any $m \in \mathcal{C}_0$. Let $S \in K_m$ and $m' \in \mathcal{C}_0 \setminus \{m\}$ satisfying $\rho_{F,S}(m') = \rho_{F,S}(m) \neq 0$. Then $(mm') = \mathcal{R}_{m,S,\mathcal{C}} = \mathcal{R}_{m',S,\mathcal{C}}$ and so S is in

$\mathcal{A}_{m,m'} := \sigma_{\mathcal{T}_m\mathcal{C}}((m m')) \cap \sigma_{\mathcal{T}_{m'}\mathcal{C}}((m m'))$. Observe that, if $\sigma_{\mathcal{T}_m\mathcal{C}}((m m')) = \sigma_{\mathcal{T}_{m'}\mathcal{C}}((m m'))$, then these lines are $(m m')$ and so $(m m')$ is stable by $\sigma_{\mathcal{T}_m\mathcal{C}}$ and by $\sigma_{\mathcal{T}_{m'}\mathcal{C}}$. But $\mathcal{T}_m\mathcal{C}$ and $\mathcal{N}_m\mathcal{C}$ are the only lines containing m which are stable by $\sigma_{\mathcal{T}_m\mathcal{C}}$. Therefore, $\sigma_{\mathcal{T}_m\mathcal{C}}((m m')) = \sigma_{\mathcal{T}_{m'}\mathcal{C}}((m m'))$, implies that $(m m') \in \{\mathcal{T}_m\mathcal{C}, \mathcal{N}_m\mathcal{C}\} \cap \{\mathcal{T}_{m'}\mathcal{C}, \mathcal{N}_{m'}\mathcal{C}\}$. If $\{\mathcal{T}_m\mathcal{C}, \mathcal{N}_m\mathcal{C}\} \cap \{\mathcal{T}_{m'}\mathcal{C}, \mathcal{N}_{m'}\mathcal{C}\} = \emptyset$, then S is the only point of $\mathcal{A}_{m,m'}$, so S is equal to

$$\tau_m(m') := \pi((m \wedge \sigma_{\mathcal{T}_m\mathcal{C}}(m')) \wedge (m' \wedge \sigma_{\mathcal{T}_{m'}\mathcal{C}}(m'))).$$

Notice that τ_m is a rational map with coordinates of degree $2d$. We obtain that S belongs to the Zariski closure of $\tau_m(\mathcal{C})$, which (according to [6, Proposition 4.4]) is contained in an algebraic curve of degree at most $\mathcal{C} \cdot \tau_m^*(H) \leq 2d^2$ (where H is the hyperplane class in \mathbb{P}^2). Otherwise, $S \in \mathcal{A}_{m,m'} = (m m') \in \{\mathcal{T}_m\mathcal{C}, \mathcal{N}_m\mathcal{C}\}$. Finally, we have $K_m \subseteq \bar{K}_m := \tau_m(\mathcal{C}) \cup \mathcal{T}_m\mathcal{C} \cup \mathcal{N}_m\mathcal{C}$ which is an algebraic curve of degree at most $2d^2 + 2$ (use for example the fundamental lemma of [7] applied with τ_m). \square

Proof of Theorem 2.1. Let us prove that $\rho_{F,S}$ is birational on \mathcal{C} for a generic S in \mathbb{P}^2 . The birationality of $\Phi_{F,S}$ will follow. Indeed, for a generic S in \mathbb{P}^2 , the caustic $\Sigma_S(\mathcal{C})$ is a curve (see for example [8]). Therefore, for generic $m, m' \in \mathcal{C}$, $\Phi_{F,S}(m) = \Phi_{F,S}(m')$ implies that $\rho_{F,S}(m) = \rho_{F,S}(m')$. With the notations of Lemma 2.2, we define $\mathcal{A}' := \bigcup_{E \subset \mathcal{C}_0: \#E < \infty} \bigcap_{m \in \mathcal{C}_0 \setminus E} \bar{K}_m$. We prove that the set $\mathcal{F} := \left\{ \bigcap_{m \in \mathcal{C}_0 \setminus E} \bar{K}_m, E \subset \mathcal{C}_0, \#E < \infty \right\}$ is inductive for the inclusion. Let $(\mathcal{F}_j := \bigcap_{m \in \mathcal{C}_0 \setminus E_j} \bar{K}_m)_{j \geq 1}$ be an increasing sequence of sets belonging to \mathcal{F} . Let us show that the union Z of these sets is also in \mathcal{F} . First $Z \subseteq \bigcap_{m \in \mathcal{C}_0 \setminus \bigcup_{i \geq 1} E_i} \bar{K}_m \subseteq \bar{K}_{m_0}$ for some fixed $m_0 \in \mathcal{C}_0 \setminus \bigcup_{i \geq 1} E_i$. Now \bar{K}_{m_0} is the union of a finite number of irreducible algebraic curves C_1, \dots, C_p . Let $i \in \{1, \dots, p\}$ and let d_i be the degree of C_i . If $C_i \subseteq Z$, then there exists $N_i \geq 1$ such that $C_i \subseteq \mathcal{F}_{N_i}$. Assume now that $C_i \not\subseteq Z$. Then $(C_i \cap \mathcal{F}_j)_{j \geq 1}$ is an increasing sequence of finite sets containing at most $d_i \times (2d^2 + 2)$ points. Therefore, there exists $N_i \geq 1$ such that $(C_i \cap Z) \subseteq \mathcal{F}_{N_i}$. We conclude that $Z = \mathcal{F}_{\max(N_1, \dots, N_p)}$ and so Z is in \mathcal{F} . So \mathcal{F} is inductive.

From the Zorn lemma, either \mathcal{F} is empty or it admits a maximal element (for the inclusion). If it is empty, then $\mathcal{A} = \mathcal{A}' = \emptyset$. If it is not empty and if $\mathcal{F}_0 := \bigcap_{m \in \mathcal{C}_0 \setminus E_0} \bar{K}_m$ (with $E_0 \subset \mathcal{C}_0$ and $\#E_0 < \infty$) is a maximal element of \mathcal{F} , then $\mathcal{A}' = \mathcal{F}_0$. Indeed, \mathcal{A}' contains \mathcal{F}_0 by definition of \mathcal{A}' . Conversely, let $S \in \mathcal{A}'$, there exists $E \subset \mathcal{C}_0$ such that $\#E < \infty$ and such that $S \in \bigcap_{m \in \mathcal{C}_0 \setminus E} \bar{K}_m$. Hence $S \in \bigcap_{m \in \mathcal{C}_0 \setminus (E \cup E_0)} \bar{K}_m$. Since we also have $\bigcap_{m \in \mathcal{C}_0 \setminus E_0} \bar{K}_m \subseteq \bigcap_{m \in \mathcal{C}_0 \setminus (E \cup E_0)} \bar{K}_m$, we conclude that $S \in \bigcap_{m \in \mathcal{C}_0 \setminus E_0} \bar{K}_m$. Therefore, in any case, \mathcal{A} is contained in an algebraic curve, this gives the S -genericity of the birationality of $\rho_{F,S}$ and so the statement of Theorem 2.1. \square

3. LIGHT GENERIC FORMULAS FOR THE DEGREE AND THE CLASS OF CAUSTICS

Let $\mathcal{C} = V(F) \subset \mathbb{P}^2$ be any irreducible algebraic curve of degree $d \geq 2$. We call **isotropic tangent** to \mathcal{C} any tangent to \mathcal{C} containing I or J . Before stating our formulas, let us introduce some notations.

For any $P \in \mathbb{P}^2$, we write $\mu_P(\mathcal{C})$ for the multiplicity of \mathcal{C} at P . We recall that $\mu_P(\mathcal{C}) = 1$ means that P is a non singular point of \mathcal{C} . For any $P \in \mathcal{C}$, we write $\text{Branch}_P(\mathcal{C})$ for the set of branches of \mathcal{C} at P . Let us write $\mathcal{E}_{\mathcal{C}}$ for the set of couples (P, \mathcal{B}) with $P \in \mathcal{C}$ and with $\mathcal{B} \in \text{Branch}_P(\mathcal{C})$. For any $(P, \mathcal{B}) \in \mathcal{E}_{\mathcal{C}}$, we write $\mathcal{T}_P\mathcal{B}$ for the tangent line to \mathcal{B} at P and $e_{\mathcal{B}}$ for the multiplicity of \mathcal{B} . We recall that $\sum_{\mathcal{B} \in \text{Branch}_P(\mathcal{C})} e_{\mathcal{B}} = \mu_P(\mathcal{C})$. For any $(P, \mathcal{B}) \in \mathcal{E}_{\mathcal{C}}$ and any algebraic curve \mathcal{C}' , we denote by $i_P(\mathcal{C}, \mathcal{C}')$ (resp. $i_P(\mathcal{B}, \mathcal{C}')$) the intersection number of \mathcal{C} (resp. \mathcal{B}) with \mathcal{C}' at P . We recall that the contact number $\Omega_{m_1}(\mathcal{C}, \mathcal{C}')$ of \mathcal{C} with \mathcal{C}' is given by $\Omega_{m_1}(\mathcal{C}, \mathcal{C}') = i_P(\mathcal{C}, \mathcal{C}') - \mu_P(\mathcal{C})\mu_P(\mathcal{C}')$. The line at infinity of \mathbb{P}^2 is written ℓ_{∞} . Combining Theorem 2.1 with the main results of [7, 8], we obtain:

Proposition 3.1. *Let $\mathcal{C} = V(F) \subseteq \mathbb{P}^2$ be any irreducible algebraic curve of degree $d \geq 2$ and of class d^{\vee} . For a generic $\mathcal{S} \in \mathbb{P}^2$, we have*

$\deg(\Sigma_{\mathcal{S}}(\mathcal{C})) = 3d + f_0 - t_I - t_J$ and $\text{class}(\Sigma_{\mathcal{S}}(\mathcal{C})) = 2d^{\vee} + d - g - \mu_I(\mathcal{C}) - \mu_J(\mathcal{C})$, where g is the contact number of \mathcal{C} with ℓ_{∞} , i.e. $g := \sum_{m_1 \in \mathcal{C} \cap \ell_{\infty}} \Omega_{m_1}(\mathcal{C}, \ell_{\infty})$, where f_0 is the number of “inflectional branches” of \mathcal{C} not tangent to the line at infinity, i.e.

$$f_0 := \sum_{(P, \mathcal{B}) \in \mathcal{E}_{\mathcal{C}} : i_P(\mathcal{B}, \mathcal{T}_P\mathcal{B}) > 2e_{\mathcal{B}}, \mathcal{T}_P\mathcal{B} \neq \ell_{\infty}} (i_P(\mathcal{B}, \mathcal{T}_P\mathcal{B}) - 2e_{\mathcal{B}}).$$

and where t_P is the number of branches of \mathcal{C} tangent at P to the line at infinity: $t_P = \sum_{\mathcal{B} \in \text{Branch}_P(\mathcal{C}) : \mathcal{T}_P\mathcal{B} = \ell_{\infty}} e_{\mathcal{B}}$.

Proof. According to Theorem 2.1, for a generic \mathcal{S} in \mathbb{P}^2 , the degree and class (with multiplicity) of $\Sigma_{\mathcal{S}}(\mathcal{C})$ are equal to its degree and class. For the degree formula, we use Theorem 20 of [7]. For the class formula, we use Theorem 2 of [8]. For a generic $\mathcal{S} \in \mathbb{P}^2$ ($\mathcal{S} \in \mathbb{P}^2 \setminus (\mathcal{C} \cup \ell_{\infty})$) not contained in an isotropic tangent to \mathcal{C} , we have $f = \mu_I(\mathcal{C}) + \mu_J(\mathcal{C})$, $f' = 0$, $g' = 0$ and $q' = 0$ (with the notations of Theorem 2 in [8]). \square

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