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EXTENDING TO THE COMPLEX LINE DULAC'S CORNER MAPS OF NON-DEGENERATE PLANAR SINGULARITIES

LOÏC TEYSSIER

ABSTRACT. We study the complex Dulac map for a holomorphic foliation of the complex plane, near a non-degenerate singularity (both eigenvalues of the linearization are nonzero) with two separatrices. Following the well-known results of Y. IL'YASHENKO we provide a geometric approach allowing to study the whole maximal domain of (geometric) definition of the Dulac map. In particular its topology and the regularity of its boundary are completely described. We also study the order of magnitude of the first non-trivial term of its asymptotic expansion and show how to compute it using path integrals supported in the leaves of the linearized foliation. Explicit bounds on the remainder are given. We perform similarly the study of the Dulac time spent around the singularity. All results are formulated in a unified framework taking no heed to the usual dynamical discrimination (*i.e.* no matter whether the singularity is formally orbitally linearizable or not and regardless of the arithmetic of the eigenvalues ratio).

1. INTRODUCTION

We consider a germ of a holomorphic vector field at the origin of the complex plane $A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y}$ admitting an isolated, non-degenerate singularity at $(0, 0)$. In other words the origin is the only local zero of the vector field, and its linear part $[\nabla A, \nabla B]$ at this point is a 2×2 matrix with two nonzero eigenvalues, of ratio $\lambda \in \mathbb{C}_{\neq 0}$. Up to choose differently the local analytic coordinates (we particularly refer to Lemma 1.5) we may assume without loss of generality that the vector field admits the following expression:

$$(X) \quad X_R = \lambda x \frac{\partial}{\partial x} + (1 + R)y \frac{\partial}{\partial y}, \quad R(0, 0) = 0, \quad R \in x^a \mathbb{C}\{x, y\}$$

for some non-negative integer a . Our study is carried out on a fixed polydisc $\mathcal{U} = \rho\mathbb{D} \times r\mathbb{D}$ small enough for the relation

$$(R) \quad \sup_{\mathcal{U}} |R| < 1$$

to hold. At some point we also use the hypothesis

$$(a) \quad \Re \left(a + \frac{1}{\lambda} \right) \geq 0.$$

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This setting encompasses almost all non-degenerate singularities, including every kind of saddle singularities ($\lambda < 0$).

We write \mathcal{F}_R the (holomorphic, singular) foliation of \mathcal{U} whose leaves are defined by the integral curves of X_R . This foliation admits two special leaves (called separatrices) each of whose adherence corresponds to a branch of $\{xy = 0\}$. We define

$$(1.1) \quad \hat{\mathcal{U}} := \mathcal{U} \setminus \{xy = 0\} .$$

Outside $\{x = 0\}$ the foliation is transverse everywhere to the fibers of the fibration

$$\Pi : (x, y) \mapsto x$$

and if (R) holds the foliation is transverse to that of $(x, y) \mapsto y$ too. Being given $(x_*, y_*) \in \hat{\mathcal{U}}$ it is thus possible (under suitable assumptions that will be detailed later on) to lift in the foliation through Π a path γ linking x to x_* , starting from the point (x, y_*) . The arrival end-point of the lifted path defines uniquely a point $(x_*, y_x) \in \Pi^{-1}(x_*)$. This construction yields a locally analytic map from the transverse disc $\{y = y_*\}$ into the transverse disc $\{x = x_*\}$, which is known as the **Dulac map**

$$\mathcal{D}_R : x \neq 0 \mapsto y_x$$

of X_R associated to (x_*, y_*) . This map is in general multivalued, and its monodromy is generated by the holonomy of \mathcal{F}_R computed on $\{x = x_*\}$ by winding around $\{x = 0\}$.

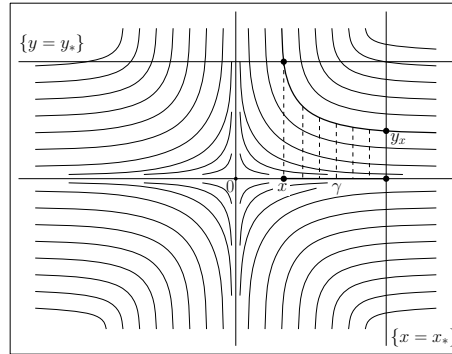


FIGURE 1.1. The Dulac map, pictured in a real slice.

Y. IL'YASHENKO carried out important works [II'84, II'85, II'91] aimed at studying the germ of a subdomain of $\{y = y_*\}$ at (x_*, y_*) on which the germ of a mapping \mathcal{D}_R is holomorphic and has «interesting» asymptotics, in connection to Dulac's conjecture and ultimately Hilbert's 16th problem. We give more contextual details in paragraphs to come, let us just say for now that Y. IL'YASHENKO proved \mathcal{D}_R is defined at least on a *standard quadratic domain* in logarithmic coordinates. By contrast, our concern here is to consider \mathcal{D}_R as a global mapping and to describe in a detailed manner the maximal Riemann surface Ω_R on which \mathcal{D}_R is defined (for which the geometric construction above can be performed¹). Our main result is the following:

¹It may happen that \mathcal{D}_R admits a bigger Riemann surface as a multivalued map, but this case is not dealt with here.

Main Theorem. *Assume conditions (X) and (R) hold.*

- (1) Ω_R is simply connected and embeds into the universal covering of $\{y = y_*, x \neq 0\}$.
- (2) Any component of the boundary $\partial\Omega_R$ is a piecewise-analytic curve.
- (3) If $\Re(\lambda) \geq 0$ then Ω_R is connected.

Remark.

- (1) Notice that the result holds as soon as the quantitative and explicit conditions (X), (R) are fulfilled: the theorem is a semi-local result.
- (2) If the condition (R) is not met then the foliation \mathcal{F}_R is likely to fail being transverse to the fibration Π at some point. The Dulac map becomes multivalued in the presence of such tangency points, meaning that Ω_R can no longer be embedded in the universal cover of the punctured transverse line. The topology of Ω_R can therefore get richer, e.g. when the tangency point corresponds to a finite branch-point: in that case $\text{adh}(\Omega_R)$ contains an orbifold point and Ω_R is not simply connected anymore.
- (3) Conclusions (1) and (3) are still valid when R is merely C^1 as a real function since their proofs actually only use elementary variational and topological arguments based on the sole knowledge of $\sup_{\mathcal{U}} |R|$, as well as the local rectification theorem for vector fields. Conclusion (3) may hold with $\partial\Omega_R$ of the same regularity as R but the local finiteness requirement in «piecewise» might be violated in some instances (see in particular Section 3.4).

1.1. **Context and known facts.** The Dulac map governs part of the dynamics of \mathcal{F}_R and has been submitted to an intense study at least in the two settings we describe now.

1.1.1. *The proof of Dulac's conjecture regarding finiteness of the number of limit cycles for analytic vector fields in the real plane.* Dulac maps are basic ingredients of the cross-section first-return map along a poly-cycle, whose attractive fixed-points correspond to limit cycles. As was noticed by Y. IL'YASHENKO [II'85] the original «proof» of H. DULAC [Dul23] crucially depends on a lemma which turned out to be false. Many powerful, if intricate, tools have been developed in the 1980's decade which finally led to a complete proof of Dulac's conjecture. Two parallel approaches evolved at the time to analyze the asymptotic expansion of the Dulac map: J. ÉCALLE [Éca92] studied it first formally using trans-series then through resurgent summation techniques, while Y. IL'YASHENKO devised an argument based on super-accurate asymptotic series in the book [II'91] dedicated to the proof of Dulac's conjecture. The key point of the argument is that a map with null asymptotic series should vanishes identically. This is not true for every possible domains, which is the reason why Dulac's attempt failed. A central class of domains for which that property does hold are called standard (quadratic) domains.

Although various authors contributed to the tale of Dulac's conjecture, the aim of this article is not to offer a comprehensive list. We make the choice to refer the reader to the textbook [IY08] for more details regarding the context in which the Dulac map intervenes, for this book has a more geometric flavor. We also mention the unpublished material [Lor10] containing many dynamical details in the complex setting.

Regularity Theorem [II'91, II'84]. Assume $\lambda < 0$. Up to reduce the value of ρ and r , the set Ω_R contains a **standard quadratic domain** for $C > 0$ big enough:

$$(1.2) \quad \Omega(C) := \varphi_C(\{z : \Re(z) > 0\})$$

image of the real half-plane $\{\Re(z) > 0\}$ by the conformal mapping

$$\varphi_C : z \mapsto z_* - z - C\sqrt{1+z}.$$

Remark.

- (1) The constant C can be made as small as wished by taking x_* close enough to 0. Viewed in the original x -variable, this domain contains germs of a sector around $\{x = 0\}$ of arbitrary aperture.
- (2) The most recent (and so far shortest) version of the proof given in [IY08, chap. IV] uses the condition (R) (or more precisely the bound $|R| < 1/2$ with $a = 0$). The approach ultimately relies on the fact that the holonomy of \mathcal{F}_R is a parabolic germ, which happens only when λ is rational, and no useful replacement estimate of the behavior of the iterates of the holonomy currently exist when λ is an irrational.
- (3) In [II'91, II'84] a proof of the statement for every $\lambda \in \mathbb{R}_{<0}$ is performed under the (non-restrictive, see Lemma 1.5) assumption that $a > 0$ is big enough.

1.1.2. *The topology of singular germs of a planar holomorphic foliation.* The study of the dynamics of a singular foliation through Seidenberg's reduction² process naturally involves Dulac maps as «corner maps» encoding the transition between different components of the exceptional divisor. They measure how the different components of the projective holonomy pseudo-group mix together. In that context D. MARÍN and J.-F. MATTEI [MM08] proved that under suitable (generic) hypothesis a germ of a singular foliation is locally incompressible: there exists an adapted base of neighborhoods of $(0,0)$ in which the (non-trivial) cycles lying in the leaves of the restricted foliation must wind around the complement of the separatrix locus, in the trail of Milnor's theorem regarding holomorphic fibrations outside the singular fibers. This study is the first step towards a complete analytical and topological classification of (generic) singular germs of a foliation [MM14].

One of the main ingredients of their proof is the control of the «roughness» of the corner maps and elements of the projective holonomy. This roughness can be read in the first two terms of the asymptotic expansion of the Dulac map. In that respect Proposition 1.2, stated further down, allows to get rid of a non-necessary technical hypothesis in Marín-Mattei's theorem, namely discarding «bad³» irrational ratios appearing in Seidenberg's reduction of the singularity. We refer to [Tey15] for a complete dealing with the more general setting, as well as the proof of:

²According to Seidenberg's algorithm (see [Sei68]) any isolated singularity of a germ of a holomorphic foliation \mathcal{F} can be «reduced» through a proper, rational map $\pi : \mathcal{M} \rightarrow (\mathbb{C}^2, 0)$, where \mathcal{M} is a conformal neighborhood of a tree $E := \pi^{-1}(0,0)$ of normally-crossing, conformal divisors $\mathbb{P}_1(\mathbb{C})$. The pulled-back foliation $\pi^*\mathcal{F}$ has only isolated, reduced singularities (located on E): either non-degenerate or of saddle-node type (exactly one nonzero eigenvalue).

³A «bad» irrational number is unusually well approximated by rational numbers (the so-called «small-divisors» problem), characterized by Brjuno's explicit arithmetic condition [Brj71] expressed in terms of the convergents of λ .

Incompressibility Theorem [Tey15]. *Take $\lambda \neq 0$ and assume that conditions (X) and (R) are fulfilled. Let $\mathcal{E} : (z, w) \mapsto (\exp z, \exp w)$ be the universal covering of $\hat{\mathcal{U}}$. Then the foliation $\mathcal{E}^* \mathcal{F}_R$ is regular and each one of its leaves is simply-connected.*

1.2. Discussion and additional results. Let us begin with formulating a few remarks.

- We provide a framework which does not depend on the usual dynamical discrimination (*i.e.* no matter whether the singularity is formally orbitally linearizable or not and regardless of the arithmetic of λ). We only distinguish between **node-like** ($\Re(\lambda) \geq 0$) and **saddle-like singularities** ($\Re(\lambda) < 0$). Notice that if $\lambda \notin \mathbb{R}$ the singularity is hyperbolic, thus orbitally linearizable by Poincaré's theorem: the genuinely difficult cases arise only for real λ (actually $\lambda < 0$), which are the cases usually studied in the literature. Yet our constructions are real-analytic with respect to λ , allowing irregular situations to be studied as limiting cases of *e.g.* hyperbolic situations.
- In the setting of real planar vector fields a **quasi-resonant saddle** ($\lambda \in \mathbb{R}_{<0} \setminus \mathbb{Q}$) is always orbitally conjugate to its linear part by a (sufficiently more regular than) C^∞ change of coordinates [II'85, II'91], which explains why part of the literature (*e.g.* [IY08]) mainly focus on **resonant saddles** ($\lambda \in \mathbb{Q}_{<0}$). This is no longer true in the complex plane in the presence of «bad» irrational ratios [Per92]. Although a quasi-resonant saddle is orbitally conjugate to its linearization in the universal covering of $\hat{\mathcal{U}}$, this conjugacy cannot extend to a C^∞ map along the lift of the separatrices. Therefore it is not sufficient to study linear foliations to encompass complex Dulac maps or their asymptotics in the case of quasi-resonant saddles.
- We express below the Dulac map as an integral (more precisely, as the integral of the differential 1-form $Rd(\log x^{-1/\lambda})$ along paths tangent to \mathcal{F}_R). This integral form, while not being strictly speaking an «integral representation», will prove useful when performing computations. Let us call this integration process the **characteristics** of R along \mathcal{F}_R .

Following the Incompressibility Theorem the natural space to study the Dulac map is the universal covering of $\hat{\mathcal{U}}$ (see (1.1)):

$$\begin{aligned} \mathcal{E} : \tilde{\mathcal{U}} &\longrightarrow \hat{\mathcal{U}} \\ (z, w) &\longmapsto (\exp z, \exp w) . \end{aligned}$$

The Main Theorem (1) asserts that \mathcal{D}_R can be understood as a local holomorphic function $z \mapsto w_z$, after having fixed once and for all a preimage $(z_*, w_*) \in \mathcal{E}^{-1}(x_*, y_*)$ and set $\mathcal{D}_R(z_*) := w_*$. Because its Riemann surface Ω_R is simply connected the map \mathcal{D}_R actually is holomorphic on an open set of $\{w = w_*\}$, still written Ω_R , corresponding to those z giving birth to a path $\gamma_R(z)$ tangent to $\mathcal{E}^* \mathcal{F}_R$ with starting endpoint (z, w_*) and landing endpoint (z_*, w_z) .

Proximity Theorem. *Assume that conditions (X) and (R) hold.*

(1) *For every $z \in \Omega_R$ we have*

$$\mathcal{D}_R(z) = w_* + \frac{z_* - z}{\lambda} + \frac{1}{\lambda} \int_{\gamma_R(z)} R \circ \mathcal{E} \, dz .$$

(2) If $\Re(\lambda) < 0$ and the condition (a) is satisfied then one has the asymptotic approximation

$$\int_{\gamma_R(z)} R \circ \mathcal{E} dz = \int_{\gamma_0(z)} R \circ \mathcal{E} dz + o(|z \exp^{-z/\lambda}|)$$

when $\Re(z)$ tends to $-\infty$ with a bounded imaginary part.

Although the Dulac map's asymptotic expansion on standard domains can be expressed formally as

$$\mathcal{D}_R(x) \simeq \sum_{n,m} D_{n,m} s_{n,m}(x), \quad D_{n,m} \in \mathbb{C},$$

where

$$s_{n,m}(x) = \begin{cases} \frac{x^{n\lambda+m} - x_*^{n\lambda+m}}{n\lambda+m} & \text{if } n\lambda + m \neq 0 \\ \log \frac{x}{x_*} & \text{otherwise} \end{cases}$$

is the Écalle-Roussarie compensator, the expansion converges if, and only if, the foliation is analytically normalizable⁴, as was proved by A. MOURTADA and R. MOUSSU [MM97, Proposition 1]. We compute the first non-trivial term in the asymptotic expansion for all saddle-like singularities. We also derive explicit bounds on the remainder in semi-infinite «horizontal» bands $\{\Re(z) < \text{cst}, |\Im(z)| < \text{cst}\}$. It is most probably possible to obtain explicit bounds on sub-standard-domains following the estimate produced in [Il'91, Il'84]. I decided not to include this computation for the sake of concision. Also these sharper bounds are not needed to deal with incompressibility of generic degenerate singularities [MM08, MM14, Tey15].

The computation of the characteristics along the model \mathcal{F}_0 in (4) can be carried out explicitly. The exact value of $\int_{\gamma_0(z)} (x^n y^m) \circ \mathcal{E} dz$ does not offer an insightful interest as such (see Section 4). We can nonetheless deduce from it the dominant part of $\int_{\gamma_0(z)} R \circ \mathcal{E} dz$, which splits into two components: the *regular part* $\int_z^{z_*} R(\exp u, 0) du$, which induces a holomorphic function in \mathcal{U} since $R(0, 0) = 0$, and the *resonant part* obtained by selecting in R only well-chosen monomials.

Definition 1.1. The **resonant support** $\text{Res}(a, \lambda)$ associated to (a, λ) is

- the empty set if $\lambda \notin \mathbb{R}_{<0}$,
- otherwise the subset of \mathbb{N}^2 defined by

$$\text{Res}(a, \lambda) := \left\{ (n, m) \in \mathbb{N}^2 : m > 0, n \geq a, |n\lambda + m| < \frac{1}{2n} \right\}.$$

For $G(x, y) = \sum_{n \geq 0, m \geq 0} G_{n,m} x^n y^m \in \mathbb{C}\{x, y\}$ we denote by G_0 the **regular part** of G

$$G_0(x) := G(x, 0)$$

and by G_{Res} its **resonant part**

$$G_{\text{Res}}(x, y) := \sum_{(n,m) \in \text{Res}(a,\lambda)} G_{n,m} x^n y^m.$$

It turns out that this support corresponds indeed to resonant or quasi-resonant monomials, according to the rationality of λ (Lemma 4.8), which carry without surprise the major part of the non-regular characteristics.

⁴Situation which arises not so often in the most interesting (quasi-)resonant cases.

Proposition 1.2. *Assume that $\Re(\lambda) < 0$ and conditions (X), (R) and (a) holds. For any $G \in \mathbb{C}\{x, y\}$ we have*

$$\int_{\gamma_0(z)} G_0 \circ \mathcal{E} dz = G_0(0)(z_* - z) + O(\exp z)$$

and if moreover $G \in x^a \mathbb{C}\{x, y\}$

$$\begin{aligned} \int_{\gamma_0(z)} G_{\text{Res}} \circ \mathcal{E} dz &= O(|z \exp^{-z/\lambda}|) \\ \int_{\gamma_0(z)} (G - G_{\text{Res}} - G_0) \circ \mathcal{E} dz &= O(|\exp^{-z/\lambda}|) \end{aligned}$$

(here again $O(\bullet)$ regards the situation when $\Re(z)$ tends to $-\infty$ while z has bounded imaginary part).

As far as I know the above result is new when $\lambda < 0$ is irrational. Notice that in this case and if G_{Res} is finitely supported then

$$\int_{\gamma_0(z)} G_{\text{Res}} \circ \mathcal{E} dz = O(|\exp^{-z/\lambda}|).$$

Remark 1.3. If the condition (a) is not fulfilled then terms of order of magnitude $\exp az$ will appear instead of $\exp^{-z/\lambda}$. This noise can be avoided by normalizing X_R further (Lemma 1.5).

1.3. Time spent near the singularity. In applications (for instance in the study of real analytic planar vector fields) it is sometimes important to estimate the Dulac time, that is the time it takes to drift from (z, w_*) to $(z_*, \mathcal{D}_R(z))$ in the flow of the vector field. In the case of X_R this time is obviously

$$\mathcal{T}_{R,1}(z, z_*) := \frac{z - z_*}{\lambda} = \int_{\gamma_R(z)} \frac{dz}{\lambda}.$$

Multiplying X_R by a holomorphic unit U does not change the underlying foliation (i.e. the Dulac map), although it does the Dulac time $\mathcal{T}_{R,U}$. The later is obtained by integrating a time-form⁵. P. MARDEŠIĆ and M. SAAVEDRA performed the complex analytic continuation of the Dulac time near resonant saddle singularities in [MS07]. Their approach follows the techniques of [IY08] while providing slightly different paths of integration $\gamma_R(z)$. This work was followed by a paper [MMV08] of P. MARDEŠIĆ, D. MARÍN and J. VILLADELPRAT, studying the asymptotic expansion of the Dulac time with a controlled bound on the remainder in families of orbitally linearizable real planar vector fields. The results presented here complete and sharpen these studies in the more general setting described in the Proximity Theorem and Proposition 1.2.

One can choose the time-form as

$$\tau(x, y) := \frac{dx}{\lambda x U(x, y)},$$

so that the next result holds.

⁵A meromorphic 1-form τ is a time-form for a vector field X when $\tau(X) = 1$.

Theorem 1.4. *Take $U \in \mathcal{O}(\mathcal{U})$ such that $U(0,0) \neq 0$. Assume that \mathcal{U} is chosen in such a way that, in addition to conditions (X) and (R), the holomorphic function $U|_{\mathcal{U}}$ never vanishes. Then the Dulac time is holomorphic on Ω_R and*

$$\mathcal{T}_{R,U}(z) = \int_{\gamma_R(z)} \frac{dz}{\lambda U \circ \mathcal{E}}.$$

If moreover $\Re(\lambda) < 0$, the condition (a) holds and $U - U_0 \in x^a y \mathbb{C}\{x, y\}$ then, as $\Re(z)$ tends to $-\infty$ with bounded imaginary part,

$$\mathcal{T}_{R,U}(z) - \int_{\gamma_0(z)} \frac{dz}{\lambda U_0 \circ \mathcal{E}} = \int_{\gamma_0(z)} \frac{dz}{\lambda U_{\text{Res}} \circ \mathcal{E}} + o(|z \exp^{-z/\lambda}|).$$

The integral subtracted on the left-hand side boils down to $\frac{z_0 - z}{\lambda U(0,0)}$ if $U - U(0,0)$ belongs to $x^a y \mathbb{C}\{x, y\}$. We mention that this situation can always be enforced by preparing the vector field:

Lemma 1.5. *Let Z be a germ of a holomorphic vector field near an isolated, non-degenerate singularity with ratio of eigenvalues $\lambda \notin \mathbb{R}_{\geq 0}$. Then there exists $a \in \mathbb{N}_{>0}$ satisfying (a) and a choice of local analytical coordinates such that $Z = UX_R$ for two germs of a function satisfying $R \in x^a y \mathbb{C}\{x, y\}$ and $U - U(0,0) \in x^a y \mathbb{C}\{x, y\}$ with $U(0,0) \neq 0$.*

This lemma is plainly trivial when $\lambda \notin \mathbb{R}_{\leq 0}$: in that setting Z is locally analytically conjugate to its linear part, corresponding to $R = 0$ and $U = \text{cst}$. When $\lambda < 0$ is irrational the vector field is formally linearizable and can be put in the sought form for any finite order $a \in \mathbb{N}$, particularly one such that $a + \frac{1}{\lambda} > 0$. When $\lambda = -\frac{p}{q}$ for p and q positive co-prime integers, resonances may appear and Z may not be even formally orbitally linearizable. These resonances correspond to pairs (n, m) of integers belonging to $(q, p)\mathbb{N}$ (those for which $n\lambda + m = 0$) in the Taylor expansion of R and $U - U(0,0)$, and as such cannot appear for an index n lesser than q or for $m = 0$. It is thus possible to cancel out formally the $(q-1)$ -jet with respect to x and the 0-jet with respect to y of the given functions, meaning we can take $a := q$. Then $a + \frac{1}{\lambda} \geq 0$. The fact that this formal transform can always be chosen convergent is well known.

1.4. Extension of the results to other singularities. Condition (X) is satisfied except for some cases when $\{\lambda, 1/\lambda\} \cap \mathbb{N} \neq \emptyset$. The heuristic is that these singularities may not possess sufficiently many separatrices: the resonant node ($\lambda \neq 0$) and the saddle-node ($\lambda = 0$, exactly one nonzero eigenvalue) admit only one in general. The former case is not very interesting since it corresponds to vector fields which can be analytically reduced to polynomial vector fields (Poincaré-Dulac normal forms [Dul09]) for which explicit computations are easily carried out. The geometry of the foliation itself is quite tame and completely understood. Save for some minor and technical complications, the framework we present can be adapted to encompass this case, although the trouble is not worth the induced lack of clarity in the exposition.

The case of the saddle-node is richer. In [Tey15] we prove that the Incompressibility Theorem holds in that case too. When the saddle-node is not divergent (*i.e.* it admits two separatrices) it can be brought in the form (X) and the Dulac map admits an integral representation as in Proximity Theorem (1)

$$\mathcal{D}_R(z) = \mathcal{D}_{\mu x^k}(z) + \int_{\gamma_R(z)} (R - \mu x^k) \circ \mathcal{E} \frac{dz}{\exp(kz)}$$

where $(k, \mu) \in \mathbb{N}_{>0} \times \mathbb{C}$ is the formal invariant of the saddle-node and $\mathcal{D}_{\mu x^k}$ is the Dulac map for the normal form which can be explicitly computed:

$$\mathcal{D}_{\mu x^k}(z) = w_* + \mu(z - z_*) + \frac{\exp(-kz) - \exp(-kz_*)}{k}.$$

When the singularity is a divergent saddle-node it is possible to obtain an integral representation as well as a sectoral asymptotic behavior. We refer also to [Lor10] for more details. Notice again that the results are quantitative and hold whenever condition (R) is satisfied.

After Seidenberg's reduction of its singularity a (germ of a) nilpotent foliation possesses singular points either of non-degenerate or of saddle-node type. As a consequence the work done here and in [Lor10] is somehow sufficient to analyze more general Dulac maps, although the difficulty of the task is huge. Yet there is a special case where it is not necessary to perform the reduction of singularities to be able to carry out some computations, which is in fact the most general formulation of the framework we introduce here, corresponding to vector fields in the form generalizing (X)

$$X_R = X_0 + RY$$

where:

- X_0 and Y are commuting, generically transverse vector fields,
- Y admits a holomorphic first-integral u with connected fibers,
- $R \in u^a \mathbb{C}\{x, y\}$ for some $a > 0$.

Being given both a transverse disc Σ meeting a common separatrix of X_R and Y at some point p_* , and a transverse Σ' corresponding to a trajectory $\{u = u_*\}$ of Y , we can define the Dulac map of X_R joining Σ to Σ' by lifting paths through the fibration $(x, y) \mapsto u(x, y)$. Then, with equality as multivalued maps on $\Sigma \setminus \{p_*\}$, we have the implicit relation

$$H_0 \circ \mathcal{D}_R = H_0 \circ \mathcal{D}_0 \times \exp \int_{\gamma_R} R\tau$$

where τ is some time-form of X_R and H_0 a first-integral⁶ of X_0 . With little additional work the techniques used here can be applied in that context, in particular regarding the shape of the domain of \mathcal{D}_R and, when applicable, its asymptotics.

1.5. Structure of the article. This paper only uses elementary techniques and is consequently self-contained.

- Section 3 is devoted to proving the Main Theorem.
- This paper goes on with Section 4 where the explicit computation of characteristics $\int_{\gamma_0(z)} G \circ \mathcal{E} dz$ are performed for the model \mathcal{F}_0 . Yet the core of the section is the integral formula (1) of the Proximity Theorem (Section 4.1) and the study of the asymptotic deviation between $\int_{\gamma_R} G \circ \mathcal{E} dz$ and $\int_{\gamma_0(z)} G \circ \mathcal{E} dz$. Immediate consequences of this estimation are (2) of the Proximity Theorem and the best part of Theorem 1.4.
- We end this paper with the proof of Proposition 1.2 in Section 4.3.3, completing Theorem 1.4.

⁶This first-integral can be multivalued, as is the case in the main situation studied here where $H_0(x, y) = x^{-1/\lambda}y$.

1.6. Notations and conventions.

- Let $K \subset \mathbb{R}^m$ be a compact set. A mapping $f : K \rightarrow \mathbb{R}^n$ will be said real-analytic if it is the restriction of a real-analytic mapping on an open neighborhood of K .
- All the paths γ we use throughout the paper are, for the sake of simplicity, piecewise real-analytic maps from some compact interval \mathbb{I} into \mathcal{U} . Its starting point (*resp.* ending point) is written γ_\star (*resp.* γ^\star). It will always be possible, though, to perturb γ slightly so that it is real-analytic everywhere when needed.
- Take a foliation \mathcal{F} defined on a domain \mathcal{U} and some subset $A \subset \mathcal{U}$. The **saturation** $\text{Sat}_{\mathcal{F}}(A) \subset \mathcal{U}$ is the union of all the leaves of \mathcal{F} intersecting A .
- The **restriction** of the foliation \mathcal{F} to a sub-domain $\mathcal{V} \subset \mathcal{U}$ is the foliation on \mathcal{V} , written $\mathcal{F} \cap \mathcal{V}$, whose leaves are the connected components of the trace on \mathcal{V} of the leaves of \mathcal{F} .
- For the sake of concision we make the convention that an object \mathcal{X} hatted with a *tilde* stands for its pull-back in logarithmic coordinates $\tilde{\mathcal{X}} := \mathcal{E}^* \mathcal{X}$

$$\begin{aligned} \mathcal{E} : \tilde{\mathcal{U}} &\longrightarrow \hat{\mathcal{U}} \\ (z, w) &\longmapsto (\exp z, \exp w). \end{aligned}$$

- If $G \in \mathcal{O}(\mathcal{U}) \cap x^a \mathbb{C}\{x, y\}$ is bounded we define its norm as

$$\|G\| := \sup_{\mathcal{U}} \left| \frac{G}{x^a} \right|.$$

- We recall that the rectifying theorem for regular points p of a foliation \mathcal{F} asserts the existence of a local analytic chart $\psi : (V, p) \rightarrow \mathbb{C}^2$ such that $\psi_*(\mathcal{F} \cap V)$ is a foliation by lines of constant direction. We call (ψ, V) a **rectifying chart**.

Definition 1.6. Let $\Sigma \subset \mathcal{U} \setminus \{(0, 0)\}$ be a cross-section, transversal everywhere to \mathcal{F}_R (for short, a **transverse** to \mathcal{F}_R).

- (1) We introduce the groupoid $\Gamma_R(\Sigma)$ of equivalence classes of paths γ tangent to \mathcal{F}_R with $\gamma_\star \in \Sigma$, up to tangential homotopy (that is, homotopy within a given leaf of \mathcal{F}_R) with fixed end-points. We call it **the tangential groupoid of \mathcal{F}_R relative to Σ** .
- (2) The tangential groupoid of \mathcal{F}_R relative to Σ is naturally endowed with a structure of a foliated complex surface, which can be understood as the foliated universal covering of the saturation $\text{Sat}_{\mathcal{F}_R}(\Sigma)$ of Σ by the leaves of \mathcal{F}_R , that is the locally biholomorphic, onto mapping

$$\begin{aligned} \sigma : \Gamma_R(\Sigma) &\longrightarrow \text{Sat}_{\mathcal{F}_R}(\Sigma) \\ \gamma &\longmapsto \gamma^\star. \end{aligned}$$

- (3) Fix a tangent path $\eta \in \Gamma_R(\Sigma)$ and define the abstract transversal set

$$\Gamma_\Sigma := \{\gamma \in \Gamma_R(\Sigma) : \Pi \circ \gamma = \Pi \circ \eta\}.$$

We call **holonomy map** of \mathcal{F}_R associated to (η, Σ) the locally biholomorphic map

$$\begin{aligned} \mathfrak{h}_\eta : \Gamma_\Sigma &\longrightarrow \Pi^{-1}(\Pi(\eta^\star)) \\ \gamma &\longmapsto \gamma^\star. \end{aligned}$$

- (4) The **Dulac map** of \mathcal{F}_R associated with (x_*, y_*) is the holomorphic function defined on

$$\Gamma_* := \{ \gamma \in \Gamma_R(\mathcal{U} \cap \{y = y_*, x \neq 0\}) : \Pi(\gamma^*) = x_* \}$$

by

$$\begin{aligned} \mathcal{D}_R : \Gamma_* &\longrightarrow \Pi^{-1}(x_*) \\ \gamma &\longmapsto \gamma^*. \end{aligned}$$

2. INCOMPRESSIBILITY OF THE LEAVES

We recall that \mathcal{U} is some polydisc centered at $(0, 0)$ on which R is bounded and holomorphic. We define $\tilde{\mathcal{U}} := \mathcal{U} \setminus \{xy = 0\}$ and $\hat{\mathcal{U}}$ its universal covering through the exponential map \mathcal{E}

$$\begin{aligned} \mathcal{E} : \tilde{\mathcal{U}} &\longrightarrow \hat{\mathcal{U}} \\ (z, w) &\longmapsto (\exp z, \exp w). \end{aligned}$$

The Incompressibility Theorem asserts the leaves of the foliation $\tilde{\mathcal{F}} := \mathcal{E}^* \mathcal{F}_R$ are simply-connected. We recall briefly the argument of the proof for two reasons: on the one hand because we need some basic estimates borrowed from said argument for the rest of the article, on the other hand because it will make apparent that the proof remains valid when R is merely C^1 as a real function.

Write

$$\mathcal{E}^* X_R = \mathcal{E}^* X_0 + R \circ \mathcal{E} \times \mathcal{E}^* \left(y \frac{\partial}{\partial y} \right)$$

where

$$\begin{aligned} \mathcal{E}^* X_0 &= \lambda \frac{\partial}{\partial z} + \frac{\partial}{\partial w} \\ \mathcal{E}^* \left(y \frac{\partial}{\partial y} \right) &= \frac{\partial}{\partial w}. \end{aligned}$$

The vector field $\mathcal{E}^* X_R$ is holomorphic and regular on the infinite complex rectangle

$$\tilde{\mathcal{U}} := \{ (z, w) \in \mathbb{C}^2 : \Re(z) < \ln \rho, \Re(w) < \ln r \}.$$

It induces a foliation $\tilde{\mathcal{F}}$ transversal to the fibers of

$$\Pi : (z, w) \longmapsto z,$$

hence the leaf $\tilde{\mathcal{L}}_{p_0}$ is everywhere locally the graph $\{w = f(z)\}$ of some unique germ of a holomorphic function defined in a neighborhood $V(p_0)$ of $\Pi(p_0)$. Because of this property the boundary of $\tilde{\mathcal{L}}_{p_0}$ is included in the boundary of the domain of study

$$\partial \tilde{\mathcal{U}} = \{ \Re(z) = \ln \rho \text{ or } \Re(w) = \ln r \}.$$

Incompressibility follows from the existence of (a family of) curves included in $\tilde{\mathcal{L}}_{p_0}$ which project by Π on line segments of constant direction. A cycle γ within $\tilde{\mathcal{L}}_{p_0}$ will therefore be pushed along those curves, as if repelled by the beam of a searchlight, resulting in a homotopy in $\tilde{\mathcal{L}}_{p_0}$ with a path $\hat{\gamma}$ whose projection bounds a region of empty interior. The foliation obtained by restricting $\tilde{\mathcal{F}}$ to the 3-space

$\Pi^{-1}(\hat{\gamma}(\Pi))$ is a 1-dimensional foliation transverse to the fibers of Π . Therefore the leaves are contractible: $\hat{\gamma}$ (and γ) is homotopic in $\tilde{\mathcal{L}}_{p_0}$ to a point.

Definition 2.1. (See Figure 2.1.) For $v \in \{\Re(z) < \ln \rho\}$, $0 < \delta < \pi$ and $\vartheta \in \mathbb{S}^1$ the domain

$$S(v, \vartheta, \delta) := \{z : \Re(z) < \ln \rho, |\arg(z - v) - \arg \vartheta| < \delta\}$$

is called a **(searchlight) beam** of aperture 2δ , direction ϑ and vertex v . If $v = \Pi(p_0)$ we say it is a **stability beam** when the real part of the lift in $\tilde{\mathcal{L}}_{p_0}$, starting from p_0 , of an outgoing ray $t \geq 0 \mapsto v + t\theta$, with $|\arg \theta / \vartheta| < \delta$, is decreasing.

Remark 2.2. This particularly means that the outgoing ray lifts *completely* in $\tilde{\mathcal{L}}_{p_0}$ as long as it does not cross $\{\Re(z) = \ln \rho\}$.

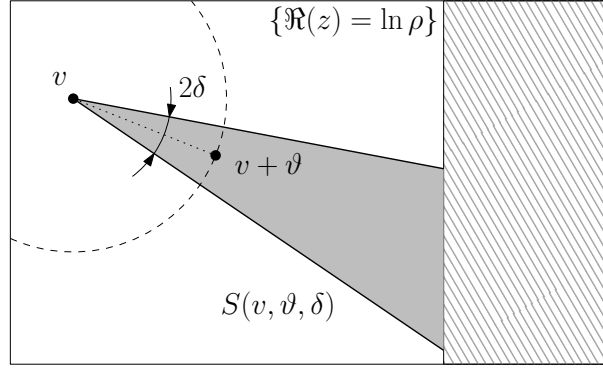


FIGURE 2.1. A searchlight beam.

Lemma 2.3. *There exist $\delta \in]0, \pi]$ and $\vartheta \in \mathbb{S}^1$ such that for all $p_0 = (z_0, w_0) \in \tilde{\mathcal{U}}$ the beam $S(z_0, \vartheta, \delta)$ is a stability beam. We can take*

$$\begin{aligned} \vartheta &:= -\frac{\lambda}{|\lambda|} \\ \delta &\in \left] 0, \arccos\left(\sup_U |R|\right) \right[, \end{aligned}$$

so that one can take δ as close to $\frac{\pi}{2}$ as one wishes by sufficiently diminishing ρ and r . Besides for any integral curve of $\mathcal{E}^* X_R$ of the form $t \mapsto (z_0 + t\theta, w(t))$ with $t \geq 0$, $\theta \in \mathbb{S}^1$ and $w(0) = w_0$ we have the estimate for $\Re(\theta) \neq 0$

$$(2.1) \quad \left| w(t) - w_0 - t \frac{\theta}{\lambda} \right| \leq \frac{\exp(a \Re(z_0))}{|\lambda \Re(\theta)| a} \|R\| |1 - \exp(at \Re(\theta))|$$

and taking the limit $\Re(\theta) \rightarrow 0$

$$(2.2) \quad \left| w(t) - w_0 \pm t \frac{i}{\lambda} \right| \leq \frac{\exp(a \Re(z_0))}{|\lambda|} t \|R\| .$$

Proof. The lift in $\tilde{\mathcal{F}}$ of a germ of a ray $z(t) = z_0 + \theta t$, with $\theta \in \mathbb{S}^1$ and $t \geq 0$, starting from p_0 is obtained as the solution to

$$\frac{\dot{w}}{\dot{z}}(t) = \frac{1 + R \circ \mathcal{E}(z(t), w(t))}{\lambda} , w(0) = w_0 ,$$

that is

$$(2.3) \quad \dot{w}(t) = \frac{\theta}{\lambda} (1 + R \circ \mathcal{E}(z_0 + \theta t, w(t))).$$

The function $t \mapsto \varphi(t) := \Re(w(t))$ is therefore solution to the differential equation

$$(2.4) \quad \dot{\varphi} = \Re\left(\frac{\theta}{\lambda} (1 + R \circ \mathcal{E} \circ (z, w))\right),$$

which particularly means that

$$\left| \dot{\varphi}(t) - \Re\left(\frac{\theta}{\lambda}\right) \right| \leq \frac{\exp(a\Re(z_0)) \|R\| \exp(at\Re(\theta))}{|\lambda|} < \frac{\rho^a}{|\lambda|} \|R\|.$$

Exploiting the cruder estimate by taking $\theta \in \vartheta \exp(i[-\delta, \delta])$ we derive

$$(2.5) \quad \dot{\varphi}(t) \leq -\frac{1}{|\lambda|} (\cos \delta - \rho^a \|R\|) < 0.$$

Since $\varphi(0) < \ln r$ it follows that $\varphi(t) < \ln r$ as long as $\Re(w(t)) < \ln r$, which is our first claim.

Integrating both sides of the estimate yields

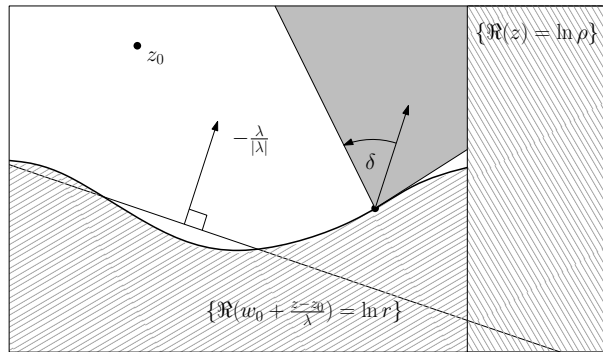
$$\left| \Re\left(w(t) - w_0 - t\frac{\theta}{\lambda}\right) \right| < \frac{\exp(a\Re(z_0)) \|R\| |1 - \exp(at\Re(\theta))|}{|\lambda \Re(\theta)| a}.$$

The study we just performed can be carried out in just the same way for the imaginary part of w , yielding

$$\left| \Im\left(w(t) - w_0 - t\frac{\theta}{\lambda}\right) \right| < \frac{\exp(a\Re(z_0)) \|R\| |1 - \exp(at\Re(\theta))|}{|\lambda \Re(\theta)| a},$$

proving the sought estimate. □

Remark 2.4. We should stress that the «roughness» of $\partial\Omega_{p_0}$ is controlled by the aperture 2δ of the stability beam, which can be taken as close to π as one wishes, and by the direction ϑ (which is that of the model). This is a kind of «conic-convexity» which forbids $\partial\Omega_{p_0}$ to be too wild. In fact the closer $\Re(z)$ is to $-\infty$, the more Ω_{p_0} looks like $\{\Re(w_0 + \frac{z-z_0}{\lambda}) < \ln r, \Re(z) < \ln \rho\}$ near z .



The previous remark can be made more quantitative if we explicitly allow δ to vary. In fact we can sharpen the estimate from the previous lemma by replacing

in (2.5) $\rho^a \|R\|$ with $M(z(t), w(t))$ where

$$\begin{aligned} M(z, w) &:= \exp(a\Re(z))(K_0 \exp \Re(z) + K_1 \exp \Re(w)), \\ K_0 &:= \sup_{\mathcal{U}} \left| \frac{\partial x^{-a} R}{\partial x} \right|, \\ K_1 &:= \sup_{\mathcal{U}} \left| \frac{\partial x^{-a} R}{\partial y} \right|, \end{aligned}$$

for

$$|R \circ \mathcal{E}(z, w)| \leq M(z, w).$$

By reducing slightly the size of \mathcal{U} we can assert that $\sup_{\mathcal{U}} M < 1$. The function M depends implicitly on the parameter $(a, K_0, K_1) \in \mathbb{N} \times \mathbb{R}_{\geq 0}^2$.

Corollary 2.5. *Define the functional space*

$$\mathbb{E} := \left\{ \epsilon \in C^0(\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}) : \sup_{\mathbb{R}_{\geq 0}} \epsilon \leq \frac{1 - M(\ln \rho, \ln r)}{|\lambda|} \text{ and } \lim_{t \rightarrow \infty} \int_0^t \epsilon(s) ds = +\infty \right\}.$$

For $(z, w) \in \tilde{\mathcal{U}}$, $\epsilon \in \mathbb{E}$ and $t \in \mathbb{R}$ set

$$\delta_\epsilon(t, z, w) := \arccos(\epsilon(|t|)|\lambda| + M(z, w))$$

(the bound enforced on ϵ guarantees that $\epsilon(\mathbb{R}_{\geq 0})|\lambda| + M(\tilde{\mathcal{U}})$ is included in $[0, 1]$). Fix $p_0 = (z_0, w_0) \in \tilde{\mathcal{U}}$. As long as the image of the path $z_\epsilon^\pm : t \in \pm\mathbb{R}_{\geq 0} \mapsto z_\epsilon^\pm(t)$, solution of

$$\begin{cases} \frac{dz_\epsilon^\pm}{dt}(t) = \vartheta \exp(\pm i \delta_\epsilon(t, z_\epsilon^\pm(t), w_0 - \int_0^{|t|} \epsilon(s) ds)) \\ z_\epsilon^\pm(0) = z_0 \end{cases},$$

stays in $\{z : \Re(z) < \ln \rho\}$ then it is contained in Ω_{p_0} . Moreover if $\Re(\lambda) > 0$ each z_ϵ^\pm is a solution for all $t \in \pm\mathbb{R}_{\geq 0}$ and

$$\left| \tan \arg \frac{dz_\epsilon^\pm}{dt}(t) \right| \underset{t \rightarrow \pm\infty}{\sim} \frac{\sqrt{1 - (\epsilon(|t|)|\lambda|)^2}}{\epsilon(|t|)|\lambda|}.$$

The estimate on $\tan \arg \frac{dz_\epsilon^\pm}{dt}$ as t goes to $\pm\infty$ controls the asymptotic direction of the image curve. If $\epsilon \rightarrow 0$ then the curve gets arbitrarily tangent to the imaginary axis.

Proof. Assume $t \geq 0$. By construction, and similarly as in (2.4), the corresponding solution w_ϵ^+ satisfies

$$\begin{aligned} \frac{d\Re(w_\epsilon^+)}{dt}(t) &\leq -\frac{1}{|\lambda|} \left(\cos \delta_\epsilon \left(t, z_\epsilon^+(t), w_0 - \int_0^t \epsilon(s) ds \right) - M \left(z_\epsilon^+(t), w_0 - \int_0^t \epsilon(s) ds \right) \right) \\ &= -\epsilon(t) < 0. \end{aligned}$$

The rest of the proof is clear. \square

Definition 2.6. We call **maximal stability beam** of parameter (a, K_0, K_1) based at p_0 the open set included in Ω_{p_0}

$$\mathcal{S}_{\max}(p_0) := \text{connected component of } p_0 \text{ in } \left(\{ \Re(z) < \ln \rho \} \cap \bigcup_{\epsilon \in \mathbb{E}} z_\epsilon^\pm(\pm\mathbb{R}_{\geq 0}) \right).$$

By taking $\frac{R}{x^\alpha}$ linear we see that a maximal stability beam is optimal for all \mathcal{F}_R with same corresponding parameter (a, K_0, K_1) . Notice that $\partial\mathcal{S}_{\max}(p_0)$ is the limit of curves parameterized by z_ϵ^\pm with $\epsilon \rightarrow 0$ in \mathbb{E} . In particular its slope gets asymptotically tangent to the imaginary axis.

3. GEOMETRY OF Ω_R AND OF ITS BOUNDARY

We continue to write \mathcal{D}_R for the Dulac map of \mathcal{F}_R associated to some couple (x_*, y_*) expressed in the coordinates \mathcal{E} (i.e. understood as a locally holomorphic function of (z, w)). We recall that for any $p_0 \in \tilde{\mathcal{U}}$ the leaf $\tilde{\mathcal{L}}_{p_0}$ of $\tilde{\mathcal{F}}$ passing through p_0 projects on

$$\Omega_{p_0} := \Pi(\tilde{\mathcal{L}}_{p_0}).$$

Proposition 3.1. *We fix a preimage $(z_*, w_*) \in \mathcal{E}^{-1}(x_*, y_*)$.*

- (1) *The Dulac map is holomorphic on the open, simply connected set*

$$\Omega := \{z \in \mathbb{C} : (z, w_*) \in \tilde{\mathcal{U}} \text{ and } z_* \in \Omega_{(z, w_*)}\}.$$

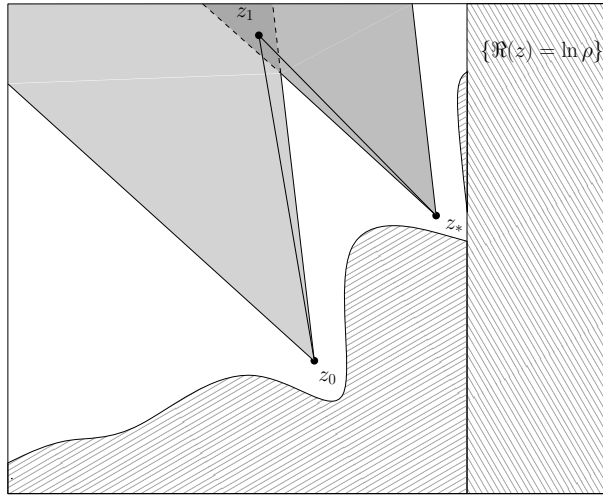
We write Ω_ the connected component containing z_* . As Riemann surfaces Ω and Ω_R are isomorphic.*

- (2) *The boundary $\partial\Omega$ is a locally finite union of piecewise real-analytic curves.*
 (3) *If $\Re(\lambda) \geq 0$ then $\Omega = \Omega_*$ and $\text{adh}(\Omega_*) \cap \{\Re(z) = \ln \rho\}$ is a nonempty line segment. For every $N \in \mathbb{N}_{>0}$ there exists $r \geq r' > 0$ such that this line segment contains at least $\ln \rho + i\text{Im}(z_*) + [-\pi iN, \pi iN]$ whenever $\Re(w_*) < \ln r'$.*
 (4) *If $\Re(\lambda) < 0$ there exists $0 < \rho' \leq \rho$ and $0 < r' \leq r$ depending only on a, λ and $\|R\|$ such that for every $\Re(z_*) < \ln \rho'$, $\Re(w_*) < \ln r'$ and $N \in \mathbb{N}_{>0}$ the domain Ω_* contains some infinite half-band $\{\Re(z) \leq \kappa', |\text{Im}(z - z_*)| \leq \pi N\}$ with $\kappa' \leq \Re(z_*)$ depending only on N, a, λ and $\|R\|$.*

Remark. In (4) one can take $\rho = \rho'$ and $r = r'$ when λ is real. Also when $a > 0$ one can guarantee that $r = r'$.

The rest of the section is devoted to proving the proposition. In doing so we build an explicit tangent path linking (z_0, w_*) to $(z_*, \mathcal{D}_R(z_0))$, see Proposition 3.2 below, which will serve in the next section to establish the asymptotic expansion of the Dulac map through the integral formula of Corollary 4.6. For saddle-like singularities the paths are built in the same fashion as in [II'84, II'91, IY08]. We underline right now the fact that the projection $\tilde{\gamma}$ of that path through Π does not depend on w_* , but only on $z_0, a, \lambda, \rho, \|R\|$ and z_* .

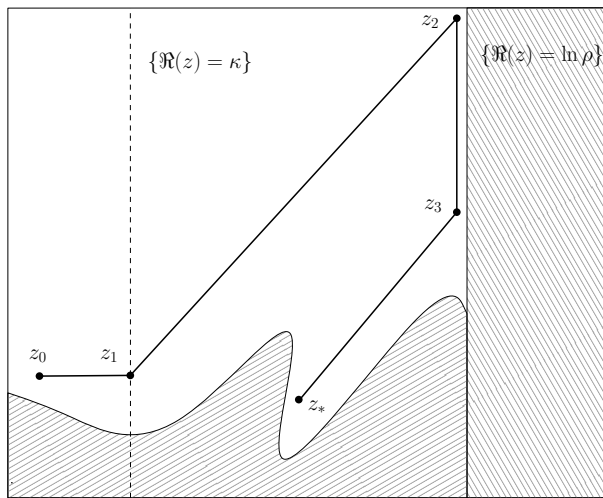
3.1. The integration path. We write $p_0 := (z_0, w_*)$. If $\Re(\lambda) \geq 0$ and $z_0 \in \Omega$ then both stability beams $S(z_0, \vartheta, \delta)$ and $S(z_*, \vartheta, \delta)$ are included in Ω_{p_0} and their intersection W is non-empty. Therefore z_0 can be linked to z_* in Ω_{p_0} by following first a ray segment of $S(z_0, \vartheta, \delta)$ from z_0 to some point z_1 in W , then from this point backwards z_* along a ray segment of $S(z_*, \vartheta, \delta)$, as illustrated in Figure 3.1 below.


 FIGURE 3.1. The path of integration $\tilde{\gamma}$ when $\Re \epsilon(\lambda) \geq 0$.

On the contrary if $\Re \epsilon(\lambda) < 0$ the candidate region W could be beyond $\{\Re(z) = \ln \rho\}$. The construction must therefore be adapted.

Proposition 3.2. *Assume $\Re \epsilon(\lambda) < 0$. There exists $\kappa \in \mathbb{R}$ depending only on $a, \lambda, \|R\|$ and z_* for which the following property holds: for every $z_0 \in \Omega$ one can choose a path $\tilde{\gamma} : z_0 \rightarrow z_*$ with image inside Ω_{ρ_0} in such a way that $\tilde{\gamma}$ is a polygonal line of ordered vertexes $(z_0, z_1, z_2, z_3, z_*)$ with (we refer also to Figure 3.2 below)*

- $z_1 = \max\{\kappa, \Re(z_0)\} + i\Im(z_0)$,
- $\arg(z_2 - z_1) = \arg \vartheta \pm \delta$,
- $\Re(z_2) = \Re(z_3) < \ln \rho$ and $|\Im(z_3 - z_2)| \leq |\Im(z_0 - z_*)| + \tan(|\arg \vartheta| + \delta)(\ln \rho - \kappa)$,
- $\arg(z_* - z_3) = \arg \vartheta \pm \delta$.


 FIGURE 3.2. The path of integration $\tilde{\gamma}$ when $\Re \epsilon(\lambda) < 0$.

Remark 3.3. We could have made a similar construction without the use of κ (i.e. by joining directly z_0 to some z_2), but we need it in order to obtain uniform bounds with respect to $\text{Im}(z_0 - z_*)$ in the next section.

Proof. Write

$$(3.1) \quad \vartheta_{\pm} := \vartheta \exp(\pm i\delta).$$

Either $\Re(\vartheta_+)$ (if $\text{Im}(\lambda) \leq 0$) or $\Re(\vartheta_-)$ (if $\text{Im}(\lambda) \geq 0$) is positive, let us assume for the sake of example that $\text{Im}(\lambda) \leq 0$, the other case being similar in every respect. There exists $\kappa \leq \Re(z_*)$ such that the ray segment $(z_0 + \mathbb{R}_{\geq 0}) \cap \{\Re(z) < \kappa\}$ is included in Ω_{p_0} . Obviously κ depends only on $a, \lambda, \|R\|$ and z_* . We take for $\tilde{\gamma}$ the polygonal line of ordered vertexes $(z_0, z_1, z_2, z_3, z_*)$ built in the following fashion.

- If $z_0 \in \{\Re(z) < \kappa\}$ then the partial ray $(z_0 + \vartheta_0 \mathbb{R}_{\geq 0}) \cap \{\Re(z) < \kappa\}$, included in Ω_{p_0} according to Lemma 2.3, leaves the region at some point z_1 with $\Re(z_1) = \kappa$. Otherwise we set $z_1 := z_0$.
- Both rays $\{z_1, z_*\} + \vartheta_+ \mathbb{R}_{\geq 0}$, included in Ω_{p_0} , intersect the line $\{\Re(z) = \ln \rho - \epsilon\}$ in, respectively, z_2 and z_3 for $\epsilon > 0$ very small.

The line segment $[z_2, z_3]$, and therefore the whole image of $\tilde{\gamma}$, is included in Ω_{p_0} thanks to the next lemma:

Lemma 3.4. *If $\Re(\lambda) \leq 0$ then $\text{adh}(\Omega_{p_0}) \cap \{\Re(z) = \ln \rho\}$ is a nonempty line segment. If $\lambda \notin \mathbb{R}_{\geq 0}$ there exists $r, \rho > 0$ such that the same property holds.*

Proof. In the case $\Re(\lambda) \leq 0$ we have $\max\{\Re(\vartheta \exp(\pm i\delta))\} > 0$; say, for the sake of example, that $\vartheta_+ := \vartheta \exp(i\delta)$ has positive real part. If λ is not a positive number this property can be secured by decreasing ρ, r and taking δ as close to $\frac{\pi}{2}$ as need be. Take a path Γ connecting two points of $\text{adh}(\Omega_{p_0}) \cap \{\Re(z) = \ln \rho\}$ (which is a non-empty set) within Ω_{p_0} . Let I be the line segment of $\{\Re(z) = \ln \rho\}$ joining those points. The ray $p - \vartheta_+ \mathbb{R}_{\geq 0}$ emitted from some $p \in I$ separates $\{\Re(z) \leq \ln \rho\}$ into two connected regions. Since Γ starts from one of them and lands in the other one, the curve must cross the ray at some point $q \in \Omega_{p_0}$. The ray $q + \vartheta_+ \mathbb{R}_{\geq 0}$ is included in Ω_{p_0} since it lies within a stability beam, while it contains p in its adherence. \square

\square

We deduce the following characterization.

Corollary 3.5. *Assume $\Re(\lambda) < 0$. The following propositions are equivalent.*

- (1) $z_0 \in \Omega$,
- (2) $z_* \in \Omega_{p_0}$,
- (3) *there exists $\epsilon > 0$ such that for all $0 < \epsilon \leq \epsilon$ the points z_2 and z_3 built in Proposition 3.2 can be taken with $\Re(z_2) = \Re(z_3) = \ln \rho - \epsilon$, meaning $[z_2, z_3] \subset \Omega_{p_0}$.*

Proof. (1) \Leftrightarrow (2) is the definition of \mathcal{D}_R and Ω while (3) \Rightarrow (2) is clear. The converse follows from the previous proposition and its proof, particularly Lemma 3.4. \square

3.2. The dual searchlight sweep.

Lemma 3.6. *If $\Re(\lambda) \geq 0$ the beam $S(z_0, -\vartheta, \delta)$ is included in Ω for any $z_0 \in \Omega$. If $\Re(\lambda) < 0$ beam $S(z_*, -\vartheta, \delta)$ is included in Ω_* .*

For any $z \in S(z_0, -\vartheta, \delta)$ we can link (z, w_*) to some point (z_0, w) with $\Re(w) \leq \Re(w_*)$ by lifting in $\tilde{\mathcal{F}}$ the line segment $[z, z_0]$. Therefore the lemma is trivial in the case where $\Re(\lambda) < 0$. On the contrary when $\Re(\lambda) \geq 0$ the lemma is a consequence of the next one.

Lemma 3.7. *Assume that $\Re(\lambda) \geq 0$, $z_0 \in \Omega$ and let $\eta := \Re(w_*)$. Then for any other choice of w_* with real part lesser or equal to η we still have $z_0 \in \Omega$ as well.*

Proof. We set up a connectedness argument. Let $B := \{w_* : \Re(w_*) \leq \eta\}$ and $A := \{w_* : w_* \in B \text{ and } z_0 \in \Omega\}$. By assumption A is not empty, and it is open in B for the same reason that Ω is open. More precisely any $w_* \in A$ admits a neighborhood V in B such that the image of $\tilde{\gamma}$ is included in $\Omega_{(z_0, w)}$ for every $w \in V$. Let now a sequence $(w_n)_{n \in \mathbb{N}} \subset A$ converge towards $w_\infty \in B$. If $\Re(\lambda) \geq 0$ then the image of $\tilde{\gamma}$ is included in the union of the two stability beams $S := S(z_0, \vartheta, \delta) \cup S(z_*, \vartheta, \delta)$ which are themselves included in every $\Omega_{(z_0, w_n)}$. Because the real analytic curves defining $\partial\Omega_{(z_0, w)}$ vary continuously when w does we have $S \subset \Omega_{(z_0, w_\infty)}$ also. In particular $z_* \in \Omega_{(z_0, w_\infty)}$ and z_0 belongs to Ω_* for $w_* := w_\infty$. The former property implies in turn that A is a closed subset of B and as such spans the whole region B . \square

Remark 3.8. When $\Re(\lambda) < 0$ the above argument does not work since the image of $\tilde{\gamma}$ must sometimes leave the (adherence of the) union of stability beams emitted by z_1 and z_* (when it visits $[z_2, z_3]$). Nothing guarantees that the limiting domain $\Omega_{(z_0, w_\infty)}$ does not disconnect $[z_2, z_3]$ at some point. Since this lemma will be used to show simple-connectedness of Ω_* in the node-like case, we will need another argument in the saddle-like case.

3.3. Proof of Proposition 3.1 (1). We first mention that Ω is clearly open since if one can link a point (z, w_*) to $(z_*, \mathcal{D}_R(z))$ with a tangent path γ , whose image is included in the open set $\tilde{\mathcal{U}}$, then surely this is again the case for a neighborhood of z (consider a covering of the compact image of γ by rectifying charts). Take now a simple loop $\Gamma_0 \subset \Omega$ bounding a relatively compact, simply-connected domain W_0 and show $W_0 \subset \Omega$ (without loss of generality we may assume that Γ_0 is smooth and real-analytic). If $\Re(\lambda) \geq 0$ Lemma 3.6 proves precisely that fact, since

$$W_0 \subset \bigcup_{z \in \Gamma_0} S(z, -\vartheta, \delta) \subset \Omega.$$

Assume next that $\Re(\lambda) < 0$ and denote by $\tilde{\gamma}_z$ the integration path built in Proposition 3.2 for $z_0 := z$, while $(z)_j$ stand for the corresponding vertex z_j of the polygonal line. We want to show that $\tilde{\gamma}_z$ can be lifted in $\tilde{\mathcal{F}}$ through Π starting from (z, w_*) all the way to $(z_*, \mathcal{D}_R(z))$, when $z \in W_0$. Along the line segments $[z, (z)_1]$ and $[(z)_1, (z)_2]$ the real part of the w -component of the lift is decreasing, therefore $\tilde{\gamma}_z$ can be lifted at least until $(z)_2$ for every $\Re(z) < \ln \rho$.

From now on we work in the 3-dimensional real slice

$$\mathcal{R} := \{(z, w) : \Re(z) = \ln \rho - \epsilon, \Re(w) < \ln r\}$$

for fixed $\epsilon > 0$, which we identify to

$$\mathbb{R} \times \mathbb{C} = \{(t, w) : t = \text{Im}(z), (z, w) \in \mathcal{R}\}.$$

We recall that we write $\tilde{\mathcal{F}} \cap \mathcal{R}$ the restriction of $\tilde{\mathcal{F}}$ to \mathcal{R} , which is a 1-dimensional real-analytic regular foliation everywhere transverse to the fibers of $\Pi|_{\mathcal{R}}$ (the lines $\{t = \text{cst}\}$).

Lemma 3.9. *We refer to Figure 3.3. Take a relatively compact, simply connected domain $W \subset \{\Re(z) < \ln \rho - \epsilon\}$ with smooth real-analytic boundary Γ . Consider the correspondence map*

$$\begin{aligned} \mathfrak{h}_{0 \rightarrow 2} : \text{adh}(W) &\longrightarrow \mathcal{R} \\ z &\longmapsto \mathfrak{h}_{[z, (z)_1, (z)_2]}(z) \end{aligned}$$

where \mathfrak{h}_γ denotes the holonomy of $\tilde{\mathcal{F}}$ associated to $(\gamma, \{w = w_*\})$ as in Definition 1.6.

- (1) $\mathfrak{h}_{0 \rightarrow 2}$ is a real-analytic, open and injective mapping.
- (2) Set

$$\begin{aligned} \Gamma_2 &:= \mathfrak{h}_{0 \rightarrow 2}(\Gamma) \subset \mathcal{R} \\ W_2 &:= \mathfrak{h}_{0 \rightarrow 2}(W) \subset \mathcal{R}. \end{aligned}$$

The compact set $\text{adh}(W_2)$ is a smoothly-embedded real-analytic disc with boundary Γ_2 .

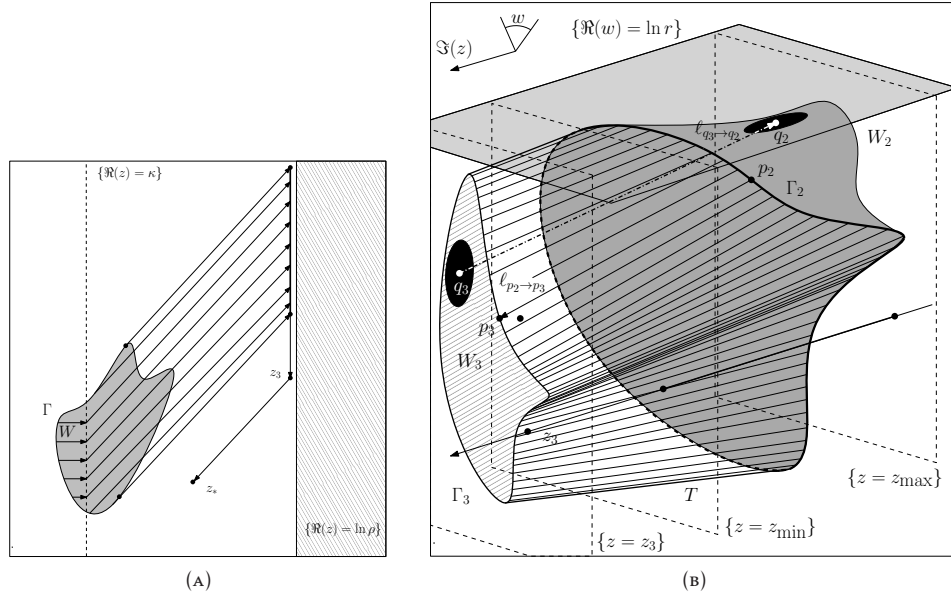


FIGURE 3.3

Proof. Write $\mathfrak{h}_{0 \rightarrow 2}(z) =: (\Im((z)_2), w_2(z))$.

- (1) The mapping $\mathfrak{h}_{0 \rightarrow 2}$ is clearly locally real-analytic and open, because w_2 is holomorphic and non-constant. In particular $\partial W_2 = \Gamma_2$. Show that $\mathfrak{h}_{0 \rightarrow 2}$ is injective. If $\text{adh}(W_2)$ contains a double-point $\mathfrak{h}_{0 \rightarrow 2}(z) = \mathfrak{h}_{0 \rightarrow 2}(z')$ with, say, $\Re(z) \leq \Re(z')$ then z and z' belong to the same polygonal line $[z, (z)_1, (z)_2]$. Since the real part of the w -component of the lift of the polygonal line is strictly decreasing we can only have $z = z'$.

- (2) The parametric tangent space of W_2 at $p = \mathfrak{h}_{0 \rightarrow 2}(z)$ is spanned by $\begin{bmatrix} 0 \\ \frac{\partial w_2}{\partial \Re(z)}(z) \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \frac{\partial w_2}{\partial \Im(z)}(z) \end{bmatrix}$ as $\frac{\partial(z)_2}{\partial \Im(z)} = i$ (we assume here for the sake of simplicity that $\Re(z) < \kappa$ so that $\frac{\partial(z)_2}{\partial \Re(z)} = 0$). From Cauchy-Riemann formula we know that $\frac{\partial w_2}{\partial \Re(z)}(z) = 0$ if, and only if, $w_2'(z) = 0$. This outcome is not possible because w_2 is locally invertible, hence $\text{adh}(W_2)$ is a smooth real-analytic surface. \square

Remark 3.10. This is the only place where we explicitly use the holomorphy of R via Cauchy-Riemann formula, although it is not necessary. In the case where R should only be assumed C^1 as a real mapping, the preceding proof can be adapted because w_2 is a C^2 -diffeomorphism and therefore $\frac{\partial w_2}{\partial \Re(z)}(z)$ cannot vanish.

Although $\tilde{\mathcal{F}} \cap \mathcal{R}$ need not be transverse everywhere to $\text{adh}(W_2)$, the tangency points are nonetheless simple as asserted by the following lemma.

Lemma 3.11. *For every $p_2 \in W_2$ there exist a neighborhood V of p_2 in \mathcal{R} such that any leaf of $\tilde{\mathcal{F}} \cap V$ intersects $\text{adh}(W_2)$ at most in a single point.*

Proof. Let $p_2 = \mathfrak{h}_{0 \rightarrow 2}(z_0) \in \text{adh}(W_2)$ be given and let γ_0 be the curve linking (z_0, w_*) to p_2 along $\tilde{\mathcal{F}}$ above $[z_0, z_2]$. We take a finite covering $C = \bigcup_{i \leq d} C_i$ of γ_0 by rectifying holomorphic charts $(\psi_i : C_i \rightarrow \mathbb{C}^2)_{i \leq d}$ of $\tilde{\mathcal{F}}$. By relabeling the collection if necessary we assume that $(z_0, w_*) \in C_0$. We may choose C_0 small enough so that for any $(z, w_*) \in (\text{adh}(W) \times \{w_*\}) \cap C_0$ the tangent curve linking (z, w_*) to $\mathfrak{h}_{0 \rightarrow 2}(z)$ above $[z, (z)_2]$ is included in C . Since $\tilde{\mathcal{F}}$ is transverse to the fibers of $(z, w) \mapsto w$ if condition (R) holds, $\text{adh}(W) \cap C_0$ is transverse to $\tilde{\mathcal{F}} \cap C_0$: for C_0 small enough the set $\text{adh}(W) \cap C_0$ meets any leaf of $\tilde{\mathcal{F}} \cap C_0$ in at most one point. The conclusion follows as $\mathfrak{h}_{0 \rightarrow 2}$ is injective and maps $\text{adh}(W)$ into $\text{adh}(W_2)$. It suffices to take V such that

$$V \cap \text{adh}(W_2) = \mathfrak{h}_{0 \rightarrow 2}(\{z : (z, w_*) \in C_0\}).$$

\square

We apply Lemma 3.9 to $W := W_0$ (so that $\Gamma = \Gamma_0 \subset \Omega$) in order to build $W_2 \cup \Gamma_2$ through the correspondence map $\mathfrak{h}_{0 \rightarrow 2}$ (we may choose ϵ independent on $z_0 := z \in W_0 \cup \Gamma_0$ in Corollary 3.5 and use this value to define $h_{0 \rightarrow 2}$). The key point to complete the proof of Proposition 3.1 (1) is that $(z)_3 = z_3$ does not actually depend on z once z_* is chosen, therefore the process of lifting $[(z)_2, z_3]$ in $\tilde{\mathcal{F}}$ takes place solely in \mathcal{R} . Because $\Gamma_0 \subset \Omega$ the map

$$\begin{aligned} \mathfrak{h}_{2 \rightarrow 3} : \Gamma_2 &\longrightarrow \{z = z_3\} \cap \mathcal{R} \\ p_2 &\longmapsto \mathfrak{h}_{[\Pi(p_2), z_3]}(p_2) \end{aligned}$$

is well-defined. Set

$$\Gamma_3 := \left\{ \mathfrak{h}_{[\Pi(p), z_3]}(p) : p \in \Gamma_2 \right\} \subset \{z = z_3\} \cap \mathcal{R}$$

(Γ_3 is a smooth, simple real-analytic loop). We need to ensure the existence of $W_3 \subset \{z = z_3\} \cap \mathcal{R}$ built in the same way from W_2 . In fact we prove the stronger statement below:

Lemma 3.12. *The map $\mathfrak{h}_{2 \rightarrow 3}$ extends to a bijective real-analytic map from $\text{adh}(W_2)$ onto the compact connected component $\text{adh}(W_3) \subset \{z = z_3\} \cap \mathcal{R}$ enclosed by Γ_3 .*

Proof. We call $\ell_{p_2 \rightarrow p_3}$ the piece of the leaf of $\tilde{\mathcal{F}} \cap \mathcal{R}$ linking $p_2 \in \Gamma_2$ to $p_3 \in \Gamma_3$ above $[\Pi(p_2), z_3]$ (for the sake of clarity the leaves of $\tilde{\mathcal{F}} \cap \mathcal{R}$ are shown as straight lines in Figure 3.3 (B)). Let T be the smooth cylinder obtained as

$$T := \bigcup_{p_2 \in \Gamma_2} \ell_{p_2 \rightarrow p_3}$$

and \hat{T} be the capped cylinder $\hat{T} := T \cup \text{adh}(W_2) \cup \text{adh}(W_3)$ which is an immersed piecewise-real-analytic sphere. The leaf ℓ_{q_3} of $\tilde{\mathcal{F}} \cap \mathcal{R}$ issued from $q_3 \in W_3$ enters into the space bounded by \hat{T} and, because it is transverse to the fibers of $\Pi|_{\mathcal{R}}$, must meet the compact \hat{T} at some point. Because $\tilde{\mathcal{F}} \cap \mathcal{R}$ is regular ℓ_{q_3} cannot intersect T (invariant by $\tilde{\mathcal{F}} \cap \mathcal{R}$) and therefore intersects both W_2 and W_3 . We therefore obtain a (connected) sub-leaf $\ell_{q_3 \rightarrow q_2} \subset \ell_{q_3}$ containing both q_3 and some $q_2 \in \ell_{q_3} \cap \text{adh}(W_2)$. If $q_3 = p_3 \in \Gamma_3$ we let $\ell_{q_3 \rightarrow q_2} := \ell_{p_2 \rightarrow p_3}$ and $q_2 := p_2$. Define the map

$$\begin{aligned} \mathfrak{h}_{3 \rightarrow 2} : \text{adh}(W_3) &\longrightarrow \text{adh}(W_2) \\ q_3 &\longmapsto q_2 \end{aligned}$$

whose restriction to Γ_3 is the inverse of $\mathfrak{h}_{2 \rightarrow 3}$. Clearly $\mathfrak{h}_{3 \rightarrow 2}$ is injective. We need to show that $\mathfrak{h}_{3 \rightarrow 2}$ is continuous, so that it will be an homomorphism onto its image, which can then only span the whole $\text{adh}(W_2)$ for $\mathfrak{h}_{3 \rightarrow 2}(\Gamma_3) = \Gamma_2$.

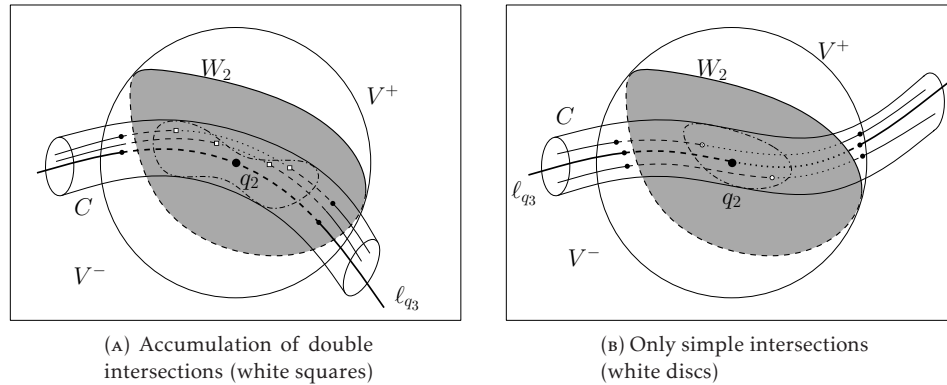


FIGURE 3.4

First notice that if q_2 is not a tangency point between $\tilde{\mathcal{F}} \cap \mathcal{R}$ and $\text{adh}(W_2)$ then $\mathfrak{h}_{3 \rightarrow 2}$ is continuous at q_3 (proceed with the argument in a finite covering of $\ell_{q_3 \rightarrow q_2}$ by rectifying charts). Now if ℓ_{q_3} is tangent to $\text{adh}(W_2)$ at q_2 then one among two mutually exclusive situations happens. In the following V is any sufficiently small neighborhood of q_2 which is separated by $\text{adh}(W_2)$ into two domains V^+ and V^- with, say, V^+ outside \hat{T} .

- (1) The leaf ℓ_{q_3} does not exit \hat{T} near q_2 , that is $\ell_{q_3} \cap V^+ = \emptyset$, as in Figure 3.4 (A).
- (2) The leaf pierces \hat{T} at q_2 , i.e. $\ell_{q_3} \cap V^\pm \neq \emptyset$, as in Figure 3.4 (B).

Let C be a small tubular neighborhood of $\ell_{q_3} \cap V$ invariant by $\tilde{\mathcal{F}} \cap \mathcal{R}$. Lemma 3.11 rules (1) out because in that case there must exist leaves visiting V^+ after leaving V^- and before entering again in V^- . Hence case (2) is the only possible event and $\mathfrak{h}_{3 \rightarrow 2}$ is continuous as any leaf must exit by V^+ after arriving through V^- . \square

Corollary 3.13. *For any $z \in W_0$ the path $\tilde{\gamma}_z$ can be lifted in $\tilde{\mathcal{F}}$ through Π starting from (z, w_*) all the way to $(z_*, \mathcal{D}_R(z))$. In other words $W_0 \subset \Omega$ and Ω is simply connected.*

Proof. Lemma 3.12 asserts that $\tilde{\gamma}_z$ can be lifted at least until z_3 . In the 3-dimensional real slice $\mathcal{R}' := [z_3, z_*] \times \mathbb{C}$ the saturation of Γ_3 by the leaves of the regular foliation $\tilde{\mathcal{F}} \cap \mathcal{R}'$ (transverse to the fibers of $\Pi|_{\mathcal{R}'}$) is a smooth cylinder T' (similar to T in the proof of the previous lemma). Therefore any leaf of $\tilde{\mathcal{F}} \cap \mathcal{R}'$ starting from W_3 cannot escape from T' and must reach the transverse line $\{z = z_3\}$. \square

3.4. Proof of Proposition 3.1 (2). A point $z \in \{\Re(z) < \ln \rho\}$ belongs to Ω if, and only if, the lift of the polygonal line $\tilde{\gamma}_z$ in $\tilde{\mathcal{F}}$ starting from (z_0, w_*) does not meet the real 3-space $\{\Re(w) = \ln r\} \subset \mathbb{C}^2$. For $z_0 \in \partial\Omega \setminus \{\Re(z) = \ln \rho\}$ we can find a compact neighborhood $W \ni z_0$ such that if r were slightly bigger we would have $W \subset \Omega$. In particular we can assume that the Dulac map is the restriction to Ω of an analytic map, written \mathcal{D} , on a neighborhood of W . For $z \in W$ we call ℓ_z the real-analytic curve obtained as the image of the lift of the path $\tilde{\gamma}_z$ in $\tilde{\mathcal{F}}$ starting from (z, w_*) and ending at $(z_*, \mathcal{D}(z))$. Then

$$A_W := \bigcup_{z \in W} \ell_z$$

is a compact real-analytic 3-manifold which intersects $\{\Re(w) = \ln r\}$ along a compact real surface \mathcal{S} (with boundary). If $z \in W \cap \partial\Omega$ the curve ℓ_z cannot meet $\{\Re(w) > \ln r\}$. Therefore ℓ_z intersects \mathcal{S} in a finite numbers of points $(p_n^z)_{1 \leq n \leq d}$ with $d = d(z) \geq 1$, all of them tangency points. According to (2.3), assuming $\Pi(p_n^z)$ lies on the line segment $[(z)_j, (z)_{j+1}]$ of direction $\theta_j \in \mathbb{S}^1$, the tangency point p_n^z lies in

$$\mathcal{T}_j := \left\{ p \in \mathbb{C}^2 : \Re \left(\frac{\theta_j}{\lambda} (1 + R \circ \mathcal{E}(p)) \right) = 0 \right\} \cap \mathcal{S}.$$

If $\mathcal{T}_j = \mathcal{S}$ then $R = 0$ and the result is clear. Otherwise $\mathcal{T} := \bigcup_j \mathcal{T}_j$ is a finite union of irreducible real-analytic curves, as is $\{z \in W : \ell_z \cap \mathcal{S} \subset \mathcal{T}\}$. The latter contains $\partial\Omega \cap W$.

Remark 3.14. In case R is merely C^1 we cannot guarantee that $\ell_z \cap \mathcal{S}$ is finite.

3.5. Proof of Proposition 3.1 (3). From Lemma 3.6 follows the fact that $\text{adh}(\Omega) \cap \{\Re(z) = \ln \rho\}$ is a nonempty line segment, as can be seen by adapting in a straightforward way the proof of Lemma 3.4. In particular Ω is connected. To prove that $\text{adh}(\Omega) \cap \{\Re(z) = \ln \rho\}$ can be arbitrarily wide provided that $\Re(w_*)$ be sufficiently small it is sufficient to invoke the fact that $\{y = 0\}$ is the adherence of a separatrix of \mathcal{F}_R , so that $\Gamma_R(\Sigma)$ contains elements winding more and more around $\{x = 0\}$.

3.6. Proof of Proposition 3.1 (4). Because $S(z_*, -\vartheta, \delta) \subset \Omega_*$ we only need to ensure that $\text{Im}(\vartheta_+)$ and $\text{Im}(\vartheta_-)$ have opposite signs, where ϑ_{\pm} is defined by (3.1). This can be enforced by taking $\Re(z_*)$ and $\Re(w_*)$ negative enough, *i.e.* by taking δ as close to $\frac{\pi}{2}$ as needed.

4. ASYMPTOTICS OF THE DULAC MAP

Definition 4.1. A **time-form** of a vector field X is a meromorphic 1-form τ such that $\tau(X) = 1$.

Because of the specific form of X_R one can always choose a time-form as

$$\tau := \frac{dx}{\lambda x}.$$

We prove first in Section 4.1 that to obtain the image of x by the Dulac map \mathcal{D}_R we need to compute the integral

$$\int_{\gamma_R(x)} R\tau, \quad ,$$

where γ_R is a path tangent to \mathcal{F}_R linking (x, y_*) to some point of $\Pi^{-1}(x_*)$. This is (1) of the Proximity Theorem. We intend in a second step (Section 4.2) to compare this value with that of

$$\int_{\gamma_0(x)} R\tau$$

which can be explicitly computed (Section 4.3). We prove more generally the quantitative result:

Theorem 4.2. *Let $N \in \mathbb{N}_{>0}$ be given. There exists a constant $M > 0$ depending only on $N, \lambda, a, \rho, \|R\|$ and δ such that for any $G \in \mathcal{O}(\mathcal{U}) \cap x^a \mathbb{C}\{x, y\}$ with $\frac{\partial G}{\partial y}$ bounded and all $x = \exp z$ with $|\operatorname{Im}(z - z_*)| \leq \pi N$ one has*

$$\left| \int_{\gamma_R(x)} G\tau - \int_{\gamma_0(x)} G\tau \right| \leq M \left\| \frac{\partial G}{\partial y} \right\| |y_*| |x|^a.$$

Remark 4.3. Under the assumption $\Re\left(a + \frac{1}{\lambda}\right) > 0$ we have $|x|^a = o(|x^{-1/\lambda} \log x|)$, proving (2) of the Proximity Theorem when $G := R$.

4.1. Integral expression of the Dulac map: proof of Proximity Theorem (1).

Lemma 4.4. *Let Σ be a transverse to \mathcal{F}_R ; we refer to Definition 1.6 for the construction of the groupoid $\Gamma_R(\Sigma)$. For given $G \in \mathcal{O}(\mathcal{U})$ the integration process*

$$F : \gamma \in \Gamma_R(\Sigma) \mapsto \int_{\gamma} G\tau$$

gives rise to a holomorphic function whose Lie derivative $X_R \cdot F$ along X_R can be computed by considering F as a local analytic function of the end-point γ^ . Then*

$$X_R \cdot F = G.$$

Proof. Outside the singular locus of X_R there exists a local rectifying system of coordinates: a one-to-one map ψ such that $\psi^* X_R = \frac{\partial}{\partial t}$. In these coordinates we have $\psi^*(G\tau) = G \circ \psi dt$. The fundamental theorem of integral calculus yields the result. \square

Notice that X_0 admits a (multivalued) first-integral with connected fibers

$$H_0(x, y) := x^{-1/\lambda} y,$$

which means that it lifts through σ to a holomorphic map, still written H_0 , constant along the leaves of $\sigma^*\mathcal{F}_0$ and whose range is in one-to-one correspondence with the space of leaves of $\sigma^*\mathcal{F}_0$.

Lemma 4.5. *Let Σ be a transverse to \mathcal{F}_R . The function*

$$\begin{aligned} H_R : \Gamma_R(\Sigma) &\longrightarrow \mathbb{C} \\ \gamma &\longmapsto H_0 \exp \int_{\gamma} -R\tau \end{aligned}$$

is a holomorphic first-integral of $\sigma^\mathcal{F}_R$ with connected fibers.*

Proof. The fact that H_R is holomorphic on $\Gamma_R(\Sigma)$ is clear enough from Lemma 4.4. It is a first integral of $\sigma^*\mathcal{F}_R$ if, and only if, the Lie derivative $X_R \cdot H_R$ vanishes. This quantity is computed as follows:

$$\begin{aligned} X_R \cdot H_R &= X_0 \cdot H_R + R \left(y \frac{\partial}{\partial y} \right) \cdot H_R \\ &= H_R \times \left(X_R \cdot \int_{\gamma} -R\tau + R \frac{\left(y \frac{\partial}{\partial y} \right) \cdot H_0}{H_0} \right). \end{aligned}$$

Since $\left(y \frac{\partial}{\partial y} \right) \cdot H_0 = H_0$ our claim holds. The fact that H_R has connected fibers is a direct consequence of both facts that H_0 also has and $H_R|_{\Sigma} = H_0$. \square

Corollary 4.6. *We have*

$$\mathcal{D}_R = \mathcal{D}_0 \times \exp \int_{\bullet} R\tau.$$

Proof. For any path $\gamma \in \Gamma^*$ we have the relation $H_R(\gamma_{\star}) = H_R(\gamma)$, that is

$$H_0(\gamma) \exp \int_{\gamma} -R\tau = H_0(\gamma_{\star}).$$

The conclusion follows since $\gamma \mapsto H_0(\gamma)$ is linear with respect to the y -coordinate of γ when x_{\star} is fixed. \square

4.2. Approximation to the formal model. We fix once and for all a preimage $(z_{\star}, w_{\star}) \in \mathcal{E}^{-1}(x_{\star}, y_{\star})$. Since \mathcal{D}_R is naturally defined on the universal covering $\tilde{\mathcal{U}}$ of $\mathcal{U} := \mathcal{U} \setminus \{xy = 0\}$ we keep on working in logarithmic coordinates

$$(x, y) = \mathcal{E}(z, w) = (\exp z, \exp w).$$

Notice that the time form τ is transformed into

$$\tilde{\tau} := \mathcal{E}^* \tau = \frac{1}{\lambda} dz.$$

We make here the hypothesis that $\Re(\lambda) < 0$. We need to compare this integral and the one obtained for the model, *i.e.* bound

$$\Delta(z_0) := \int_{\tilde{\gamma}(z_0)} \tilde{G}(z, w_R(z, z_0)) - \tilde{G}(z, w_0(z, z_0)) dz$$

where $\tilde{\gamma}(z_0)$ is a path linking z_0 to z_* within $\Omega_{(z_0, w_*)}$ and $z \mapsto w_R(z, z_0)$ is its lift in $\tilde{\mathcal{F}}_R$ starting from (z_0, w_*) . We mention that

$$w_0(z, z_0) = w_* + \frac{z - z_0}{\lambda}.$$

For any $(z, w_j) \in \tilde{\mathcal{U}}$ we have the estimate

$$\left| \tilde{G}(z, w_2) - \tilde{G}(z, w_1) \right| \leq |\exp(az)| \left\| \frac{\partial G}{\partial y} \right\| |\exp w_2 - \exp w_1|$$

so that

$$|\Delta(z_0)| \leq \left\| \frac{\partial G}{\partial y} \right\| \int_{\tilde{\gamma}} |\exp(az + w_0(z, z_0)) (\exp(w_R(z, z_0) - w_0(z, z_0)) - 1) dz|.$$

Setting

$$D_R(z, z_0) := |w_R(z, z_0) - w_0(z, z_0)|$$

and taking $|\exp z - 1| \leq |z| \exp |z|$ into account we derive

$$|\Delta(z_0)| \leq \left\| \frac{\partial G}{\partial y} \right\| \int_{\tilde{\gamma}} \exp \Re(az + w_0(z, z_0)) D_R(z, z_0) \exp D_R(z, z_0) |dz|.$$

The proof is done when the next lemma is established:

Lemma 4.7. *There exists a constant $K > 0$, depending only on $N, \lambda, a, \rho, \|R\|, z_*$ and δ , such that*

$$\sup_t D_R(\tilde{\gamma}(t), z_0) \leq K$$

where $\tilde{\gamma}$ is the integration path built in Proposition 3.2. The values of K is explicitly, if crudely, determined in the proof to come.

Proof. Invoking the estimate (2.1) from Lemma 2.3 and setting

$$\begin{aligned} C_1 &:= \frac{\|R\|}{a|\lambda|} \\ C_2 &:= \frac{C_1}{\Re(\vartheta_+)} \\ C_3 &:= a\rho^a C_1 \end{aligned}$$

we know that, using the number κ obtained in Proposition 3.2,

$$\begin{aligned} \sup_{z \in [z_0, z_1]} D_R(z, z_0) &\leq K_1 := C_1(\rho^a + \exp(a\kappa)) \\ \sup_{z \in [z_1, z_2]} D_R(z, z_0) &\leq K_2 := K_1 + C_2(\rho^a + \exp(a\kappa)) \\ \sup_{z \in [z_2, z_3]} D_R(z, z_0) &\leq K_3 := K_2 + C_3(2\pi N + \tan(|\arg \vartheta_+|)(\ln \rho - \kappa)) \\ \sup_{z \in [z_3, z_*]} D_R(z, z_0) &\leq K := K_3 + C_2(\rho^a + \exp \Re(az_*)). \end{aligned}$$

□

We conclude now the proof starting from

$$|\Delta(z_0)| \leq K \left\| \frac{\partial G}{\partial y} \right\| \exp \Re(K + w_* - z_0/\lambda) \int_{\bar{\gamma}} \exp \Re\left(\left(a + \frac{1}{\lambda}\right)z\right) |dz|.$$

Let $\alpha := \left|a + \frac{1}{\lambda}\right|$. We bound each partial integral $I_{\star \rightarrow \bullet} := \int_{[z_*, z_\bullet]} \exp \Re\left(\left(a + \frac{1}{\lambda}\right)z\right) |dz|$ in the following manner:

$$\begin{aligned} I_{0 \rightarrow 1} &\leq \exp \Re\left(\left(a + \frac{1}{\lambda}\right)z_0\right) \int_0^{\kappa - \Re(z_0)} \exp\left(t \Re\left(a + \frac{1}{\lambda}\right)\right) dt \\ &\leq \frac{\exp \Re\left(\left(a + \frac{1}{\lambda}\right)\kappa\right) - \exp \Re\left(\left(a + \frac{1}{\lambda}\right)z_0\right)}{\Re\left(a + \frac{1}{\lambda}\right)}, \\ I_{1 \rightarrow 2} &\leq \exp \Re\left(\left(a + \frac{1}{\lambda}\right)z_1\right) \int_0^{\ln \rho - \kappa} \exp\left(t \Re\left(\left(a + \frac{1}{\lambda}\right)\vartheta_+\right)\right) dt \\ &\leq \exp(\alpha |z_1|) \frac{\exp(\alpha(\ln \rho - \kappa)) - 1}{\alpha} \\ &\leq \exp\left(\alpha\left(\sqrt{\kappa^2 + \pi^2 N^2} + \ln \rho - \kappa\right)\right), \\ I_{2 \rightarrow 3} &\leq \exp(\alpha |z_2|) \int_0^{\operatorname{Im}(z_3 - z_2)} \exp\left(t \left|\operatorname{Im}\left(\frac{1}{\lambda}\right)\right|\right) |dt| \\ &\leq \exp\left(\alpha |z_2| + \left|\frac{1}{\lambda}\right| |\operatorname{Im}(z_3 - z_2)|\right) \\ &\leq \exp\left(\alpha(|z_*| + \pi N + \tan(|\arg \vartheta| + \delta)(\ln \rho - \kappa))\right) \\ &\quad \times \exp\left(\left|\frac{1}{\lambda}\right| (2\pi N + \tan(|\arg \vartheta| + \delta)(\ln \rho - \kappa))\right), \\ I_{3 \rightarrow * } &\leq \exp\left(\alpha\left(\sqrt{\Re(z_*)^2 + \pi^2 N^2} + \ln \rho - \Re(z_*)\right)\right). \end{aligned}$$

In particular the dominant integral in the above list is $I_{0 \rightarrow 1}$, so that there exists a constant M , satisfying the required dependency properties, with

$$|\Delta(z_0)| \leq M \left\| \frac{\partial G}{\partial y} \right\| \exp \Re(w_* + az_0).$$

Since $\Re\left(a + \frac{1}{\lambda}\right) > 0$ we have

$$\Delta(z_0) = o\left(|z_0| \exp \Re\left(\frac{-z_0}{\lambda}\right)\right)$$

as expected.

4.3. Study of the model.

4.3.1. *Explicit computation.* We want to compute for $n, m \in \mathbb{N}$ the functions defined by

$$\begin{aligned} T_{n,m}(z) &:= \int_{\bar{\gamma}(z)} \exp(nu + mw_0(u, z_0)) du \\ &= \exp\left(m\left(w_* - \frac{z}{\lambda}\right)\right) \times \int_z^{z_*} \exp\left(\left(n + \frac{m}{\lambda}\right)u\right) du. \end{aligned}$$

If $n + m/\lambda = 0$ then $\lambda = -p/q$, with p and q co-prime positive integers, and $(n, m) = k(q, p)$ with $k \in \mathbb{N}$. In that case, and when $k > 0$,

$$(4.1) \quad T_{kq, kp}(z) = (z_* - z) \exp(k(pw_* + qz)) = O(|z \exp \Re(az)|).$$

The other case $n + m/\lambda \neq 0$ is not more difficult:

$$T_{n,m}(z) = \exp(mw_* + nz) \frac{\exp((n + m/\lambda)(z_* - z)) - 1}{n + m/\lambda}.$$

One can see easily that as $n + m/\lambda$ tends to zero (which may happen if, and only if, λ is a negative irrational) the function $T_{n,m}$ grows in modulus. The dominant support introduced in Definition 1.1 allows to discriminate between two kinds of growth rate.

4.3.2. *Resonant support.* We show now that the resonant support consists of (quasi-)resonant monomials only.

Lemma 4.8. *Assume that $\lambda < 0$ and $a + \frac{1}{\lambda} > 0$.*

(1) *If $\lambda = -p/q < 0$ is a rational number then*

$$\text{Res}(a, \lambda) = \{k(q, p) : k \in \mathbb{N}, kq \geq a\}.$$

(2) *If λ is a negative irrational we denote by $(-p_k/q_k)_{k \in \mathbb{N}}$ its sequence of convergents. Then*

$$\text{Res}(a, \lambda) = \{(q_k, p_k) : k \in \mathbb{N}, q_k \geq a\}.$$

Proof.

(1) Because we have $n \geq a \geq q$ the relation $|n\lambda + m| < \frac{1}{2n}$ becomes

$$|np - mq| < \frac{q}{2n} < 1.$$

Hence $np = mq$ and since p and q are co-prime the conclusion follows.

(2) This is a consequence of the well-known result in continued-fraction theory: if $\frac{p}{q} \in \mathbb{Q}_{>0}$ is given such that $\left| \frac{p}{q} + \lambda \right| < q^{-2/2}$ then (p, q) is one of the convergents of $|\lambda|$. □

4.3.3. *Dominant terms: proof of Proposition 1.2.* Nothing needs to be proved for G_0 so we assume that G expands into a power series $G(x, y) = \sum_{n \geq a, m > 0} G_{n,m} x^n y^m$ convergent on a closed polydisc of poly-radii at least $(\rho + \epsilon, r + \epsilon)$. Because of the Cauchy formula, for all n, m

$$|G_{n,m}| \leq C(\rho + \epsilon)^{-n} (r + \epsilon)^{-m}$$

where $C := \sup_{|x|=\rho+\epsilon, |y|=r+\epsilon} |G(x, y)|$.

If λ is not real then

$$\inf_{(n,m) \in \mathbb{N}^2 \setminus \{(0,0)\}} |n\lambda + m| \geq a |\text{Im}(\lambda)| > 0.$$

Let $z - z_*$ be given with imaginary part bounded by $N\pi$ for some integer $N > 0$ and with real part lesser than

$$\mu := - \left| \frac{\text{Im}(\lambda)}{\Re(\lambda)} \right| N\pi.$$

By construction of μ we have

$$\begin{aligned}\Re\left(\frac{m}{\lambda}(z_* - z)\right) &= \frac{m}{|\lambda|^2}(\Re(\lambda)\Re(z_* - z) + \Im(\lambda)\Im(z_* - z)) < 0 \\ &\leq \Re\left(\frac{1}{\lambda}(z_* - z)\right)\end{aligned}$$

so that we derive at last

$$|T_{n,m}(z)| \leq \frac{2|\lambda|r^m\rho^{n+\Re(1/\lambda)}}{a|\Im(\lambda)|}\exp\Re\left(-\frac{z}{\lambda}\right).$$

Now

$$\begin{aligned}\left|\int_z^{z_*} G \circ \mathcal{E} dz\right| &= \left|\sum_{n \geq a, m > 0} G_{n,m} T_{n,m}(z)\right| \\ &\leq \frac{2|\lambda|\rho^{\Re(1/\lambda)}C}{a|\Im(\lambda)|} \times \frac{(r+\epsilon)(\rho+\epsilon)}{\epsilon^2} \times \exp\Re\left(-\frac{z}{\lambda}\right) \\ &= O(|\exp -z/\lambda|),\end{aligned}$$

ending the proof for the non-real case. In fact this reasoning goes on holding even when $\lambda < 0$ as long as (n, m) belongs not to $\text{Res}(a, \lambda)$, since in that case

$$|n + m/\lambda| > \frac{|\lambda|}{2n}$$

has strictly sub-geometric inverse.

Take now λ negative real and $G = G_{\text{Res}}$. If $\lambda = -p/q$ is a negative rational number then (4.1) provides what remains to be proved. Assume now that λ is irrational. Because $|\exp z - 1| \leq |z|$ when $\Re(z) < 0$ we have for $(n, m) \in \text{Res}(a, \lambda)$

$$|T_{n,m}(z)| \leq |(z - z_*) \exp(nz)| r^m.$$

Therefore

$$\begin{aligned}\left|\int_{\gamma_0(z)} G_{\text{Res}} \circ \mathcal{E} dz\right| &\leq C \sum_{(n,m) \in \text{Res}(a,\lambda)} (\rho+\epsilon)^{-n} (r+\epsilon)^{-m} |T_{n,m}(z)| \\ &\leq C |z_* - z| \sum_{(n,m) \in \text{Res}(a,\lambda)} \left(\frac{\exp \Re(z)}{\rho+\epsilon}\right)^n \left(\frac{r}{r+\epsilon}\right)^m \\ &\leq C \frac{(r+\epsilon)(\rho+\epsilon)}{\epsilon^2} |z_* - z| \exp \Re(az) \\ &= O(|z \exp -z/\lambda|)\end{aligned}$$

as expected.

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