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Abstract. We study arc complexes of surfaces in the most general setting of surfaces with marked points in the interior and on the boundary. In particular, we prove that except a few cases every automorphism is induced by a homeomorphism of the surface which fixes the marked points setwise and the isomorphism type of these arc complexes determines the topological data of the underlying surface. Our proofs are based on a combinatorial approach which leads to new information on the geometry of these objects and are independent of all the other well-known combinatorial rigidity results.

1. Introduction

Combinatorics of arcs play a key-role in the study of the moduli space of Riemann surfaces. In [6, 7] Harer defined the arc complex of a punctured surface in order to study some homological properties of the mapping class group. In [3] Bowditch and Epstein showed that appropriate quotients of the arc complex give a combinatorial compactification of the moduli space of punctured hyperbolic surfaces. In [20, 18, 15], Penner described some other features of this combinatorial compactification in the context of his decorated Teichmüller theory. In [16, 19] Penner suggests an approach to mapping class group and moduli space problems through the combinatorics of arcs and arc complexes.

In this paper we deal with arc complexes in the most general setting of surfaces with bondary, marked points on the boundary and in the interior. Some topological features (i.e. connectedness, contractibility) of arc complexes in this setting have been studied by Hatcher in [8]. In [17, 19] Penner determines the topological type of some of their quotients, he also sketches interesting relations with the decorated moduli space for surfaces with boundary.

In this paper we study the problem of combinatorial rigidity of arc complexes for surfaces with boundary, marked points on the boundary and in the interior. We say that an arc complex is rigid if its automorphism group is isomorphic to the mapping class group of the underlying surface. We denote by $(S^s_{g,b}, \mathbf{p})$ an orientable surface of genus g with $b \geq 0$ boundary components, $p_i \geq 1$ marked points on each boundary component whenever $b > 0$ (with $\mathbf{p} = (p_1, \dots p_b)$), and $s \geq 0$ marked points in the interior of S. We denote by $A(S^s_{g,b}, \mathbf{p})$ its arc complex, $A_\sharp(S^s_{g,b}, \mathbf{p})$ the subcomplex spanned by arcs with endpoints on the boundary and Aut $A(S^s_{g,b}, \mathbf{p})$, Aut $A_\sharp(S^s_{g,b}, \mathbf{p})$ their automorphism group (for more precise definitions, see Section 2). The main theorems we prove are the following:

Theorem 1.1. *Let $b + s > 0$ and $A(S^s_{g,b}, \mathbf{p}) \neq \varnothing$. If $(S^s_{g,b}, \mathbf{p}) \neq (S^0_{0,2}, (1, 1)), (S^1_{1,0}, \varnothing), (S^3_{0,0}, \varnothing)$, and $\dim A(S^s_{g,b}, \mathbf{p}) \geq 1$, then $A(S^s_{g,b}, \mathbf{p})$ is rigid. In the exceptional cases, the natural homomorphisms $\mathrm{MCG}(S^s_{g,b}, \mathbf{p}) \to \mathrm{Aut}\ A(S^s_{g,b}, \mathbf{p})$ is surjective, but not injective.*

Theorem 1.2. *Let $b > 0$ and $p_i \geq 1$ for $i = 1, \dots, b$. If $(S^s_{g,b}, \mathbf{p}) \neq (S^0_{0,2}, (1, 1))$, then $A_\sharp(S^s_{g,b}, \mathbf{p})$ is rigid. In the exception cases the natural homomorphism $\mathrm{MCG}(S^0_{0,2}, (1, 1)) \to \mathrm{Aut}\ A_\sharp(S^0_{0,2}, (1, 1))$ is surjective, but not injective.*

Theorem 1.3. *If $A(S^s_{g,b}, \mathbf{p})$ is not empty and isomorphic to $A(S^{s'}_{g',b'}, \mathbf{p}')$, then $s = s'$, $b = b'$, $g = g'$ and $p_i = p'_i$ for all i.*

The exceptional cases are listed and studied in Section 2. As a corollary of the first theorem in the case $b = 0$ we have a new proof of a theorem of Irmak-McCarthy [5].

As far as we know, the first example of simplicial complex with the rigidity property is the curve complex. The rigidity theorem for the curve complex of a punctured surface was first stated by Ivanov [11] for surfaces of genus greater than 1, then proved in low genus by Korkmaz [9] and finally reproved in the most general case by Luo [12]. Applications of this result include a new proof of Royden's theorem for the isometries of the Teichmüller space for punctured surfaces and the study of finite index subgroups of mapping class groups (see for instance [11, 10]).
Rigidity properties of simplicial complexes built from surfaces have been investigated in the past by many different authors; a survey of these results and their applications can be found in [14]. Most of these proofs are based on (non-trivial) reductions to the rigidity theorem for the curve complex. Our proof for arc complexes is instead completely independent from this and all the other results, and it leads to new information on the geometry of our complexes. Our result is also useful in the proof of Royden's theorem concerning Teichmüller isometries for surfaces with boundary [1].

1.1. **Structure of the paper.** The structure of the paper is the following. In Section 2 we introduce the notation and restate the main theorems as Theorem A, B and C. We also discuss some new results about the combinatorics of arc complexes, including several invariance lemmas which will be used throughout the paper. Finally we prove Theorem A. In Section 3 we discuss examples and give a proof of Theorem B. Section 4 is devoted to the proof of Theorem C.

2. Combinatorics of arc complexes

Let us first fix the notation. Let $S = S^s_{g,b}$ be a compact orientable surface with genus $g \geq 0$, $b \geq 0$ ordered boundary components and s distinguished points in the interior of the surface. When $b > 0$ we shall fix a finite set $\mathscr{P}$ of distinguished points on ∂S and denote by $\mathbf{p} = (p_1, \ldots, p_b)$ the vector whose component p_i is the number of distinguished points on the i-th boundary component of S. Finally we denote by $(S^s_{g,b}, \mathbf{p})$ or $(S, \mathbf{p})$ (when no ambiguity occurs) the pair given by a surface $S^s_{g,b}$ and a vector $\mathbf{p}$ related to the distinguished points on its boundary as here described.

We shall now recall the definition of mapping class group of the pair $(S, \mathbf{p})$.
Let $\mathscr{S}$ be the set of the s distinguished points in the interior of S. Let $\mathrm{Homeo}(S, \mathbf{p})$ be the group of homeomorphisms of S fixing $\mathscr{P} \cup \mathscr{S}$ as a set. Let $\mathrm{Homeo}_0(S, \mathbf{p}) \subseteq \mathrm{Homeo}(S, \mathbf{p})$ be the normal subgroup of homeomorphisms isotopic to the identity through isotopies fixing $\mathscr{S} \cup \mathbf{p}$. The *mapping class group of the pair* $(S, \mathbf{p})$ is the group $\mathrm{MCG}^*(S, \mathbf{p}) = \mathrm{Homeo}(S, \mathbf{p})/\mathrm{Homeo}_0(S, \mathbf{p})$. The *pure mapping class group of the pair*$(S, \mathbf{p})$ is the subgroup $P\mathrm{MCG}^*(S, \mathbf{p}) < \mathrm{MCG}^*(S, \mathbf{p})$ generated by the homeomorphisms fixing $\mathscr{S} \cup \mathscr{P}$ pointwise.

Let $\mathscr{B}_i$ be the i-th boundary component of S with p_i marked points on it. We will introduce here the definition of $\frac{2\pi}{p_i}$-*rotation* around $\mathscr{B}_i$.
First consider the annulus $A = S^1 \times [0, 1]$ in $\mathbb{R}^2$ (equipped with polar coordinates (θ, r)) with marked points $\{(\frac{2\pi}{p_i} j, 1)\}_{j=0,\ldots,p_i-1}$. The $\frac{2\pi}{p_i}$-*rotation* map of A is the map $R : A \to A$ defined as $R(\theta, r) = (\theta + \frac{2\pi}{p_i} t, t)$. Remark that R is orientaton-preserving, the restriction $R_| : S^1 \times \{1\} \to S^1 \times \{1\}$ is a rotation of angle $\frac{2\pi}{p_i}$, the restriction $R_| : S^1 \times \{0\} \to S^1 \times \{0\}$ is the identity and the power R^{p_i} is the right Dehn-twist around the core curve of the annulus.

Let $\{P_j\}_{j=0,\ldots,p_i-1}$ be the set of marked points on $\mathscr{B}_i$. Let N be the closure of a regular neighborhood of $\mathscr{B}_i$, and choose a homeomorphism $\phi : N \to A$ such that $\phi(P_j) = (\frac{2\pi j}{p_i}, 1)$ for all $j = 0, \ldots, p_i - 1$. We consider the homeomorphism $\tilde{R}_i : (S, \mathbf{p}) \to (S, \mathbf{p})$ defined as follows:

$$\tilde{R}_i(x) = \begin{cases} \phi^{-1} \circ R \circ \phi(x) & \text{for } x \in N \\ x & \text{for } x \in S \setminus N \end{cases}$$

The map $\tilde{R}_i$ depends on the choice of ϕ and N, but the equivalence class modulo isotopies which fix $\mathscr{P}$ pointwise doesn't depend on such choices and gives a well-defined non-trivial element $\rho_{\frac{2\pi}{p_i}} = [\tilde{R}_i]$ in $\mathrm{MCG}^*(S, \mathbf{p})$. We call such an element the $\rho_{\frac{2\pi}{p_i}}$-*rotation* around the i-th boundary component $\mathscr{B}_i$.

We remark that the group $R_{\mathbf{p}} = \langle \rho_{\frac{2\pi}{p_1}}, \dots, \rho_{\frac{2\pi}{p_b}} \rangle$, generated by all the rotations around the boundary components of S is abelian of rank b.
Let us denote by Σ_n be the symmetric group on n elements. For every $i = 1, \dots, b$, let r_i be the number of boundary components having exactly p_i marked points. Finally we denote by $P\mathrm{MCG}^*(S)$ the subgroup of $\mathrm{MCG}^*(S, \mathbf{p})$ generated by mapping classes fixing pointwise $\mathscr{S} \cup \partial S$. The following two propositions are not difficult to prove.

Proposition 2.1. *There is a short non-split exact sequence:*

$$0 \to P\mathrm{MCG}(S, \mathbf{p}) \to \mathrm{MCG}(S, \mathbf{p}) \to \oplus_{i=1}^{b} (\Sigma_{r_i} \ltimes \mathbb{Z}_{p_i}) \oplus \Sigma_s \to 0.$$

If $s = 0$ and the p_i are all distincts, $\mathrm{MCG}(S, \mathbf{p})$ *is generated by* $R_{\mathbf{p}}$ *and Dehn twists about simple closed curves non-parallel to* ∂S.

Proposition 2.2. *The following holds:*

(1) *If there exist p_i such that $p_i \geq 3$, then* $P\mathrm{MCG}^*(S, \mathbf{p}) = P\mathrm{MCG}(S, \mathbf{p})$.
(2) *If $s = 0$ and for all $i = 1, \dots, b$ $|p_i| \leq 2$, then* $P\mathrm{MCG}^*(S, \mathbf{p})$ *is generated by* $\langle P\mathrm{MCG}^*(S), i \rangle$, *where i an involution which fixes every point in* $\mathbf{p}$;
(3) *In any other case,* $P\mathrm{MCG}^*(S)$ *is isomorphic to* $P\mathrm{MCG}^*(S, \mathbf{p})$.

2.1. **Arc complexes $A(S, \mathbf{p})$.** In this section we will define arc complexes and give some examples in low dimension.

We denote by $A(S, \mathbf{p})$ the simplicial complex whose vertices are the equivalence classes of arcs with endpoints on $\mathscr{P} \cup \mathscr{S}$ modulo isotopies fixing $\mathscr{P} \cup \mathscr{S}$ pointwise. A set of vertices $\langle a_1, \dots, a_k \rangle$ spans a $k-1$-simplex if and only if all the vertices can be realized simultaneously as disjoint arcs.
We shall denote by $A_\sharp(S, \mathbf{p})$ the subcomplex of $A(S, \mathbf{p})$ spanned by isotopy classes of arcs with both endpoints on $\mathscr{P}$. If $s = 0$, we have $A_\sharp(S, \mathbf{p}) = A(S, \mathbf{p})$. In general $A_\sharp(S, \mathbf{p})$ has codimension s in $A(S, \mathbf{p})$. The definitions here also make sense when $\mathscr{P} = \varnothing$. In this case we shall use the notation $A(S_g^s)$ instead of $A(S_{g,0}^b, \varnothing)$.

The following remarks illustrate some basic properties of these complexes. Their proofs easily follow from our definitions.

Remark 2.3. *Let $g, s \geq 0$, $b \geq 1$ and $\mathbf{p} = (p_1, \dots, p_b) \in (\mathbb{N} \setminus \{0\})^b$.*
The following holds:

(1) *$A(S, \mathbf{p}) = \varnothing$ if and only if $(g, b, s) = (0, 1, 0)$ and $p_1 \in \{1, 2, 3\}$.*
(2) *$A(S, \mathbf{p})$ has a finite number of vertices if and only if $g = 0$, $b = 1$ and $s \leq 1$. In particular, $A(S, \mathbf{p})$ is a single point if and only if $g = 0$, $b = 1$, $s = 1$ and $\mathbf{p} = (1)$.*

Remark 2.4. *If $b = 0$, $g \geq 0$ and $s \geq 1$, the following holds:*

(1) *$A(S_{g,0}^s) = \varnothing$ if and only if $(g, s) = (0, 1)$;*
(2) *$A(S_{g,0}^s)$ has a finite number of vertices if and only if $g = 0, s \leq 3$. In particular $A(S_{g,0}^s)$ is a single point if and only if $g = 0$ and $s = 2$, and $A(S_{0,0}^3)$ is homeomorphic to a disk having 6 vertices and 4 2-simplices.*

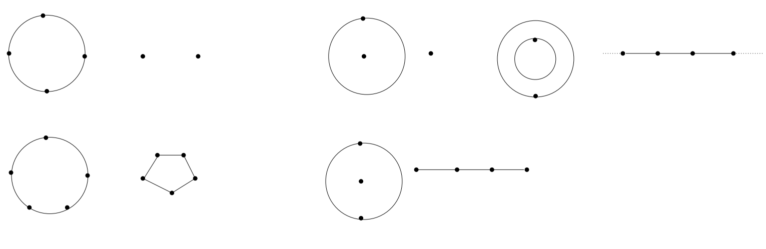

FIGURE 1. Surfaces and their respective arc complexes in Remark 2.5

Remark 2.5 (Low dimensional cases)**.** *Let $g, s \geq 0$, $b \geq 1$ and $\mathbf{p} = (p_1, \dots, p_b) \in (\mathbb{N} \setminus \{0\})^b$. The following holds:*

(1) $(g, b, s, \mathbf{p}) \in \{(0, 1, 0; (4)), (0, 1, 1; (1))\}$ *if and only if* $A(S, \mathbf{p})$ *has dimension* 0.
In particular, $A(S^1_{0,1}; (1))$ is a single point and $A(S^0_{0,1}, (4))$ consists of two disjoint vertices (see Figure 1).

(2) $(g, b, s, \mathbf{p}) \in \{(0, 2, 0; (1, 1)), (0, 1, 0; (5)), (0, 1, 1; (2))\}$ *if and only if* $A(S, \mathbf{p})$ *has dimension* 1.
In particular, $A(S^0_{0,2}, (1, 1))$ is isomorphic to $\mathbb{R}$, $A(S^0_{0,1}; (5))$ has diameter 2 and $A(S^1_{0,1}; (2))$ has diameter 3.

(3) $(g, b, s, \mathbf{p}) \in \{(0, 1, 0; (6)), (0, 1, 1; (3)), (0, 2, 0; (1, 2)\}$ *if and only if* $A(S, \mathbf{p})$ *has dimension* 2.

In Figure 1 we show some some examples of surfaces together with their arc complexes in the low dimensional cases.

By an elementary Euler characteristic argument, we find that the dimension of simplices in the complexes is bounded from above, in particular $A(S, \mathbf{p})$ and $A_\sharp(S, \mathbf{p})$ have dimension respectively $6g + 3b + 3s + |\mathbf{p}| - 7$ and $6g + 3b + 2s + |\mathbf{p}| - 7$. We remark that in both $A(S, \mathbf{p})$ and $A_\sharp(S, \mathbf{p})$ each simplex of maximal dimension corresponds to a collection of disjoint non-homotopic arcs which is maximal with respect to inclusion on the surface. Indeed, a maximal simplex in $A(S, \mathbf{p})$ corresponds to a triangulation of S with vertices in $\mathscr{P} \cup \mathscr{S}$, and the complement on $(S, \mathbf{p})$ of a maximal simplex in $A_\sharp(S, \mathbf{p})$ corresponds to a union of once-punctured discs (with punctures in $\mathscr{S}$) and (immersed) triangles with vertices in $\mathscr{P}$.
A proof of the following lemma can be found in [8].

Proposition 2.6 (Hatcher [8])**.** *If $A(S, \mathbf{p})$ has dimension at least 1, then $A(S, \mathbf{p})$ is arcwise connected. Moreover, except when S is a disk or an annulus with $s = 0$, $A(S, \mathrm{p})$ is contractible. In the exceptional cases, $A(S, \mathbf{p})$ is homeomorphic to a sphere.*

The case when the surface is a disk has also been studied by Braun and Ehrenborg in [3].

In [17] Penner studies the topology of quotients of these arc complexes through the action of the pure mapping class group, suggesting a deep connection with the topology of the moduli space. In particular he proves the following result:

Theorem 2.7 (Penner [17])**.** *Let $(S^s_{g,b}, \mathbf{p})$ be a compact orientable surface with genus g, $b \geq 1$ boundary components, s marked points in the interior and $\mathbf{p} = (p_1, \dots, p_b)$ marked points on the boundary, with $p_i \geq 1$ for all i. The quotient $Q(S^s_{g,b}, \mathbf{p})$ of $A_\sharp(S^s_{g,b}, \mathbf{p})$ by the action of the pure mapping class group $P\mathrm{MCG}(S^s_{g,b}, \mathbf{p})$ is a sphere only in the cases*

$$\begin{array}{ll} Q(S^s_{0,1}, \mathbf{p}) \textit{ for } s \geq 0; & Q(S^1_{0,2}, \mathbf{p}) \textit{ for } p_1 + p_2 \geq 2\ ; \\ Q(S^1_{1,1}, \mathbf{p}) \textit{ for } p_1 \geq 1; & Q(S^0_{0,2}, \mathbf{p}) \textit{ for } p_1 + p_2 \geq 2\ ; \\ Q(S^1_{0,1}, \mathbf{p}) \textit{ for } p_1 \geq 1; & Q(S^0_{0,3}, \mathbf{p}) \textit{ for } p_1 + p_2 + p_3 \geq 3\ . \end{array}$$

Furthermore, $Q(S^s_{g,b}, \mathbf{p})$ is a PL-manifold but not a sphere if and only if $p_i = 1$ for all i and $(g, b, s) \in \{(0, 2, 2), (0, 3, 1), (1, 3, 1), (1, 2, 0)\}$. In all other cases the quotient $Q(S^s_{g,b}, \mathbf{p})$ is not a PL-manifold.

We remark that $A_\sharp(S^s_{g,b}, \mathbf{p})$ and $A(S^s_{g,b}, \mathbf{p})$ coincide when $s = 0$. The topology of the non-spherical quotients is still unknown.
Using the notation we have introduced, we restate the main results Theorem 1.1 and 1.2 in the following equivalent forms:

Theorem A. *If $A(S^s_{g,b}, \mathbf{p})$ is not empty and isomorphic to $A(S^{s'}_{g',b'}, \mathbf{p}')$, then $s = s'$, $b = b'$, $g = g'$ and $p_i = p'_i$ for all i.*

Theorem B. *If $b + s > 0$, $(S^s_{g,b}, \mathbf{p}) \neq (S^0_{0,2}, (1, 1)), S^1_1$, and $\dim A(S^s_{g,b}, \mathbf{p}) \geq 1$, then $A(S^s_{g,b}, \mathbf{p})$ is rigid. In the exceptional cases, the natural homomorphism $\mathrm{MCG}(S^s_{g,b}, \mathbf{p}) \to \mathrm{Aut}\ A(S^s_{g,b}, \mathbf{p})$ is surjective, but not injective.*

Theorem C. *If $b > 0$, $(S^s_{g,b}, \mathbf{p}) \neq (S^0_{0,2}, (1,1))$, and* $\dim A_\sharp(S^s_{g,b}, \mathbf{p}) \geq 1$*, then $A_\sharp(S^s_{g,b}, \mathbf{p})$ is rigid. In the exceptional cases, the natural homomorphism* $\mathrm{MCG}(S^s_{g,b}, \mathbf{p}) \to \mathrm{Aut}\ A_\sharp(S^s_{g,b}, \mathbf{p})$ *is surjective, but not injective.*

We are going to prove Theorem A in this section, Theorem B in Section 3 and Theorem C in Section 4.

2.2. **Intersection numbers.** Let v_1, v_2 be two vertices in $A(S, \mathbf{p})$. We recall the usual notion of *intersection number* $i(v_1, v_2) = \min|\mathring{\alpha} \cap \mathring{\beta}|$, where α, β are essential arcs on S with α in the homotopy class v_1 and β in the homotopy class v_2, and $\mathring{\alpha}$, $\mathring{\beta}$ their interior.

Definition 2.8. *Let τ and σ be two simplices in $A(S, \mathbf{p})$ with the same dimension. We say that σ and τ are obtained from each other by a* flip *if there exists vertices $v_1 \in \tau$ and $v_2 \in \sigma$ (called* flippable*) such that the following properties hold:*

- $i(v_1, v_2) = 1$*;*
- $i(v_1, w) = 0$ *for every* $w \in \sigma \setminus v_2$*;*
- $i(v_2, z) = 0$ *for every* $z \in \tau \setminus v_1$*.*

A proof of the following lemma can be found in [8] or in [13].

Lemma 2.9. *Let α, β be two maximal simplices in $A(S, \mathbf{p})$. Then there exists a finite sequence $\tau_0, \dots, \tau_n$ of maximal simplices such that $\tau_0 = \alpha$, $\tau_n = \beta$ and for any $i = 0, \dots, n-1$ τ_{i+1} is obtained by τ_i by a flip.*

The following lemmas are adapted from Ivanov's paper in [11].

Invariance Lemma 2.10 (Intersection number)**.** *Let us denote by $\mathscr{A}$ the arc complex A or $A_\sharp$. Let $\mathscr{A}(S, \mathbf{p})$ and $\mathscr{A}(S', \mathbf{p}')$ have dimension greater than 1, and $\phi : \mathscr{A}(S, \mathbf{p}) \to \mathscr{A}(S', \mathbf{p}')$ be an isomorphism. For any $\alpha_1, \alpha_2 \in \mathscr{A}(S, \mathbf{p})$ such that $i(\alpha_1, \alpha_2) = 1$, we have $i(\phi(\alpha_1), \phi(\alpha_2)) = 1$.*

Proof. Let us first consider the case when $\mathscr{A} = A$. Since ϕ is an isomorphism, $\dim A(S, \mathbf{p}) = \dim A(S', \mathbf{p}')$ and ϕ sends maximal simplices (that is, triangulations of $(S, \mathbf{p})$) into maximal simplices (that is, triangulations of $(S', \mathbf{p}')$). Let α and β be arcs intersecting exactly once, we can extend α to a triangulation τ_α such that the set of arcs $\tau_\beta := (\tau_\alpha \setminus \{\alpha\}) \cup \beta$ is also a triangulation of S. Let τ be the simplex of $A(S, \mathbf{p})$ defined as $\tau = \tau_\alpha \cap \tau_\beta = \tau_\alpha \setminus \alpha = \tau_\beta \setminus \beta$, it has codimension 1. Now $\phi(\tau_\alpha)$ and $\phi(\tau_\beta)$ are triangulations of $(S', \mathbf{p}')$, and $\phi(\tau) = \phi(\tau_\alpha) \cap \phi(\tau_\beta) = \phi(\tau_\alpha) \setminus \phi(\alpha) = \phi(\tau_\beta) \setminus \phi(\beta)$ has codimension 1. Hence, one can pass from $\phi(\tau_\alpha)$ to $\phi(\tau_\beta)$ with one elementary move. We have necessarily $i(\phi(\alpha), \phi(\beta)) = 1$.

Let us adapt the argument for $A_\sharp(S, \mathbf{p})$. Let $\mathscr{V}$ be the set of all vertices of $A_\sharp(S, \mathbf{p})$ which correspond to simple closed loops around exactly one point in $\mathscr{S}$. It is easy to see that any maximal simplex σ of $A_\sharp(S, \mathbf{p})$ contains exactly s disjoint elements of $\mathscr{V}$. Now let $\alpha_1, \alpha_2 \in A_\sharp(S, \mathbf{p})$ be such that $i(\alpha_1, \alpha_2) = 1$. Notice that for each $v \in \mathscr{V}$ we have $i(v, \alpha) \neq 1$ for all $\alpha \in A_\sharp(S, \mathbf{p})$, so nor α_1 nor α_2 are elements in $\mathscr{V}$. Let us extend α_1, α_2 to maximal simplices $\sigma_{\alpha_1}, \sigma_{\alpha_2}$ such that $\sigma_{\alpha_2} = \langle \sigma_{\alpha_1} \setminus \alpha_1, \alpha_2 \rangle$ is the simplex spanned by $\sigma_{\alpha_1} \setminus \alpha_1$ and α_2. Let us define $\sigma_0 = \sigma_{\alpha_1} \cap \sigma_{\alpha_2}$, it is a simplex of codimension 1. Both $\phi(\sigma_{\alpha_1}) = \langle \phi(\sigma_{\alpha_0}), \phi(\alpha_1) \rangle$ and $\phi(\sigma_{\alpha_2}) = \langle \phi(\sigma_{\alpha_0}), \phi(\alpha_2) \rangle$ are maximal simplices in $A_\sharp(S, \mathbf{p})$. Now let us realize $\phi(\sigma_0)$ and look at its complement on S. Since $\phi(\sigma_0)$ has codimension 1, its complement contains at most one element of $\mathscr{V}$. If the complement contains exactly one element $v \in \mathscr{V}$, then we would have $v = \phi(\alpha_1) = \phi(\alpha_2)$, in contradiction with the injectivity: in fact the simplices $\phi(\sigma_{\alpha_1}), \phi(\sigma_{\alpha_2})$ being both maximal simplices, both of them have the same number s of elements of $\mathscr{V}$. Thus all the complementary regions of $\phi(\sigma_0)$ are open triangles except one open square which should contain both $\phi(\alpha_1)$ and $\phi(\alpha_2)$. We then conclude that $i(\phi(\alpha_1), \phi(\alpha_2)) = 1$. □

The following lemma, which gives a useful criterion to establish whether two automorphisms coincide or not, follows easily from the above Invariance Lemma.

Lemma 2.11. *Let $\phi_1, \phi_2 \in \mathrm{Aut}\ A(S, \mathbf{p})$. If there exists a maximal simplex $\sigma = \langle a_1, \dots, a_M \rangle$ in $A(S, \mathbf{p})$ such that $\phi_1(a_i) = \phi_2(a_i)$ for all $i = 1, \dots, M$, then $\phi_1(v) = \phi_2(v)$ for all $v \in A(S, \mathbf{p})$.*

2.3. **Proof of Theorem A.** The goal of this section is to state some Invariance Lemmas which will be used throughout the paper and to prove Theorem A.

Let us first recall some well-known definitions of simplicial topology that we will use in the rest of the paper (a classical reference is [4]).

Let K be a nonempty simplicial complex and let σ be one of its simplices. The *link* $\mathrm{Lk}(\sigma, K)$ of σ is the subcomplex of K whose simplices are the simplices τ such that $\sigma \cap \tau = \varnothing$ and $\sigma \cup \tau$ is a simplex of K. Let K_1 and K_2 be two simplicial complexes whose vertex sets V_1 and V_2 are disjoint. The *join of* K_1 *and* K_2 is a simplicial complex $K_1 \star K_2$ with vertex set $V_1 \cup V_2$; a subset of $V_1 \cup V_2$ is a simplex of $K_1 \star K_2$ if and only if it is a simplex of K_1, a simplex of K_2 or the union of a simplex of K_1 and a simplex of K_2. We have $\dim(K_1 \star K_2) = \dim K_1 + \dim K_2 + 1$. The *cone* $C(K)$ over K is the join of K with only one vertex $\{w_0\}$.

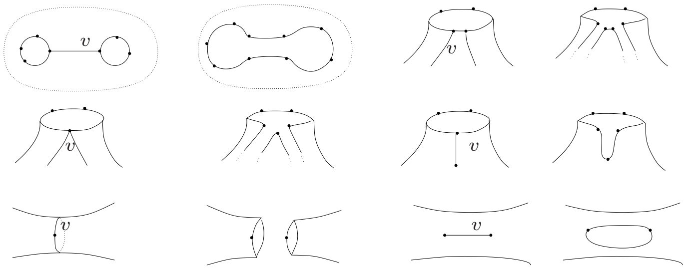

FIGURE 2. Cutting along v on S

It is important to remark that arc complexes of surfaces in this setting are "stable" under simplicial constructions. Indeed, the link of a simplex in the arc complex of a surface can be described in term of the arc complexes of surfaces (with boundary and marked points) of lower complexity. The link of a vertex is the (join of) arc complex(es) of the subsurface(s) obtained by cutting along the arc which corresponds to the vertex. Such subsurface(s) will have one or more boundary components (with new marked points on it) according on the geometric properties of the arc (separating, non-separating, joining one or two distinct boundary components, etc.). The same holds for the links of simplices. In Figure 2 we illustrate the new surface one gets cutting along the arc v and how to add marked points on its boundary components according to the type of v. When v is non-separating, $Lk(v)$ is isomorphic to the arc complex of the new surface. We can thus restate Lemma 2.11 in the following equivalent form:

Lemma 2.12. *Let* $v \in A(S^s_{g,b}, \mathbf{p})$ *be a vertex. If* $\phi, \psi \in \mathrm{Aut}\, A(S^s_{g,b}, \mathbf{p})$ *fix* v *and coincide on each vertex of* $\mathrm{Lk}(v)$*, then* $\phi = \psi$.

Let us now introduce some useful vocabulary.

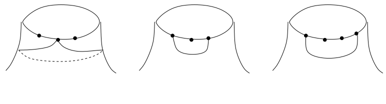

FIGURE 3. A 3-leaf, a 3-petal and a 4-petal

Let $S = (S^s_{g,b}, \mathbf{p})$, and $\mathscr{B}_i$ be the i-th boundary component of S. If $p_i \geq 4$ (resp. $p_i \geq 3$) we call a 4-*petal* (resp. a 3-*petal*) an arc which runs parallel to $\mathscr{B}_i$, joins two distinct marked points and bounds a disc containing exactly 4 (resp. 3) marked points on its boundary (see Figure 3). If $p_i \geq 2$, we call a p_i-leaf any loop based at one marked point and running parallel to $\mathscr{B}_i$. By the previous discussion we have for instance that if l is a p_1-leaf, then $\mathrm{Lk}(l) = A(S^0_{0,1}, (p_1 + 1)) \star A(S^s_{g,b}, (1, p_2, \ldots, p_b))$, if m is a j-petal around the first boundary component then $\mathrm{Lk}(m) = A(S^0_{0,1}, (j)) \star A(S^s_{g,b}, (p_1 - j + 2, p_2, \ldots, p_b))$.

The following remarks are immediate and very useful.

Remark 2.13. *The following holds:*

(1) $Lk(v, A(S,\mathbf{p}))$ *consists of two disjoint vertices if and only if* $(g,s,b,\mathbf{p}) = (0,1,0,(4))$.
(2) $Lk(v, A_\sharp(S,\mathbf{p}))$ *consists of two disjoint vertices if and only if* $(g,s,b,\mathbf{p}) \in \{(0,1,0,(4)),(0,1,1,(2))\}$.
(3) *Let* $\dim A_\sharp(S,\mathbf{p}) \geq 1$, *and let* v_1, v_2 *be two vertices in* $A(S,\mathbf{p})$. $Lk(v_1) = Lk(v_2)$ *as subsets of* $A(S,\mathbf{p})$ *if and only if* $v_1 = v_2$. *The same statement holds for* $A_\sharp(S,\mathbf{p})$.
(4) $A(S,\mathbf{p})$ *is a cone if and only if* $(g,s,b,\mathbf{p}) = (0,1,1,(1))$, *namely if* $A(S,\mathbf{p})$ *is a point.*
(5) *The join of two arc complexes is a cone if and only if one of the two arc complexes is* $A(S^1_{0,1},(1))$.

Remark 2.14. *The following holds:*

(1) $\mathrm{diam}A_\sharp(S^s_{g,b},(1_b)) \geq \mathrm{diam}A(S^s_{g,b},(1_b)) \geq \mathrm{diam}A(S_{g,b+s})$. *In particular, if* $\mathrm{diam}A(S_{g,b+s})$ *is infinite,* $\mathrm{diam}A_\sharp(S^s_{g,b},(1_b))$ *and* $\mathrm{diam}A(S^s_{g,b},(1_b))$ *are infinite as well.*
(2) *If* $\mathrm{diam}A(S_{g,b+s}) = \infty$, *then* $A(S^s_{g,b},\mathbf{p})$ *has infinite diameter or it contains a simplex* σ *with* $\mathrm{Lk}(\sigma) \cong A(S^s_{g,b},(1_b))$ *which has infinite diameter. The same statement holds for* $A_\sharp(S^s_{g,b},\mathbf{p})$.
(3) *If there exists* i *such that* $p_i \geq 5$, *then* $\mathrm{diam}A(S^s_{g,b},\mathbf{p}) = 2$.

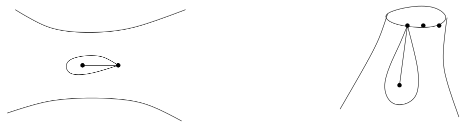

Figure 4. Lemma 2.16

For brevity, we will say that an arc l on $(S,\mathbf{p})$ is a *drop* if it is a simple loop based on a point bounding a disc with a puncture (see Figure 4). An edge $\langle l, v\rangle$ in $A(S,\mathbf{p})$ is an *edge-drop* if it is the arc complex of a once-punctured disc embedded in S as in Figure 4. An arc on $(S,\mathbf{p})$ is *properly separating* if it is separating and is not a 3-petal, a 2-leaf or a drop. The following remark directly follows from these definitions and easily implies the invariance lemma below.

Remark 2.15. *The following holds:*

(1) *an arc* v *is properly separating if and* $Lk(v) = A_1 \star A_2$ *where both* A_1 *and* A_2 *are two arc complexes with more than one vertex.*
(2) *an arc* l *is a drop if and only if* $Lk(l)$ *in* $\mathscr{A}(S^s_{g,b},\mathbf{p})$ *is a cone.*
(3) *an arc* v *is a 4-petal or a 3-leaf if and only if* $Lk(v) = A_1 \star A_2$, *with* A_1 *consisting of two disjoint vertices, and* A_2 *the arc complex of some surface.*

Invariance Lemma 2.16 (Separating arcs)**.** *Let us denote by* $\mathscr{A}(S^s_{g,b},\mathbf{p})$ *the arc complex* $A(S^s_{g,b},\mathbf{p})$ *or* $A_\sharp(S^s_{g,b},\mathbf{p})$. *Assume* $\dim\mathscr{A}(S^s_{g,b},\mathbf{p}) \geq 2$.
The following holds:

(1) *Let* $\phi : \mathscr{A}(S^s_{g,b},\mathbf{p}) \to \mathscr{A}(S^{s'}_{g',b'},\mathbf{p}')$ *be an isomorphism. If* l *is a properly separating arc, then* $\phi(l)$ *is a properly separating arc on* S'.
(2) *Let* $\phi : A(S^s_{g,b},\mathbf{p}) \to A(S^{s'}_{g',b'},\mathbf{p}')$ *be an isomorphism. If* l *is a drop, then* $\phi(l)$ *is a drop on* S'.
(3) *Let* $\phi : A(S^s_{g,b},\mathbf{p}) \to A(S^{s'}_{g',b'},\mathbf{p}')$ *be an isomorphism. If* $\langle l, v\rangle$ *is an edge-drop , then also the edge* $\langle\phi(l),\phi(v)\rangle$ *is an edge-drop.*

Proposition 2.17. *Let* $\mathscr{S}$ *be the set of marked points in the interior of* S, *with* $|\mathscr{S}| = s$. *we have:*

(1) *If* $A(S^s_{g,b},\mathbf{p})$ *is isomorphic to* $A(S^{s'}_{g',b'},\mathbf{p}')$, *then* $s' = s$.
(2) *If* α *is an arc joining two points in* $\mathscr{S}$, *then the image of* α *through an isomorphism is also an arc joining two points in* $\mathscr{S}$.

Proof. By Remark 2.16, drops are simplicial invariants. Since the maximal dimension of a simplex spanned by drops is s, isomorphic arc complexes have the same number of marked points in the interior, as stated in (1). In order to prove (2) it is sufficient to remark that the link of such an arc is the arc complex of a surface with $s-1$ or $s-2$ marked points in the interior. The conclusion follows from (1). □

Lemma 2.18. *The following holds:*

(1) *Let $K_1 = \{a, b\}$ (isomorphic to $A(S^0_{0,1}, (4))$ or $A_\sharp(S^1_{0,1}, (2))$) and K_2 is an arc complex of dimension at least 1.*
If K'_1 and K'_2 are arc complexes such that $K_1 \star K_2$ is isomorphic to $K'_1 \star K'_2$, then up to reordering K'_1 is isomorphic K_1 and K'_2 is somorphic K_2.

(2) *Let $K_1 = A(S^0_{0,2}, (1, 1)) = \mathbb{R}$ and K_2 be an arc complex with infinite vertices.*
If K'_1 and K'_2 are arc complexes such that $K_1 \star K_2$ is isomorphic $K'_1 \star K'_2$, then up to reordering K'_1 is isomorphic K_1 and K'_2 is isomorphic to K_2.

Proof. 1. The pair $\{a, b\}$ is the unique pair of vertices in $K_1 \star K_2$ whose links coincide. Let $\phi : K_1 \star K_2 \to K'_1 \star K'_2$ be an isomorphism; $\phi(a)$ and $\phi(b)$ are necessarily in the same K'_i (otherwise they would be connected by an edge). Since K'_i is an arc complex as well, it contains two vertices with the same link if and only if it is isomorphic to K_1 (Proposition 2.13).
2. Let $v \in K_1 = A(S^0_{0,2}, (1, 1)) = \mathbb{R}$, and let $\phi : K_1 \star K_2 \to K'_1 \star K'_2$ be an isomorphism. Assume that $\phi(v) \in K'_1$; we have

$$\{a, b\} \star K_2 = Lk(v, K_1 \star K_2) \cong Lk(\phi(v), K'_1 \star K'_2) = Lk(\phi(v), K'_1) \star K'_2.$$

Now $Lk(\phi(v), K'_1)$ is either isomorphic to a join of arc complexes $A_1 \star A_2$ or is an arc complex itself. An argument similar to the one used above allows us to exclude the first case. Thus $Lk(\phi(v), K'_1)$ is an arc complex. By (1) we conclude either $Lk(\phi(v), K'_1) \cong \{a, b\}$ (and $K'_2 \cong K_2$) or $K'_2 \cong \{a, b\}$ (and $Lk(\phi(v), K'_1) = K_2$. In both cases the conclusion follows from the application of (1). □

Invariance Lemma 2.19 (Petals)**.** *Let $\mathscr{A}(S^s_{g,b}, \mathbf{p})$ be $A(S^s_{g,b}, \mathbf{p})$ or $A_\sharp(S^s_{g,b}, \mathbf{p})$ of dimension at least 2, and $\phi : \mathscr{A}(S^s_{g,b}, \mathbf{p}) \to \mathscr{A}(S^{s'}_{g',b'}, \mathbf{p}')$ be an isomorphism.*
The following holds:

(1) *If l_1 is a 3-leaf, then $\phi(l_1)$ is a 3-leaf.*
(2) *If l_2 is a 4-petal, then $\phi(l_2)$ is a 4-petal.*
(3) *If l_3 is a 3-petal, then $\phi(l_3)$ is a 3-petal.*

Moreover, if l_i is based on a boundary component having exactly $p \geq 3$ marked points, $\phi(l_i)$ is also based on a boundary component with the same number of marked points, and $\sum_{p_i \geq 3} p_i = \sum_{p'_j \geq 3} p'_j$.

Proof. 1. and 2. By Remark 2.15 (3) we just have to prove that in the case $A(S^s_{g,b}, \mathbf{p})$ contains both a 3-leaf l_1 and a 4-petal l_2, $\phi(l_1)$ cannot be a 4-petal and $\phi(l_2)$ cannot be a 3-leaf of $A(S^{s'}_{g',b'}, \mathbf{p}')$.
Without loss of generality, we assume that l_1 is based on the first boundary component $\mathscr{B}_1$, and l_2 is based on $\mathscr{B}_2$ (resp. $p_1 = 3$ and $p_2 \geq 4$), namely $Lk(l_1, A(S^s_{g,b}, \mathbf{p})) \cong A(S^0_{0,1}, (4)) \star A(S^s_{g,b}, (1, p_2, \ldots, p_b))$ and $Lk(l_2, A(S^s_{g,b}, \mathbf{p})) \cong A(S^0_{0,1}, (4)) \star A(S^s_{g,b}, (3, p_2 - 2, \ldots, p_b))$.
Let ρ_1, ρ_2 be respectively the $\frac{2\pi}{3}$-rotation around $\mathscr{B}_1$ and the $\frac{2\pi}{p_2}$-rotation around $\mathscr{B}_2$. We remark that for any $i = 0, 1, 2$ the arcs represented by $\rho_1^i(l_1)$ and $\rho_2^i(l_2)$ are respectively a 3-leaf and a 4-petal. The intersection patterns for these families of arcs are $i(\rho_1^i(l_1), \rho_1^{j\pm1}(l_1)) = 2\delta_{ij}$ for $i, j = 0, 1, 2$, and $i(\rho_2^h(l_2), \rho_2^{k\pm1}(l_2)) = \delta_{hk}$ for $h, k = 0, \ldots, p_2 - 1$.
By the simplicial invariance of their intersection number (Invariance Lemma 2.10), we deduce that the arcs $\{\phi(\rho_2^j(l_2))\}_{j=0,\ldots p_2-1}$ are all based on the same boundary component of S', and the arcs are all of the same type (i.e. either they are all 3-leaves or they are all 4-petals). Since $p_2 \geq 4$, we deduce that they are necessarily 4-petals, hence $\phi(l_1)$ is necessarily a 3-leaf.

3. Remark that for every 3-petal l_3 based on the i-th boundary component there exists a 4-petal (or a 3-leaf, in the case $p_i = 3$) l_4 based on that same component such that $\mathrm{Lk}(l_4, A(S, \mathbf{p})) = \{l_3, \rho_i(l_3)\} \star A(S^{s'}_{g',b'}, \mathbf{p}') \cong A(S_{0,1,0}, (4)) \star A(S^{s'}_{g',b'}, \mathbf{p}')$. By Lemma 2.18 and the previous case, we deduce that $\phi(l_4)$ is also a 4-petal (or a 3-leaf when $p_i = 3$), and the same holds for all the $\rho_i^j(l_4)$'s as well. The number p_i of points on the i-th boundary component of S is necessarily equal to the number of 3-petals based on it. Since $i(\rho^j(l_3), \rho^{j\pm1}(l_3)) = 1$ for all $j = 0, \ldots, p_i - 1$, our conclusion follows by simpliciality as in the previous case.
The last statement follows directly from the arguments used here. □

Following the usual definition, we say that a *non-separating* arc on S is an arc which does not disconnect the surface.

Invariance Lemma 2.20 (Non-separating arcs)**.** *Let $\mathscr{A}(S^s_{g,b}, \mathbf{p})$ be $A(S^s_{g,b}, \mathbf{p})$ or $A_\sharp(S^s_{g,b}, \mathbf{p})$, and assume* $\dim\mathscr{A}(S^s_{g,b}, \mathbf{p}) \geq 2$. *Let $\phi : \mathscr{A}(S^s_{g,b}, \mathbf{p}) \to \mathscr{A}(S^{s'}_{g',b'}, \mathbf{p}')$ be an isomorphism. The following holds:*

1. *if v is a non-separating arc, then $\phi(v)$ is also a non-separating arc;*
2. *if w is a 2-leaf, then also $\phi(w)$ is a 2-leaf.*

Proof. Without loss of generality, we assume that v joins the first and the second boundary component, hence $Lk(v) = A(S^s_{g',b-1}, (p_1 + p_2 + 2, p_3, \ldots, p_b))$. By Lemma 2.19 $\phi(v_1)$ is either a non-separating arc or a 2-petal. Now by simpliciality we have $Lk(\phi(v)) \cong A(S^s_{g',b-1}, (p_1 + p_2 + 2, p_3, \ldots, p_b))$, with $p_1 + p_2 + 2 \geq 4$. If $\phi(v)$ were a 2-petal, $Lk(\phi(v)) = A(S^s_{g',b}, \mathbf{p}')$, with $p'_i = 1$ for some i and $p'_j = p_j$ for all $j \neq i$. Now $\sum_{p'_h \geq 3} p'_h = \sum_{p_h \geq 3} p_h < p_1 + p_2 + 2 + \sum_{p_h \geq 3, h \geq 3} p_h$ in contradiction with Invariance Lemma 2.19. The same argument also proves that $\phi(w)$ is necessarily a 2-leaf. □

The arguments in Lemma 2.19 easily prove the following:

Corollary 2.21. *Let $\phi \in \mathrm{Aut}\ A(S, \mathbf{p})$ be an automorphism. the following holds:*

1. *For every boundary component $\mathscr{B}$ of S there exists $f \in \mathrm{MCG}^\star(S, \mathbf{p})$ such that $f_\star \circ \phi$ fixes every 3-petal (or every 2-leaf) on $\mathscr{B}$.*
2. *If $f \in \mathrm{MCG}^*(S, \mathbf{p})$ fixes two intersecting 3-petal (or 2-leaves), then ϕ fixes every 3-petal (or 2-leaves).*

Invariance Lemma 2.22 (Leaves)**.** *Let $\mathscr{A}(S^s_{g,b}, \mathbf{p})$ be $A(S^s_{g,b}, \mathbf{p})$ or $A_\sharp(S^s_{g,b}, \mathbf{p})$. Let l be an n-leaf on $\mathscr{A}(S^s_{g,1}, (n))$, then $\phi(l)$ is an n-leaf.*

Proof. Notice that there exists a unique 3-petal v which intersects l, and there is no non-separating arc α such that $i(\alpha, l) = i(\alpha, v) = 0$. By simpliciality and Lemma 2.20, the same properties hold for $\phi(l)$. By Lemma 2.16 $\phi(l)$ is a separating loop. If both the connected components bounded by $\phi(l)$ were different from $(S_{0,1,0}, (n+1))$, there would be a non-separating arc disjoint from both the 3-petal $\phi(v)$ and $\phi(l)$, and we would get a contradiction. □

When $p_i = 1$ for all $i = 1, \ldots, b$, we will use the notation (1_b) or $\mathbf{1}$ to refer to the vector $\mathbf{p} = (1, \ldots, 1)$. We say that an edge $\langle l, v \rangle$ of $A(S^s_{g,b}, \mathbf{1})$ is an *edge bridge* if v corresponds to a non-separating arc connecting two distinct boundary component and l corresponds to a separating loop wrapping around v as in Figure 5. We remark that $Lk(l) \cong A(S^0_{0,2}, (1,1)) \star A(S^s_{g,b}, (2, 1_{b-1})) \cong \mathbb{R} \star A(S^s_{g,b}, (2, 1_{b-1}))$, and v is a vertex of $A(S^0_{0,2}, (1,1))$. The following lemma is an immediate application of Lemma 2.18 (2).

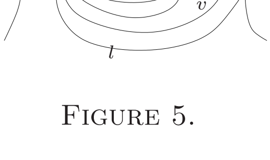

Figure 5.

Invariance Lemma 2.23. *Let $\langle l, v\rangle$ be an edge bridge of $A(S^s_{g,b}, \mathbf{1})$, and $\phi : A(S^s_{g,b}, \mathbf{1}) \to A(S^{s'}_{g',b'}, \mathbf{1})$ an isomorphism. The edge $\phi(\langle l, v\rangle)$ is an edge bridge as well.*

It is now immediate to deduce Theorem A.

Theorem A. *If $A(S^s_{g,b}, \mathbf{p})$ is not empty and isomorphic to $A(S^{s'}_{g',b'}, \mathbf{p}')$, then $s = s'$, $b = b'$, $g = g'$ and $p_i = p'_i$ for all i.*

Proof. The proof in the low dimensional cases follows by Remark 2.5. The equality $s = s'$ has already been proved in Proposition 2.17. The equality $p_i = p'_i$ for all i follows immediately from Lemmas 2.19 and 2.20. To prove $b = b'$ it is sufficient to remark that by Lemma 2.23 the maximal number of disjoint edge-drops on a surface depends only on b and is a simplicial invariant. Finally $g = g'$ follows by the equalities between the dimensions of the arc complexes in the hypothesis of this theorem. □

3. Proof of Theorem B

In this section we deal with the proof of Theorem B. Invariance lemmas developed in Section 2 imply some suitable reduction lemmas, which allow an inductive approach to the proof of theorem B. The structure of the section is the following: in Subsection 3.1 we deal with the case $g = 0$, in Subsection 3.2 we deal with the cases $b = 0$ and $b = 1$, and in Subsection 3.3 we prove the reduction lemmas and complete the proof of Theorem B.

By sake of brevity, we will introduce the following definitions.

Definition 3.1. *Let $(S^s_{g,b}, \mathbf{p})$ be a surface such that its arc complex $A(S^s_{g,b}, \mathbf{p})$ is not empty. We say that $A(S^s_{g,b}, \mathbf{p})$ is* rigid *if its automorphism group* Aut $A(S^s_{g,b}, \mathbf{p})$ *is isomorphic to the mapping class group* $\mathrm{MCG}^*(S^s_{g,b}, \mathbf{p})$. *We shall say that $A(S^s_{g,b}, \mathbf{p})$ is* weakly rigid *if the natural homomorphism* $\mathrm{MCG}^*(S^s_{g,b}, \mathbf{p}) \to \mathrm{Aut}\ A(S^s_{g,b}, \mathbf{p})$ *is surjective.*

We will show later that when there is enough topology, the two notions of rigidity and weak rigidity are equivalent.

3.1. Basic case: genus 0.

Here we prove Theorem B for some genus 0 surfaces.

Figure 6. Fans and chords

3.1.1. *Polygon $(S^0_{0,1}, (n))$, with $n \geq 4$.* Let $(S^0_{0,1}, (n))$ be a polygon with a set $\mathscr{P} = \{P_0, \ldots, P_{n-1}\}$ of $n \geq 4$ marked points on its boundary (enumerated with respect to the order induced by the orientation of ∂S), and we denote by $\rho_{\frac{2\pi}{n}}$ the rotation of angle $\frac{2\pi}{n}$ around the first boundary component. We recall the following is a well-known fact. A proof can be found for instance in [3].

Theorem 3.2. *$A(S^0_{0,1}, (n))$ is PL-homeomorphic to $\mathbb{S}^{n-4}$.*

For any point $P \in \mathscr{P}$, we define the *fan based in P* to be the triangulation F_P as in Figure 6, and we define the *chord* based at P the 3-petal c_P joining the two marked points adjacent to P. If P is the i-th point in $\mathscr{P}$, we shall also use the notation c_i when referring to c_P.
It is immediate to remark that a triangulation T is a fan if and only if there exists $P \in \mathscr{P}$ such that $Lk(c_P, A(S, (n))) \cap T = \varnothing$. If $\mathscr{C} = \{c_i\}$ is the set of chords of $(S^0_{0,1}, (n))$, then according to

our notation $i(c_i, c_{i\pm1}) = 1$ for $i = 0, \ldots, n-1$ and $i(c_i, c_j) = 0$ for $|i-j| \neq 1$. Let $F_{P_0} = \{\gamma_i\}$ be the fan based at P_0, where γ_i is the arc which connects P_0 to the i-th point of $\mathscr{P}$. According to our notation, we have $i(c_i, \gamma_j) = \delta_{ij}$ and $i(c_0, \gamma_j) = 1$ for all γ_j. Since these intersection patterns are simplicial invariant by Lemma 2.10, we easily deduce the following:

Lemma 3.3. *Let $\phi : A(S^0_{0,1}, (n)) \to A(S^0_{0,1}, (n))$ be an automorphism. The following holds:*

1. *If $\mathscr{C}$ is the set of chords of S, then $\phi(\mathscr{C}) = \mathscr{C}$, and ϕ either preserves or reverses the cyclic order of the chords.*
2. *Let $F_P = \{\gamma_i\}$ be the fan based at P, where γ_i is the arc which connects P to the i-th point of $\mathscr{P}$ according to the (cyclic) order on $\mathscr{P}$. There exists $P' \in \mathscr{P}$ such that the triangulation $\phi(F_P) = \{\phi(\gamma_i)\}$ is a fan triangulation $F_{P'}$. The map ϕ either preserves or reverses the order of arcs in F_P.*

The following holds:

Theorem 3.4 (Weak rigidity of polygons)**.** *For $n \geq 4$, $A(S^0_{0,1}, (n))$ is weakly rigid.*

Proof. By Lemma 2.11 it is sufficient to prove that if F_P is a fan and $\phi(F_P)$ is its image through ϕ, then there exists a homeomorphism $\tilde{\phi}$ such that $\tilde{\phi}_*$ agrees with ϕ on each arc of F_P.
Up to precomposition with a rotation $\rho_{\frac{2\pi}{n}}$, we assume $\phi(F_P) = F_P$. By Lemma 3.3, the order of arcs in F_P is either preserved or reversed. Up to precomposition with a reflection, we can assume that ϕ preserves the order of arcs in F_P. Up to isotopies, we also assume that ϕ fixes each arc pointwise. By Lemma 2.12, we can conclude by extending ϕ to a homeomorphism of the disc by the identity on the inner triangles.

□

3.1.2. *Annuli.* In the following section we shall study the annuli $(S^0_{0,2}, (p_1, p_2))$. We denote by ρ_1 and ρ_2 the two rotations (respectively of $\frac{2\pi}{p_1}$ and $\frac{2\pi}{p_2}$) around the two boundary components of S, and by i the inversion which exchanges the two boundary components of the surface.

Example 3.5 (Annulus $(S^0_{0,2}, (1, 1))$)**.**

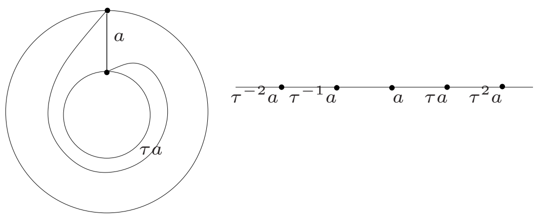

Figure 7. Annulus

If a is an arc as in Figure 7, then $\mathrm{MCG}^*(S^0_{0,2}, (1, 1))$ is generated by $\langle \tau, r, i \rangle$, where τ is the Dehn twist along the core curve of the annulus , r is the reflection with respect to a, and i is the inversion which exchanges the two boundary components of S. Since for any arc α in $A(S^0_{0,2}, (1, 1))$ we have $i(\alpha, \tau\alpha) = 0$, $A(S^0_{0,2}, (1, 1))$ is isomorphic to the real line.
Notice that the natural homomorphism $\mathrm{MCG}^*(S^0_{0,2}, (1, 1)) \to \mathrm{Aut}\, A(S^0_{0,2}, (1, 1))$ is surjective but not injective: r and i have the same image.

Example 3.6 (Annulus $(S^0_{0,2}, (1, 2))$)**.**

Let τ be the Dehn twist around the core of the annulus, let ρ be the π-rotation which exchanges the two marked points and let r be the reflection which fixes the three marked points. It is easy to see that the group $\mathrm{MCG}^*(S^0_{0,2}, (1, 2))$ is generated by the elements τ, ρ, r.
Let a, a' be arcs as in Figure 8. Let l be the loop around the upper point on the outer boundary component of S, and let l' be the loop around the lower point of the boundary component of S.

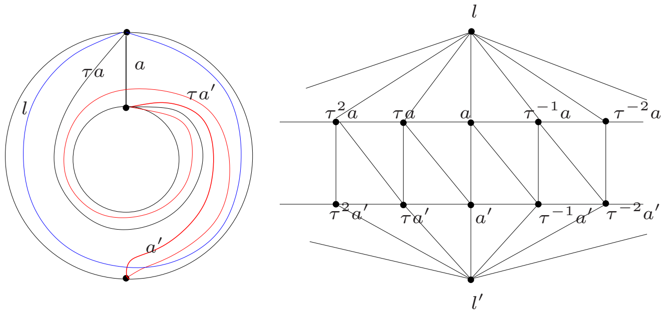

FIGURE 8.

It is not too difficult to see that the complex $A(S^0_{0,2},(1,2))$ looks like the strip in Figure 8. It is not too difficult to use this configuration to deduce directly that the natural homomorphism $\mathrm{MCG}^*(S^0_{0,2},(1,2)) \to \mathrm{Aut}\ A(S^0_{0,2},(1,2))$ is surjective.

Theorem 3.7 (Annuli)**.** *For any $p_1, p_2 \in \mathbb{N}$, $A(S^0_{0,2},(p_1,p_2))$ is weakly rigid. If $p_1 = p_2 = 1$, $A(S^0_{0,2},(p_1,p_2))$ is not rigid.*

Proof. Let a be an arc joining the two boundary components. Let ϕ be an automorphism of $A(S^s_{0,2},(p_1,p_2))$. By Lemma 2.20, we can assume $\phi(a) = a$ and by Lemma 2.21 we can assume that ϕ fixes every 3-leaf in the first boundary component. Cutting the surface along a, we find a new surface $(S^s_{0,1},(p_1+p_2+2))$. The map ϕ induces by restriction an automorphism $\phi_|$ which fixes at least two intersecting 3-petals. By Lemma 2.21, $\phi_|$ fixes any other 3-petal. By Theorem 3.4, $\phi_|$ is induced by a homeomorphism of the surface, which restricts to the identity on the boundary. We can just glue back the two pieces of the boundary coming form the cut along a and get a homeomorphism of the annulus, which induces ϕ by Lemma 2.12.

To prove the second statement just notice that if $p_1 = p_2 = 1$, then r and i have the same image. □

3.2. **Basic case:** $b = 1$**.** Here we prove Theorem B for surfaces with one boundary component. Let us first work on the pair $(S^0_{g,1},(1)) = (S^0_{g,1},P)$ and we denote by P the unique marked point on the boundary of S. This subsection is structured as follows: in Paragraph 1 we shall study the properties of a natural forgetful map between $A(S^0_{g,1},(1))$ and $A(S^1_g)$; in Paragraph 2 we shall introduce a useful reduction lemma and prove Theorem B for surface with $b = 1$; in Paragraph 3 we shall deal with the case $b = 0$, providing a new proof of a result by Irmak-McCarthy.

3.2.1. *The forgetful map.* Recall that the Dehn-twist τ around the boundary of S is not the identity in $\mathrm{MCG}^*(S^s_{g,1},P)$. Let a be a simple closed arc based at P on S, and let $a^-, a^+ = \tau a^-$ be the arcs obtained from a twisting only one of its two endpoints (see Figure 9). The natural inclusion $(S^s_{g,1},P) \to S^s_{g,1}$, which "forgets" about P, induces a natural *forgetful* map $f : A(S^s_{g,1},P) \to A(S^s_{g,1}) \cong A(S^{s+1}_g)$, where the vertex $[a]_P \in A(S,P)$, which corresponds to a, is mapped to the corresponding vertex $[a] \in A(S^s_{g,1})$ (forgetting about P). We remark that $f([\tau^n a]_P) = f([\tau^n a^-]_P) = f([\tau^n a^+]_P)$.

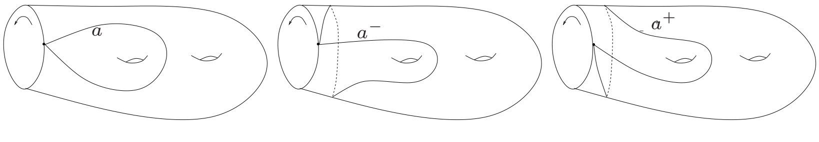

FIGURE 9.

Lemma 3.8. *Let $f : A(S,P) \ni [a]_P \to [a] \in A(S)$ be the natural forgetful map. The following holds:*

(1) *f is well-defined and surjective, and for every $[a] \in A(S)$, $f^{-1}([a])$ is a 1-dimensional simplicial complex isomorphic to $\mathbb{R}$.*
(2) *If $\phi \in \mathrm{Aut}\,(A(S,P))$ is an automorphism induced by an element of $\mathrm{MCG}^*(S,P)$, then for every $a \in A(S,P)$ the restriction of ϕ is an isomorphism: $\phi_| : f^{-1}([a]) \to f^{-1}([\phi(a)])$. Moreover there is a well-defined simplicial map $f(\phi) : A(S) \ni [a] \mapsto A(S) \ni f([\phi(a)])$ which is also an automorphism.*
(3) *If $\tau : (S,P) \to (S,P)$ is the Dehn twist around ∂S, then $\tau_* : A(S,P) \to A(S,P)$ is a 2-translation on all fibers $f^{-1}([a])$, and $f(\phi) : A(S) \to A(S)$ is the identity.*

The following two lemmas clarify some of the properties of f.

Lemma 3.9. *Let $\sigma : A(S,P) \to A(S,P)$ be an automorphism such that $f(\sigma) : A(S) \to A(S)$ is well-defined and is the identity. Then, either σ is the identity $id_{A(S,P)}$ or σ is induced by a power τ^k of a Dehn twist around ∂S.*

Proof. Claim 1: There does not exist $[a] \in A(S)$ such that $\sigma_| : f^{-1}([a]) \to f^{-1}([a])$ is a 1-translation.

By contradiction, let $[a] \in A(S)$ be such an element. Let us fix a hyperbolic metric on S such that the boundary of S is geodesic. Remember that any vertex of $A(S)$ has exactly one geodesic representative in its isotopy class; geodesic representatives always intersect each other minimally and are always transverse to the boundary. Let $\bar{a}$ be geodesic representative for $[a]$.

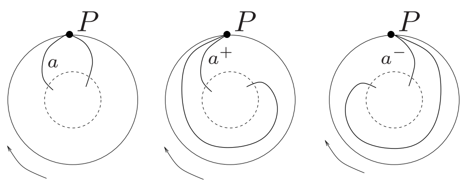

FIGURE 10. a, a^+, a^-

We can then define "preferred" classes $a, a^+, a^- \in A(S,P)$ just taking the relative isotopy classes of the loops obtained joining the endpoints of $\bar{a}$ to P as in Figure 10. Remark that $\tau a^- = a^+$. Moreover, we can similarly define a "base point" b on every other fiber $f^{-1}([b])$, $[b] \in Lk([a], A(S))$ and describe completely the links between the fibers $f^{-1}([a])$ and $f^{-1}([b])$.

We have then 3 cases: Figures 11, 12, 13.

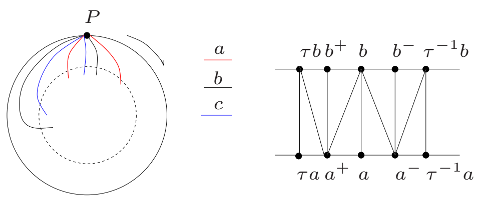

FIGURE 11. Case 1

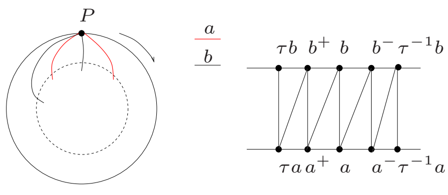

FIGURE 12. Case 2

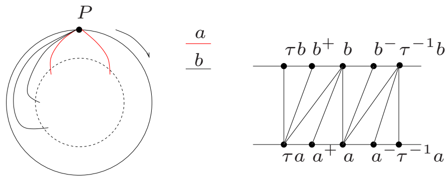

FIGURE 13. Case 3

Notice that in Case 1 we have $|Lk(a, A(S,P)) \cap f^{-1}([b])| = 1$ and $|Lk(a^{\pm}, A(S,P)) \cap f^{-1}([b])| = 3$. This is enough to prove Claim 1: for if σ restricts to a 1-translation on $f^{-1}([a])$, then $a \mapsto a^{\pm}$ and $|Lk(a, A(S,P)) \cap f^{-1}([b])| = |Lk(\sigma(a), A(S,P)) \cap \sigma(f^{-1}([b]))| = |Lk(a^{\pm}, A(S,P)) \cap f^{-1}([b])|$.

Claim 2: There does not exist $[a] \in A(S)$ such that $\sigma_| : f^{-1}([a]) \to f^{-1}([a])$ is a reflection.

We will prove the claim by contradiction. In the same setting of the proof of the previous claim, the simplicial definition of the reflection of $f^{-1}([a])$ is the following:

$$\rho_a : \begin{cases} a & \mapsto & a \\ a^- & \mapsto & \tau^{-1}a^- \\ \tau^k a & \mapsto & \tau^{-k}a & \text{for } k \in \mathbb{Z} \\ \tau^k a^- & \mapsto & \tau^{-k-1}a^- & \text{for } k \in \mathbb{N} \end{cases}$$

Now assume that $\sigma : A(S,P) \to A(S,P)$ extends $\rho_a : f^{-1}([a]) \to f^{-1}([a])$. Recall from the proof of the previous claim that for every $[b] \in Lk([a], A(S,P))$ the fibers $f^{-1}([a])$ and $f^{-1}([b])$ can be linked in three different ways (Figures 11, 12, 13). It is not difficult to verify that:

- $\sigma_| = \rho_b : f^{-1}([b]) \to f^{-1}([b])$ in Case 1;
- $\sigma_| = \sigma_b \circ \rho_b : f^{-1}([b]) \to f^{-1}([b])$ in Case 2;
- $\sigma_| = \sigma_b \circ \rho_b \circ \sigma_b^{-1} : f^{-1}([b]) \to f^{-1}([b])$ in Case 3.

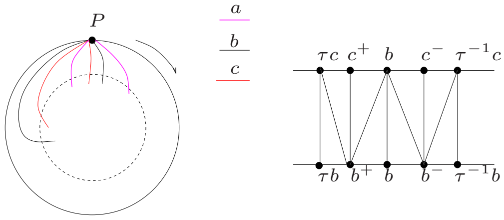

FIGURE 14. Simplicial relations between $f^{-1}([b])$ and $f^{-1}([c])$

Let now $[b]$, $[c] \in A(S)$ such that $\langle [a], [b], [c] \rangle$ is a 2-simplex in $A(S)$ and they both are in Case 2. It is not difficult to verify that the simplicial relations between $f^{-1}([b])$ and $f^{-1}([c])$ in Figure 14 are not compatible with the definitions of $\sigma_| : f^{-1}([b]) \to f^{-1}([b])$ and $\sigma_| : f^{-1}([c]) \to f^{-1}([c])$. We get to a contradiction, and this proves Claim 2.

By Claim 1 and 2, σ must coincide with some τ^{k_a} on each fiber $f^{-1}([a])$. The discussion in Claim 1 about the connections between the fibers $f^{-1}([a])$ and $f^{-1}([b])$ when $\langle [a], [b] \rangle$ is an edge on $A(S)$ and the connectedness of $A(S,P)$ proves that k_a is the same for all the fibers. □

Lemma 3.10. *Let $g \geq 1$. If $\phi : A(S,P) \to A(S,P)$ is a simplicial automorphism, then $f(\phi) : A(S) \ni [a] \mapsto f([\phi(a)]) \in A(S)$ is well-defined and it is an automorphism.*

Proof. Remark that if $\langle a, b \rangle$ is an edge of $A(S,P)$, then either $\langle f(a), f(b) \rangle$ is an edge in $A(S)$ or $f(a) = f(b)$ and $b = a^{\pm} \in f^{-1}([a])$ according to the above description of the fiber of $[a] \in A(S)$. Moreover if $\langle a_1, \ldots a_M \rangle$ is a maximal simplex in $A(S,P)$ (that is, if it corresponds to a triangulation of (S,P)) then the set $\{f(a_1), \ldots, f(a_M)\}$ spans a maximal simplex in $A(S)$, and there are exactly two indices $i \neq j$ such that $f(a_i) = f(a_j)$ (that is $a_j = a_i^{\pm}$).

By contradiction now assume that there exists $\phi \in \text{Aut } A(S, P)$ such $f(\phi)$ is not well defined or simplicial. Hence, there are two cases:

(1) there exists an edge $\langle a, b\rangle \in A(S, P)$ such that $\langle f(a), f(b)\rangle$ is an edge in $A(S)$, but $f(\phi(a)) = f(\phi(b)) \in A(S)$;
(2) there exists an edge $\langle a, a^{\pm}\rangle \in A(S, P)$ such that $f(a) = f(a^{\pm})$, $f(\phi(a^{\pm})) \neq f(\phi(a))$ and $\langle f(\phi(a^{\pm})), f(\phi(a))\rangle$ is an edge in $A(S)$.

Claim 1: Let $\langle a, b\rangle$ be an edge of $A(S, P)$ as in the case 1. Then there does not exist $c \in A(S, P)$ such that $\langle a, b, c\rangle$ is a 2-simplex in $A(S, P)$, $\langle f(a), f(b), f(c) >$ is a 2-simplex in $A(S)$ and $f(\phi(a)) = f(\phi(b)) = f(\phi(c))$.

By contradiction, let c be such a vertex, and let δ_{abc} be a maximal simplex in $A(S, P)$ which extends the 2-simplex $\langle a, b, c\rangle$. By simpliciality $\phi(\delta_{abc})$ is a maximal simplex in $A(S, P)$ which contains the simplex $\langle \phi(a), \phi(b), \phi(c)\rangle$, and $f(\phi(\delta_{abc}))$ spans a maximal simplex in $A(S)$. Then by the previous remark, at most two elements in the set $\{f(\phi(a)), f(\phi(b)), f(\phi(c))\}$ can coincide.

Claim 2: Let $\langle a, b\rangle$ be an edge as in the case 1. Then $\langle a, a^{\pm}\rangle$ spans an edge of $A(S, P)$ as in the case 2.

Consider the 2-simplex $\langle a, a^{\pm}, b\rangle$ and extend it to a maximal simplex $\delta_{aa^{\pm}b}$ of $A(S, P)$. Notice that Then $\phi(\delta_{aa^{\pm}b})$ is a maximal simplex of $A(S, P)$, and by the above remark exactly two of its vertices must have the same image through f. Now it follows from the hypothesis that if $f(\phi(a)) = f(\phi(b))$, then necessarily $f(\phi(a)) \neq f(\phi(a^{\pm}))$, and $\langle a, a^{\pm}\rangle$ is an edge of $A(S, P)$ in the case 2.

Claim 3: Let $\langle a, a^{\pm}\rangle$ be an edge as in the case 2, and let $\delta_{aa^{\pm}}$ be a maximal simplex of $A(S, P)$ extending it. Then $\delta_{aa^{\pm}}$ contains a unique vertex b^{δ} such that $\langle a, b^{\delta}\rangle$ is an edge as in the case 1.

By simpliciality $\phi(\delta_{aa^+})$ is a maximal simplex in $A(S, P)$. It follows from the hypothesis that $f(\phi(a)) \neq f(\phi(a^+))$, then by the above remark there exists $b \in \delta_{aa^+}$ such that $f(\phi(b)) = f(\phi(a))$. Now $f(a) = f(a^+)$, then by Claim 1 necessarily $f(b) \neq f(a)$. The uniqueness of b follows from the same argument.

Without loss of generality we can assume that $\langle a, a^+\rangle$ is an edge as in the case 2 (Claim 2 guarantees that such an edge exists).

In the genus 1 case the proof is direct. Remark that in $(S^0_{1,1}, (1))$ there is only one orbit of arcs through the action of the mapping class group. Up to precomposing with a simplicial automorphism induced by a mapping class, we can assume $\phi(a) = a$. The map ϕ restricts to a simplicial automorphism of the annulus $(S^0_{0,2}, (1, 2))$ obtained by cutting S along a. We remark that the two arcs a^+ and a^- correspond to the two 2-leaves of the annulus. By Lemma 2.20, ϕ preserves the set of 2-leaves, hence $\phi(a^+) \in \{a^+, a^-\}$ and $f(\phi(a^+)) = f(a)$, we get to a contradiction.

Let us now focus on the case $g \geq 2$. Let $\delta^1_{aa^+}$ be a maximal simplex of $A(S, P)$ extending $\langle a, a^+\rangle$. Let b^1 be the unique vertex in $\delta^1_{aa^+}$ as in Claim 3. Now flip $\delta^1_{aa^+}$ on b^1, and let $\delta^2_{aa^+}$ be the new triangulation and b^2 be the new side. By Claims 1 and 3 the edge $\langle a, b^2\rangle$ is necessarily as in the case 1. Now since $g \geq 2$ the situation looks like in Figure 15 and b^2 bounds a triangle on (S, P) where at least one of the other two sides is neither a nor a^+. Performing another flip on that arc, we find another maximal simplex $\delta^3_{aa^+}$ still containing a, a^+ and b^2. Now flipping again on b^2, we obtain a new maximal simplex $\delta^4_{aa^+}$ containing a, a^+ and a new arc b^3 (not contained in $\delta^3_{aa^+}$). By Claim 1 and 3, the edge $\langle a, b^3\rangle$ is in the case 1, and $\langle b^1, a, b^3\rangle$ spans a 2-simplex (see again 15), but this contradicts Claim 1.

□

We can summarize the results of the previous lemmas in the following proposition.

Proposition 3.11. *The forgetful map $f : A(S, P) \to A(S)$ is simplicial an induces a homomorphism $f_* : \text{Aut } A(S, P) \to \text{Aut } A(S)$ whose kernel is generated by Dehn twists around ∂S.*

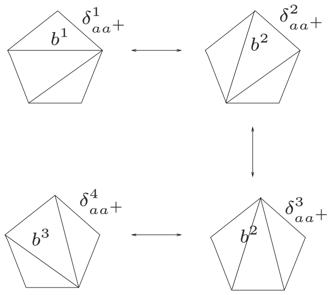

FIGURE 15.

3.2.2. *Proof of Theorem B for $b = 1$.* Let us now complete the proof of Theorem B for surfaces with one boundary component. We will prove another Reduction Lemma, which will be used together with Reduction Lemma 3.15 as a key ingredient in our proof.

Lemma 3.12. *Let $S^0_{g,1}$ be a surface of genus $g \geq 1$ with one boundary component. If $A(S^0_{g,1})$ is weakly rigid, then also $A(S^0_{g,1}, P)$ is weakly rigid.*

Proof. Let $\phi \in \text{Aut } A(S^0_{g,1}, P)$ be an automorphism. By Lemma 3.10 $f(\phi) \in \text{Aut } A(S_{g,1})$, and it follows from the hypothesis that there exists a mapping class $\text{MCG}^*(S_{g,1})$ which induces ϕ. Let $\bar{\phi} : (S, P) \to (S, P)$ be a homeomorphism in such a class and let $\bar{\phi}_* : A(S, P) \to A(S, P)$ be the induced map. We have $id = f(\bar{\phi}_*^{-1} \circ \phi) : A(S_{g,1}) \to A(S_{g,1})$; then there exists $k \in \mathbb{Z}$ such that $\bar{\phi}_*^{-1} \circ \phi = \tau_*^k$, hence ϕ is induced by $\phi \circ \tau^k$. □

The following proposition can be also regarded as a particular case of a theorem of Irmak and McCarthy [5]. We will postpone its proof to the next section.

Proposition 3.13. *If $S^0_{g,1}$ is a compact orientable surface of genus $g \geq 2$ with one boundary component, then the natural homomorphism $\text{MCG}^*(S^0_{g,1}) \to \text{Aut } A(S^0_{g,1})$ is surjective.*

An easy application of Lemma 3.12 and the previous proposition proves the following:

Proposition 3.14. *Let $(S^0_{g,1}, (n))$ be a surface of genus $g \geq 1$ with one boundary component and n marked points on it. Then $A(S^0_{g,1}, (n))$ is weakly rigid.*

Proof. The case $n = 1$ easily follows from Lemma 3.12.
Now let us use an inductive argument. By Lemma 2.21, we can assume that ϕ fixes every 3-petal (or 2-leaf). Let v be a 3-petal (or 2-leaf), cutting S along v we find two surfaces $(S^0_{g,1}, (1))$ and $(S^0_{0,1}, (3))$, and ϕ induces an automorphism $\phi_|$ of the arc complex of $(S^0_{g,1}, (n-1))$. By induction $\phi_|$ is induced by a homeomorphism ϕ_1 of $(S^0_{g,1}, (n-1))$ which fixes every point on the boundary. Lemma 2.12 ensures that the homeomorphim obtained by glueing ϕ_1 to a suitable homeomorphism of $(S^0_{0,1}, (3))$ induces ϕ on the whole $A(S^0_{g,1}, (n))$. □

Here is the first Reduction Lemma.

Reduction Lemma 3.15. *For any $s \geq 0$ and $g \geq 0$, if $A(S^0_{g,1}, (1))$ is weakly rigid, then also $A(S^s_{g,1}, (1))$ is weakly rigid.*

Proof. Let $\phi \in \text{Aut } A(S^s_{g,1}, (1))$ be an automorphism. For every $i = 1, \ldots, b$, let $\langle l_i, v_i \rangle$ be an edge as in Lemma 2.16 (3), corresponding to the i-th puncture. Without loss of generality, we can assume that the set of all pairs $\{l_i, v_i\}_i$ spans a simplex on $A(S^s_{g,1}, (1))$ and, by Lemma 2.16, $\phi(l_i) = l_i$ and $\phi(v_i) = v_i$ for all $i = 1, \ldots, s$. By restriction, ϕ induces an automorphism $\phi_|$ on $Lk(\sigma) = A(S^0_{0,1}, (1))$. It follows from the hypothesis that $\phi_|$ is induced by a homeomorphism $\tilde{\phi} : (S^0_{0,1}, (s+1)) \to (S^0_{0,1}, (s+1))$.

We claim that $\tilde{\phi}$ restricts to the identity on the boundary of $(S^0_{0,1}, (s+1))$. By Lemma 2.20 we can equivalently show that $\tilde{\phi}$ fixes every 3-petal on $(S^0_{0,1}, (s+1))$. Let us denote by l_{ii+1} the

FIGURE 16. Reduction Lemma 3.15 in genus 0 case

3-petal of $(S^0_{g,1},(s+1))$ which joins the i-th and the $i+1$-th marked point on the boundary of $(S^0_{0,1},(s+1))$. Let a_{ii+1} be the arc joining the i-th and the $i+1$-th puncture of $(S^0_{0,1},(s+1))$ as it is shown in the Figure 16. The intersection pattern of the a_{ii+1}'s, of the l_j's and l_{ii+1}'s is the following:

$i(a_{i,i+1},l_i) = i(a_{ii+1},l_{i+1}) = 1$
$i(a_{ii+1},l_k) = 0$ for $i \neq k$
$i(a_{i,i+1},l_{i+1,i+2}) = i(l_{i+1,i+2},a_{i+2,i+3}) = 1$
$i(a_{h,h+1},l_{k+1,k+2}) = i(l_{k+1,k+2},a_{h+2,h+3}) = 0$ for $h \neq k$

Using Lemmas 2.17-(2), 2.19 and the automorphism invariance of the intersection patterns given above, we immediately deduce that necessarily $\phi(l_{ii+1}) = l_{ii+1}$ for all i, and the claim is proved. By the claim, we can extend $\tilde{\phi}$ to a homeomorphism of the surface inducing ϕ just glueing back the punctured discs bounded by the l_i's. □

As an immediate application of Propositions 3.14 and 3.15, we have:

Theorem 3.16 (Weak rigidity for $b=1$). *Let $(S^s_{g,1},(1))$ be an orientable surface of genus $g \geq 1$. Then $A(S^s_{g,1},(1))$ is weakly rigid.*

3.2.3. *Proof of Proposition 3.13 and $b=0$.* In this section we will use Lemma 3.12 to give an independent proof of Proposition 3.13.

Lemma 3.17. *Let $S^1_{g,0}$ be a closed orientable surface of genus $g \geq 2$ with one marked point P. Let $c \in A(S^1_{g,0})$ be a vertex corresponding to an arc separating S in two connected components $S = S'_c \cup S''_c$.*
For any $\phi \in$ Aut $A(S^1_{g,0})$, $\phi(c)$ corresponds to an arc which separates S in two connected components $S = S'_{\phi(c)} \cup S''_{\phi(c)}$, with $S'_{\phi(c)}$ homeomorphic to S'_c and $S''_{\phi(c)}$ homeomorphic to S''_c. Moreover, ϕ restricts to isomorphisms $\phi_| : A(S'_c,P) \to A(S'_{\phi(c)},P)$ and $\phi_| : A(S''_c,P) \to A(S''_{\phi(c)},P)$.

Proof. By simpliciality, $Lk(c, A(S^1_{g,0})) = A(S'_c,P) \star A(S''_c,P) \cong Lk(\phi(c), A(S^1_{g,0}))$ has diameter 2. If $\phi(c)$ were non-separating, then $Lk(\phi(c), A(S^1_{g,0})) \cong A(S^0_{g-1,2},(1,1))$ has infinite diameter (Remark 2.14). Thus, $\phi(c)$ separates S into two connected components $S'_{\phi(c)}$ and $S''_{\phi(c)}$. We remark that in this setting the following conditions are equivalent:

(1) $S'_{\phi(c)}, S''_{\phi(c)}$ are respectively homeomorphic to S'_c, S''_c;
(2) $(\text{genus}(S'_{\phi(c)}), \text{genus}(S''_{\phi(c)})) = (\text{genus}(S'_c), \text{genus}(S''_c))$;
(3) $\dim \mathrm{A}(\mathrm{S}'_c,\mathrm{P}) = \dim \mathrm{A}(\mathrm{S}'_{\phi(c)},\mathrm{P})$;
(4) the number of arcs of a triangulation of S'_c is equal to the number of arcs of a triangulation of $S'_{\phi(c)}$.

Without loss of generality, we assume $g(S'_c) = \max\{g(S'_c), g(S''_c), g(S'_{\phi(c)}), g(S''_{\phi(c)})\}$. Let μ_c be a maximal simplex in $A(S_c,P)$, that is $\dim\mu_c = \dim \mathrm{Lk}(c, A(S))$. Let $\mathscr{I}(\mu_c)$ be the set of simplices of $Lk(c,A(S))$ obtained from μ_c by an elementary move. Since μ_c corresponds to a triangulation of S_c, we have $|\mathscr{I}(\mu_c)| = \dim\, \mu_c + 1 = \dim \mathrm{A}(\mathrm{S}'_c,\mathrm{P}) + 1$. By simpliciality, $\phi(\mathscr{I}(\mu_c))$ corresponds precisely to the set of simplices in $Lk(\phi(c), A(S))$ obtained from $\phi(\mu_c)$ by an elementary move, and we have $|\phi(\mathscr{I}(\mu_c))| = |\mathscr{I}(\mu_c)|$. We write $\phi(\mu_c)$ as $\phi(\mu_c) = \langle \mu'_{\phi(c)}, \mu''_{\phi(c)} \rangle$, where $\mu'_{\phi(c)}$ is the empty set or simplex in $A(S'_{\phi(c)}, P)$, and the same holds for $\mu''_{\phi(c)}$ in $A(S''_{\phi(c)}, P)$, and we remark that $\dim \mu'_{\phi(c)} + \dim \mu''_{\phi(c)} + 2 = \dim \phi(\mu_c) + 1 = \dim \mu_c + 1 = \dim \mathrm{A(S'_c, P)} + 1 = |\mathscr{I}(\mu_c)|$.

By contradiction assume that $0 \leq \dim \mu'_{\phi(c)} < \dim \mathrm{A(S'_{\phi(c)}, P)}$, that is $\mu'_{\phi(c)}$ is neither empty nor a triangulation of $(S'_{\phi(c)}, P)$. Since $g \geq 2$, there are at least two different ways to extend $\mu'_{\phi(c)}$

FIGURE 17. Two ways of flipping v in $\mu'_{\phi(c)}$

to a triangulation of $S'_{\phi(c)}$ and, since $(S'_{\phi(c)}, P)$ has only one boundary component, there exists at least one vertex of $\mu'_{\phi(c)}$ flippable in at least two different ways (see Figure 17). It follows that $|\mathscr{I}(\phi(\mu_c))| \geq \dim \mu''_{\phi(c)} + 1 + \dim \mu'_{\phi(c)} + 2 > |\mathscr{I}(\mu_c)|$, and we get to a contradiction. The same argument holds if we assume $0 \leq \dim \mu''_{\phi(c)} < \dim \mathrm{A(S'_{\phi(c)}, P)}$.

We deduce that either $\dim \mu'_{\phi(c)} = \dim \mathrm{A(S'_{\phi(c)}, P)}$ (and $\mu''_{\phi(c)} = \varnothing$) or $\dim \mu''_{\phi(c)} = \dim \mathrm{A(S''_{\phi(c)}, P)}$ (and $\mu'_{\phi(c)} = \varnothing$). In the first case $\phi(\mu_c) = \phi(\mu'_c) \subset A(S'_c, P)$ has maximal dimension. Similarly, in the second case, $\phi(\mu_c) = \phi(\mu''_c) \subset A(S''_c, P)$ has maximal dimension. The conclusion easily follows from the equivalence of the above conditions 1 and 2. □

This lemma actually gives a proof of Proposition 3.13:

Proposition 3.18. *Let $S^1_{g,0}$ be an orientable surface of genus $g \geq 1$ with one marked point P. Then the natural representation* $\mathrm{MCG}^*(S^1_{g,0}) \to \mathrm{Aut}\ A(S^1_{g,0})$ *is surjective.*

Proof. We recall that this result is well known for $g = 1$, since $A(S^1_1)$ is isomorphic to the Farey graph.
Let $\phi \in \mathrm{Aut}\ A(S^1_{g,0})$ be a simplicial automorphism, and let $c \in A(S^1_{g,0})$ be an arc which separates S in two subsurfaces $(S^0_{1,1}, P)$ of genus 1 and $(S^0_{g_2,1}, P)$ of genus $g_2 \geq 1$. Up to precomposing ϕ with an automorphism induced by $\mathrm{MCG}^*(S^1_{g,0})$, we can assume $\phi(c) = c$, and ϕ restricts to automorphisms ϕ_1 and ϕ_2, respectively of $A(S^0_{1,1}, P)$ and $A(S^0_{g_2,1}, P)$. By the genus 1 case, ϕ_1 is induced by a homeomorphism $f_1 : (S^0_{1,1}, P) \to (S^0_{1,1}, P)$.
If $g_2 = 1$, let $f_2 : (S^0_{1,1}, P) \to (S^0_{1,1}, P)$ be the homeomorphism which induces ϕ_2. We glue f_2 to f_1, and the resulting homeomorphism $f : S^1_{1,0} \to S^1_{1,0}$ induces ϕ (Lemma 2.12).
An inductive argument on g_2 allows us to conclude. □

We shall now complete our proof with the $b = 0$ case. An analogous version of this case in the slightly different context of injective simplicial maps and arc complexes of surfaces with boundary (and no marked points) has also been achieved by Irmak and Mc Carthy. Their proof is based on an extensive study of all the possible reciprocal configurations of quintuplets of arcs connecting two boundary components (see [5]). By sake of completeness we show here that our indirect approach leads to a new proof that each simplicial automorphism of $A(S^s_g)$ is induced by a mapping class.

The following lemma can be proved with the same argument as Proposition 3.18.

Lemma 3.19. *Let $S^{s+1}_{g,0}$ be a compact orientable surface of genus $g \geq 2$ with $s + 1$ marked points. Let $c_1 \in A(S^{s+1}_{g,0})$ be a vertex corresponding to a separating arc which decomposes S as $S = S'_{c_1} \cup S''_{c_1}$ where $S'_{c_1} = (S^0_{g'+1,1}, (1))$ and $S''_{c_1} = (S^s_{g''+1,1}, (1))$.*
For any $\phi \in \mathrm{Aut}\ A(S^{s+1}_{g,0})$, $\phi(c_1)$ is a separating arc whose induced decomposition is $S =$

$S'_{\phi(c_1)} \cup S''_{\phi(c_1)}$, *with* $S'_{\phi(c_1)}$ *homeomorphic to* S'_{c_1} *and* $S''_{\phi(c_1)}$ *homeomorphic to* S''_{c_1}. *Moreover,* ϕ *induces isomorphisms* $\phi_| : A(S'_{c_1}, P) \to \phi(A(S'_{c_1}, P)) = A(S'_{\phi(c_1)}, P)$ *and* $\phi_| : A(S''_{c_1}, P) \to \phi(A(S''_{c_1}, P)) = A(S''_{\phi(c_1)}, P)$ *by restriction.*

Theorem 3.20. *Let* $S^s_{g,0}$ *be an orientable surface of genus* $g \geq 2$ *with* $s \geq 1$ *punctures. Then the natural homomorphism* $\mathrm{MCG}^*(S^s_{g,0}) \to \mathrm{Aut}\ A(S^s_{g,0})$ *is surjective.*

Proof. Let $\phi \in \mathrm{Aut}\ A(S^s_{g,0})$ and let c be a simple closed loop based at the puncture P on S such that c disconnects the surface into the two subsurfaces $(S^{s-1}_{1,1}, P)$ (of genus 1) and $(S^0_{g_2,1}, P)$ (of genus $g_2 \geq 1$). By Lemma 3.19, up to precomposition with an element of the mapping class group, ϕ restricts to automorphisms of $A(S^{s-1}_{1,1}, P)$ and $A(S^0_{g_2,1}, P)$. Hence, by Proposition 3.14 in the genus 1 case, Proposition 3.18 and Lemma 3.12 both automorphisms are induced by homeomorphims of the respective surfaces. Glueing them, we get a homeomorphism of S inducing ϕ by Lemma 2.12. □

3.3. **General case.** The Invariance Lemmas proved in Section 2 imply the following Reduction Lemmas.

Reduction Lemma 3.21. *Let* $g \geq 1$. *If* $A(S^s_{g,b}, \mathbf{1})$ *is weakly rigid, then for every* $\mathbf{p} = (p_1, \ldots, p_b) \in \mathbb{N}^b_0$ $A(S^s_{g,b}, \mathbf{p})$ *is weakly rigid.*

Proof. By an inductive argument it is sufficient to show that the surjectivity of $\mathrm{MCG}^\star(S^s_{g,b}, (p_1 - 1, \ldots, p_b)) \to \mathrm{Aut}\ A(S^s_{g,b}, (p_1 - 1, \ldots, p_b))$ implies the surjectivity of $\mathrm{MCG}^\star(S^s_{g,b}, (p_1 - 1, \ldots, p_b)) \to \mathrm{Aut}\ A(S^s_{g,b}, (\mathbf{p}))$. Let $\phi \in \mathrm{Aut}\ A(S^s_{g,b}, \mathbf{1}))$ be an automorphism.
Assume first that $p_1 \geq 3$. By Lemma 2.19 we can assume that ϕ fixes every 3-petal (or 3-leaf) on the first boundary component, up to precomposition with an automorphism induced by an element of the mapping class group. Let v_1, v_2 be two 3-petals such that $i(v_1, v_2) = 0$. Let us cut along v_1, ϕ induces an automorphism $\phi_|$ of the arc complex of the surface $(S^s_{g,b}, (p_1 - 1, \ldots, p_b))$ obtained cutting along v_1. Our hypothesis implies that $\phi_|$ is induced by a homeomorphism $\tilde{\phi}_|$ of $(S^s_{g,b}, (p_1 - 1, \ldots, p_b))$. Since $\tilde{\phi}_|(v_2) = v_2$, $\tilde{\phi}_|$ is the identity on the boundary of its first component, it agrees with the identity on $(S^0_{0,1}, (3))$ and, by glueing, it gives a homeomorphism on $(S^s_{g,b}, \mathbf{p})$ which induces ϕ. Lemma 2.20 ensures us that the same argument holds for $p_i = 2$ using 2-leaves instead of 3-petals. □

Reduction Lemma 3.22. *Let* $b \geq 2$. *If* $A(S^s_{g,b-1}, \mathbf{1})$ *is weakly rigid, then* $A(S^s_{g,b}, \mathbf{1})$ *is weakly rigid.*

Proof. Let $\langle l, v \rangle$ be an edge as in Lemma 2.23. Without loss of generality assume l is based on the first boundary component and v joins the first and the second boundary component. Let $\phi \in \mathrm{Aut}\ A(S^s_{g,b}, \mathbf{1})$ be an automorphism. Up to precomposition with an element induced by a mapping class, we can assume $\phi(l) = l$ and $\phi(v) = v$. Cutting along v, ϕ restricts to an automorphism $\phi_|$ of the arc complex of the surface $(S^s_{g,b-1}, (4, 1_{b-2}))$. Our hypothesis and Lemma 3.21 imply that $\phi_|$ is induced by a homeomorphism $\tilde{\phi}_| : (S^s_{g,b-1}, (4, 1_{b-2})) \to (S^s_{g,b-1}, (4, 1_{b-2}))$. Since $\tilde{\phi}_|(l) = l$, ϕ preserves the segment of the first boundary components of $(S^s_{g,b-1}, (4, 1_{b-2}))$ which corresponds to the cut along v, we can thus glue back and get a homeomorphism of $(S^s_{g,b}, \mathbf{1})$. □

Let us now prove that the two definitions of rigid and weakly rigid are equivalent.

Proposition 3.23. *Let* $A(S^s_{g,b}, \mathbf{p})$ *be not empty, with* $(g, b, s, \mathbf{p}) \neq (0, 2, 0, (1, 1))$. *If* $A(S^s_{g,b}, \mathbf{p})$ *is weakly rigid, then it is also rigid.*

Proof. This is equivalent to showing that if $f \in \mathrm{MCG}^*(\mathrm{S}^{\mathrm{s}}_{\mathrm{g,b}}, \mathbf{p})$ is such that the automorphism $f_\star : A(S^s_{g,b}, \mathbf{p}) \to A(S^s_{g,b}, \mathbf{p})$ is the identity, then f is isotopic to the identity.

Let us assume first f exchanges at least two boundary components of S, say $\mathscr{B}_1$ and $\mathscr{B}_2$. We have necessarily $b = 2$ and $s = 0$. Indeed, otherwise f would send an arc connecting $\mathscr{B}_1$ and

$\mathscr{B}_3$ (or the puncture $\mathscr{S}$) to an arc connecting $\mathscr{B}_2$ and $\mathscr{B}_3$ (or the puncture $\mathscr{S}$), in contradiction with the assumption $f_\star = id$. Moreover, we have $p_1 = p_2 = 1$ otherwise $f_\star$ would exchange 2-leaves around different boundary components. Finally, $g = 0$ ($s = 0$) otherwise $f_\star$ would exchange loops based on different boundary components and running around a handle. The non-rigidity in the case $(g, b, s, \mathbf{p}) = (0, 2, 0, (1, 1))$ has been proved in Theorem B for annuli.

Now assume that f fixes all the boundary components. Of course, f cannot be a rotation. If $f_{|\mathscr{B}_1}$ is homotopic to a reflection, then $p_1 \leq 2$ otherwise two 3-petals around $\mathscr{B}_1$ would be exchanged. If $g \geq 1$ there would exist some loop α based on $\mathscr{B}_1$ such that $f_\star(\alpha^-) = \alpha^+$, according to the notation used in Subsection 3.2.1. The same argument also excludes the case $g = 0$ with $b > 2$ or $b = 1$ and $s > 0$. The three remaining cases $(g, b, s, \mathbf{p}) = (0, 2, 0, (1, 1))$, $(0, 2, 0, (2, 2))$, $(0, 2, 0, (1, 2))$ have been discussed in Section 3.1.2.

If f is now homotopic to the identity, $f_\star = id$ implies that f fixes each arc of a triangulation. If the triangulation of $(S^s_{g,b}, \mathbf{p})$ consists of at least 4 arcs (i.e. $\dim A(S^s_{g,b}, \mathbf{p}) \geq 3$), then f necessary fixes each triangle as well, hence f is isotopic to the identity. Low dimensional cases have been checked in Proposition 2.5. □

We finally deduce Theorem B:

Theorem B. *Let $(S^s_{g,b}, \mathbf{p})$ be an orientable surface of genus $g \geq 1$ with s marked points in the interior, b ordered boundary components, and let $\mathbf{p} = (p_1, \ldots, p_b)$ be the vector whose i-th component p_i is the number of marked points on the i-th boundary component of S. If $A(S^s_{g,b}, \mathbf{p})$ is not empty, then $A(S^s_{g,b}, \mathbf{p})$ is weakly rigid. If furthermore $(g, b, s, \mathbf{p}) \neq (0, 2, 0, (1, 1))$, then $A(S^s_{g,b}, \mathbf{p})$ is rigid.*

Proof. Reduction Lemmas 3.21 3.22 and 3.15 allows us to reduce to the cases genus 0 and one boundary component surfaces with positive genus. We can thus conclude by Theorems 3.4, 3.7, 3.16 and 3.20. □

4. Proof of Theorem C

Since there will be no ambiguity, in this section we shall use the notation $A_\sharp$ for $A_\sharp(S^s_{g,b}, \mathbf{p})$ and A for $A(S^s_{g,b}, \mathbf{p})$. Here we shall prove the following result

Theorem C. *Let $(S^s_{g,b}, \mathbf{p})$ be an orientable surface of genus g with $b \geq 1$ boundary components, s punctures and $p_i \geq 1$ for all $i = 1, \ldots, b$. If $(S^s_{g,b}, \mathbf{p}) \neq (S^0_{0,2}, (1, 1))$, then $A_\sharp(S^s_{g,b}, \mathbf{p})$ is rigid. Moreover, in the exceptional cases the natural homomorphism* $\mathrm{MCG}(S^0_{0,2}, (1, 1)) \to \mathrm{Aut}\ A_\sharp(S^0_{0,2}, (1, 1))$ *is surjective, but not injective.*

Proof. In order to prove Theorem C, we shall first prove that any automorphism $\phi : A_\sharp \to A_\sharp$ extends to an automorphism $\tilde{\phi} : A \to A$.

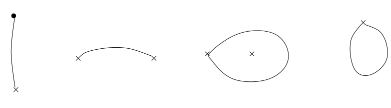

FIGURE 18.

Step 1: Extending ϕ on the vertices of $A \setminus A_\sharp$. We shall define an extension $\tilde{\phi}$ of ϕ on A_0. We classify the vertices of $A \setminus A_\sharp$ in 4 types, as in Figure 18: arcs α joining a marked point on the boundary to a puncture inside, arcs β joining two punctures, loops γ based at a puncture wrapping around another puncture, loops δ based on a puncture (different from type γ).
Let α be an arc joining a marked point on the boundary to a puncture inside, and let us complete α to the edge-drop $\langle \alpha, l_\alpha \rangle$ (we can do it in a unique way). By Lemma 2.16, $\phi(l_\alpha)$ is an arc of the same type. Hence, we can define $\tilde{\phi}(\alpha)$ as the unique complement of $\phi(l_\alpha)$ to an edge-drop in A. The following lemma is an easy consequence of Lemmas 2.17 and 2.18.

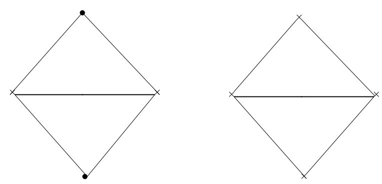

FIGURE 19.

Lemma 4.1. *The configuration of arcs in a square like in Figure 19-left is invariant through the action of* ϕ *(and* $\tilde{\phi}$*, defined above).*

Let $\beta \in A$ be an arc joining two punctures. Let us choose vertices $v_1, \ldots, v_4 \in A$ represented by disjoint arcs as in the case above, such that they form a square on S whose diagonal is β (as in Figure 19-left). Let us denote by β^* the other diagonal of this square, and remark that $\beta^* \in A_\sharp$. By Lemma 4.1, the arcs $\tilde{\phi}(v_1), \ldots, \tilde{\phi}(v_4)$ form a square with diagonal $\phi(\beta^*)$. Let us then define $\tilde{\phi}(\beta) := \phi(\beta^*)^*$, the other diagonal of this new square. We remark that at this step this definition depends only the choice of $v_1, \ldots, v_4$. We shall see later that it is actually natural.

Lemma 4.2. *The configuration of arcs in a square joining punctures in a square like Figure 19-right is invariant through the action of* ϕ *(and* $\tilde{\phi}$*, as defined above).*

Let γ be a loop around β, and let α be one of the arcs not intersecting γ used in the definition of $\tilde{\phi}(\beta)$. By definition, $\tilde{\phi}(\beta)$ is an arc of the same type of β, and $\tilde{\phi}(\alpha)$ is an arc of the same type of α. By the above lemma $\tilde{\phi}(\alpha)$ and $\tilde{\phi}(\beta)$ share a (unique) common endpoint.
We can thus define $\tilde{\phi}(\gamma)$ as the loop based at this end and running close around $\tilde{\phi}(\beta)$. We remark that this definition depends only on the definition of $\tilde{\phi}(\beta)$. Let γ be a loop based at a

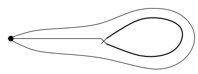

FIGURE 20.

puncture. Let us choose α_γ an arc disjoint from γ which connects the puncture to a marked point on the boundary, and let l_γ be the loop boundary of $\alpha_\gamma \cup \gamma$ as in Figure 19-right. As in the above lemmas, it is not difficult to prove that the relative configuration of $\alpha_\gamma \cup l_\gamma$ is invariant through the action of $\tilde{\phi}$.
We can then define $\tilde{\phi}(\gamma)$ as the loop parallel to $\phi(l_\gamma)$ based at the same marked point on ∂S as $\tilde{\phi}(\alpha_\gamma)$. We remark that this definition depends only on the choice of α_γ.

Step 2: Simpliciality of $\tilde{\phi}$. We can resume Lemmas 4.1, 4.2 used above in the following lemma, whose proof is an immediate consequence of Lemmas 2.16 and 2.18.

Lemma 4.3. *The maps* ϕ *and* $\tilde{\phi}$ *preserve squares and their diagonals.*

It is not difficult to see that $\tilde{\phi}$ is simplicial if and only if for any maximal simplex T in A $\tilde{\phi}(T)$ is a maximal simplex as well. For any $T_\sharp$ maximal simplex in $A_\sharp$, there is a natural way to extend $T_\sharp$ to a maximal simplex $\tilde{T}_\sharp$ by adding arcs of type α as above. By definition of $\tilde{\phi}$, the simultaneous disjointness of all such arcs is preserved in this case, and $\tilde{T}_\sharp$ is a maximal simplex in A as well. To prove the statement in full generality, just recall that any two triangulations of A are connected by flips and Lemma 4.3 ensures us that simpliciality is preserved through flips.

Step 3: Surjectivity of $\tilde{\phi}$. We remark that $\tilde{\phi}$ preserves the types of arcs in Figure 18, moreover it is clearly surjective on arcs α. Surjectivity on arcs γ clearly follows from surjectivity on arcs β. Surjectivity on β and δ follows from Lemma 4.3. Let w be such an arc, there exists a square (whose sides $v_1, \dots, v_4$ are arcs of type α) on S having w as a diagonal (see for instance Figure 19-left for β). By surjectivity on sides of type α, $v_i = \tilde{\phi}(u_i)$. By Lemma 4.3, $u_1, \dots, u_4$ is a square as well and its diagonals are the preimages of diagonals of the $v_1, \dots, v_4$.

Step 4: Injectivity of $\tilde{\phi}$. Injectivity on arcs of type α follows by definition, injectivity on arcs of type β and γ follows by construction. Imagine $\tilde{\phi}(\delta_1) = \tilde{\phi}(\delta_2)$.

Step 5: Good definition of $\tilde{\phi}$. By the above steps, all the possible extensions $\tilde{\phi}$ are automorphisms of A. By the remark in Step 2, the maps $\tilde{\phi}$ are all canonically determined on a triangulation $\tilde{T}_\sharp$. Hence, by Lemma 2.11, they all coincide and the definition of $\tilde{\phi}$ doesn't depend on any choice.

To conclude the proof, let us just remark that the restriction map $\beta : \text{Aut } A \to \text{Aut } A_\sharp$ defined as $\beta(\phi) := \phi_|$ is well-defined and is a group homomorphism, and so is $\alpha : \text{Aut } A_\sharp \to \text{Aut } A$ defined as $\alpha(\phi) := \tilde{\phi}$. Moreover $\alpha \circ \beta = id_{\text{Aut } A}$ and $\beta \circ \alpha = id_{\text{Aut } A_\sharp}$, hence $\text{Aut} A_\sharp \cong \text{Aut } A$, and Theorem B is proved. □

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IRMA, UMR 7501 7 rue René-Descartes 67000 Strasbourg
E-mail address: disarlo@math.unistra.fr