On quaternionic functions

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Abstract. Several sets of quaternionic functions are described and studied. Residue current of the right inverse of a quaternionic function is introduced in particular cases.

1. Introduction

We will work with the definition of quaternions using pairs of complex numbers and with a modified Cauchy-Fueter operator that have been introduced in [CLSSS 07]. We will only use right multiplication; the (right) inverse of a nonzero quaternion is defined. We will consider (for simplicity) C^∞ $\mathbb{H}$-valued quaternionic functions defined on an open set U of $\mathbb{H}$ containing 0. If such a function does not vanish over U, it has an (algebraic) inverse which is defined almost everywhere on U. Examples are given (section 2).

The origin of this research is a tentative of extension to right inverse of a quaternionic function of the notion of residue current of a meromorphic differential 1-form of one complex variable, which will be developed in section 5. In one complex variable, if the given function is holomorphic, with isolated zeros of finite multiplicity, its inverse is meromorphic, then holomorphic outside the set of poles; so it is natural to search when this property extends to hyperholomorphic functions.

In section 3, we characterize the quaternionic functions which are hyperholomorphic and whose inverses are hyperholomorphic almost everywhere, on U, as the solutions of a system of two non linear PDE. We only find non trivial examples of a solution, showing that the considered space of functions is significant: we will call theses functions hypermeromorphic. This also defines a space of germs of functions at 0.

In section 4, we try to describe a subspace $\mathcal{H}_U$ of hyperholomorphic and hypermeromorphic functions defined almost everywhere on U, having "good properties for addition and multiplication"; we obtain again systems

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of non linear PDE, and we give first results on, mainly unknown, spaces of functions.

In section 5, we first recall Cauchy principal value and residue current in $\mathbb{C}$, locally at 0. Afterwards, we define and study, locally, Cauchy principal value and residue current for the inverse of a quaternionic function, in very particular cases, and in relation with the classical theory in two complex variables.

This paper is a first announcement of a more complete one in progress.

2. Quaternions. $\mathbb{H}$-valued functions. [CLSSS 07]

2.1. **Quaternions.** If $q \in \mathbb{H}$, then $q = z_1 + z_2\mathbf{j}$ where $z_1, z_2 \in \mathbb{C}$, hence $\mathbb{H} \cong \mathbb{C}^2 \cong \mathbb{R}^4$ as complex or real vector space. We have: $z_1\mathbf{j} = \mathbf{j}\overline{z}_1$ (by computation in real coordinates); by definition, the *modulus* of q is $\parallel q \parallel = (|z_1|^2 + |z_2|^2)^{\frac{1}{2}}$.

The *conjugate* of q is $\overline{q} = \overline{z}_1 - z_2\mathbf{j}$. Let * denote the *(right) multiplication* in $\mathbb{H}$:

$q * \overline{q} = (z_1 + z_2\mathbf{j}) * (\overline{z}_1 - z_2\mathbf{j}) = |z_1|^2 - z_1z_2\mathbf{j} + z_2\mathbf{j}\overline{z}_1 - z_2\mathbf{j}z_2\mathbf{j} = |z_1|^2 + |z_2|^2$, then: the *(right) inverse* of $q = z_1 + z_2\mathbf{j}$ is: $(|z_1|^2 + |z_2|^2)^{-1}\overline{q} = (|z_1|^2 + |z_2|^2)^{-1}(\overline{z}_1 - z_2\mathbf{j})$. Moreover: $(|z_1|^2 + |z_2|^2)^{-1}(\overline{z}_1 - z_2\mathbf{j}) * (z_1 + z_2\mathbf{j}) = 1$, so the right inverse of q^{-1} is q.

2.2. **Quaternionic functions.** Let U be an open set of $\mathbb{H} \cong \mathbb{C}^2$ and $f \in C^\infty(U, \mathbb{H})$, then $f = f_1 + f_2\mathbf{j}$, where $f_1, f_2 \in C^\infty(U, \mathbb{C})$. The complex valued functions f_1, f_2 will be called the *components* of f.

Remark that $\mathbb{H}$ is a real vector space in which real analysis is valid, in particular differential forms, distributions and currents are defined in $\mathbb{H}$.

Remark that $\dfrac{\partial f_1}{\partial \overline{z}_1}\mathbf{j} = \mathbf{j}\dfrac{\partial \overline{f}_1}{\partial z_1}$ and analogous relations.

2.3. **Modified Cauchy-Fueter operator $\mathcal{D}$. Hyperholomorphic functions. ([CLSSS 07], [F 39]).** For $f \in C^\infty(U, \mathbb{H})$, with $f = f_1 + f_2\mathbf{j}$, where $f_1, f_2 \in C^\infty(U, \mathbb{C})$,

$\mathcal{D}f(q) = \dfrac{1}{2}\Big(\dfrac{\partial}{\partial \overline{z}_1} + \mathbf{j}\dfrac{\partial}{\partial \overline{z}_2}\Big)f(q) = \dfrac{1}{2}\Big(\dfrac{\partial f_1}{\partial \overline{z}_1} - \dfrac{\partial \overline{f}_2}{\partial z_2}\Big)(q) + \mathbf{j}\dfrac{1}{2}\Big(\dfrac{\partial f_1}{\partial \overline{z}_2} + \dfrac{\partial \overline{f}_2}{\partial z_1}\Big)(q)$.

A function $f \in C^\infty(U, \mathbb{H})$ s said to be *hyperholomorphic* if $\mathcal{D}f = 0$.

Characterization of the hyperholomorphic function f on U: $\dfrac{\partial f_1}{\partial \overline{z}_1} - \dfrac{\partial \overline{f}_2}{\partial z_2} = 0; \dfrac{\partial f_1}{\partial \overline{z}_2} + \dfrac{\partial \overline{f}_2}{\partial z_1} = 0$, on U.

The conditions:f_1 is holomorphic and: f_2 is holomorphic are equivalent. So *holomorphic functions* will be identified with hyperholomorphic functions f such that $f_2 = 0$.

Let $f' = f'_1 + f'_2\mathbf{j}$, $f" = f"_1 + f"_2\mathbf{j}$ be two hyperholomorphic functions.

For every $\alpha \in \mathbb{HI}$, $\mathcal{D}(f'\alpha) = 0$, $\mathcal{D}(f' + f") = \mathcal{D}f' + \mathcal{D}f" = 0$.

Proposition 2.1. *The set $\mathcal{H}$ of almost everywhere defined hyperholomorphic functions is an $\mathbb{HI}$-right vector space.*

Proposition 2.2. *Let f', $f"$ be two hyperholomorphic functions. Then, their product $f' * f"$ satisfies:*

$$\mathcal{D}(f' * f") = \mathcal{D}f' * \mathbf{j}f" + \big(f'(\frac{\partial}{\partial\overline{z}_1}) + \overline{f}'\mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)f"$$

Proof. $f' = f'_1 + f'_2\mathbf{j}$, $f" = f"_1 + f"_2\mathbf{j}$ be two hyperholomorphic functions.

We have: $f'*f" = (f'_1+f'_2\mathbf{j})(f"_1+f"_2\mathbf{j}) = f'_1f"_1-f'_2\overline{f}"_2+(f'_1f"_2+f'_2\overline{f}"_1)\mathbf{j}$

Compute

$$\frac{1}{2}\big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)\big(f'_1f"_1 - f'_2\overline{f}"_2 + (f'_1f"_2 + f'_2\overline{f}"_1)\mathbf{j}\big)$$

By derivation of the first factors of the sum $f' * f"$, we get the first term:

$$\frac{1}{2}\big(\frac{\partial f'_1}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial f'_1}{\partial\overline{z}_2}\big)(f"_1 + f"_2\mathbf{j}) + \frac{1}{2}\big(\frac{\partial f'_2}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial f'_2}{\partial\overline{z}_2}\big)\mathbf{jj}(\overline{f}"_2 - \overline{f}"_1\mathbf{j})$$

$$= \frac{1}{2}\big(\frac{\partial f'_1}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial f'_1}{\partial\overline{z}_2}\big)(f"_1 + f"_2\mathbf{j}) + \frac{1}{2}\big(\frac{\partial f'_2\mathbf{j}}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial f'_2\mathbf{j}}{\partial\overline{z}_2}\big)\mathbf{j}(f"_2\mathbf{j} + f"_1) = \mathcal{D}f' * \mathbf{j}f"$$

By derivation in

$$\frac{1}{2}\big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)\big(f'_1f"_1 + f'_2\mathbf{j}f"_2\mathbf{j} + (f'_1f"_2\mathbf{j} + f'_2\mathbf{j}f"_1)\big)$$

of the second factors of the sum $f'*f"$, we get the second term (up to factor $\frac{1}{2}$):

$$f'_1\frac{\partial f"_1}{\partial\overline{z}_1}+\overline{f}'_1\mathbf{j}\frac{\partial f"_1}{\partial\overline{z}_2}+f'_1\frac{\partial f"_2}{\partial\overline{z}_1}\mathbf{j}+\overline{f}'_1\mathbf{j}\frac{\partial f"_2}{\partial\overline{z}_2}\mathbf{j}+f'_2\mathbf{j}\frac{\partial f"_2}{\partial\overline{z}_1}\mathbf{j}+\overline{f}'_2\mathbf{j}\frac{\partial f"_2}{\partial\overline{z}_2}+f'_2\mathbf{j}\frac{\partial f"_1}{\partial\overline{z}_1}+\overline{f}'_2\mathbf{jj}\frac{\partial f"_1}{\partial\overline{z}_2}$$

$$= (f'_1+f'_2\mathbf{j})(\frac{\partial}{\partial\overline{z}_1})(f"_1+f"_2\mathbf{j})+(\overline{f}'_1+\overline{f}'_2\mathbf{j})\mathbf{j}\frac{\partial}{\partial\overline{z}_2}(f"_1+f"_2\mathbf{j}) = \big((f'_1+f'_2\mathbf{j})(\frac{\partial}{\partial\overline{z}_1})+(\overline{f}'_1+\overline{f}'_2\mathbf{j})\mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)(f"_1+f"_2\mathbf{j})$$

$$= \big(f'(\frac{\partial}{\partial\overline{z}_1}) + \overline{f}'\mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)f"$$

□

If the components of f' and $f"$ are real, the second term is:

$$\frac{1}{2}(f'_1 + f'_2\mathbf{j})(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2})(f"_1 + f"_2\mathbf{j}) = f' * \mathcal{D}f"$$

i.e.

Corollary 2.3. *The set $\mathcal{H}_{I\!R}$ of almost everywhere defined hyperholomorphic functions whose components are real is an $I\!R$-right algebra.*

2.4. **Null set and inverse of a quaternionic function.** We call *inverse* of a function $f : q \mapsto f(q)$, the function $f^{-1} : q \mapsto f(q)^{-1}$. Let $f = f_1 + f_2\mathbf{j}$ be a quaternionic function on U. The null set $Z(f)$ satisfies: $f_1 = 0; f_2 = 0$, then $Z(f)$ is of measure 0 in U. Ex.: $f_1 = \overline{z}_1; f_2 = \overline{z}_2$, then $Z(f) = \{0\}$. Note that if f is holomorphic, then, $f_2 \equiv 0$ and $Z(f)$ is a complex hypersurface in $\mathbb{C}^2$.

Inversion and hyperholomorphy. The inverse of the quaternionic function f is the peculiar quaternionic function defined almost everywhere on U:

$$\frac{1}{f} = (|f_1|^2 + |f_2|^2)^{-1}(\overline{f}_1 - f_2\mathbf{j}) = |f|^{-1}\overline{f}$$

where $\overline{f}$ is the (quaternionic) conjugate of f.

Assume f to be hyperholomorphic and $Z(f) = \{0\}$, then $\dfrac{1}{f}$ is not necessarily hyperholomorphic outside $\{0\}$.

Ex.: $f = \overline{z}_1 + \overline{z}_2\mathbf{j}$, then

$$\frac{1}{f} = (z_1\overline{z}_1 + z_2\overline{z}_2)^{-1}(z_1 - \overline{z}_2\mathbf{j});\ \ \mathcal{D}(\frac{1}{f}) \neq 0,$$

Example of a function hyperholomorphic outside 0.

$H(q) = (z_1\overline{z}_1 + z_2\overline{z}_2)^{-2}(\overline{z}_1 - \overline{z}_2\mathbf{j})$ *is hyperholomorphic* since:
$\mathcal{D}H(q) = \dfrac{1}{2}(z_1\overline{z}_1 + z_2\overline{z}_2)^{-3}(-2z_1\overline{z}_1 + z_1\overline{z}_1 + z_2\overline{z}_2 - 2z_2\overline{z}_2 + (z_1\overline{z}_1 + z_2\overline{z}_2)) = 0.$
But $F = z_1 + \overline{z}_2\mathbf{j}$ *is not hyperholomorphic*: the conjugate of F is $\overline{F} = \overline{z}_1 - \overline{z}_2\mathbf{j}$; $(z_1 + \overline{z}_2\mathbf{j}) * (\overline{z}_1 - \overline{z}_2\mathbf{j}) = z_1\overline{z}_1 + z_2\overline{z}_2$. So $F^{-1} = (\overline{z}_1 - \overline{z}_2\mathbf{j})(z_1\overline{z}_1 + z_2\overline{z}_2)^{-1}$, and $H(q) = F^{-1}(z_1\overline{z}_1 + z_2\overline{z}_2)^{-1} = \dfrac{F^{-1}}{|F|}$.

(H *is the Cauchy kernel* for the modified Cauchy-Fueter operator $\mathcal{D}$).
Inverse of a holomorphic function.

Let $f = f_1 + 0\mathbf{j}$ be a hyperholomorphic function. Then $f^{-1} = f_1^{-1} + 0\mathbf{j}$ and f^{-1} is hyperholomorphic outside of the complex hypersurface $Z(f)$. Remark that $Z(f)$ is a subvariety of complex dimension 1, then of measure zero, in U.

We will consider almost everywhere defined hyperholomorphic functions on U. Ex.: holomorphic, meromorphic functions.

3. Hyperholomorphic functions whose inverses are hyperholomorphic almost everywhere.

Proposition 3.1. *The following conditions are equivalent*

(*i*) *the function* $f = f_1 + f_2\mathbf{j}$ *and its right inverse are hyperholomorphic, when they are defined;*

(*ii*) *we have the equations:*

$$(\overline{f}_1 - f_1)\frac{\partial \overline{f}_1}{\partial z_1} - \overline{f}_2\frac{\partial f_2}{\partial z_1} - f_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2} = 0, \tag{3}$$

$$\overline{f}_2\frac{\partial f_1}{\partial z_1} + \frac{\partial \overline{f}_2}{\partial z_1}(\overline{f}_1 - f_1) - f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2} = 0. \tag{4}$$

Proof. Let $f = f_1 + f_2\mathbf{j}$ be a hyperholomorphic function and $g = g_1 + g_2\mathbf{j} = |f|^{-1}(\overline{f}_1 - f_2\mathbf{j})$ its inverse; so $g_1 = |f|^{-1}\overline{f}_1$; $g_2 = -|f|^{-1}f_2$, where $|f| = (f_1\overline{f}_1 + f_2\overline{f}_2)$.

$$\mathcal{D}g(q) = \frac{1}{2}\Big(\frac{\partial}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial}{\partial \overline{z}_2}\Big)g(q) = \frac{1}{2}\Big(\frac{\partial g_1}{\partial \overline{z}_1} - \frac{\partial \overline{g}_2}{\partial z_2}\Big)(q) + \mathbf{j}\frac{1}{2}\Big(\frac{\partial g_1}{\partial \overline{z}_2} + \frac{\partial \overline{g}_2}{\partial z_1}\Big)(q)$$

$$\frac{\partial g_1}{\partial \overline{z}_1} = |f|^{-1}\frac{\partial \overline{f}_1}{\partial \overline{z}_1} - |f|^{-2}\overline{f}_1\Big(\frac{\partial f_1}{\partial \overline{z}_1}\overline{f}_1 + f_1\frac{\partial \overline{f}_1}{\partial \overline{z}_1} + \frac{\partial f_2}{\partial \overline{z}_1}\overline{f}_2 + f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_1}\Big), \text{etc}$$

$$(**)\ \ 2|f|^2\mathcal{D}g = (f_1\overline{f}_1 + f_2\overline{f}_2)\Big(\frac{\partial \overline{f}_1}{\partial \overline{z}_1} + \frac{\partial \overline{f}_2}{\partial z_2}\Big) - \overline{f}_1 f_1\frac{\partial \overline{f}_1}{\partial \overline{z}_1} - \overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial \overline{z}_1} - \overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_1}$$

$$-\overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial \overline{z}_1} - \overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_2} - f_1\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_2}. - \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_2} - f_2\overline{f}_2\frac{\partial \overline{f}_2}{\partial z_2}$$

$$+\mathbf{j}\Big((f_1\overline{f}_1 + f_2\overline{f}_2)\Big(\frac{\partial \overline{f}_1}{\partial \overline{z}_2} - \frac{\partial \overline{f}_2}{\partial z_1}\Big) - \overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial \overline{z}_2} - \overline{f}_1 f_1\frac{\partial \overline{f}_1}{\partial \overline{z}_2} - \overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial \overline{z}_2} - \overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}$$

$$+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1} + f_1\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_1} + \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1} + f_2\overline{f}_2\frac{\partial \overline{f}_2}{\partial z_1}\Big)$$

f being hyperholomorphic, g hyperholomorphic is equivalent to:

$$(**)\ \ +f_1\overline{f}_1\frac{\partial \overline{f}_2}{\partial z_2} + f_2\overline{f}_2\frac{\partial \overline{f}_1}{\partial \overline{z}_1} - \overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial \overline{z}_1} - \overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_1} - \overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_2} - \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_2}$$

$$+\overline{f}_2\frac{\partial f_2}{\partial \overline{z}_1}(f_1 - \overline{f}_1) = 0$$

$$+f_2\overline{f}_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2} - \overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial \overline{z}_2} - \overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2} + \overline{f}_1\frac{\partial f_1}{\partial \overline{z}_2}(f_1 - \overline{f}_1) + \overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1} + f_1\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_1}$$

$$+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1} = 0$$

After conjugaison of the first equation, and using

$$\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f}_2}{\partial z_2} = 0; \frac{\partial f_1}{\partial \overline{z}_2} + \frac{\partial \overline{f}_2}{\partial z_1} = 0, \tag{1}$$

we get:

$$(**)$$
$$+f_2\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_1(\overline{f}_1-f_1)\frac{\partial \overline{f}_1}{\partial z_1}-f_1\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\frac{\partial \overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_1f_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}-f_2f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}=0$$

$$+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}+(f_1-\overline{f}_1)\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_1}+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}+\overline{f}_1\frac{\partial f_1}{\partial \overline{z}_2}(f_1-\overline{f}_1)+f_2\overline{f}_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}.-\overline{f}_1f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}=0$$

Assume $f_1 \neq 0$, $f_2 \neq 0$. After multiplication of the first equation by f_1 and of the second by $-f_2$, and sum, we get

$$(\overline{f}_1 - f_1)\frac{\partial \overline{f}_1}{\partial z_1} - \overline{f}_2\frac{\partial f_2}{\partial z_1} - f_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2} = 0$$

By an analogous process, we get:

$$\overline{f}_2\frac{\partial f_1}{\partial z_1} + \frac{\partial \overline{f}_2}{\partial z_1}(\overline{f}_1 - f_1) - f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2} = 0$$

□

Corollary 3.2. *If f satisfies the conditions of the Proposition, the same is true for αf with $\alpha \in \mathbb{R}$.*

Let $f = f_1 + 0\mathbf{j}$ be an almost everywhere holomorphic function, then the condition (ii) of Proposition 3.1 is satisfied.

Now give another example of quaternionic function satisfying the conditions of Proposition 3.1:

Proposition 3.3. *Let $f = f_1 + f_2\mathbf{j}$, with $f_1 = z_1 + \overline{z}_1 + z_2 + \overline{z}_2 + A$, $f_2 = -z_1 - \overline{z}_1 + z_2 + \overline{z}_2 + B$, $A, B \in \mathbb{R}$, then: f and f^{-1} outside the zero set of f, are hyperholomorphic The null set of $f = f_1 + f_2\mathbf{j}$, for f_1, f_2 as above, for $A = B = 0$, is:*

$$z_1 + \overline{z}_1 + z_2 + \overline{z}_2 = 0; -z_1 - \overline{z}_1 + z_2 + \overline{z}_2 = 0$$

i.e, by difference and sum: $\overline{z}_1 + z_1 = 0; z_2 + \overline{z}_2 = 0$, i.e. $x_1 = 0; x_2 = 0$ in $\mathbb{R}^4$.

Proof. $f_1 = z_1 + \overline{z}_1 + z_2 + \overline{z}_2 + A$, A constant; then:

$$-\overline{f}_2\frac{\partial f_2}{\partial z_1} - f_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2} = 0, \text{ i.e. } -\overline{f}_2\frac{\partial f_2}{\partial z_1} - f_2 = 0, \tag{3}$$

Try: f_2 *real.* Then $\dfrac{\partial f_2}{\partial z_1} = -1$ and $f_2 = -z_1 + C(\overline{z}_1, z_2, \overline{z}_2) = -z_1 - \overline{z}_1 + C'(\overline{z}_1, z_2, \overline{z}_2)$, with C' real and $\dfrac{\partial C'}{\partial z_1} = 0$.

From (4), $\dfrac{\partial f_2}{\partial \overline{z}_2} = \dfrac{\partial C'}{\partial \overline{z}_2} = 1$, and $C' = z_2 + \overline{z}_2 + C"(\overline{z}_1, z_2)$, with $C"$ real. $f_2 = -z_1 - \overline{z}_1 + z_2 + \overline{z}_2 + C"(\overline{z}_1, z_2)$, with $C"$ real, and $\dfrac{\partial C"}{\partial \overline{z}_2} = 0$, $\dfrac{\partial C"}{\partial z_1} = 0$.

$C"$ being holomorphic in z_2 and $\overline{z}_1$ is a constant B.

Hence: $f_2 = -z_1 - \overline{z}_1 + z_2 + \overline{z}_2 + B$, with $B \in I\!R$. □

4. On the space of hyperalgebraic functions

4.1. **Definition.** Let U be an open neighborhood of 0 in $I\!H\!I \cong \mathbb{C}^2$. *From now on,we will only consider the quaternionic functions* $f = f_1 + f_2\mathbf{j}$ *having the following properties*:

(i) when f_1 and f_2 are not holomorphic, the set $Z(f_1) \cap Z(f_2)$ is *discrete* on U;

(ii) for every $q \in Z(f_1) \cap Z(f_2)$, $J_q^{\alpha}(.)$ denoting the *jet of order* α *at* q [M 66], let $m_i = \sup_{\alpha_i} J_q^{\alpha_i}(f_i) = 0$; m_i, $i = 1, 2$, is finite.

Define: $m_q = \inf_i m_i$.

Remark that, in this paper, the considered peculiar examples of quaternionic functions f satisfy: the set $Z(f_1) \cap Z(f_2)$ is reduced to one point and that $\alpha_i = 1$.

Let $f = f_1 + f_2\mathbf{j}$ be a quaternonic function on U and $g = g_1 + g_2\mathbf{j} = |f|^{-1}(\overline{f}_1 - f_2\mathbf{j})$ its inverse; so $g_1 = |f|^{-1}\overline{f}_1$; $g_2 = -|f|^{-1}f_2$, where $|f| = (f_1\overline{f}_1 + f_2\overline{f}_2)$.

The right inverse of g is $h = h_1 + h_2\mathbf{j}$, with $h_1 = |g|^{-1}\overline{g}_1$; $h_2 = -|g|^{-1}g_2$; $|g| = |f|^{-2}(\overline{f}_1 f_1 + \overline{f}_2 f_2) = |f|^{-1}$; then: $h_1 = |g|^{-1}\overline{g}_1 = f_1$, ... So *the right inverse of* g *is* f.

4.2. **Definition.** We will call *hypermeromorphic function*, on U, any almost everywhere defined hyperholomorphic function whose right inverse is hyperholomorphic almost everywhere.

Thanks to Definition 4.1, meromorphic functions in one complex varable are hypermeromorphic.

From Proposition 3.3, the set $\mathcal{M}$ of hypermeromorphic functions is not reduced to the space of meromorphic functions in one complex variable.

Let $\mathcal{M}_0$ be the set of elements f of $\mathcal{M}$ described in Proposition 3.3. $\mathcal{M}_0$ is an $I\!R$-vector space; it also contains f^{-1} and the products $f * f^{-1} = 1$.

Proposition 4.1. *Let f , g be two hypermeromorphic functions on U, then the following conditions are equivalent:*

(i) the product $f * g$ is hypermeremorphic;

(ii) f and g satisfy the system of PDE:

$$g_1(\frac{\partial f_1}{\partial \overline{z}_1} + \frac{\partial \overline{f}_2}{\partial z_2}) + (f_1 - \overline{f}_1)\frac{\partial g_1}{\partial \overline{z}_1} + \overline{f}_2\frac{\partial g_1}{\partial z_2} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_1} = 0$$

$$g_1(\frac{\partial f_1}{\partial \overline{z}_2} - \frac{\partial \overline{f}_2}{\partial z_1}) + (f_1 - \overline{f}_1)\frac{\partial g_1}{\partial \overline{z}_2} - \overline{f}_2\frac{\partial g_1}{\partial z_1} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_2} = 0$$

Proof. Let $f = f_1 + f_2\mathbf{j}$ and $g = g_1 + g_2\mathbf{j}$ two hypermeromorphic functions and $f * g = f_1g_1 - f_2\overline{g}_2 + (f_1g_2 - f_2\overline{g}_1)\mathbf{j}$ their product, then $\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f}_2}{\partial z_2} = 0; \frac{\partial f_1}{\partial \overline{z}_2} + \frac{\partial \overline{f}_2}{\partial z_1} = 0$ and..., on U and, the conditions for the product to be hyperholomorphic are:

$$\frac{\partial(f_1g_1 - f_2\overline{g}_2)}{\partial \overline{z}_1} - \frac{\partial(\overline{f}_1\overline{g}_2 - \overline{f}_2g_1)}{\partial z_2} =$$

$$g_1(\frac{\partial f_1}{\partial \overline{z}_1} + \frac{\partial \overline{f}_2}{\partial z_2}) + f_1\frac{\partial g_1}{\partial \overline{z}_1} - \overline{f}_1\frac{\partial \overline{g}_2}{\partial z_2} + \overline{f}_2\frac{\partial g_1}{\partial z_2} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_1} = 0$$

$$\frac{\partial(f_1g_1 - f_2\overline{g}_2)}{\partial \overline{z}_2} + \frac{\partial(\overline{f}_1\overline{g}_2 - \overline{f}_2g_1)}{\partial z_1} =$$

$$g_1(\frac{\partial f_1}{\partial \overline{z}_2} - \frac{\partial \overline{f}_2}{\partial z_1}) + f_1\frac{\partial g_1}{\partial \overline{z}_2} + \overline{f}_1\frac{\partial \overline{g}_2}{\partial z_1} - \overline{f}_2\frac{\partial g_1}{\partial z_1} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_2} = 0$$

□

Corollary 4.2. *Let f, g be two hypermeromorphic functions on U,whose components are real, then the following conditions are equivalent:*

(i) the product $f * g$ is hypermeremorphic;

(ii) f and g satisfy te system of PDE:

$$g_1(\frac{\partial f_1}{\partial \overline{z}_1} + \frac{\partial f_2}{\partial z_2}) + f_2\frac{\partial g_1}{\partial z_2} - f_2\frac{\partial g_2}{\partial \overline{z}_1} = 0$$

$$g_1(\frac{f_1}{\partial \overline{z}_2} - \frac{\partial f_2}{\partial z_1}) - f_2\frac{\partial g_1}{\partial z_1} - f_2\frac{\partial g_2}{\partial \overline{z}_2} = 0$$

4.3. **Definition.** We will call *hyperalgebraic* the hypermeromorphic functions whose sum and product are hypermeromorphic: see section 4.2.

Proposition 4.3. *The set $\mathcal{M}$ of hypermeromorphic functions on U is a subalgebra of the algebra of quaternionic functions.*

Proposition 4.4. *The set $\mathcal{A}$ of hyperalgebraic functions on U is a "field" with only associativity of the multiplication.*

5. About residue current in quaternionic analysis: particular cases.

5.1. **Residue current in an open set of $\mathbb{C}$.** [D10a],[D10b]

Let $\omega = g(z)dz$ be a meromorphic 1-form on a small enough open set $0 \in U \subset \mathbb{C}$ having 0 as unique pole, with multiplicity k:

$$g = \sum_{l=1}^{k} \frac{a_{-l}}{z^l} + \text{holomorphic function}$$

Note that ω is d-closed.

Let $\psi = \psi_0 d\overline{z} \in \mathcal{D}^1(U)$ be a 1-test form. In general $g\psi$ is not integrable, but the Cauchy principal value

$$Vp[\omega](\psi) = \lim_{\epsilon \to 0} \int_{|z| \geq \epsilon} \omega \wedge \psi$$

exists as a current, and $dVp[\omega] = d"Vp[\omega] = \text{Res}[\omega]$ is the residue current of ω. For any test function φ on U,

$$\text{Res}[\omega](\varphi) = \lim_{\epsilon \to 0} \int_{|z| = \epsilon} \omega \wedge \varphi$$

Then $\text{Res}[\omega] = 2\pi i \ \text{res}_0(\omega)\delta_0 + dB = \sum_{j=0}^{k-1} b_j \frac{\partial^j}{\partial z^j}\delta_0$ where $\text{res}_0(\omega) = a_{-1}$ is the Cauchy residue. We remark that δ_0 is the integration current on the subvariety $\{0\}$ of U, that $D = \sum_{j=0}^{k-1} b_j \frac{\partial^j}{\partial z^j}$ with $b_j = \lambda_j a_{-j}$ where the λ_j are universal constants.

Conversely, given the subvariety $\{0\}$ and the differential operator D, then the meromorphic differential form ω is equal to gdz, up to holomorphic form; hence the residue current $\text{Res}[\omega] = D\delta_0$, can be constructed.

5.2. **Cauchy principal value of a quaternionic 1-form.** Let f be a quaternionic function on an open neighborhood U of 0 in $\mathbb{HI}$ satisfying the conditions of Definition 4.1, with $m_i = 1$, $i = 1, 2$.

Let $\omega = \frac{df}{f} = \frac{1}{f}(df_1 + df_2\mathbf{j})$. We want to extend ω into a current of degree 1; first, consider the part of type (1,0). Let $\psi = \psi_1 d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 + \psi_2 dz_1 \wedge d\overline{z}_1 \wedge d\overline{z}_2$ be a test form.

Define:

$$Vp[\omega](\psi)=\lim_{\varepsilon\to 0}\int_{|f|\geq\varepsilon}(|f_1|^2+|f_2|^2)^{-1}(\overline{f}_1-f_2\mathbf{j})(df_1+df_2\mathbf{j})\wedge(\psi_1 d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2+\psi_2 dz_1\wedge d\overline{z}_1\wedge d\overline{z}_2)$$

We have to prove the existence of $Vp[\omega]$ at least if f is hyperholomorphic (Check the proof in the classical case where f is holomorphic in one complex variable or in two complex variables where the proof is less easy, but don't need the resolution of singularities).

$$(df_1+df_2\mathbf{j})\wedge(\psi_1 d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2+\psi_2 dz_1\wedge d\overline{z}_1\wedge d\overline{z}_2)$$

$$=\big(\frac{\partial f_1}{\partial z_1}\psi_1+\frac{\partial f_1}{\partial z_2}\psi_2-(\frac{\partial f_2}{\partial\overline{z}_1}\overline{\psi}_1-\frac{\partial f_2}{\partial\overline{z}_2}\overline{\psi}_2)\mathbf{j}\big)dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2$$

Take polar coordinates: $\lambda=\parallel q\parallel=(|z_1|^2+|z_2|^2)^{\frac{1}{2}}$ and the spherical coordinates on $\lambda\mathbf{S}^3$. Let $d\sigma$ be the volume element on $\mathbf{S}^3$, and K a convenient universal constant, then:

$$dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2=K\lambda d\lambda\wedge d\sigma$$

$$Vp[\omega](\psi)=\lim_{\varepsilon\to 0}\int_{|f|\geq\varepsilon}(|f_1|^2+|f_2|^2)^{-1}(\overline{f}_1-f_2\mathbf{j})\big(\frac{\partial f_1}{\partial z_1}\psi_1+\frac{\partial f_1}{\partial z_2}\psi_2-(\frac{\partial f_2}{\partial\overline{z}_1}\overline{\psi}_1-\frac{\partial f_2}{\partial\overline{z}_2}\overline{\psi}_2)\mathbf{j}\big)K\lambda d\lambda\wedge d\sigma$$

Same result for the part of type ((0,1) of $Vp[\omega]$.

We will prove the existence of $Vp[\omega]$ in a particular case:

5.2.2. *Particular case*: $f_1=\overline{z}_1$; $f_2=\overline{z}_2$. Then: $\mathcal{D}f=0$.

$$Vp[\omega](\psi)=\lim_{\varepsilon\to 0}\int_{|f|\geq\varepsilon}(|\overline{z}_1|^2+|\overline{z}_2|^2)^{-1}(\overline{z}_1-z_2\mathbf{j})(d\overline{z}_1+d\overline{z}_2\mathbf{j})\wedge(\psi_1 d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2+\psi_2 dz_1\wedge d\overline{z}_1\wedge d\overline{z}_2)$$

$$Vp[\omega](\psi)=\lim_{\varepsilon\to 0}\int_{|f|\geq\varepsilon}(|\overline{z}_1|^2+|\overline{z}_2|^2)^{-1}(\overline{z}_1-z_2\mathbf{j})(\overline{\psi}_2 dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2\mathbf{j})$$

$$Vp[\omega](\psi)=\lim_{\varepsilon\to 0}\int_{\lambda\geq\varepsilon}\lambda^{-2}(\overline{z}_1-z_2\mathbf{j})(K\overline{\psi}_2\lambda d\lambda\wedge d\sigma\mathbf{j})$$

Same result for the part of type ((0,1) of $Vp[\omega]$.

This defines a current of order 0 on $I\!H\!I$.

5.2.3. If $f_2=0$ and $f=f_1$ is *holomorphic*, then

$$Vp[\omega](\psi)=\lim_{\varepsilon\to 0}\int_{|f|\geq\varepsilon}\frac{df}{f}\wedge(\psi_1 d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2+\psi_2 dz_1\wedge d\overline{z}_1\wedge d\overline{z}_2)$$

$$=Vp[\omega](\psi)=\lim_{\varepsilon\to 0}\int_{|f|\geq\varepsilon}\frac{1}{f}(\frac{\partial f}{\partial z_1}\psi_1+\frac{\partial f}{\partial z_2}\psi_2)dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2$$

5.3. **Residue.**

5.3.1. *Assume:* $Z(f)=\{0\}$. We want to define a current, $\mathrm{Res}[\omega]$, and first its part of type $(1,1)$, on a test form $\varphi=\varphi_{11}dz_1\wedge d\overline{z}_1+\varphi_{12}dz_1\wedge d\overline{z}_2+\varphi_{21}dz_2\wedge d\overline{z}_1+\varphi_{22}dz_2\wedge d\overline{z}_2$, as follows:

$$\mathrm{Res}[\omega](\varphi)=\lim_{\varepsilon\to 0}\int_{|f|=\varepsilon}(|f_1|^2+|f_2|^2)^{-1}(\overline{f}_1-f_2\mathbf{j})(df_1+df_2\mathbf{j})(\varphi) \tag{5}$$

$$(df_1+df_2\mathbf{j})(\varphi)=$$

$$=\big(\frac{\partial f_1}{\partial z_1}dz_1+\frac{\partial f_1}{\partial z_2}dz_2-(\frac{\partial f_2}{\partial \overline{z}_1}d\overline{z}_1-\frac{\partial f_2}{\partial \overline{z}_2}d\overline{z}_2)\mathbf{j}\big)(\varphi_{11}dz_1\wedge d\overline{z}_1+\varphi_{12}dz_1\wedge d\overline{z}_2+\varphi_{21}dz_2\wedge d\overline{z}_1+\varphi_{22}dz_2\wedge d\overline{z}_2)$$

$$=\frac{\partial f_1}{\partial z_1}dz_1\wedge(\varphi_{21}dz_2\wedge d\overline{z}_1+\varphi_{22}dz_2\wedge d\overline{z}_2)+\frac{\partial f_1}{\partial z_2}dz_2\wedge(\varphi_{11}dz_1\wedge d\overline{z}_1+\varphi_{12}dz_1\wedge d\overline{z}_2)$$

$$-\frac{\partial f_2}{\partial \overline{z}_1}d\overline{z}_1\mathbf{j}\wedge(\varphi_{21}dz_2\wedge d\overline{z}_1+\varphi_{22}dz_2\wedge d\overline{z}_2)+\frac{\partial f_2}{\partial \overline{z}_2}d\overline{z}_2\mathbf{j}\wedge(\varphi_{11}dz_1\wedge d\overline{z}_1+\varphi_{12}dz_1\wedge d\overline{z}_2)$$

$$=(-\frac{\partial f_1}{\partial z_1}\varphi_{21}+\frac{\partial f_1}{\partial z_2}\varphi_{11})dz_1\wedge d\overline{z}_1\wedge dz_2+(\frac{\partial f_1}{\partial z_1}\varphi_{22}-\frac{\partial f_1}{\partial z_2}\varphi_{12})dz_1\wedge dz_2\wedge d\overline{z}_2$$

$$+\big(\frac{\partial f_2}{\partial \overline{z}_1}\overline{\varphi}_{21}-\frac{\partial f_2}{\partial \overline{z}_2}\overline{\varphi}_{11}\big)\mathbf{j}dz_1\wedge d\overline{z}_1\wedge dz_2+\big(-\frac{\partial f_2}{\partial \overline{z}_1}\overline{\varphi}_{22}-\frac{\partial f_2}{\partial \overline{z}_2}\overline{\varphi}_{12}\big)\mathbf{j}dz_1\wedge dz_2\wedge d\overline{z}_2$$

$$=(A+B\mathbf{j})dz_1\wedge d\overline{z}_1\wedge dz_2+(C+D\mathbf{j})dz_1\wedge dz_2\wedge d\overline{z}_2$$

Take polar coordinates: $\lambda=\parallel q\parallel=(|z_1|^2+|z_2|^2)^{\frac{1}{2}}$ and the spherical coordinates on $\mathbf{S}^3$. Let $d\sigma$ the volume element on $\mathbf{S}^3$, and K_j, $(j=1,2)$ a convenient universal constant:

$$dz_1\wedge d\overline{z}_1\wedge dz_2|_{\mathbf{S}^3}=K_1d\lambda\wedge d\sigma$$
$$dz_1\wedge dz_2\wedge d\overline{z}_2|_{\mathbf{S}^3}=K_2d\lambda\wedge d\sigma$$

Same result for the part of any type of $\mathrm{Res}[\omega]$.

5.3.2. *Particular case.* $f_1=\overline{z}_1$; $f_2=\overline{z}_2$. Then: $\mathcal{D}f=0$.

$$(d\overline{z}_1+d\overline{z}_2\mathbf{j})(\varphi)=\varphi_{22}d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2+d\overline{z}_2\mathbf{j}\wedge\varphi_{12}dz_1\wedge d\overline{z}_2$$
$$=\varphi_{22}d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2+\overline{\varphi}_{12}d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2\mathbf{j}$$

$$\mathrm{Res}[\omega](\varphi)=\lim_{\varepsilon\to 0}\int_{|f|=\varepsilon}(|z_1|^2+|z_2|^2)^{-1}(z_1-\overline{z}_2\mathbf{j})(\varphi_{22}d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2+\overline{\varphi}_{12}d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2\mathbf{j})$$
$$=\lim_{\varepsilon\to 0}\int_{\|f\|=\varepsilon}(|z_1|^2+|z_2|^2)^{-1}(z_1-\overline{z}_2\mathbf{j})(\varphi_{22}-\overline{\varphi}_{12}\mathbf{j})d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2)$$

$$= \lim_{\varepsilon\to 0}\int_{\mathbf{S}^3}(|z_1|^2+|z_2|^2)^{-1}(z_1-\overline{z}_2\mathbf{j})(\varphi_{22}-\overline{\varphi}_{12}\mathbf{j})\varepsilon d\sigma$$

$$= \mathrm{res}[\omega](\varphi_{22}-\overline{\varphi}_{12}\mathbf{j})(0) = \delta_0(\varphi_{22}-\overline{\varphi}_{12}\mathbf{j})$$

$\int_{\|f\|=\varepsilon}$ means the integration on $\varepsilon\mathbf{S}^3$, where $\mathbf{S}^3$ is the unit 3-sphere and $d\sigma$ the volume element on $\mathbf{S}^3$, res$[\omega]$, a constant playing the part of the Cauchy residue, and δ_0 the Dirac measure at 0.

Same result for the part of any type of Res$[\omega]$.

5.3.3. *f holomorphic.* If $f_2 = 0$ and $f = f_1$ is holomorphic,

$$\mathrm{Res}[\omega](\varphi) = \lim_{\varepsilon\to 0}\int_{|f_1|=\varepsilon}\frac{df_1}{f_1}(\varphi)$$

Assume $f_1 = z_1$, then: $z_1 = \varepsilon^{i\theta}$, and

$$\mathrm{Res}[\omega](\varphi) = \lim_{\varepsilon\to 0}\int_0^{2\pi} id\theta\varphi_{2,2}(\varepsilon e^{i\theta}, z_2)dz_2\wedge d\overline{z}_2 = 2\pi i[Z(z_1)](\varphi)$$

Same result for any holomorphic f_1 since $f_1 = \varepsilon e^{i\theta}$.

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