SPECTRAL ESTIMATES ON THE SPHERE

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Abstract. In this article we establish optimal estimates for the first eigenvalue of Schrödinger operators on the d-dimensional unit sphere. These estimates depend on L^p norms of the potential, or of its inverse, and are equivalent to interpolation inequalities on the sphere. We also characterize a semi-classical asymptotic regime and discuss how our estimates on the sphere differ from those on the Euclidean space.

1. Introduction

Let Δ be the Laplace-Beltrami operator on the unit d-dimensional sphere $\mathbb{S}^d$. Our first result is concerned with the sharp estimate of the first negative eigenvalue $\lambda_1 = \lambda_1(-\Delta - V)$ of the Schrödinger operator $-\Delta - V$ on $\mathbb{S}^d$ (with potential $-V$) in terms of L^p-norms of V.

The literature on spectral estimates for the negative eigenvalues of Schrödinger operators on manifolds is limited. We can quote two papers of P. Federbusch and O.S. Rothaus, [16, 33], which establish a link between logarithmic Sobolev inequalities and the ground state energy of Schrödinger operators. The Rozenbljum-Lieb-Cwikel inequality (case $\gamma = 0$ with standard notations: see below) on manifolds has been studied in [25, Section 5]; we may also refer to [26] for the semi-classical regime, and to [24, 31] for more recent results in this direction. In two articles (see [20, 21]) on Lieb-Thirring type inequalities (also see [24, 31] for other results on manifolds), A. Ilyin considers Schrödinger operators on unit spheres restricted to the space of functions orthogonal to constants and uses the original method of E. Lieb and W. Thirring in [27]. The exclusion of the zero mode of the Laplace-Beltrami operator results in *semi-classical* estimates similar to those for negative eigenvalues of Schrödinger operators in Euclidean spaces.

The results in this paper are somewhat complementary. We show that if the L^p-norm of V is smaller than an explicit value, then the first eigenvalue $\lambda_1(-\Delta - V)$ cannot satisfy the semi-classical inequality and thus it is impossible to obtain standard Lieb-Thirring type inequalities for the whole negative spectrum. However, we show that if the L^p-norm of the potential is large then the first eigenvalue behaves semi-classically and the best constant in the inequality asymptotically coincides with the best constants $L^1_{\gamma,d}$ of the corresponding inequality in the Euclidean space of same dimension (see below). In this regime the first eigenfunction is concentrated around some point on $\mathbb{S}^d$ and can be identified with an eigenfunction of the Schrödinger operator on the tangent space, up to a small error. In Appendix A, we illustrate the transition between the small L^p-norm regime and the asymptotic, semi-classical regime by numerically computing the optimal estimates for the eigenvalue $\lambda_1(-\Delta - V)$ in terms of the norms $\|V\|_{L^p(\mathbb{S}^d)}$.

In order to formulate our first theorem let us introduce the measure $d\omega$ induced by Lebesgue's measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ and the uniform probability measure $d\sigma = d\omega/|\mathbb{S}^d|$ with $|\mathbb{S}^d| = \omega(\mathbb{S}^d)$. We shall denote by $\|\cdot\|_{L^q(\mathbb{S}^d)}$ the quantity $\|u\|_{L^q(\mathbb{S}^d)} = \big(\int_{\mathbb{S}^d} |u|^q \, d\sigma\big)^{1/q}$ for any $q > 0$ (hence including in the case $q \in (0,1)$, for which $\|\cdot\|_{L^q(\mathbb{S}^d)}$ is not anymore a norm, but only a quasi-norm). Because of the normalization of $d\sigma$, when making comparisons with corresponding results in the Euclidean space, we will need the constant

$$\kappa_{q,d} := |\mathbb{S}^d|^{1-\frac{2}{q}} .$$

The well-known optimal constant $L^1_{\gamma,d}$ in the one bound state Keller-Lieb-Thirring inequality is defined as follows: for any function ϕ on $\mathbb{R}^d$, if $\lambda_1(-\Delta - \phi)$ denotes the lowest negative eigenvalue of the Schrödinger

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operator $-\Delta-\phi$ (with potential $-\phi$) when it exists, and 0 otherwise, we have

$$|\lambda_1(-\Delta-\phi)|^\gamma \le \mathrm{L}^1_{\gamma,d}\int_{\mathbb{R}^d}\phi_+^{\gamma+\frac{d}{2}}\,dx\,, \tag{1}$$

provided $\gamma\ge 0$ if $d\ge 3$, $\gamma>0$ if $d=2$, and $\gamma\ge 1/2$ if $d=1$. Notice that only the positive part ϕ_+ of ϕ is involved in the right-hand side of the above inequality. Assuming that $\gamma>1-d/2$ if $d=1$ or 2, we shall consider the exponents

$$q=2\,\frac{2\,\gamma+d}{2\,\gamma+d-2}\quad\text{and}\quad p=\frac{q}{q-2}=\gamma+\frac{d}{2}\,,$$

which are therefore such that $2<q=\frac{2\,p}{p-1}\le 2^*$ with $2^*:=\frac{2\,d}{d-2}$ if $d\ge 3$, and $q=\frac{2\,p}{p-1}\in(2,+\infty)$ if $d=1$ or 2. To simplify notations, we adopt the convention $2^*:=\infty$ if $d=1$ or 2. It is also convenient to introduce the notation

$$\alpha_*:=\frac14\,d\,(d-2)\,.$$

In Section 2 we shall prove the following result.

Theorem 1. *Let $d\ge 1$, $p\in\big(\max\{1,d/2\},+\infty\big)$. Then there exists a convex increasing function $\alpha:\mathbb{R}^+\to\mathbb{R}^+$ with $\alpha(\mu)=\mu$ for any $\mu\in\big[0,\frac{d}{2}\,(p-1)\big]$ and $\alpha(\mu)>\mu$ for any $\mu\in\big(\frac{d}{2}\,(p-1),+\infty\big)$, such that*

$$|\lambda_1(-\Delta-V)|\le\alpha\big(\|V\|_{\mathrm{L}^p(\mathbb{S}^d)}\big) \tag{2}$$

for any nonnegative $V\in\mathrm{L}^p(\mathbb{S}^d)$. Moreover, for large values of μ, we have

$$\alpha(\mu)^{p-\frac{d}{2}}=\mathrm{L}^1_{p-\frac{d}{2},d}\,(\kappa_{q,d}\,\mu)^p\,(1+o(1))\,.$$

The estimate (2) is optimal in the sense that there exists a nonnegative function V such that $\mu=\|V\|_{\mathrm{L}^p(\mathbb{S}^d)}$ and $|\lambda_1(-\Delta-V)|=\alpha(\mu)$ for any $\mu\in\big(\frac{d}{2}\,(p-1),+\infty\big)$. If $\mu\le\frac{d}{2}\,(p-1)$, equality in (2) is achieved by constant potentials.

If $p=d/2$ and $d\ge 3$, then (2) is satisfied with $\alpha(\mu)=\mu$ only for $\mu\in[0,\alpha_*]$. If $d=p=1$, then (2) is also satisfied for some nonnegative, convex function α on $\mathbb{R}^+$ such that $\mu\le\alpha(\mu)\le\mu+\pi^2\,\mu^2$ for any $\mu\in(0,+\infty)$, equality in (2) is achieved and $\alpha(\mu)=\pi^2\,\mu^2(1+o(1))$ as $\mu\to+\infty$.

Since $\lambda_1(-\Delta-V)$ is nonpositive for any nonnegative, nontrivial V, inequality (2) is a lower estimate. We have indeed found that

$$0\ge\lambda_1(-\Delta-V)\ge-\alpha\big(\|V\|_{\mathrm{L}^p(\mathbb{S}^d)}\big)\,.$$

If V changes sign, the above inequality still holds if V is replaced by the positive part V_+ of V, provided the lowest eigenvalue is negative. We can then write

$$|\lambda_1(-\Delta-V)|\le\alpha\left(\|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)}\right)\quad\forall\,V\in\mathrm{L}^p(\mathbb{S}^d)\,. \tag{3}$$

The expression of $\mathrm{L}^1_{\gamma,d}$ is not explicit (except in the case $d=1$: see [27, p. 290]) but can be given in terms of an optimal constant in some Gagliardo-Nirenberg-Sobolev inequality (see [27], and (9)–(10) below in Section 2.1). In case $d=p=1$, notice that $\mathrm{L}^1_{1/2,1}=1/2$ (see Appendix B.2) and $\kappa_{\infty,1}=2\pi$ so that our formula in the asymptotic regime $\mu\to+\infty$ is consistent with the other cases.

The reader is invited to check that Theorem 1 can be reformulated in a more standard language of spectral theory as follows. We recall that $\gamma=p-d/2$ and that $d\omega$ is the standard measure induced on the unit sphere $\mathbb{S}^d$ by Lebesgue's measure on $\mathbb{R}^{d+1}$.

Corollary 2. *Let $d\ge 1$ and consider a nonnegative function V. For $\mu=\|V\|_{\mathrm{L}^{\gamma+\frac{d}{2}}(\mathbb{S}^d)}$ large, we have*

$$|\lambda_1(-\Delta-V)|^\gamma\lesssim\mathrm{L}^1_{\gamma,d}\int_{\mathbb{S}^d}V^{\gamma+\frac{d}{2}}\,d\omega \tag{4}$$

if either $\gamma>\max\{0,1-d/2\}$ or $\gamma=1/2$ and $d=1$. However, if $\mu=\|V\|_{\mathrm{L}^{\gamma+\frac{d}{2}}(\mathbb{S}^d)}\le\frac14\,d\,(2\,\gamma+d-2)$, then we have

$$|\lambda_1(-\Delta-V)|^{\gamma+\frac{d}{2}}\le\int_{\mathbb{S}^d}V^{\gamma+\frac{d}{2}}\,d\omega \tag{5}$$

for any $\gamma\ge\max\{0,1-d/2\}$ and this estimate is optimal.

Here the notation $f \lesssim g$ as $\mu \to +\infty$ means that $f \le c(\mu)\, g$ with $\lim_{\mu\to\infty} c(\mu) = 1$. The limit case $\gamma = \max\{0, 1 - d/2\}$ in (5) is covered by approximations. We may also notice that optimality in (5) is achieved by constant potentials. Let us give some details.

If we consider a sequence of constant functions $(V_n)_{n\in\mathbb{N}}$ uniformly converging towards 0, for instance $V_n = 1/n$, then we get that

$$\lim_{n\to\infty} \frac{|\lambda_1(-\Delta - V_n)|^\gamma}{\int_{\mathbb{S}^d} V_n^{\gamma+\frac{d}{2}}\, d\omega} = +\infty$$

which clearly forbids the possibility of an inequality of the same type as (4) for small values of $\int_{\mathbb{S}^d} V^{\gamma+\frac{d}{2}}\, d\omega$. This is however compatible with the results of A. Ilyin in dimension $d = 2$. In [21, Theorem 2.1], the author states that if P is the orthogonal projection defined by $P\,u := u - \fint_{\mathbb{S}^2} u\, d\omega$, then the negative eigenvalues $\lambda_k(P\,(-\Delta - V)\,P)$ satisfy the semi-classical inequality

$$\sum_k |\lambda_k(P\,(-\Delta - V)\,P)| \le \frac{3}{8} \int_{\mathbb{S}^2} V^2\, d\omega\,.$$

Another way of seeing that inequalities like (4) are incompatible with small potentials is based on the following observation. Inequality (5) shows that

$$|\lambda_1(-\Delta - V)| \le \Big(\int_{\mathbb{S}^2} V^2\, d\omega \Big)^{1/2}$$

if the L^2-norm of V is smaller than 1. Since such an inequality is sharp, the semi-classical Lieb-Thirring inequalities for the Schrödinger operator on the sphere $\mathbb{S}^2$ are therefore impossible for small potentials and can be achieved only in a semi-classical asymptotic regime, that is, when the norm $\|V\|_{\mathrm{L}^2(\mathbb{S}^2)}$ is large.

Our second main result is concerned with the estimates from below for the first eigenvalue of Schrödinger operators with positive potentials. In this case, by analogy with (1), it is convenient to introduce the constant $\mathrm{L}^1_{-\gamma,d}$ with $\gamma > d/2$ which is the optimal constant in the inequality:

$$\lambda_1(-\Delta + \phi)^{-\gamma} \le \mathrm{L}^1_{-\gamma,d} \int_{\mathbb{R}^d} \phi^{\frac{d}{2}-\gamma}\, dx\,, \tag{6}$$

where ϕ is any positive potential on $\mathbb{R}^d$ and $\lambda_1(-\Delta+\phi)$ denotes the lowest positive eigenvalue if it exists, or $+\infty$ otherwise. Inequality (6) is less standard than (1): we refer to [15, Theorem 12] for a statement and a proof. As in Theorem 1, we shall also introduce exponents p and q such that

$$q = 2\,\frac{2\,\gamma - d}{2\,\gamma - d + 2} \quad \text{and} \quad p = \frac{q}{2-q} = \gamma - \frac{d}{2}\,,$$

so that p (resp. $q = \frac{2\,p}{p+1}$) takes arbitrary values in $(0, +\infty)$ (resp. $(0, 2)$). With these notations, we have the counterpart of Theorem 1 in the case of positive potentials.

Theorem 3. *Let $d \ge 1$, $p \in (0, +\infty)$. There exists a concave increasing function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ with $\nu(\beta) = \beta$ for any $\beta \in \big[0, \frac{d}{2}\,(p+1)\big]$ if $p > 1$, $\nu(\beta) \le \beta$ for any $\beta > 0$ and $\nu(\beta) < \beta$ for any $\beta \in \big(\frac{d}{2}\,(p+1), +\infty\big)$, such that*

$$\lambda_1(-\Delta + W) \ge \nu\big(\beta\big) \quad \textit{with} \quad \beta = \|W^{-1}\|^{-1}_{\mathrm{L}^p(\mathbb{S}^d)}\,, \tag{7}$$

for any positive potential W such that $W^{-1} \in \mathrm{L}^p(\mathbb{S}^d)$. Moreover, for large values of β, we have

$$\nu(\beta)^{-\,(p+\frac{d}{2})} \lesssim \mathrm{L}^1_{-(p+\frac{d}{2}),d}\,(\kappa_{q,d}\,\beta)^{-p}\,.$$

The estimate (7) is optimal in the sense that there exists a nonnegative potential W such that $\beta^{-1} = \|W^{-1}\|_{\mathrm{L}^p(\mathbb{S}^d)}$ and $\lambda_1(-\Delta+W) = \nu(\beta)$ for any positive β and p. If $\beta \le \frac{d}{2}\,(p+1)$ and $p > 1$, equality in (7) is achieved by constant potentials.

Again the expression of $\mathrm{L}^1_{-\gamma,d}$ is not explicit when $d \ge 2$ but can be given in terms of an optimal constant in some Gagliardo-Nirenberg-Sobolev inequality (see [15], and (17)–(18) below in Section 4).

We can rewrite Theorem 3 in terms of $\gamma = p + d/2$ and explicit integrals involving W.

Corollary 4. *Let $d \ge 1$ and $\gamma > d/2$. For $\beta = \|W^{-1}\|^{-1}_{\mathrm{L}^{\gamma-\frac{d}{2}}(\mathbb{S}^d)}$ large, we have*

$$\big(\lambda_1(-\Delta + W)\big)^{-\gamma} \lesssim \mathrm{L}^1_{-\gamma,d} \int_{\mathbb{S}^d} W^{\frac{d}{2}-\gamma}\, d\omega\,.$$

However, if $\gamma \ge \frac{d}{2}+1$ and if $\beta = \|W^{-1}\|^{-1}_{\mathrm{L}^{\gamma-\frac{d}{2}}(\mathbb{S}^d)} \le \frac{1}{4}\,d\,(2\,\gamma - d + 2)$, then we have

$$\big(\lambda_1(-\Delta + W)\big)^{\frac{d}{2}-\gamma} \le \int_{\mathbb{S}^d} W^{\frac{d}{2}-\gamma}\, d\omega\,,$$

and this estimate is optimal.

This paper is organized as follows. Section 2 contains various results on interpolation inequalities; the most important one for our purpose is stated in Lemma 5. Theorem 1, Corollary 2 and various spectral estimates for Schrödinger operators with negative potentials are established in Section 3. Section 4 deals with the case of positive potentials and contains the proofs of Theorem 3 and Corollary 4. Section 5 is devoted to the threshold case ($q = 2$, that is, p, $\gamma \to +\infty$) of exponential estimates for eigenvalues or, in terms of interpolation inequalities, to logarithmic Sobolev inequalities. Finally numerical and technical results have been collected in two appendices.

2. Interpolation inequalities and consequences for negative potentials

2.1. **Inequalities in the Euclidean space.** Let us start by some considerations on inequalities in the Euclidean space, which play a crucial role in the semi-classical regime.

We recall that we denote by 2^* the Sobolev critical exponent $\frac{2d}{d-2}$ if $d \ge 3$ and consider Sobolev's inequality on $\mathbb{R}^d$, $d \ge 3$,

$$\|v\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \le \mathsf{S}_d\, \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, v \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{8}$$

where S_d is the optimal constant and $\mathcal{D}^{1,2}(\mathbb{R}^d)$ is the Beppo-Levi space obtained by completion of smooth compacty supported functions with respect to the norm $v \mapsto \|\nabla v\|_{\mathrm{L}^2(\mathbb{R}^d)}$. See Appendix B.4 for details and comments on the expression of S_d.

Assume now that $d \ge 1$ and recall that $2^* = +\infty$ if $d = 1$ or 2. In the subcritical case, that is, $q \in (2, 2^*)$, let

$$\mathsf{K}_{q,d} := \inf_{v \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}}$$

be the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality

$$\mathsf{K}_{q,d}\, \|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \le \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, v \in \mathrm{H}^1(\mathbb{R}^d)\,. \tag{9}$$

The optimal constant $\mathrm{L}^1_{\gamma,d}$ in the one bound state Keller-Lieb-Thirring inequality is such that

$$\mathrm{L}^1_{\gamma,d} := (\mathsf{K}_{q,d})^{-\,p} \quad \text{with} \quad p = \gamma + \frac{d}{2}\,, \quad q = 2\,\frac{2\,\gamma + d}{2\,\gamma + d - 2}\,. \tag{10}$$

See Appendix B.5 for a proof and references, and [27] for a detailed discussion. Also see [27, Appendix A. Numerical studies, by J.F. Barnes] for numerical values of $\mathsf{K}_{q,d}$.

We shall also define the exponent

$$\vartheta := d\,\frac{q-2}{2\,q}$$

which plays an important role in the scale invariant form of the Gagliardo-Nirenberg-Sobolev interpolation inequalities associated to $\mathsf{K}_{q,d}$: see Appendix B.1 for details.

2.2. **Interpolation inequalities on the sphere.** Using the inverse stereographic projection (see Appendix B.3), it is possible to relate interpolation inequalities on $\mathbb{R}^d$ with interpolation inequalities on $\mathbb{S}^d$. In this section we consider the case of the sphere. Notice that $\alpha_* = d/(q-2)$ when $q = 2^* = 2\,d/(d-2)$, $d \ge 3$.

Lemma 5. *Let $q \in (2, 2^*)$. Then there exists a concave increasing function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ with the following properties*

$$\mu(\alpha) = \alpha \quad \forall\, \alpha \in \left[0, \tfrac{d}{q-2}\right] \quad \text{and} \quad \mu(\alpha) < \alpha \quad \forall\, \alpha \in \left(\tfrac{d}{q-2}, +\infty\right),$$

$$\mu(\alpha) = \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\, \alpha^{1-\vartheta}\, (1 + o(1)) \quad \text{as} \quad \alpha \to +\infty\,,$$

such that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \alpha\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\alpha)\, \|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)\,. \tag{11}$$

If $d \ge 3$ and $q = 2^*$, the inequality also holds for any $\alpha > 0$ with $\mu(\alpha) = \min\{\alpha, \alpha_*\}$.

The remainder of this section is mostly devoted to the proof of Lemma 5. A fundamental tool is a *rigidity* result proved by M.-F. Bidaut-Véron and L. Véron in [9, Theorem 6.1] for $q > 2$, which goes as follows. Any positive solution of

$$-\,\Delta f + \alpha\, f = f^{q-1} \tag{12}$$

has a unique solution $f \equiv \alpha^{1/(q-2)}$ for any $0 < \alpha \le d/(q-2)$. A straightforward consequence of this rigidity result is the following interpolation inequality (see [9, Corollary 6.2]):

$$\int_{\mathbb{S}^d} |\nabla u|^2\, d\sigma \ge \frac{d}{q-2} \left[\left(\int_{\mathbb{S}^d} |u|^q\, d\sigma \right)^{2/q} - \int_{\mathbb{S}^d} |u|^2\, d\sigma \right] \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma)\,. \tag{13}$$

Inequality (13) holds for any $q \in [1,2) \cup (2, 2^*]$ if $d \ge 3$ and for any $q \in [1,2) \cup (2, \infty)$ if $d = 1$ or 2. An alternative proof of (13) has been established in [5] for $q > 2$ using previous results by E. Lieb in [28] and the Funk-Hecke formula (see [17, 19]). The whole range $p \in [1,2) \cup (2, 2^*)$ was covered in the case of the ultraspherical operator in [7, 8]. Also see [4, 23] for the *carré du champ* method, and [14] for an elementary proof. Inequality (13) is *tight* as defined by D. Bakry in [3, Section 2], in the sense that equality is achieved only by constants.

Remark 6. *Inequality* (13) *is equivalent to*

$$\inf_{u \in \mathrm{H}^1(\mathbb{S}^d) \setminus \{0\}} \frac{(q-2)\, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} = d\,.$$

Although we will not make use of them in this paper, we may notice that the following properties hold true:

(i) *If $q < 2^*$, the above infimum is not achieved in $\mathrm{H}^1(\mathbb{S}^d) \setminus \{0\}$ but*

$$\lim_{\varepsilon \to 0_+} \frac{(q-2)\, \|\nabla u_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u_\varepsilon\|^2_{\mathrm{L}^q(\mathbb{S}^d)} - \|u_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} = d$$

if $u_\varepsilon := 1 + \varepsilon\, \varphi$, where φ is a non-trivial eigenfunction of the Laplace-Beltrami operator corresponding the first nonzero eigenvalue (see below Section 2.3).

(ii) *If $q = 2^*$, $d \ge 3$, there are non-trivial optimal functions for* (13)*, due to the conformal invariance. Alternatively, these solutions can be constructed from the family of Aubin-Talenti optimal functions for Sobolev's inequality, using the inverse stereographic projection.*

(iii) *If $\alpha > \alpha_*$ and $q = 2^*$, $d \ge 3$, there are no optimal functions for* (11)*, since otherwise $\alpha \mapsto \mu(\alpha)$ would not be constant on (α_*, α): see Proposition 7 below.*

2.3. **Properties of the function $\alpha\mapsto\mu(\alpha)$ in the subcritical case.** Assume that $q\in(2,2^*)$. For any $\alpha>0$, consider

$$\mathcal{Q}_\alpha[u]:=\frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d,d\sigma)\,.$$

It is a standard result of the calculus of variations that

$$\inf_{\substack{u\,\in\,\mathrm{H}^1(\mathbb{S}^d,\,d\sigma)\\ \int_{\mathbb{S}^d}|u|^q\,d\sigma\,=\,1}}\mathcal{Q}_\alpha[u]:=\mu(\alpha)$$

is achieved by a minimizer $u\in\mathrm{H}^1(\mathbb{S}^d,d\sigma)$ which solves the Euler-Lagrange equations

$$-\,\Delta u+\alpha\,u-\mu(\alpha)\,u^{q-1}=0\,. \tag{14}$$

Indeed we know that there is a Lagrange multiplier associated to the constraint $\int_{\mathbb{S}^d}|u|^q\,d\sigma=1$, and multiplying (14) by u and integrating on $\mathbb{S}^d$, we can identify it with $\mu(\alpha)$. As a corollary, we have shown that (11) holds. The fact that the Lagrange multiplier can be identified so easily is a consequence of the fact that all terms in (11) are two-homogeneous.

We can now list some basic properties of the function $\alpha\mapsto\mu(\alpha)$.

(1) For any $\alpha>0$, $\mu(\alpha)$ is positive, since the infimum is achieved by a nonnegative function u and $u=0$ is incompatible with the constraint $\int_{\mathbb{S}^d}|u|^q\,d\sigma=1$. By taking a constant test function, we see that $\mu(\alpha)\le\alpha$, for all $\alpha>0$. The function $\alpha\mapsto\mu(\alpha)$ is monotone nondecreasing since for a given $u\in\mathrm{H}^1(\mathbb{S}^d,d\sigma)\setminus\{0\}$, the function $\alpha\mapsto\mathcal{Q}_\alpha[u]$ is monotone increasing. It is actually strictly monotone. Indeed if $\mu(\alpha_1)=\mu(\alpha_2)$ with $\alpha_1<\alpha_2$, then one can notice that $\mathcal{Q}_{\alpha_1}[u_2]<\mu(\alpha_1)$ if u_2 is a minimizer of $\mathcal{Q}_{\alpha_2}$ satisfying the constraint $\int_{\mathbb{S}^d}|u_2|^q\,d\sigma=1$, which provides an obvious contradiction.

(2) We have

$$\mu(\alpha)=\alpha\quad\forall\,\alpha\in\left(0,\tfrac{d}{q-2}\right]\,.$$

Indeed, if u is a solution of (14), then $f=\mu(\alpha)^{1/(q-2)}\,u$ solves (12) and is therefore a constant function if $\alpha\le d/(q-2)$ according to [9, Theorem 6.1], and so is u as well. Because of the normalization constraint $\|u\|_{\mathrm{L}^q(\mathbb{S}^d)}=1$, we get that $u=1$, which proves the statement.

On the contrary, we have

$$\mu(\alpha)<\alpha\quad\forall\,\alpha>\frac{d}{q-2}\,.$$

Let us prove this. Let φ be a non-trivial eigenfunction of the Laplace-Beltrami operator corresponding the first nonzero eigenvalue:

$$-\,\Delta\varphi=d\,\varphi\,.$$

If $x=(x_1,x_2,...x_d,z)$ are cartesian coordinates of $x\in\mathbb{R}^{d+1}$ so that $\mathbb{S}^d\subset\mathbb{R}^{d+1}$ is characterized by the condition $\sum_{i=1}^d x_i^2+z^2=1$, then a simple choice of such a function φ is $\varphi(x)=z$. By orthogonality with respect to the constants, we know that $\int_{\mathbb{S}^d}\varphi\,d\sigma=0$. We may now Taylor expand $\mathcal{Q}_\alpha$ around $u=1$ by considering $u=1+\varepsilon\,\varphi$ as $\varepsilon\to0$ and obtain that

$$\mu(\alpha)\le\mathcal{Q}_\alpha[1+\varepsilon\,\varphi]=\frac{(d+\alpha)\,\varepsilon^2\int_{\mathbb{S}^d}|\varphi|^2\,d\sigma+\alpha}{\left(\int_{\mathbb{S}^d}|1+\varepsilon\,\varphi|^q\,d\sigma\right)^{2/q}}=\alpha+\left[d+\alpha\,(2-q)\right]\varepsilon^2\int_{\mathbb{S}^d}|\varphi|^2\,d\sigma+o(\varepsilon^2)\,.$$

By taking ε small enough, we get $\mu(\alpha)<\alpha$ for all $\alpha>d/(q-2)$. Optimizing on the value of $\varepsilon>0$ (not necessarily small) provides an interesting test function: see Section A.1.

(3) The function $\alpha\mapsto\mu(\alpha)$ is concave, because it is the minimum of a family of affine functions.

2.4. **More estimates on the function $\alpha \mapsto \mu(\alpha)$.** We first consider the critical case $q = 2^*$, $d \geq 3$. As in the subcritical case $q < 2^*$, we have $\mu(\alpha) = \alpha$ for $\alpha \leq \alpha^*$. For $\alpha > \alpha^*$, the function $\alpha \mapsto \mu(\alpha)$ is constant:

Proposition 7. *With the notations of Lemma 5, if $d \geq 3$ and $q = 2^*$, then*

$$\mu(\alpha) = \alpha_* \quad \forall\, \alpha > \alpha_* = \frac{d}{q-2} = \frac{1}{4}\, d\,(d-2)\,.$$

Proof. Consider the Aubin-Talenti optimal functions for Sobolev's inequality and more specifically, let us choose the functions

$$v_\varepsilon(x) := \left(\tfrac{\varepsilon}{\varepsilon^2+|x|^2}\right)^{\frac{d-2}{2}} \qquad \forall\, x \in \mathbb{R}^d\,, \quad \forall\, \varepsilon > 0\,,$$

which are such that $\|v_\varepsilon\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = \|v_1\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ is independent of ε. With standard notations (see Appendix B.3), let $\mathrm{N} \in \mathbb{S}^d$ be the North Pole. Using the stereographic projection Σ, *i.e.* for the functions defined for any $y \in \mathbb{S}^d \setminus \{\mathrm{N}\}$ by

$$u_\varepsilon(y) = \left(\tfrac{|x|^2+1}{2}\right)^{\frac{d-2}{2}} v_\varepsilon(x) \quad \text{with} \quad x = \Sigma(y)\,,$$

we find that $\|u_\varepsilon\|_{\mathrm{L}^{2^*}(\mathbb{S}^d)} = \|v_1\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ for any $\varepsilon > 0$, so that

$$\mu(\alpha) \leq \mathcal{Q}_\alpha[u_\varepsilon] = \frac{\|\nabla v_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + (\alpha-\alpha_*)\int_{\mathbb{R}^d} |v_\varepsilon|^2 \left(\frac{2}{1+|x|^2}\right)^2 dx}{\kappa_{2^*,d}\, \|v_\varepsilon\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}} = \alpha_* + 4\,|\mathbb{S}^d|^{1-\frac{2}{d}}\,(\alpha-\alpha_*)\,\frac{\delta(d,\varepsilon)}{\|v_1\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}$$

where we have used the fact that $\kappa_{2^*,d}\,\mathsf{S}_d = 1/\alpha_*$ (see Appendix B.4) and

$$\delta(d,\varepsilon) := \int_0^\infty \left(\frac{\varepsilon}{\varepsilon^2+r^2}\right)^{d-2} \frac{r^{d-1}}{(1+r^2)^2}\, dr = \varepsilon^2 \int_0^\infty \left(\frac{1}{1+s^2}\right)^{d-2} \frac{s^{d-1}}{(1+\varepsilon^2\,s^2)^2}\, ds\,.$$

One can check that $\lim_{\varepsilon\to 0_+} \delta(d,\varepsilon) = 0$ since

$$\delta(d,\varepsilon) \leq \varepsilon^2 \int_0^\infty \frac{s^{d-1}}{(1+s^2)^{d-2}}\, ds \quad \text{if} \quad d \geq 5 \quad \text{and} \quad \delta(d,\varepsilon) \leq \varepsilon\, c_d \int_0^{+\infty} \frac{ds}{(1+s^2)^2} \quad \text{if} \quad d = 3 \text{ or } 4\,,$$

with $c_3 = 1$ and $c_4 = 3\sqrt{3}/16$. □

The next step is devoted to a lower estimate for the function $\alpha \mapsto \mu(\alpha)$ in the subcritical case, which shows that $\lim_{\alpha\to+\infty} \mu(\alpha) = +\infty$ in contrast with the critical case.

Proposition 8. *With the notations of Lemma 5, if $d \geq 3$ and $q \in (2, 2^*)$, then for any $\alpha > \alpha_*$ we have*

$$\alpha > \mu(\alpha) \geq \alpha_*^\vartheta\, \alpha^{1-\vartheta}\,,$$

with $\vartheta = d\,\frac{q-2}{2\,q}$. For every $s \in (2, 2^*)$ if $d \geq 3$, or every $s \in (2, +\infty)$ if $d = 1$ or 2, such that $s > q$, we also have that

$$\alpha > \mu(\alpha) \geq \left(\tfrac{d}{s-2}\right)^\theta \alpha^{1-\theta}\,,$$

for any $\alpha > d/(s-2)$ and $\theta = \theta(s,q,d) := \frac{s\,(q-2)}{q\,(s-2)}$.

Proof. The first case can be seen as a limit case of the second one as $s \to 2^*$ and $\vartheta = \theta(2^*, q, d)$. Using Hölder's inequality, we can estimate $\|u\|_{\mathrm{L}^q(\mathbb{S}^d)}$ by

$$\|u\|_{\mathrm{L}^q(\mathbb{S}^d)} \leq \|u\|^\theta_{\mathrm{L}^s(\mathbb{S}^d)}\, \|u\|^{1-\theta}_{\mathrm{L}^2(\mathbb{S}^d)}$$

and get the result using

$$\mathcal{Q}_\alpha[u] \geq \left(\frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \alpha\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^s(\mathbb{S}^d)}}\right)^\theta \left(\frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \alpha\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)^{1-\theta} \geq \left(\tfrac{d}{s-2}\right)^\theta \alpha^{1-\theta}\,.$$

□

Proposition 9. *With the notations of Lemma 5, for every $q \in (2, 2^*)$ we have*

$$\limsup_{\alpha\to+\infty} \alpha^{\vartheta-1}\mu(\alpha) \leq \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\,.$$

Proof. Let v be an optimal function for $\mathsf{K}_{q,d}$ and define for any $x \in \mathbb{R}^d$ the function

$$v_\alpha(x) := v\left(2\sqrt{\alpha-\alpha_*}\,x\right)$$

with $\alpha_* = \frac14\,d\,(d-2)$ and $\alpha > \alpha_*$, so that

$$\int_{\mathbb{R}^d} |\nabla v_\alpha|^2\,dx = 2^{2-d}\,(\alpha-\alpha_*)^{1-\frac{d}{2}} \int_{\mathbb{R}^d} |\nabla v|^2\,dx\,,$$

$$\int_{\mathbb{R}^d} |v_\alpha|^q \left(\tfrac{2}{1+|x|^2}\right)^{d-(d-2)\frac{q}{2}}\,dx = 2^{-(d-2)\frac{q}{2}}\,(\alpha-\alpha_*)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |v|^q \left(1+\tfrac{|x|^2}{4\,(\alpha-\alpha_*)}\right)^{-d+(d-2)\frac{q}{2}}\,dx\,.$$

Now we observe that the function $u_\alpha(y) := \left(\frac{|x|^2+1}{2}\right)^{(d-2)/2} v_\alpha(x)$, where $y = \Sigma^{-1}(x)$ and Σ is the stereographic projection (see Appendix B.3), is such that

$$\mathcal{Q}_\alpha[u_\alpha] = \frac{1}{\kappa_{q,d}}\,\frac{\int_{\mathbb{R}^d} |\nabla v_\alpha|^2\,dx + (\alpha-\alpha_*)\int_{\mathbb{R}^d} |v_\alpha|^2 \left(\frac{2}{1+|x|^2}\right)^2 dx}{\left[\int_{\mathbb{R}^d} |v_\alpha|^q \left(\frac{2}{1+|x|^2}\right)^{d-(d-2)\frac{q}{2}}\,dx\right]^{\frac{2}{q}}}\,.$$

Passing to the limit as $\alpha \to +\infty$, we get

$$\lim_{\alpha\to+\infty} \int_{\mathbb{R}^d} |v|^q \left(1+\tfrac{|x|^2}{4\,(\alpha-\alpha_*)}\right)^{-d+(d-2)\frac{q}{2}}\,dx = \int_{\mathbb{R}^d} |v|^q\,dx$$

by Lebesgue's theorem of dominated convergence. The limit also holds with q replaced by 2. This proves that

$$\mathcal{Q}_\alpha[u_\alpha] = (\alpha-\alpha_*)^{1-\frac{d}{2}+\frac{d}{q}} \left(\frac{\mathsf{K}_{q,d}}{\kappa_{q,d}} + o(1)\right) \quad \text{as} \quad \alpha \to +\infty$$

which concludes the proof because $\vartheta = d\,(q-2)/(2\,q)$. □

2.5. **The semi-classical regime: behavior of the function $\alpha \mapsto \mu(\alpha)$ as $\alpha \to +\infty$.** Assume that $q \in (2, 2^*)$. If we combine the results of Propositions 8 and 9, we know that $\mu(\alpha) \sim \alpha^{1-\vartheta}$ as $\alpha \to +\infty$ if $d \ge 3$. If $d = 1$ or 2, we know that $\lim_{\alpha\to+\infty} \mu(\alpha) = +\infty$ with a growth at least equivalent to $\alpha^{2/q-\varepsilon}$ with $\varepsilon > 0$, arbitrarily small, according to Proposition 8, and at most equivalent to $\alpha^{1-\vartheta}$ by Proposition 9. To complete the proof of Lemma 5, it remains to determine the precise behavior of $\mu(\alpha)$ as $\alpha \to +\infty$.

Proposition 10. *With the notations of Lemma 5, for every $q \in (2, 2^*)$, with $\vartheta = d\,\frac{q-2}{2\,q}$ we have*

$$\mu(\alpha) = \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\,\alpha^{1-\vartheta}(1+o(1)) \quad \textit{as} \quad \alpha \to +\infty\,.$$

Proof. Suppose by contradiction that there is a positive constant η and a sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty} \alpha_n^{\vartheta-1}\mu(\alpha_n) \le \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}} - \eta\,. \tag{15}$$

Consider a sequence $(u_n)_{n\in\mathbb{N}}$ of functions in $\mathrm{H}^1(\mathbb{S}^d)$ such that $\mathcal{Q}_{\alpha_n}[u_n] = \mu(\alpha_n)$ and $\|u_n\|_{\mathrm{L}^q(\mathbb{S}^d)} = 1$ for any $n \in \mathbb{N}$. From (15), we know that

$$\alpha_n\,\|u_n\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le \mathcal{Q}_{\alpha_n}[u_n] = \mu(\alpha_n) \le \alpha_n^{1-\vartheta} \left(\frac{\mathsf{K}_{q,d}}{\kappa_{q,d}} - \eta\right)(1+o(1)) \quad \text{as} \quad n \to +\infty$$

that is

$$\limsup_{n\to+\infty} \alpha_n^{\vartheta}\,\|u_n\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}} - \eta\,.$$

The normalization $\|u_n\|_{\mathrm{L}^q(\mathbb{S}^d)} = 1$ for any $n \in \mathbb{N}$ and the limit $\lim_{n\to+\infty} \|u_n\|_{\mathrm{L}^2(\mathbb{S}^d)} = 0$ mean that the sequence $(u_n)_{n\in\mathbb{N}}$ concentrates: there exists a sequence $(y_i)_{i\in\mathbb{N}}$ of points in $\mathbb{S}^d$ (eventually finite) and two sequences of positive numbers $(\zeta_i)_{i\in\mathbb{N}}$ and $(r_{i,n})_{i,n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} r_{i,n} = 0$, $\Sigma_{i\in\mathbb{N}}\zeta_i = 1$ and $\int_{\mathbb{S}^d\cap B(y_i,r_{i,n})} |u_{i,n}|^q\,d\sigma = \zeta_i + o(1)$, where $u_{i,n} \in \mathrm{H}^1(\mathbb{S}^d)$, $u_{i,n} = u_n$ on $\mathbb{S}^d \cap B(y_i, r_{i,n})$ and $\operatorname{supp} u_{i,n} \subset \mathbb{S}^d \cap B(y_i, 2\,r_{i,n})$. Here $o(1)$ means that uniformly with respect to i, the remainder term converges towards

0 as $n \to +\infty$. A computation similar to those of the proof of Proposition 9, we can *blow up* each function $u_{i,n}$ and prove

$$(\alpha_n - \alpha_*)^{\vartheta-1} \int_{\mathbb{S}^d} \left(|\nabla u_{i,n}|^2 + \alpha_n\, |u_{i,n}|^2\right) d\sigma \geq \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\, \zeta_i^{2/q} + o(1) \quad \forall\, i\,.$$

Let us choose an integer N such that $\left(\Sigma_{i=1}^N \zeta_i\right)^{2/q} > 1 - \frac{\kappa_{q,d}\,\eta}{2\,\mathsf{K}_{q,d}}$. Then we find that

$$(\alpha_n - \alpha_*)^{\vartheta-1} \int_{\mathbb{S}^d} \left(|\nabla u_n|^2 + \alpha_n\, |u_n|^2\right) d\sigma \geq \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\, \Sigma_1^N \zeta_i^{2/q} + o(1) \geq \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\, \left(\Sigma_1^N \zeta_i\right)^{2/q} + o(1) \geq \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}} - \frac{\eta}{2} + o(1)\,,$$

a contradiction with (15). □

For details on the behavior of $\mathsf{K}_{q,d}$ as q varies, see Proposition 15. Collecting all results of this section, this completes the proof of Lemma 5.

3. Spectral estimates for the Schrödinger operator on the sphere

This section is devoted to the proof of Theorem 1. As a consequence of the results of Lemma 5, the function $\alpha \mapsto \mu(\alpha)$ is invertible, of inverse $\mu \mapsto \alpha(\mu)$, if $d = 1$, 2 or $d \geq 3$ and $q < 2^*$, and we have the inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2\, d\sigma - \mu \left(\int_{\mathbb{S}^d} |u|^q\, d\sigma\right)^{\frac{2}{q}} \geq -\,\alpha(\mu) \int_{\mathbb{S}^d} |u|^2\, d\sigma \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma)\,, \quad \forall\, \mu > 0\,. \tag{16}$$

Moreover, the function $\mu \mapsto \alpha(\mu)$ is monotone increasing, convex, satisfies $\alpha(\mu) = \mu$ for any $\mu \in (0, \frac{d}{q-2}]$ and $\alpha(\mu) > \mu$ for any $\mu > d/(q-2)$.

Consider the Schrödinger operator $-\Delta - V$ for some function $V \in \mathrm{L}^p(\mathbb{S}^d)$ and the corresponding energy functional

$$\mathcal{E}[u] \;:=\; \int_{\mathbb{S}^d} |\nabla u|^2\, d\sigma - \int_{\mathbb{S}^d} V\, |u|^2\, d\sigma\,.$$

Let

$$\lambda_1(-\Delta - V) \quad := \quad \inf_{\substack{u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma) \\ \int_{\mathbb{S}^d} |u|^2\, d\sigma = 1}} \mathcal{E}[u]\,.$$

By Hölder's inequality, we have

$$\mathcal{E}[u] \geq \int_{\mathbb{S}^d} |\nabla u|^2\, d\sigma - \|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)}\, \|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}\,,$$

with $\frac{1}{p} + \frac{2}{q} = 1$. From Section 2, with $\mu = \|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)}$, we deduce

$$\mathcal{E}[u] \geq -\,\alpha(\mu)\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma)\,, \quad \forall\, V \in \mathrm{L}^p(\mathbb{S}^d)\,,$$

which amounts to a Keller-Lieb-Thirring inequality on the sphere (3), or equivalently

$$\int_{\mathbb{S}^d} |\nabla u|^2\, d\sigma - \int_{\mathbb{S}^d} V\, |u|^2\, d\sigma + \alpha\left(\|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)}\right) \int_{\mathbb{S}^d} |u|^2\, d\sigma \geq 0 \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma)\,, \quad \forall\, V \in \mathrm{L}^p(\mathbb{S}^d)\,.$$

Notice that this inequality contains simultaneously (3) and (16), by optimizing either on u or on V.

Optimality in (3) still needs to be proved. This can be done by taking an arbitrary $\mu \in (0, \infty)$ and considering an optimal function for (16), for which we have

$$\int_{\mathbb{S}^d} |\nabla u|^2\, d\sigma - \mu \left(\int_{\mathbb{S}^d} |u|^q\, d\sigma\right)^{\frac{2}{q}} = \alpha(\mu) \int_{\mathbb{S}^d} |u|^2\, d\sigma\,.$$

Because the above expression is homogeneous of degree two, there is no restriction to assume that $\int_{\mathbb{S}^d} |u|^q\, d\sigma = 1$ and, since the solution is optimal, it solves the Euler-Lagrange equation

$$-\,\Delta u - V\, u = \alpha(\mu)\, u$$

with $V = \mu\, u^{q-2}$, such that

$$\|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)} = \mu\, \|u\|^{q/p}_{\mathrm{L}^q(\mathbb{S}^d)} = \mu\,.$$

Hence such a function V realizes the equality in (3).

Taking into account Lemma 5 and (10), this completes the proof of Theorem 1 in the general case. The case $d = 1$ and $\gamma = 1/2$ has to be treated specifically. Using $u \equiv 1$ as a test function, we know that $|\lambda_1(-\Delta - V)| \le \mu = \int_{\mathbb{S}^1} V\,dx$. On the other hand consider $u \in \mathrm{H}^1(\mathbb{S}^1)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{S}^1)} = 1$. Since $\mathrm{H}^1(\mathbb{S}^1)$ is embedded into $C^{0,1/2}(\mathbb{S}^1)$, there exists $x_0 \in \mathbb{S}^1 \approx [0, 2\,\pi)$ such that $u(x_0) = 1$ and

$$|u(x)|^2 - 1 \,= 2\int_{x_0}^{x} u(y)\,u'(y)\,dy = 2\int_{x_0+2\pi}^{x} u(y)\,u'(y)\,dy$$

can be estimated by

$$\begin{aligned}\big|\,|u(x)|^2 - 1\,\big| \le 2\int_{x_0}^{x} |u(y)|\,|u'(y)|\,dy &= 2\int_{x_0+2\pi}^{x} |u(y)|\,|u'(y)|\,dy\\ &\le \int_0^{2\pi} |u(y)|\,|u'(y)|\,dy \le \left(\int_0^{2\pi} |u(y)|^2\,dy \int_0^{2\pi} |u'(y)|^2\,dy\right)^{1/2}\end{aligned}$$

using the Cauchy-Schwarz inequality, that is

$$\big|\,|u(x)|^2 - 1\,\big| \le 2\,\pi\,\|u'\|_{\mathrm{L}^2(\mathbb{S}^1)}\,,$$

since $\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \frac{1}{2\,\pi}\int_0^{2\pi} |u'(y)|^2\,dy$ and $\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \frac{1}{2\,\pi}\int_0^{2\pi} |u(y)|^2\,dy = 1$ (recall that $d\sigma$ is a probability measure). Thus we get

$$|u(x)|^2 \le 1 + 2\,\pi\,\|u'\|_{\mathrm{L}^2(\mathbb{S}^1)}\,,$$

from which it follows that

$$\lambda_1(-\Delta - V) \ge \|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \mu\left(1 + 2\,\pi\,\|u'\|_{\mathrm{L}^2(\mathbb{S}^1)}\right) \ge -\mu - \pi^2\,\mu^2\,.$$

This shows that $\mu \le \alpha(\mu) \le \mu + \pi^2\,\mu^2$. By the Arzelà-Ascoli theorem, the embedding of $\mathrm{H}^1(\mathbb{S}^1)$ into $C^{0,1/2}(\mathbb{S}^1)$ is compact. When $d = 1$ and $\gamma = 1/2$, the proof of the asymptotic behavior of $\alpha(\mu)$ as $\mu \to +\infty$ can then be completed as in the other cases.

4. Spectral inequalities in the case of positive potentials

In this section we address the case of Schrödinger operators $-\Delta + W$ where W is a positive potential on $\mathbb{S}^d$ and we derive estimates from below for the first eigenvalue of such operators. In order to do so, we first study interpolation inequalities in the Euclidean space $\mathbb{R}^d$, like those studied in Section 2 (for $q > 2$).

For this purpose, let us define for $q \in (0, 2)$ the constant

$$\mathsf{K}^*_{q,d} := \inf_{v \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\,,$$

that is the the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality

$$\mathsf{K}^*_{q,d}\,\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \quad \forall\, v \in \mathrm{H}^1(\mathbb{R}^d) \tag{17}$$

(with the convention that the r.h.s. is infinite if $|v|^q$ is not integrable).

The optimal constant $\mathrm{L}^1_{-\gamma,d}$ in (6) is such that

$$\mathrm{L}^1_{-\gamma,d} := \left(\mathsf{K}^*_{q,d}\right)^{-\gamma} \quad \text{with} \quad q = 2\,\frac{2\,\gamma - d}{2\,\gamma - d + 2}\,. \tag{18}$$

See Appendix B.6 for a proof. Let us define the exponent

$$\delta := \frac{2\,q}{2\,d - q\,(d-2)}\,.$$

Lemma 11. *Let $q \in (0, 2)$ and $d \ge 1$. Then there exists a concave increasing function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ with the following properties:*

$$\nu(\beta) \le \beta \quad \forall\,\beta > 0 \quad \textit{and} \quad \nu(\beta) < \beta \quad \forall\,\beta \in \left(\tfrac{d}{2-q}, +\infty\right),$$

$$\nu(\beta) = \beta \quad \forall\,\beta \in \left[0, \tfrac{d}{2-q}\right] \quad \textit{if} \quad q \in [1, 2)\,, \quad \textit{and} \quad \lim_{\beta\to 0_+} \frac{\nu(\beta)}{\beta} = 1 \quad \textit{if} \quad q \in (0, 1)\,,$$

$$\nu(\beta) = \mathsf{K}^*_{q,d}\,\left(\kappa_{q,d}\,\beta\right)^{\delta}\,(1 + o(1)) \quad \textit{as} \quad \beta \to +\infty\,,$$

such that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \beta\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \geq \nu(\beta)\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)\,. \tag{19}$$

Proof. Inequality (19) is obtained by minimizing the l.h.s. under the constraint $\|u\|_{\mathrm{L}^2(\mathbb{S}^d)} = 1$: there is a minimizer which satisfies

$$-\Delta u + \beta\,u^{q-1} - \nu(\beta)\,u = 0\,.$$

Case $q \in (1,2)$. The proof is very similar to that of Lemma 5, so we leave it to the reader. Written for the optimal value of $\nu(\beta)$, inequality (19) is optimal in the following sense:

(i) If $0 < \beta \leq d/(2-q)$, equality is achieved by constants. See [14] for rigidity results on $\mathbb{S}^d$.

(ii) If $\beta = d/(2-q)$, the sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n := 1 + \frac{1}{n}\,\varphi$ where φ is an eigenfunction of the Laplace-Beltrami operator, is a minimizing sequence of the quotient to the l.h.s. of (19) divided by the r.h.s. which converges to the optimal value of $\nu(\beta) = \beta = d/(2-q)$, that is,

$$\lim_{n\to\infty} \frac{\|\nabla u_n\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u_n\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u_n\|^2_{\mathrm{L}^q(\mathbb{S}^d)}} = \frac{d}{2-q}\,.$$

(iii) If $\beta > d/(2-q)$, there exists a non-constant positive function $u \in \mathrm{H}^1(\mathbb{S}^d) \setminus \{0\}$ such that equality holds in (19).

Case $q \in (0,1]$. In this case, since $\mathbb{S}^d$ is compact, the case $q \leq 1$ does not differ from the case $q \in (1,2)$ as far as the existence of $\nu(\beta)$ is concerned. The only difference is that there is no known rigidity result for $q < 1$. However we can prove that

$$\lim_{\beta\to 0_+} \frac{\nu(\beta)}{\beta} = 1\,.$$

Indeed, let us notice that $\nu(\beta) \leq \beta$ (use constants as test functions). On the other hand, let $u_\beta = c_\beta + v_\beta$ be a minimizer for $\nu(\beta)$ such that $c_\beta = \int_{\mathbb{S}^d} u_\beta\ d\sigma$ and, as a consequence, $\int_{\mathbb{S}^d} v_\beta\ d\sigma = 0$. Without l.o.g. we can set $\int_{\mathbb{S}^d} |c_\beta + v_\beta|^2\ d\sigma = c_\beta^2 + \int_{\mathbb{S}^d} |v_\beta|^2\ d\sigma = 1$. Using the Poincaré inequality, we know that $\|\nabla v_\beta\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq d\,\|v_\beta\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$ and hence

$$d\,\|v_\beta\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \beta\,\|c_\beta + v_\beta\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \leq \|\nabla v_\beta\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \beta\,\|c_\beta + v_\beta\|^2_{\mathrm{L}^q(\mathbb{S}^d)} = \nu(\beta) \leq \beta$$

which shows that $\lim_{\beta\to 0_+} \|v_\beta\|_{\mathrm{L}^2(\mathbb{S}^d)} = 0$ and $\lim_{\beta\to 0_+} c_\beta = 1$. As a consequence, $\|c_\beta + v_\beta\|^2_{\mathrm{L}^q(\mathbb{S}^d)} = c_\beta^2\,(1 + o(1))$ as $\beta \to 0_+$ and we obtain that

$$\beta\,(1 + o(1)) = \beta\,c_\beta^2\,(1 + o(1)) \leq \nu(\beta)\,,$$

which concludes the proof.

Asymptotic behavior of $\nu(\beta)$. Finally, the asymptotic behavior of $\nu(\beta)$ when β is large can be investigated using concentration-compactness methods similar to those used in the proofs of Propositions 8, 9 and 10. Details are left to the reader. □

Proof of Theorem 3. By Hölder's inequality we have

$$\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} = \left(\int_{\mathbb{S}^d} W^{-\frac{q}{2}}\,\left(W\,|u|^2\right)^{\frac{q}{2}}\,d\sigma\right)^{2/q} \leq \|W^{-1}\|_{\mathrm{L}^{\frac{q}{2-q}}(\mathbb{S}^d)}\ \int_{\mathbb{S}^d} W\,|u|^2\,d\sigma\,.$$

Using (19), we get

$$\int_{\mathbb{S}^d} |\nabla u|^2\,d\sigma + \int_{\mathbb{S}^d} W\,|u|^2\,d\sigma \geq \int_{\mathbb{S}^d} |\nabla u|^2\,d\sigma + \|W^{-1}\|^{-1}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \geq \nu\left(\|W^{-1}\|^{-1}_{\mathrm{L}^p(\mathbb{S}^d)}\right)\int_{\mathbb{S}^d} |u|^2\,d\sigma$$

with $p = q/(2-q)$, which proves (7). Then Theorem 3 is an easy consequence of Lemma 11. □

5. The threshold case: $q = 2$

The limiting case $q = 2$ in the interpolation inequality (13) corresponds to the logarithmic Sobolev inequality

$$\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\sigma \le \frac{2}{d}\int_{\mathbb{S}^d} |\nabla u|^2\, d\sigma \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma)$$

which has been studied, *e.g.,* in [5, 12, 11]. For earlier results on the sphere, see [16, 33, 30] and references therein (in particular for the circle). Now, if we consider inequality (11), in the limiting case $q = 2$ we obtain the following interpolation inequality.

Lemma 12. *For any $p > \max\{1, d/2\}$, there exists a concave nondecreasing function $\xi : (0\,, +\infty) \to \mathbb{R}$ with the properties*

$$\xi(\alpha) = \alpha \quad \forall\, \alpha \in (0, \alpha_0) \quad \textit{and} \quad \xi(\alpha) < \alpha \quad \forall\, \alpha > \alpha_0$$

for some $\alpha_0 \in \left[\frac{d}{2}\,(p-1), \frac{d}{2}\,p\right]$, and

$$\xi(\alpha) \sim \alpha^{1-\frac{d}{2\,p}} \quad \textit{as} \quad \alpha \to +\infty$$

such that

(20)

$$\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\sigma + p\,\log\left(\tfrac{\xi(\alpha)}{\alpha}\right) \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le p\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,\log\left(1 + \frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)\,.$$

Proof. Consider Hölder's inequality: $\|u\|_{\mathrm{L}^r(\mathbb{S}^d)} \le \|u\|^{\theta}_{\mathrm{L}^2(\mathbb{S}^d)}\,\|u\|^{1-\theta}_{\mathrm{L}^q(\mathbb{S}^d)}$, with $2 \le r < q$ and $\theta = \frac{2}{r}\,\frac{q-r}{q-2}$. To emphasize the dependence of θ in r, we shall write $\theta = \theta(r)$. By taking the logarithm of both sides of the inequality, we find that

$$\frac{1}{r}\log\int_{\mathbb{S}^d} |u|^r\, d\sigma \le \frac{\theta(r)}{2}\,\log\int_{\mathbb{S}^d} |u|^2\, d\sigma + \frac{1-\theta(r)}{q}\,\log\int_{\mathbb{S}^d} |u|^q\, d\sigma\,.$$

The inequality becomes an equality when $r = 2$, so that we may differentiate at $r = 2$ and get, with $q = \frac{2\,p}{p-1} < 2^*$, *i.e.* $p = \frac{q}{q-2}$, the *logarithmic Hölder* inequality

$$\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\sigma \le p\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,\log\left(\frac{\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)\,.$$

We may now use inequality (11) to estimate

$$\frac{\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \le \frac{\alpha}{\mu(\alpha)}\,\left(1 + \frac{1}{\alpha}\,\frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)$$

where $\mu = \mu(\alpha)$ is the constant which appears in Lemma 5. Thus we get

$$\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\sigma + p\,\log\left(\tfrac{\mu(\alpha)}{\alpha}\right) \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le p\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,\log\left(1 + \frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)\,,$$

which proves that the inequality

$$\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\sigma + p\,\log\xi(\alpha)\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le p\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,\log\left(\alpha + \frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)$$

holds for some optimal constant $\xi(\alpha) \ge \mu(\alpha)$, which is therefore concave and such that $\lim_{\alpha\to+\infty} \xi(\alpha) = +\infty$. This establishes (20). The fact that equality is achieved for every $\alpha > 0$ follows from the method of [13, Proposition 3.3].

Testing (20) with constant functions, we find that $\xi(\alpha) \le \alpha$ for any $\alpha > 0$. On the other hand, $\xi(\alpha) \ge \mu(\alpha) = \alpha$ for any $\alpha \le \frac{d}{q-2} = \frac{d}{2}\,(p-1)$. Testing (20) with $u = 1 + \varepsilon\,\varphi$, we find that $\xi(\alpha) < \alpha$ if $\alpha > \frac{d}{2}\,p$.

By Proposition 10, we know that $\xi(\alpha) \ge \mu(\alpha) \sim \alpha^{1-\vartheta}$ with $\vartheta = d\,\frac{q-2}{2\,q} = \frac{d}{2\,p}$ as $\alpha \to +\infty$. As in the proof of Propositions 9 and 10, let us consider an optimal function u_α for (20). Then we have

$$p\,\log\left(\tfrac{\xi(\alpha)}{\alpha}\right) = p\,\log\left(1+\frac{1}{\alpha}\,\|\nabla u_\alpha\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) - \int_{\mathbb{S}^d} |u_\alpha|^2\log|u_\alpha|^2\,d\sigma \sim \frac{p}{\alpha}\,\|\nabla u_\alpha\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \int_{\mathbb{S}^d} |u_\alpha|^2\log|u_\alpha|^2\,d\sigma$$

as $\alpha \to +\infty$ and u_α concentrates at a single point like in the case $q > 2$ so that, after a stereographic projection which transforms u_α into v_α, the function v_α is, up to higher order terms, optimal for the Euclidean logarithmic Sobolev inequality

$$\int_{\mathbb{R}^d} |v|^2\log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\right)dx + \frac{d}{2}\,\log(\pi\,\varepsilon\,e^2)\,\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \varepsilon\,\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

which holds for any $\varepsilon > 0$ and any $v \in \mathrm{H}^1(\mathbb{R}^d)$. Here we have of course $\varepsilon = p/\alpha$ and find that

$$p\,\log\left(\tfrac{\xi(\alpha)}{\alpha}\right) = \tfrac{d}{2}\,\log\left(\pi\,\tfrac{p}{\alpha}\,e^2\right)(1+o(1)) \quad \text{as} \quad \alpha \to +\infty\,,$$

which concludes the proof. □

Corollary 13. *With the notations of Lemma 12, for any $\alpha > 0$ we have*

$$\frac{\alpha}{p}\int_{\mathbb{S}^d} |u|^2\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)d\sigma + \alpha\,\log\left(\tfrac{\xi(\alpha)}{\alpha}\right)\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)\,.$$

Proof. This is a straightforward consequence of Lemma 12 using the fact that $\log(1+x) \le x$ for any $x > 0$. □

As in the case $q \neq 2$, Corollary 13 provides some spectral estimates. Let $u \in \mathrm{H}^1(\mathbb{S}^d)$ be such that $\|u\|_{\mathrm{L}^2(\mathbb{S}^d)} = 1$. A straightforward optimization with respect to an arbitrary function W shows that

$$\inf_W \left[\int_{\mathbb{S}^d} W\,|u|^2\,d\sigma + \mu\,\log\left(\int_{\mathbb{S}^d} e^{-W/\mu}\,d\sigma\right)\right] = -\,\mu\int_{\mathbb{S}^d} |u|^2\,\log|u|^2\,d\sigma\,,$$

with optimality case achieved by W such that

$$|u|^2 = \frac{e^{-W/\mu}}{\int_{\mathbb{S}^d} e^{-W/\mu}\,d\sigma}\,.$$

Notice that, up to the addition of a constant, we can always assume that $\int_{\mathbb{S}^d} e^{-W/\mu}\,d\sigma = 1$, which uniquely determines the optimal W. Now, by Corollary 13 applied with $\mu = \alpha/p$, we find that

$$\int_{\mathbb{S}^d} |\nabla u|^2\,d\sigma + \int_{\mathbb{S}^d} W\,|u|^2\,d\sigma \ge \alpha\,\log\left(\tfrac{\xi(\alpha)}{\alpha}\right) - \frac{\alpha}{p}\,\log\left(\int_{\mathbb{S}^d} e^{-\,p\,W/\alpha}\,d\sigma\right).$$

This leads us to the following statement.

Corollary 14. *Let $d \ge 1$. With the notations of Lemma 12, we have the following estimate*

$$e^{-\,\lambda_1(-\Delta - W)/\alpha} \le \frac{\alpha}{\xi(\alpha)}\,\left(\int_{\mathbb{S}^d} e^{-\,p\,W/\alpha}\,d\sigma\right)^{1/p}$$

for any function W such that $e^{-\,p\,W/\alpha}$ is integrable. This estimate is optimal in the sense that there exists a nonnegative function W for which the inequality becomes an equality.

Appendix A. Further estimates and numerical results

A.1. **A refined upper estimate.** Let $q \in (2, 2^*)$. For $\alpha > d/(q-2)$, we can give an upper estimate of the optimal constant $\mu(\alpha)$ in inequality (11) of Lemma 5. Consider functions which depend only on z, with the notations of Section 2.3. Then (11) is equivalent to an inequality that can be written as

$$\mathsf{F}_\alpha[f] := \frac{\int_{-1}^1 |f'|^2\,\nu\,d\nu_d + \alpha\int_{-1}^1 |f|^2\,d\nu_d}{\left(\int_{-1}^1 |f|^q\,d\nu_d\right)^{2/q}} \ge \mu(\alpha)$$

where $d\nu_d$ is the probability measure defined by

$$\nu_d(z)\,dz = d\nu_d(z) := Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz \quad \text{with} \quad \nu(z) := 1-z^2\;, \quad Z_d := \sqrt{\pi}\,\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}\,.$$

See [14] for details. To get an estimate, it is enough to take a well chosen test function: consider $f_\varepsilon(z) := 1+\varepsilon\,\varphi(z)$ and as in Section 2.3 we can choose $\varphi(z)=z$. Then one can optimize $h_\alpha(\varepsilon) = \mathsf{F}_\alpha[f_\varepsilon]$ with respect to $\varepsilon\in(0,1)$, and observe that $\int_{-1}^1 |f_\varepsilon'|^2\,\nu\,d\nu_d = d\,\varepsilon^2\int_{-1}^1 z^2\,d\nu_d$, so that $h_\alpha(\varepsilon)$ can be written as

$$h_\alpha(\varepsilon) = \frac{\alpha+(d+\alpha)\,\varepsilon^2\int_{-1}^1 z^2\,d\nu_d}{\left(\int_{-1}^1 |1+\varepsilon\,z|^q\,d\nu_d\right)^{2/q}} \ge \mu(\alpha)\,.$$

When $\varepsilon\to 0_+$, we recover that $h_\alpha(\varepsilon)-\alpha \sim \big[d-\alpha\,(q-2)\big]\,\varepsilon^2\int_{-1}^1 z^2\,d\nu_d < 0$ if $\alpha > d/(q-2)$, but a better estimate can be achieved simply by considering $\mu_+(\alpha) := \inf_{\varepsilon\in(0,1)} h_\alpha(\varepsilon)$ so that $\mu(\alpha)\le\mu_+(\alpha)<\alpha$. The function $\alpha\mapsto\mu_+(\alpha)$ can be computed explicitly (using hypergeometric functions) and is shown in Fig. 1.

A.2. **Numerical results.** In this section, we illustrate the various estimates obtained in this paper by numerical computations done in the special case $d=3$ and $q=3$. See Fig. 1 for the computation of the curve $\alpha\mapsto\mu(\alpha)$ and how it behaves compared to the theoretical estimates obtained in this paper. We emphasize that our upper and lower estimates $\alpha\mapsto\mu_\pm(\alpha)$ bifurcate from the line $\mu=\alpha$ precisely at $\alpha = d/(q-2)$ if $q\in(2,2^*)$ (and at $\alpha=d/(2-q)$ if $q\in(1,2)$). The curve corresponding to the asymptotic regime is also plotted, but gives relevant information only as $\alpha\to\infty$.

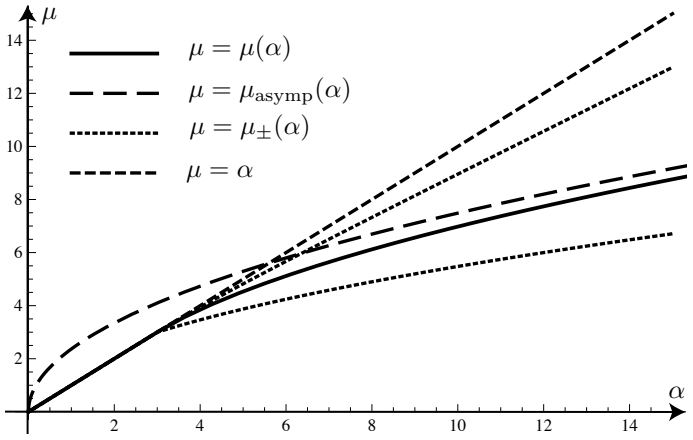


FIGURE 1. *In the case $q>2$, the optimal constant is given by $\mu=\alpha$ for $\alpha\le d/(q-2)$ and the curve $\mu=\mu(\alpha)$ for $\alpha>d/(q-2)$. An upper estimate is given by the curve $\mu=\mu_+(\alpha)$ obtained by optimizing the function $h_\alpha(\varepsilon)$ in terms of $\varepsilon\in(0,1)$ while a lower estimate, namely $\mu=\mu_-(\alpha)=\alpha_*^\vartheta\,\alpha^{1-\vartheta}$ has been established in Proposition 8. The asymptotic regime is governed by $\mu(\alpha)\sim\mu_{\rm asymp}(\alpha)=\mathsf{K}_{q,d}\,\kappa_{q,d}^{-1}\,\alpha^{1-\vartheta}$ as $\alpha\to+\infty$ according to Lemma 5. The above plot shows the various curves in the special case $d=3$ and $q=3$.*

The convergence towards the asymptotic regime is illustrated in Fig. 2 which shows the convergence of $\mu(\alpha)/\mu_{\rm asymp}(\alpha)$ towards 1 as $\alpha\to+\infty$ in the special case $d=3$ and $q=3$. In terms of spectral properties, for large potentials, eigenvalues of the Schrödinger operator can be estimated according to Theorem 1 by the Euclidean Keller-Lieb-Thirring constant that has been numerically computed for instance in [27, Appendix A. Numerical studies, by J.F. Barnes].

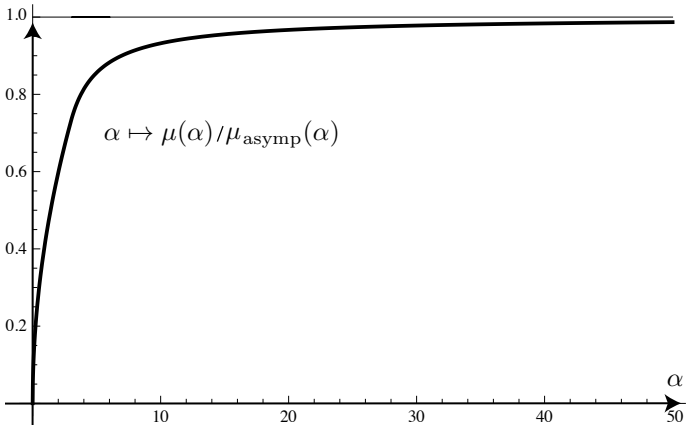


FIGURE 2. *The asymptotic regime corresponding to $\alpha \to +\infty$ has the interesting feature that, up to a dependence in $\alpha^{1-\vartheta}$ and a normalization factor proportional to $\kappa_{q,d}$, the optimal constant $\mu(\alpha)$ behaves like the optimal constant in the Euclidean space, as has been established in Proposition 10.*

APPENDIX B. CONSTANTS ON THE EUCLIDEAN SPACE

B.1. **Scaling of the Gagliardo-Nirenberg-Sobolev inequality.** Let $q > 2$ and denote by $\mathsf{K}_{\rm GN}(q)$ the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality, given by

$$\mathsf{K}_{\rm GN}(q) := \inf_{u \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla u\|^{2\,\vartheta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|u\|^{2\,(1-\vartheta)}_{\mathrm{L}^2(\mathbb{R}^d)}}{\|u\|^2_{\mathrm{L}^q(\mathbb{R}^d)}} \quad \text{with} \quad \vartheta = \vartheta(q,d) = d\,\frac{q-2}{2\,q}\,.$$

An optimization of the quotient in the definition of $\mathsf{K}_{q,d}$, which has been defined in Section 2, allows to relate this constant with $\mathsf{K}_{\rm GN}(q)$. Indeed, if we optimize $\mathcal{N}[u] := \int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} |u|^2\,dx$ under the scaling $\lambda \mapsto u_\lambda(x) := \lambda^{d/q}\,u(\lambda\,x)$, then we find that

$$\mathcal{N}[u_\lambda] = \lambda^{2\,(1-\vartheta)} \int_{\mathbb{R}^d} |\nabla u|^2\,dx + \lambda^{-\,2\,\vartheta} \int_{\mathbb{R}^d} |u|^2\,dx$$

achieves its minimum at

$$\lambda_\star = \sqrt{\frac{\vartheta}{1-\,\vartheta}}\,\frac{\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}}{\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}}\,,$$

so that

$$\mathcal{N}[u_{\lambda_\star}] = \vartheta^{-\,\vartheta}\,(1-\,\vartheta)^{-\,(1-\vartheta)}\,\|\nabla u\|^{2\,\vartheta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|u\|^{2\,(1-\vartheta)}_{\mathrm{L}^2(\mathbb{R}^d)}\,,$$

thus proving that $\mathsf{K}_{q,d}$ can be computed in terms of $\mathsf{K}_{\rm GN}(q)$ as

$$\mathsf{K}_{q,d} = \vartheta^{-\,\vartheta}\,(1-\,\vartheta)^{-\,(1-\vartheta)}\,\mathsf{K}_{\rm GN}(q)\,.$$

B.2. **Asymptotic regimes in Gagliardo-Nirenberg-Sobolev inequalities.** Let $q > 2$ and consider the constant $\mathsf{K}_{q,d}$ as above. To handle the case of dimension $d = 1$, we may observe that for any smooth compactly supported function u on $\mathbb{R}$, we can write either

$$|u(x)|^2 = 2\left|\int_{-\infty}^x u(y)\,u'(y)\,dy\right| \le \|u\|^2_{\mathrm{L}^2(-\infty,x)} + \|u'\|^2_{\mathrm{L}^2(-\infty,x)} \quad \forall\, x \in \mathbb{R}$$

or

$$|u(x)|^2 = 2\left|\int_x^{+\infty} u(y)\,u'(y)\,dy\right| \le \|u\|^2_{\mathrm{L}^2(x,+\infty)} + \|u'\|^2_{\mathrm{L}^2(x,+\infty)} \quad \forall\, x \in \mathbb{R}$$

thus proving that

$$|u(x)|^2 \le \frac{1}{2}\left(\|u\|^2_{\mathrm{L}^2(\mathbb{R})} + \|u'\|^2_{\mathrm{L}^2(\mathbb{R})}\right) \quad \forall\, x \in \mathbb{R}\,,$$

that is, the Agmon inequality

$$\frac{\|u\|^2_{\mathrm{L}^2(\mathbb{R})} + \|u'\|^2_{\mathrm{L}^2(\mathbb{R})}}{\|u\|^2_{\mathrm{L}^\infty(\mathbb{R})}} \ge 2\,,$$

and hence $\mathsf{K}_{\infty,1} \ge 2$. Equality is achieved by the function $u(x) = e^{-|x|}$, $x \in \mathbb{R}$, and we have shown that

$$\mathsf{K}_{\infty,1} = 2\,.$$

Proposition 15. *Assume that $q > 2$. For all $d \ge 1$,*

$$\lim_{q\to 2_+} \mathsf{K}_{q,d} = 1$$

and, for all $d \ge 3$,

$$\lim_{q\to 2^*} \mathsf{K}_{q,d} = \mathsf{S}_d$$

where S_d is the best constant in inequality (8)*. If $d = 1$, then $\lim_{q\to+\infty} \mathsf{K}_{q,1} = \mathsf{K}_{\infty,1}$.*

Proof. For any $v \in \mathrm{H}^1(\mathbb{R}^d)$ and $d \ge 3$, we have

$$\lim_{q\to 2^*} \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}} \ge \lim_{q\to 2^*} \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}} = \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}} \ge \mathsf{S}_d\,,$$

thus proving that $\lim_{q\to 2^*} \mathsf{K}_{q,d} \ge \mathsf{S}_d$. On the other hand, we may use the Aubin-Talenti function

$$\overline{u}(x) = (1+|x|^2)^{-\frac{d-2}{2}} \quad \forall\, x \in \mathbb{R}^d \tag{21}$$

as test function for $\mathsf{K}_{q,d}$ if $d \ge 5$, *i.e.*

$$\mathsf{K}_{q,d} \le \vartheta^{-\,\vartheta}\,(1-\,\vartheta)^{-\,(1-\,\vartheta)}\,\frac{\|\nabla \overline{u}\|^{2\,\vartheta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|\overline{u}\|^{2\,(1-\,\vartheta)}_{\mathrm{L}^2(\mathbb{R}^d)}}{\|\overline{u}\|^2_{\mathrm{L}^q(\mathbb{R}^d)}}$$

and observe that the right-hand side converges to S_d since $\lim_{q\to 2^*} \vartheta(q,d) = 1$. If $d = 3$ or 4, standard additionnal truncations are needed. The case corresponding to $q \to \infty$, $d = 1$ is dealt with as above.

Now we investigate the limit as $q \to 2_+$. For any $v \in \mathrm{H}^1(\mathbb{R}^d)$, we have

$$\lim_{q\to 2_+} \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}} \ge \lim_{q\to 2_+} \frac{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}} = 1\,,$$

thus proving that $\lim_{q\to 2_+} \mathsf{K}_{q,d} \ge 1$, and for any $v \in \mathrm{H}^1(\mathbb{R}^d)$, the right-hand side in

$$\mathsf{K}_{q,d} \le \vartheta^{-\,\vartheta}\,(1-\,\vartheta)^{-\,(1-\,\vartheta)}\,\frac{\|\nabla v\|^{2\,\vartheta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|v\|^{2\,(1-\,\vartheta)}_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}}$$

converges to 1 as $q \to 2_+$. This completes the proof. □

B.3. **Stereographic projection.** On $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, we can introduce the coordinates $y = (\rho\,\phi, z) \in \mathbb{R}^d \times \mathbb{R}$ such that $\rho^2 + z^2 = 1$, $z \in [-1,1]$, $\rho \ge 0$ and $\phi \in \mathbb{S}^{d-1}$, and consider the stereographic projection $\Sigma : \mathbb{S}^d \backslash \{\mathrm{N}\} \to \mathbb{R}^d$ defined by $\Sigma(y) = x$ where, using the above notations, $x = r\,\phi$ with $r = \sqrt{(1+z)/(1-z)}$ for any $z \in [-1,1)$. In this setting the *North Pole* N corresponds to $z = 1$ (and is formally sent at infinity) while the *equator* (corresponding to $z = 0$) is sent onto the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Hence $x \in \mathbb{R}^d$ is such that $r = |x|$, $\phi = \frac{x}{|x|}$, and we have the useful formulae

$$z = \frac{r^2-1}{r^2+1} = 1 - \frac{2}{r^2+1}\,, \quad \rho = \frac{2\,r}{r^2+1}\,.$$

With these notations in hand, we can transform any function u on $\mathbb{S}^d$ into a function v on $\mathbb{R}^d$ using

$$u(y) = \left(\tfrac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\tfrac{r^2+1}{2}\right)^{\frac{d-2}{2}} v(x) = (1-z)^{-\frac{d-2}{2}}\,v(x)$$

and a painful but straightforward computation shows that, with $\alpha_* = \frac{1}{4}\,d\,(d-2)$,

$$\int_{\mathbb{S}^d} |\nabla u|^2\,d\omega + \alpha_* \int_{\mathbb{S}^d} |u|^2\,d\omega = \int_{\mathbb{R}^d} |\nabla v|^2\,dx \quad \text{and} \quad \int_{\mathbb{S}^d} |u|^q\,d\omega = \int_{\mathbb{R}^d} |v|^q \left(\tfrac{2}{1+|x|^2}\right)^{d-(d-2)\frac{q}{2}}\,dx\,.$$

As a consequence, Inequalities (11) and (19) are transformed respectively into

$$\int_{\mathbb{R}^d} |\nabla v|^2\,dx + 4\,(\alpha-\alpha_*) \int_{\mathbb{R}^d} |v|^2\,\frac{dx}{(1+|x|^2)^2} \ge \mu(\alpha)\,\kappa_{q,d} \left[\int_{\mathbb{R}^d} |v|^q \left(\tfrac{2}{1+|x|^2}\right)^{d-(d-2)\frac{q}{2}}\,dx\right]^{\frac{2}{q}} \quad \forall\, v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

if $q \in (2, 2^*)$ and $\alpha \ge \alpha_*$, and

$$\int_{\mathbb{R}^d} |\nabla v|^2 \, dx + \beta \, \kappa_{q,d} \left[\int_{\mathbb{R}^d} |v|^q \left(\tfrac{2}{1+|x|^2} \right)^{d-(d-2)\frac{q}{2}} dx \right]^{\frac{2}{q}} \ge 4 \left(\nu(\beta) + \alpha_* \right) \int_{\mathbb{R}^d} |v|^2 \, \frac{dx}{(1+|x|^2)^2} \quad \forall \, v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

if $q \in (1, 2)$ and $\beta > 0$.

B.4. **Sobolev's inequality: expression of the constant and references.** The proof that Sobolev's inequality (8) becomes an equality if and only if $u = \overline{u}$ given by (21) up to a multiplication by a constant, a translation and a scaling is due to T. Aubin and G. Talenti: see [2, 34]. However, G. Rosen in [32] showed (by linearization) that the function given by (21) is a local minimum when $d = 3$ and computed the critical value.

Much earlier, G. Bliss in [10] (also see [18]) established that, among radial functions, the following inequality holds

$$\left(\int_{\mathbb{R}^d} |f|^p \, |x|^{r+1-d-p} \, dx \right)^{\frac{2}{p}} \le \mathsf{C}_{\rm Bliss} \int_{\mathbb{R}^d} |\nabla f|^2 \, |x|^{1-d} \, dx$$

when $r = \frac{p}{2} - 1$. With the change of variables $f(x) = v\left(|x|^{-\frac{1}{d-2}} \, \frac{x}{|x|} \right)$, the inequality is changed into

$$\left(\int_{\mathbb{R}^d} |v|^{\frac{2d}{d-2}} \, dx \right)^{\frac{d-2}{d}} \le \frac{\mathsf{C}_{\rm Bliss}}{(d-2)^{2\frac{d-1}{d}}} \int_{\mathbb{R}^d} |\nabla v|^2 \, dx$$

if $p = 2^*$ and it is a straightforward consequence of [10] that the equality is achieved with $v = \overline{u}$.

According to the duplication formula (see for instance [1]) for the Γ function, we know that

$$\Gamma(x) \, \Gamma\left(x + \tfrac{1}{2}\right) = 2^{1-2\,x} \, \sqrt{\pi} \, \Gamma(2\,x) \, .$$

As a consequence, the best constant in Sobolev's inequality (8) can be written either as

$$\mathsf{S}_d = \frac{4}{d\,(d-2)\,|\mathbb{S}^d|^{2/d}}$$

where the surface of the d-dimensional unit sphere is given by $|\mathbb{S}^d| = 2\,\pi^{\frac{d+1}{2}} / \Gamma\left(\frac{d+1}{2}\right)$ (see for instance [5]), or as

$$\mathsf{S}_d = \frac{1}{\pi \, d \, (d-2)} \left(\frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \right)^{\frac{2}{d}}$$

according to [2, 10, 32, 34]. This last expression can easily be recovered using the fact that optimality in (8) is achieved by $\overline{u}$ defined in (21), while the first one, namely $1/\mathsf{S}_d = \frac{1}{4} \, d \, (d-2) \, \kappa_{2^*,d}$, is an easy consequence of the stereographic projection and the computations of Section B.3 with $\alpha = \alpha_*$ and $q = 2^*$.

B.5. **A proof of** (10)**.** Assume that $q > 2$ and let us relate the optimal constant $\mathrm{L}^1_{\gamma,d}$ in the one bound state Keller-Lieb-Thirring inequality (1) with the optimal constant $\mathsf{K}_{q,d}$ in the Gagliardo-Nirenberg-Sobolev inequality (9). In this case, recall that $p = \frac{q}{q-2} = \gamma + \frac{d}{2}$. For any nonnegative function ϕ defined on $\mathbb{R}^d$ such that $\|\phi\|_{\mathrm{L}^p(\mathbb{R}^d)} = \mathsf{K}_{q,d}$, using Hölder's inequality we can write that

$$\int_{\mathbb{R}^d} \left(|\nabla v|^2 - \phi \, |v|^2 \right) dx \ge \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|\phi\|_{\mathrm{L}^p(\mathbb{R}^d)} \, \|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}$$

for any $v \in \mathrm{H}^1(\mathbb{R}^d)$. Using (9), namely

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{K}_{q,d} \, \|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \ge - \, \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \, ,$$

this proves that

$$|\lambda_1(-\Delta - \phi)| \le 1 \quad \forall \, \phi \in \mathrm{L}^p(\mathbb{R}^d) \quad \text{such that} \quad \|\phi\|_{\mathrm{L}^p(\mathbb{R}^d)} = \mathsf{K}_{q,d} \, . \tag{22}$$

Next one can observe that inequality (1) can be rephrased as

$$\mathrm{L}^1_{\gamma,d} = \sup_{\phi \in \mathrm{L}^p(\mathbb{S}^d)} \sup_{v \in \mathrm{H}^1(\mathbb{R}^d) \setminus \{0\}} \left(\mathcal{R}[v,\phi] \right)^\gamma \quad \text{with} \quad \mathcal{R}[v,\phi] := \frac{\int_{\mathbb{R}^d} \left(\phi \, |v|^2 - |\nabla v|^2 \right) dx}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \, \|\phi\|^{\frac{2\,p}{2\,p-d}}_{\mathrm{L}^p(\mathbb{R}^d)}}$$

where $p=\gamma+d/2$ so that the exponent $\frac{2\,p}{2\,p-d}$ is precisely the one for which we get the scaling invariance of $\mathcal{R}$. Indeed, with $v_\lambda(x):=v(\lambda\,x)$ and $\phi_\lambda(x):=\phi(\lambda\,x)$, we get that $\mathcal{R}[v_\lambda,\lambda^2\,\phi_\lambda]=\mathcal{R}[v,\phi]$ for any $\lambda>0$. Hence we find that

$$\sup_{v\in\mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}}\mathcal{R}[v,\phi]=\frac{|\lambda_1(-\Delta-\phi)|}{\|\phi\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\frac{2\,p}{2\,p-d}}}=\sup_{v\in\mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}}\mathcal{R}[v_\lambda,\lambda^2\,\phi_\lambda]=\frac{|\lambda_1(-\Delta-\lambda^2\,\phi_\lambda)|}{\|\lambda^2\,\phi_\lambda\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\frac{2\,p}{2\,p-d}}}$$

and if we choose λ such that

$$\lambda^{\frac{2\,p-d}{p}}\,\|\phi\|_{\mathrm{L}^p(\mathbb{R}^d)}=\|\lambda^2\,\phi_\lambda\|_{\mathrm{L}^p(\mathbb{R}^d)}=\mathsf{K}_{q,d}\,,$$

we obtain

$$\frac{|\lambda_1(-\Delta-\phi)|}{\|\phi\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\frac{2\,p}{2\,p-d}}}\le\frac{1}{\mathsf{K}_{q,d}^{\frac{2\,p}{2\,p-d}}}$$

using (22), which proves that $\mathrm{L}^1_{\gamma,d}\le(\mathsf{K}_{q,d})^{-\,p}$ with $p=\gamma+\frac{d}{2}$. Since optimality can be preserved at each step, this actually proves (10).

See [22, 27, 35, 36, 6, 15] for further details. In the Euclidean case, notice that the equivalence can be extended to the case of systems on the one hand and to Lieb-Thirring inequalities on the other hand: see [27, 29, 15].

B.6. **A proof of** (18)**.** As in [15], we can also relate $\mathrm{L}^1_{-\gamma,d}$ and $\mathsf{K}^*_{q,d}$ when $q=2\,\frac{2\,\gamma-d}{2\,\gamma-d+2}$ takes values in $(0,2)$. The method is similar to that of Appendix B.5. For any function $v\in\mathrm{H}^1(\mathbb{R}^d)$ such that v^q is integrable and any positive potential ϕ such that ϕ^{-1} is in $\mathrm{L}^p(\mathbb{R}^d)$ with $p=q/(2-q)$, we can use Hölder's inequality as in the proof of Theorem 3 and get

$$\int_{\mathbb{R}^d}\left(|\nabla v|^2+\phi\,|v|^2\right)\,dx\ge\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+\frac{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}}{\|\phi^{-1}\|_{\mathrm{L}^p(\mathbb{R}^d)}}\,.$$

Using (17), namely $\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}\ge\mathsf{K}^*_{q,d}\,\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, this proves that

$$\lambda_1(-\Delta+\phi)\ge\mathsf{K}^*_{q,d}\quad\forall\,\phi\in\mathrm{L}^p(\mathbb{R}^d)\quad\text{such that}\quad\|\phi^{-1}\|_{\mathrm{L}^p(\mathbb{R}^d)}=1\,.$$

Inequality (6) can be rephrased as

$$\mathrm{L}^1_{-\,\gamma,d}=\sup_{\phi\in\mathrm{L}^p(\mathbb{S}^d)}\sup_{v\in\mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}}\left(\mathcal{R}[v,\phi]\right)^{-\gamma}\quad\text{with}\quad\mathcal{R}[v,\phi]:=\frac{\int_{\mathbb{R}^d}\left(|\nabla v|^2+\phi\,|v|^2\right)\,dx}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\,\|\phi^{-1}\|^{p/\gamma}_{\mathrm{L}^p(\mathbb{R}^d)}$$

with $\gamma=p+\frac{d}{2}$. The same scaling as in Appendix B.5 applies: with $v_\lambda(x):=v(\lambda\,x)$ and $\phi_\lambda(x):=\phi(\lambda\,x)$, we get that $\mathcal{R}[v_\lambda,\lambda^2\,\phi_\lambda]=\mathcal{R}[v,\phi]$ for any $\lambda>0$ and hence

$$\mathrm{L}^1_{-\,\gamma,d}=\left(\mathsf{K}^*_{q,d}\right)^{-\gamma}\,,$$

which completes the proof of (18).

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