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# Convection and total variation flow

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## Abstract

We deal with a nonlinear hyperbolic scalar conservation law, regularised by the total variation flow operator (or 1-Laplacian). We give an entropy weak formulation, for which we prove the existence and the uniqueness of the solution. The existence result is established using the convergence of a numerical approximation (a splitting scheme where the hyperbolic flow is treated with finite volumes and the total variation flow with finite elements). Some numerical simulations are also presented.

**Keywords:** Bingham fluid, hyperbolic scalar conservation law, total variation flow, 1-Laplacian, entropy formulation, finite volumes, finite elements

## 1 Introduction

A Bingham fluid, also called rigid viscoplastic fluid, is a material that behaves as a rigid solid below a certain stress yield and as a viscous fluid above this yield; a familiar example of such a material is the tooth paste. For a  $d$ -dimensional Bingham fluid, the relation between the stress tensor  $\sigma$ , seen as a  $d \times d$  matrix, the pressure  $p$  and the velocity  $u$  is

$$\sigma = -pI_d + g \frac{\mathbf{D}(u)}{|\mathbf{D}(u)|_F} + \nu \mathbf{D}(u), \quad (1)$$

where  $g$  and  $\nu$  are positive constants,  $I_d$  is the  $d \times d$  identity matrix,  $\mathbf{D}(u)$  is the  $d \times d$  matrix such that  $\mathbf{D}(u)_{ij} := \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ , and  $|\cdot|_F$  denotes the Frobenius matrix norm. The term  $g \frac{\mathbf{D}(u)}{|\mathbf{D}(u)|_F}$  enforces the plastic behaviour,  $g$  being the plasticity yield, while the term  $\nu \mathbf{D}(u)$  enforces the viscous behaviour,  $\nu$  being the viscosity parameter. The mathematical analysis of Bingham fluid flows dates back to the work of [13], where the problems are formulated as variational inequalities in Sobolev spaces. The numerical approximation of a Bingham fluid flow is usually treated with finite element techniques; we refer to [11] for a recent review.

When the viscosity becomes negligible ( $\nu = 0$ ), the analytical and numerical framework described above is no longer suitable – let us mention however an existence result in 2D obtained by [26]. Although the study of inviscid Bingham fluids has been initiated in [5] with the case of an unsteady flow without convection term, the presence of a nonlinear convection term is naturally issued from the inertial term in the momentum conservation equation. Unfortunately, the study of this problem seems to be out of reach in the actual state of the art, and we only consider here a simplified model of unsteady Bingham flow with convection. This simplified model is scalar and consists in seeking  $u : \mathbb{R}^d \times (0, T) \rightarrow \mathbb{R}$  such that

$$\partial_t u + \operatorname{div} F(x, t, u) - \operatorname{div} (\operatorname{Sgn}(\nabla u)) = 0, \quad \text{on } \mathbb{R}^d \times (0, T), \quad (2)$$

$$u(x, 0) = u_{\text{ini}}(x), \quad \text{on } \mathbb{R}^d, \quad (3)$$

where  $d \in \mathbb{N}^*$ ,  $F : \mathbb{R}^d \times (0, T) \times \mathbb{R} \rightarrow \mathbb{R}^d$  is divergence-free with respect to the space variables and  $u_{\text{ini}}$  is a given function from  $\mathbb{R}^d$  to  $\mathbb{R}$ . We denote by  $\operatorname{Sgn}$  the vector sign function, which is a set-valued map from  $\mathbb{R}^d$  to  $\mathcal{P}(\mathbb{R}^d)$  such that

$$\lambda \in \operatorname{Sgn}(\mu) \Leftrightarrow \begin{cases} |\lambda| \leq 1 & \text{if } \mu = 0 \\ \lambda = \frac{\mu}{|\mu|} & \text{if } \mu \neq 0, \end{cases}$$

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where  $|\cdot|$  denotes the euclidean norm in  $\mathbb{R}^d$ . In equation (2), the term  $\operatorname{div} F$  describes the convection in the fluid regime, while the term  $\operatorname{div}(\operatorname{Sgn}(\nabla u))$  enforces the plastic behaviour (the plasticity yield is taken equal to 1 for simplicity).

Problem (2)-(3) is considered in this work under the following hypotheses, denoted by Hypotheses (HC) in the following.

(HC1) For a given  $T > 0$ , we define  $Q_T := \mathbb{R}^d \times (0, T)$ .

(HC2) The initial datum  $u_{\text{ini}}$  is assumed to belong to  $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ . The essential infimum and supremum of  $u_{\text{ini}}$  are denoted by  $a_0$  and  $b_0$ , respectively.

(HC3) The flux function  $F \in C^1(Q_T \times \mathbb{R}, \mathbb{R}^d)$  is assumed to be divergence-free with respect to the space variables, that is

$$\operatorname{div}_x F(x, t, u) := \sum_{i=1}^d \frac{\partial F_i}{\partial x_i}(x, t, u) = 0, \quad \forall (x, t, u) \in Q_T \times \mathbb{R}. \quad (4)$$

Furthermore,  $\frac{\partial F}{\partial u}$  is assumed to be locally Lipschitz continuous and such that, for all compact set  $K \subset \mathbb{R}$ ,  $|\frac{\partial F}{\partial u}| \leq C_K$  a.e. on  $Q_T \times K$ , where  $C_K$  is a constant depending on  $K$ .

Problem (2)-(3) can be viewed as a nonlinear hyperbolic scalar conservation law regularised by the total variation flow operator (or the 1-Laplacian). For nonlinear hyperbolic conservation laws, it is well known that the standard weak formulation fails to ensure the uniqueness of the solution and must be replaced by an entropy formulation; see, e.g., [22] or [27]. With some types of regularisation, as for instance the viscous regularisation, the uniqueness is recovered. That is not the case with the total variation flow regularisation, which has no spatial smoothing effect and does not prevent the formation of shocks.

Nonlinear hyperbolic problems are usually approximated with finite volumes [16, 23, 25]. Unfortunately, finite volumes are not suitable for the approximation of the total variation flow: indeed, if a sequence  $(u_k)_{k \in \mathbb{N}}$  of piece-wise constant functions converges to  $u$  in  $L^1$ , the total variation of  $u_k$  does not converge in general to the total variation of  $u$  (see [3] for an example). The total variation flow must be approximated in  $W^{1,1}$ -conforming discrete spaces, such as  $P_1$  finite element spaces [2, 20, 21]. Numerical schemes combining finite volumes and finite element schemes have already been considered for scalar conservation laws with a diffusion term [19] and for degenerate parabolic equations [18].

In the present article, we first give an entropy formulation for Problem (2)-(3) and prove the uniqueness of the entropy solution using the doubling variable technique. Note that our entropy formulation of the total variation term is similar to the one developed in [4] to study the total variation flow with  $L^1_{\text{loc}}$  initial data (without hyperbolic term).

The existence of the entropy solution follows from the convergence of a numerical approximation, based on a splitting scheme. The hyperbolic flow is treated with finite volumes and the total variation flow is treated with  $P_1$  finite elements. The finite volume mesh is built as a dual mesh of the finite element mesh, which makes simple the interpolation step between the two meshes. For the hyperbolic step (or finite volume step), we choose an explicit time discretisation for sake of simplicity. For the total variation flow step (or finite element step), we are led to define an implicit scheme accounting for the nonregularity of the total variation flow operator. To guarantee the maximum principle, which is essential for the stability of the scheme, we use a nonobtuse finite element mesh. A parabolic regularisation term had to be included in this step in order to control the convergence of the scheme.

The convergence proof of the numerical approximation relies on the Kolmogorov-Riesz compactness theorem, which provides us with the strong convergence in  $L^1_{\text{loc}}(\mathbb{R}^d \times (0, T))$  of the discrete solutions. It requires uniform estimates on the space and time translates of the discrete solutions. To establish these estimates, the total variation term is crucial. For scalar conservation laws without total variation flow regularisation, these estimates are not true and the convergence study of the numerical approximations must be carried out with other tools [10, 8, 16].

The article is organised as follows. In section 2, the concept of entropy solution for Problem (2)-(3) is defined and its uniqueness is proved. Section 3 describes the numerical approximation and its first properties (well-posedness, maximum principle). A priori estimates on the discrete solutions are provided in Section 4 and a discrete entropy formulation is established in Section 5. The convergence of the numerical approximation (and thus the existence of an entropy solution) is finally proved in Section 6 using the results of the two previous sections. In the last section, some numerical simulations in 1D and 2D are presented.

## 2 Entropy formulation for nonlinear hyperbolic equation with total variation flow

### 2.1 Functions with bounded variation

Let us first recall basic properties concerning the functions with bounded variation. For a comprehensive presentation, we refer to [1], [15] or [31].

- Let  $\Omega$  be an open subset of  $\mathbb{R}^d$ . The total variation of a function  $u \in L^1(\Omega)$  is defined by

$$TV_{\Omega}(u) := \sup \left\{ \int_{\Omega} u \operatorname{div} \phi \, dx; \phi \in C_c^1(\Omega; \mathbb{R}^d) \text{ with } \|\phi\|_{L^{\infty}(\Omega)} \leq 1 \right\}.$$

In particular, the total variation of a function  $u \in W^{1,1}(\Omega)$  is equal to

$$TV_{\Omega}(u) = \int_{\Omega} |\nabla u| \, dx.$$

The space of functions over  $\Omega$  with bounded variation, denoted by  $BV(\Omega)$ , is the set of functions  $u \in L^1(\Omega)$  such that  $TV_{\Omega}(u) < +\infty$ . Equipped with the norm  $\|\cdot\|_{BV(\Omega)} := \|\cdot\|_{L^1(\Omega)} + TV_{\Omega}(\cdot)$ , the space  $BV(\Omega)$  is a (nonreflexive) Banach space.

- The distributional derivative of  $u \in BV(\Omega)$ , denoted by  $Du$ , is the vector Radon measure such that

$$\int_{\Omega} u \operatorname{div} \phi \, dx = - \int_{\Omega} \phi \, Du, \quad \forall \phi \in C_c^1(\Omega, \mathbb{R}^d).$$

- The norm of the vector measure  $Du$  is denoted by  $|Du|$ . It is a positive Radon measure and there is a measurable function  $h : \mathbb{R} \rightarrow \mathbb{R}^d$ , with  $|h(x)| = 1$  for all  $x \in \mathbb{R}$ , such that  $Du = h|Du|$ .
- The norm of  $Du$  is linked to the total variation by the identity  $|Du|(\Omega) = TV_{\Omega}(u)$ .
- The total variation is lower semi-continuous relatively to the convergence in  $L^1_{\text{loc}}$ . In other words, if  $(u_k)_{k \in \mathbb{N}}$  is a sequence in  $BV(\Omega)$  converging to  $u$  in  $L^1_{\text{loc}}(\Omega)$ , then

$$\liminf_{k \rightarrow \infty} |Du_k|(\Omega) \geq |Du|(\Omega).$$

- If  $u \in BV(\Omega)$  and  $f \in C^1(\mathbb{R})$ , then  $f(u) \in BV(\Omega)$ .
- The space  $L^1(0, T; BV(\Omega))$  is the set of measurable functions  $u : (0, T) \rightarrow BV(\Omega)$  such that  $\int_0^T \|u(t)\|_{BV(\Omega)} \, dt < +\infty$ . Equipped with the norm  $\|\cdot\|_{L^1(0, T; BV(\Omega))} := \int_0^T \|\cdot\|_{BV(\Omega)} \, dt$ , it is a (nonreflexive) Banach space. For more information on functions valued in Banach spaces, we refer for instance to [28, Chapter III].

### 2.2 Definition of entropy solutions

In the usual entropy formulations of scalar conservation laws, the admissible entropies are the  $C^1$  convex functions or the so-called Kruzhkov entropies. Let us recall that the Kruzhkov entropies are the functions  $|\cdot - \kappa|$  with  $\kappa \in \mathbb{R}$ , the corresponding entropy fluxes being the functions  $F(\cdot \top \kappa) - F(\cdot \perp \kappa)$ , where  $a \top b$  denotes the maximum of  $a$  and  $b$  and  $a \perp b$  denotes the minimum of  $a$  and  $b$ .

The entropy formulation of the problem (2)-(3), owing to the term  $\operatorname{div} \operatorname{Sgn}(\nabla u)$ , requires more regular entropies.

**Definition 2.1:** Under Hypotheses (HC), an admissible entropy is a convex Lipschitz continuous function  $\eta \in C^{\infty}(\mathbb{R})$ . The corresponding entropy flux is the locally Lipschitz continuous function  $\Phi \in C^0(Q_T \times \mathbb{R}, \mathbb{R}^d)$  such that

$$\Phi(x, t, u) = \frac{1}{2} \int_{a_0}^u \eta'(\kappa) \frac{\partial F}{\partial u}(x, t, \kappa) \, d\kappa + \frac{1}{2} \int_{b_0}^u \eta'(\kappa) \frac{\partial F}{\partial u}(x, t, \kappa) \, d\kappa + \frac{1}{2} (\eta'(a_0)F(x, t, a_0) + \eta'(b_0)F(x, t, b_0)),$$

and thus  $\frac{\partial \Phi}{\partial u}(x, t, u) = \eta'(u) \frac{\partial F}{\partial u}(x, t, u)$ .

**Remark 2.2 (Consistency with Kruzhkov entropy pairs):** The definition of the entropy flux could be done up to any divergence free function. Definition 2.1 is such that, if  $\eta$  converges in some sense to the Kruzhkov entropy  $|\cdot - \kappa|$  for  $a_0 < \kappa < b_0$ , then  $\Phi$  converges to  $F(x, t, \cdot \top \kappa) - F(x, t, \cdot \perp \kappa)$ . Indeed, for  $\eta \in C^2(\mathbb{R})$ , we have the identity

$$\begin{aligned} \eta(x) &= \frac{1}{2} \int_{a_0}^{b_0} \eta''(\kappa) |x - \kappa| d\kappa \\ &\quad + \frac{1}{2} ((\eta'(a_0) + \eta'(b_0))x + \eta(a_0) - \eta'(a_0)a_0 + \eta(b_0) - \eta'(b_0)b_0), \quad \forall x \in [a_0, b_0], \end{aligned} \quad (5)$$

and we have as well

$$\Phi(x, t, u) := \frac{1}{2} \int_{a_0}^{b_0} \eta''(\kappa) (F(x, t, u \top \kappa) - F(x, t, u \perp \kappa)) d\kappa + \frac{\eta'(a_0) + \eta'(b_0)}{2} F(x, t, u). \quad (6)$$

Since the flux function  $F$  is divergence-free with respect to the space variables, the entropy flux is divergence-free as well.

**Definition 2.3:** Under Hypotheses (HC), a function  $u \in L^\infty(Q_T) \cap L^1(0, T; BV(\mathbb{R}^d))$  is said to be an entropy solution of (2)-(3) if there exists  $\lambda \in L^\infty(Q_T)^d$ , with  $|\lambda| \leq 1$  almost everywhere on  $Q_T$ , such that, for all admissible entropy-entropy flux pairs  $(\eta, \Phi)$  in the sense of Definition 2.1 and all nonnegative test functions  $\varphi \in C_c^\infty(Q_T)$ ,

$$\int_{Q_T} \left( \eta(u) \partial_t \varphi + (\Phi(x, t, u) - \lambda \eta'(u)) \cdot \nabla \varphi \right) dx dt - \int_{Q_T} \varphi |D[\eta'(u)]| dt + \int_{\mathbb{R}^d} \eta(u_{\text{ini}}(x)) \varphi(x, 0) dx \geq 0. \quad (7)$$

Since  $\eta'$  is in  $C^1(\mathbb{R})$ , the function  $\eta'(u)$  is in  $L^1(0, T; BV(\mathbb{R}^d))$ . Therefore, the term  $\int_{Q_T} \varphi |D[\eta'(u)]| dt$  is meaningful. The function  $\lambda$ , which is not necessarily unique, is called a multiplier by analogy with a Lagrange multiplier.

### 2.3 Formal derivation of the entropy formulation

In order to enlighten the link between the strong formulation and the entropy formulation, we present below a formal derivation of the entropy formulation by the vanishing viscosity method. Let us consider a viscous regularisation of the equation (2):

$$\partial_t u_\epsilon + \operatorname{div} F(x, t, u_\epsilon) - \operatorname{div} \operatorname{Sgn}(\nabla u_\epsilon) - \epsilon \Delta u_\epsilon = 0, \quad (8)$$

where  $\epsilon$  is a positive constant. One can justify that this problem is well-posed and, owing to the term  $\operatorname{div} \operatorname{Sgn}(\nabla u_\epsilon)$ , we can expect that  $u_\epsilon$  tends to  $u$  in  $L^1_{\text{loc}}(Q_T)$  when  $\epsilon \rightarrow 0$ . For a given admissible entropy-entropy flux pairs  $(\eta, \Phi)$  in the sense of Definition 2.1, multiplying (8) by  $\eta'(u_\epsilon)$ , we find

$$\partial_t \eta(u_\epsilon) + \operatorname{div} \left( \Phi(x, t, u_\epsilon) - \operatorname{Sgn}(\nabla u_\epsilon) \eta'(u_\epsilon) \right) + \nabla \eta'(u_\epsilon) \cdot \operatorname{Sgn}(\nabla u_\epsilon) - \epsilon \Delta \eta(u_\epsilon) + \epsilon \eta''(u_\epsilon) |\nabla u_\epsilon|^2 = 0. \quad (9)$$

The entropy  $\eta$  being convex, we have

$$\nabla \eta'(u_\epsilon) \cdot \operatorname{Sgn}(\nabla u_\epsilon) = \eta''(u_\epsilon) \nabla u_\epsilon \cdot \operatorname{Sgn}(\nabla u_\epsilon) \geq 0,$$

whence

$$\nabla \eta'(u_\epsilon) \cdot \operatorname{Sgn}(\nabla u_\epsilon) = |\nabla \eta'(u_\epsilon)|.$$

Still by convexity of  $\eta$ , we have

$$\epsilon \eta''(u_\epsilon) |\nabla u_\epsilon|^2 \geq 0.$$

Let us now multiply (9) by a nonnegative test function  $\varphi \in C_c^\infty(Q_T)$  and integrate over  $Q_T$ . We thus obtain

$$\int_{Q_T} \left( \eta(u_\epsilon) \partial_t \varphi + (\Phi(x, t, u_\epsilon) - \operatorname{Sgn}(\nabla u_\epsilon) \eta'(u_\epsilon)) \cdot \nabla \varphi - |\nabla \eta'(u_\epsilon)| \varphi \right) dx dt + \int_{\mathbb{R}^d} \eta(u_{\text{ini}}(x)) \varphi(x, 0) dx \geq 0. \quad (10)$$

Since  $u_\epsilon \rightarrow u$  in  $L^1_{\text{loc}}(Q_T)$ , it follows from the semi-continuity of the total variation that

$$\liminf_{\epsilon \rightarrow 0} \int_{Q_T} |\nabla \eta'(u_\epsilon)| \varphi dx dt \geq \int_{Q_T} \varphi |D[\eta'(u)]| dt.$$

Since the family  $\{\text{Sgn}(\nabla u_\epsilon)\}_{\epsilon>0}$  is bounded, there exists  $\lambda \in L^\infty(Q_T)^d$  such that, up to a subsequence,

$$\text{Sgn}(\nabla u_\epsilon) \rightharpoonup \lambda \text{ weakly-* in } L^\infty(Q_T)^d.$$

Finally, letting  $\epsilon \rightarrow 0$  in (10), we obtain (7).

## 2.4 Existence and uniqueness

**Theorem 2.4:** Under Hypotheses (HC), there exists one and only one entropy solution of (2)-(3) in the sense of Definition 2.3.

**Proof** The existence of an entropy solution is proved by the convergence of a numerical approximation; see Theorem 6.2 below. To prove the uniqueness, we use the doubling variable technique.

Step 1. Let  $u$  be an entropy solution and  $\lambda_u$  a corresponding multiplier. We consider the entropy  $\eta(\cdot - \kappa)$ , where  $\eta \in C^\infty(\mathbb{R})$  is an even convex function and  $\kappa \in \mathbb{R}$ . Let  $\Phi_\kappa$  be the corresponding entropy flux. Then, by definition of an entropy solution, for all nonnegative test functions  $\varphi \in C_c^\infty(Q_T)$ ,

$$\begin{aligned} \int_{Q_T} \left( \eta(u(x, t) - \kappa) \partial_t \varphi + (\Phi_\kappa(x, t, u(x, t)) - \lambda_u(x, t) \eta'(u(x, t) - \kappa)) \cdot \nabla_x \varphi \right) dx dt \\ - \int_{Q_T} \varphi |D_x[\eta'(u(x, t) - \kappa)]| dt + \int_{\mathbb{R}^d} \eta(u_{\text{ini}}(x) - \kappa) \varphi(x, 0) dx \geq 0. \end{aligned} \quad (11)$$

Let  $v$  be another entropy solution and  $\lambda_v$  a corresponding multiplier. Since  $\eta$  is an even function,  $\eta(\cdot - \kappa) = \eta(\kappa - \cdot)$  and  $\eta'(\cdot - \kappa) = -\eta'(\kappa - \cdot)$ . Then, denoting by  $y$  and  $s$  the space and time variables, for all nonnegative test functions  $\varphi \in C_c^\infty(Q_T)$ ,

$$\begin{aligned} \int_{Q_T} \left( \eta(\kappa - v(y, s)) \partial_s \varphi + (\Phi_\kappa(y, s, v(y, s)) + \lambda_v(y, s) \eta'(\kappa - v(y, s))) \cdot \nabla_y \varphi \right) dy ds \\ - \int_{Q_T} \varphi |D_y[\eta'(\kappa - v(y, s))]| dt + \int_{\mathbb{R}^d} \eta(u_{\text{ini}}(y) - \kappa) \varphi(y, 0) dy \geq 0. \end{aligned} \quad (12)$$

Step 2. We now introduce well-chosen test functions. Let  $\{\rho_\epsilon\}_{\epsilon>0}$  be a family of mollifiers in  $\mathbb{R}^d$  such that  $\text{supp } \rho_\epsilon \subset B(0, \epsilon)$  and  $\{\bar{\rho}_\epsilon\}_{\epsilon>0}$  be a family of mollifiers in  $\mathbb{R}$  such that  $\text{supp } \bar{\rho}_\epsilon \subset [-\epsilon, 0]$ . Let  $r > 0$ ,  $\tau > 0$ , and  $\psi \in C_c^\infty(\mathbb{R}, \mathbb{R}^+)$ . We define  $\phi : Q_T \times Q_T \rightarrow \mathbb{R}^+$  such that

$$\phi(x, t, y, s) = \psi(t) \rho_r(x - y) \bar{\rho}_\tau(t - s), \quad \forall (x, t, y, s) \in Q_T \times Q_T. \quad (13)$$

We take  $\kappa = v(y, s)$  and  $\varphi = \phi(\cdot, \cdot, y, s)$  in (11). Next, integrating with respect to  $y$  and  $s$  over  $Q_T$ , and noticing that  $\phi(x, 0, y, s) = 0$  for all  $s > 0$ , we obtain

$$\begin{aligned} \int_{Q_T \times Q_T} \left( \eta(u(x, t) - v(y, s)) \partial_t \phi + (\Phi_{v(y, s)}(x, t, u(x, t)) - \lambda_u(x, t) \eta'(u(x, t) - v(y, s))) \cdot \nabla_x \phi \right) dx dt dy ds \\ - \int_{Q_T \times Q_T} \phi |D_x[\eta'(u(x, t) - v(y, s))]| dt dy ds \geq 0. \end{aligned} \quad (14)$$

Similarly, taking  $\kappa = u(x, t)$  and  $\varphi = \phi(x, t, \cdot, \cdot)$  in (12), then integrating with respect to  $x$  and  $t$  over  $Q_T$ , we obtain

$$\begin{aligned} \int_{Q_T \times Q_T} \left( \eta(u(x, t) - v(y, s)) \partial_s \phi + (\Phi_{u(x, t)}(y, s, v(y, s)) + \lambda_v(y, s) \eta'(u(x, t) - v(y, s))) \cdot \nabla_y \phi \right) dx dt dy ds \\ - \int_{Q_T \times Q_T} \phi |D_y[\eta'(u(x, t) - v(y, s))]| ds dx dt + \int_{\mathbb{R}^d \times Q_T} \eta(u_{\text{ini}}(y) - u(x, t)) \phi(x, t, y, 0) dy dx dt \geq 0. \end{aligned} \quad (15)$$

Adding the above relations (14) and (15), we find

$$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 \geq 0,$$

where

$$\begin{aligned}
A_1 &:= \int_{Q_T \times Q_T} \eta(u(x, t) - v(y, s))(\partial_t \phi + \partial_s \phi) \, dx dt dy ds, \\
A_2 &:= \int_{Q_T \times Q_T} \Phi_{v(y, s)}(x, t, u(x, t)) \cdot \nabla_x \phi \, dx dt dy ds, \\
A_3 &:= \int_{Q_T \times Q_T} \Phi_{u(x, t)}(y, s, v(y, s)) \cdot \nabla_y \phi \, dx dt dy ds, \\
A_4 &:= \int_{Q_T \times Q_T} (\lambda_u(x, t) - \lambda_v(y, s)) \eta'(u(x, t) - v(y, s)) \cdot (\nabla_x \phi + \nabla_y \phi) \, dx dt dy ds, \\
A_5 &:= \int_{Q_T \times Q_T} \lambda_v(y, s) \eta'(u(x, t) - v(y, s)) \cdot \nabla_x \phi \, dx dt dy ds - \int_{Q_T \times Q_T} \phi |D_x[\eta'(u(x, t) - v(y, s))]| \, dt dy ds, \\
A_6 &:= \int_{Q_T \times Q_T} -\lambda_u(x, t) \eta'(u(x, t) - v(y, s)) \cdot \nabla_y \phi \, dx dt dy ds - \int_{Q_T \times Q_T} \phi |D_y[\eta'(u(x, t) - v(y, s))]| \, ds dx dt, \\
A_7 &:= \int_{\mathbb{R}^d \times Q_T} \eta(u_{\text{ini}}(y) - u(x, t)) \phi(x, t, y, 0) \, dy dx dt.
\end{aligned}$$

Step 3. We now remark that we can get rid of  $A_4$ ,  $A_5$  and  $A_6$ , the terms arising from the total variation. The term  $A_4$  vanishes owing to identity  $\nabla_x \phi + \nabla_y \phi = 0$ . The term  $A_5$  is nonpositive. Indeed, integrating by parts with respect to  $x$  the first term of  $A_5$ , we obtain

$$A_5 = \int_{Q_T \times Q_T} -\lambda_v(y, s) \phi D_x[\eta'(u(x, t) - v(y, s))] \, dt dy ds - \int_{Q_T \times Q_T} \phi |D_x[\eta'(u(x, t) - v(y, s))]| \, dt dy ds. \quad (16)$$

Since  $|\lambda_v(y, s)| \leq 1$  for all  $(y, s) \in Q_T$ , we deduce that  $A_5 \leq 0$ . With the same argument, we prove that  $A_6 \leq 0$ . Step 4. The resulting equation is now

$$A_1 + A_2 + A_3 + A_7 \geq 0. \quad (17)$$

Let  $\kappa \in \mathbb{R}$ . Consider the sequence of entropies  $(\eta_k)_{k \in \mathbb{N}^*}$  such that  $\eta_k(x - \kappa) := \sqrt{1/k + (x - \kappa)^2}$ . This sequence of entropies converges uniformly to the Kruzhkov entropy  $|\cdot - \kappa|$ , and the entropy flux converges as well to the Kruzhkov entropy flux. Passing to the limit in (17) with this sequence of entropies, we obtain, thanks to Remark 2.2,

$$A_{10} + A_{20} + A_{30} + A_{70} \geq 0, \quad (18)$$

where

$$\begin{aligned}
A_{10} &:= \int_{Q_T \times Q_T} |u(x, t) - v(y, s)|(\partial_t \phi + \partial_s \phi) \, dx dt dy ds, \\
A_{20} &:= \int_{Q_T \times Q_T} [F(x, t, u(x, t)) \top v(y, s) - F(x, t, u(x, t)) \perp v(y, s)] \cdot \nabla_x \phi \, dx dt dy ds, \\
A_{30} &:= \int_{Q_T \times Q_T} [F(y, s, v(y, s)) \top u(x, t) - F(y, s, v(y, s)) \perp u(x, t)] \cdot \nabla_y \phi \, dx dt dy ds, \\
A_{70} &:= \int_{\mathbb{R}^d \times Q_T} |u_{\text{ini}}(y) - u(x, t)| \phi(x, t, y, 0) \, dy dx dt.
\end{aligned}$$

Step 5. The remaining of the proof is identical to the one of Lemma 5 in [8], since its starting point is precisely (18).  $\square$

### 3 Numerical approximation

#### 3.1 Notation and hypotheses

The finite element mesh, denoted by  $\mathcal{T}_h$ , is a conforming simplicial mesh of  $\mathbb{R}^d$  of size  $h$ . In order to ensure the maximum principle, each element of  $\mathcal{T}_h$  is assumed to be nonobtuse; we recall that a simplex is said to be nonobtuse if the angles between any two facets are less than or equal to  $\pi/2$ . The family of the vertices of the mesh is denoted by  $(x_p)_{p \in \mathbb{N}}$ , and for any  $K \in \mathcal{T}_h$ , we denote by  $\mathcal{V}_K \subset \mathbb{N}$  the set of the  $d + 1$  indices of the vertices of  $K$ , and by  $\mathcal{E}_K$  the set of the  $d + 1$  faces of  $K$ .

The finite volume mesh, denoted by  $\mathcal{D}_h$ , is a polyhedral mesh of  $\mathbb{R}^d$  such that the interface between two cells is a finite union of faces. The mesh  $\mathcal{D}_h$  is a dual mesh of  $\mathcal{T}_h$  in the sense that each cell of  $\mathcal{D}_h$  contains one and only one node of  $\mathcal{T}_h$ . The nodes of  $\mathcal{T}_h$  and the corresponding cells of  $\mathcal{D}_h$  are denoted by  $(x_p)_{p \in \mathbb{N}}$  and  $(Q_p)_{p \in \mathbb{N}}$ , respectively. Let us introduce some additional notation about  $\mathcal{D}_h$ :  $\mathcal{N}_p$  is the set containing the indices of the neighbouring cells of  $Q_p$ ,  $\mathcal{E}_h$  is the set of couples  $(p, q)$  such that  $Q_p$  and  $Q_q$  are neighbours,  $\sigma_{p,q}$  is the interface between two neighbour cells  $Q_p$  and  $Q_q$ ,  $\nu_{p,q}$  is the unit normal vector to  $\sigma_{p,q}$  pointing toward  $Q_q$ ,  $m_p$  is the measure of  $Q_p$ ,  $m_{p,q}$  is the measure of  $\sigma_{p,q}$ .

**Remark 3.1:** Since a square can be divided into two right triangles and a cube can be divided into six nonobtuse tetrahedra, it is easy to build nonobtuse simplicial meshes of  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . In fact, it is possible to generate nonobtuse simplicial meshes on any polygonal or polyhedral domain; see [6] and references therein.

**Remark 3.2:** The dual mesh can be defined, for any  $d \geq 1$ , by the following procedure. For any  $p \in \mathbb{N}$  and any  $K \in \mathcal{T}_h$  with  $p \in \mathcal{V}_K$ , the set  $Q_p^K$  is defined as the set of all points  $x \in K$  under the form  $x = \sum_{q \in \mathcal{V}_K} \lambda_q x_q$ , with  $\sum_{q \in \mathcal{V}_K} \lambda_q = 1$  and  $\lambda_p \geq \lambda_q$  for all  $q \in \mathcal{V}_K$ , and then define  $Q_p$  as the union of all  $Q_p^K$ , such that  $p \in \mathcal{V}_K$ . In  $\mathbb{R}^2$ , for every triangle  $K \in \mathcal{T}_h$  containing the node  $x_p$ , a part of the boundary of  $Q_p^K$  is the union of the segments joining the centre of gravity of  $K$  with the midpoint of the two edges of  $K$  incident to  $x_p$ . This yields a closed polygonal line which delimits a cell  $Q_p$  associated to  $x_p$ .

Let us define the discrete functional spaces

$$\begin{aligned} V_h &:= \{\widehat{v}_h \in C^0(\mathbb{R}^d); \widehat{v}_h|_K \text{ is affine for each } K \in \mathcal{T}_h\}, \\ M_h &:= \{\bar{v}_h \in L^1_{\text{loc}}(\mathbb{R}^d); \bar{v}_h|_Q \text{ is constant for each } Q \in \mathcal{D}_h\}, \\ \Lambda_h &:= \{\mu_h \in L^\infty(\mathbb{R}^d)^d; \mu_h|_K \text{ is constant for each } K \in \mathcal{T}_h\}. \end{aligned}$$

In our scheme, the multiplier  $\lambda$  is approximated in  $\Lambda_h$  and the velocity  $u$  is approximated simultaneously in  $V_h$  and in  $M_h$ . To handle easily this double discretisation, we define the discrete velocity as a collection of values  $(u_p)_{p \in \mathbb{N}}$ , to which we associate the functions  $\widehat{u}_h \in V_h$  and  $\bar{u}_h \in M_h$  such that

$$\widehat{u}_h(x_p) = u_p, \quad \text{and} \quad \bar{u}_h|_{Q_p} = u_p \quad \forall p \in \mathbb{N}.$$

We now define

$$X_h = \{v_h \in \mathbb{R}^{\mathbb{N}}; \nabla \widehat{v}_h \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \bar{v}_h \in L^2(\mathbb{R}^d)\}. \quad (19)$$

Letting  $K \in \mathcal{T}_h$ , if we denote by  $\{x_p\}_{p \in \mathcal{V}_K}$  the vertices of  $K$  and by  $\{\phi_p\}_{p \in \mathcal{V}_K}$  the corresponding Lagrange basis function, we can write, for any  $u_h, v_h \in X_h$ ,

$$\nabla \widehat{v}_h|_K \cdot \nabla \widehat{u}_h|_K = \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} v_p u_q \nabla \phi_p|_K \cdot \nabla \phi_q|_K.$$

Using the fact that  $\sum_{p \in \mathcal{V}_K} \phi_p|_K = 1$ , and thus  $\sum_{p \in \mathcal{V}_K} \nabla \phi_p|_K = 0$ , the preceding equation can be rewritten as

$$\nabla \widehat{v}_h|_K \cdot \nabla \widehat{u}_h|_K = \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} u_q (v_p - v_q) \nabla \phi_p|_K \cdot \nabla \phi_q|_K.$$

Exchanging the roles of  $p$  and  $q$ , we get

$$\nabla \widehat{v}_h|_K \cdot \nabla \widehat{u}_h|_K = \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} u_p (v_q - v_p) \nabla \phi_p|_K \cdot \nabla \phi_q|_K.$$

Adding the two preceding relations provides

$$2 \nabla \widehat{v}_h|_K \cdot \nabla \widehat{u}_h|_K = - \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} (u_p - u_q) (v_p - v_q) \nabla \phi_p|_K \cdot \nabla \phi_q|_K,$$

and therefore

$$\int_K \nabla \widehat{v}_h(x) \cdot \nabla \widehat{u}_h(x) dx = \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (u_p - u_q) (v_p - v_q) \quad \text{with} \quad T_{pq}^K = -\frac{1}{2} \int_K \nabla \phi_p(x) \cdot \nabla \phi_q(x) dx. \quad (20)$$

Since the simplex  $K$  is nonobtuse, we have the standard inequality (see for example [6])

$$\nabla\phi_p|_K \cdot \nabla\phi_q|_K \leq 0 \text{ and } T_{pq}^K \geq 0, \quad \forall p, q \in \mathcal{V}_K, p \neq q. \quad (21)$$

Let us observe that, for any  $u_h \in \mathbb{R}^N$ , we have, denoting for a.e.  $x \in \mathbb{R}^d$  by  $p(x) \in \mathbb{N}$  such that  $x \in Q_p$ ,

$$\text{for a.e. } x \in \mathbb{R}^d, |\widehat{u}_h(x) - \bar{u}_h(x)| = |(x - x_{p(x)}) \cdot \nabla \widehat{u}_h(x)| \leq h |\nabla \widehat{u}_h(x)|, \quad (22)$$

which implies, if  $\nabla \widehat{u}_h \in L^1(\mathbb{R}^d)^d$ ,

$$\|\bar{u}_h - \widehat{u}_h\|_{L^1(\mathbb{R}^d)} \leq h \|\nabla \widehat{u}_h\|_{L^1(\mathbb{R}^d)^d}. \quad (23)$$

For the finite volume step we need to define numerical fluxes between neighbour cells at each time step. The numerical flux from the cell  $Q_p$  to the cell  $Q_q$  at time  $t^n$  is denoted by  $F_{p,q}^n$ . We require that the family of numerical fluxes  $(F_{p,q}^n)_{p,q,n \in \mathbb{N}}$  is admissible and consistent with the flux  $F$  in the sense of the two definitions below. The Godunov scheme [25, see, e.g.,] provides, for instance, such numerical fluxes.

**Definition 3.3:** A family of numerical fluxes  $(F_{p,q}^n)_{p,q,n \in \mathbb{N}}$  is said to be admissible if

- $F_{p,q}^n \in C^0([a_0, b_0]^2, \mathbb{R})$  and there exists  $L > 0$  such that, for all  $(p, q) \in \mathcal{E}_h$  and for all  $n \in \{0, \dots, N-1\}$ , the function  $F_{p,q}^n$  is Lipschitz continuous with the constant  $m_{p,q}L$  with respect to each of its variables.
- $F_{p,q}^n$  is monotone, i.e. nondecreasing with respect to its first variable and nonincreasing with respect to its second variable,
- $F_{p,q}^n$  is conservative, i.e.  $F_{p,q}^n(u, v) = -F_{q,p}^n(v, u)$  for all  $(u, v) \in [a_0, b_0]^2$ .

**Definition 3.4:** Let  $F$  be a flux function. A family of numerical fluxes  $(F_{p,q}^n)_{p,q,n \in \mathbb{N}}$  is said to be consistent with  $F$  if

$$F_{p,q}^n(u, u) = \frac{1}{\delta x} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} F(x, t, u) \cdot \nu_{p,q} \, ds(x) dt, \quad \forall u \in [a_0, b_0]. \quad (24)$$

The following lemma, which is a discrete version of the divergence theorem, will be used in several places in the rest of the article.

**Lemma 3.5:** Let  $(F_{p,q}^n)_{p,q,n \in \mathbb{N}}$  be a family of numerical fluxes consistent with a flux function  $F$ . If  $F$  is divergence-free, then

$$\forall u \in \mathbb{R}, \sum_{q \in \mathcal{N}_p} F_{p,q}(u, u) = 0. \quad (25)$$

**Remark 3.6:** The consistency hypothesis for the numerical fluxes could be slightly weakened (see, for instance [8, Section 2.1]).

## 3.2 Description of the numerical scheme and well-posedness

We consider a family of discretisations  $(\mathcal{F}_{h,\delta x})_{h,\delta x > 0}$  – by discretisation, we mean a finite element mesh  $\mathcal{T}_h$ , a finite volume mesh  $\mathcal{D}_h$ , a time step  $\delta x$ , a family of numerical fluxes  $(F_{p,q}^n)_{p,q,n \in \mathbb{N}}$ . We assume that the following hypotheses, denoted in the following by Hypotheses (HD), are satisfied uniformly by any element  $\mathcal{F}_{h,\delta x}$  of the family.

- (HD1) There exists  $\alpha > 0$  such that  $m_p \geq \alpha h^d$  and  $|\partial Q_p| \leq \frac{1}{\alpha} h^{d-1}$  for all  $p \in \mathbb{N}$ .
- (HD2) There exists a constant  $\beta$  such that  $\text{diam}(Q_p) \leq \beta h$  for all  $p \in \mathbb{N}$ , where  $\text{diam}(Q_p)$  is the diameter of  $Q_p$ .
- (HD3) There exists  $\rho > 0$  such that  $|K| \geq \rho h^d$ , for all  $K \in \mathcal{T}_h$ .
- (HD4) There exists an admissible family of numerical fluxes  $(F_{p,q}^n)_{p,q,n \in \mathbb{N}}$  in the sense of Definition 3.3, which is consistent with  $F$  in the sense of Definition 3.4. The constant  $L$  in Definition 3.3 is assumed to be independent of the discretisation.

(HD5) The time interval  $[0, T]$  is divided into  $N$  equal intervals of length  $\delta t$ , such that the following CFL condition holds

$$\delta t \leq \frac{1}{2L} \inf_{(p,q) \in \mathcal{E}_h} \frac{m_p}{m_{p,q}}. \quad (26)$$

The scheme for approximating (2)-(3) is given by:

- Initialisation of  $(u_p^0)_{p \in \mathbb{N}}$  such that  $\bar{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ :

$$u_p^0 = \frac{1}{m_p} \int_{Q_p} u_{\text{ini}}(x) dx, \quad \forall p \in \mathbb{N}. \quad (27)$$

- Finite volume step. Letting  $(u_p^n)_{p \in \mathbb{N}}$  such that  $\bar{u}_h^n \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ , seek  $(u_p^{n+\frac{1}{2}})_{p \in \mathbb{N}}$  such that  $\bar{u}_h^{n+\frac{1}{2}} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and

$$m_p \frac{u_p^{n+\frac{1}{2}} - u_p^n}{\delta t} + \sum_{q \in \mathcal{N}_p} F_{p,q}^n(u_p^n, u_q^n) = 0, \quad \forall p \in \mathbb{N}. \quad (28)$$

- Finite element step. Seek  $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$  such that

$$\int_{\mathbb{R}^d} \frac{\bar{u}_h^{n+1} - \bar{u}_h^{n+\frac{1}{2}}}{\delta t} \bar{v}_h dx + \int_{\mathbb{R}^d} (\lambda_h^{n+1} + \theta(h) \nabla \hat{u}_h^{n+1}) \cdot \nabla \hat{v}_h dx = 0, \quad \forall v_h \in X_h, \quad (29)$$

$$\lambda_h^{n+1} \in \text{Sgn}(\nabla \hat{u}_h^{n+1}), \quad (30)$$

where  $\theta \in C^0((0, +\infty))$  is a positive function such that

$$\lim_{h \rightarrow 0} \theta(h) = \lim_{h \rightarrow 0} \frac{h}{\theta(h)} = 0. \quad (31)$$

An example of such a function  $\theta$  is  $\theta(h) = h^\gamma$  with  $0 < \gamma < 1$ .

**Remark 3.7:** The explicit time discretisation of the hyperbolic step could be replaced by an implicit time discretisation. In this case, the CFL condition (26) on the time step would not be necessary.

For each discretisation  $\mathcal{F}_{h,\delta t}$ , we define the approximate solutions  $\hat{u}_{h,\delta t} : Q_T \rightarrow \mathbb{R}$ ,  $\bar{u}_{h,\delta t} : Q_T \rightarrow \mathbb{R}$ , and  $\lambda_{h,\delta t} : Q_T \rightarrow \mathbb{R}$  such that

$$\begin{aligned} \hat{u}_{h,\delta t}(\cdot, t) &:= \hat{u}_h^{n+1} & \text{if } t \in (t^n, t^{n+1}], \\ \bar{u}_{h,\delta t}(\cdot, t) &:= \bar{u}_h^{n+1} & \text{if } t \in (t^n, t^{n+1}], \\ \lambda_{h,\delta t}(\cdot, t) &:= \lambda_h^{n+1} & \text{if } t \in (t^n, t^{n+1}]. \end{aligned}$$

The proposition below proves that the scheme has at least one solution, which is unique with respect to the velocity.

**Proposition 3.8:** Let us assume Hypotheses (HC) and Hypotheses (HD). Then there exists at least one solution to Scheme (27)-(30) such that  $\bar{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and, for all  $n \in \mathbb{N}$ ,  $\bar{u}_h^{n+\frac{1}{2}}, \bar{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and  $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$ . Moreover,  $\bar{u}_h^0$  and, for all  $n \in \mathbb{N}$ ,  $\bar{u}_h^{n+\frac{1}{2}}$  and  $u_h^{n+1}$  are unique and, for all  $p \in \mathbb{N}$ ,  $a_0 \leq u_p^{n+\frac{1}{2}} \leq b_0$  and  $a_0 \leq u_p^{n+1} \leq b_0$ .

**Proof** Thanks to (27), we get from Hypothesis (HC2) that

$$\|\bar{u}_h\|_{L^1(\mathbb{R}^d)} \leq \|u_{\text{ini}}\|_{L^1(\mathbb{R}^d)},$$

and  $a_0 \leq u_p^0 \leq b_0$  for all  $p \in \mathbb{N}$ , which completes the proof that  $\bar{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ .

For any  $n \in \mathbb{N}$ , we get from Proposition 3.10, assuming  $\bar{u}_h^n \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ , that  $\bar{u}_h^{n+\frac{1}{2}} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  with  $a_0 \leq u_p^{n+\frac{1}{2}} \leq b_0$  for all  $p \in \mathbb{N}$ . Using Proposition 3.11, we get the existence of  $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$  such that (30) holds, and we get that  $u_h^{n+1}$  is unique. It now suffices to apply Proposition 3.13, for obtaining that  $\bar{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  with  $a_0 \leq u_p^{n+1} \leq b_0$  for all  $p \in \mathbb{N}$ .  $\square$

Let us now state and prove the propositions used in the proof of the preceding result. We have first the following result.

**Proposition 3.9:** Let us assume Hypotheses (HC) and Hypotheses (HD). Let  $(\eta, \Phi)$  be an entropy-entropy flux pair in the sense of Definition 2.1, and let  $n \in \mathbb{N}$  be given. Then, the family  $(\Phi_{p,q}^n)_{p,q,n \in \mathbb{N}}$  of admissible numerical fluxes defined by

$$\Phi_{p,q}^n(x, y) := \frac{1}{2} \int_{a_0}^{b_0} \eta''(\kappa) (F_{p,q}^n(x \top \kappa, y \top \kappa) - F_{p,q}^n(x \perp \kappa, y \perp \kappa)) d\kappa + \frac{\eta'(a_0) + \eta'(b_0)}{2} F_{p,q}^n(x, y), \quad (32)$$

is consistent with  $\Phi$  in the sense of Definition 3.4, and is such that, if  $u_h^{n+\frac{1}{2}} \in \mathbb{R}^{\mathbb{N}}$  is obtained from  $u_h^n \in \mathbb{R}^{\mathbb{N}}$  by (28) with  $a_0 \leq u_p^n \leq b_0$ , for all  $p \in \mathbb{N}$ , then

$$m_p \frac{\eta(u_p^{n+\frac{1}{2}}) - \eta(u_p^n)}{\delta t} + \sum_{q \in \mathcal{N}_p} \Phi_{p,q}^n(u_p^n, u_q^n) \leq 0, \quad \forall n \in \{0, \dots, N-1\}, \quad \forall p \in \mathbb{N}. \quad (33)$$

Furthermore, there is a constant  $L'$ , depending only on  $L, \eta, a_0$  and  $b_0$ , such that, for all  $(p, q) \in \mathcal{E}_h$  and for all  $n \in \{0, \dots, N-1\}$ , the function  $\Phi_{p,q}^n$  is Lipschitz continuous with respect of each of its variables with the constant  $m_{p,q} L'$ .

**Proof** Consider first the Kruzhkov entropies, that are the functions  $|\cdot - \kappa|$ , with  $\kappa \in \mathbb{R}$ . The corresponding entropy fluxes are the functions  $F(\cdot \top \kappa) - F(\cdot \perp \kappa)$ . The derivation of the discrete entropy inequality involving the Kruzhkov entropies for the finite volume scheme (28) is standard and can be found for instance in [8, Lemma 3] or [16, Lemma 27.1]. This inequality, which holds thanks to Hypothesis (HD5), reads

$$m_p \frac{|u_p^{n+\frac{1}{2}} - \kappa| - |u_p^n - \kappa|}{\delta t} + \sum_{q \in \mathcal{N}_p} (F_{p,q}^n(u_p^n \top \kappa, u_q^n \top \kappa) - F_{p,q}^n(u_p^n \perp \kappa, u_q^n \perp \kappa)) \leq 0, \quad (34)$$

for all  $n \in \{0, \dots, N-1\}$  and  $p \in \mathbb{N}$ .

Using this identity, together with (5), (32) and (34), we deduce the inequality

$$m_p \frac{\eta(u_p^{n+\frac{1}{2}}) - \eta(u_p^n)}{\delta t} + \sum_{q \in \mathcal{N}_p} \Phi_{p,q}^n(u_p^n, u_q^n) \leq 0. \quad (35)$$

□

From the above result, we get the following one.

**Proposition 3.10:** Let us assume Hypotheses (HC) and Hypotheses (HD). Assume that, for a given  $n \in \mathbb{N}$ ,  $a_0 \leq u_p^n \leq b_0$  for all  $p \in \mathbb{N}$  and  $\bar{u}_h^n \in L^1(\mathbb{R}^d)$ . Then  $\bar{u}_h^{n+\frac{1}{2}} \in L^1(\mathbb{R}^d)$ ,  $a_0 \leq u_p^{n+\frac{1}{2}} \leq b_0$  for all  $p \in \mathbb{N}$ , and for all entropy  $\eta$  in the sense of Definition 2.1 such that  $\eta(0) = 0$  and  $\eta'(0) = 0$ ,

$$\int_{\mathbb{R}^d} \eta(\bar{u}_h^{n+\frac{1}{2}}(x)) dx \leq \int_{\mathbb{R}^d} \eta(\bar{u}_h^n(x)) dx. \quad (36)$$

**Proof** We get, from (33) and applying Lemma 3.5,

$$m_p \frac{\eta(u_p^{n+\frac{1}{2}}) - \eta(u_p^n)}{\delta t} + \sum_{q \in \mathcal{N}_p} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(0, 0)) \leq 0, \quad \forall n \in \{0, \dots, N-1\}, \quad \forall p \in \mathbb{N},$$

For  $\varepsilon > 0$ , we multiply the preceding inequality by  $\delta t \exp(-\varepsilon|x_p|)$  and we sum the result on  $p \in \mathbb{N}$ . We get

$$\sum_{p \in \mathbb{N}} m_p (\eta(u_p^{n+\frac{1}{2}}) - \eta(u_p^n)) \exp(-\varepsilon|x_p|) + \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(0, 0)) \exp(-\varepsilon|x_p|) \leq 0.$$

Introducing, for any  $(p, q) \in \mathcal{E}_h$ , the point  $x_{pq} = \frac{1}{2}(x_p + x_q)$  and the property  $\Phi_{p,q}^n(u, v) = -\Phi_{q,p}^n(v, u)$  for all  $(u, v) \in [a_0, b_0]^2$ , we get

$$\sum_{p \in \mathbb{N}} m_p (\eta(u_p^{n+\frac{1}{2}}) - \eta(u_p^n)) \exp(-\varepsilon|x_p|) + \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} A_{pq}^n \leq 0,$$

with

$$A_{pq}^n = (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(0, 0)) (\exp(-\varepsilon|x_p|) - \exp(-\varepsilon|x_{pq}|)).$$

Observing that Proposition 3.9 implies

$$|\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(0, 0)| \leq L' m_{pq} (|u_p^n| + |u_q^n|),$$

we get that

$$A_{pq}^n \geq -\frac{1}{2} L' m_{pq} (|u_p^n| + |u_q^n|) \varepsilon h \geq -\frac{1}{2} CL' (m_p |u_p^n| + m_q |u_q^n|) \varepsilon.$$

This leads to

$$\sum_{p \in \mathbb{N}} m_p (\eta(u_p^{n+\frac{1}{2}}) - \eta(0)) \exp(-\varepsilon|x_p|) \leq \sum_{p \in \mathbb{N}} m_p (\eta(u_p^n) - \eta(0)) \exp(-\varepsilon|x_p|) + \frac{1}{2} \varepsilon L' \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} C(m_p |u_p^n| + m_q |u_q^n|).$$

Since a simplex has  $d + 1$  faces, we get

$$\sum_{p \in \mathbb{N}} m_p (\eta(u_p^{n+\frac{1}{2}}) - \eta(0)) \exp(-\varepsilon|x_p|) \leq \sum_{p \in \mathbb{N}} m_p (\eta(u_p^n) - \eta(0)) + (d + 1) \varepsilon LC \sum_{p \in \mathbb{N}} m_p |u_p^n|.$$

Letting  $\varepsilon \rightarrow 0$ , we get by monotonous convergence (recall that thanks to the hypothesis  $\bar{u}_h^n \in L^1(\mathbb{R}^d)$ , we also have  $\eta(\bar{u}_h^n) \in L^1(\mathbb{R}^d)$  and  $\eta(s) \geq \eta(0) = 0$ ),

$$\sum_{p \in \mathbb{N}} m_p \eta(u_p^{n+\frac{1}{2}}) \leq \sum_{p \in \mathbb{N}} m_p \eta(u_p^n),$$

which concludes the proof of (36).

Letting  $\eta$  tend to  $\eta(s) = s \top b_0 - b_0$ , we get, since  $\eta(u_p^{n+\frac{1}{2}}) - \eta(0) = 0$  for all  $p \in \mathbb{N}$ , that  $\eta(u_p^{n+\frac{1}{2}}) - \eta(0) = 0$  for all  $p \in \mathbb{N}$ , which shows that  $u_p^{n+\frac{1}{2}} \leq b_0$  for all  $p \in \mathbb{N}$ . The same reasoning holds with  $\eta(s) = a_0 - s \perp a_0$ , which shows that  $u_p^{n+\frac{1}{2}} \geq a_0$ . Finally, letting  $\eta$  tend to  $\eta(s) = |s|$ , we conclude that  $\bar{u}_h^{n+\frac{1}{2}} \in L^1(\mathbb{R}^d)$ .

□

The proposition below proves that the finite element step is well-posed, provided that  $\bar{u}_h^{n+\frac{1}{2}} \in L^2(\mathbb{R}^d)$ , which holds if  $\bar{u}_h^{n+\frac{1}{2}} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ , and gives a variational characterisation of  $u_h^{n+1}$ .

**Proposition 3.11:** Let us assume Hypotheses (HC) and Hypotheses (HD). Let  $n \in \mathbb{N}$  be given and let us assume that  $\bar{u}_h^{n+\frac{1}{2}} \in L^2(\mathbb{R}^d)$ . Then, equations (29)-(30) admit a solution  $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$ . Furthermore,  $u_h^{n+1}$  is unique and is the minimiser of the functional  $J_h^{n+1} : X_h \rightarrow \mathbb{R}$  defined by

$$J_h^{n+1}(v_h) := \frac{1}{2\delta} \int_{\mathbb{R}^d} (\bar{v}_h - \bar{u}_h^{n+\frac{1}{2}})^2 dx + \int_{\mathbb{R}^d} (|\nabla \hat{v}_h| + \frac{1}{2} \theta(h) |\nabla \hat{v}_h|^2) dx. \quad (37)$$

**Proof** We remark that, defining the following norm on the Banach space  $X_h$ ,

$$\|v_h\|_{X_h} = \|\bar{v}_h\|_{L^2(\mathbb{R}^d)} + \|\nabla \hat{v}_h\|_{L^1(\mathbb{R}^d)} + \|\nabla \hat{v}_h\|_{L^2(\mathbb{R}^d)},$$

we obtain that the stricly convex function  $J_h^{n+1}$  is such that

$$\lim_{\|v_h\|_{X_h} \rightarrow \infty} J_h^{n+1}(v_h) = +\infty.$$

Hence  $J_h^{n+1}$  reaches its unique minimum value at some point  $u_h^{n+1} \in X_h$ . Let us prove that this point is characterised by (29)-(30).

Let us first assume that there exists  $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$ , solution to (29)-(30). Writing, for any  $w_h \in X_h$ ,  $J_h^{n+1}(w_h) = J_h^{n+1}(u_h^{n+1} + v_h)$  with  $v_h = w_h - u_h^{n+1}$ , we have using (29)

$$J_h^{n+1}(w_h) - J_h^{n+1}(u_h^{n+1}) = \frac{1}{2\delta} \int_{\mathbb{R}^d} \bar{v}_h(x)^2 dx + A + \int_{\mathbb{R}^d} \frac{1}{2} \theta(h) |\nabla \hat{v}_h(x)|^2 dx,$$

with

$$\begin{aligned} A &= \int_{\mathbb{R}^d} (|\nabla(\widehat{u}_h^{n+1} + \widehat{v}_h)(x)| - |\nabla\widehat{u}_h(x)| - \lambda_h^{n+1}(x) \cdot \nabla\widehat{v}_h(x)) dx \\ &= \sum_{K \in \mathcal{T}_h} (|\nabla(\widehat{u}_h^{n+1} + \widehat{v}_h)|_K - |\nabla\widehat{u}_h^{n+1}|_K - \lambda_K^{n+1} \cdot (\nabla\widehat{v}_h)|_K). \end{aligned}$$

Recall that, if  $(\nabla\widehat{u}_h^{n+1})|_K \neq 0$ , then  $\lambda_K^{n+1} = \frac{(\nabla\widehat{u}_h^{n+1})|_K}{|(\nabla\widehat{u}_h^{n+1})|_K|}$  and otherwise that  $|\lambda_K^{n+1}| \leq 1$ .

Using that for any  $a, b \in \mathbb{R}^d$  with  $a \neq 0$ , we have  $|a+b| - |a| - \frac{a \cdot b}{|a|} = \frac{|a||a+b| - a \cdot (a+b)}{|a|} \geq 0$ , and  $|b| - \lambda_K^{n+1} \cdot b \geq 0$ , we get  $A \geq 0$ , which proves that  $J_h^{n+1}(w_h) - J_h^{n+1}(u_h^{n+1}) \geq 0$ , and therefore that  $J_h^{n+1}$  reaches its minimum value at  $u_h^{n+1}$ .

Reciprocally, let us assume that  $J_h^{n+1}$  reaches its minimum value at some point  $u_h^{n+1} \in X_h$ . Let us prove that there exists  $\lambda_h^{n+1} \in \Lambda_h$  such that (29)-(30) holds. We denote by  $\mathcal{T}_{h,0}^{n+1} = \{K \in \mathcal{T}_h, (\nabla\widehat{u}_h^{n+1})|_K = 0\}$ , and we define the linear form  $\mathcal{L}_h^{n+1} : X_h \rightarrow \mathbb{R}$  by

$$\mathcal{L}_h^{n+1}(v_h) := \frac{1}{\delta t} \int_{\mathbb{R}^d} (\bar{u}_h^{n+1} - \bar{u}_h^{n+\frac{1}{2}}) \bar{v}_h dx + \int_{\nabla\widehat{u}_h^{n+1} \neq 0} \frac{\nabla\widehat{u}_h^{n+1}}{|\nabla\widehat{u}_h^{n+1}|} \cdot \nabla\widehat{v}_h dx + \theta(h) \int_{\mathbb{R}^d} \nabla\widehat{u}_h^{n+1} \cdot \nabla\widehat{v}_h dx.$$

For a given  $\varepsilon \in \mathbb{R}$  with  $\varepsilon \neq 0$  and for any  $v_h \in X_h$ , we write that

$$J_h^{n+1}(u_h^{n+1} + \varepsilon v_h) - J_h^{n+1}(u_h^{n+1}) \geq 0.$$

Assuming  $\varepsilon > 0$ , dividing the above equation by  $\varepsilon$  and letting  $\varepsilon \rightarrow 0$ , we get

$$\mathcal{L}_h^{n+1}(v_h) + \int_{\nabla\widehat{u}_h^{n+1}=0} |\nabla\widehat{v}_h| dx \geq 0.$$

Assuming  $\varepsilon < 0$ , dividing the above equation by  $\varepsilon$  and letting  $\varepsilon \rightarrow 0$ , we get

$$\mathcal{L}_h^{n+1}(v_h) - \int_{\nabla\widehat{u}_h^{n+1}=0} |\nabla\widehat{v}_h| dx \leq 0.$$

Hence we get that

$$\forall v_h \in X_h, |\mathcal{L}_h^{n+1}(v_h)| \leq \int_{\nabla\widehat{u}_h^{n+1}=0} |\nabla\widehat{v}_h| dx = \sum_{K \in \mathcal{T}_{h,0}^{n+1}} |K| |\nabla\widehat{v}_h|_K. \quad (38)$$

We consider the set  $E$  of all functions  $f$  from  $\mathcal{T}_{h,0}^{n+1} \rightarrow \mathbb{R}^d$  which are bounded for the norm

$$\|f\|_E = \sum_{K \in \mathcal{T}_{h,0}^{n+1}} |K| |f_K|.$$

We observe that, defining  $F = \{f \in E, \exists v_h \in X_h, f = \nabla\widehat{v}_h\}$ , the linear form  $B : F \rightarrow \mathbb{R}$ , such that  $B(f) = \mathcal{L}_h^{n+1}(v_h)$  is well defined (since if  $\nabla\widehat{v}_h = \nabla\widehat{w}_h$  on all  $K \in \mathcal{T}_{h,0}^{n+1}$ , then we get from (38) that  $\mathcal{L}_h^{n+1}(v_h) = \mathcal{L}_h^{n+1}(w_h)$ ) and continuous (again from (38)). Applying the Hahn-Banach theorem, this linear form can be prolonged on  $E$  by a linear form, again denoted by  $B$ , with the same norm. Hence there exists  $(\lambda_K^{n+1})_{K \in \mathcal{T}_{h,0}^{n+1}}$  with  $|\lambda_K^{n+1}| \leq 1$  for all  $K \in \mathcal{T}_{h,0}^{n+1}$  and

$$\forall f \in E, B(f) = - \sum_{K \in \mathcal{T}_{h,0}^{n+1}} |K| \lambda_K^{n+1} \cdot f_K.$$

Therefore we have

$$\forall v_h \in X_h, \mathcal{L}_h^{n+1}(v_h) + \sum_{K \in \mathcal{T}_{h,0}^{n+1}} |K| \lambda_K^{n+1} \cdot (\nabla\widehat{v}_h)|_K = 0,$$

which concludes, denoting  $\lambda_K^{n+1} = \frac{(\nabla\widehat{u}_h^{n+1})|_K}{|(\nabla\widehat{u}_h^{n+1})|_K|}$  if  $(\nabla\widehat{u}_h^{n+1})|_K \neq 0$ , the proof that  $\lambda_h^{n+1} \in \Lambda_h$  is such that (29)-(30) holds.  $\square$

**Remark 3.12:** Solving the saddle-point problem (29)-(30) or, which is equivalent, minimising the functional  $J_h^{n+1}$  is not an easy task. For instance, if one tries to minimise  $J_h^{n+1}$  (or more precisely a regularisation of  $J_h^{n+1}$ , since  $J_h^{n+1}$  is nondifferentiable) with conventional algorithms, such as the gradient or Newton algorithms, one obtains very slow convergences. Motivated by the applications in image denoising where this kind of problem arises, specific algorithms have been developed since the 1990s. Let us mention in particular the lagged diffusivity algorithm [30], saddle-point algorithms [9, 2] or semi-smooth Newton methods [24].

**Proposition 3.13:** Let us assume Hypotheses (HC) and Hypotheses (HD). Let  $n \in \mathbb{N}$ . Let  $\eta \in C^2(\mathbb{R})$  such that  $\eta(0) = 0$ ,  $\eta'(0) = 0$  and there exists  $M \in \mathbb{R}^+$  with  $\eta''(s) \in [0, M]$  for all  $s \in \mathbb{R}$ . Assume that  $u_h^{n+\frac{1}{2}} \in \mathbb{R}^{\mathbb{N}}$  is such that

$$\sum_{p \in \mathbb{N}} m_p \eta(u_p^{n+\frac{1}{2}}) < \infty.$$

Then

$$\sum_{p \in \mathbb{N}} m_p \eta(u_p^{n+1}) \leq \sum_{p \in \mathbb{N}} m_p \eta(u_p^{n+\frac{1}{2}}). \quad (39)$$

As a consequence, if  $a_0 \leq u_p^{n+\frac{1}{2}} \leq b_0$  for all  $p \in \mathbb{N}$ , then  $a_0 \leq u_p^{n+1} \leq b_0$  for all  $p \in \mathbb{N}$ , and if  $\bar{u}_h^{n+\frac{1}{2}} \in L^1(\mathbb{R}^d)$  then  $\bar{u}_h^{n+1} \in L^1(\mathbb{R}^d)$ .

**Proof** We remark that  $v_h$ , defined by  $v_p = \eta'(u_p^{n+1})$ , is such that  $v_h \in X_h$ . Indeed, there holds  $|v_p| \leq M|u_p^{n+1}|$ , and, using (20)-(21), we have

$$\|\nabla \hat{v}_h\|_{L^2(\mathbb{R}^d)}^2 = \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (\eta'(u_p^{n+1}) - \eta'(u_q^{n+1}))^2 \leq M^2 \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (u_p^{n+1} - u_q^{n+1})^2 = \|\nabla \hat{u}_h^{n+1}\|_{L^2(\mathbb{R}^d)}^2,$$

which implies  $v_h \in X_h$ . We now remark that, for any  $K \in \mathcal{T}_h$ , if  $\nabla \hat{u}_h^{n+1}|_K = 0$ , then  $u_p^{n+1} = u_q^{n+1}$  for any  $p, q \in \mathcal{V}_K$ , and therefore  $\nabla \hat{v}_h|_K = 0$ . So one can write

$$\lambda_{h|K}^{n+1} \cdot \nabla \hat{v}_h|_K = 0 = \nabla \hat{u}_h^{n+1}|_K \cdot \nabla \hat{v}_h|_K.$$

If  $\nabla \hat{u}_h^{n+1}|_K \neq 0$ , we then have

$$\lambda_{h|K}^{n+1} \cdot \nabla \hat{v}_h|_K = \frac{1}{|\nabla \hat{u}_h^{n+1}|_K} \nabla \hat{u}_h^{n+1}|_K \cdot \nabla \hat{v}_h|_K.$$

Hence, defining  $\alpha_K^{n+1}$  by  $\alpha_K^{n+1} = 1$  if  $\nabla \hat{u}_h^{n+1}|_K = 0$ , and  $\alpha_K^{n+1} = \frac{1}{|\nabla \hat{u}_h^{n+1}|_K}$  otherwise, we get

$$\int_{\mathbb{R}^d} (\lambda_h^{n+1} + \theta(h) \nabla \hat{u}_h^{n+1}) \cdot \nabla \hat{v}_h \, dx = \sum_{K \in \mathcal{T}_h} (\alpha_K^{n+1} + \theta(h)) \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (u_p^{n+1} - u_q^{n+1}) (\eta'(u_p^{n+1}) - \eta'(u_q^{n+1})) \geq 0.$$

Hence we can write from (30)

$$\sum_{p \in \mathbb{N}} m_p (u_p^{n+1} - u_p^{n+\frac{1}{2}}) \eta'(u_p^{n+1}) \leq 0.$$

Applying  $\eta(b) - \eta(a) = \eta'(a)(b-a) + \eta''(c) \frac{(b-a)^2}{2}$  for  $c$  between  $a$  and  $b$ , we get

$$\sum_{p \in \mathbb{N}} m_p (\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})) \leq \sum_{p \in \mathbb{N}} m_p (u_p^{n+1} - u_p^{n+\frac{1}{2}}) \eta'(u_p^{n+1}) \leq 0,$$

which concludes the proof of (39).

We then conclude in the same way as the finite volume explicit step. Then assuming  $u_p^{n+\frac{1}{2}} \leq b_0$  for all  $p \in \mathbb{N}$  and letting  $\eta$  tend to  $\eta(s) = s^\top b_0 - b_0$ , we get that  $\eta(u_p^{n+1}) = 0$ , which shows that  $u_p^{n+1} \leq b_0$ . The same reasoning holds with  $\eta(s) = a_0 - s^\top a_0$ , which shows that  $u_p^{n+1} \geq a_0$ . Finally, letting  $\eta$  tend to  $\eta(s) = |s|$ , we conclude that  $\bar{u}_h^{n+1} \in L^1(\mathbb{R}^d)$ .  $\square$

## 4 A priori estimates on the approximate solutions

A  $L^\infty(Q_T)$  estimate on the family of approximate velocities  $(\bar{u}_{h,\delta})_{h,\delta>0}$  has already been proved in Proposition 3.8. The aim of this section is to establish additional estimates on  $(\hat{u}_{h,\delta})_{h,\delta>0}$ , namely a  $L^1(0, T; BV(\mathbb{R}^d))$  estimate, and  $L^1_{\text{loc}}(Q_T)$  estimates on the space and time translates. The estimates on the space and time translates are deduced from the  $L^1(0, T; BV(\mathbb{R}^d))$  estimate.

**Remark 4.1:** Hypotheses (HD1) and (HD2) imply that each cell of  $\mathcal{D}_h$  has a finite number of neighbours (and this number is bounded independently of  $\mathcal{F}_{h,\delta}$ ).

**Remark 4.2:** Throughout this section and the next one,  $C$  denotes a generic constant independent of the discretisation  $\mathcal{F}_{h,\delta}$ .

### 4.1 $L^1(0, T; BV(\mathbb{R}^d))$ estimate

**Proposition 4.3:** Let us assume Hypotheses (HC) and Hypotheses (HD). Let  $\bar{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and, for all  $n \in \mathbb{N}$ ,  $\bar{u}_h^{n+\frac{1}{2}}, \bar{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and  $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$  be a solution to Scheme (27)-(30). Then there holds

$$\frac{1}{2} \|\bar{u}_{h,\delta}\|_{L^\infty(0,T;L^2(\mathbb{R}^d))}^2 + \sum_{n=1}^N \delta t \int_{\mathbb{R}^d} |\nabla \hat{u}_h^n| dx \leq \frac{1}{2} \|u_{\text{ini}}\|_{L^2(\mathbb{R}^d)}^2, \quad (40)$$

and

$$\sum_{n=0}^{N-1} \delta t \int_{\mathbb{R}^d} \theta(h) |\nabla \hat{u}_h^{n+1}|^2 dx = \theta(h) \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (u_p^{n+1} - u_q^{n+1})^2 \leq \frac{1}{2} \|u_{\text{ini}}\|_{L^2(\mathbb{R}^d)}^2. \quad (41)$$

**Proof** Thanks to Proposition 3.10, letting  $\eta(s) = s^2$ , we have

$$\int_{\mathbb{R}^d} \bar{u}_h^{n+\frac{1}{2}}(x)^2 dx \leq \int_{\mathbb{R}^d} \bar{u}_h^n(x)^2 dx. \quad (42)$$

We test (29) with  $v_h = \delta \hat{u}_h^{n+1}$  and, since  $\lambda_h^{n+1} \in \text{Sgn}(\nabla u_h^{n+1})$ , we obtain

$$\sum_{p \in \mathbb{N}} m_p u_p^{n+1} \left( u_p^{n+1} - u_p^{n+\frac{1}{2}} \right) + \delta t \int_{\mathbb{R}^d} (|\nabla \hat{u}_h^{n+1}| + \theta(h) |\nabla \hat{u}_h^{n+1}|^2) dx = 0.$$

This leads to

$$\sum_{p \in \mathbb{N}} m_p \left( \frac{1}{2} (u_p^{n+1})^2 + \frac{1}{2} (u_p^{n+1} - u_p^{n+\frac{1}{2}})^2 - \frac{1}{2} (u_p^{n+\frac{1}{2}})^2 \right) + \delta t \int_{\mathbb{R}^d} (|\nabla \hat{u}_h^{n+1}| + \theta(h) |\nabla \hat{u}_h^{n+1}|^2) dx = 0,$$

hence giving, thanks to (42)

$$\sum_{p \in \mathbb{N}} m_p \left( \frac{1}{2} (u_p^{n+1})^2 - \frac{1}{2} (u_p^n)^2 \right) + \delta t \int_{\mathbb{R}^d} (|\nabla \hat{u}_h^{n+1}| + \theta(h) |\nabla \hat{u}_h^{n+1}|^2) dx \leq 0. \quad (43)$$

Summing (43) over  $n \in \{0, \dots, m\}$  for any  $m = 1, \dots, N-1$  we obtain the inequality

$$\frac{1}{2} \sum_{p \in \mathbb{N}} m_p (u_p^{m+1})^2 + \sum_{n=0}^m \delta t \int_{\mathbb{R}^d} (|\nabla \hat{u}_h^{n+1}| + \theta(h) |\nabla \hat{u}_h^{n+1}|^2) dx \leq \sum_{p \in \mathbb{N}} m_p \frac{1}{2} (u_p^0)^2. \quad (44)$$

We conclude the proof of the lemma since the above inequality holds for any  $m$ .  $\square$

## 4.2 Time translate estimate

**Proposition 4.4:** Let us assume Hypotheses (HC) and Hypotheses (HD). Let  $\bar{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and, for all  $n \in \mathbb{N}$ ,  $\bar{u}_h^{n+\frac{1}{2}}, \bar{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and  $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$  be a solution to Scheme (27)-(30). For all  $R > 0$ , there exists a constant  $C$ , independent of the family  $(\mathcal{F}_{h,\delta})_{h,\delta>0}$ , such that,

$$\int_0^{T-s} \int_{B(0,R)} |\bar{u}_{h,\delta}(x, t+s) - \bar{u}_{h,\delta}(x, t)| \, dx dt \leq C\sqrt{s}, \quad \forall s \in [0, T]. \quad (45)$$

**Proof 1.** Let  $s \in [0, T]$ . Using the Cauchy-Schwarz inequality, we have

$$\int_0^{T-s} \int_{B(0,R)} |\bar{u}_{h,\delta}(x, t+s) - \bar{u}_{h,\delta}(x, t)| \, dx dt \leq \left( CR^d T \int_0^{T-s} \int_{\mathbb{R}^d} (\bar{u}_{h,\delta}(x, t+s) - \bar{u}_{h,\delta}(x, t))^2 \, dx dt \right)^{\frac{1}{2}}. \quad (46)$$

We define the function  $\nu : \mathbb{R} \rightarrow \mathbb{Z}$  such that  $\nu(t) = n+1$  if  $t \in (n\delta, (n+1)\delta]$  and the function  $\chi_n(t, s) : \mathbb{R} \times \mathbb{R} \rightarrow \{0, 1\}$  such that  $\chi_n(t, s) = 1$  if  $\nu(t) \leq n < \nu(t+s)$  and  $\chi_n(t, s) = 0$  otherwise; so that

$$\begin{aligned} \int_0^{T-s} \int_{\mathbb{R}^d} (\bar{u}_{h,\delta}(x, t+s) - \bar{u}_{h,\delta}(x, t))^2 \, dx dt &= \int_0^{T-s} \int_{\mathbb{R}^d} \left( \bar{u}_h^{\nu(t+s)} - \bar{u}_h^{\nu(t)} \right)^2 \, dx dt \\ &= \int_0^{T-s} \int_{\mathbb{R}^d} \left( \bar{u}_h^{\nu(t+s)} - \bar{u}_h^{\nu(t)} \right) \sum_{n=1}^{N-1} \chi_n(t, s) (\bar{u}_h^{n+1} - \bar{u}_h^n) \, dx dt. \end{aligned} \quad (47)$$

2. We denote  $\tau = 0$  or  $\tau = s$ , and we multiply (28), for  $n \geq 1$ , by  $\delta u_p^{\nu(t+\tau)}$  and sum over all  $p \in \mathbb{N}$ :

$$\sum_{p \in \mathbb{N}} m_p u_p^{\nu(t+\tau)} \left( u_p^{n+\frac{1}{2}} - u_p^n \right) + \delta \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} u_p^{\nu(t+\tau)} \left( F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n) \right) = 0. \quad (48)$$

We test (29) with  $v_h = \delta \hat{u}_h^{\nu(t+\tau)}$ :

$$\sum_{p \in \mathbb{N}} m_p u_p^{\nu(t+\tau)} \left( u_p^{n+1} - u_p^{n+\frac{1}{2}} \right) + \delta \int_{\mathbb{R}^d} (\lambda_h^{n+1} \cdot \nabla \hat{u}_h^{\nu(t+\tau)} + \theta(h) \nabla \hat{u}_h^{n+1} \cdot \nabla \hat{u}_h^{\nu(t+\tau)}) \, dx = 0. \quad (49)$$

Adding the above equalities (48) and (49), we find

$$\begin{aligned} \sum_{p \in \mathbb{N}} m_p u_p^{\nu(t+\tau)} (u_p^{n+1} - u_p^n) + \delta \sum_{p \in \mathbb{N}} u_p^{\nu(t+\tau)} \sum_{q \in \mathcal{N}_p} (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) \\ + \delta \int_{\mathbb{R}^d} (\lambda_h^{n+1} \cdot \nabla \hat{u}_h^{\nu(t+\tau)} + \theta(h) \nabla \hat{u}_h^{n+1} \cdot \nabla \hat{u}_h^{\nu(t+\tau)}) \, dx = 0. \end{aligned} \quad (50)$$

3. Let  $p \in \mathbb{N}$ ,  $q \in \mathcal{N}_p$  and  $K$  an element of  $\mathcal{T}_h$  for which  $x_p$  and  $x_q$  are vertices. In view of (HD1) and (HD4), for  $n \geq 1$ ,

$$|F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)| \leq 2m_{p,q}L|u_p^n - u_q^n| \leq 2m_{p,q}Lh|\nabla \hat{u}_h^n|_K \leq Ch^d |\nabla \hat{u}_h^n|_K. \quad (51)$$

Therefore, owing to (HD3) and the finite number of neighbours in  $\mathcal{D}_h$ ,

$$\left| \sum_{p \in \mathbb{N}} u_p^{\nu(t+\tau)} \sum_{q \in \mathcal{N}_p} (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) \right| \leq C \int_{\mathbb{R}^d} |\nabla \hat{u}_h^n| \, dx. \quad (52)$$

The property  $|\lambda_h^{n+1}| \leq 1$  gives immediately

$$\left| \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla \hat{u}_h^{\nu(t+\tau)} \, dx \right| \leq \int_{\mathbb{R}^d} |\nabla \hat{u}_h^{\nu(t+\tau)}| \, dx. \quad (53)$$

From (50), (52) and (53), it follows

$$\left| \int_{\mathbb{R}^d} \bar{u}_h^{\nu(t+\tau)} (\bar{u}_h^{n+1} - \bar{u}_h^n) dx \right| \leq C \delta t \left( \int_{\mathbb{R}^d} (|\nabla \hat{u}_h^n| + |\nabla \hat{u}_h^{\nu(t+\tau)}| + \frac{\theta(h)}{2} (|\nabla \hat{u}_h^{n+1}|^2 + |\nabla \hat{u}_h^{\nu(t+\tau)}|^2)) dx \right). \quad (54)$$

4. Using the above estimate (54), we have

$$\left| \int_0^{T-s} \int_{\mathbb{R}^d} \bar{u}_h^{\nu(t+\tau)} \sum_{n=1}^{N-1} \chi_n(t, s) (\bar{u}_h^{n+1} - \bar{u}_h^n) dx dt \right| \leq C \sum_{n=1}^N \int_0^{T-s} \chi_n(t, s) \delta t (A^n + B^{\nu(t+\tau)}) dt,$$

with, for all  $n = 1, \dots, N$ ,

$$A^n = \int_{\mathbb{R}^d} (|\nabla \hat{u}_h^n| + \frac{\theta(h)}{2} |\nabla \hat{u}_h^{n+1}|^2) dx$$

and

$$B^n = \int_{\mathbb{R}^d} (|\nabla \hat{u}_h^n| + \frac{\theta(h)}{2} |\nabla \hat{u}_h^n|^2) dx.$$

We then apply [17, Lemma 4.6], which leads to

$$\left| \int_0^{T-s} \int_{\mathbb{R}^d} \bar{u}_h^{\nu(t+\tau)} \sum_{n=1}^{N-1} \chi_n(t, s) (\bar{u}_h^{n+1} - \bar{u}_h^n) dx dt \right| \leq C s \sum_{n=1}^N \delta t (A^n + B^n). \quad (55)$$

5. Recalling (40) and (41) which provide bounds on  $\sum_{n=1}^N \delta t (A^n + B^n)$ , and collecting (47) and (55), we obtain the desired estimate (45).  $\square$

### 4.3 Space translate estimate

**Proposition 4.5:** Let us assume Hypotheses (HC) and Hypotheses (HD). Let  $\bar{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and, for all  $n \in \mathbb{N}$ ,  $\bar{u}_h^{n+\frac{1}{2}}, \bar{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and  $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$  be a solution to Scheme (27)-(30). There exists a constant  $C$ , independent of the family  $(\mathcal{F}_{h,\delta})_{h,\delta>0}$ , such that

$$\int_0^T \int_{\mathbb{R}^d} |\hat{u}_{h,\delta}(x+y, t) - \hat{u}_{h,\delta}(x, t)| dx dt \leq C|y|, \quad \forall y \in \mathbb{R}^d. \quad (56)$$

and, for any  $R > 0$ , there exists  $C'$ , independent of the family  $(\mathcal{F}_{h,\delta})_{h,\delta>0}$ ,

$$\int_0^T \int_{B(0,R)} |\bar{u}_{h,\delta}(x+y, t) - \bar{u}_{h,\delta}(x, t)| dx dt \leq C(|y| + 2CTR^d h), \quad \forall y \in \mathbb{R}^d. \quad (57)$$

**Proof** For a given element  $K \in \mathcal{T}_h$  and a given couple of points  $(a, b) \in \mathbb{R}^d \times \mathbb{R}^d$ , we denote by  $\chi_K(a, b)$  the length of the segment  $[a, b] \cap \bar{K}$ . In particular, if  $[a, b]$  does not intersect  $\bar{K}$ , then  $\chi_K(a, b) = 0$ . Let  $n \in \{1, \dots, N\}$  and  $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ . By applying the mean value theorem on each element intersected by the segment  $[x, x+y]$ , we find the inequality

$$|\hat{u}_h^n(x+y) - \hat{u}_h^n(x)| \leq \sum_{K \in \mathcal{T}_h} \chi_K(x, x+y) |\nabla \hat{u}_h^n|_K. \quad (58)$$

Next, an integration with respect to  $x$  yields

$$\int_{\mathbb{R}^d} |\hat{u}_h^n(x+y) - \hat{u}_h^n(x)| dx \leq \sum_{K \in \mathcal{T}_h} |\nabla \hat{u}_h^n|_K \int_{\mathbb{R}^d} \chi_K(x, x+y) dx. \quad (59)$$

For any  $K \in \mathcal{T}_h$ , the function  $x \mapsto \chi_K(x, x+y)$  is bounded by  $\min(h, |y|)$  and is zero outside a domain of measure lower than  $h^{d-1}(h + |y|)$ . Therefore,

$$\int_{\mathbb{R}^d} \chi_K(x, x+y) dx \leq h^{d-1}(h + |y|) \min(h, |y|) \leq 2h^d |y|, \quad (60)$$

and

$$\int_{\mathbb{R}^d} |\hat{u}_h^n(x+y) - \hat{u}_h^n(x)| dx \leq C|y| \sum_{K \in \mathcal{T}_h} h^d |\nabla \hat{u}_h^n|_K. \quad (61)$$

Using the hypothesis (HD3), we obtain

$$\int_{\mathbb{R}^d} \sum_{K \in \mathcal{T}_h} h^d |\nabla \widehat{u}_h^n|_K \leq C \int_{\mathbb{R}^d} |\nabla \widehat{u}_h^n| \, dx. \quad (62)$$

Summing the above inequality (62) over  $\{1, \dots, N\}$ , and using (40), we find the desired estimate (56). We then deduce (57) using (22).  $\square$

## 5 Entropy formulation for the approximate solutions

The aim of this section is to establish an entropy formulation, similar to (7), for the approximate solutions. We first prove a discrete entropy inequality for the finite volume step (Proposition 3.9). Then, we take into account the finite element step to obtain the complete discrete entropy formulation (Proposition 5.1). Error terms occur in this formulation. Proposition 5.1 ensures that they tend to zero when the meshsize and the time step tend to zero.

**Proposition 5.1:** Let us assume Hypotheses (HC) and Hypotheses (HD). Let  $\bar{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and, for all  $n \in \mathbb{N}$ ,  $\bar{u}_h^{n+\frac{1}{2}}, \bar{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and  $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$  be a solution to Scheme (27)-(30). Let  $(\eta, \Phi)$  be an entropy-entropy flux pair. Then, for all nonnegative test functions  $\varphi \in C_c^\infty(Q_T)$ ,

$$\begin{aligned} & \int_{Q_T} \eta(\bar{u}_{h,\delta}) \partial_t \varphi \, dx dt + \int_{Q_T} \Phi(x, t, \bar{u}_{h,\delta}) \cdot \nabla \varphi \, dx dt - \int_{Q_T} \eta'(\widehat{u}_{h,\delta}) \lambda_{h,\delta} \cdot \nabla \varphi \, dx dt \\ & - \int_{Q_T} \varphi |\nabla \eta'(\widehat{u}_{h,\delta})| \, dx dt + \int_{\mathbb{R}^d} \eta(\bar{u}_h^0) \varphi(x, 0) \, dx + e_{h,\delta}^1 + e_{h,\delta}^2 + e_{h,\delta}^3 + e_{h,\delta}^4 + e_{h,\delta}^5 + e_{h,\delta}^6 \geq 0, \end{aligned} \quad (63)$$

where  $e_{h,\delta}^1, e_{h,\delta}^2, e_{h,\delta}^3, e_{h,\delta}^4, e_{h,\delta}^5$ , and  $e_{h,\delta}^6$ , defined by the relations (72), (76), (78), (79), (73), and (74), respectively, satisfy

$$\lim_{h \rightarrow 0, \delta \rightarrow 0} (|e_{h,\delta}^1| + |e_{h,\delta}^2| + |e_{h,\delta}^3| + |e_{h,\delta}^4| + |e_{h,\delta}^5| + |e_{h,\delta}^6|) = 0. \quad (64)$$

### Proof First step: proof of (63)

1. Let  $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}_+)$ . We set  $v^{n+1} := \eta'(\widehat{u}_h^{n+1})\psi$ . We test (29) with the function  $v_h^{n+1} \in V_h$  such that  $v_p^{n+1} = \eta'(u_p^{n+1})\psi(x_p)$ , for all  $p \in \mathbb{N}$ . We get:

$$\sum_{p \in \mathbb{N}} m_p \frac{u_p^{n+1} - u_p^{n+\frac{1}{2}}}{\delta} \eta'(u_p^{n+1}) \psi(x_p) + \int_{\mathbb{R}^d} (\lambda_h^{n+1} \cdot \nabla v_h^{n+1} + \theta(h) \nabla \widehat{u}_h^{n+1} \cdot \nabla v_h^{n+1}) \, dx = 0. \quad (65)$$

Since  $\eta$  is convex,

$$(u_p^{n+1} - u_p^{n+\frac{1}{2}}) \eta'(u_p^{n+1}) \geq \eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}}) \quad \forall p \in \mathbb{N},$$

and thus (65) leads to

$$\sum_{p \in \mathbb{N}} m_p \frac{\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})}{\delta} \psi(x_p) + \int_{\mathbb{R}^d} (\lambda_h^{n+1} \cdot \nabla v_h^{n+1} + \theta(h) \nabla \widehat{u}_h^{n+1} \cdot \nabla v_h^{n+1}) \, dx \leq 0. \quad (66)$$

The second term of (66) can be rewritten as

$$\begin{aligned} & \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla v_h^{n+1} \, dx = \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla v^{n+1} \, dx + \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (v_h^{n+1} - v^{n+1}) \, dx \\ & = \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (\eta'(\widehat{u}_h^{n+1})) \psi \, dx + \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla \psi \eta'(\widehat{u}_h^{n+1}) \, dx + \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (v_h^{n+1} - v^{n+1}) \, dx. \end{aligned} \quad (67)$$

We have

$$\int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (\eta'(\widehat{u}_h^{n+1})) \psi \, dx = \int_{\mathbb{R}^d} \eta''(\widehat{u}_h^{n+1}) \lambda_h^{n+1} \cdot \nabla (\widehat{u}_h^{n+1}) \psi \, dx = \int_{\mathbb{R}^d} |\nabla \eta'(\widehat{u}_h^{n+1})| \psi \, dx. \quad (68)$$

The third term of (66) can be rewritten as

$$\int_{\mathbb{R}^d} \theta(h) \nabla \widehat{u}_h^{n+1} \cdot \nabla v_h^{n+1} dx = \theta(h) \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (u_p^{n+1} - u_q^{n+1}) (\eta'(u_p^{n+1}) \psi(x_p) - \eta'(u_q^{n+1}) \psi(x_q)) = T_1 + T_2,$$

with

$$\begin{aligned} T_1 &= \theta(h) \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (u_p^{n+1} - u_q^{n+1}) (\eta'(u_p^{n+1}) - \eta'(u_q^{n+1})) \frac{\psi(x_p) + \psi(x_q)}{2} \geq 0 \\ T_2 &= \theta(h) \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (u_p^{n+1} - u_q^{n+1}) (\psi(x_p) - \psi(x_q)) \frac{\eta'(u_p^{n+1}) + \eta'(u_q^{n+1})}{2}. \end{aligned} \quad (69)$$

Collecting (66), (67), (69) and (68), we obtain

$$\begin{aligned} & \sum_{p \in \mathbb{N}} m_p \frac{\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})}{\delta t} \psi(x_p) + \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla \psi \eta'(\widehat{u}_h^{n+1}) dx + \int_{\mathbb{R}^d} |\nabla \eta'(\widehat{u}_h^{n+1})| \psi dx \\ & \leq \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (v^{n+1} - v_h^{n+1}) dx + \theta(h) \sum_{p \in \mathbb{N}} \sum_{q \in \mathbb{N}} T_{pq} |u_p^{n+1} - u_q^{n+1}| |\psi(x_p) - \psi(x_q)| \frac{\eta'(u_p^{n+1}) + \eta'(u_q^{n+1})}{2}. \end{aligned} \quad (70)$$

Taking  $\psi = \varphi(\cdot, t)$  in the above inequality, then integrating on  $(t^n, t^{n+1})$  and summing over  $n \in \{0, \dots, N-1\}$ , we eventually find

$$\begin{aligned} & \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})}{\delta t} \int_{t^n}^{t^{n+1}} m_p \varphi(x_p, t) dt + \int_{Q_T} \eta'(\widehat{u}_{h,\delta}) \lambda_{h,\delta} \cdot \nabla \varphi dx dt \\ & \quad + \int_{Q_T} \varphi |\nabla \eta'(\widehat{u}_{h,\delta})| dx dt \leq e_{h,\delta}^1 + e_{h,\delta}^5 + e_{h,\delta}^6, \end{aligned} \quad (71)$$

where

$$e_{h,\delta}^1 := \int_0^T \inf_{v_h \in V_h} \int_{\mathbb{R}^d} |\nabla (\eta'(\widehat{u}_{h,\delta}) \varphi - v_h)| dx dt, \quad (72)$$

$$e_{h,\delta}^5 := \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}^d} \lambda_h^{n+1}(x) \cdot \nabla (v_h^{n+1}(x, t) - v^{n+1}(x, t)) dx dt, \quad (73)$$

and

$$e_{h,\delta}^6 := \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \theta(h) \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K |u_p^{n+1} - u_q^{n+1}| |\varphi(x_p, t) - \varphi(x_q, t)| \frac{\eta'(u_p^{n+1}) + \eta'(u_q^{n+1})}{2} dt. \quad (74)$$

2. Multiplying (33) by  $\varphi_p^n := \int_{t^n}^{t^{n+1}} \int_{Q_p} \varphi(x, s) dx ds$ , then adding to (71), we obtain

$$\begin{aligned} & \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^n)}{\delta t} \varphi_p^n + \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{1}{m_p} \sum_{q \in \mathcal{N}_p} \Phi_{p,q}^n(u_p^n, u_q^n) \varphi_p^n \\ & \quad + \int_{Q_T} \eta'(\widehat{u}_{h,\delta}) \lambda_{h,\delta} \cdot \nabla \varphi dx dt + \int_{Q_T} \varphi |\nabla \eta'(\widehat{u}_{h,\delta})| dx dt - e_{h,\delta}^1 - e_{h,\delta}^5 - e_{h,\delta}^6 - e_{h,\delta}^2 \leq 0, \end{aligned} \quad (75)$$

where

$$e_{h,\delta}^2 := \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})}{\delta t} \left( \varphi_p^n - \int_{t^n}^{t^{n+1}} m_p \varphi(x_p, t) dt \right). \quad (76)$$

Now, observing that

$$\int_{Q_T} \eta(\widehat{u}_{h,\delta}) \partial_t \varphi dx dt = - \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^n)}{\delta t} \int_{t^n}^{t^{n+1}} \int_{Q_p} \varphi(x, t^{n+1}) dx dt - \sum_{p \in \mathbb{N}} \eta(u_p^0) \int_{Q_p} \varphi(x, 0) dx$$

we can rewrite (75) as

$$\begin{aligned} & \int_{Q_T} \eta(\bar{u}_{h,\delta}) \partial_t \varphi \, dx dt + \int_{Q_T} \Phi(x, t, \bar{u}_{h,\delta}) \cdot \nabla \varphi \, dx dt - \int_{Q_T} \eta'(\hat{u}_{h,\delta}) \lambda_{h,\delta} \cdot \nabla \varphi \, dx dt \\ & - \int_{Q_T} \varphi |\nabla \eta'(\hat{u}_{h,\delta})| \, dx dt + \sum_{p \in \mathbb{N}} \eta(u_p^0) \int_{Q_p} \varphi(x, 0) \, dx + e_{h,\delta}^1 + e_{h,\delta}^5 + e_{h,\delta}^6 + e_{h,\delta}^2 + e_{h,\delta}^3 + e_{h,\delta}^4 \geq 0, \end{aligned} \quad (77)$$

where

$$e_{h,\delta}^3 := \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^n)}{\delta t} \left( \int_{t^n}^{t^{n+1}} \int_{Q_p} \varphi(x, t^{n+1}) \, dx dt - \varphi_p^n \right), \quad (78)$$

$$e_{h,\delta}^4 := - \int_{Q_T} \Phi(x, t, \bar{u}_{h,\delta}) \cdot \nabla \varphi \, dx dt - \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{1}{m_p} \sum_{q \in \mathcal{N}_p} \Phi_{p,q}^n(u_p^n, u_q^n) \varphi_p^n. \quad (79)$$

## Second step: proof of (64)

### 1. Study of $e^1$

Setting  $w(t) := \eta'(\hat{u}_h(t)) \varphi(t)$ , we can write  $e_{h,\delta}^1 = \int_0^T \inf_{v_h \in V_h} |w(t) - v_h|_{W^{1,1}(\mathbb{R}^d)} \, dt$ , where  $|\cdot|_{W^{1,1}(\mathbb{R}^d)}$  denotes the semi-norm of the Sobolev space  $W^{1,1}(\mathbb{R}^d)$ . The gradient of  $w(t)$  is

$$\nabla w(t) = \eta'(\hat{u}_h(t)) \nabla \varphi(t) + \varphi(t) \eta''(\hat{u}_h(t)) \nabla \hat{u}_h(t).$$

Therefore, using Proposition 3.8 and (40), we obtain

$$|w(t)|_{W^{1,1}(\mathbb{R}^d)} \leq C,$$

where the constant  $C$  depends only on  $\eta$ ,  $\varphi$ ,  $a_0$ , and  $b_0$ . Then, the family of meshes being shape-regular (hypothesis (HD3)), a standard approximation result [14, see, e.g.] ensures that

$$\lim_{h \rightarrow 0} \inf_{v_h \in V_h} |w(t) - v_h|_{W^{1,1}(\mathbb{R}^d)},$$

and thus

$$\lim_{h \rightarrow 0, \delta \rightarrow 0} e_{h,\delta}^1 = 0.$$

### 2. Study of $e^2$ and $e^3$

Since  $\eta$  is  $C^1$  and the discrete velocity is bounded between  $a_0$  and  $b_0$ , we have

$$\begin{aligned} |e_{h,\delta}^2| & \leq \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} |\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})| \int_{t^n}^{t^{n+1}} \int_{Q_p} |\nabla_x \varphi(x, t)| \, dx dt \\ & \leq C \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} m_p \delta t |u_p^{n+1} - u_p^{n+\frac{1}{2}}|. \end{aligned}$$

Employing the same arguments as in the proof of Proposition 4.4, we can derive the estimate

$$\sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} m_p |u_p^{n+1} - u_p^{n+\frac{1}{2}}| \leq C.$$

Hence  $|e_{h,\delta}^2| \leq C \delta$ . Similarly,  $|e_{h,\delta}^3| \leq C \delta$ .

### 3. Study of $e^4$

Set

$$I := \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{1}{m_p} \sum_{q \in \mathcal{N}_p} \Phi_{p,q}^n(u_p^n, u_q^n) \varphi_p^n \quad \text{and} \quad I^* := - \int_{Q_T} \Phi(x, t, \bar{u}_{h,\delta}) \cdot \nabla \varphi \, dx dt,$$

so that  $e_{h,\delta}^4 = -I + I^*$ . From (3.5), it follows  $\sum_{q \in \mathcal{N}_p} \Phi_{p,q}^n(u, u) = 0$  and thus

$$I = \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{1}{m_p} \sum_{q \in \mathcal{N}_p} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_p^n, u_p^n)) \varphi_p^n = I_1 + I_2,$$

where

$$I_1 := \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \frac{1}{m_p} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_p^n, u_p^n)) \varphi_p^n, \quad I_2 := \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \frac{1}{m_q} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_q^n, u_q^n)) \varphi_q^n.$$

Applying the divergence theorem on each cell in the expression of  $I^*$ , we find

$$\begin{aligned} I^* &= - \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \int_{t^n}^{t^{n+1}} \int_{\partial Q_p} \Phi(\gamma, t, u_p^n) \cdot \nu_{p,q} \varphi(\gamma, t) \, d\gamma dt \\ &= - \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} (\Phi(\gamma, t, u_p^n) \cdot \nu_{p,q} - \Phi(\gamma, t, u_q^n) \cdot \nu_{p,q}) \varphi(\gamma, t) \, d\gamma dt \\ &= I_1^* - I_2^* \end{aligned}$$

where

$$\begin{aligned} I_1^* &:= \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left( \frac{1}{m_{p,q}} \Phi_{p,q}^n(u_p^n, u_q^n) - \Phi(\gamma, t, u_p^n) \cdot \nu_{p,q} \right) \varphi(\gamma, t) \, d\gamma dt, \\ I_2^* &:= \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left( \frac{1}{m_{p,q}} \Phi_{p,q}^n(u_p^n, u_q^n) - \Phi(\gamma, t, u_q^n) \cdot \nu_{p,q} \right) \varphi(\gamma, t) \, d\gamma dt. \end{aligned}$$

Let us now estimate  $|I_1 - I_1^*|$ . We first rewrite  $I_1^*$  as

$$\begin{aligned} I_1^* &= \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_p^n, u_p^n)) \frac{1}{m_{p,q}} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \varphi(\gamma, t) \, dt d\gamma \\ &\quad + \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left( \frac{1}{m_{p,q}} \Phi_{p,q}^n(u_p^n, u_p^n) - \Phi(\gamma, s, u_p^n) \cdot \nu_{p,q} \right) \varphi(\gamma, t) \, dt d\gamma. \end{aligned}$$

Then, the consistency of the family  $\{\Phi_{p,q}^n\}$  with  $\Phi$  allows us to turn the above equation into

$$\begin{aligned} I_1^* &= \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_p^n, u_p^n)) \frac{1}{m_{p,q}} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \varphi(\gamma, t) \, dt d\gamma \\ &+ \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left( \Phi(\gamma, t, u_p^n) \cdot \nu_{p,q} - \frac{1}{m_{p,q}} \Phi_{p,q}^n(u_p^n, u_p^n) \right) \frac{1}{\delta m_{p,q}} \left( \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \varphi(\gamma, t) - \varphi(\xi, s) \, ds d\xi \right) dt d\gamma. \end{aligned}$$

Hence

$$\begin{aligned} I_1 - I_1^* &= \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \delta (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_p^n, u_p^n)) \frac{1}{\delta^2 m_p m_{p,q}} \int_{t^n}^{t^{n+1}} \int_{Q_p} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \varphi(x, t) - \varphi(\gamma, s) \, dt dx ds d\gamma \\ &+ \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left( \frac{1}{m_{p,q}} \Phi_{p,q}^n(u_p^n, u_p^n) - \Phi(\gamma, t, u_p^n) \cdot \nu_{p,q} \right) \frac{1}{\delta m_{p,q}} \left( \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \varphi(\gamma, t) - \varphi(\xi, s) \, ds d\xi \right) dt d\gamma. \end{aligned}$$

From the mean value theorem and the uniform Lipschitz continuity of  $\{\Phi_{p,q}^n\}$ , we thus derive the estimate

$$|I_1 - I_1^*| \leq C(h + \delta) \left( \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \delta m_{p,q} |u_p^n - u_q^n| + \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \delta (h + \delta) m_{p,q} \right).$$

Using the finite number of neighbours in  $\mathcal{D}_h$ , hypothesis (HD5), and estimate (40), we obtain the bound

$$\sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \delta m_{p,q} |u_p^n - u_q^n| \leq C \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} \delta h^d |\nabla \widehat{u}_h^n|_K \leq C \sum_{n=0}^{N-1} \delta \int_{\mathbb{R}^d} |\nabla \widehat{u}_h^n| dx \leq C. \quad (80)$$

Using (HD1) and (HD5), we also obtain

$$\sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \delta (h + \delta) m_{p,q} \leq C. \quad (81)$$

Finally we deduce the estimate  $|I_1 - I_1^*| \leq C(h + \delta)$ . The term  $I_2 - I_2^*$  can be treated similarly, which yields  $|\epsilon_{h,\delta}^4| \leq C(h + \delta)$ .

#### 4. Study of $e^5$

We let, for  $t \in [t^n, t^{n+1}]$ ,  $\psi = \varphi(\cdot, t)$ . Defining  $\chi_K = 0$  if  $\psi(x) = 0$  on  $K$  and 1 otherwise, we observe that

$$\int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (v^{n+1} - v_h^{n+1}) dx = \sum_{K \in \mathcal{T}_h} \chi_K \lambda_h^{n+1} \cdot \int_K \nabla (v^{n+1} - v_h^{n+1}) dx,$$

and we have

$$\int_K \nabla (v^{n+1} - v_h^{n+1}) dx = \sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} (\eta'(u_h^{n+1})\psi(x) - (\eta'(u_h^{n+1})\psi)_h(x)) ds(x).$$

We use the fact that, for all  $f \in C^2(\bar{\sigma})$  and for all  $x \in \sigma$ , we have  $|f(x) - f_h(x)| \leq \max_{x \in \sigma} |D^2 f(x)| h^2$ , denoting by  $f_h(x)$  the affine function equal to  $f$  at the vertices of the simplex  $\sigma$ . Moreover,  $u_h^{n+1}$  is affine on  $\sigma$  with tangential gradient bounded by  $|\nabla u_h^{n+1}|_K$ . We then get, for all  $x \in \sigma$ , letting  $C_3^\eta$  be a bound of  $|\eta'|$ ,  $|\eta''|$  and  $|\eta'''|$  and  $C_2^\psi$  be a bound of  $\psi$ ,  $|\nabla \psi|$  and  $|D^2 \psi|$ ,

$$|\eta'(u_h^{n+1})\psi(x) - (\eta'(u_h^{n+1})\psi)_h(x)| \leq h^2 C_3^\eta C_2^\psi (|\nabla u_h^{n+1}|_K^2 + 2|\nabla u_h^{n+1}|_K + 1).$$

Since the above expression is integrated over  $\sigma$ , we get

$$\int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (v^{n+1} - v_h^{n+1}) dx \leq \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \chi_K |\sigma| h^2 C_3^\eta C_2^\psi (|\nabla u_h^{n+1}|_K^2 + 2|\nabla u_h^{n+1}|_K + 1).$$

Thanks to the geometrical hypotheses and to (41), we get

$$\sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \chi_K |\sigma| h^2 |\nabla u_h^{n+1}|_K^2 \leq Ch \sum_{K \in \mathcal{T}_h} |K| |\nabla u_h^{n+1}|_K^2 \leq \frac{h}{\theta(h)} C.$$

Besides, we have, thanks to (40),

$$\sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \chi_K |\sigma| h^2 |\nabla u_h^{n+1}|_K \leq Ch \sum_{K \in \mathcal{T}_h} |K| |\nabla u_h^{n+1}|_K \leq hC.$$

Finally

$$\sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \chi_K |\sigma| h^2 \leq Ch \sum_{K \in \mathcal{T}_h} \chi_K |K| \leq Ch |\text{supp} \psi|.$$

Hence each of the above terms tends to 0 with  $h$  thanks to Hypotheses (HD).

#### 5. Study of $e^6$

We again let, for  $t \in [t^n, t^{n+1}[$ ,  $\psi = \varphi(\cdot, t)$ . Letting  $C_\psi$  be a Lipschitz constant for  $\psi$ , and  $\bar{C}$  a bound for  $\eta'(u_p^{n+1})$ , we get, thanks to the Cauchy-Schwarz inequality

$$\begin{aligned} & \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K |u_p^{n+1} - u_q^{n+1}| |\psi(x_p) - \psi(x_q)| \frac{\eta'(u_p^{n+1}) + \eta'(u_q^{n+1})}{2} \\ & \leq \bar{C} C_\psi \left( \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (u_p^{n+1} - u_q^{n+1})^2 \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K \chi_K |x_p - x_q|^2 \right)^{1/2}. \end{aligned}$$

Thanks to the geometrical hypotheses and to (41), we get

$$\theta(h) \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K |u_p^{n+1} - u_q^{n+1}| |\psi(x_p) - \psi(x_q)| \frac{\eta'(u_p^{n+1}) + \eta'(u_q^{n+1})}{2} \leq \bar{C} C_\psi \theta(h) \left( \frac{C}{\theta(h)} C |\text{supp} \psi| \right)^{1/2},$$

which tends to 0 with  $h$  thanks to the hypotheses (HD).  $\square$

## 6 Convergence of the approximate solutions

The convergence proof follows a fairly standard path. We consider a sequence of discretisations  $(\mathcal{F}_{h_k, \delta_k})_{k \in \mathbb{N}}$  whose meshsize and time step tend to zero. In addition, we assume that  $(\mathcal{F}_{h_k, \delta_k})_{k \in \mathbb{N}}$  satisfies uniformly the conditions (HD1)-(HD5) and that the sequence of initial conditions  $(\bar{u}_{h_k}^0)_{k \in \mathbb{N}}$  converges to  $u_{\text{ini}}$  in  $L^1(\mathbb{R}^d)$ . There is no difficulty to build such a sequence of discretisations. The approximate solutions on  $(\mathcal{F}_{h_k, \delta_k})_{k \in \mathbb{N}}$  are denoted by  $(\bar{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$ ,  $(\hat{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$  and  $(\lambda_{h_k, \delta_k})_{k \in \mathbb{N}}$ .

In a first step (Proposition 6.1), we extract converging subsequences from the sequences  $(\bar{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$ ,  $(\hat{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$  and  $(\lambda_{h_k, \delta_k})_{k \in \mathbb{N}}$ , using compactness arguments, notably the Kolmogorov-Riesz theorem. We then pass to the limit in the discrete entropy formulation (63) to deduce the convergence of  $(\hat{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$  toward the entropy solution (Theorem 6.2). We can conclude to the convergence of the whole sequence  $(\hat{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$  by uniqueness of the entropy solution.

**Proposition 6.1:** Let us assume Hypotheses (HC) and Hypotheses (HD). Let  $\bar{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and, for all  $n \in \mathbb{N}$ ,  $\bar{u}_h^{n+\frac{1}{2}}, \bar{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and  $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$  be a solution to Scheme (27)-(30). Then there exists  $u \in L^\infty(Q_T) \cap L^1(0, T; BV(\mathbb{R}^d))$ , and  $\lambda \in L^\infty(Q_T)^d$ , with  $|\lambda| \leq 1$  almost everywhere on  $Q_T$ , such that, up to a subsequence,

$$\hat{u}_{h_k, \delta_k} \rightarrow u \quad \text{in } L^1_{\text{loc}}(Q_T), \quad \bar{u}_{h_k, \delta_k} \rightarrow u \quad \text{in } L^1_{\text{loc}}(Q_T), \quad \lambda_{h_k, \delta_k} \rightharpoonup \lambda \quad \text{weakly-* in } L^\infty(Q_T). \quad (82)$$

**Proof 1.** The estimates (45) and (56) allow us to apply Kolmogorov theorem to the sequence  $(\bar{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$ . Thus, there exists  $u \in L^1(Q_T)$  such that, up to a subsequence,  $(\bar{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$  converges to  $u$  in  $L^1_{\text{loc}}(Q_T)$ . Applying (23) and (40), we get that  $\hat{u}_{h_k, \delta_k} \rightarrow u$  in  $L^1_{\text{loc}}(Q_T)$  as well. Furthermore,  $u \in L^\infty(Q_T) \cap L^1(0, T; BV(\mathbb{R}^d))$ , since the sequence  $(\hat{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$  is uniformly bounded in  $L^\infty(Q_T)$  and  $L^1(0, T; BV(\mathbb{R}^d))$ .

2. As the sequence  $(\lambda_{h_k, \delta_k})_{k \in \mathbb{N}}$  is bounded in  $L^\infty(Q_T)$ , there exists  $\lambda \in L^\infty(Q_T)^d$  such that, up to a subsequence,  $(\lambda_{h_k, \delta_k})_{k \in \mathbb{N}}$  converges weakly-\* to  $\lambda$  in  $L^\infty(Q_T)$ . Furthermore,  $|\lambda| \leq 1$  almost everywhere on  $Q_T$ , since  $|\lambda_{h_k, \delta_k}| < 1$  on  $Q_T$ .  $\square$

**Theorem 6.2:** Let us assume Hypotheses (HC) and Hypotheses (HD). Let  $\bar{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and, for all  $n \in \mathbb{N}$ ,  $\bar{u}_h^{n+\frac{1}{2}}, \bar{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and  $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$  be a solution to Scheme (27)-(30). Let  $u \in L^\infty(Q_T) \cap L^1(0, T; BV(\mathbb{R}^d))$ , and  $\lambda \in L^\infty(Q_T)^d$ , with  $|\lambda| \leq 1$  almost everywhere on  $Q_T$ , such that, up to a subsequence, the convergences (82) hold. Then  $u$  is the entropy solution of (2)-(3).

**Proof** Fix an entropy  $\eta$  and a test function  $\varphi$ . Let  $(\bar{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$ ,  $(\hat{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$  and  $(\lambda_{h_k, \delta_k})_{k \in \mathbb{N}}$  denote the subsequences extracted in Proposition 6.1, and  $u$  and  $\lambda$  denote their limits. The sequence  $(\bar{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$  is bounded and converges in  $L^1_{\text{loc}}(Q_T)$ , therefore

$$\int_{Q_T} \eta(\bar{u}_{h_k, \delta_k}) \partial_t \varphi \, dx dt + \int_{Q_T} \Phi(x, t, \bar{u}_{h_k, \delta_k}) \cdot \nabla \varphi \, dx dt \rightarrow \int_{Q_T} \left( \eta(u) \partial_t \varphi + \Phi(x, t, u) \cdot \nabla \varphi \right) \, dx dt.$$

Since  $\widehat{u}_{h_k, \delta_k} \rightarrow u$  in  $L^1_{\text{loc}}(Q_T)$  and  $\lambda_{h_k, \delta_k} \rightharpoonup \lambda$  weakly- $*$  in  $L^\infty(Q_T)$ ,

$$\int_{Q_T} \eta'(\widehat{u}_{h_k, \delta_k}) \lambda_{h_k, \delta_k} \cdot \nabla \varphi \, dt dx \rightarrow \int_{Q_T} \eta'(u) \lambda \cdot \nabla \varphi \, dt dx.$$

By lower semi-continuity of the total variation relatively to the  $L^1$  convergence, we obtain

$$\liminf_{k \rightarrow +\infty} \int_{Q_T} \varphi |\nabla \eta'(\widehat{u}_{h_k, \delta_k})| \, dt dx \geq \int_{Q_T} \varphi |D[\eta'(u)]| \, dt.$$

From (27), we get that  $\bar{u}_{h_k}^0 \rightarrow u_{\text{ini}}$  in  $L^1(\mathbb{R}^d)$ , and thus

$$\int_{\mathbb{R}^d} \eta(\bar{u}_h^0) \varphi(x, 0) \, dx \rightarrow \int_{\mathbb{R}^d} \eta(u_{\text{ini}}) \varphi(x, 0) \, dx.$$

Finally, using the above limits and Proposition 5.1, we can pass to the limit in (63) and find

$$\int_{Q_T} \left( \eta(u) \partial_t \varphi + (\Phi(x, t, u) - \lambda \eta'(u)) \cdot \nabla \varphi \right) \, dx dt - \int_{Q_T} \varphi |D[\eta'(u)]| \, dt + \int_{\mathbb{R}^d} \eta(u_{\text{ini}}) \varphi(x, 0) \, dx \geq 0,$$

which proves that  $u$  is the entropy solution. Owing to the uniqueness of the entropy solution (proved in Section 1.4), we conclude that, in fact, the whole sequence  $(\widehat{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$  converges to  $u$ .  $\square$

**Remark 6.3:** In the present article, we have proposed a finite volume-finite element approximation of a simplified scalar model of inviscid Bingham flow and proved its convergence. The convergence analysis relies on  $L^\infty(Q_T)$  and  $L^1(0, T; BV(\mathbb{R}^d))$  estimates of the discrete solution. The  $L^\infty(Q_T)$  estimate is obtained by a maximum principle in both the hyperbolic flow step and the total variation flow step.

The extension of the finite volume-finite element approximation to the vector-valued inviscid Bingham flow is quite straightforward. We refer to [23] for the finite volume approximation of hyperbolic systems and [7, 5] for the approximation of the vectorial total variation flow. However, the convergence analysis in this case seems out of reach. The difficulties do not come from the vectorial total variation flow: the  $BV$  estimate is still valid and, although there is no longer a maximum principle, a  $L^\infty$  estimate can still be proven [5]. On the contrary, most of the tools used to prove the convergence of a finite volume approximation of a nonlinear hyperbolic scalar equation (maximum principle,  $L^\infty$  estimate, weak  $BV$  estimates) are no longer valid in the vectorial case.

## 7 Numerical examples

To illustrate the behaviour of the numerical scheme, simulations are performed on two examples: a one-dimensional problem where the flux function  $F$  describes a Burgers flow (we use Scilab for these simulations) and a two-dimensional problem where the flux function  $F$  describes a mere advection (we use FreeFem++ for these simulations). We numerically solve the two-dimensional problem posed on  $\mathbb{R}^2$  by considering periodic boundary conditions. Since in this case, an analytical solution is known, a quantitative study of the convergence is provided. In both cases, we do not implement the additional term in  $\theta(h)$ , which allows for keeping a compact support for the approximation of  $u$  in the numerical tests.

### 7.1 Burgers flow and total variation flow in 1D

We consider the equation

$$\partial_t u + \frac{1}{2} \partial_x (u^2) - g \partial_x \text{Sgn}(\partial_x u) = 0, \quad (x, t) \in \mathbb{R} \times (0, T), \quad (83)$$

with  $g = 5 \cdot 10^{-4}$ . The initial datum is a bump (see Figure 1, left). The computational domain is taken large enough for the solution not to touch the boundary in the course of the simulation. The finite element and finite volume meshes, of size  $h$ , are staggered. The finite volume step is advanced using the Godunov scheme [25, see, e.g.,], which is stable under the CFL condition  $\sup_p |u_p^n| \delta t \leq \frac{1}{2} h$ , for all  $n \in \{0, \dots, N-1\}$ . As in the previous test case, the finite element step is solved using the lagged diffusivity algorithm

The numerical simulation is performed with  $h = 0.002$ ,  $\delta t = 0.001$ ,  $\epsilon = 10^{-8}$  and 20 iterations at each time step for the lagged diffusivity algorithm. In Figure 1, the computed solution is represented at different times. As in a Burgers flow (without total variation flow), a shock wave forms at the right end, while a rarefaction wave forms at the left end. The main difference introduced by the total variation flow is the progressive clipping of the solution.

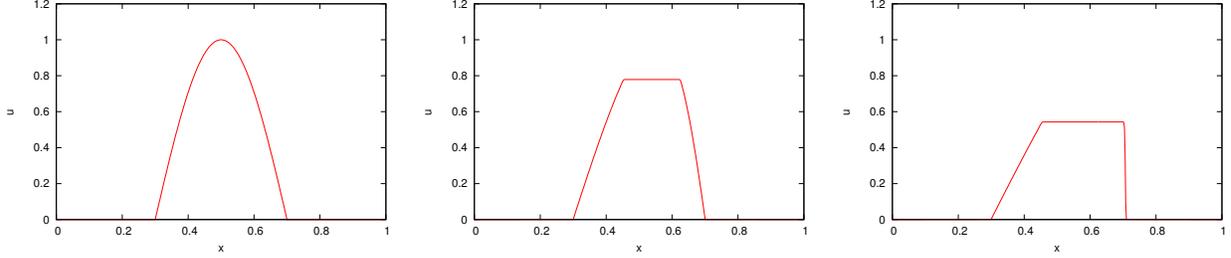


Fig. 1: Burgers flow and total variation flow. Numerical solution at times  $t = 0$ ,  $t = 0.05$ , and  $t = 0.15$ .

## 7.2 Advection and total variation flow in 2D

We consider the problem

$$\partial_t u + c_x \partial_x u + c_y \partial_y u - g \operatorname{div} \operatorname{Sgn}(\nabla u) = 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, T), \quad (84)$$

where  $c_x$ ,  $c_y$ , and  $g$  are constants. The initial condition is

$$u(x, y, 0) = 1_{D_0}(x, y), \quad (x, y) \in \mathbb{R}^2, \quad (85)$$

where  $1_{D_0}$  denotes the indicator function of  $D_0$ , the disk of radius  $r_0$  centred at point  $(x_0, y_0)$ . It is possible to determine the exact solution to this problem. Indeed, for some bounded sets  $S$  of finite perimeter in  $\mathbb{R}^2$ , including disks, it is proved in [4] that the solution of

$$\begin{aligned} \partial_t u - \operatorname{div} \operatorname{Sgn}(\nabla u) &= 0, & (x, y, t) &\in \mathbb{R}^2 \times (0, T), \\ u(x, y, 0) &= 1_S(x, y), & (x, y) &\in \mathbb{R}^2, \end{aligned}$$

is the function  $u : (x, y, t) \mapsto (1 - |\partial S|/|S|t)^+ 1_S(x, y)$ . The proof can be easily adapted to deal with the advection and the exact solution of (84)-(85) is found to be

$$u : (x, y, t) \mapsto \left(1 - \frac{2gt}{r_0}\right)^+ 1_{D_0}(x - c_x t, y - c_y t).$$

The numerical simulations use the following parameters:  $(x_0, y_0) = (0.3, 0.3)$ ,  $r_0 = 0.2$ ,  $c_x = 0.8$ ,  $c_y = 0.2$ ,  $g = 0.03$ . To mimic the space  $\mathbb{R}^2$ , square domains with periodic boundary conditions are used for both the finite volume step and the finite element step. The finite element domain, denoted by  $\Omega$ , is meshed with squares of size  $h$  divided into two triangles. The finite volume domain, slightly larger than the finite element domain, is meshed with square cells of size  $h$ , centred at each node of the finite element mesh. Such a pair of meshes is represented in Figure 2. The advection step is solved using the so-called corner-transport upwind scheme [25,

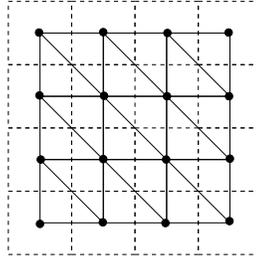


Fig. 2: Example of meshes used for the two-dimensional simulations. The finite element mesh is in solid line and the finite volume mesh is in dashed line.

see, e.g.,], which is slightly more accurate than the standard upwind scheme. This scheme is stable under the CFL condition  $\delta t \leq \min\left(\frac{h}{c_x}, \frac{h}{c_y}\right)$ . The finite element step is solved using the lagged diffusivity algorithm, whose principle is recalled in Appendix A.

Remark 7.1: The corner-transport upwind scheme does not fit exactly into the theoretical framework defined in Section 3.1 since there is a numerical flux between the cells that touch at corner. However, the well-posedness and convergence results remain valid (the proofs can be readily adapted).

The first simulation is performed with  $\Omega = (0, 1) \times (0, 1)$ ,  $h = 0.01$ ,  $\text{CFL} = 1$ ,  $\epsilon = 10^{-6}$  and 20 iterations at each time step for the lagged diffusivity algorithm. The numerical solution, represented at different times in Figure 3, is in good agreement with the exact solution: the contours are sharp and the support is little deformed. Interestingly, the total variation minimisation in the finite element step limits the diffusion due to the finite volume scheme; see Figure 4. Table 1 collects the relative error between the exact solution and the discrete solution in  $L^1$ -norm for various meshsizes (keeping a CFL equal to 1). Precisely, the relative error is given by the expression

$$\frac{\|u^{per} - \hat{u}_{h,\delta}\|_{L^1(\Omega \times (0,T))}}{\|u^{per}\|_{L^1(\Omega \times (0,T))}}$$

where  $u^{per}$  is the periodic version of  $u$  and the norms are computed with a second-order accurate quadrature formula. The results show, as expected, a decrease of the error when the mesh becomes finer. The computed convergence rate is clearly sublinear (0.28 between the first mesh and the second mesh, 0.21 between the second mesh and the third mesh). It is probably due to the discontinuity in the solution. Indeed, the optimal  $L^1$ -norm convergence rate for a monotone scheme applied to a linear advection problem with discontinuous data is only  $O(h^{\frac{1}{2}})$  [29]. The convergence rate for the total variation minimisation with finite elements seems also to be sublinear in the case of a discontinuous source term (the best  $L^2$ -norm convergence rate proved in the literature is  $O(h^{\frac{1}{6}})$  [2]).

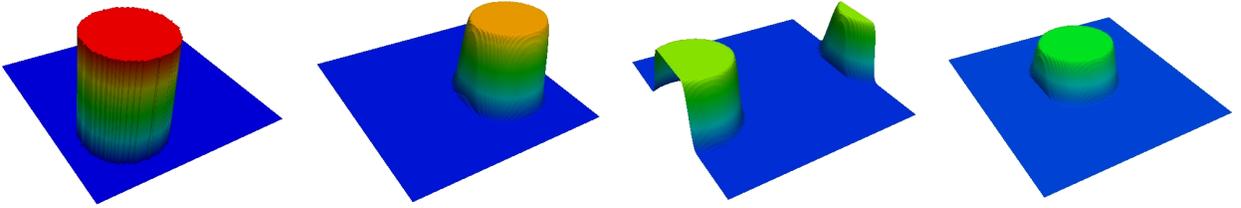


Fig. 3: Advection and total variation flow. Numerical solution at times  $t = 0$ ,  $t = 0.5$ ,  $t = 1$ ,  $t = 1.5$ . The mesh is warped according to the value of the solution and the color scale goes from blue (0) to red (1).

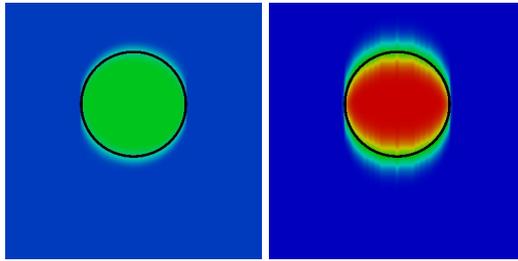


Fig. 4: Numerical solution of the advection and total variation flow (left) and numerical solution of the advection without total variation flow (right) at time  $t = 1.5$ . The color scale goes from blue (0) to red (1). The black circle indicates the support of the exact solution.

Remark 7.2: The regularisation parameter  $\epsilon$  is taken very small and the number of iterations is relatively large in order to make negligible the error due to the lagged diffusivity algorithm. It is actually possible to choose a larger regularisation parameter (and thus a smaller number of iterations) without deteriorating the convergence rate of the finite element step. We refer to [21] for the scaling law relating the meshsize to the regularisation parameter.

$h$	relative error
0.02	0.618
0.01	0.5099
0.005	0.4403

Tab. 1: Relative error between the discrete and exact solutions in  $L^1$ -norm.

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## A Lagged diffusivity algorithm

The lagged diffusivity algorithm is a simple and efficient algorithm for solving the discrete total variation minimisation problem [30, 12]. Under the notation of Section 3.1, consider the minimisation over  $Y_h$  of the functional  $J_h : Y_h \rightarrow \mathbb{R}$ , defined by

$$J_h(v_h) := \int_{\mathbb{R}^d} \left( \frac{1}{2\delta t} (\bar{v}_h(x) - \bar{z}_h(x))^2 + |\nabla \hat{v}_h(x)| \right) dx,$$

where  $z_h \in \mathbb{R}^N$  is such that  $\bar{z}_h \in L^2(\mathbb{R}^d)$ ,  $\delta t > 0$  are given and where

$$Y_h = \{v_h \in \mathbb{R}^N; \nabla \hat{v}_h \in L^1(\mathbb{R}^d), \bar{v}_h \in L^2(\mathbb{R}^d)\}.$$

The lagged diffusivity algorithm replaces the nondifferentiable functional  $J_h$  by the regularised functional  $J_h^\epsilon : Y_h \rightarrow \mathbb{R}$ , defined by

$$J_h^\epsilon(v_h) := \int_{\mathbb{R}^d} \left( \frac{1}{2\delta t} (\bar{v}_h(x) - \bar{z}_h(x))^2 + \sqrt{\epsilon(x)^2 + |\nabla \hat{v}_h(x)|^2} \right) dx,$$

where  $\epsilon(x) \in L^1(\mathbb{R}^d)$  is a small positive function on  $\mathbb{R}^d$ . Then the respective minimisers  $u_h$  and  $u_h^\epsilon$  are the respective solutions of the following variational inequalities:

$$\int_{\mathbb{R}^d} \left( \frac{1}{\delta t} (\bar{u}_h(x) - \bar{z}_h(x))(\bar{v}_h(x) - \bar{u}_h(x)) + |\nabla \hat{v}_h(x)| - |\nabla \hat{u}_h(x)| \right) dx \geq 0, \quad \forall v_h \in Y_h,$$

and

$$\int_{\mathbb{R}^d} \left( \frac{1}{\delta t} (\bar{u}_h^\epsilon(x) - \bar{z}_h(x))(\bar{v}_h(x) - \bar{u}_h^\epsilon(x)) + \sqrt{\epsilon(x)^2 + |\nabla \hat{v}_h(x)|^2} - \sqrt{\epsilon(x)^2 + |\nabla \hat{u}_h^\epsilon(x)|^2} \right) dx \geq 0, \quad \forall v_h \in Y_h.$$

Letting  $v_h = u_h^\epsilon$  in the first one and  $v_h = u_h$  in the second one, adding the resulting equations, using  $|\nabla \hat{u}_h^\epsilon(x)| - \sqrt{\epsilon(x)^2 + |\nabla \hat{u}_h^\epsilon(x)|^2} \leq 0$ , we obtain

$$\int_{\mathbb{R}^d} \left( -\frac{1}{\delta t} (\bar{u}_h(x) - \bar{u}_h^\epsilon(x))^2 + \sqrt{\epsilon(x)^2 + |\nabla \hat{u}_h(x)|^2} - |\nabla \hat{u}_h(x)| \right) dx \geq 0.$$

We now use

$$\forall a, b \geq 0, \quad \sqrt{a^2 + b^2} \leq a + b.$$

We get

$$\delta t \int_{\mathbb{R}^d} \epsilon(x) dx \geq \int_{\mathbb{R}^d} (\bar{u}_h(x) - \bar{u}_h^\epsilon(x))^2 dx,$$

which proves that  $\bar{u}_h^\epsilon$  tends to  $\bar{u}_h$  in  $L^2(\mathbb{R}^d)$  as  $\epsilon \rightarrow 0$  in  $L^1(\mathbb{R}^d)$ . We now remark that the minimiser  $u_h^\epsilon$  of  $J_h^\epsilon$  is also solution of the variational problem

$$\int_{\mathbb{R}^d} \left( \frac{1}{\delta t} (\bar{u}_h^\epsilon(x) - \bar{z}_h(x)) \bar{v}_h(x) + \frac{\nabla \hat{u}_h^\epsilon(x) \cdot \nabla \hat{v}_h(x)}{\sqrt{\epsilon(x)^2 + |\nabla \hat{u}_h^\epsilon(x)|^2}} \right) dx = 0, \quad \forall v_h \in Y_h.$$

The preceding problem is then computed with a fixed-point algorithm. The algorithm is initialized with  $u_h^{(0)} = z_h$ . At the iteration  $k + 1$ , the approximate solution  $u_h^{(k+1)}$  is computed by solving the linear diffusion problem

$$\int_{\mathbb{R}^d} \left( \frac{1}{\delta t} (\bar{u}_h^{(k+1)}(x) - \bar{z}_h(x)) \bar{v}_h(x) + \frac{\nabla \hat{u}_h^{(k+1)}(x) \cdot \nabla \hat{v}_h(x)}{\sqrt{\epsilon(x)^2 + |\nabla \hat{u}_h^{(k)}(x)|^2}} \right) dx = 0, \quad \forall v_h \in Y_h.$$

The diffusivity field in the above diffusion problem depends on  $u_h^{(k)}$ , whence the name lagged diffusivity.