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## Convection and total variation flow

FRANÇOIS BOUCHUT<sup>†</sup>, DAVID DOYEN<sup>‡</sup> AND ROBERT EYMARD<sup>§</sup>

*Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées (UMR 8050),  
UPEMLV, UPEC, CNRS, F-77454, Marne-la-Vallée, France*

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We deal with a nonlinear hyperbolic scalar conservation law, regularized by the total variation flow operator (or 1-Laplacian). We give an entropy weak formulation, for which we prove the existence and the uniqueness of the solution. The existence result is established using the convergence of a numerical approximation (a splitting scheme where the hyperbolic flow is treated with finite volumes and the total variation flow with finite elements). Some numerical simulations are also presented.

*Keywords:* Bingham fluid, hyperbolic scalar conservation law, total variation flow, 1-Laplacian, entropy formulation, finite volumes, finite elements

A Bingham fluid, also called rigid viscoplastic fluid, is a material that behaves as a rigid solid below a certain stress yield and as a viscous fluid above this yield; a familiar example of such a material is the tooth paste. For a  $d$ -dimensional Bingham fluid, the relation between the stress tensor  $\sigma$ , seen as a  $d \times d$  matrix, the pressure  $p$  and the velocity  $u$  is

$$\sigma = -pI_d + g \frac{\mathbf{D}(u)}{|\mathbf{D}(u)|_F} + \nu \mathbf{D}(u), \quad (0.1)$$

where  $g$  and  $\nu$  are positive constants,  $I_d$  is the  $d \times d$  identity matrix,  $\mathbf{D}(u)$  is the  $d \times d$  matrix such that  $\mathbf{D}(u)_{ij} := \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ , and  $|\cdot|_F$  denotes the Frobenius matrix norm. The term  $g \frac{\mathbf{D}(u)}{|\mathbf{D}(u)|_F}$  enforces the plastic behavior,  $g$  being the plasticity yield, while the term  $\nu \mathbf{D}(u)$  enforces the viscous behavior,  $\nu$  being the viscosity parameter. The mathematical analysis of Bingham fluid flows dates back to the work of Duvaut & Lions (1976), where the problems are formulated as variational inequalities in Sobolev spaces. The numerical approximation of a Bingham fluid flow is usually treated with finite element techniques; we refer to Dean *et al.* (2007) for a recent review.

When the viscosity becomes negligible ( $\nu = 0$ ), the analytical and numerical framework described above is no longer suitable – let us mention however an existence result in 2D obtained by Lions (1972). Although the study of inviscid Bingham fluids has been initiated in Bouchut *et al.* (2012) with the case of an unsteady flow without convection term, the presence of a nonlinear convection term is naturally issued from the inertial term in the momentum conservation equation. Unfortunately, the study of this problem seems to be out of reach in the actual state of the art, and we only consider here a simplified model of unsteady Bingham flow with convection. This simplified model is scalar and consists in

<sup>†</sup>Email: francois.bouchut@univ-mlv.fr

<sup>‡</sup>Corresponding author. Email: david.doyen@univ-mlv.fr

<sup>§</sup>Email: robert.eynard@univ-mlv.fr

seeking  $u : \mathbb{R}^d \times (0, T) \rightarrow \mathbb{R}$ , with compact support in space, such that

$$\partial_t u + \operatorname{div} F(x, t, u) - \operatorname{div} (\operatorname{Sgn}(\nabla u)) = 0, \quad \text{on } \mathbb{R}^d \times (0, T), \quad (0.2)$$

$$u(x, 0) = u_{\text{ini}}(x), \quad \text{on } \mathbb{R}^d, \quad (0.3)$$

where  $d \in \{1, 2, 3\}$ ,  $F : \mathbb{R}^d \times (0, T) \times \mathbb{R} \rightarrow \mathbb{R}^d$  is divergence-free with respect to the space variables,  $\operatorname{Sgn}$  denotes the vector sign function, and  $u_{\text{ini}}$  is a function with compact support. We recall that the  $\operatorname{Sgn}$  function is a set-valued map from  $\mathbb{R}^d$  to  $\mathcal{P}(\mathbb{R}^d)$  such that

$$\lambda \in \operatorname{Sgn}(\mu) \Leftrightarrow \begin{cases} |\lambda| \leq 1 & \text{if } \mu = 0 \\ \lambda = \frac{\mu}{|\mu|} & \text{if } \mu \neq 0 \end{cases}$$

where  $|\cdot|$  denotes the euclidean norm in  $\mathbb{R}^d$ . In equation (0.2), the term  $\operatorname{div} F$  describes the convection in the fluid regime, while the term  $\operatorname{div} (\operatorname{Sgn}(\nabla u))$  enforces the plastic behavior (the plasticity yield is taken equal to 1 for simplicity).

Problem (0.2)-(0.3) can be viewed as a nonlinear hyperbolic scalar conservation law regularized by the total variation flow operator (or the 1-Laplacian). Let  $\Omega$  be an open subset of  $\mathbb{R}^d$ , the total variation of a function  $u \in L^1(\Omega)$  is

$$TV_{\Omega}(u) := \sup \left\{ \int_{\Omega} u \operatorname{div} \phi \, dx; \phi \in C_c^1(\Omega; \mathbb{R}^d) \text{ with } \|\phi\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

In particular, the total variation of a function  $u \in W^{1,1}(\Omega)$  is

$$TV_{\Omega}(u) = \int_{\Omega} |\nabla u| \, dx.$$

For nonlinear hyperbolic conservation laws, it is well known that the standard weak formulation fails to ensure the uniqueness of the solution and must be replaced by an entropy formulation; see, e.g., Godlewski & Raviart (1991) or Serre (1999). With some types of regularizations, as for instance the viscous regularization, the uniqueness is recovered. That is not the case with the total variation flow regularization, which has no spatial smoothing effect and does not prevent the formation of shocks.

Nonlinear hyperbolic problems are usually approximated with finite volumes (Eymard *et al.*, 2000; Godlewski & Raviart, 1996; LeVeque, 2002). Unfortunately, finite volumes are not suitable for the approximation of the total variation flow: indeed, if a sequence  $(u_k)_{k \in \mathbb{N}}$  of piecewise constant functions converges to  $u$  in  $L^1$ , the total variation of  $u_k$  does not converge in general to the total variation of  $u$  (see Bělík & Luskin (2003) for an example). The total variation flow must be approximated in  $W^{1,1}$ -conforming discrete spaces, such as  $P_1$  finite element spaces (Bartels, 2012; Feng & Prohl, 2003; Feng *et al.*, 2005). Numerical schemes combining finite volumes and finite element schemes have already been considered for scalar conservation laws with a diffusion term (Feistauer *et al.*, 1999) and for degenerate parabolic equations (Eymard *et al.*, 2006).

In the present article, we first give an entropy formulation for Problem (0.2)-(0.3) and prove the uniqueness of the entropy solution using the doubling variable technique. Note that our entropy formulation of the total variation term is similar to the one developed in Bellettini *et al.* (2002) to study the total variation flow with  $L^1_{loc}$  initial data (without hyperbolic term).

The existence of the entropy solution follows from the convergence of a numerical approximation, based on a splitting scheme. The hyperbolic flow is treated with finite volumes and the total variation

flow is treated with  $P_1$  finite elements. The finite volume mesh is built as a dual mesh of the finite element mesh, which makes simple the interpolation step between the two meshes. For the hyperbolic step (or finite volume step), we choose an explicit time discretization for sake of simplicity. For the total variation flow step (or finite element step), we are led to define an implicit scheme accounting for the nonregularity of the total variation flow operator. To guarantee the maximum principle, which is essential for the stability of the scheme, we use a nonobtuse finite element mesh.

The convergence proof of the numerical approximation relies on the Kolmogorov-Riesz compactness theorem, which provides us with the strong convergence in  $L^1(\mathbb{R}^d \times (0, T))$  of the discrete solutions. It requires uniform estimates on the space and time translates of the discrete solutions. To establish these estimates, the total variation term is crucial. For scalar conservation laws without total variation flow regularization, these estimates are not true and the convergence study of the numerical approximations must be carried out with other tools (Cockburn *et al.*, 1995; Chainais-Hillairet, 1999; Eymard *et al.*, 2000).

The article is organized as follows. In section 1, the concept of entropy solution for Problem (0.2)-(0.3) is defined and its uniqueness is proved. Section 2 describes the numerical approximation and its first properties (well-posedness, maximum principle). A priori estimates on the discrete solutions are provided in Section 3 and a discrete entropy formulation is established in Section 4. The convergence of the numerical approximation (and thus the existence of an entropy solution) is finally proved in Section 5 using the results of the two previous sections. In the last section, some numerical simulations in 1D and 2D are presented.

## 1. Entropy formulation for nonlinear hyperbolic equation with total variation flow

### 1.1 Notation and hypotheses

Let us first recall basic properties concerning the functions with bounded variation. For a comprehensive presentation, we refer to Ambrosio *et al.* (2000), Evans & Gariepy (1992) or Ziemer (1989).

- Let  $\Omega$  be an open subset of  $\mathbb{R}^d$ . The space of functions over  $\Omega$  with bounded variation, denoted by  $BV(\Omega)$ , is the set of functions  $u \in L^1(\Omega)$  such that  $TV_\Omega(u) < +\infty$ . Equipped with the norm  $\|\cdot\|_{BV(\Omega)} := \|\cdot\|_{L^1(\Omega)} + TV_\Omega(\cdot)$ , the space  $BV(\Omega)$  is a (nonreflexive) Banach space.
- The distributional derivative of  $u \in BV(\Omega)$ , denoted by  $Du$ , is the vector Radon measure such that

$$\int_{\Omega} u \operatorname{div} \phi \, dx = - \int_{\Omega} \phi \, Du, \quad \forall \phi \in C_c^1(\Omega, \mathbb{R}^d).$$

- The norm of the vector measure  $Du$  is denoted by  $|Du|$ . It is a positive Radon measure and there is a measurable function  $h : \mathbb{R} \rightarrow \mathbb{R}^d$ , with  $|h(x)| = 1$  for all  $x \in \mathbb{R}$ , such that  $Du = h|Du|$ .
- The norm of  $Du$  is linked to the total variation by the identity  $|Du|(\Omega) = TV_\Omega(u)$ .
- The total variation is lower semi-continuous relatively to the convergence in  $L^1_{loc}$ . In other words, if  $(u_k)_{k \in \mathbb{N}}$  is a sequence in  $BV(\Omega)$  converging to  $u$  in  $L^1_{loc}(\Omega)$ , then

$$\liminf_{k \rightarrow \infty} |Du_k|(\Omega) \geq |Du|(\Omega).$$

- If  $u \in BV(\Omega)$  and  $f \in C^1(\mathbb{R})$ , then  $f(u) \in BV(\Omega)$ .

- The space  $L^1(0, T; BV(\Omega))$  is the set of measurable functions  $u : (0, T) \rightarrow BV(\Omega)$  such that  $\int_0^T \|u(t)\|_{BV(\Omega)} dt < +\infty$ . Equipped with the norm  $\|\cdot\|_{L^1(0, T; BV(\Omega))} := \int_0^T \|\cdot\|_{BV(\Omega)} dt$ , it is a (nonreflexive) Banach space. For more information on functions valued in Banach spaces, we refer for instance to Showalter (1997, Chapter III).

Let us now introduce some notation and specify the hypotheses on the initial datum and the flux function. We set  $Q_T := \mathbb{R}^d \times (0, T)$ . The initial datum  $u_{\text{ini}}$  is assumed to belong to  $BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  and to have a compact support. The essential infimum and supremum of  $u_{\text{ini}}$  are denoted by  $a_0$  and  $b_0$ , respectively. The flux function  $F$  is in  $C^1(Q_T \times \mathbb{R}, \mathbb{R}^d)$  and is assumed to be divergence-free with respect to the space variables, that is

$$\operatorname{div}_x F(x, t, u) := \sum_{i=1}^d \frac{\partial F}{\partial x_i}(x, t, u) = 0, \quad \forall (x, t, u) \in Q_T \times \mathbb{R}. \quad (1.1)$$

Furthermore,  $\frac{\partial F}{\partial u}$  is assumed to be locally Lipschitz continuous and such that, for all compact set  $K \subset \mathbb{R}$ ,  $\left| \frac{\partial F}{\partial u} \right| \leq C_K$  a.e. on  $Q_T \times K$ , where  $C_K$  is a constant depending on  $K$ .

## 1.2 Definition of entropy solutions

In the usual entropy formulations of scalar conservation laws, the admissible entropies are the  $C^1$  convex functions or the so-called Kruzhkov entropies ( $|\cdot - \kappa|$  with  $\kappa \in \mathbb{R}$ ). The entropy formulation of the problem (0.2)-(0.3), owing to the term  $\operatorname{div} \operatorname{Sgn}(\nabla u)$ , requires more regular entropies.

**DEFINITION 1.1** An admissible entropy is a convex function  $\eta \in C^2(\mathbb{R})$ . The corresponding entropy flux is the function  $\Phi \in C^0(Q_T \times \mathbb{R}, \mathbb{R}^d)$  such that

$$\Phi(u, x, t) := \int_0^u \eta'(v) \frac{\partial F}{\partial u}(x, t, v) dv.$$

Since the flux function  $F$  is divergence-free with respect to the space variables, the entropy flux is divergence-free as well.

**DEFINITION 1.2** A function  $u \in L^\infty(Q_T) \cap L^1(0, T; BV(\mathbb{R}^d))$ , with compact support in space, is said to be an entropy solution of (0.2)-(0.3) if there exists  $\lambda \in L^\infty(Q_T)^d$ , with  $|\lambda| \leq 1$  almost everywhere on  $Q_T$ , such that, for all admissible entropy-entropy flux pairs  $(\eta, \Phi)$  in the sense of Definition 1.1 and all nonnegative test functions  $\varphi \in C_c^\infty(Q_T)$ ,

$$\int_{Q_T} \left( \eta(u) \partial_t \varphi + (\Phi(x, t, u) - \lambda \eta'(u)) \cdot \nabla \varphi \right) dx dt - \int_{Q_T} \varphi |D[\eta'(u)]| dt + \int_{\mathbb{R}^d} \eta(u_{\text{ini}}(x)) \varphi(x, 0) dx \geq 0. \quad (1.2)$$

Since  $\eta'$  is in  $C^1(\mathbb{R})$ , the function  $\eta'(u)$  is in  $L^1(0, T; BV(\mathbb{R}^d))$ . Therefore, the term  $\int_{Q_T} \varphi |D[\eta'(u)]| dt$  is meaningful. The function  $\lambda$ , which is not necessarily unique, is called a multiplier by analogy with a Lagrange multiplier.

**REMARK 1.1** In the above definition, the entropy solution  $u$  is supposed to be in  $L^1(0, T; BV(\mathbb{R}^d))$ . Owing to the formal identity (neglecting the terms in  $\partial F / \partial x_i$ )

$$\partial_t |Du| + \operatorname{div} \left( |Du| \frac{\partial F}{\partial u} \right) - \operatorname{div}(\operatorname{Sgn}(\nabla u) \operatorname{div}(\operatorname{Sgn}(\nabla u))) + (\operatorname{div}(\operatorname{Sgn}(\nabla u)))^2 = 0,$$

and the fact that  $u_{\text{ini}} \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , we actually expect a better regularity :  $u \in L^\infty(0, T; BV(\mathbb{R}^d))$ . However, our existence proof, based on the convergence of a numerical scheme, does not allow us to obtain this higher regularity (for general meshes, finite volume schemes are not generally total variation diminishing).

### 1.3 Formal derivation of the entropy formulation

In order to enlight the link between the strong formulation and the entropy formulation, we present below a formal derivation of the entropy formulation by the vanishing viscosity method. Let us consider a viscous regularization of the equation (0.2):

$$\partial_t u_\varepsilon + \operatorname{div} F(x, t, u_\varepsilon) - \operatorname{div} \operatorname{Sgn}(\nabla u_\varepsilon) - \varepsilon \Delta u_\varepsilon = 0, \quad (1.3)$$

where  $\varepsilon$  is a positive constant. One can justify that this problem is well-posed and, owing to the term  $\operatorname{div} \operatorname{Sgn}(\nabla u_\varepsilon)$ , we can expect that  $u_\varepsilon$  tends to  $u$  in  $L^1_{\text{loc}}(Q_T)$  when  $\varepsilon \rightarrow 0$ . Multiplying (1.3) by  $\eta'(u_\varepsilon)$ , we find

$$\partial_t \eta(u_\varepsilon) + \operatorname{div} \left( \Phi(x, t, u_\varepsilon) - \operatorname{Sgn}(\nabla u_\varepsilon) \eta'(u_\varepsilon) \right) + \nabla \eta'(u_\varepsilon) \cdot \operatorname{Sgn}(\nabla u_\varepsilon) - \varepsilon \Delta \eta(u_\varepsilon) + \varepsilon \eta''(u_\varepsilon) |\nabla u_\varepsilon|^2 = 0. \quad (1.4)$$

The entropy  $\eta$  being convex, we have

$$\nabla \eta'(u_\varepsilon) \cdot \operatorname{Sgn}(\nabla u_\varepsilon) = \eta''(u_\varepsilon) \nabla u_\varepsilon \cdot \operatorname{Sgn}(\nabla u_\varepsilon) \geq 0,$$

whence

$$\nabla \eta'(u_\varepsilon) \cdot \operatorname{Sgn}(\nabla u_\varepsilon) = |\nabla \eta'(u_\varepsilon)|.$$

Still by convexity of  $\eta$ , we have

$$\varepsilon \eta''(u_\varepsilon) |\nabla u_\varepsilon|^2 \geq 0.$$

Let us now multiply (1.4) by a nonnegative test function  $\varphi \in C_c^\infty(Q_T)$  and integrate over  $Q_T$ . We thus obtain

$$\int_{Q_T} \left( \eta(u_\varepsilon) \partial_t \varphi + (\Phi(x, t, u_\varepsilon) - \operatorname{Sgn}(\nabla u_\varepsilon) \eta'(u_\varepsilon)) \cdot \nabla \varphi - |\nabla \eta'(u_\varepsilon)| \varphi \right) dx dt + \int_{\mathbb{R}^d} \eta(u_{\text{ini}}(x)) \varphi(x, 0) dx \geq 0. \quad (1.5)$$

Since  $u_\varepsilon \rightarrow u$  in  $L^1_{\text{loc}}(Q_T)$ , it follows from the semicontinuity of the total variation that

$$\liminf_{\varepsilon \rightarrow 0} \int_{Q_T} |\nabla \eta'(u_\varepsilon)| \varphi dx dt \geq \int_{Q_T} \varphi |D[\eta'(u)]| dt.$$

Since the family  $\{\operatorname{Sgn}(\nabla u_\varepsilon)\}_{\varepsilon > 0}$  is bounded, there exists  $\lambda \in L^\infty(Q_T)^d$  such that, up to a subsequence,

$$\operatorname{Sgn}(\nabla u_\varepsilon) \rightharpoonup \lambda \text{ weakly-* in } L^\infty(Q_T)^d.$$

Finally, letting  $\varepsilon \rightarrow 0$  in (1.5), we obtain (1.2).

### 1.4 Existence and uniqueness

**THEOREM 1.3** There exists one and only one entropy solution of (0.2)-(0.3).

*Proof.* The existence of an entropy solution is proved by the convergence of a numerical approximation; see Theorem 5.3 below. To prove the uniqueness, we use the doubling variable technique.

Step 1. Let  $u$  be an entropy solution and  $\lambda_u$  a corresponding multiplier. We consider the entropy  $\eta(\cdot - \kappa)$ , where  $\eta \in C^2(\mathbb{R})$  is an even convex function and  $\kappa \in \mathbb{R}$ . Let  $\Phi_\kappa$  be the corresponding entropy flux. Then, by definition of an entropy solution, for all nonnegative test functions  $\varphi \in C_c^\infty(Q_T)$ ,

$$\begin{aligned} & \int_{Q_T} \left( \eta(u(x,t) - \kappa) \partial_t \varphi + (\Phi_\kappa(x,t, u(x,t)) - \lambda_u(x,t) \eta'(u(x,t) - \kappa)) \cdot \nabla_x \varphi \right) dx dt \\ & - \int_{Q_T} \varphi |D_x[\eta'(u(x,t) - \kappa)]| dt + \int_{\mathbb{R}^d} \eta(u_{\text{ini}}(x) - \kappa) \varphi(x,0) dx \geq 0. \end{aligned} \quad (1.6)$$

Let  $v$  be another entropy solution and  $\lambda_v$  a corresponding multiplier. Since  $\eta$  is an even function,  $\eta(\cdot - \kappa) = \eta(\kappa - \cdot)$  and  $\eta'(\cdot - \kappa) = -\eta'(\kappa - \cdot)$ . Then, denoting by  $y$  and  $s$  the space and time variables, for all nonnegative test functions  $\varphi \in C_c^\infty(Q_T)$ ,

$$\begin{aligned} & \int_{Q_T} \left( \eta(\kappa - v(y,s)) \partial_s \varphi + (\Phi_\kappa(y,s, v(y,s)) + \lambda_v(y,s) \eta'(\kappa - v(y,s))) \cdot \nabla_y \varphi \right) dy ds \\ & - \int_{Q_T} \varphi |D_y[\eta'(\kappa - v(y,s))]| dt + \int_{\mathbb{R}^d} \eta(u_{\text{ini}}(y) - \kappa) \varphi(y,0) dy \geq 0. \end{aligned} \quad (1.7)$$

Step 2. We now introduce well-chosen test functions. Let  $\{\rho_\varepsilon\}_{\varepsilon>0}$  be a family of mollifiers in  $\mathbb{R}^d$  such that  $\text{supp } \rho_\varepsilon \subset B(0, \varepsilon)$  and  $\{\bar{\rho}_\varepsilon\}_{\varepsilon>0}$  be a family of mollifiers in  $\mathbb{R}$  such that  $\text{supp } \bar{\rho}_\varepsilon \subset [-\varepsilon, 0]$ . Let  $r > 0$ ,  $\tau > 0$ , and  $\psi \in C_c^\infty(\mathbb{R}, \mathbb{R}^+)$ . We define  $\phi : Q_T \times Q_T \rightarrow \mathbb{R}^+$  such that

$$\phi(x,t,y,s) = \psi(t) \rho_r(x-y) \bar{\rho}_\tau(t-s), \quad \forall (x,t,y,s) \in Q_T \times Q_T. \quad (1.8)$$

We take  $\kappa = v(y,s)$  and  $\varphi = \phi(\cdot, \cdot, y, s)$  in (1.6). Next, integrating with respect to  $y$  and  $s$  over  $Q_T$ , and noticing that  $\phi(x,0,y,s) = 0$  for all  $s > 0$ , we obtain

$$\begin{aligned} & \int_{Q_T \times Q_T} \left( \eta(u(x,t) - v(y,s)) \partial_t \phi + (\Phi_{v(y,s)}(x,t, u(x,t)) - \lambda_u(x,t) \eta'(u(x,t) - v(y,s))) \cdot \nabla_x \phi \right) dx dt dy ds \\ & - \int_{Q_T \times Q_T} \phi |D_x[\eta'(u(x,t) - v(y,s))]| dt dy ds \geq 0. \end{aligned} \quad (1.9)$$

Similarly, taking  $\kappa = u(x,t)$  and  $\varphi = \phi(x,t, \cdot, \cdot)$  in (1.7), then integrating with respect to  $x$  and  $t$  over  $Q_T$ , we obtain

$$\begin{aligned} & \int_{Q_T \times Q_T} \left( \eta(u(x,t) - v(y,s)) \partial_s \phi + (\Phi_{u(x,t)}(y,s, v(y,s)) + \lambda_v(y,s) \eta'(u(x,t) - v(y,s))) \cdot \nabla_y \phi \right) dx dt dy ds \\ & - \int_{Q_T \times Q_T} \phi |D_y[\eta'(u(x,t) - v(y,s))]| ds dx dt + \int_{\mathbb{R}^d \times Q_T} \eta(u_{\text{ini}}(y) - u(x,t)) \phi(x,t,y,0) dy dx dt \geq 0. \end{aligned} \quad (1.10)$$

Adding the above relations (1.9) and (1.10), we find

$$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 \geq 0,$$

where

$$\begin{aligned}
 A_1 &:= \int_{Q_T \times Q_T} \eta(u(x,t) - v(y,s)) (\partial_t \phi + \partial_s \phi) \, dx dt dy ds, \\
 A_2 &:= \int_{Q_T \times Q_T} \Phi_{v(y,s)}(x,t, u(x,t)) \cdot \nabla_x \phi \, dx dt dy ds, \\
 A_3 &:= \int_{Q_T \times Q_T} \Phi_{u(x,t)}(y,s, v(y,s)) \cdot \nabla_y \phi \, dx dt dy ds, \\
 A_4 &:= \int_{Q_T \times Q_T} (\lambda_u(x,t) - \lambda_v(y,s)) \eta'(u(x,t) - v(y,s)) \cdot (\nabla_x \phi + \nabla_y \phi) \, dx dt dy ds, \\
 A_5 &:= \int_{Q_T \times Q_T} \lambda_v(y,s) \eta'(u(x,t) - v(y,s)) \cdot \nabla_x \phi \, dx dt dy ds - \int_{Q_T \times Q_T} \phi |D_x[\eta'(u(x,t) - v(y,s))]| \, dt dy ds, \\
 A_6 &:= \int_{Q_T \times Q_T} -\lambda_u(x,t) \eta'(u(x,t) - v(y,s)) \cdot \nabla_y \phi \, dx dt dy ds - \int_{Q_T \times Q_T} \phi |D_y[\eta'(u(x,t) - v(y,s))]| \, ds dx dt, \\
 A_7 &:= \int_{\mathbb{R}^d \times Q_T} \eta(u_{\text{ini}}(y) - u(x,t)) \phi(x,t, y, 0) \, dy dx dt.
 \end{aligned}$$

Step 3. We now remark that we can get rid of  $A_4$ ,  $A_5$  and  $A_6$ , the terms arising from the total variation. The term  $A_4$  vanishes owing to identity  $\nabla_x \phi + \nabla_y \phi = 0$ . The term  $A_5$  is nonpositive. Indeed, integrating by parts with respect to  $x$  the first term of  $A_5$ , we obtain

$$A_5 = \int_{Q_T \times Q_T} -\lambda_v(y,s) \phi D_x[\eta'(u(x,t) - v(y,s))] \, dt dy ds - \int_{Q_T \times Q_T} \phi |D_x[\eta'(u(x,t) - v(y,s))]| \, dt dy ds. \quad (1.11)$$

Since  $|\lambda_v(y,s)| \leq 1$  for all  $(y,s) \in Q_T$ , we deduce that  $A_5 \leq 0$ . With the same argument, we prove that  $A_6 \leq 0$ .

Step 4. The resulting equation is now

$$A_1 + A_2 + A_3 + A_7 \geq 0. \quad (1.12)$$

If we let  $\eta$  tend to the absolute value function, that is, if we let the entropy  $\eta(\cdot - \kappa)$  tend to the Kruzhkov entropy  $|\cdot - \kappa|$ , we obtain

$$A_{10} + A_{20} + A_{30} + A_{70} \geq 0, \quad (1.13)$$

where

$$\begin{aligned}
 A_{10} &:= \int_{Q_T \times Q_T} |u(x,t) - v(y,s)| (\partial_t \phi + \partial_s \phi) \, dx dt dy ds, \\
 A_{20} &:= \int_{Q_T \times Q_T} (F(x,t, u(x,t)) \top v(y,s) - F(x,t, u(x,t)) \perp v(y,s)) \cdot \nabla_x \phi \, dx dt dy ds, \\
 A_{30} &:= \int_{Q_T \times Q_T} (F(y,s, v(y,s)) \top u(x,t) - F(y,s, v(y,s)) \perp u(x,t)) \cdot \nabla_y \phi \, dx dt dy ds, \\
 A_{70} &:= \int_{\mathbb{R}^d \times Q_T} |u_{\text{ini}}(y) - u(x,t)| \phi(x,t, y, 0) \, dy dx dt.
 \end{aligned}$$

Step 5. The remaining of the proof is identical to the uniqueness proof done in Chainais-Hillairet (1999), since its starting point is precisely (1.13).  $\square$



## 2. Numerical approximation

### 2.1 Notation and hypotheses

The finite element mesh, denoted by  $\mathcal{T}_h$ , is a conforming simplicial mesh of  $\mathbb{R}^d$  of size  $h$ . In order to ensure the maximum principle, each element of  $\mathcal{T}_h$  is assumed to be nonobtuse; we recall that a simplex is said to be nonobtuse if the angles between any two facets are less than or equal to  $\pi/2$ . The finite volume mesh, denoted by  $\mathcal{D}_h$ , is a polyhedral mesh of  $\mathbb{R}^d$  such that the interface between two cells is a finite union of faces. The mesh  $\mathcal{D}_h$  is a dual mesh of  $\mathcal{T}_h$  in the sense that each cell of  $\mathcal{D}_h$  contains one and only one node of  $\mathcal{T}_h$ . The nodes of  $\mathcal{T}_h$  and the corresponding cells of  $\mathcal{D}_h$  are denoted by  $(x_p)_{p \in \mathbb{N}}$  and  $(Q_p)_{p \in \mathbb{N}}$ , respectively. Let us introduce some additional notation about  $\mathcal{D}_h$ :  $\mathcal{N}_p$  is the set containing the indices of the neighboring cells of  $Q_p$ ,  $\mathcal{E}_h$  is the set of couples  $(p, q)$  such that  $Q_p$  and  $Q_q$  are neighbors,  $\sigma_{p,q}$  is the interface between two neighbor cells  $Q_p$  and  $Q_q$ ,  $\nu_{p,q}$  is the unit normal vector to  $\sigma_{p,q}$  pointing toward  $Q_q$ ,  $m_p$  is the measure of  $Q_p$ ,  $m_{p,q}$  is the measure of  $\sigma_{p,q}$ .

REMARK 2.1 Since a square can be divided into two right triangles and a cube can be divided into six nonobtuse tetrahedra, it is easy to build nonobtuse simplicial meshes of  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . In fact, it is possible to generate nonobtuse simplicial meshes on any polygonal or polyhedral domain; see Brandts *et al.* (2009) and references therein.

REMARK 2.2 In  $\mathbb{R}^2$ , with a general unstructured triangular mesh  $\mathcal{T}_h$ , the dual mesh  $\mathcal{D}_h$  can always be built as follows. Let  $x_p$  a node of  $\mathcal{T}_h$ . For every triangle  $K \in \mathcal{T}_h$  containing the node  $x_p$ , join the centroid of  $K$  with the midpoint of the two edges of  $K$  incident to  $x_p$ . This yields a closed polygonal line which delimits a cell  $Q_p$  associated to  $x_p$ . Obviously, the cells  $(Q_p)_{p \in \mathbb{N}}$  built with this method form a mesh of  $\mathbb{R}^2$ . In  $\mathbb{R}^3$ , there is no such general procedure. It is nevertheless quite simple to build a dual mesh for a structured tetrahedral mesh.

Let us define the discrete functional spaces

$$\begin{aligned} V_h &:= \{v_h \in C^0(\mathbb{R}^d); v_h|_K \text{ is affine for each } K \in \mathcal{T}_h\}, \\ M_h &:= \{v_h \in L^\infty(\mathbb{R}^d); v_h|_Q \text{ is constant for each } Q \in \mathcal{D}_h\}, \\ \Lambda_h &:= \{\mu_h \in L^\infty(\mathbb{R}^d)^d; \mu_h|_K \text{ is constant for each } K \in \mathcal{T}_h\}. \end{aligned}$$

In our scheme, the multiplier  $\lambda$  is approximated in  $\Lambda_h$  and the velocity  $u$  is approximated alternately in  $V_h$  (finite element step) and in  $M_h$  (finite volume step). To handle easily this double discretization, we define the discrete velocity as a collection of values  $(u_p)_{p \in \mathbb{N}}$ , to which we associate the functions  $\hat{u}_h \in V_h$  and  $\bar{u}_h \in M_h$  such that

$$\hat{u}_h(x_p) = u_p, \quad \text{and} \quad \bar{u}_h|_{Q_p} = u_p \quad \forall p \in \mathbb{N}.$$

The time interval  $[0, T]$  is divided into  $N$  equal intervals of length  $\delta t$  and the approximate velocity at time  $t^n := n\delta t$  is denoted by  $(u_p^n)_{p \in \mathbb{N}}$ . The initial discrete velocity  $(u_p^0)_{p \in \mathbb{N}}$  can be built with various interpolation procedures, depending on the regularity of  $u_{\text{ini}}$ . For now, we only assume that the initial discrete velocity has a compact support.

In the finite element step, the Sgn function is approximated by the set-valued map  $\text{Sgn}_h : \Lambda_h \rightarrow \mathcal{P}(\Lambda_h)$  such that

$$\lambda_h \in \text{Sgn}_h(\mu_h) \Leftrightarrow \begin{cases} |\lambda_{h|K}| \leq 1 & \text{for } K \in \mathcal{T}_h \text{ such that } \mu_{h|K} = 0 \\ \lambda_{h|K} = \frac{\mu_{h|K}}{|\mu_{h|K}|} & \text{for } K \in \mathcal{T}_h \text{ such that } \mu_{h|K} \neq 0 \end{cases}$$

Note that, for all  $\mu_h \in \Lambda_h$  and  $\lambda_h \in \text{Sgn}_h(\mu_h)$ ,

$$\int_{\mathbb{R}^d} \lambda_h \cdot \mu_h \, dx = \int_{\mathbb{R}^d} |\mu_h| \, dx.$$

We also need a lumped  $L^2$  scalar product on  $V_h$  to guarantee the maximum principle. Let  $(\cdot, \cdot)_h$  be the scalar product on  $V_h$  defined by

$$(v_h, w_h)_h := \sum_{p \in \mathbb{N}} \omega_p v_h(x_p) w_h(x_p),$$

with  $\omega_p := \sum_{K \in \mathcal{T}_{h,p}} |K| / (d+1)$ ,  $\mathcal{T}_{h,p}$  being the set of elements for which  $x_p$  is a vertex.

**REMARK 2.3** In  $\mathbb{R}^2$ , for a dual mesh built as explained in Remark 2.2, we have  $\omega_p = m_p$ . Indeed, for any node  $x_p$  of a triangle  $K \in \mathcal{T}_h$ , the area of  $Q_p \cap K$  is equal to the third of the area of  $K$ . This follows from properties of the centroid and medians in a triangle.

For the finite volume step we need to define numerical fluxes between neighbor cells at each time step. The numerical flux from the cell  $Q_p$  to the cell  $Q_q$  at time  $t^n$  is denoted by  $F_{p,q}^n$ . We require that the family of numerical fluxes  $\{F_{p,q}^n\}$  is admissible and consistent with the flux  $F$  in the sense of the two definitions below. The Godunov scheme (see, e.g., LeVeque, 2002) provides, for instance, such numerical fluxes.

**DEFINITION 2.1** A family of numerical fluxes  $\{F_{p,q}^n\}$  is said to be admissible if

- $F_{p,q}^n \in C^2([a_0, b_0]^2, \mathbb{R})$ ,
- $F_{p,q}^n$  is monotone, i.e. nondecreasing with respect to its first variable and nonincreasing with respect to its second variable,
- $F_{p,q}^n$  is conservative, i.e.  $F_{p,q}^n(u, v) = -F_{q,p}^n(v, u)$  for all  $(u, v) \in [a_0, b_0]^2$ .

**DEFINITION 2.2** Let  $F$  be a flux function. A family of numerical fluxes  $\{F_{p,q}^n\}$  is said to be consistent with  $F$  if

$$F_{p,q}^n(u, u) = \frac{1}{\delta t} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} F(\gamma, t, u) \cdot \nu_{p,q} \, d\gamma dt, \quad \forall u \in [a_0, b_0]. \quad (2.1)$$

The following lemma, which is a discrete version of the divergence theorem, will be used in several places in the rest of the article.

**LEMMA 2.1** Let  $\{F_{p,q}^n\}$  be a family of numerical fluxes consistent with a flux function  $F$ . If  $F$  is divergence-free, then

$$\sum_{q \in \mathcal{N}_p} F_{p,q}(u, u) = 0. \quad (2.2)$$

To guarantee the stability of the finite volume step, which is explicit, the time step must satisfy a CFL condition. Let  $L$  be a constant such that, for all  $(p, q) \in \mathcal{E}_h$  and for all  $n \in \{0, \dots, N-1\}$ , the function  $F_{p,q}^n$  is Lipschitz continuous with the constant  $m_{p,q}L$  with respect to each of its variables. We prescribe the CFL condition

$$\delta t \leq \frac{1}{2L} \inf_{p \in \mathbb{N}} m_p. \quad (2.3)$$

## 2.2 Description of the numerical scheme and well-posedness

The scheme for solving (0.2)-(0.3) is initialized with the velocity  $(u_p^0)_{p \in \mathbb{N}}$ . The velocity  $(u_p^{n+1})_{p \in \mathbb{N}}$  is computed in two steps.

- Finite volume step. Seek  $(u_p^{n+\frac{1}{2}})_{p \in \mathbb{N}}$  such that

$$m_p \frac{u_p^{n+\frac{1}{2}} - u_p^n}{\delta t} + \sum_{q \in \mathcal{N}_p} F_{p,q}^n(u_p^n, u_q^n) = 0, \quad \forall p \in \mathbb{N}. \quad (2.4)$$

- Finite element step. Seek  $\hat{u}_h^{n+1} \in V_h$  and  $\lambda_h^{n+1} \in \Lambda_h$  such that

$$\left( \frac{\hat{u}_h^{n+1} - \hat{u}_h^{n+\frac{1}{2}}}{\delta t}, v_h \right)_h + \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla v_h \, dx = 0, \quad \forall v_h \in V_h, \quad (2.5)$$

$$\lambda_h^{n+1} \in \text{Sgn}_h(\nabla \hat{u}_h^{n+1}). \quad (2.6)$$

The finite volume step, which is explicit, is obviously well-posed, provided that  $u_p^n \in [a_0, b_0]$  for all  $p \in \mathbb{N}$ . The proposition below proves that the finite element step is also well-posed, provided that  $\hat{u}_h^{n+\frac{1}{2}} \in BV(\mathbb{R}^d)$ , and gives a variational characterization of  $\hat{u}_h^{n+1}$ .

**PROPOSITION 2.3** Let  $n \in \{0, \dots, N-1\}$ . Assume  $\hat{u}_h^{n+\frac{1}{2}} \in BV(\mathbb{R}^d)$ . Then, equations (2.5)-(2.6) admit a solution  $(\hat{u}_h^{n+1}, \lambda_h^{n+1}) \in V_h \times \Lambda_h$ . Furthermore,  $\hat{u}_h^{n+1}$  is unique and is the minimizer of the functional  $J_h^{n+1} : V_h \rightarrow \mathbb{R}$  defined by

$$J_h^{n+1}(v_h) := \frac{1}{2\delta t} \sum_{p \in \mathbb{N}} \omega_p \left( v_h(x_p) - u_p^{n+\frac{1}{2}} \right)^2 + \int_{\mathbb{R}^d} |\nabla v_h| \, dx. \quad (2.7)$$

*Proof.* Define the convex-concave function  $\mathcal{L}^{n+1} : V_h \times \Lambda_h \rightarrow \overline{\mathbb{R}}$  such that

$$\mathcal{L}_h^{n+1}(v_h, \mu_h) := \frac{1}{2\delta t} \sum_{p \in \mathbb{N}} \omega_p \left( v_h(x_p) - u_p^{n+\frac{1}{2}} \right)^2 + \int_{\mathbb{R}^d} \mu_h \cdot \nabla v_h \, dx - \chi(\mu_h),$$

where  $\chi(\mu_h) = 0$  if  $|\mu_h| \leq 1$  and  $\chi(\mu_h) = +\infty$  otherwise. Equations (2.5)-(2.6) are equivalent to

$$\begin{aligned} u_h^{n+1} &= \arg \min_{v_h \in V_h} \mathcal{L}_h^{n+1}(v_h, \lambda_h^{n+1}), \\ \lambda_h^{n+1} &= \arg \max_{\mu_h \in \Lambda_h} \mathcal{L}_h^{n+1}(u_h^{n+1}, \mu_h). \end{aligned}$$

Consequently, any solution of (2.5)-(2.6) is a saddle-point of  $\mathcal{L}^{n+1}$ , and conversely. Since the function  $\mathcal{L}^{n+1}$  is proper, closed and convex-concave, it admits at least one saddle-point (see, e.g., Rockafellar, 1970), and there exists at least one solution of (2.5)-(2.6). Furthermore, we notice that

$$\arg \min_{v_h \in V_h} \mathcal{L}_h^{n+1}(v_h, \lambda_h^{n+1}) = \arg \min_{v_h \in V_h} J_h^{n+1}(v_h),$$

which implies that  $u_h^{n+1}$  is also the unique minimizer of the strictly convex functional  $J_h^{n+1}$ .  $\square$

REMARK 2.4 Solving the saddle-point problem (2.5)-(2.6) or, which is equivalent, minimizing the functional  $J_h^{n+1}$  is not an easy task. For instance, if one tries to minimize  $J_h^{n+1}$  (or more precisely a regularization of  $J_h^{n+1}$ , since  $J_h^{n+1}$  is nondifferentiable) with conventional algorithms, such as the gradient or Newton algorithms, one obtains very slow convergences. Motivated by the applications in image denoising where this kind of problem arises, specific algorithms have been developed since the 1990s. Let us mention in particular the lagged diffusivity algorithm (Vogel & Oman, 1996), saddle-point algorithms (Chambolle & Pock, 2011; Bartels, 2012) or semismooth Newton methods (Hintermüller & Kunisch, 2004).

The following propositions prove that the finite volume step and the finite element step satisfy the maximum and minimum principles and that the finite element step does not expand the support of the solution.

PROPOSITION 2.4 Let  $n \in \{0, \dots, N-1\}$ . Assume that  $a_0 \leq u_p^n \leq b_0$ , for all  $p \in \mathbb{N}$ . Then,  $a_0 \leq u_p^{n+\frac{1}{2}} \leq b_0$ , for all  $p \in \mathbb{N}$ .

*Proof.* From (2.5) and (2.2), it follows

$$u_p^{n+\frac{1}{2}} = u_p^n - \frac{\delta t}{m_p} \sum_{q \in \mathcal{N}_p, u_p^n \neq u_q^n} \frac{F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)}{u_p^n - u_q^n} (u_p^n - u_q^n).$$

This relation shows that  $u_p^{n+\frac{1}{2}}$  is a convex combination of  $u_p^n$  and  $(u_q^n)_{q \in \mathcal{N}_p}$  (the nonnegativity of the coefficients of the combination is ensured by the monotony of the numerical fluxes and the CFL condition). Therefore,  $a_0 \leq u_p^{n+\frac{1}{2}} \leq b_0$ , for all  $p \in \mathbb{N}$ .  $\square$

PROPOSITION 2.5 Let  $n \in \{0, \dots, N-1\}$ . Assume  $\hat{u}_h^{n+\frac{1}{2}} \in BV(\mathbb{R}^d)$  and  $a_0 \leq u_p^{n+\frac{1}{2}} \leq b_0$  for all  $p \in \mathbb{N}$ . Then  $a_0 \leq \hat{u}_h^{n+1} \leq b_0$ .

*Proof.* Assume that there is at least one  $p \in \mathbb{N}$  such that  $u_p^{n+1} \notin [a_0, b_0]$ . Define  $v_h \in V_h$  such that

$$v_h(x_p) := \begin{cases} u_p^{n+1} & \text{if } a_0 \leq u_p^{n+1} \leq b_0 \\ b_0 & \text{if } u_p^{n+1} > b_0 \\ a_0 & \text{if } u_p^{n+1} < a_0 \end{cases}.$$

The idea is to prove that  $J_h^{n+1}(v_h) < J_h^{n+1}(\hat{u}_h^{n+1})$ , which is a contradiction since  $\hat{u}_h^{n+1}$  is defined as the minimizer of  $J_h^{n+1}$ . Let  $K \in \mathcal{T}_h$ . If we denote by  $\{x_p\}_{p \in \mathcal{Y}_K}$  the vertices of  $K$  and by  $\{\phi_p\}_{p \in \mathcal{Y}_K}$  the corresponding Lagrange basis function, we can write

$$\nabla_{v_h|_K} \cdot \nabla \hat{u}_{h|_K}^{n+1} = \sum_{p \in \mathcal{Y}_K} \sum_{q \in \mathcal{Y}_K} v_h(x_p) u_q^{n+1} \nabla \phi_{p|_K} \cdot \nabla \phi_{q|_K}. \quad (2.8)$$

Using the fact that  $\sum_{p \in \mathcal{Y}_K} \phi_{p|_K} = 1$ , and thus  $\sum_{p \in \mathcal{Y}_K} \nabla \phi_{p|_K} = 0$ , (2.8) can be rewritten as

$$\nabla_{v_h|_K} \cdot \nabla \hat{u}_{h|_K}^{n+1} = \sum_{p \in \mathcal{Y}_K} \sum_{q \in \mathcal{Y}_K, q \neq p} u_q^{n+1} (v_h(x_p) - v_h(x_q)) \nabla \phi_{p|_K} \cdot \nabla \phi_{q|_K}. \quad (2.9)$$

Since the simplex  $K$  is nonobtuse, we have the standard inequality (see, e.g., Brandts *et al.*, 2009)

$$\nabla \phi_{p|_K} \cdot \nabla \phi_{q|_K} \leq 0, \quad \forall p, q \in \mathcal{Y}_K, p \neq q. \quad (2.10)$$

Observe that, for all  $p \in \mathbb{N}$ , and for all  $z \in [a_0, b_0]$ ,  $(u_p^{n+1} - v_h(x_p))(z - v_h(x_p)) \leq 0$ . In particular,

$$(u_p^{n+1} - v_h(x_p))(v_h(x_q) - v_h(x_p)) \leq 0, \quad \forall p, q \in \mathcal{V}_K. \quad (2.11)$$

Using (2.10) and (2.11) in (2.9), we find

$$\begin{aligned} \nabla v_{h|K} \cdot \nabla u_{h|K}^{n+1} &\geq \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K, q \neq p} v_h(x_p)(v_h(x_q) - v_h(x_p)) \nabla \phi_{p|K} \cdot \nabla \phi_{q|K} \\ &= \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} v_h(x_p)v_h(x_q) \nabla \phi_{p|K} \cdot \nabla \phi_{q|K} \\ &= |\nabla v_{h|K}|^2 \end{aligned}$$

Applying the Cauchy-Schwarz inequality, we obtain  $|\nabla v_{h|K}| \leq |\nabla u_{h|K}^{n+1}|$ . Hence  $J_h^{n+1}(v_h) < J_h^{n+1}(\hat{u}_h^{n+1})$ , which finishes the proof.  $\square$

**PROPOSITION 2.6** Let  $n \in \{0, \dots, N-1\}$ . Assume  $\hat{u}_h^{n+\frac{1}{2}} \in BV(\mathbb{R}^d)$ . Then,  $\text{supp}(\hat{u}_h^{n+1}) \subset \text{supp}(\hat{u}_h^{n+\frac{1}{2}})$ .

*Proof.* Assume that there is at least one  $p \in \mathbb{N}$  such that  $x_p \notin \text{supp}(\hat{u}_h^{n+\frac{1}{2}})$  and  $u_p^{n+1} \neq 0$ . We define  $v_h \in V_h$  such that  $v_h(x_p) = u_p^{n+1}$  if  $x_p \in \text{supp}(\hat{u}_h^{n+\frac{1}{2}})$  and  $v_h(x_p) = 0$  otherwise. Clearly,  $J_h^{n+1}(v_h) < J_h^{n+1}(\hat{u}_h^{n+1})$ . This contradiction proves that  $\text{supp}(\hat{u}_h^{n+1}) \subset \text{supp}(\hat{u}_h^{n+\frac{1}{2}})$ .  $\square$

Combining the above results and the fact that a finite volume step keeps the support of the discrete solution compact, we can conclude that the scheme is globally well-posed and satisfies the maximum and minimum principles.

**PROPOSITION 2.7** For all  $k \in \{0, \frac{1}{2}, \dots, N - \frac{1}{2}, N\}$ , the discrete velocity  $(u_p^k)_{p \in \mathbb{N}}$  is well-defined and satisfies

$$a_0 \leq u_p^k \leq b_0, \quad \forall p \in \mathbb{N}. \quad (2.12)$$

### 3. A priori estimates on the approximate solutions

In this section and the next one, we consider a family of discretizations  $\{\mathcal{F}_{h,\delta t}\}$  – by discretization, we mean a finite element mesh  $\mathcal{T}_h$ , a finite volume mesh  $\mathcal{D}_h$ , a time step  $\delta t$ , a family of numerical fluxes  $\{F_{p,q}^n\}$ , and a discrete initial condition  $(u_p^0)_{p \in \mathbb{N}}$ . We assume that the following conditions are satisfied uniformly by  $\{\mathcal{F}_{h,\delta t}\}$ .

- (C1) There exists  $\alpha > 0$  such that  $m_p \geq \alpha h^d$  and  $|\partial Q_p| \leq \frac{1}{\alpha} h^{d-1}$  for all  $p \in \mathbb{N}$ .
- (C2) There exists a constant  $\beta$  such that  $\text{diam}(Q_p) \leq \beta h$  for all  $p \in \mathbb{N}$ , where  $\text{diam}(Q_p)$  is the diameter of  $Q_p$ .
- (C3) There exists  $\rho > 0$  such that  $|K| \geq \rho h^d$ , for all  $K \in \mathcal{T}_h$ .
- (C4) For all  $R > 0$ , there exists a constant  $c_R$  such that, for any  $v_h \in V_h$  with support in  $B(0, R)$ ,

$$\left| \sum_{p \in \mathbb{N}} (\omega_p - m_p) v_h(x_p) \right| \leq c_R h \|v_h\|_{L^\infty(\mathbb{R}^d)}. \quad (3.1)$$

(C5) There exists a constant  $L$  such that, for all  $(p, q) \in \mathcal{E}_h$  and for all  $n \in \{0, \dots, N-1\}$ , the function  $F_{p,q}^n$  is Lipschitz continuous with respect to each of its variables with the constant  $m_{p,q}L$ .

(C6) There exists  $\zeta > 0$  such that  $\delta t \geq \zeta h$ .

(C7) The time step satisfies the CFL condition

$$\delta t \leq \frac{\alpha h^d}{2L}. \quad (3.2)$$

(C8) There exist constants  $R^0$  and  $M^0$  such that  $\text{supp}(\hat{u}_h^0) \subset B(0, R^0)$  and  $\int_{\mathbb{R}^d} |\nabla \hat{u}_h^0| dx \leq M^0$ .

For each discretization  $\mathcal{F}_{h,\delta t}$ , we define the approximate solutions  $\hat{u}_{h,\delta t} : Q_T \rightarrow \mathbb{R}$ ,  $\bar{u}_{h,\delta t} : Q_T \rightarrow \mathbb{R}$ , and  $\lambda_{h,\delta t} : Q_T \rightarrow \mathbb{R}$  such that

$$\begin{aligned} \hat{u}_{h,\delta t}(\cdot, t) &:= \hat{u}_h^{n+1} && \text{if } t \in (t^n, t^{n+1}], \\ \bar{u}_{h,\delta t}(\cdot, t) &:= \bar{u}_h^n && \text{if } t \in [t^n, t^{n+1}), \\ \lambda_{h,\delta t}(\cdot, t) &:= \lambda_h^{n+1} && \text{if } t \in (t^n, t^{n+1}]. \end{aligned}$$

A  $L^\infty(Q_T)$  estimate on the approximate velocities  $\{\hat{u}_{h,\delta t}\}$  has already been proved in Proposition 2.7. The aim of this section is to establish additional estimates on  $\{\hat{u}_{h,\delta t}\}$ , namely an estimate on the support size, a  $L^1(0, T; BV(\mathbb{R}^d))$  estimate, and  $L^1(Q_T)$  estimates on the space and time translates. The estimates on the space and time translates are deduced from the  $L^1(0, T; BV(\mathbb{R}^d))$  estimate.

REMARK 3.1 Hypotheses (C1) and (C2) imply that each cell of  $\mathcal{D}_h$  has a finite number of neighbors (and this number is bounded independently of  $\mathcal{F}_{h,\delta t}$ ). The inverse CFL condition (C6) is needed to control the expansion of the support of the discrete solution and, together with (C4), the accumulation of errors due to the interpolation steps. Observe also that the CFL condition (C7) is slightly more stringent than the CFL condition (2.3).

REMARK 3.2 Throughout this section and the next one,  $C$  denotes a generic constant independent of the discretization  $\mathcal{F}_{h,\delta t}$ .

### 3.1 Estimate on the support size

PROPOSITION 3.1 There exists a constant  $R$ , independent of  $\mathcal{F}_{h,\delta t}$ , such that, for all  $k \in \{0, \frac{1}{2}, \dots, N - \frac{1}{2}, N\}$ ,

$$\text{supp} \hat{u}_h^k \subset B(0, R). \quad (3.3)$$

*Proof.* By hypothesis (C8),  $\text{supp}(\hat{u}_h^0) \subset B(0, R^0)$ . Let  $n \in \{0, \dots, N-1\}$ . Assume that there is a constant  $R^n$  such that  $\text{supp}(\hat{u}_h^n) \subset B(0, R^n)$ . Then, a simple computation yields  $\text{supp}(\hat{u}_h^{n+\frac{1}{2}}) \subset B(0, R^n + (2\beta + 1)h)$  and thus  $\text{supp}(\hat{u}_h^{n+1}) \subset B(0, R^n + (2\beta + 1)h)$ , according to Proposition 2.6. By induction, using the inverse CFL condition (C6), we conclude that there exists a constant  $R$ , depending only on  $R^0$ ,  $T$ ,  $\beta$  and  $\zeta$ , such that  $\text{supp} \hat{u}_h^k \subset B(0, R)$  for all  $k \in \{0, \frac{1}{2}, \dots, N - \frac{1}{2}, N\}$ .  $\square$

3.2  $L^1(0, T; BV(\mathbb{R}^d))$  estimate

PROPOSITION 3.2 There exists a constant  $C$ , independent of  $\mathcal{F}_{h, \delta t}$ , such that

$$\|\hat{u}_{h, \delta t}\|_{L^1(0, T; BV(\mathbb{R}^d))} = \|\hat{u}_{h, \delta t}\|_{L^1(Q_T)} + \sum_{n=1}^N \delta t \int_{\mathbb{R}^d} |\nabla \hat{u}_h^n| \, dx \leq C. \quad (3.4)$$

*Proof.* 1. We multiply (2.4) by  $\delta t u_p^n$  and subtract  $\sum_{q \in \mathcal{N}_p} F_{p,q}^n(u_p^n, u_q^n)$ , which is zero according to (2.2). Then, we sum over all  $p \in \mathbb{N}$  to find

$$\sum_{p \in \mathbb{N}} m_p u_p^n (u_p^{n+\frac{1}{2}} - u_p^n) + \delta t \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} u_p^n (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) = 0. \quad (3.5)$$

We test (2.5) with  $v_h = \delta t \hat{u}_h^{n+1}$  and, since  $\lambda_h^{n+1} \in \text{Sgn}_h(\nabla u_h^{n+1})$ , we obtain

$$\sum_{p \in \mathbb{N}} \omega_p u_p^{n+1} (u_p^{n+1} - u_p^{n+\frac{1}{2}}) + \delta t \int_{\mathbb{R}^d} |\nabla \hat{u}_h^{n+1}| \, dx = 0. \quad (3.6)$$

Adding (3.5) and (3.6), then rearranging, we find

$$\begin{aligned} & \sum_{p \in \mathbb{N}} m_p \left( \frac{1}{2} (u_p^{n+1})^2 + \frac{1}{2} (u_p^{n+1} - u_p^{n+\frac{1}{2}})^2 - \frac{1}{2} (u_p^{n+\frac{1}{2}} - u_p^n)^2 - \frac{1}{2} (u_p^n)^2 \right) \\ & + \delta t \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} u_p^n (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) + \delta t \int_{\mathbb{R}^d} |\nabla \hat{u}_h^{n+1}| \, dx + \sum_{p \in \mathbb{N}} (\omega_p - m_p) u_p^{n+1} (u_p^{n+1} - u_p^{n+\frac{1}{2}}) = 0. \end{aligned} \quad (3.7)$$

Summing (3.7) over  $n \in \{0, \dots, N-1\}$ , and eliminating the positive terms in the left-hand side, we obtain the inequality

$$\begin{aligned} & - \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} m_p \frac{1}{2} (u_p^{n+\frac{1}{2}} - u_p^n)^2 + \delta t \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) \\ & + \sum_{n=0}^{N-1} \delta t \int_{\mathbb{R}^d} |\nabla \hat{u}_h^{n+1}| \, dx + \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} (\omega_p - m_p) u_p^{n+1} (u_p^{n+1} - u_p^{n+\frac{1}{2}}) \leq \sum_{p \in \mathbb{N}} m_p \frac{1}{2} (u_p^0)^2. \end{aligned} \quad (3.8)$$

The estimates (2.12) and (3.3), together with hypotheses (C4) and (C6), yield the bound

$$\sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} (\omega_p - m_p) u_p^{n+1} (u_p^{n+1} - u_p^{n+\frac{1}{2}}) \leq \sum_{n=0}^{N-1} Ch \leq C. \quad (3.9)$$

Hence

$$\begin{aligned} \sum_{n=1}^N \delta t \int_{\mathbb{R}^d} |\nabla \hat{u}_h^n| \, dx & \leq C + \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} m_p \frac{1}{2} (u_p^{n+\frac{1}{2}} - u_p^n)^2 \\ & \quad - \delta t \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} u_p^n (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)). \end{aligned} \quad (3.10)$$

2. Setting  $\mathcal{E}_h^n := \{(p, q) \in \mathcal{E}_h; u_p^n > u_q^n\}$ , we can write

$$\begin{aligned} \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} u_p^n (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) = \\ \sum_{(p,q) \in \mathcal{E}_h^n} (u_p^n (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) - u_q^n (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_q^n, u_q^n))), \end{aligned} \quad (3.11)$$

where the sum over the cells has been turned into a sum over the interfaces. For all  $(p, q) \in \mathcal{E}_h$ , and for all  $n \in \{0, \dots, N-1\}$ , we define the function  $G_{p,q}^n : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$G_{p,q}^n(x) := \int_0^x s \left( \frac{\partial F_{p,q}^n}{\partial u}(s, s) + \frac{\partial F_{p,q}^n}{\partial v}(s, s) \right) ds = \int_0^x s \frac{d}{ds} (F_{p,q}^n(s, s)) ds, \quad \forall x \in [a_0, b_0]. \quad (3.12)$$

An integration by parts yields, for all  $(a, b) \in [a_0, b_0]^2$ ,

$$G_{p,q}^n(b) - G_{p,q}^n(a) = b (F_{p,q}^n(b, b) - F_{p,q}^n(a, b)) - a (F_{p,q}^n(a, a) - F_{p,q}^n(a, b)) - \int_a^b (F_{p,q}^n(s, s) - F_{p,q}^n(a, b)) ds. \quad (3.13)$$

Hence (3.11) becomes

$$\begin{aligned} \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} u_p^n (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) = \\ \sum_{(p,q) \in \mathcal{E}_h^n} (G_{p,q}^n(u_p^n) - G_{p,q}^n(u_q^n)) + \sum_{(p,q) \in \mathcal{E}_h^n} \int_{u_q^n}^{u_p^n} (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(s, s)) ds. \end{aligned} \quad (3.14)$$

From (2.2), it follows

$$\sum_{(p,q) \in \mathcal{E}_h^n} (G_{p,q}^n(u_p^n) - G_{p,q}^n(u_q^n)) = \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} G_{p,q}^n(u_p^n) = \sum_{p \in \mathbb{N}} \int_0^{u_p^n} s \frac{d}{ds} \left( \sum_{q \in \mathcal{N}_p} F_{p,q}^n(s, s) \right) ds = 0, \quad (3.15)$$

and Lemma 3.1 yields

$$\begin{aligned} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{u_q^n}^{u_p^n} (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(s, s)) ds \geq \\ \frac{\delta t}{4L} \sum_{(p,q) \in \mathcal{E}_h^n} \frac{1}{m_{p,q}} \left( \max_{u_q^n \leq c \leq d \leq u_p^n} (F_{p,q}^n(d, c) - F_{p,q}^n(d, d))^2 + \max_{u_q^n \leq c \leq d \leq u_p^n} (F_{p,q}^n(d, c) - F_{p,q}^n(c, c))^2 \right). \end{aligned} \quad (3.16)$$

Collecting (3.14), (3.15) and (3.16), we deduce the inequality

$$\begin{aligned} \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} u_p^n (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) \geq \\ \frac{\delta t}{4L} \sum_{(p,q) \in \mathcal{E}_h^n} \frac{1}{m_{p,q}} \left( \max_{u_q^n \leq c \leq d \leq u_p^n} (F_{p,q}^n(d, c) - F_{p,q}^n(d, d))^2 + \max_{u_q^n \leq c \leq d \leq u_p^n} (F_{p,q}^n(d, c) - F_{p,q}^n(c, c))^2 \right). \end{aligned} \quad (3.17)$$



3. Using (2.2), the equation (2.4) can be rewritten as

$$\left(u_p^{n+\frac{1}{2}} - u_p^n\right) = \frac{\delta t}{m_p} \left( \sum_{q \in \mathcal{N}_p} \frac{1}{\sqrt{m_{p,q}}} (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) \right). \quad (3.18)$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\left(u_p^{n+\frac{1}{2}} - u_p^n\right)^2 \leq \frac{\delta t^2}{m_p^2} \left( \sum_{q \in \mathcal{N}_p} m_{p,q} \right) \left( \sum_{q \in \mathcal{N}_p} \frac{1}{m_{p,q}} (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n))^2 \right). \quad (3.19)$$

Then, the CFL condition (3.2) and hypothesis (C1) yield the bound

$$\left(u_p^{n+\frac{1}{2}} - u_p^n\right)^2 \leq \frac{\delta t}{2L} \left( \sum_{q \in \mathcal{N}_p} \frac{1}{m_{p,q}} (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n))^2 \right). \quad (3.20)$$

We finally sum the above inequality over all  $p \in \mathbb{N}$  and turn the right-hand side into a sum over the interfaces to find

$$\sum_{p \in \mathbb{N}} \left(u_p^{n+\frac{1}{2}} - u_p^n\right)^2 \leq \frac{\delta t}{2L} \sum_{(p,q) \in \mathcal{E}_h^n} \frac{1}{m_{p,q}} \left( \max_{u_q^n \leq c \leq d \leq u_p^n} (F_{p,q}^n(d, c) - F_{p,q}^n(d, d))^2 + \max_{u_q^n \leq c \leq d \leq u_p^n} (F_{p,q}^n(d, c) - F_{p,q}^n(c, c))^2 \right). \quad (3.21)$$

4. Collecting (3.10), (3.17) and (3.21), we obtain

$$\sum_{n=1}^N \delta t \int_{\mathbb{R}^d} |\nabla \hat{u}_h^n| \, dx \leq C. \quad (3.22)$$

The complete estimate (3.4) follows readily from (2.12).  $\square$

LEMMA 3.1 Let  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ . Assume that  $f$  is nondecreasing and Lipschitz continuous with a constant  $M_1$  with respect to its first variable and nonincreasing and Lipschitz continuous with a constant  $M_2$  with respect to its second variable. Then, for all  $(a, b) \in \mathbb{R}^2$  such that  $a \leq b$ ,

$$\int_a^b (f(b, a) - f(s, s)) \, ds \geq \frac{1}{2(M_1 + M_2)} \left( \max_{a \leq c \leq d \leq b} (f(d, c) - f(d, d))^2 + \max_{a \leq c \leq d \leq b} (f(d, c) - f(c, c))^2 \right).$$

*Proof.* See proof of Lemma 25.2 in Eymard *et al.* (2000).  $\square$

### 3.3 Time translate estimate

PROPOSITION 3.3 There exists a constant  $C$ , independent of  $\mathcal{F}_{h, \delta t}$ , such that

$$\int_0^{T-s} \int_{\mathbb{R}^d} |\hat{u}_{h, \delta t}(x, t+s) - \hat{u}_{h, \delta t}(x, t)| \, dx dt \leq Cs, \quad \forall s \in [0, T]. \quad (3.23)$$

*Proof.* 1. Let  $s \in [0, T]$ . Using the Cauchy-Schwarz inequality, and remembering that  $\text{supp}(\hat{u}_{h,\delta t}) \subset B(0, R)$ , we have

$$\int_0^{T-s} \int_{\mathbb{R}^d} |\hat{u}_{h,\delta t}(x, t+s) - \hat{u}_{h,\delta t}(x, t)| \, dx dt \leq \left( 2\pi RT \int_0^{T-s} \int_{\mathbb{R}^d} (\hat{u}_{h,\delta t}(x, t+s) - \hat{u}_{h,\delta t}(x, t))^2 \, dx dt \right)^{\frac{1}{2}}. \quad (3.24)$$

We define the function  $v : (0, T) \rightarrow \mathbb{N}$  such that  $v(t) = n+1$  if  $t \in (t^n, t^{n+1}]$  and the function  $\chi_n(t, s) : (0, T) \times (0, T) \rightarrow \{0, 1\}$  such that  $\chi_n(t, s) = 1$  if  $v(t) \leq n < v(t+s)$  and  $\chi_n(t, s) = 0$  otherwise; so that

$$\begin{aligned} \int_0^{T-s} \int_{\mathbb{R}^d} (\hat{u}_{h,\delta t}(x, t+s) - \hat{u}_{h,\delta t}(x, t))^2 \, dx dt &= \int_0^{T-s} \int_{\mathbb{R}^d} (\hat{u}_h^{v(t+s)} - \hat{u}_h^{v(t)})^2 \, dx dt \\ &= \int_0^{T-s} \int_{\mathbb{R}^d} (\hat{u}_h^{v(t+s)} - \hat{u}_h^{v(t)}) \sum_{n=1}^{N-1} \chi_n(t, s) (\hat{u}_h^{n+1} - \hat{u}_h^n) \, dx dt. \end{aligned} \quad (3.25)$$

2. We multiply (2.4) by  $\delta t u_p^{v(t+s)}$  and sum over all  $p \in \mathbb{N}$ :

$$\sum_{p \in \mathbb{N}} m_p u_p^{v(t+s)} \left( u_p^{n+\frac{1}{2}} - u_p^n \right) + \delta t \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} u_p^{v(t+s)} (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)). \quad (3.26)$$

We test (2.5) with  $v_h = \delta t \hat{u}_h^{v(t+s)}$ :

$$\sum_{p \in \mathbb{N}} \omega_p u_p^{v(t+s)} \left( u_p^{n+1} - u_p^{n+\frac{1}{2}} \right) + \delta t \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla \hat{u}_h^{v(t+s)} \, dx = 0. \quad (3.27)$$

Adding the above equalities (3.26) and (3.27), and rearranging, we find

$$\begin{aligned} \sum_{p \in \mathbb{N}} m_p u_p^{v(t+s)} (u_p^{n+1} - u_p^n) + \delta t \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} u_p^{v(t+s)} (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) \\ + \delta t \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla \hat{u}_h^{v(t+s)} \, dx + \sum_{p \in \mathbb{N}} (\omega_p - m_p) u_p^{v(t+s)} \left( u_p^{n+1} - u_p^{n+\frac{1}{2}} \right) = 0. \end{aligned} \quad (3.28)$$

3. Let  $p \in \mathbb{N}$ ,  $q \in \mathcal{N}_p$  and  $K$  an element of  $\mathcal{T}_h$  for which  $x_p$  and  $x_q$  are vertices. In view of (C1) and (C5),

$$|F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)| \leq 2m_{p,q}L|u_p^n - u_q^n| \leq 2m_{p,q}Lh|\nabla \hat{u}_h^n| \leq Ch^d |\nabla \hat{u}_h^n|_K. \quad (3.29)$$

Therefore, owing to (C3) and the finite number of neighbors in  $\mathcal{T}_h$ ,

$$\left| \sum_{p \in \mathbb{N}} \sum_{q \in \mathcal{N}_p} (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) \right| \leq C \int_{\mathbb{R}^d} |\nabla \hat{u}_h^n| \, dx. \quad (3.30)$$

The property  $|\lambda_h^{n+1}| \leq 1$  gives immediately

$$\left| \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla \hat{u}_h^{v(t+s)} \, dx \right| \leq \int_{\mathbb{R}^d} |\nabla \hat{u}_h^{v(t+s)}| \, dx. \quad (3.31)$$

Estimates (2.12) and (3.3), together with hypothesis (C4), yield the bound

$$\sum_{p \in \mathbb{N}} (\omega_p - m_p) u_p^{v(t+s)} \left( u_p^{n+1} - u_p^{n+\frac{1}{2}} \right) \leq Ch. \quad (3.32)$$

From (3.28), (3.30), (3.31) and (3.32), it follows

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \hat{u}_h^{v(t+s)} (\hat{u}_h^{n+1} - \hat{u}_h^n) dx \right| &\leq Ch + \left| \sum_{p \in \mathbb{N}} m_p u_p^{v(t+s)} (u_p^{n+1} - u_p^n) \right| \\ &\leq C \left( h + \delta t \int_{\mathbb{R}^d} |\nabla \hat{u}_h^n| dx + \delta t \int_{\mathbb{R}^d} |\nabla \hat{u}_h^{v(t+s)}| dx \right). \end{aligned} \quad (3.33)$$

4. Using (C6) and the above estimate (3.33), we have

$$\left| \int_0^{T-s} \int_{\mathbb{R}^d} \hat{u}_h^{v(t+s)} \sum_{n=1}^{N-1} \chi_n(t, s) (\hat{u}_h^{n+1} - \hat{u}_h^n) dx dt \right| \leq C \sum_{n=1}^N \delta t \left( 1 + \int_{\mathbb{R}^d} |\nabla \hat{u}_h^n| dx \right) \int_0^{T-s} \chi_n(t, s) dt$$

For a given  $n \in \mathbb{N}$  and a given  $s \in [0, T]$ , we note that  $\chi_n(t, s) = 0$  for  $t \notin (t^n - s, t^n]$ , and thus  $\int_0^{T-s} \chi_n(t, s) dt \leq s$ . Hence, recalling (3.4),

$$\left| \int_0^{T-s} \int_{\mathbb{R}^d} \hat{u}_h^{v(t+s)} \sum_{n=1}^{N-1} \chi_n(t, s) (\hat{u}_h^{n+1} - \hat{u}_h^n) dx dt \right| \leq Cs \quad (3.34)$$

Similarly, we can derive

$$\left| \int_0^{T-s} \int_{\mathbb{R}^d} \hat{u}_h^{v(t)} \sum_{n=1}^{N-1} \chi_n(t, s) (\hat{u}_h^{n+1} - \hat{u}_h^n) dx dt \right| \leq Cs. \quad (3.35)$$

4. Collecting (3.25), (3.34) and (3.35), we obtain the desired estimate (3.23).  $\square$

### 3.4 Space translate estimate

PROPOSITION 3.4 There exists a constant  $C$ , independent of  $\mathcal{F}_{h, \delta t}$ , such that

$$\int_0^T \int_{\mathbb{R}^d} |\hat{u}_{h, \delta t}(x+y, t) - \hat{u}_{h, \delta t}(x, t)| dt dx \leq C|y|, \quad \forall y \in \mathbb{R}^d. \quad (3.36)$$

*Proof.* For a given element  $K \in \mathcal{F}_h$  and a given couple of points  $(a, b) \in \mathbb{R}^d \times \mathbb{R}^d$ , we denote by  $\chi_K(a, b)$  the length of the segment  $[a, b] \cap \bar{K}$ . In particular, if  $[a, b]$  does not intersect  $\bar{K}$ , then  $\chi_K(a, b) = 0$ . Let  $n \in \{1, \dots, N\}$  and  $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ . By applying the mean value theorem on each element intersected by the segment  $[x, x+y]$ , we find the inequality

$$|\hat{u}_h^n(x+y) - \hat{u}_h^n(x)| \leq \sum_{K \in \mathcal{F}_h} \chi_K(x, x+y) |\nabla \hat{u}_{h|K}^n|. \quad (3.37)$$

Next, an integration with respect to  $x$  yields

$$\int_{\mathbb{R}^d} |\hat{u}_h^n(x+y) - \hat{u}_h^n(x)| dx \leq \sum_{K \in \mathcal{F}_h} |\nabla \hat{u}_{h|K}^n| \int_{\mathbb{R}^d} \chi_K(x, x+y) dx. \quad (3.38)$$

For any  $K \in \mathcal{T}_h$ , the function  $x \mapsto \chi_K(x, x+y)$  is bounded by  $\min(h, |y|)$  and is zero outside a domain of measure lower than  $h^{d-1}(h+|y|)$ . Therefore,

$$\int_{\mathbb{R}^d} \chi_K(x, x+y) dx \leq h^{d-1}(h+|y|) \min(h, |y|) \leq 2h^d |y|, \quad (3.39)$$

and

$$\int_{\mathbb{R}^d} |\hat{u}_h^n(x+y) - \hat{u}_h^n(x)| dx \leq C|y| \sum_{K \in \mathcal{T}_h} h^d |\nabla \hat{u}_h^n|_K. \quad (3.40)$$

Using the hypothesis (C3), we obtain

$$\int_{\mathbb{R}^d} \sum_{K \in \mathcal{T}_h} h^d |\nabla \hat{u}_h^n|_K \leq C \int_{\mathbb{R}^d} |\nabla \hat{u}_h^n| dx. \quad (3.41)$$

Summing the above inequality (3.41) over  $\{1, \dots, N\}$ , and using (3.4), we find the desired estimate (3.36).  $\square$

#### 4. Entropy formulation for the approximate solutions

The aim of this section is to establish an entropy formulation, similar to (1.2), for the approximate solutions. We first prove a discrete entropy inequality for the finite volume step (Proposition 4.1). Then, we take into account the finite element step to obtain the complete discrete entropy formulation (Proposition 4.2). Error terms occur in this formulation. Proposition 4.3 ensures that they tend to zero when the meshsize and the time step tend to zero.

**PROPOSITION 4.1** Let  $(\eta, \Phi)$  be an entropy-entropy flux pair. Then, there exists a family  $\{\Phi_{p,q}^n\}$  of admissible numerical fluxes, consistent with  $\Phi$ , such that

$$\frac{\eta(u_p^{n+\frac{1}{2}}) - \eta(u_p^n)}{\delta t} + \frac{1}{m_p} \sum_{q \in \mathcal{N}_p} \Phi_{p,q}^n(u_p^n, u_q^n) \leq 0, \quad \forall n \in \{0, \dots, N-1\}, \quad \forall p \in \mathbb{N}. \quad (4.1)$$

Furthermore, there is a constant  $L'$ , depending only on  $L, \eta, a_0$  and  $b_0$ , such that, for all  $(p, q) \in \mathcal{E}_h$  and for all  $n \in \{0, \dots, N-1\}$ , the function  $\Phi_{p,q}^n$  is Lipschitz continuous with respect of each of its variables with the constant  $m_{p,q} L'$ .

*Proof.* 1. Consider first the Kruzhkov entropies, that is the functions  $|\cdot - \kappa|$ , with  $\kappa \in \mathbb{R}$ . The corresponding entropy fluxes are the functions  $F(\cdot \top \kappa) - F(\cdot \perp \kappa)$ , where  $a \top b$  denotes the maximum of  $a$  and  $b$  and  $a \perp b$  denotes the minimum of  $a$  and  $b$ . The derivation of the discrete entropy inequality involving the Kruzhkov entropies for the finite volume scheme (2.4) is standard and can be found for instance in Chainais-Hillairet (1999, Lemma 3) or Eymard *et al.* (2000, Lemma 27.1). This inequality reads

$$\frac{|u_p^{n+\frac{1}{2}} - \kappa| - |u_p^n - \kappa|}{\delta t} + \frac{1}{m_p} \sum_{q \in \mathcal{N}_p} (F_{p,q}^n(u_p^n \top \kappa, u_q^n \top \kappa) - F_{p,q}^n(u_p^n \perp \kappa, u_q^n \perp \kappa)) \leq 0, \quad (4.2)$$

for all  $n \in \{0, \dots, N-1\}$  and  $p \in \mathbb{N}$ .

2. Let  $(\eta, \Phi)$  be an entropy-entropy flux pair. Using simply the fact that  $\eta \in C^2(\mathbb{R})$ , we have the identity

$$\eta(x) = \frac{1}{2} \int_{a_0}^{b_0} \eta''(\kappa) |x - \kappa| d\kappa + \frac{\eta'(a_0) + \eta'(b_0)}{2} x + c, \quad \forall x \in [a_0, b_0], \quad (4.3)$$

where  $c$  is a constant depending on  $\eta$ . Using this identity, together with (4.2) and (2.4), we deduce the inequality

$$\frac{\eta(u_p^{n+\frac{1}{2}}) - \eta(u_p^n)}{\delta t} + \frac{1}{m_p} \sum_{q \in \mathcal{N}_p} \tilde{\Phi}_{p,q}^n(u_p^n, u_q^n) \leq 0, \quad (4.4)$$

where  $\tilde{\Phi}_{p,q}^n$  is a numerical flux defined by

$$\tilde{\Phi}_{p,q}^n(x, y) := \frac{1}{2} \int_{a_0}^{b_0} \eta''(\kappa) (F_{p,q}^n(x \top \kappa, y \top \kappa) - F_{p,q}^n(x \perp \kappa, y \perp \kappa)) d\kappa + \frac{\eta'(a_0) + \eta'(b_0)}{2} F_{p,q}^n(x, y). \quad (4.5)$$

3. We next define the continuous flux  $\tilde{\Phi}$  such that

$$\tilde{\Phi}(x, t, u) := \frac{1}{2} \int_{a_0}^{b_0} \eta''(\kappa) (F(x, t, u \top \kappa) - F(x, t, u \perp \kappa)) d\kappa + \frac{\eta'(a_0) + \eta'(b_0)}{2} F(x, t, u). \quad (4.6)$$

After some calculations, we find

$$\begin{aligned} \tilde{\Phi}(x, t, u) &= \frac{1}{2} \int_{a_0}^u \eta'(\kappa) \frac{\partial F}{\partial u}(x, t, \kappa) d\kappa + \frac{1}{2} \int_{b_0}^u \eta'(\kappa) \frac{\partial F}{\partial u}(x, t, \kappa) d\kappa \\ &\quad + \frac{1}{2} (\eta'(a_0)F(x, t, a_0) + \eta'(b_0)F(x, t, b_0)), \end{aligned}$$

and thus  $\frac{\partial \tilde{\Phi}}{\partial u}(x, t, u) = \eta'(u) \frac{\partial F}{\partial u}(x, t, u)$ . Therefore, by definition of the entropy flux, there is a continuous function  $c : \mathbb{R}^d \times (0, T) \rightarrow \mathbb{R}$  such that  $\Phi(x, t, u) = \tilde{\Phi}(x, t, u) + c(x, t)$ . Let  $\{\Phi_{p,q}^n\}$  be the family of numerical fluxes such that

$$\Phi_{p,q}^n := \tilde{\Phi}_{p,q}^n + \frac{1}{\delta t} \int_t^{t^{n+1}} \int_{\sigma_{p,q}} c(\gamma, t) \cdot \nu_{p,q} d\gamma dt. \quad (4.7)$$

It is easy to verify that this family of numerical fluxes is admissible, consistent with  $\Phi$ , uniformly Lipschitz continuous and satisfy the entropy inequality (4.1).  $\square$

**PROPOSITION 4.2** Let  $(\eta, \Phi)$  be an entropy-entropy flux pair. Then, for all nonnegative test functions  $\varphi \in C_c^\infty(Q_T)$ ,

$$\begin{aligned} \int_{Q_T} \eta(\bar{u}_{h,\delta t}) \partial_t \varphi dx dt + \int_{Q_T} \Phi(x, t, \bar{u}_{h,\delta t}) \cdot \nabla \varphi dx dt - \int_{Q_T} \eta'(\hat{u}_{h,\delta t}) \lambda_{h,\delta t} \cdot \nabla \varphi dx dt \\ - \int_{Q_T} \varphi |\nabla \eta'(\hat{u}_{h,\delta t})| dx dt + \int_{\mathbb{R}^d} \eta(\bar{u}_h^0) \varphi(x, 0) dx + e_{h,\delta t}^1 + e_{h,\delta t}^2 + e_{h,\delta t}^3 + e_{h,\delta t}^4 \geq 0, \quad (4.8) \end{aligned}$$

where  $e_{h,\delta t}^1$ ,  $e_{h,\delta t}^2$ ,  $e_{h,\delta t}^3$ , and  $e_{h,\delta t}^4$  are defined by the relations (4.15), (4.17), (4.19), and (4.20), respectively.

*Proof.* 1. Let  $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}_+)$ . We set  $v^{n+1} := \eta'(\hat{u}_h^{n+1})\psi$  and  $v_h^{n+1} := I_h v^{n+1}$ , where  $I_h$  denotes the Lagrange interpolation operator on  $V_h$ . We test (2.5) with  $v_h^{n+1}$ :

$$\sum_{p \in \mathbb{N}} \omega_p \frac{u_p^{n+1} - u_p^{n+\frac{1}{2}}}{\delta t} \eta'(u_p^{n+1}) \psi(x_p) + \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla v_h^{n+1} dx = 0. \quad (4.9)$$

Since  $\eta$  is convex,

$$\left(u_p^{n+1} - u_p^{n+\frac{1}{2}}\right) \eta'(u_p^{n+1}) \geq \eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}}) \quad \forall p \in \mathbb{N},$$

and thus (4.9) becomes

$$\sum_{p \in \mathbb{N}} \omega_p \frac{\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})}{\delta t} \psi(x_p) + \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla v_h^{n+1} \, dx \leq 0. \quad (4.10)$$

The second term of (4.10) can be rewritten as

$$\begin{aligned} \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla v_h^{n+1} \, dx &= \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla v^{n+1} \, dx + \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (v_h^{n+1} - v^{n+1}) \, dx \\ &= \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (\eta'(\hat{u}_h^{n+1})) \psi \, dx + \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla \psi \eta'(\hat{u}_h^{n+1}) \, dx + \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (v_h^{n+1} - v^{n+1}) \, dx \end{aligned} \quad (4.11)$$

By convexity of  $\eta$ ,

$$\int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (\eta'(\hat{u}_h^{n+1})) \psi \, dx = \int_{\mathbb{R}^d} \eta''(\hat{u}_h^{n+1}) \lambda_h^{n+1} \cdot \nabla (\hat{u}_h^{n+1}) \psi \, dx \geq 0,$$

whence

$$\int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (\eta'(\hat{u}_h^{n+1})) \psi \, dx = \int_{\mathbb{R}^d} |\nabla \eta'(\hat{u}_h^{n+1})| \psi \, dx. \quad (4.12)$$

Collecting (4.10), (4.11) and (4.12), we obtain

$$\begin{aligned} \sum_{p \in \mathbb{N}} \omega_p \frac{\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})}{\delta t} \psi(x_p) + \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla \psi \eta'(\hat{u}_h^{n+1}) \, dx + \int_{\mathbb{R}^d} |\nabla \eta'(\hat{u}_h^{n+1})| \psi \, dx \\ + \int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla (v_h^{n+1} - v^{n+1}) \, dx \leq 0. \end{aligned} \quad (4.13)$$

Taking  $\psi = \varphi(\cdot, t)$  in the above inequality, then integrating on  $(t^n, t^{n+1})$  and summing over  $n \in \{0, \dots, N-1\}$ , we eventually find

$$\begin{aligned} \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})}{\delta t} \int_{t^n}^{t^{n+1}} \omega_p \varphi(x_p, t) \, dt + \int_{Q_T} \eta'(\hat{u}_{h,\delta t}) \lambda_{h,\delta t} \cdot \nabla \varphi \, dx dt \\ + \int_{Q_T} \varphi |\nabla \eta'(\hat{u}_{h,\delta t})| \, dx dt - e_{h,\delta t}^1 \leq 0, \end{aligned} \quad (4.14)$$

where

$$e_{h,\delta t}^1 := \int_{Q_T} \nabla (\eta'(\hat{u}_{h,\delta t}) \varphi - I_h(\eta'(\hat{u}_{h,\delta t}) \varphi)) \, dx dt. \quad (4.15)$$

2. Multiplying (4.1) by  $\varphi_p^n := \int_{t^n}^{t^{n+1}} \int_{Q_p} \varphi(x, s) \, dx ds$ , then adding to (4.14), we obtain

$$\begin{aligned} \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^n)}{\delta t} \varphi_p^n + \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{1}{m_p} \sum_{q \in \mathcal{N}_p} \Phi_{p,q}^n(u_p^n, u_q^n) \varphi_p^n \\ + \int_{Q_T} \eta'(\hat{u}_{h,\delta t}) \lambda_{h,\delta t} \cdot \nabla \varphi \, dx dt + \int_{Q_T} \varphi |\nabla \eta'(\hat{u}_{h,\delta t})| \, dx dt - e_{h,\delta t}^1 - e_{h,\delta t}^2 \leq 0, \end{aligned} \quad (4.16)$$

where

$$e_{h,\delta t}^2 := \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})}{\delta t} \left( \varphi_p^n - \int_{t^n}^{t^{n+1}} \omega_p \varphi(x_p, t) dt \right). \quad (4.17)$$

Now, observing that

$$\int_{Q_T} \eta(\bar{u}_{h,\delta t}) \partial_t \varphi dx dt = - \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^n)}{\delta t} \int_{t^n}^{t^{n+1}} \int_{Q_p} \varphi(x, t^{n+1}) dx dt - \sum_{p \in \mathbb{N}} \eta(u_p^0) \int_{Q_p} \varphi(x, 0) dx$$

we can rewrite (4.16) as

$$\begin{aligned} & \int_{Q_T} \eta(\bar{u}_{h,\delta t}) \partial_t \varphi dx dt + \int_{Q_T} \Phi(x, t, \bar{u}_{h,\delta t}) \cdot \nabla \varphi dx dt - \int_{Q_T} \eta'(\hat{u}_{h,\delta t}) \lambda_{h,\delta t} \cdot \nabla \varphi dx dt \\ & - \int_{Q_T} \varphi |\nabla \eta'(\hat{u}_{h,\delta t})| dx dt + \sum_{p \in \mathbb{N}} \eta(u_p^0) \int_{Q_p} \varphi(x, 0) dx + e_{h,\delta t}^1 + e_{h,\delta t}^2 + e_{h,\delta t}^3 + e_{h,\delta t}^4 \geq 0, \end{aligned} \quad (4.18)$$

where

$$e_{h,\delta t}^3 := \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^n)}{\delta t} \left( \int_{t^n}^{t^{n+1}} \int_{Q_p} \varphi(x, t^{n+1}) dx dt - \varphi_p^n \right), \quad (4.19)$$

$$e_{h,\delta t}^4 := - \int_{Q_T} \Phi(x, t, \bar{u}_{h,\delta t}) \cdot \nabla \varphi dx dt - \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{1}{m_p} \sum_{q \in \mathcal{N}_p} \Phi_{p,q}^n(u_p^n, u_q^n) \varphi_p^n. \quad (4.20)$$

□

**PROPOSITION 4.3** There exists a constant  $C$ , independent of  $\mathcal{T}_{h,\delta t}$ , such that

$$|e_{h,\delta t}^1| + |e_{h,\delta t}^2| + |e_{h,\delta t}^3| + |e_{h,\delta t}^4| \leq C(h + \delta t). \quad (4.21)$$

*Proof.* 1. Setting  $w(t) := \eta'(\hat{u}_h(t))\varphi(t)$ , we can write  $e_{h,\delta t}^1 = \int_0^T |w(t) - I_h w(t)|_{W^{1,1}(\mathbb{R}^d)} dt$ , where  $|\cdot|_{W^{1,1}(\mathbb{R}^d)}$  denotes the semi-norm of the Sobolev space  $W^{1,1}(\mathbb{R}^d)$ . A standard interpolation result (see, e.g., Ern & Guermond, 2004) ensures that there exists  $c$  depending only on  $\rho$  such that

$$|w(t) - I_h w(t)|_{W^{1,1}(K)} \leq ch |w(t)|_{W^{2,1}(K)}, \quad \forall K \in \mathcal{T}_h.$$

The explicit computation of  $|w(t)|_{W^{2,1}(K)}$  yields

$$|w(t) - I_h w(t)|_{W^{1,1}(K)} \leq Ch \left( h^d + \int_K |\nabla \hat{u}_h(t)| dx \right), \quad \forall K \in \mathcal{T}_h,$$

where the constant  $C$  depends only on  $\eta$ ,  $\phi$ ,  $a_0$ ,  $b_0$  and  $\rho$ . Finally, using hypothesis (C3) and estimate (3.4), we obtain  $|e_{h,\delta t}^1| \leq Ch$ .

2. Since  $\eta$  is  $C^1$  and the discrete velocity is bounded between  $a_0$  and  $b_0$ , we have

$$\begin{aligned} |e_{h,\delta t}^2| & \leq \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} |\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})| \int_{t^n}^{t^{n+1}} \int_{Q_p} |\nabla_x \varphi(x, t)| dx dt \\ & \leq C \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} m_p \delta t |u_p^{n+1} - u_p^{n+\frac{1}{2}}|. \end{aligned}$$

Employing the same arguments as in the proof of Proposition 3.3, we can derive the estimate

$$\sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} m_p |u_p^{n+1} - u_p^{n+\frac{1}{2}}| \leq C.$$

Hence  $|e_{h,\delta t}^2| \leq C\delta t$ . Similarly,  $|e_{h,\delta t}^3| \leq C\delta t$ .

3. Set

$$I := \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{1}{m_p} \sum_{q \in \mathcal{N}_p} \Phi_{p,q}^n(u_p^n, u_q^n) \varphi_p^n \quad \text{and} \quad I^* := - \int_{Q_T} \Phi(x, t, \bar{u}_{h,\delta t}) \cdot \nabla \varphi \, dx dt,$$

so that  $e_{h,\delta t}^4 = -I + I^*$ . From (2.1), it follows  $\sum_{q \in \mathcal{N}_p} \Phi_{p,q}^n(u, u) = 0$  and thus

$$I = \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \frac{1}{m_p} \sum_{q \in \mathcal{N}_p} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_p^n, u_p^n)) \varphi_p^n = I_1 + I_2,$$

where

$$I_1 := \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \frac{1}{m_p} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_p^n, u_p^n)) \varphi_p^n, \quad I_2 := \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \frac{1}{m_q} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_q^n, u_q^n)) \varphi_q^n.$$

Applying the divergence theorem on each cell in the expression of  $I^*$ , we find

$$\begin{aligned} I^* &= - \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}} \int_{t^n}^{t^{n+1}} \int_{\partial Q_p} \Phi(\gamma, t, u_p^n) \cdot \nu_{p,q} \varphi(\gamma, t) \, d\gamma dt \\ &= - \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} (\Phi(\gamma, t, u_p^n) \cdot \nu_{p,q} - \Phi(\gamma, t, u_q^n) \cdot \nu_{p,q}) \varphi(\gamma, t) \, d\gamma dt \\ &= I_1^* - I_2^* \end{aligned}$$

where

$$\begin{aligned} I_1^* &:= \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left( \frac{1}{m_{p,q}} \Phi_{p,q}^n(u_p^n, u_q^n) - \Phi(\gamma, t, u_p^n) \cdot \nu_{p,q} \right) \varphi(\gamma, t) \, d\gamma dt, \\ I_2^* &:= \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left( \frac{1}{m_{p,q}} \Phi_{p,q}^n(u_p^n, u_q^n) - \Phi(\gamma, t, u_q^n) \cdot \nu_{p,q} \right) \varphi(\gamma, t) \, d\gamma dt. \end{aligned}$$

Let us now estimate  $|I_1 - I_1^*|$ . We first rewrite  $I_1^*$  as

$$\begin{aligned} I_1^* &= \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_p^n, u_p^n)) \frac{1}{m_{p,q}} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \varphi(\gamma, t) \, dt d\gamma \\ &\quad + \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left( \frac{1}{m_{p,q}} \Phi_{p,q}^n(u_p^n, u_p^n) - \Phi(\gamma, s, u_p^n) \cdot \nu_{p,q} \right) \varphi(\gamma, t) \, dt d\gamma. \end{aligned}$$



Then, the consistency of the family  $\{\Phi_{p,q}^n\}$  with  $\Phi$  allows us to turn the above equation into

$$I_1^* = \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_p^n, u_p^n)) \frac{1}{m_{p,q}} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \varphi(\gamma, t) \, dt \, d\gamma$$

$$+ \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left( \Phi(\gamma, t, u_p^n) \cdot \nu_{p,q} - \frac{1}{m_{p,q}} \Phi_{p,q}^n(u_p^n, u_p^n) \right) \frac{1}{\delta t m_{p,q}} \left( \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \varphi(\gamma, t) - \varphi(\xi, s) \, ds \, d\xi \right) \, dt \, d\gamma.$$

Hence

$$I_1 - I_1^* = \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \delta t (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_p^n, u_p^n)) \frac{1}{\delta t^2 m_p m_{p,q}} \int_{t^n}^{t^{n+1}} \int_{Q_p} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \varphi(x, t) - \varphi(\gamma, s) \, dt \, dx \, ds \, d\gamma$$

$$+ \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left( \frac{1}{m_{p,q}} \Phi_{p,q}^n(u_p^n, u_p^n) - \Phi(\gamma, t, u_p^n) \cdot \nu_{p,q} \right) \frac{1}{\delta t m_{p,q}} \left( \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \varphi(\gamma, t) - \varphi(\xi, s) \, ds \, d\xi \right) \, dt \, d\gamma.$$

From the mean value theorem and the uniform Lipschitz continuity of  $\{\Phi_{p,q}^n\}$ , we thus derive the estimate

$$|I_1 - I_1^*| \leq C(h + \delta t) \left( \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \delta t m_{p,q} |u_p^n - u_q^n| + \sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \delta t (h + \delta t) m_{p,q} \right).$$

Using the finite number of neighbors in  $\mathcal{D}_h$ , hypotheses (C7) and (C8), and estimate (3.4), we obtain the bound

$$\sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \delta t m_{p,q} |u_p^n - u_q^n| \leq C \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} \delta t h^d |\nabla \hat{u}_h^n|_K \leq C \sum_{n=0}^{N-1} \delta t \int_{\mathbb{R}^d} |\nabla \hat{u}_h^n| \, dx \leq C. \quad (4.22)$$

Using (C1) and (C7), we also obtain

$$\sum_{n=0}^{N-1} \sum_{(p,q) \in \mathcal{E}_h^n} \delta t (h + \delta t) m_{p,q} \leq C. \quad (4.23)$$

Finally we deduce the estimate  $|I_1 - I_1^*| \leq C(h + \delta t)$ . The term  $I_2 - I_2^*$  can be treated similarly, which yields  $|e_{h,\delta t}^4| \leq C(h + \delta t)$ .  $\square$

## 5. Convergence of the approximate solutions

The main tool used in the convergence proof of the numerical approximation is the Kolmogorov-Riesz theorem. This theorem, also called the Fréchet-Kolmogorov theorem, yields a necessary and sufficient condition for a subset of  $L^p(\mathbb{R}^d)$  ( $1 \leq p < \infty$ ) to be compact. It can be seen as a generalization and a consequence of the Arzelà-Ascoli theorem. We state below a version of the Kolmogorov-Riesz theorem tailored to our case. We refer to Brezis (2011) for a general statement and a proof.

**THEOREM 5.1** Let  $(f_k)_{k \in \mathbb{N}}$  be a sequence of functions of  $L^1(\mathbb{R}^d \times (0, T))$ . Assume that (i)  $(f_k)_{k \in \mathbb{N}}$  is bounded in  $L^1(\mathbb{R}^d \times (0, T))$ ;

- (ii) there is a bounded set  $\omega \subset \mathbb{R}^d$  such that, for all  $k \in \mathbb{N}$  and for all  $t \in (0, T)$ ,  $\text{supp} f_k(\cdot, t) \subset \omega$ ;  
 (iii) for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, for all  $k \in \mathbb{N}$ ,

$$\int_0^{T-s} \int_{\mathbb{R}^d} |f_k(x, t+s) - f_k(x, t)| \, dx dt \leq \varepsilon, \quad \forall s \in [0, \delta), \quad (5.1)$$

and

$$\int_0^T \int_{\mathbb{R}^d} |f_k(x+y, t) - f_k(x, t)| \, dx dt \leq \varepsilon, \quad \forall y \in B(0, \delta), \quad (5.2)$$

where  $B(0, \delta)$  denotes the  $d$ -dimensional open ball with center 0 and radius  $\delta$ .

Then, there exists  $f \in L^1(\mathbb{R}^d \times (0, T))$  such that, up to a subsequence,  $f_k \rightarrow f$  strongly in  $L^1(\mathbb{R}^d \times (0, T))$ . Furthermore, for all  $t \in (0, T)$ ,  $\text{supp} f(\cdot, t) \subset \omega$ .

The convergence proof follows a fairly standard path. We consider a sequence of discretizations  $(\mathcal{F}_{h_k, \delta t_k})_{k \in \mathbb{N}}$  whose meshsize and time step tend to zero. In addition, we assume that  $(\mathcal{F}_{h_k, \delta t_k})_{k \in \mathbb{N}}$  satisfies uniformly the conditions (C1)-(C8) and that the sequence of initial conditions  $(\bar{u}_{h_k}^0)_{k \in \mathbb{N}}$  converges to  $u_{\text{ini}}$  in  $L^1(\mathbb{R}^d)$ . There is no difficulty to build such a sequence of discretizations. The approximate solutions on  $(\mathcal{F}_{h_k, \delta t_k})_{k \in \mathbb{N}}$  are denoted by  $(\bar{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$ ,  $(\hat{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$  and  $(\lambda_{h_k, \delta t_k})_{k \in \mathbb{N}}$ .

In a first step (Proposition 5.2), we extract converging subsequences from the sequences  $(\bar{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$ ,  $(\hat{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$  and  $(\lambda_{h_k, \delta t_k})_{k \in \mathbb{N}}$ , using compactness arguments, notably the Kolmogorov-Riesz theorem. We then pass to the limit in the discrete entropy formulation (4.8) to deduce the convergence of  $(\hat{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$  toward the entropy solution (Theorem 5.3). We can conclude to the convergence of the whole sequence  $(\hat{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$  by uniqueness of the entropy solution.

**PROPOSITION 5.2** There exists  $u \in L^\infty(Q_T) \cap L^1(0, T; BV(\mathbb{R}^d))$ , with compact support in space, and  $\lambda \in L^\infty(Q_T)^d$ , with  $|\lambda| \leq 1$  almost everywhere on  $Q_T$ , such that, up to a subsequence,

$$\hat{u}_{h_k, \delta t_k} \rightarrow u \quad \text{in } L^1(Q_T), \quad \bar{u}_{h_k, \delta t_k} \rightarrow u \quad \text{in } L^1(Q_T), \quad \lambda_{h_k, \delta t_k} \rightharpoonup \lambda \quad \text{weakly-* in } L^\infty(Q_T).$$

*Proof.* 1. The estimates (2.12), (3.3), (3.23) and (3.36) allow us to apply Theorem 5.1 to the sequence  $(\hat{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$ . Thus, there exists  $u \in L^1(Q_T)$  such that, up to a subsequence,  $(\hat{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$  converges to  $u$  in  $L^1(Q_T)$ . By definition of  $\bar{u}_{h_k, \delta t_k}$  and  $\hat{u}_{h_k, \delta t_k}$ , it is clear that if  $\hat{u}_{h_k, \delta t_k} \rightarrow u$  in  $L^1(Q_T)$ , then  $\bar{u}_{h_k, \delta t_k} \rightarrow u$  in  $L^1(Q_T)$  as well. Furthermore,  $u \in L^\infty(Q_T) \cap L^1(0, T; BV(\mathbb{R}^d))$ , since the sequence  $(\hat{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$  is uniformly bounded in  $L^\infty(Q_T)$  and  $L^1(0, T; BV(\mathbb{R}^d))$ .

2. As the sequence  $(\lambda_{h_k, \delta t_k})_{k \in \mathbb{N}}$  is bounded in  $L^\infty(Q_T)$ , there exists  $\lambda \in L^\infty(Q_T)^d$  such that, up to a subsequence,  $(\lambda_{h_k, \delta t_k})_{k \in \mathbb{N}}$  converges weakly-\* to  $\lambda$  in  $L^\infty(Q_T)$ . Furthermore,  $|\lambda| \leq 1$  almost everywhere on  $Q_T$ , since  $|\lambda_{h_k, \delta t_k}| < 1$  on  $Q_T$ .  $\square$

**THEOREM 5.3** The sequence  $(\hat{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$  converges strongly in  $L^1(Q_T)$  to the entropy solution of (0.2)-(0.3).

*Proof.* Fix an entropy  $\eta$  and a test function  $\varphi$ . Let  $(\bar{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$ ,  $(\hat{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$  and  $(\lambda_{h_k, \delta t_k})_{k \in \mathbb{N}}$  denote the subsequences extracted in Proposition 5.2, and  $u$  and  $\lambda$  denote their limits. The sequence  $(\bar{u}_{h_k, \delta t_k})_{k \in \mathbb{N}}$  is bounded, has a compact support and converges in  $L^1(Q_T)$ , therefore

$$\int_{Q_T} \eta(\bar{u}_{h_k, \delta t_k}) \partial_t \varphi \, dx dt + \int_{Q_T} \Phi(x, t, \bar{u}_{h_k, \delta t_k}) \cdot \nabla \varphi \, dx dt \rightarrow \int_{Q_T} \left( \eta(u) \partial_t \varphi + \Phi(x, t, u) \cdot \nabla \varphi \right) \, dx dt.$$

Since  $\hat{u}_{h_k, \delta_k} \rightarrow u$  in  $L^1(Q_T)$  and  $\lambda_{h_k, \delta_k} \rightharpoonup \lambda$  weakly-\* in  $L^\infty(Q_T)$ ,

$$\int_{Q_T} \eta'(\hat{u}_{h_k, \delta_k}) \lambda_{h_k, \delta_k} \cdot \nabla \varphi \, dt \, dx \rightarrow \int_{Q_T} \eta'(u) \lambda \cdot \nabla \varphi \, dt \, dx.$$

By lower semi-continuity of the total variation relatively to the  $L^1$  convergence, we obtain

$$\liminf_{k \rightarrow +\infty} \int_{Q_T} \varphi |\nabla \eta'(\hat{u}_{h_k, \delta_k})| \, dt \, dx \geq \int_{Q_T} \varphi |D[\eta'(u)]| \, dt.$$

By hypothesis,  $\bar{u}_{h_k}^0 \rightarrow u_{\text{ini}}$  in  $L^1(\mathbb{R}^d)$ , and thus

$$\int_{\mathbb{R}^d} \eta(\bar{u}_h^0) \varphi(x, 0) \, dx \rightarrow \int_{\mathbb{R}^d} \eta(u_{\text{ini}}) \varphi(x, 0) \, dx.$$

Finally, using the above limits and Proposition 4.3, we can pass to the limit in (4.8) and find

$$\int_{Q_T} \left( \eta(u) \partial_t \varphi + (\Phi(x, t, u) - \lambda \eta'(u)) \cdot \nabla \varphi \right) \, dx \, dt - \int_{Q_T} \varphi |D[\eta'(u)]| \, dt + \int_{\mathbb{R}^d} \eta(u_{\text{ini}}) \varphi(x, 0) \, dx \geq 0,$$

which proves that  $u$  is the entropy solution. Owing to the uniqueness of the entropy solution (proved in Section 1.4), we conclude that, in fact, the whole sequence  $(\hat{u}_{h_k, \delta_k})_{k \in \mathbb{N}}$  converges to  $u$ .  $\square$

## 6. Numerical examples

To illustrate the behavior of the numerical scheme, simulations are performed on two examples: a two-dimensional problem where the flux function  $F$  describes a mere advection and a one-dimensional problem where the flux function  $F$  describes a Burgers flow. For the two-dimensional problem, for which an analytical solution is known, a quantitative study of the convergence is provided. The simulations are performed using FreeFem++ (for the two-dimensional problem) and Scilab (for the one-dimensional problem).

### 6.1 Advection and total variation flow in 2D

We consider the problem

$$\partial_t u + c_x \partial_x u + c_y \partial_y u - g \operatorname{div} \operatorname{Sgn}(\nabla u) = 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, T), \quad (6.1)$$

where  $c_x$ ,  $c_y$ , and  $g$  are constants. The initial condition is

$$u(x, y, 0) = 1_{D_0}(x, y), \quad (x, y) \in \mathbb{R}^2, \quad (6.2)$$

where  $1_{D_0}$  denotes the indicator function of  $D_0$ , the disk of radius  $r_0$  centered at point  $(x_0, y_0)$ . It is possible to determine the exact solution to this problem. Indeed, for some bounded sets  $S$  of finite perimeter in  $\mathbb{R}^2$ , including disks, it is proved in Belletini *et al.* (2002) that the solution of

$$\begin{aligned} \partial_t u - \operatorname{div} \operatorname{Sgn}(\nabla u) &= 0, & (x, y, t) &\in \mathbb{R}^2 \times (0, T), \\ u(x, y, 0) &= 1_S(x, y), & (x, y) &\in \mathbb{R}^2, \end{aligned}$$

is the function  $u : (x, y, t) \mapsto (1 - |\partial S|/|S|t)^+ 1_S(x, y)$ . The proof can be easily adapted to deal with the advection and the exact solution of (6.1)-(6.2) is found to be

$$u : (x, y, t) \mapsto \left(1 - \frac{2gt}{r_0}\right)^+ 1_{D_0}(x - c_x t, y - c_y t).$$

The numerical simulations use the following parameters:  $(x_0, y_0) = (0.3, 0.3)$ ,  $r_0 = 0.2$ ,  $c_x = 0.8$ ,  $c_y = 0.2$ ,  $g = 0.03$ . To mimick the space  $\mathbb{R}^2$ , square domains with periodic boundary conditions are used for both the finite volume step and the finite element step. The finite element domain, denoted by  $\Omega$ , is meshed with squares of size  $h$  divided into two triangles. The finite volume domain, slightly larger than the finite element domain, is meshed with square cells of size  $h$ , centered at each node of the finite element mesh. Such a pair of meshes is represented in Figure 1. The advection step is solved using the

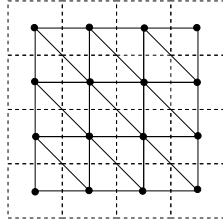


FIG. 1. Example of meshes used for the two-dimensional simulations. The finite element mesh is in solid line and the finite volume mesh is in dashed line.

so-called corner-transport upwind scheme (see, e.g., LeVeque, 2002), which is slightly more accurate than the standard upwind scheme. This scheme is stable under the CFL condition  $\delta t \leq \min\left(\frac{h}{c_x}, \frac{h}{c_y}\right)$ . The finite element step is solved using the lagged diffusivity algorithm, whose principle is recalled in Appendix A.

**REMARK 6.1** The corner-transport upwind scheme does not fit exactly into the theoretical framework defined in Section 2.1 since there is a numerical flux between the cells that touch at corner. However, the well-posedness and convergence results remain valid (the proofs can be readily adapted).

The first simulation is performed with  $\Omega = (0, 1) \times (0, 1)$ ,  $h = 0.01$ ,  $\text{CFL} = 1$ ,  $\varepsilon = 10^{-6}$  and 20 iterations at each time step for the lagged diffusivity algorithm. The numerical solution, represented at different times in Figure 2, is in good agreement with the exact solution: the contours are sharp and the support is little deformed. Interestingly, the total variation minimization in the finite element step limits the diffusion due to the finite volume scheme; see Figure 3. Table 1 collects the relative error between the exact solution and the discrete solution in  $L^1$ -norm for various meshsizes (keeping a CFL equal to 1). Precisely, the relative error is given by the expression

$$\frac{\|u^{per} - \hat{u}_{h, \delta t}\|_{L^1(\Omega \times (0, T))}}{\|u^{per}\|_{L^1(\Omega \times (0, T))}}$$

where  $u^{per}$  is the periodized version of  $u$  and the norms are computed with a second-order accurate quadrature formula. The results show, as expected, a decrease of the error when the mesh becomes finer. The computed convergence rate is clearly sublinear (0.28 between the first mesh and the second

$h$	relative error
0.02	0.618
0.01	0.5099
0.005	0.4403

Table 1. Relative error between the discrete and exact solutions in  $L^1$ -norm.

mesh, 0.21 between the second mesh and the third mesh). It is not surprising to obtain a sublinear rate with a solution presenting a discontinuity.

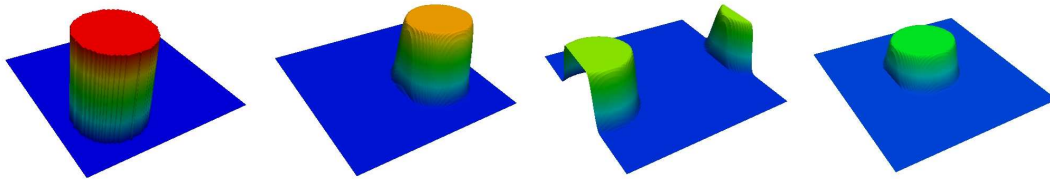


FIG. 2. Advection and total variation flow. Numerical solution at times  $t = 0, t = 0.5, t = 1, t = 1.5$ . The mesh is warped according to the value of the solution and the color scale goes from blue (0) to red (1).

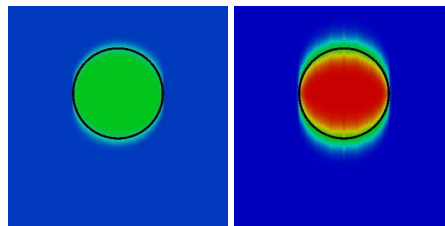


FIG. 3. Numerical solution of the advection and total variation flow (left) and numerical solution of the advection without total variation flow (right) at time  $t = 1.5$ . The color scale goes from blue (0) to red (1). The black circle indicates the support of the exact solution.

## 6.2 Burgers flow and total variation flow in 1D

We consider the equation

$$\partial_t u + \frac{1}{2} \partial_x (u^2) - g \partial_x \text{Sgn}(\partial_x u) = 0, \quad (x, t) \in \mathbb{R} \times (0, T), \quad (6.3)$$

with  $g = 5 \cdot 10^{-4}$ . The initial datum is a bump (see Figure 4, left). The computational domain is taken large enough for the solution not to touch the boundary in the course of the simulation. The finite element and finite volume meshes, of size  $h$ , are staggered. The finite volume step is advanced using the

Godunov scheme (see, e.g., LeVeque, 2002), which is stable under the CFL condition  $\sup_p |u_p^n| \delta t \leq \frac{1}{2}h$ , for all  $n \in \{0, \dots, N-1\}$ . As in the previous test case, the finite element step is solved using the lagged diffusivity algorithm

The numerical simulation is performed with  $h = 0.002$ ,  $\delta t = 0.001$ ,  $\varepsilon = 10^{-8}$  and 20 iterations at each time step for the lagged diffusivity algorithm. In Figure 4, the computed solution is represented at different times. As in a Burgers flow (without total variation flow), a shock wave forms at the right end, while a rarefaction wave forms at the left end. The main difference introduced by the total variation flow is the progressive clipping of the solution.

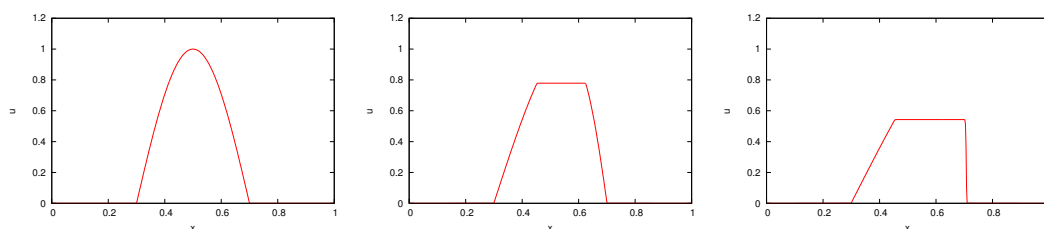


FIG. 4. Burgers flow and total variation flow. Numerical solution at times  $t = 0$ ,  $t = 0.05$ , and  $t = 0.15$ .

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### A. Lagged diffusivity algorithm

The lagged diffusivity algorithm is a simple and efficient algorithm for solving the total variation minimization problem (Vogel & Oman, 1996; Dobson & Vogel, 1997). Let us describe this algorithm in a  $P_1$  finite element setting, with Neumann boundary conditions. Let  $\Omega$  be an open bounded set of  $\mathbb{R}^d$  and  $\mathcal{T}_h$  a simplicial mesh over  $\Omega$ , whose nodes are denoted by  $\{x_p\}_{1 \leq p \leq N_p}$ . The finite element space over  $\mathcal{T}_h$  is denoted by  $V_h$ . The  $L^2$  scalar product is lumped using weights  $\{\omega_p\}_{1 \leq p \leq N_p}$ . Consider the minimization over  $V_h$  of the functional  $J_h : V_h \rightarrow \mathbb{R}$ , defined by

$$J_h(v_h) := \sum_{p=1}^{N_p} \omega_p (v_h(x_p) - z_h(x_p))^2 + \alpha \int_{\Omega} |\nabla v_h| \, dx,$$

where  $z_h \in V_h$  and  $\alpha > 0$  are given. The lagged diffusivity algorithm replaces the nondifferentiable functional  $J_h$  by the regularized functional  $J_{\varepsilon,h} : V_h \rightarrow \mathbb{R}$ , defined by

$$J_{\varepsilon,h}(v_h) := \sum_{p=1}^{N_p} \omega_p (v_h(x_p) - z_h(x_p))^2 + \alpha \int_{\Omega} \sqrt{\varepsilon + |\nabla v_h|^2} \, dx,$$

where  $\varepsilon$  is a small positive parameter. Since the functional  $J_{\varepsilon,h}$  is strictly convex and differentiable, its unique minimizer satisfies an Euler-Lagrange equation, namely

$$\sum_{p=1}^{N_p} \omega_p u_h(x_p) v_h(x_p) + \alpha \int_{\Omega} \frac{\nabla u_h \cdot \nabla v_h}{\sqrt{\varepsilon + |\nabla u_h|^2}} \, dx = \sum_{p=1}^{N_p} \omega_p z_h(x_p) v_h(x_p), \quad \forall v_h \in V_h.$$

The minimizer of  $J_{\varepsilon,h}$  is then computed with a fixed-point algorithm. The algorithm is initialized with  $u_h^{(0)} = z_h$ . At the iteration  $k+1$ , the approximate solution  $u_h^{(k+1)}$  is computed by solving the linear diffusion problem

$$\sum_{p=1}^{N_p} \omega_p u_h^{(k+1)}(x_p) v_h(x_p) + \alpha \int_{\Omega} \frac{\nabla u_h^{(k+1)} \cdot \nabla v_h}{\sqrt{\varepsilon + |\nabla u_h^{(k)}|^2}} \, dx = \sum_{p=1}^{N_p} \omega_p z_h(x_p) v_h(x_p), \quad \forall v_h \in V_h.$$

The diffusivity field in the above diffusion problem depends on  $u^{(k)}$ , whence the name lagged diffusivity.