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# Generalised continuation by means of right limits

David Sauzin and Giulio Tiozzo

April 18, 2014

## Abstract

Several theories have been proposed to generalise the concept of analytic continuation to holomorphic functions of the disc for which the circle is a natural boundary. Elaborating on Breuer-Simon's work on *right limits* of power series, Baladi-Marmi-Sauzin recently introduced the notion of *renascent right limit* and *rrel-continuation*.

We discuss a few examples and consider particularly the classical example of *Poincaré simple pole series* in this light. These functions are represented in the disc as series of infinitely many simple poles located on the circle; they appear for instance in small divisor problems in dynamics. We prove that any such function admits a unique rrel-continuation, which coincides with the function obtained outside the disc by summing the simple pole expansion. We also discuss the relation with monogenic regularity in the sense of Borel.

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# 1 Introduction

When one is given a function  $g$  holomorphic in the unit disc  $\mathbb{D}$ , one can ask whether  $g$  is related in some way to a holomorphic function defined outside the disc. A first answer to the question comes from Weierstrass's notion of analytic continuation. Given a point  $\lambda$  on the unit circle, if there exists a neighbourhood  $V$  and a holomorphic function on  $V$  whose restriction to  $V \cap \mathbb{D}$  is  $g$ , then we say that  $\lambda$  is a *regular point* and the restriction of  $g$  to the outer part of  $V$  is an analytic continuation. If there is no regular point on the unit circle, then we say that the unit circle is a *natural boundary* for  $g$ , but is it the end of the story?

It is the purpose of "generalised analytic continuation" to investigate this situation and suggest other ways in which an outer function can be related to the inner function  $g$ . The reader is referred to the monograph [RS02] for a panorama of various theories which have been proposed to go beyond Weierstrass's point of view on analytic continuation.

In this paper, we shall explore a new type of generalised analytic continuation, called *rrel-continuation*, which is based on the notion of *right limits* introduced in [BS11] as a tool unifying various classical criteria to detect a natural boundary (an earlier, related approach is due to [Ag49]). The notion of rrel-continuation was put forward in the recent article [BMS12] to deal with the natural boundary of Ruelle's susceptibility function; the goal of this paper is to start developing a general theory of rrel-continuation, see how it applies to some classical cases and compare it to other theories of analytic continuation.

We start by giving a few definitions and analysing some basic examples. In particular, we shall define the class of *rrel-continuable* functions as functions on the unit disc whose Taylor expansion at the origin admits a re-nascent right limit (see section 2.2). Each re-nascent right limit determines a function on the complement of  $\mathbb{D}$  which we call an *rrel-continuation* of  $g$ .

Let now  $g$  be an rrel-continuable function: if  $g$  is continuable outside  $\mathbb{D}$  in the traditional sense, then its analytic continuation across any arc coincides with its rrel-continuation (and thus must be unique); on the other hand, if  $g$  has more than one rrel-continuation, then the unit circle must be a natural boundary for  $g$  (Proposition 2.7). Particularly interesting is thus the case of functions called *uniquely rrel-continuable*, that admit a unique rrel-continuation: indeed, in this case  $\mathbb{D}$  may or may not be a natural boundary, but the rrel-continuation is a canonically defined function outside the disc which we may think of as continuation of  $g$ .

In the present paper we construct uniquely rrel-continuable functions in several contexts and explore the relationship between the function and its rrel-continuation(s).

As we shall see, the notion of rrel-continuation is well suited to study power series generated by dynamical systems; in particular, we analyse in

this light Hecke's example  $g_{\mathbb{H}}(z) = \sum_{k \geq 1} \{k\theta\} z^k$ , where  $\theta \in \mathbb{R} \setminus \mathbb{Q}$  and  $\{\cdot\}$  denotes the fractional part function. It was shown in [BS11] how the theory of right limits implies that  $g_{\mathbb{H}}$  has a natural boundary on the unit circle; we show that it has a unique rrl-continuation given by  $-\sum_{n < 0} \{n\theta\} z^n = g_{\mathbb{H}}(z^{-1}) + (1 - z)^{-1}$  for  $|z| > 1$ .

In the second part of the paper, we apply the theory to the classical situation, first considered by Poincaré in 1883, where  $g(z)$  is defined for  $|z| < 1$  as a series of simple poles

$$g(z) := \sum_{n \geq 0} \frac{\rho_n}{z - e^{i\theta_n}}$$

where the points  $e^{i\theta_n}$  are dense on the unit circle and the nonzero complex numbers  $\rho_n$  form an absolutely convergent series. For such a function (which we call *Poincaré simple pole series*, or *PSP-series* for short) the unit circle is a natural boundary in the classical sense; however, there is a natural candidate for the outer function, namely the sum  $h(z)$  of the simple pole series for  $|z| > 1$ . We will prove that every simple pole series  $g(z)$  inside the disc has a unique rrl-continuation, which coincides with the outer function:

**Theorem 1.1.** *Let  $g \in \mathcal{O}(\mathbb{D})$  be an inner PSP-series. Then  $g$  is uniquely rrl-continuable and its rrl-continuation is the associated outer PSP-series.*

This result was announced in [BMS12, Appendix A.2]. As an unexpected byproduct, we obtain that, for any  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ , the function  $g(z) = \sum_{k \geq 0} \text{dist}(k\theta, \mathbb{Z}) z^k$ , which is somewhat similar to Hecke's example, has a natural boundary on the unit circle and a unique rrl-continuation.

The theory of rrl-continuation makes sense more generally for functions with values in an arbitrary complex Banach space  $B$ , and in fact we shall prove a more general version of Theorem 1.1 for PSP-series with values in  $B$ .

As an application, PSP-series with values in a Banach space arise in a dynamical context as solutions to the cohomological equation for small divisor problems [MS03].

We will also compare the concept of rrl-continuity with Borel's concept of monogenic regularity as developed in [MS03], [MS11]: in particular, we shall see (Theorem 4.8) that a large class of PSP-series is monogenic, and their continuation in the sense of monogenic functions coincides with the outer series. This fact raises the question whether any monogenic function admits a unique rrl-continuation.

## 2 Continuation by renascent right limits

### 2.1 Preliminaries

We are interested in holomorphic functions defined in the unit disc by power series of the form

$$g(z) = \sum_{k=0}^{\infty} a_k z^k \quad (2.1)$$

with a bounded sequence of complex coefficients  $(a_k)_{k \geq 0}$ . Our aim is to investigate the possibility of defining “generalised analytic continuations” for  $|z| > 1$  when the unit circle is a natural boundary. We will accept what is called a strong natural boundary in [BS11]:

**Definition 2.1.** A function  $g(z)$  holomorphic in the unit disc is said to have a *strong natural boundary on the unit circle* if, for every nonempty interval  $(\omega_1, \omega_2)$ ,

$$\sup_{0 < r < 1} \int_{\omega_1}^{\omega_2} |g(re^{i\omega})| d\omega = \infty. \quad (2.2)$$

Clearly, if the unit circle is a strong natural boundary for  $g$ , then the unit circle is a natural boundary in the usual sense, since the function is not even bounded in any sector  $\{re^{i\omega} \mid r \in (0, 1), \omega \in (\omega_1, \omega_2)\}$ .

The article [BS11] provides a remarkable criterium to detect strong natural boundaries (Theorem 2.5 below), based on the notion of right limit that we now recall.

**Definition 2.2.** (i) Let  $\underline{a} = (a_k)_{k \geq 0}$  be a sequence in a topological space  $E$ .

A *right limit of  $\underline{a}$*  is any two-sided sequence  $\underline{b} = (b_n)_{n \in \mathbb{Z}}$  of  $E$  for which there exists a strictly increasing sequence of positive integers  $(k_j)_{j \geq 1}$  such that

$$\lim_{j \rightarrow \infty} a_{n+k_j} = b_n \quad \text{for every } n \in \mathbb{Z}. \quad (2.3)$$

(ii) Let  $g$  be a holomorphic function of the unit disc. We say that  $\underline{b}$  is a *right limit of  $g$*  if the sequence  $\underline{a}$  formed by the Taylor coefficients at the origin,  $a_k := g^{(k)}(0)/k!$ , is bounded and  $\underline{b}$  is a right limit of  $\underline{a}$ .

In view of (2.3), each  $b_n$  must be an accumulation point of  $\underline{a}$ . When  $E$  is a compact metric space, every sequence  $\underline{a}$  admits right limits; given  $\ell \in \mathbb{Z}$  and  $c$  accumulation point of  $\underline{a}$ , one can always find a right limit  $\underline{b}$  such that  $b_\ell = c$  (see *e.g.* [BMS12, Lemma 2.1]).

In the case of a function  $g$  with bounded Taylor coefficients, each right limit gives rise to two generating series which will play an important role when investigating the boundary behaviour of  $g$ :

**Definition 2.3.** Given a two-sided bounded sequence of complex numbers  $\underline{b} = (b_n)_{n \in \mathbb{Z}}$ , we define the *inner and outer functions associated with  $\underline{b}$*  as

$$g_{\underline{b}}^+(z) = \sum_{n \geq 0} b_n z^n, \quad z \in \mathbb{D},$$

$$g_{\underline{b}}^-(z) = - \sum_{n < 0} b_n z^n, \quad z \in \mathbb{E},$$

where  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$  is the unit disc and  $\mathbb{E} = \{z \in \mathbb{C} \mid |z| > 1\} \cup \{\infty\}$  is a disc centred at  $\infty$  in the Riemann sphere  $\widehat{\mathbb{C}}$ .

**Definition 2.4.** Given an arc  $J$  of the unit circle,  $\underline{b} = (b_n)_{n \in \mathbb{Z}}$  is said to be *reflectionless on  $J$*  if  $g_{\underline{b}}^+$  has an analytical continuation in a neighbourhood  $U$  of  $J$  in  $\mathbb{C}$  and this analytical continuation coincides with  $g_{\underline{b}}^-$  on  $U \cap \mathbb{E}$ .

Note that this terminology stems from the spectral theory of Jacobi matrices, it is not related to the Schwarz reflection principle.

**Theorem 2.5** (Breuer–Simon, [BS11]). *Let  $g$  be holomorphic in  $\mathbb{D}$  with bounded Taylor coefficients at 0.*

- (i) *Consider a nonempty interval  $(\omega_1, \omega_2)$  and the corresponding arc of the unit circle  $J = \{e^{i\omega} \mid \omega \in (\omega_1, \omega_2)\}$ , and assume that (2.2) is violated. Then every right limit of  $g$  is reflectionless on  $J$ .*
- (ii) *If  $\underline{b} = (b_n)_{n \in \mathbb{Z}}$  and  $\tilde{\underline{b}} = (\tilde{b}_n)_{n \in \mathbb{Z}}$  are two distinct right limits of  $g$  and if there exists  $N \in \mathbb{Z}$  such that either  $b_n = \tilde{b}_n$  for all  $n \geq N$  or  $b_n = \tilde{b}_n$  for all  $n \leq N$ , then the unit circle is a strong natural boundary for  $g$ .*

## 2.2 The rrl-continuable functions

In [BMS12], motivated by Breuer–Simon’s work, right limits were used to define a type of generalised analytic continuation as follows:

**Definition 2.6.** (i) A *renascent right limit* of a sequence  $\underline{a}$  in a topological space is any right limit  $\underline{b} = (b_n)_{n \in \mathbb{Z}}$  of  $\underline{a}$  such that  $b_n = a_n$  for all  $n \geq 0$ .

(ii) An *rrl-continuable function* is a holomorphic function  $g$  which admits a renascent right limit  $\underline{b}$ ; then  $g_{\underline{b}}^+ = g$  in  $\mathbb{D}$  and the function  $g_{\underline{b}}^-$ , which is holomorphic in  $\mathbb{E}$  and vanishes at  $\infty$ , is said to be an *rrl-continuation of  $g$* .

(iii) An rrl-continuable function  $g$  is said to be *uniquely rrl-continuable* if it has a unique rrl-continuation; in the opposite case, it is said to be *polygenous*.

As a consequence of Theorem 2.5, the situation for rrl-continuable functions is simpler than for arbitrary functions:

**Proposition 2.7.** *Let  $g$  be an rrl-continuable function. Then*

(i) *either there is an arc of the unit circle through which  $g$  admits analytic continuation; then  $g$  is uniquely rrl-continuable and all the analytic continuations of  $g$  through arcs of the unit circle match and coincide with the rrl-continuation of  $g$ ;*

(ii) *or the unit circle is a strong natural boundary for  $g$ .*

*If  $g$  is polygenous, then the unit circle is a strong natural boundary for  $g$ .*

*Proof.* Let  $\underline{b}$  be a renascent right-limit of  $g$ . Suppose there exists a closed arc  $J = \{e^{i\omega} \mid \omega \in [\omega_1, \omega_2]\}$  in the neighbourhood of which  $g$  admits an analytic continuation  $h_J$ . Then

$$\sup_{0 < r < 1} \int_{\omega_1}^{\omega_2} |g(re^{i\omega})| d\omega < \infty,$$

hence, by Theorem 2.5(i),  $\underline{b}$  is reflectionless on  $J$ ; since  $g = g_{\underline{b}}^+$ , this means that  $h_J = g_{\underline{b}}^-$ , independently of the choice of  $J$ . The uniqueness of the renascent right-limit follows too.

If on the contrary there is no analytic continuation for  $g$  across any arc of the unit circle, then  $\underline{b}$  is not reflectionless on any arc thus Theorem 2.5(i) entails that the unit circle is a strong natural boundary.

The last statement follows from Theorem 2.5(ii). □

When an rrl-continuable function  $g$  has a natural boundary on the unit circle, we may still think of the rrl-continuations of  $g$  as being somewhat “connected” to  $g$  and consider them as a kind of generalised analytic continuation, and the case of a unique rrl-continuation may then be particularly interesting.

**Example 2.8.** In the case of a preperiodic sequence,  $g(z) = \sum_{k=0}^{\infty} a_k z^k$  with  $a_k = a_{k+p}$  for all  $k \geq m$ , one checks easily that there is no renascent right limit unless  $m = 0$ , *i.e.* the sequence is periodic, in which case  $g(z) = (a_0 + a_1 z + \dots + a_{p-1} z^{p-1}) / (1 - z^p)$  is rational and uniquely rrl-continuable. More generally, *any rational function which is regular on the Riemann sphere minus the unit circle and whose poles are simple is uniquely rrl-continuable*; this follows from Theorem 4.4 below (we shall see that one can even afford for an infinite set of “poles” on the unit circle—we use quotation marks because the function is then no longer rational). Notice that we restrict ourselves to simple poles because we consider only the case of bounded Taylor coefficients.

We emphasize that a holomorphic function  $g$  with bounded Taylor coefficients may have no rrl-continuation at all, independently of whether the unit circle is a natural boundary or not. For instance, if the sequence of Taylor coefficients of  $g$  at the origin tends to 0, then the only right limit of  $g$  is  $b_n \equiv 0$  and  $g$  cannot be rrl-continuable unless  $g(z) \equiv 0$ ; the previous example also shows that no polynomial is rrl-continuable except the trivial one. Observe also that if two holomorphic functions of  $\mathbb{D}$  differ by a function  $h$  which is holomorphic in a disc  $\{|z| < R\}$  with  $R > 1$ , then they have the same right limits; for instance, for any such  $h$ , the function  $h(z) + (1-z)^{-1}$  has only one right limit, the constant sequence  $b_n \equiv 1$ , but only when  $h(z) \equiv 0$  is this right limit a reascent one.

Notice that with the usual analytic continuation it may happen that, for a given  $g \in \mathcal{O}(\mathbb{D})$ , there are several arcs through which analytic continuation is possible but leads to different results. Think *e.g.* of  $(1+z)^{1/2}(1-z)^{-1/2}$ . However, in view of Proposition 2.7, such examples are not rrl-continuable.

Note also that, given  $r \geq 1$  and  $g \in \mathcal{O}(\mathbb{D})$  divisible by  $z^r$ , one has

$h$  is an rrl-continuation of  $g \implies z^{-r}h(z)$  is an rrl-continuation of  $z^{-r}g(z)$

as a consequence of (2.3), but the converse is not necessarily true: it may be that  $z^{-r}g(z)$  is rrl-continuable but not  $g$  itself (think *e.g.* of  $g(z) = z(1-z)^{-1}$ ).<sup>1</sup>

### 2.3 Functions with values in complex Banach spaces

Part of the theory can be extended to analytic functions of the disc with values in an arbitrary complex Banach space. Let  $B$  a complex Banach space, and  $(a_n)_{n \geq 0}$  a bounded sequence of elements of  $B$ . Then the definition of right limit (Definition 2.2) and reascent right limit (Definition 2.6) of  $(a_n)_{n \geq 0}$  still makes sense, and the power series

$$f(z) := \sum a_n z^n$$

defines a function holomorphic in  $\mathbb{D}$  with values in  $B$ , and one can ask whether  $f$  is rrl-continuable and what its rrl-continuations are, since Definition 2.6 still makes sense (the only difference being that in the infinite-dimensional case it is not necessarily true that any bounded sequence of coefficients  $(a_n)_{n \geq 0}$  has at least one right limit).

In this more general context, rrl-continuity still has consequences in terms of natural boundaries. In particular, one has the following weaker form of Proposition 2.7 where “strong natural boundary” is replaced by “natural boundary”:

---

<sup>1</sup>But if  $g(z)$  is divisible by  $z$  and  $z^{-1}g(z)$  admits as rrl-continuation  $h(z) = -\sum_{n < 0} b_n z^n$ , then  $b_{-1} + g(z)$  admits as rrl-continuation  $b_{-1} + zh(z)$ .



**Proposition 2.9.** *Let  $g$  be an rrl-continuable function with values in a complex Banach space  $B$ . Then*

(i) *either there is an arc of the unit circle through which  $g$  admits analytic continuation; then  $g$  is uniquely rrl-continuable and all the analytic continuations of  $g$  through arcs of the unit circle match and coincide with the rrl-continuation of  $g$ ;*

(ii) *or the unit circle is a natural boundary for  $g$ .*

*If  $g$  is polygenous, then the unit circle is a natural boundary for  $g$ .*

*Proof.* The proof proceeds exactly as the proof of Proposition 2.7, using instead of Theorem 2.5 the weaker form ([BS11], Theorem 1.3). In fact, the proof of ([BS11], Theorem 1.3) generalizes verbatim, since it uses only the maximum principle and Vitali's convergence theorem, which still hold for vector-valued holomorphic functions (see *e.g.* [HP57]).  $\square$

On the other hand, the statement for strong natural boundaries does not appear to immediately generalize; in fact, the proof of Theorem 2.5 ([BS11], Theorem 1.4) ultimately uses the existence of radial limits for functions in the Hardy space  $H^1$ , which does not hold in general.

### 3 Dynamical examples

#### 3.1 Power series generated by dynamical systems

A first interesting class of power series which arises in connection with dynamical systems is as follows:

**Definition 3.1.** Let  $E$  be a metric space and  $T: E \rightarrow E$  be a continuous map. Given  $\gamma \in E$ , we consider its orbit  $(\gamma_k)_{k \geq 0} = (T^k(\gamma))_{k \geq 0}$ . Then, for any complex Banach space  $B$  and for any bounded function  $\varphi: E \rightarrow B$ , we say that the sequence

$$a_k := \varphi(T^k(\gamma)), \quad k \geq 0$$

is *generated by the dynamical system  $T$*  (in that situation  $\varphi$  is called an *observable*).

To determine the right limits of the power series  $\sum_{k=0}^{\infty} \varphi(T^k(\gamma)) z^k$ , one may try to determine first the right limits of the orbit  $(T^k(\gamma))$  itself (and then to exploit continuity or discontinuity properties of the observable  $\varphi$ ). The following result is a generalisation of [BMS12, Lemma 2.4].

**Lemma 3.2.** *The right limits of the orbit  $(T^k(\gamma))_{k \geq 0}$  are exactly the full orbits of  $T$  which are contained in the  $\omega$ -limit set  $\omega(\gamma, T)$ .*

*Proof.* Recall that the full orbits of  $T$  are the two-sided sequences  $(\gamma_n)_{n \in \mathbb{Z}}$  of  $E$  such that  $\gamma_{n+1} = T(\gamma_n)$  for all  $n \in \mathbb{Z}$ . Suppose  $(\xi_n)_{n \in \mathbb{Z}}$  is a right limit of  $(T^k(\gamma))_{k \geq 0}$ . Then there exists a sequence  $k_j \rightarrow \infty$  such that, for each  $n$ ,  $T^{n+k_j}(\gamma) \rightarrow \xi_n$ , hence each  $\xi_n$  belongs to  $\omega(\gamma, T)$ . Moreover, by continuity of  $T$ , for each  $n$  we have  $T^{n+1+k_j}(\gamma) = T(T^{n+k_j}(\gamma)) \rightarrow T(\xi_n) = \xi_{n+1}$ , hence  $(\xi_n)_{n \in \mathbb{Z}}$  is a full orbit.

Conversely, suppose  $(\xi_n)_{n \in \mathbb{Z}}$  is a full orbit of  $T$  contained in  $\omega(\gamma, T)$ . By continuity of  $T$ , for each  $j \geq 1$  we can choose a neighborhood  $U_j$  of  $\xi_{-j}$  such that, if  $x \in U_j$ , then  $d(T^k(x), T^k(\xi_{-j})) \leq 1/j$  for all  $k \in \{0, 1, \dots, 2j\}$ . Since each  $\xi_{-j}$  belongs to  $\omega(\gamma, T)$ , we can choose an increasing sequence  $k_j \rightarrow \infty$  such that  $T^{k_j}(\gamma)$  belongs to  $U_j$  for each  $j$ . Now fix  $n \in \mathbb{Z}$ ; for any  $j \geq |n|+1$ , we have  $0 < n + j < 2j$ , thus

$$d(T^{n+k_j+j}(\gamma), \xi_n) = d(T^{n+j}(T^{k_j}(\gamma)), T^{n+j}(\xi_{-j})) \leq 1/j,$$

whence  $\lim_{j \rightarrow \infty} T^{n+k_j+j}(\gamma) = \xi_n$  and the claim is proven.  $\square$

One finds in [BMS12, Theorem 2] an example of a sequence  $(a_k)_{k \geq 0}$  of the form  $(\varphi(T^k(\gamma)))_{k \geq 0}$  which has uncountably many reascent right limits: its generating series is highly polygenous (in that example the observable  $\varphi$  is continuous but the dynamics  $T$  is a non-invertible map of a compact interval of  $\mathbb{R}$ ; the non-invertibility helps construct a huge set of full orbits).

**Example 3.3.** The arithmetic example due to Hecke

$$g_H(z) = \sum_{k=1}^{\infty} \{k\theta\} z^k, \quad z \in \mathbb{D}, \quad (3.1)$$

where  $\theta \in \mathbb{R} \setminus \mathbb{Q}$  and  $\{\cdot\}$  denotes the fractional part, was shown to have a strong natural boundary in [BS11]. This can be viewed as a series generated by the translation  $x \mapsto x + \theta$  on  $\mathbb{R}/\mathbb{Z}$  for a discontinuous observable. We shall see in Proposition 3.6 that  $g_H(z)$  is uniquely rrl-continuable and that  $z^{-1}g_H(z)$  has exactly two rrl-continuations.

### 3.2 The case of symbolic dynamics

Another class of examples arises from symbolic dynamics: if  $E = \bigcup_{k=1}^n P_k$  is a partition of the phase space in a finite number of sets, we can define the piecewise constant observable

$$\varphi(x) := c_k \quad \text{for } x \in P_k,$$

for some choice of complex constants  $c_1, \dots, c_n$ . Then, given a point  $x$ , the corresponding sequence generated by a dynamical system  $T: E \rightarrow E$  is called *itinerary* of  $x$ :

$$\mathbf{itin}(x) := (\varphi(T^k(x)))_{k \geq 0}.$$

A powerful application is Milnor–Thurston’s *kneading theory* [MT88]. Let  $T : [0, 1] \rightarrow [0, 1]$  be a continuous, unimodal map, with  $T(0) = T(1) = 0$  and a critical point  $c \in (0, 1)$  which we assume non-periodic for simplicity; we consider the piecewise constant observable  $\varphi$  which takes the value 1 on  $[0, c]$  and  $-1$  on  $(c, 1]$ . The *kneading sequence*  $(\epsilon_k)_{k \geq 0}$  of  $T$  is defined to be the itinerary of  $c$ . The *kneading determinant* is the power series

$$D(z) := 1 + \sum_{k \geq 1} \epsilon_1 \cdots \epsilon_k z^k.$$

One of the applications of the kneading determinant is to find the topological entropy of the map. Indeed, if  $s$  is the smallest positive real zero of  $D(z)$ , then the entropy of  $T$  equals  $-\log s$  ([MT88], Theorem 6.3).

**Example 3.4.** As an example, consider  $T(z) := z^2 + c$  with  $c$  the Feigenbaum parameter ( $c \cong -1.401155189\dots$ ). Then its kneading determinant is known to be

$$D(z) = \sum_{k=0}^{\infty} (-1)^{\tau_k} z^k,$$

where  $\tau := (01101001\dots)$  is the *Thue-Morse sequence* generated by the substitution  $0 \rightarrow 01, 1 \rightarrow 10$ , starting with 0. Notice that, by the defining relation of  $\tau$ , it is not hard to prove that

$$D(z) = \prod_{m=0}^{\infty} (1 - z^{2^m})$$

(from which it follows that the entropy of  $T$  is 0). One can check that  $D(z)$  has precisely two renascent right limits, hence the unit circle is a strong natural boundary.

A thorough investigation of the applications to symbolic dynamics will be the object of a forthcoming article.

### 3.3 Circle maps and the rrl-continuability of Hecke’s example

The following result is a variant of a theorem proved in [BS11] and used there to show that Hecke’s example has a strong natural boundary on the unit circle. It deals with the series generated by a dynamical system on the circle, with a special kind of observable:

**Theorem 3.5.** *Let  $f : \mathbb{T} \rightarrow \mathbb{T}$  be a homeomorphism of the circle  $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ , and  $x^* \in \mathbb{T}$  a point with dense forward orbit under  $f$ . Moreover, let  $B$  be a complex Banach space,  $\varphi : \mathbb{T} \rightarrow B$  a bounded function, and  $\Delta \subset \mathbb{T}$  a subset of the torus with empty interior. Assume that:*

- $\varphi$  is continuous on  $\mathbb{T} \setminus \Delta$ ,
- each point of  $\Delta$  is a point of discontinuity for  $\varphi$  at which right and left limits exist and  $\varphi$  is either right- or left-continuous.

Then the function

$$g(z) := \sum_{k=0}^{\infty} \varphi(f^k(x^*)) z^k, \quad z \in \mathbb{D}$$

has the following properties:

- (i) If  $f^k(x^*) \notin \Delta$  for all  $k > 0$ , then  $g$  is rrl-continuable.
- (ii) If  $f^k(x^*) \notin \Delta$  for all  $k \geq 0$  and there exists  $n < 0$  such that  $f^n(x^*) \in \Delta$ , then  $g$  has at least two different rrl-continuations.

*Proof.* Let us use the notation

$$y_j \xrightarrow{>} y^*, \quad \text{resp.} \quad y_j \xrightarrow{<} y^*,$$

if  $(y_j)_{j \geq 1}$  is a sequence and  $y^*$  is a point in  $\mathbb{T}$  for which there exist lifts  $(\tilde{y}_j)_{j \geq 1}$  and  $\tilde{y}^*$  in  $\mathbb{R}$  such that  $\lim_{j \rightarrow \infty} \tilde{y}_j = \tilde{y}^*$  and, for  $j$  large enough,  $\tilde{y}^* < \tilde{y}_j < \tilde{y}^* + \frac{1}{2}$ , resp.  $\tilde{y}^* - \frac{1}{2} < \tilde{y}_j < \tilde{y}^*$ . We set

$$x_n := f^n(x^*), \quad n \in \mathbb{Z}$$

and notice that, by the density of  $\{x_k\}_{k \geq 0}$  in  $\mathbb{T}$ , for every  $y^* \in \mathbb{T}$  one can find increasing integer sequences  $(k_j^+)_{j \geq 1}$  and  $(k_j^-)_{j \geq 1}$  such that  $x_{k_j^\pm} \xrightarrow{\cong} y^*$ .

Suppose first that  $f^k(x^*) \notin \Delta$  for all  $k > 0$ . Let us choose an increasing integer sequence  $(k_j)_{j \geq 1}$  such that  $x_{k_j} \xrightarrow{\epsilon} x^*$  with  $\epsilon$  standing for ‘>’, unless  $x^* \in \Delta$  and  $\varphi$  is left-continuous at  $x^*$ , in which case  $\epsilon$  stands for ‘<’. Then, for each  $n \in \mathbb{Z}$ ,  $x_{n+k_j} = f^n(x_{k_j}) \xrightarrow{\epsilon_n} f^n(x^*) = x_n$  with  $\epsilon_n$  standing for ‘>’ or ‘<’ according as  $f^n$  preserves or reverses orientation, and  $b_n := \lim_{j \rightarrow \infty} \varphi(x_{n+k_j})$  exists because  $\varphi$  has both left and right limits at  $x_n$ . Now, for  $n > 0$ , we have  $x_n \notin \Delta$ , hence  $b_n = \varphi(x_n)$ ; for  $n = 0$ , we also have  $b_0 = \varphi(x_0)$  even if  $x_0 = x^* \in \Delta$  thanks to our choice of  $(k_j)$ ; therefore we have found a renascent right limit for  $(\varphi(x_k))_{k \geq 0}$ .

Suppose now that  $f^k(x^*) \notin \Delta$  for all  $k \geq 0$  and that one can pick  $\ell > 0$  such that  $f^{-\ell}(x^*) \in \Delta$ . Let us choose increasing integer sequences  $(k_j^+)_{j \geq 1}$  and  $(k_j^-)_{j \geq 1}$  such that  $x_{k_j^\pm} \xrightarrow{\cong} x_{-\ell}$ . For each  $n \in \mathbb{Z}$ , we have  $x_{n+\ell+k_j^\pm} \xrightarrow{\epsilon_n^\pm} x_n$ , with  $\epsilon_n^\pm$  depending on whether  $f^{n+\ell}$  preserves or reverses orientation, and  $b_n^\pm := \lim_{j \rightarrow \infty} \varphi(x_{n+\ell+k_j^\pm})$  exists because  $\varphi$  has left and right limits at  $x_n$ .

For  $n \geq 0$ ,  $\varphi$  is continuous at  $x_n$ , thus  $b_n^+ = b_n^- = \varphi(x_n)$ , but for  $n = -\ell$  we have

$$b_{-\ell}^+ = \lim_{x \xrightarrow{>} x_{-\ell}} \varphi(x) \neq b_{-\ell}^- = \lim_{x \xrightarrow{<} x_{-\ell}} \varphi(x),$$

which means that we have two different renascent right limits.  $\square$

We now discuss the rrl-continuability of Hecke's example.

**Proposition 3.6.** *Let us fix  $\theta \in \mathbb{R} \setminus \mathbb{Q}$  and denote the fractional part function by  $\{\cdot\}$ . Then the function*

$$g_H(z) := \sum_{k=1}^{\infty} \{k\theta\} z^k$$

has a unique rrl-continuation, which is

$$g_H^-(z) = - \sum_{n < 0} \{n\theta\} z^n = g_H(z^{-1}) + (1-z)^{-1}, \quad z \in \mathbb{E}. \quad (3.2)$$

Moreover, the function  $z^{-1}g_H(z) = \sum_{k=0}^{\infty} \{(k+1)\theta\} z^k$  has exactly two rrl-continuations, namely

$$- \sum_{n < 0} \{(n+1)\theta\} z^n = z^{-1}g_H^-(z) \quad \text{and} \quad -z^{-1} + z^{-1}g_H^-(z), \quad (3.3)$$

and the unit circle is a strong natural boundary both for  $g_H(z)$  and  $z^{-1}g_H(z)$ .

*Proof.* Hecke's example falls into case (i) of Theorem 3.5: denoting by  $\pi: \mathbb{R} \rightarrow \mathbb{T}$  the canonical projection, we can define the homeomorphism  $f$  by  $f \circ \pi(\tilde{x}) = \pi(\tilde{x} + \theta)$  and the discontinuous observable  $\varphi^+$  by  $\varphi^+ \circ \pi(\tilde{x}) = \tilde{x} - [\tilde{x}]$ , then  $g_H(z) = \sum_{k=0}^{\infty} a_k z^k$  with  $a_k := \varphi^+(f^k(x_H^*))$  and  $x_H^* := \pi(0)$ , and  $\Delta$  is reduced to  $\{x_H^*\}$  in that case, with  $\varphi^+$  right-continuous. Therefore, since  $\theta$  is irrational, the forward orbit of  $x_H^*$  does not hit  $x_H^*$  again, hence  $g_H$  has at least one rrl-continuation.

To show its uniqueness, we observe that  $\varphi^+: \mathbb{T} \rightarrow [0, 1)$  is a section of  $\pi: \mathbb{R} \rightarrow \mathbb{T}$  and the restriction of the canonical projection  $\pi^+: [0, 1) \rightarrow \mathbb{T}$  is a right inverse for  $\varphi^+$ . Let us thus consider a renascent right limit  $\underline{b} = (b_n)_{n \in \mathbb{Z}}$  of  $(a_k)_{k \geq 0}$ , and set  $y_n := \pi(b_n)$ .

The continuity of  $\pi$  entails that  $(y_n)$  is a renascent right limit of  $(\pi(a_k)) = (f^k(x_H^*))$ , thus  $y_n = f^n(y_0)$  by Lemma 3.2. But  $y_0 = x_H^*$  because the right limit is renascent, hence  $y_n \in \mathbb{T} \setminus \{x_H^*\}$  for all  $n \in \mathbb{Z}^*$ . Since we already knew that  $b_n \in [0, 1]$  (because  $a_k \in [0, 1)$ ) and  $\pi(b_n) = y_n$ , we deduce that, for  $n \in \mathbb{Z}^*$ ,  $b_n \in (0, 1)$  and  $b_n = \varphi^*(y_n)$ . We thus obtain (3.2) (the representation of  $g_H^-$  as  $g_H(z^{-1}) + (1-z)^{-1}$  stems from  $\tilde{x} \in \mathbb{R} \setminus \mathbb{Z} \implies -\{-\tilde{x}\} = \{\tilde{x}\} - 1$ ).

On the other hand, the function  $g(z) := \sum_{k=0}^{\infty} \{(k+1)\theta\} z^k$  falls into case (ii) of Theorem 3.5. Indeed, the only difference with the previous

case is the initial condition,  $x_g^* = \pi(\theta)$ . Therefore,  $g$  has at least two rrl-continuations. By Theorem 2.5(ii), it follows that the unit circle is a strong natural boundary for  $g$ , and thus also for  $g_H$ .

Arguing as above, we see that any renascent right limit of  $g$  is of the form  $\underline{b} = (b_n)_{n \in \mathbb{Z}}$  with  $y_n := \pi(b_n) = f^n(x_g^*)$ ; now  $y_n \in \mathbb{T} \setminus \{\pi(0)\}$  only for  $n \in \mathbb{Z} \setminus \{-1\}$ , while  $y_{-1} = \pi(0)$ . Since  $b_n \in [0, 1]$  and  $\pi(b_n) = y_n$  for all  $n \in \mathbb{Z}$ , there are only two possibilities:  $b_{-1} = 0$  or  $1$  and  $b_n = \varphi^+(y_n) = \varphi^+(f^n(x_g^*))$  for  $n \in \mathbb{Z} \setminus \{-1\}$ . Both cases are possible (if not there would be only one renascent right limit), thus we find the two rrl-continuations indicated in (3.3).  $\square$

**Corollary 3.7.** *With the same assumptions and notations as in Proposition 3.6, defining*

$$g_{H,\gamma}(z) := \sum_{k=0}^{\infty} \{\gamma + k\theta\} z^k, \quad z \in \mathbb{D} \quad (3.4)$$

for every  $\gamma \in \mathbb{R} \setminus (\mathbb{Z} + \theta\mathbb{Z})$ , one gets a unique rrl-continuation for  $g_{H,\gamma}$ :

$$g_{H,\gamma}^-(z) = - \sum_{n<0} \{\gamma + n\theta\} z^n = g_{H,-\gamma}(z^{-1}) + z(1-z)^{-1} + \{\gamma\}, \quad (3.5)$$

and the unit circle is a strong natural boundary.

*Proof.* The existence of the rrl-continuation is guaranteed by Theorem 3.5(i), exactly as in the proof of Proposition 3.6. One finds that (3.5) is the only possible rrl-continuation by following the same lines.

Since  $0$  is an accumulation point of the sequence  $(a_k)_{k \geq 0} = (\{\gamma + k\theta\})_{k \geq 0}$  of  $(0, 1)$ , we can find a right limit  $(b_n)_{n \in \mathbb{Z}}$  such that  $b_0 = 0$ , and  $b_n \in [0, 1]$  for every  $n \in \mathbb{Z}$ . With the same notations for  $\pi$ ,  $f$  and  $\varphi^+$  as in the proof of Proposition 3.6, since  $\pi$  is continuous, it maps  $(b_n)_{n \in \mathbb{Z}}$  onto a right limit  $(y_n)_{n \in \mathbb{Z}}$  of the sequence  $(\pi(a_k))_{k \geq 0} = (f^k \circ \pi(\gamma))_{k \geq 0}$ , which is necessarily of the form  $y_n = f^n(y_0)$  (by Lemma 3.2). Since  $y_0 = \pi(0)$ , we see that, for  $n \in \mathbb{Z}^*$ ,  $y_n \in \mathbb{T} \setminus \{\pi(0)\}$  and  $\pi(b_n) = y_n$ , hence  $b_n \in (0, 1)$  and  $b_n = \varphi^+(y_n)$ .

We just obtained that  $(\{n\theta\})_{n \in \mathbb{Z}}$  is a right limit of  $g_{H,\gamma}$ . By virtue of Proposition 3.6, this right limit is not reflectionless on any arc, Theorem 2.5(i) thus implies that the unit circle is a strong natural boundary.  $\square$

*Remark 3.8.* There is a relationship between the arithmetical properties of  $\theta$  and the functions  $g_{H,\gamma}$ : namely, for any  $0 \leq \gamma_1 < \gamma_2 < 1$  one has

$$g_{H,\gamma_1}(z) - g_{H,\gamma_2}(z) - \frac{\gamma_2 - \gamma_1}{1 - z} = \sum_{k \in \mathcal{N}(\gamma_1, \gamma_2)} z^k,$$

where  $\mathcal{N}(\gamma_1, \gamma_2)$  is the set of the occurrence times of the sequence  $(\{k\theta\})_{k \geq 0}$  in  $[1 - \gamma_2, 1 - \gamma_1)$ , that is  $\mathcal{N}(\gamma_1, \gamma_2) := \{k \geq 0 \mid k\theta \in [-\gamma_2, -\gamma_1) + \mathbb{Z}\}$ .

*Remark 3.9.* Let  $\theta \in \mathbb{R} \setminus \mathbb{Q}$  and  $\gamma \in \mathbb{R}$ . Replacing the fractional part function with  $t \in \mathbb{R} \mapsto |t|_{\mathbb{Z}} := \text{dist}(t, \mathbb{Z})$ , we get an example which looks similar:

$$g_{\gamma}^{+}(z) := \sum_{k=0}^{\infty} |\gamma + k\theta|_{\mathbb{Z}} z^k, \quad z \in \mathbb{D}.$$

However, the corresponding observable is continuous (it is the function  $\varphi: \mathbb{T} \rightarrow \mathbb{R}$  defined by  $\varphi(\pi(\tilde{x})) = |\gamma + \tilde{x}|_{\mathbb{Z}}$ ), hence the study of  $g_{\gamma}^{+}$  requires completely different techniques. We shall prove unique rrl-continuability of  $g_{\gamma}^{+}$  in Corollary 4.7.

## 4 Poincaré simple pole series and generalised continuation

The second half of this article is dedicated to what is probably the simplest non-trivial situation in which one might wish to test the notion of rrl-continuability. Namely, we will prove that every Poincaré simple pole series is uniquely rrl-continuable (Theorem 4.4); this result was announced without proof in [BMS12, Appendix A.2].

### 4.1 Poincaré simple pole series and rrl-continuability

We will use the same notations as in Definition 2.3 for  $\widehat{\mathbb{C}}$ ,  $\mathbb{D}$  and  $\mathbb{E}$ , and denote by  $\mathbb{S}$  the unit circle viewed as a subset of  $\mathbb{C} \subset \widehat{\mathbb{C}}$ .

For any real number  $\nu > 0$  and complex Banach space  $(B, \|\cdot\|)$ , we denote by  $\ell^{\nu}(\mathbb{S}, B)$  the set of all functions  $\rho: \mathbb{S} \rightarrow B$  such that the family  $(\|\rho(\lambda)\|^{\nu})_{\lambda \in \mathbb{S}}$  is summable, *i.e.*

$$\sum_{\lambda \in \mathbb{S}} \|\rho(\lambda)\|^{\nu} < \infty.$$

Given  $\rho \in \ell^{\nu}(\mathbb{S}, B)$ , its *support* is the set (finite or countably infinite)

$$\text{supp } \rho := \{\lambda \in \mathbb{S} \mid \rho(\lambda) \neq 0\}.$$

**Definition 4.1.** Let  $\rho \in \ell^1(\mathbb{S}, B)$ . The *Poincaré simple pole series* (PSP-series for short) associated with  $\rho$  is the  $B$ -valued function  $\Sigma(\rho)$  defined by

$$\Sigma(\rho)(z) := \sum_{\lambda \in \mathbb{S}} \frac{\rho(\lambda)}{z - \lambda}, \quad z \in \widehat{\mathbb{C}} \setminus \overline{\text{supp } \rho}. \quad (4.1)$$

The series (4.1) converges normally on every compact subset of  $\widehat{\mathbb{C}} \setminus \overline{\text{supp } \rho}$ , thus in particular it defines a  $B$ -valued holomorphic function on  $\mathbb{D} \cup \mathbb{E}$ . We

shall call respectively *inner* and *outer* PSP-series the restrictions of  $\Sigma(\rho)$  to the inside and outside the unit disc, and denote them by

$$\Sigma(\rho)^+ := \Sigma(\rho)|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D}, B), \quad \Sigma(\rho)^- := \Sigma(\rho)|_{\mathbb{E}} \in \mathcal{O}(\mathbb{E}, B).$$

We say that the outer PSP-series  $\Sigma(\rho)^-$  is *associated* with the inner PSP-series  $\Sigma(\rho)^+$ .

The class of inner PSP-series is an interesting class of functions for which we will prove unique rrl-continuability. It clearly contains the rational functions which are regular on  $\widehat{\mathbb{C}} \setminus \mathbb{S}$  and whose poles are simple, but we are more interested in the case where  $\mathbb{S}$  is a natural boundary. Our terminology is motivated by Poincaré's article [Po83] (see also [Po92]), where he studies this kind of series; assuming that the support of  $\rho$  is dense in  $\mathbb{S}$ , Poincaré proves that the unit circle is a natural boundary for  $\Sigma(\rho)^\pm$  and he discusses the relationship between the two functions. Later Borel, Wolff and Denjoy studied such series, considering also more general distributions of poles  $\lambda$  (not restricted to lie on  $\mathbb{S}$ ). The subclass of PSP-series  $\Sigma(\rho)$  with  $\text{supp } \rho$  contained in the set of roots of unity was studied in [MS03] for dynamical reasons, in relation with small divisor problems.

Any inner PSP-series uniquely determines the associated outer PSP-series, because  $\rho$  and thus  $\Sigma(\rho)^-$  are uniquely determined by  $\Sigma(\rho)^+$ :

**Lemma 4.2.** *Let  $B$  be a complex Banach space and  $\rho \in \ell^1(\mathbb{S}, B)$ . Then*

$$\rho(\lambda) = \lim (z - \lambda) \Sigma(\rho)^+(z) \quad \text{as } z \rightarrow \lambda \text{ radially}$$

for every  $\lambda \in \mathbb{S}$ . Therefore the map  $\rho \in \ell^1(\mathbb{S}, B) \mapsto \Sigma(\rho)^+ \in \mathcal{O}(\mathbb{D}, B)$  is injective.

*Proof.* Given  $\lambda^* \in \mathbb{S}$ , we can write

$$(z - \lambda^*) \Sigma(\rho)^+(z) - \rho(\lambda^*) = \sum_{\lambda \in \mathbb{S} \setminus \{\lambda^*\}} \rho(\lambda) \frac{z - \lambda^*}{z - \lambda}, \quad z \in \mathbb{D}.$$

For each  $\lambda \in \mathbb{S} \setminus \{\lambda^*\}$ , we have  $z \in [0, \lambda^*] \Rightarrow |z - \lambda^*| < |z - \lambda|$ , thus  $\|\rho(\lambda) \frac{z - \lambda^*}{z - \lambda}\| \leq \|\rho(\lambda)\|$ . Now, since  $\frac{z - \lambda^*}{z - \lambda} \rightarrow 0$  as  $z \rightarrow \lambda^*$ , the result follows by dominated convergence.  $\square$

*Remark 4.3.* In fact, one even has

$$\rho(\lambda) = \lim (z - \lambda) \Sigma(\rho)^+(z) \quad \text{as } z \rightarrow \lambda \text{ non-tangentially}$$

for every  $\lambda \in \mathbb{S}$ .

Lemma 4.2 shows that, if  $\lambda \in \text{supp } \rho$ , the function  $\Sigma(\rho)^+$  is not bounded on the ray  $[0, \lambda]$ , hence  $\lambda$  is necessarily a singular point of the function. This entails a dichotomy:



- (a) either  $\text{supp } \rho$  is not dense in  $\mathbb{S}$ ; then  $\widehat{\mathbb{C}} \setminus \overline{\text{supp } \rho}$  is connected and  $\mathbb{S} \setminus \overline{\text{supp } \rho}$  is a countable union of open arcs of the unit circle, in the neighbourhood of which  $\Sigma(\rho)$  is holomorphic; we can thus view  $\Sigma(\rho)^+$  and  $\Sigma(\rho)^-$  as the analytic continuation of each other through any of these arcs;
- (b) or  $\text{supp } \rho$  is dense in the unit circle,  $\mathbb{D}$  and  $\mathbb{E}$  are the two connected components of  $\widehat{\mathbb{C}} \setminus \overline{\text{supp } \rho} = \widehat{\mathbb{C}} \setminus \mathbb{S}$  and the unit circle is a natural boundary for both  $\Sigma(\rho)^+$  and  $\Sigma(\rho)^-$ .

Our main theorem about inner PSP-series is in terms of the notion of generalised analytic continuation discussed in Section 2: the outer PSP-series is the unique rrl-continuation of the inner PSP-series with which it is associated. This result was stated in the introduction as Theorem 1.1 in the case where  $B = \mathbb{C}$ , we restate it now in full generality:

**Theorem 4.4.** *Let  $B$  a complex Banach space. Suppose that  $g = \Sigma(\rho)^+ \in \mathcal{O}(\mathbb{D}, B)$  is an inner PSP-series, with  $\rho \in \ell^1(\mathbb{S}, B)$ . Then  $g$  is uniquely rrl-continuable and its rrl-continuation is the associated outer PSP-series  $\Sigma(\rho)^-$ .*

The proof will be given in Section 5. Together with Proposition 2.7, this immediately yields

**Corollary 4.5.** *Assume that  $B = \mathbb{C}$ . Suppose that the unit circle is a natural boundary for an inner PSP-series  $g = \Sigma(\rho)^+ \in \mathcal{O}(\mathbb{D})$  (i.e. the support of the corresponding  $\rho \in \ell^1(\mathbb{S}, \mathbb{C})$  is dense in  $\mathbb{S}$ ). Then the unit circle is a strong natural boundary for  $g$ .*

One might be tempted to enlarge the framework by considering any series of rational functions which are regular in  $\widehat{\mathbb{C}} \setminus \mathbb{S}$  rather than series of simple poles only. However, one must be aware that such an expansion may represent a constant function in  $\mathbb{D}$  and another constant function in  $\mathbb{E}$ , a phenomenon which certainly does not fit with our intuition of what generalised analytic continuation should be (see [Po92], [RS02]).

## 4.2 An example based on harmonic analysis

With the help of Theorem 4.4 we easily get

**Proposition 4.6.** *Let  $B$  be a complex Banach space. Let  $\varphi: \mathbb{R} \rightarrow B$  be continuous and 1-periodic and assume that the sequence  $(\hat{\varphi}(j))_{j \in \mathbb{Z}}$  of its Fourier coefficients is absolutely convergent:*

$$\sum_{j \in \mathbb{Z}} |\hat{\varphi}(j)| < \infty. \tag{4.2}$$

Then, for any  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ , the sum of the convergent power series

$$g^+(z) := \sum_{n \geq 0} \varphi(n\theta) z^n, \quad z \in \mathbb{D} \quad (4.3)$$

is an inner PSP-series, thus uniquely rrl-continuable, with associated outer PSP-series given by

$$g^-(z) := - \sum_{n < 0} \varphi(n\theta) z^n, \quad z \in \mathbb{E}. \quad (4.4)$$

Moreover, if  $\varphi$  is not a trigonometric polynomial, then the unit circle is a natural boundary for  $g^+$  (a strong one if  $B = \mathbb{C}$ ).

*Proof.* Consider the pairwise distinct points  $\lambda_j = e^{-2\pi i j \theta}$ ,  $j \in \mathbb{Z}$ , and define  $\rho: \mathbb{S} \rightarrow \mathbb{C}$  by

$$\rho(\lambda_j) = -\lambda_j \hat{\varphi}(j), \quad j \in \mathbb{Z},$$

and  $\rho(\lambda) = 0$  if  $\lambda$  is not an integer power of  $e^{-2\pi i \theta}$ ; then  $\rho \in \ell^1(\mathbb{S}, B)$  and  $-\sum \rho(\lambda) \lambda^{-n-1} = \sum \hat{\varphi}(j) e^{2\pi i j n \theta} = \varphi(n\theta)$  for every  $n \in \mathbb{Z}$ , hence the formula (5.2) for the Taylor coefficients at 0 of an inner PSP-series and the coefficients at  $\infty$  of the associated outer PSP-series yields  $g^\pm = \Sigma(\rho)^\pm$ . The rest follows from Theorem 4.4.  $\square$

The previous result is reminiscent of Hecke's example: the formula (3.2) that we obtained for its rrl-continuation looks like an echo of (4.4). In Hecke's example, however, the observable  $\varphi^+$  violates condition (4.2) (since its Fourier coefficients decay only as  $\frac{1}{j}$ ) and is not continuous. Here is an example, similar in nature to Hecke's example or its generalisation (3.4), but satisfying the assumptions of Proposition 4.6:

**Corollary 4.7.** *We set, for any  $t \in \mathbb{R}$ ,  $|t|_{\mathbb{Z}} := \text{dist}(t, \mathbb{Z})$  the distance from  $t$  to the closest integer. Let  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ . Then, for each  $\gamma \in \mathbb{R}$ , the sum of the convergent power series*

$$g_\gamma^+(z) := \sum_{n \geq 0} |\gamma + n\theta|_{\mathbb{Z}} z^n, \quad z \in \mathbb{D} \quad (4.5)$$

is an inner PSP-series, thus uniquely rrl-continuable, with rrl-continuation given by

$$g_\gamma^-(z) := - \sum_{n < 0} |\gamma + n\theta|_{\mathbb{Z}} z^n = -g_{-\gamma}^+(z^{-1}) + |\gamma|_{\mathbb{Z}}, \quad z \in \mathbb{E} \quad (4.6)$$

and the unit circle is a strong natural boundary for  $g_\gamma^+$ .

*Proof.* The Fourier coefficients of  $\varphi_\gamma(t) := |\gamma + t|_{\mathbb{Z}}$  are  $O(1/j^2)$  and  $\varphi_\gamma(-t) = \varphi_{-\gamma}(t)$ .  $\square$

### 4.3 Poincaré simple pole series and monogenic regularity

In this section we will compare the notion of rrl-continuability developed so far to the property of *monogenic regularity* introduced by Borel [Bo17]. Monogenic regularity is an alternative way to generalised analytic continuation, and spaces of monogenic functions enjoy quasianalyticity properties that we will recall in this section. We will prove that, at least for certain PSP-series, the two theories of continuation overlap, since the outer function is the continuation of the inner function according to both definitions.

Let  $B$  be a complex Banach space. Given  $K' \subset K \subset \widehat{\mathbb{C}}$  and a linear space  $E$  of  $B$ -valued functions defined on  $K$ , we say that  $K'$  is a *uniqueness set* for  $E$  if the only function of  $E$  which vanishes on  $K'$  is the zero function, and we say that  $E$  is  $\mathcal{H}^1$ -*quasianalytic relatively to  $K'$*  if any subset of  $K'$  of positive one-dimensional Hausdorff measure is a uniqueness set for  $E$ .

For a closed subset  $K$  of  $\widehat{\mathbb{C}}$ , we denote by  $\mathcal{O}(K, B)$  the space of all  $B$ -valued functions which are continuous on  $K$  and holomorphic in the interior of  $K$ ; and we denote by  $\mathcal{C}_{hol}^1(K, B) \subset \mathcal{O}(K, B)$  the Banach space of all  $B$ -valued functions which are  $C^1$ -holomorphic on  $K$  (*i.e.* Whitney-differentiable in the complex sense—see [MS03] or [CMS14] for a precise definition).

Let  $(K_j)_{j \geq 0}$  be a monotonic non-decreasing sequence of closed subsets of  $\widehat{\mathbb{C}}$ . The space of Borel monogenic functions  $\mathcal{M}((K_j), B)$  is defined as the projective limit

$$\mathcal{M}((K_j), B) := \varprojlim \mathcal{C}_{hol}^1(K_j, B).$$

It is proven in [MS11] that, for certain sequences  $(K_j)$ , the space  $\mathcal{M}((K_j), B)$  is  $\mathcal{H}^1$ -quasianalytic relatively to  $K := \bigcup K_j$ . The next proposition shows that PSP-series enjoy this quasianalyticity property, at least for  $\rho \in \ell^{1/4}(\mathbb{S}, B)$ .

**Theorem 4.8.** *Let  $\rho \in \ell^{1/4}(\mathbb{S}, B)$ . Then there exists an increasing sequence  $(K_j)$  of compact subsets of  $\widehat{\mathbb{C}}$  such that*

- *the set  $K := \bigcup K_j$  has its complement contained in  $\overline{\text{supp } \rho} \subset \mathbb{S}$  and of zero Haar measure;*
- *the function  $\Sigma(\rho)$  has a unique continuous extension to  $K$ ;*
- *the space of Borel monogenic functions associated with  $(K_j)$ ,*

$$\mathcal{M}((K_j), B),$$

*contains this extension of  $\Sigma(\rho)$  and is  $\mathcal{H}^1$ -quasianalytic relatively to  $K$ .*

The  $\mathcal{H}^1$ -quasianalyticity property means that any function  $g$  of the space  $\mathcal{M}((K_j), B)$  is determined by its restriction to any subset of  $K$  which has positive linear Hausdorff measure; in particular it is determined by its

inner restriction  $g_{\mathbb{D}}$ . In the case of  $g = \Sigma(\rho)$  as in Theorem 4.8, this yields a totally different way of recovering the outer function  $g_{\mathbb{E}} = \Sigma(\rho)^-$  from the inner function  $\Sigma(\rho)^+$ . In view of Theorem 4.4, one may wonder whether it is true that  $g_{\mathbb{E}}$  is the only rrl-continuation of  $g_{\mathbb{D}}$  for any  $g \in \mathcal{M}((K_j), B)$ .

*Proof of Theorem 4.8.* For each  $j \geq 1$ , define

$$K_j := \left\{ z \in \mathbb{C} \mid \text{for each } \lambda \in \mathbb{S}, |z - \lambda| \geq \frac{1}{j} \|\rho(\lambda)\|^{1/4} \right\} \cup \{\infty\}.$$

Each  $K_j$  is a compact subset of the Riemann sphere, with  $K_j \subset K_{j+1}$ , and the fact that  $\rho$  is bounded implies that the complement of  $K := \bigcup_{j \geq 1} K_j$  is contained  $\overline{\text{supp } \rho}$  and thus in  $\mathbb{S}$ ; it has zero Haar measure because  $\rho \in \ell^{1/4}(\mathbb{S}, B)$ .

Let us check that the space  $\mathcal{M}((K_j), B)$  is  $\mathcal{H}^1$ -quasianalytic. Denoting by  $\Gamma_j^{(i)} := \partial(K_j \cap \mathbb{D})$  and  $\Gamma_j^{(e)} := \partial(K_j \cap \mathbb{E})$ , we observe that  $\Gamma_j^{(i)}$  and  $\Gamma_j^{(e)}$  are rectifiable Jordan curves and, for each  $j$  sufficiently large, the set  $\Gamma_j^{(i)} \cap \Gamma_j^{(e)} = K_j \cap \mathbb{S}$  has positive one-dimensional Hausdorff measure, since

$$\left| \{ \theta \in [0, 2\pi] \mid e^{i\theta} \in \mathbb{S} \setminus K_j \} \right| \leq \sum_{\lambda \in \mathbb{S}} 4 \arcsin \left( \frac{\|\rho(\lambda)\|^{1/4}}{j} \right) \leq 2\pi \sum_{\lambda \in \mathbb{S}} \frac{\|\rho(\lambda)\|^{1/4}}{j}.$$

Thus, in the language of [MS11, Definition 4], each pair  $(\Gamma_j^{(i)}, \Gamma_j^{(e)})$  is a nested pair and  $K_j = K(\Gamma_j^{(i)}, \Gamma_j^{(e)})$ , so the space  $\mathcal{M}((K_j), B)$  of monogenic functions is  $\mathcal{H}^1$ -quasianalytic by [MS11, Corollary A].

Let us now check that this space contains  $\Sigma(\rho)$ . The normal convergence properties

$$z \in K_j \Rightarrow \begin{cases} \sum_{\lambda \in \mathbb{S}} \frac{\|\rho(\lambda)\|}{|z - \lambda|} \leq j \sum_{\lambda \in \mathbb{S}} \|\rho(\lambda)\|^{3/4} < \infty, \\ \sum_{\lambda \in \mathbb{S}} \frac{\|\rho(\lambda)\|}{|z - \lambda|^2} \leq j^2 \sum_{\lambda \in \mathbb{S}} \|\rho(\lambda)\|^{1/2} < \infty, \end{cases}$$

which hold for each  $j$ , show that the series  $f(z) := \sum_{\lambda \in \mathbb{S}} \frac{\rho(\lambda)}{z - \lambda}$  and  $f^{(1)}(z) := \sum_{\lambda \in \mathbb{S}} \frac{\rho(\lambda)}{(z - \lambda)^2}$  define continuous functions in  $K$ . Moreover, given  $j \geq 1$ , we have for any  $\varepsilon > 0$

$$\begin{aligned} \left| f(z_2) - f(z_1) - f^{(1)}(z_1)(z_2 - z_1) \right| &= \left| (z_1 - z_2)^2 \sum_{\lambda \in \mathbb{S}} \frac{\rho(\lambda)}{(z_1 - \lambda)^2(z_2 - \lambda)} \right| \\ &\leq |z_1 - z_2|^2 j^3 \sum_{\lambda \in \mathbb{S}} \|\rho(\lambda)\|^{1/4} \leq \varepsilon |z_1 - z_2| \end{aligned}$$

as soon as  $z_1, z_2 \in K_j$  and  $|z_1 - z_2|$  is sufficiently small, hence  $f$  is  $C^1$ -holomorphic on  $K_j$ . We conclude that  $f \in \mathcal{M}((K_j), B)$ . Clearly,  $\Sigma(\rho)$  coincides with the restriction of  $f$  to the set  $\widehat{\mathbb{C}} \setminus \overline{\text{supp } \rho}$ ; the closure of the latter set is  $\widehat{\mathbb{C}}$  thus, by continuity,  $f$  is unique.  $\square$

*Remark 4.9.* If we do not suppose  $\rho \in \ell^{1/4}(\mathbb{S}, B)$  but only  $\rho \in \ell^{1/2}(\mathbb{S}, B)$ , then the same construction as above is sufficient to get an extension of  $\Sigma(\rho)$  which belongs to  $\mathcal{O}(K_j, B)$  for each  $j$ . Observe that the spaces  $\mathcal{O}(K_j, B)$  too are  $\mathcal{H}^1$ -quasianalytic by virtue of [MS11, Corollary A].

**Example 4.10.** Let  $G(w) = \sum_{m=1}^{\infty} G_m w^m$  be holomorphic for  $|w| < r_0$  (with values in  $\mathbb{C}$ ). A particular case of Theorem 4.8 occurs when solving the ‘‘cohomological equation’’

$$f(qw) - f(w) = G(w). \quad (4.7)$$

One then gets a vector-valued PSP-series, which was studied under the name ‘‘Borel-Wolff-Denjoy series’’ in [MS03]. Indeed, for any  $r < r_0$ , the functional equation (4.7) has solution

$$f_q(w) = \sum_{m=1}^{\infty} G_m \frac{w^m}{q^m - 1} \quad (4.8)$$

in the space  $B = \mathbb{B}_r$  of all bounded holomorphic functions of the disc  $\{w \in \mathbb{C} \mid |w| < r\}$ . Taking into account the dependence on the ‘‘multiplier’’  $q$  thus defines a map  $q \in \mathbb{D} \mapsto f_q(\cdot) \in \mathbb{B}_r$ , which turns out to be a PSP-series with values in  $\mathbb{B}_r$  (where the role of the variable  $z$  of the present article is played by  $q$ ). Indeed, one can rewrite (4.8) as

$$f_q = \sum_{\lambda \in \mathbb{S}} \frac{\rho(\lambda)}{q - \lambda}$$

where the map  $\rho: \mathbb{S} \rightarrow \mathbb{B}_r$  has its support contained in the set of all roots of unity and is defined by

$$\rho(\lambda)(w) := \lambda \sum_{k=1}^{\infty} G_{km} \frac{w^{km}}{km} \quad (4.9)$$

for  $\lambda$  primitive root of unity of order  $m$  (see also [MS03]). An easy estimate shows that  $\|\rho(\lambda)\|$  decays at least geometrically with respect to  $m$ , thus  $\rho \in \ell^{1/4}(\mathbb{S}, \mathbb{B}_r)$  and Theorem 4.8 immediately entails

**Proposition 4.11.** *The function  $q \in \mathbb{D} \mapsto f_q(\cdot) \in \mathbb{B}_r$  is an inner PSP-series which has an  $\mathcal{H}^1$ -quasianalytic monogenic continuation through the unit circle, given by the sum of the right-hand side of (4.8) for  $|q| > 1$ .*

If  $G$  is not a polynomial, then the support of  $\rho$  is dense in  $\mathbb{S}$  (because  $G_m \neq 0$  implies that  $\rho(\lambda) \neq 0$  for all  $\lambda$  such that  $\lambda^m = 1$ ) and the unit circle is thus a natural boundary for this function. As a consequence of Corollary 4.5, we also get

**Proposition 4.12.** *If  $G$  is not a polynomial and  $w_* \in \mathbb{C}$  satisfies  $0 < |w_*| < r_0$ , then the function  $q \in \mathbb{D} \mapsto f_q(w_*) \in \mathbb{C}$  has a strong natural boundary on the unit circle.*

*Proof.* We set  $G_n^* = G_n w_*^n / n$  for every  $n \in \mathbb{N}^*$  and suppose that the unit circle is not a strong natural boundary. Since  $q \mapsto f_q(w_*) = \Sigma(\rho(\cdot)(w_*))^+(q)$  is a scalar inner PSP-series, Corollary 4.5 entails that the support of the function  $\lambda \in \mathbb{S} \mapsto \rho(\lambda)(w_*)$  is not dense in  $\mathbb{S}$ . In view of (4.9), this says that there exists  $m_0 \in \mathbb{N}^*$  such that the complex numbers

$$\rho_m^* := \sum_{k \geq 1} G_{km}^*, \quad m \in \mathbb{N}^*,$$

vanish for  $m \geq m_0$ . But, by Möbius inversion,  $G_n^* = \sum_{d \geq 1} \mu(d) \rho_{dn}^*$  for every  $n \in \mathbb{N}^*$  (where  $\mu(d)$  is the sum of all primitive roots of unity of order  $d$ ), hence  $G_n = 0$  for  $n \geq m_0$ .  $\square$

## 5 Proof of Theorem 4.4

Let  $B$  be a complex Banach space. Let  $\rho \in \ell^1(\mathbb{S}, B)$ . The Taylor expansion of  $\Sigma(\rho)^+$  at the origin and the Taylor expansion of  $\Sigma(\rho)^-$  at  $\infty$  are easily computed by expanding the geometric series  $\frac{1}{z-\lambda} = -\lambda^{-1}(1 - \lambda^{-1}z)^{-1} = z^{-1}(1 - \lambda z^{-1})^{-1}$  and permuting sums; one can write the result as

$$\Sigma(\rho)^+(z) = \sum_{n \geq 0} b_n z^n, \quad \Sigma(\rho)^-(z) = - \sum_{n < 0} b_n z^n, \quad (5.1)$$

$$b_n := - \sum_{\lambda \in \mathbb{S}} \rho(\lambda) \lambda^{-n-1} \quad \text{for } n \in \mathbb{Z}. \quad (5.2)$$

### 5.1 The associated outer PSP-series is an rrl-continuation

We prove in this section the first part of Theorem 4.4, namely that  $\Sigma(\rho)^-$  is an rrl-continuation of  $\Sigma(\rho)^+$ . It is enough to find an unbounded integer sequence  $(k_j)_{j \geq 1}$  such that

$$\lim_{j \rightarrow \infty} b_{n+k_j} = b_n \quad \text{for every } n \in \mathbb{Z} \quad (5.3)$$

(indeed, from any such unbounded sequence, one can extract an increasing sequence for which (5.3) still holds, showing that  $\underline{b} = (b_n)_{n \in \mathbb{Z}}$  is a right limit of  $(b_k)_{k \geq 0}$ , with  $\Sigma(\rho)^+ = g_{\underline{b}}^+$  and  $\Sigma(\rho)^- = g_{\underline{b}}^-$ ).

We have  $b_{n+k_j} = -\sum_{\lambda \in \mathbb{S}} \lambda^{-k_j} \cdot \rho(\lambda) \lambda^{-n-1}$  and  $\|\lambda^{-k_j} \cdot \rho(\lambda) \lambda^{-n-1}\| = \|\rho(\lambda)\|$ . Thus, by the dominated convergence theorem, if the sequence  $(k_j)_{j \geq 1}$  satisfies

$$\lim_{j \rightarrow \infty} \lambda^{k_j} = 1 \quad \text{for each } \lambda \in \text{supp } \rho \quad (5.4)$$

then (5.3) holds. The problem thus reduces to finding an unbounded integer sequence  $(k_j)_{j \geq 1}$  satisfying (5.4).

1st case

Assume that  $\text{supp } \rho$  is contained in the set of all roots of unity. Then one can take  $k_j := j!$ , since  $\lambda^{j!} = 1$  for any  $j \geq$  order of  $\lambda$  as a root of unity.

2nd case

Assume that  $\text{supp } \rho$  is not contained in the set of all roots of unity. We write  $\text{supp } \rho = \{\lambda_1, \lambda_2, \dots\}$ . For each  $j \geq 1$ , we set

$$V_j := \{e^{2\pi i \omega} \mid 0 \leq \omega < 1/j\} \subset \mathbb{S}$$

and we consider the ‘‘cells’’

$$W_{\ell_1, \ell_2, \dots, \ell_j} := e^{2\pi i \ell_1 / j} V_j \times e^{2\pi i \ell_2 / j} V_j \times \dots \times e^{2\pi i \ell_j / j} V_j \subset \mathbb{S}^j$$

(where  $e^{2\pi i \ell_r / j} V_j$  is short-hand for  $\{e^{2\pi i \omega} \mid \ell_r / j \leq \omega < (\ell_r + 1) / j\}$ ) for each integer  $j$ -tuple  $(\ell_1, \ell_2, \dots, \ell_j)$  with  $0 \leq \ell_1, \ell_2, \dots, \ell_j \leq j - 1$ ; these are  $j^j$  cells which cover the torus  $\mathbb{S}^j$ . Now consider the  $j^j + 1$  points

$$\Lambda_{j,m} := (\lambda_1^m, \lambda_2^m, \dots, \lambda_j^m) \in \mathbb{S}^j, \quad \text{for } m = 0, 1, \dots, j^j.$$

Out of them, at least two belong to the same cell, we thus can find  $m_j < m'_j$  such that  $\Lambda_{j,m_j}$  and  $\Lambda_{j,m'_j}$  belong to the same cell  $W_{\ell_1, \ell_2, \dots, \ell_j}$ ; this means that

$$\lambda_r^{m_j}, \lambda_r^{m'_j} \in e^{2\pi i \ell_r / j} V_j$$

for all  $r = 1, 2, \dots, j$ . This implies

$$\lambda_r^{k_j} \in V_j, \quad \text{where } k_j := m'_j - m_j \quad (5.5)$$

for all  $r \leq j$ . Keeping  $r$  fixed but arbitrary, we thus get  $\lim_{j \rightarrow \infty} \lambda_r^{k_j} = 1$ . Therefore we have obtained (5.4) with the sequence  $(k_j)_{j \geq 1}$  defined by (5.5). Now this sequence cannot be bounded because, if it were, (5.4) would imply that each element of  $\text{supp } \rho$  is a root of unity. This ends the first part of the proof of Theorem 4.4.

## 5.2 Uniqueness of the rrl-continuation for an inner PSP-series

We now prove the second part of Theorem 4.4, namely that there is no rrl-continuation for  $\Sigma(\rho)^+$  other than  $\Sigma(\rho)^-$ .

**Proposition 5.1.** *Suppose that a strictly increasing sequence of positive integers  $(k_j)_{j \geq 1}$  satisfies*

$$\lim_{j \rightarrow \infty} b_{n+k_j} = b_n \quad \text{for each } n \geq 0, \quad (5.6)$$

then

$$\lim_{j \rightarrow \infty} \lambda^{k_j} = 1 \quad \text{for each } \lambda \in \text{supp } \rho. \quad (5.7)$$

*Proof of Proposition 5.1.* Let  $(k_j)_{j \geq 1}$  be as in the statement. For each  $j \geq 1$  we define a map  $\delta_j: \mathbb{S} \rightarrow B$  by

$$\delta_j(\lambda) := \rho(\lambda)\lambda^{-k_j} - \rho(\lambda) \quad \text{for each } \lambda \in \mathbb{S}.$$

We just need to prove that  $\lim_{j \rightarrow \infty} \delta_j(\lambda) = 0$  for each  $\lambda \in \mathbb{S}$ . Notice that

$$\|\delta_j(\lambda)\| \leq 2\|\rho(\lambda)\|, \quad (5.8)$$

hence  $\delta_j \in \ell^1(\mathbb{S}, B)$ . By (5.1)–(5.2), we have

$$\Sigma(\delta_j)^+(z) = \sum_{n \geq 0} c_{j,n} z^n \quad \text{for } z \in \mathbb{D}$$

with  $c_{j,n} = -\sum_{\lambda \in \mathbb{S}} (\rho(\lambda)\lambda^{-k_j} - \rho(\lambda))\lambda^{-n-1} = b_{n+k_j} - b_n$ . The assumption (5.6) implies  $\lim_{j \rightarrow \infty} c_{j,n} = 0$  for each  $n \geq 0$ . Since  $\|c_{j,n} z^n\| \leq 2\|\rho\|_{\ell^1(\mathbb{S}, B)} |z|^n$ , the dominated convergence theorem yields

$$\lim_{j \rightarrow \infty} \Sigma(\delta_j)^+(z) = 0 \quad \text{for each } z \in \mathbb{D}. \quad (5.9)$$

Fix  $\lambda^* \in \mathbb{S}$ . By the same computation as in the proof of Lemma 4.2, for each  $j \geq 1$  and  $z \in [0, \lambda^*)$  we have

$$\begin{aligned} \left\| \delta_j(\lambda^*) - (z - \lambda^*)\Sigma(\delta_j)^+(z) \right\| &= \left\| \sum_{\lambda \neq \lambda^*} \delta_j(\lambda) \frac{z - \lambda^*}{z - \lambda} \right\| \\ &\leq \varepsilon^*(z) := 2 \sum_{\lambda \neq \lambda^*} \left\| \rho(\lambda) \frac{z - \lambda^*}{z - \lambda} \right\| \end{aligned}$$

(using (5.8)). By dominated convergence,

$$\lim \varepsilon^*(z) = 0 \quad \text{as } z \rightarrow \lambda^* \text{ radially} \quad (5.10)$$



(because  $z \in [0, \lambda^*] \Rightarrow |z - \lambda^*| < |z - \lambda|$ , thus  $\|\rho(\lambda) \frac{z - \lambda^*}{z - \lambda}\| \leq \|\rho(\lambda)\|$ ).

Let  $\epsilon > 0$ . For every  $j \geq 1$  and  $z \in [0, \lambda^*)$ , we have

$$\|\delta_j(\lambda^*)\| \leq \|(z - \lambda^*)\Sigma(\delta_j)^+(z)\| + \epsilon^*(z). \quad (5.11)$$

By (5.10), we can choose  $z \in [0, \lambda^*)$  such that  $\epsilon^*(z) \leq \frac{\epsilon}{2}$ . Using (5.9), for that particular  $z$ , we can find  $J \geq 1$  such that for all  $j \geq J$  also the first term in the right-hand side of (5.11) is  $\leq \frac{\epsilon}{2}$ . This shows that  $\lim_{j \rightarrow \infty} \delta_j(\lambda^*) = 0$  for every  $\lambda^* \in \mathbb{S}$ , which was the desired conclusion.  $\square$

*Proposition 5.1 allows to complete the proof of Theorem 4.4.* Suppose that  $h(z) = -\sum_{n < 0} \beta_n z^n$  is an rrl-continuation of  $\Sigma(\rho)^+$ . We must show that  $h$  coincides with  $\Sigma(\rho)^-$ , i.e. that  $\beta_n = b_n$  for each  $n < 0$ .

By hypothesis, there is a strictly increasing sequence of positive integers  $(k_j)_{j \geq 1}$  such that  $\beta_n = \lim_{j \rightarrow \infty} b_{n+k_j}$  for each  $n < 0$  and  $b_n = \lim_{j \rightarrow \infty} b_{n+k_j}$  for each  $n \geq 0$ . Proposition 5.1 entails

$$\lim_{j \rightarrow \infty} \lambda^{k_j} = 1 \quad \text{for each } \lambda \in \text{supp } \rho,$$

therefore, for each  $n < 0$ ,  $b_{n+k_j} = -\sum_{\lambda \in \mathbb{S}} \rho(\lambda) \lambda^{-n-1} \cdot \lambda^{-k_j} \xrightarrow{j \rightarrow \infty} b_n$  by the dominated convergence theorem (since  $\|\rho(\lambda) \lambda^{-n-1} \cdot \lambda^{-k_j}\| \leq \|\rho(\lambda)\|$ ), whence  $b_n = \beta_n$ .  $\square$

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## References

- [Ag49] S. Agmon, “Sur les séries de Dirichlet”, *Ann. Sci. École Norm. Sup. (3)* **66** (1949), 263–310.
- [BMS12] V. Baladi, S. Marmi, and D. Sauzin, “Natural boundary for the susceptibility function of generic piecewise expanding unimodal maps”, to appear in *Ergodic Theory & Dynamical Systems*, 24 pages. Available on CJO FirstView Article / January 2013, pp. 1-24. <http://dx.doi.org/10.1017/etds.2012.161>
- [Bo17] E. Borel, “Leçons sur les fonctions monogènes uniformes d’une variable complexe. Rédigées par G. Julia”, Gauthier-Villars, Paris, 1917.
- [BS11] J. Breuer and B. Simon, “Natural boundaries and spectral theory”, *Adv. Math.* **226** (2011) 4902–4920.
- [CMS14] C. Carminati, S. Marmi and D. Sauzin, “There is only one KAM curve”, *Nonlinearity* **27** (2014), 28 pages (*in print*).
- [Ca57] J. W. S. Cassels, “An introduction to Diophantine approximation”, Cambridge Tracts in Mathematics and Mathematical Physics **45**, Cambridge University Press, New York, 1957.
- [HP57] E. Hille and R.S. Phillips, “Functional analysis and semi-groups”, American Mathematical Society Colloquium Publications **31**, rev. ed, American Mathematical Society, Providence, 1957.
- [MS03] S. Marmi and D. Sauzin, “Quasianalytic monogenic solutions of a cohomological equation”, *Mem. Amer. Math. Soc.* **164** (2003), no. 780.
- [MS11] S. Marmi and D. Sauzin, “A quasianalyticity property for monogenic solutions of small divisor problems”, *Bulletin of the Brazilian Mathematical Society, New Series* **42**, 1 (2011), 45–74. <http://dx.doi.org/10.1007/s00574-011-0003-x>

- [MT88] J. Milnor and W. Thurston, “On iterated maps of the interval”, in *Dynamical systems (College Park, MD, 1986–87)*, Lecture Notes in Math. **1342**, 465–563, Springer, Berlin, 1988.
- [Po83] H. Poincaré, “Sur les fonctions à espaces lacunaires”, *Acta Societatis scientiarum Fennicae*, **12** (1883), 343–350.
- [Po92] H. Poincaré, “Sur les fonctions à espaces lacunaires”, *American Journal of Mathematics*, **14** (1892), no. 3, 201–221.
- [RS02] W. T. Ross and H. S. Shapiro, *Generalized analytic continuation*, University Lecture Series, 25. American Mathematical Society, Providence, RI, 2002.

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