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# PROBABILISTIC STRUCTURAL
# MODELING IN LINEAR DYNAMIC ANALYSIS
# OF COMPLEX MECHANICAL SYSTEMS
# II. NUMERICAL ANALYSIS AND APPLICATIONS

by

F. CHABAS    , A. DESANTI    and C. SOIZE

ABSTRACT

This paper concerns the numerical developments of a probabilistic modeling of structural fuzzy in complex mechanical systems, the theoretical foundations of which are laid down in the first part of the paper.

We present further numerical analysis needed to implant the method in a finite element solver, along with certain elements identifying the behavioral law of structural fuzzy we have built.

We present a numerical simulation of the method on standard structures (beam-plate-shell). We show that the first-order approximation of the random solution is sufficient for dispersion parameters of the fuzzy up to at least 50%, which illustrates the advantage of the method from a practical point of view.

# I. – INTRODUCTION

In the first part of this paper we laid down the theoretical foundations of the probabilistic modeling of structural fuzzy in complex structures. We recall from that discussion that, in the linear dynamic analysis of complex systems, the theory proposed provides a way of expressing elements or parts of the structure that contribute effectively to the system dynamics but which are either inaccessible or nearly so by conventional finite element modeling. The method consists of introducing these elements globally in the form of a probabilistic surface impedance constructed from a probabilistic law of fuzzy behavior, under which conditions the state variables of the system are modeled by random variables and we thus have to solve a linear equation with random operators.

In this second section of our paper we present additional developments and elements for validating the proposed theory numerically. After a brief review of a method for analyzing structures in the medium-frequency domain, we explain the application of this method to the solution of the linear vibration equation of a master structure with its fuzzy.

We then go into a numerical parametric analysis of the fuzzy constitutive equation itself, to identify roughly the fuzzy class it defines.

Finally, in the third section, the theory is validated directly on standard beam, plate and shell structures using a numerical simulation which establishes a domain of validity for the method along with the conditions under which it can be applied in practice.

This second part of our paper will use everywhere the same notation and certain relations as in the first part, but without explaining them again to avoid repetition.

# II. – FURTHER ELEMENTS IN THE DEVELOPMENT

## II.1. – REVIEW OF A METHOD OF ANALYSIS IN THE MEDIUM FREQUENCY DOMAIN

Let us consider the following linear equation over $\mathbb{C}^N$ in the Fourier space:

$$(-\omega^2 \mathcal{M}(\omega) + i\omega \mathcal{C}(\omega)$$
$$+ K(\omega)) \hat{U}(\omega) = \hat{F}(\omega), \qquad \omega \geq 0 \quad (1)$$

where, for any $\omega \geq 0$, $\mathcal{M}(\omega)$, $\mathcal{C}(\omega)$ and $K(\omega) \in \text{Mat}(N, N)$ are real definite, positive $N \times N$ matrices, where $t \mapsto F(t) = (F_1(t), \dots, E_N(t))$ [resp. $t \mapsto U(t) = (U_1(t), \dots, U_N(t))$] is a function defined over $\mathbb{R}$, with values in $\mathbb{C}^N$, having a Fourier transform denoted $\omega \mapsto \hat{F}(\omega) = (\hat{F}_1(\omega), \dots, \hat{F}_N(\omega))$. [resp. $\omega \mapsto \hat{U}(\omega) = (\hat{U}_1(\omega), \dots, \hat{U}_N(\omega))$], defined over $\mathbb{R}$ with values in $\mathbb{C}^N$.

The function $U$ represents the mechanical variables, for example the displacements, and $F$ is given and corresponds for example to the forces. We further assume that the applications $\omega \mapsto \mathcal{M}(\omega)$, $\mathcal{C}(\omega)$ and $K(\omega)$ are piecewise continuous over $\mathbb{R}^+ = [0, +\infty]$.

According to these hypotheses, if $F \in H_B(\mathbb{R}, \mathbb{C}^N)$ with $B$ being a bounded ramdom closed interval of $\mathbb{R}^+$ then $U \in H_B(\mathbb{R}, \mathbb{C}^N)$ (*see* Part I).

### II,1.1. – *Band of analysis and MF band*

Let $B_n = \left[\Omega_n - \dfrac{\Delta\omega}{2}; \Omega_n + \dfrac{\Delta\omega}{2}\right]$ be a bounded closed interval of $\mathbb{R}^+$ with a central frequency $\Omega_n > 0$, of bandwidth $\Delta\omega > 0$ such that $\Omega_n - \dfrac{\Delta\omega}{2} > 0$. We denote the open interval associated with $B_n$ as $\mathring{B}_n$.

A narrow MF band is an interval $B_n$ such that $\Delta\omega/\Omega_n \ll 1$. The centered LF band associated with $B_n$ will be denoted $B_0 = \left[-\dfrac{\Delta\omega}{2}; \dfrac{\Delta\omega}{2}\right]$.

Let $B = [\omega_I; \omega_F]$ $(0 < \omega_I < \omega_F)$ be a closed bounded interval of $\mathbb{R}^+$. This interval, *a priori* of random width, will be called the band of analysis.

In the MF method, this band is partitioned into a finite number $N_B$ of narrow disjunct MF bands $\left(B = \bigcup\limits_{n=1}^{N_B} B_n\right)$ and the MF analysis is carried out on each $B_n$ separately. We will further assume that the partitioning is done in such a way that the applications $\omega \mapsto \mathcal{M}(\omega)$, $\omega \mapsto \mathcal{C}(\omega)$ and $\omega \mapsto K(\omega)$ are continuous over each $B_n$ and are thus also bounded in consideration of the hypothesis of section II,1 above.

Consequently, for the MF algorithm, the solution of (1) for $F \in H_B(\mathbb{R}, \mathbb{C}^N)$ comes down to solving $N_B$ independent equations of the type:

$$(-\omega^2 \mathcal{M}(\omega) + i\omega \mathcal{C}(\omega) + K(\omega)) \hat{U}_{(n)}(\omega) = \hat{F}_n(\omega). \quad (2)$$

with $F_n \in H_{B_n}(\mathbb{R}, \mathbb{C}^N)$ and $\hat{U}_{(n)}(\omega) = \hat{U}(\omega) \mathbf{1}_{B_n}(\omega)$, where $\omega \mapsto \mathbf{1}_{B_n}(\omega)$ is the function indicating the interval $B_n$.

### II,1.2. – *Approximate solution over $B_n$*

Let $\omega \mapsto A(\omega)$ be a continuous, bounded application of $\mathring{B}_n$ in $\text{Mat}(N, N)$. We can then define the mean matrix $\Delta_n$ over $B_n$ such that

$$A_n = \frac{1}{\Delta\omega} \int_{B_n} A(\omega) \, d\omega. \quad (3)$$

Denoting the mean matrices over $B_n$ associated with $\mathcal{M}(\omega)$, $\mathcal{C}(\omega)$ and $K(\omega)$ (which exist, considering the

hypothesis) as $\mathscr{M}_n$, $\mathscr{C}_n$ and $K_n$, we define the approximate solution $\hat{U}_n(\omega)$ of equation (2) relative to the band $B_n$, which is such that for $F \in H_{B_n}(\mathbb{R}, \mathbb{C}^N)$:

$$(-\omega^2 \mathscr{M}_n + i\omega \mathscr{C}_n + K_n)\, \hat{U}_n(\omega) = \hat{F}_n(\omega). \qquad (4)$$

We can easily check that, as $\Delta\omega \to 0$, $\hat{U}_n(\omega)$ tends toward the true solution $\hat{U}_{(n)}(\omega)$ for the norm of $L^2(\mathbb{R}, \mathbb{C}^N)$. It is nonetheless clear that the approximation is of interest in practice only if the convergence does not require us to take $\Delta\omega$ too small. This problem is addressed in [78] and we will not come back to it.

II,1.3. — *Expression of the approximate solution over $B_n$*

The MF algorithm method consists of introducing two time scales, one short time scale $\tau_c$ associated with the central frequency of the band and one long time scale $\tau_1$ associated with the bandwidth $B_n$. We thus associate with the functions $t \mapsto F_n(t)$ and $t \mapsto U_n(t)$ of $H_{B_n}(\mathbb{R}, \mathbb{C}^N)$ the functions $t \mapsto F_0(t)$ and $t \mapsto U_0(t)$ defined by:

$$F_0(t) = F_n(t)\, e^{-i\Omega_n t} \qquad (5)$$

$$U_0(t) = U_n(t)\, e^{-i\Omega_n t}. \qquad (6)$$

Trivially we can verify that $F_0$, $U_0$, $\dot{U}_0$, $\ddot{U}_0$ (where $\dot{U}$ designates the derivative of $U$ with respect to $t$) are in $H_{B_0}(\mathbb{R}, \mathbb{C}^N)$. Applying the sampling theorem and by Fourier transform we then get the expression for the approximate solution $U_n$ and its derivatives:

$$\hat{U}_n(\omega) = \mathbf{1}_{B_n}(\omega)\, \tau_l \sum_{m \in \mathbb{Z}} U_0(m\tau_l)\, e^{-im\tau_l(\omega - \Omega_n)} \qquad (7)$$

$$\hat{\dot{U}}_n(\omega) = \mathbf{1}_{B_n}(\omega)\, \tau_l \sum_{m \in \mathbb{Z}} (\dot{U}_0(m\tau_l)$$
$$+ i\Omega_n U_0(m\tau_l))\, e^{-im\tau_l(\omega - \Omega_n)} \qquad (8)$$

$$\hat{\ddot{U}}_n(\omega) = \mathbf{1}_{B_n}(\omega)\, \tau_l \sum_{m \in \mathbb{Z}} (\ddot{U}_0(m\tau_l)$$
$$+ 2i\Omega_n \dot{U}_0(m\tau_l)$$
$$- \Omega_n^2 U_0(m\tau_l))\, e^{-im\tau_l(\omega - \Omega_n)} \qquad (9)$$

where $\tau_l = \dfrac{2\pi}{\Delta\omega}$, the series being convergent in $L^2(\mathbb{R}, \mathbb{C}^N)$.

These relations show that the solution $U_n$ will be entirely determined once we know $U_0$, $\dot{U}_0$, $\ddot{U}_0$ at times $m\tau_l$, $m \in \mathbb{Z}$.

*Note:* We can observe that the algebraic treatment of the high-order frequency components of the $F_n$ and $U_n$ signals [relations (5) and (6)] introduce no approximation in the method, the only errors of a numerical order being related to the computation of the LF solution $U_0$.

II,1.4. — *Computation of the Low Frequency Solution $U_0$*

Let us consider the time equation associated with (4) that is written

$$\mathscr{M}_n \ddot{U}_n(t) + \mathscr{C}_n \dot{U}_n(t) + K_n U_n(t) = F_n(t). \qquad (10)$$

Substituting (5) and (6) in (10), we verify that $U_0$ is the solution of

$$\mathscr{M}_n \ddot{U}_0 + \mathbb{C}_n \dot{U}_0(t) + \mathbb{K}_n U_0(t) = F_0(t) \qquad (11)$$

with

$$\mathbb{C}_n = \mathscr{C}_n + 2i\Omega_n \mathscr{M}_n \qquad (12)$$

$$\mathbb{K}_n = K_n + i\Omega_n \mathscr{C}_n - \Omega_n^2 \mathscr{M}_n. \qquad (13)$$

Since $\operatorname{supp} \hat{U}_0 = \operatorname{supp} \hat{F}_0 = \left[-\dfrac{\Delta\omega}{2}; \dfrac{\Delta\omega}{2}\right]$, (11) is an equation in the low frequency domain and can be solved by any unconditional stable implicit numerical integration method. We chose a modified Newmark scheme [78].

Also, as $F_0 \in H_{B_0}(\mathbb{R}, \mathbb{C}^N)$, $\forall \varepsilon_0 > 0$, $\exists m_0 \in \mathbb{Z}$ such that

$$\int_{-\infty}^{m_0 \tau_l} \| F_0(t) \|_N^2 \, dt < \varepsilon_0. \qquad (14)$$

Similarly, $U_0 \in H_{B_0}(\mathbb{R}, \mathbb{C}^N)$, thus $\forall \varepsilon_1 > 0$, $\exists m_1 \in \mathbb{Z}$ such that

$$\int_{m_1 \tau_l}^{+\infty} \| U_0(t) \|_N^2 \, dt < \varepsilon_1. \qquad (15)$$

Then, taking the initial conditions $U_0(t) = \dot{U}_0(t) = 0$ for $t \leq m_0 \tau_l$, the numerical integration is carried out between times $m_0 \tau_l$ and $m_1 \tau_l$. Of course under these conditions, the summations in relations (7) to (9) must be replaced by $\displaystyle\sum_{m=m_0}^{m_1}$.

II,2. — COMPUTATION OF THE RANDOM SOLUTION OVER $B_n$

The equation of the vibrations over $B_n$ of the master structure with its fuzzy, discretized with $N$ degrees of freedom, is written

$$(Z_{sF}(\omega) + \mathbf{Z}_F(\omega, \lambda(\omega)))\, \hat{U}_n(\omega) = \hat{F}_n(\omega) \qquad (16)$$

where $Z_{sF}$ is a deterministic impedance, $\mathbb{Z}_F$ is a random impedance defined over $(\mathscr{A}, \mathscr{T}, \mathscr{P})$ with values Mat $(N, N)$, centered, and where $F_n \in H_{B_n}(\mathbb{R}, \mathbb{C}^N)$.

We have shown in part one that under certain hypotheses, which we will assume to be verified here,

the solution $\hat{U}_n$ of (16) is written $\mathscr{P}$ almost surely (or "a. s.") for $d\omega$ almost all $\omega$ in $B_n$:

$$\hat{U}_n(\omega) = \hat{U}_n^{(0)}(\omega)$$

$$+ \sum_{k=1}^{+\infty} \left\{ \sum_{l_1=1}^{L} \cdots \sum_{l_k=1}^{L} X_{l_1} \times \cdots \right.$$

$$\left. \times X_{l_k} \hat{U}_{l_1 \ldots l_k, n}^{(k)}(\omega) \right\} \quad (17)$$

where $\hat{U}_n^{(0)}$ and $\hat{U}_{l_1 \ldots l_k, n}^{(k)}$ $(k \in \mathbb{N}^*)$ belong to $H_{B_n}(\mathbb{R}, \mathbb{C}^N)$ and where the $X_{l_k}$ terms are random variables defined over $(\mathscr{A}, \mathscr{T}, \mathscr{P})$, with real values, generally independent, having a uniform probability with support $[-\sqrt{3}, \sqrt{3}]$. They are thus centered and have a variance of unity.

### II,2.1. — *Computation of the mean solution*

According to section I, $\hat{U}_n^{(0)}$ is such that:

$$\begin{rcases} (-\omega^2 \mathscr{M}_n + i\omega \mathscr{C}_n + K_n) \hat{U}_n^{(0)}(\omega) = \hat{F}_n(\omega), \\ F_n \in H_{B_n}(\mathbb{R}, \mathbb{C}^N). \end{rcases} \quad (18)$$

Considering the hypotheses, (18) is analogous to the standard MF equation (4) and all of the results in section II,1, can be applied directly in computing $\hat{U}_n^{(0)}$.

### II,2.2. — *Computation of the random fluctuation terms*

According to part I, the $\hat{U}_n^{(0)}$ terms are given by the following recurrence:

$$(-\omega^2 \mathscr{M}_n + i\omega \mathscr{C}_n + K_n) \hat{U}_{l_1 \ldots l_k, n}^{(k)}(\omega) = \hat{Q}_{l_1 \ldots l_k, n}(\omega) \quad (19)$$

where

$$\begin{rcases} \hat{Q}_{l_1 \ldots l_k, n}(\omega) = -(-\omega^2 R_{l_k, n} \\ \qquad + i\omega I_{l_k, n}) \hat{U}_{l_1 \ldots l_{k-1}, n}^{(k-1)}(\omega), \\ Q_{l_1 \ldots l_k, n} \in H_{B_n}(\mathbb{R}, \mathbb{C}^N). \end{rcases} \quad (20)$$

Here again, equation (19) is of the same form as (4) and the MF method can be applied.

Let $U_{l_1 \ldots l_k, n_0}^{(k)}$ and $Q_{l_1 \ldots l_k, n_0}$ be the low frequency of the $U_{l_1 \ldots l_k, n}^{(k)}$ and $Q_{l_1 \ldots l_k, n}$ signals, defined by:

$$U_{l_1 \ldots l_k, n_0}^{(k)}(t) = U_{l_1 \ldots l_k, n}^{(k)}(t) e^{-i\Omega_n t} \quad (21)$$

$$Q_{l_1 \ldots l_k, n_0}(t) = Q_{l_1 \ldots l_k, n}(t) e^{-i\Omega_n t}. \quad (22)$$

Supposing the solution is known at the $(k-1)$ order of the recurrence, *i.e.* in fact the $U_{l_1 \ldots l_k, n_0}^{(k-1)}(m\tau_l), m \in \mathbb{Z}$, we have to compute $Q_{l_1 \ldots l_k, n_0}$ from (20) to solve the low frequency equation at order $k$:

$$\mathscr{M}_n \ddot{U}_{l_1 \ldots l_k, n_0}^{(k)}(t) + \mathbb{C}_n \dot{U}_{l_1 \ldots l_k, n_0}^{(k)}(t)$$

$$+ \mathbb{K}_n U_{l_1 \ldots l_k, n_0}^{(k)}(t) = Q_{l_1 \ldots l_k, n_0}(t). \quad (23)$$

A simple calculation gives:

$$Q_{l_1 \ldots l_k, n_0}(t) = -\frac{\tau_l}{2\pi} \sum_{m \in \mathbb{Z}} U_{l_1 \ldots l_k, n_0}^{(k-1)}(m\tau_l)$$

$$\times (-R_{l_k, n}^{(m)}(t) + i I_{l_k, n}^{(m)}(t)) \quad (24)$$

where

$$R_{l_k, n}^{(m)}(t) = R_{l_k, n} [\mathscr{K}_m(t) + 2\Omega_n \mathscr{J}_m(t) + \Omega_n^2 \mathscr{I}_m(t)] \quad (25)$$

$$I_{l_k, n}^{(m)}(t) = I_{l_k, n} [\mathscr{J}_m(t) + \Omega_n \mathscr{I}_m(t)] \quad (26)$$

with

$$\begin{rcases} \mathscr{I}_m(t) = 2 \dfrac{\sin(t - m\tau_l)(\Delta\omega/2)}{(t - m\tau_l)} & \text{if } t \neq m\tau_l \\ \mathscr{I}_m(m\tau_l) = \Delta\omega \end{rcases} \quad (27)$$

$$\begin{rcases} \mathscr{J}_m(t) = \dfrac{1}{i(t - m\tau_l)} \left\{ \Delta\omega \cos\left[(t - m\tau_l)\dfrac{\Delta\omega}{2}\right] \right. \\ \qquad \left. - \mathscr{I}_m(t) \right\} & \text{if } t \neq m\tau_l \\ \mathscr{J}_m(m\tau_l) = 0 \end{rcases} \quad (28)$$

$$\begin{rcases} \mathscr{K}_m(t) = \dfrac{1}{i(t - m\tau_l)} \\ \quad \times \left\{ \dfrac{i}{2}\Delta\omega^2 \sin\left[(t - m\tau_l)\dfrac{\Delta\omega}{2}\right] \right. \\ \qquad \left. - 2\mathscr{J}_m(t) \right\} & \text{if } t \neq m\tau_l \\ \mathscr{K}_m(m\tau_l) = \dfrac{2}{3}\left(\dfrac{\Delta\omega}{2}\right)^3. \end{rcases} \quad (29)$$

The $k$ order solution $\hat{U}_{l_1 \ldots l_k, n}^{(k)}$ of the recurrence is then given by the relation (7) to (9).

*Note:* Relation (24) uses only the values of $\hat{U}_{l_1 \ldots l_k, n_0}^{(k-1)}$ at the sampling points and not those of its derivatives. This is to minimize the numerical errors due to the integration scheme.

### II,3. — OBSERVATIONS OF THE MECHANICAL SYSTEM

### II,3.1. — *Frequency response functions*

The displacement frequency response of function $T_n^d(\omega)$ is the random variable defined over $(\mathscr{A}, \mathscr{T}, \mathscr{P})$ with values in Mat $(N, N)$ such that, for $F_n \in H_{B_n}(\mathbb{R}, \mathbb{C}^N)$:

$$\hat{U}_n(\omega) = T_n^d(\omega) \hat{F}_n(\omega) \quad (30)$$

where $\hat{U}_n$ is the solution of (16) constructed in II,2.

Let $j \in \{1, 2, \ldots, N\}$ be a fixed degree of freedom of the structure and $F_n^j$ be the application of $H_{B_n}(\mathbb{R}, \mathbb{C}^N)$ such that

$$\left.\begin{array}{l} [F_n^j(t)]_k = 0 \quad \text{si} \quad k \neq j \\ [F_n^j(t)]_j = \dfrac{1}{\pi t} \sin \dfrac{t \Delta \omega}{2} e^{i\Omega_n t} \end{array}\right\} \quad (31)$$

$F_n^j$ is then such that $[\hat{F}_n^j(\omega)]_j = \mathbf{1}_{B_n}(\omega)$.
Let $U_n^j \in H_{B_n}(\mathbb{R}, \mathbb{C}^N)$ be such that:

$$\hat{U}_n^j(\omega) = T_n^d(\omega) \hat{F}_n^j(\omega). \quad (32)$$

The relations (17) and (32) show that the displacement crossed frequency response function relative to the degree of freedom $q$ is written $\mathscr{P}$-a. s. for $d\omega$-almost all $\omega$ in $B_n$:

$$[T_n^d(\omega)]_{qj} = [\hat{U}_n^{j, (0)}(\omega)]_q$$

$$+ \sum_{k=1}^{+\infty} \left\{ \sum_{l_1=1}^{L} \cdots \sum_{l_k=1}^{L} X_{l_1} \times \cdots \right.$$

$$\left. \times X_{l_k} [\hat{U}_{l_1 \ldots l_k, n}^{j, (k)}(\omega)]_q \right\} \quad (33)$$

where $\hat{U}_n^{j, (0)}$ and $\hat{U}_{l_1 \ldots l_k, n}^{j, (k)} \ (k \in \mathbb{N}^*)$ are constructed as in section II,2. by setting $F_n = F_n^j$ in (16).

The direct frequency response functions are found for $q = j$ in (33).

We can also define the velocity and acceleration frequency response functions, written respectively ($\mathscr{P}$ a. s. for $dw$-almost all $\omega$ in $B_n$):

$$[T_n^v(\omega)]_{qj} = [\hat{\dot{U}}_n^{j, (0)}(\omega)]_q$$

$$+ \sum_{k=1}^{+\infty} \left\{ \sum_{l_1=1}^{L} \cdots \sum_{l_k=1}^{L} X_{l_1} \times \cdots \right.$$

$$\left. \times X_{l_k} [\hat{\dot{U}}_{l_1 \ldots l_k, n}^{j, (k)}(\omega)]_q \right\}. \quad (34)$$

$$[T_n^a(\omega)]_q = [\hat{\ddot{U}}_n^{j, (0)}(\omega)]_q$$

$$+ \sum_{k=1}^{+\infty} \left\{ \sum_{l_1=1}^{L} \cdots \sum_{l_k=1}^{L} X_{l_1} \times \cdots \right.$$

$$\left. \times X_{l_k} [\hat{\ddot{U}}_{l_1 \ldots l_k, n}^{j, (k)}(\omega)]_q \right\}. \quad (35)$$

### II,3.2. — Boundary condition impedances

Let $j_1, j_2, \ldots, J_I$ be the $I$ degrees of freedom selected with $\forall i \in \{1, 2, \ldots, I\}$, $j_i \in \{1, 2, \ldots, N\}$. Then $I \leq N$ (and even, most often, $I \ll N$).

For any $j_i$, $i \in \{1, 2, \ldots, N\}$ we consider the excitation $F_n^{j_i}$ of $H_{B_n}(\mathbb{R}, \mathbb{C}^N)$ defined by (31) for $j = j_i$. Following the method explained in section II,3.1, we compute for each fixed $j_i$ the random response $\hat{U}_n^{j_i}(\omega)$ defined over $(\mathscr{A}, \mathscr{T}, \mathscr{P})$ that is due to the excitation $F_n^{j_i}$ and we get from this, for all

$k \in \{1, 2, \ldots, I\}$, the direct and crossed replacement frequency response functions that are written, according to (34):

$$[T_n^d(\omega)]_{j_k j_i} = [\hat{U}_n^{j_i, (0)}(\omega)]_{j_k}$$

$$+ \sum_{k'=1}^{+\infty} \left\{ \sum_{l_1=1}^{L} \cdots \sum_{l_{k'}=1}^{L} X_{l_1} \times \cdots \right.$$

$$\left. \times X_{l_{k'}} [\hat{U}_{l_1 \ldots l_{k'}, n}^{j_i, (k')}(\omega)]_{j_k} \right\}. \quad (36)$$

Considering all of the excitations $F_n^{j_i}$ as $i$ describes $\{1, 2, \ldots, I\}$, we thereby construct a random $I \times I$ matrix $\theta_n(\omega)$, defined over $(\mathscr{A}, \mathscr{T}, \mathscr{P})$, that is complex symmetrical and such that, for any $i$ and any $k$ in $\{1, 2, \ldots, I\}$:

$$[\theta_n(\omega)]_{ki} = [T_n^d(\omega)]_{j_k j_i}. \quad (37)$$

Considering (36), the random matrix $\theta_n(\omega)$ is written:

$$\theta_n(\omega) = \theta_n^{(0)}(\omega) + \tilde{\theta}_n(\omega) \quad (38)$$

with

$$\tilde{\theta}_n(\omega) = \sum_{k'=1}^{+\infty} \left\{ \sum_{l_1=1}^{L} \cdots \sum_{l_{k'}=1}^{L} X_{l_1} \times \cdots \right.$$

$$\left. \times X_{l_{k'}} \theta_{l_1 \ldots l_{k'}, n}^{(k')}(\omega) \right\} \quad (39)$$

where $\theta_n^{(0)}(\omega)$ and $\theta_{l_1 \ldots l_{k'}, n}^{(k')}(\omega)$ (for all $k'$ in $\mathbb{N}$ and ell the multi-indices $l_1 \ldots l_{k'}$) are deterministic, complex, symmetrical $I \times I$ matrices such that, for any $i$ and any $k$ in $\{1, 2, \ldots, I\}$:

$$[\theta_n^{(0)}(\omega)]_{ki} = [\hat{U}_n^{j_i, (0)}(\omega)]_{j_k} \quad (40)$$

$$[\theta_{l_1 \ldots l_{k'}, n}^{(k')}(\omega)]_{ki} = [\hat{U}_{l_1 \ldots l_{k'}, n}^{j_i, (k')}(\omega)]_{j_k}. \quad (41)$$

Let us also suppose that all of the degrees of freedom $j_1, j_2 \ldots, j_I$ selected are free. Then for any $\omega$ in $B_n$, $\theta_n^{(0)}(\omega)^{-1}$ exists and there exists a $\tilde{\lambda}_F > 0$ such that, for $\mathrm{Sup}_{\omega \in B_n} \| \lambda(\omega) \|_\infty \leq \tilde{\lambda}_F$, $\theta_n(\omega)^{-1}$, $\mathscr{P}$ almost surely exists.

Under these conditions, the $I \times I$ random boundary condition impedance matrix defined over $(\mathscr{A}, \mathscr{T}, \mathscr{P})$ relative to the degrees of freedom $j_1, j_2 \ldots, j_I$ is written:

$$\Theta_n(\omega) = \theta_n(\omega)^{-1}. \quad (42)$$

We also have the following result:
The boundary condition impedance matrix relative to the degrees of freedom $j_1, j_2 \ldots, j_I$ is written $\mathscr{P}$ a. s. for $d\omega$-almost all $\omega$ in $B_n$:

$$\Theta_n(\omega) = \theta_n^{(0)}(\omega)^{-1}$$

$$+ \sum_{k=1}^{+\infty} (-1)^k [\theta_n^{(0)}(\omega)^{-1} \tilde{\theta}_n(\omega)]^k \theta_n^{(0)}(\omega)^{-1} \quad (43)$$

where $\theta_n^{(0)}(\omega)$ and $\tilde{\theta}_n(\omega)$ are given by (38), (39) and (40), the series being $\mathscr{P}$ a. s. convergent.

The proof involves the same process as the construction of the random solution established in section I, and will not be explained here.

II,3.3. — *Second order characteristics of the mechanical observations associated with a first order expansion of the random solution*

In this section we discuss the case where the random solution $\hat{U}_n$ is expanded to the first order, which is the only case where the second order characteristics of the observations can be obtained reasonably in algebraic form.

According to (17) the first order expansion of $\hat{U}_n$ is written:

$$\hat{U}_n(\omega) = \underline{\hat{U}}_n(\omega) + \sum_{l=1}^{L} X_l \hat{U}_l(\omega) \qquad (44)$$

where we let $\hat{U}_n^{(0)}(\omega) = \underline{U}_n(\omega)$.

Under these conditions, relations (33) to (35) become, with obvious notation:

$$[T_n^d(\omega)]_{qj} = [\underline{\hat{U}}_n^j(\omega)]_q + \sum_{l=1}^{L} X_l [\hat{U}_l^j(\omega)]_q \qquad (45)$$

$$[T_n^v(\omega)]_{qj} = [\underline{\hat{U}}_n^j(\omega)]_q + \sum_{l=1}^{L} X_l [\hat{U}_l^j(\omega)]_q \qquad (46)$$

$$[T_n^a(\omega)]_{qj} = [\underline{\hat{U}}_n^j(\omega)]_q + \sum_{l=1}^{L} X_l [\hat{U}_l^j(\omega)]_q \qquad (47)$$

and relation (43) is written:

$$\Theta_n(\omega) = \underline{\theta}_n(\omega)^{-1}$$

$$+ \sum_{l=1}^{L} - X_l \underline{\theta}_n(\omega)^{-1} \underline{\theta}_l(\omega) \underline{\theta}_n(\omega)^{-1} \qquad (48)$$

with, for any $i$ and $k$ in $\{1, 2, \ldots, I\}$:

$$[\underline{\theta}_n(\omega)]_{ki} = [\underline{\hat{U}}_n^{j_i}(\omega)]_{jk}$$

and

$$[\theta_l(\omega)]_{ki} = [\hat{U}_l^{j_i}(\omega)]_{jk}. \qquad (49)$$

As relations (44) to (48) are all of the same form, we introduce the random variable $Y(\omega)$ with values in $\mathbb{C}$ such that:

$$\hat{Y}(\omega) = \underline{\hat{Y}}(\omega) + \sum_{l=1}^{L} X_l \hat{y}_l(\omega) \qquad (50)$$

where $Y$ and $y_l (l = 1, 2, \ldots, L)$ are in $H_{B_n}(\mathbb{R}, \mathbb{C})$ and where the $X_l (l = 1, 2, \ldots, L)$ are uniform orthonormed random variables in $L^2(\mathscr{A}, \mathbb{R})$ (*see* section I).

The following are the second order characteristics [mean ($\mathscr{E}$) and variance ($\sigma^2$)] of the quantities related to $Y(\omega)$. The computations are simple but sometimes fastidious and are not reproduced here.

*Real part*

$$\mathscr{E}\{\operatorname{R\acute{e}}(\hat{Y}(\omega))\} = \operatorname{R\acute{e}}(\underline{\hat{Y}}(\omega)) \qquad (51)$$

$$\sigma^2_{\operatorname{R\acute{e}}(\hat{Y}(\omega))} = \sum_{l=1}^{L} [\operatorname{R\acute{e}}(\hat{Y}_l(\omega))]^2. \qquad (52)$$

*Imaginary part*

$$\mathscr{E}\{\operatorname{Im}(\hat{Y}(\omega))\} = \operatorname{Im}\{\underline{\hat{Y}}(\omega)\} \qquad (53)$$

$$\sigma^2_{\operatorname{Im}(\hat{Y}(\omega))} = \sum_{l=1}^{L} [\operatorname{Im}(\hat{Y}_l(\omega))]^2. \qquad (54)$$

*Square of the modulus*

$$\mathscr{E}(|\hat{Y}(\omega)|^2) = |\underline{\hat{Y}}(\omega)|^2 + \sum_{l=1}^{L} |\hat{Y}_l(\omega)|^2 \qquad (55)$$

$$\sigma^2_{|\hat{Y}(\omega)|^2} = 4 \sum_{l=1}^{L} [\operatorname{R\acute{e}}(\underline{\hat{Y}}(\omega) \overline{\hat{Y}}_l(\omega))]^2$$

$$- \frac{6}{5} \sum_{l=1}^{L} |\hat{Y}_l(\omega)|^4$$

$$+ 2 \sum_{l=1}^{L} \sum_{l'=1}^{L} [\operatorname{R\acute{e}}(\hat{Y}_l(\omega) \overline{\hat{Y}}_{l'}(\omega))]^2. \qquad (56)$$

*Energy in a sub-band* $B_n^i (B_n^i \subset B_n)$

This is the random variable defined by:

$$E_{B_n^i} = \frac{1}{2\pi} \int_{B_n^i} |\hat{Y}(\omega)|^2 \, d\omega. \qquad (57)$$

we have

$$\mathscr{E}(E_{B_n^i}) = \frac{1}{2\pi} \left\{ \int_{B_n^i} |\underline{\hat{Y}}(\omega)|^2 \right.$$

$$\left. d\omega + \sum_{l=1}^{L} \int_{B_n^i} |\hat{Y}_l(\omega)|^2 \, d\omega \right\} \qquad (58)$$

$$\sigma^2_{E_{B_n^i}} = \frac{1}{4\pi^2} \left\{ 4 \sum_{l=1}^{L} \left[ \int_{B_n^i} \operatorname{R\acute{e}}(\underline{\hat{Y}}(\omega) \overline{\hat{Y}}_l(\omega)) \, d\omega \right]^2 \right.$$

$$- \frac{6}{5} \sum_{l=1}^{L} \left( \int_{B_n^i} |\hat{Y}_l(\omega)|^2 \, d\omega \right)^2$$

$$\left. + 2 \sum_{l=1}^{L} \sum_{l'=1}^{L} \left[ \int_{B_n^i} \operatorname{R\acute{e}}(\hat{Y}_l(\omega) \overline{\hat{Y}}_{l'}(\omega)) \, d\omega \right]^2 \right\}. \qquad (59)$$

We recall that if $B_n^i = B_n$, the Plancherel theorem yields the following, since supp $\hat{Y} = B_n$:

$$E_{B_n} = \int_{\mathbb{R}} |Y(t)|^2 \, dt. \qquad (60)$$

*Note* 1: We cannot obtain the exact algebraic expression for the second-order characteristics of the modulus and of the phase. We could approximate by computing a Taylor expansion of these random variables; but the numerical findings show that the order of expansion required to achieve good accuracy is too large for the algebraic form of the results to remain simple. Recourse to a numerical simulation is necessary in this case, and this is the general method that should be used anyway for all observations when the random solution $\hat{U}$ is expanded to an order greater than one.

*Note* 2: The formulas (51) to (56) apply for a fixed $\omega$ in $B$. In other words, the expression (50) represents the general form of the relations (44) to (48) when the analysis in the frequency domain is made for discrete values of $\omega$.

In fact we may also be interested in knowing the means of the quantities computed over frequency band elements measuring $\delta\omega$ (it should be noted that $\delta\omega$ has no *a priori* relationship with the MF band $\Delta\omega$, but in general $\delta\omega < \Delta\omega$). Under these conditions, the mean operator applied to (50) yields:

$$\langle Y \rangle_\omega = \langle \underline{Y} \rangle_\omega + \sum_{l=1}^{L} \langle Y_l \rangle_\omega \qquad (61)$$

where, for any application $f$ defined over $\mathbb{R}$, we have let:

$$\langle f \rangle_\omega = \frac{1}{\delta\omega} \int_{\omega-(\delta\omega/2)}^{\omega+(\delta\omega/2)} f(\omega)\, d\omega. \qquad (62)$$

The $\omega$-dependent quantities in relations (51) to (56) then have to be replaced by their averaged values given by (62).

## III. – COMPUTER DEVELOPMENTS

All of the computer developments related to the fuzzy theory are implanted in the ADINA-ONERA software, which is a general finite element structural computation code for static and dynamic computations in linear and nonlinear analysis. In what concerns us here, for a deterministic or random steady excitation in time, this software can compute the linear LF or MF vibrations of a bounded elastic structure with or without fuzzy, in the presence or absence of an internal or internal (infinite or semi-infinite) compressible fluid. Substructuring is possible, even with finite elements of fuzzy in the substructures.

These developments conserve the optimization of the code, which means vectorizing the computations as much as possible and minimizing the disk transfers.

### III,1. – FINITE ELEMENTS OF FUZZY

For the constitutive equations of isotropic, orthotropic or anisotropic locally homogeneous fuzzy, the finite elements, with isoparametric formulations, are of the curvilinear type with two or three nodes and of the surface type with four to eight nodes.

We can emphasize that these elements generate a consistent and intrinsic damping matrix from the givens in the fuzzy constitutive equations.

### III,2. – COMPUTATION OF THE RANDOM SOLUTION

The program computes the random solution of the zero, first and second orders. For the zero order, only the mean part of the response is computed.

To do this, we use a "multicase excitation" computation sequence which, for a given $B_n$, solves equation (4) for different right-hand members simultaneously. Throughout the whole computation, the matrix system has to be triangularized only once, as the solution for each of the cases of excitation comes down to simply descending the matrix.

This sequence is used here for computing the fluctuating parts of the random solution [equation (20)] and for computing boundary condition matrices for which the structure must be excited at various points in succession. It has nonetheless been implanted in quite a general framework and can be used, for example, in analyzing the steady random vibrations by functional reduction of the excitation field [80].

The first- and second-order moments of the random observation variable are computed either by applying the relations of section I,3.3 or by a numerical estimation.

## IV. – NUMERICAL ANALYSIS OF THE FUZZY CONSTITUTIVE EQUATION

In this section we offer a few elements for identifying the probabilistic fuzzy constitutive equation described in part I of our publication.

The objective here is essentially to verify that this law does not involve any *a priori* incompatibility with the dynamics of the actual mechanical systems. In other words, and although the equation is apparently quite general in its application, we want to make sure that there is no fundamental reason it cannot actually represent the mechanical behavior of certain real fuzzies. This does not mean we are trying to list the types of fuzzy that might belong to the class of fuzzy defined by the proposed constitutive equation. This problem involves rather difficult analysis that will be addressed elsewhere.

For this identification, we have analyzed numerically the terms involved in the matrix of the constitutive equation and how they evolve as the various parameters supplying it vary. We refer the reader to the first part of this paper for the explicit algebraic form of these terms [relations (92) to (106)].

To simplify the computations involved in this parametric analysis, we take the scalar case (fuzzy with one degree of freedom) and a zero cutoff frequency. We also introduce the following hypotheses:

(a) The equivalent mean mass $\mu(\omega)$ of the fuzzy is constant in $\omega$. We can give it a value of unity since it enters the computation only in the form of a proportionality factor.

(b) The function defining the dispersions $\omega \mapsto \lambda(\omega) = (\lambda_1(\omega), \lambda_2(\omega), \lambda_3(\omega))$ at values in $\mathbb{R}^3$ is constant in $\omega$ such that $\lambda(\omega) = (\lambda, \lambda, \lambda)$, $(\lambda > 0)$ for any $\omega$ in $\mathbb{R}^+$. This hypothesis, which perhaps does not correspond to reality, is interesting because it brings out the relative effects of each of the three fluctuating parts of the probabilistic impedance.

(c) The mean damping rate $\omega \mapsto \xi(\omega)$ is constant in $\omega$.

For all of the computations the frequency domain is from 1 to 10,000 Hz. From a numerical point of view the quadratures are made by a two-dimensional trapezoid method and all of the guarantees have been made concerning the discretization of the integration domains.

The quantities shown in figures 1 to 6 are the mean functions of the impedance $\omega \mapsto R(\omega)$ and $\omega \mapsto I(\omega)$ and the standard deviations $\omega \mapsto \sigma_R(\omega)$ and $\omega \mapsto \sigma_I(\omega)$ such that:

$$\sigma_R(\omega) = [R_1(\omega)^2 + R_2(\omega)^2 + R_3(\omega)^2]^{1/2} \quad (63)$$

$$\sigma_I(\omega) = [I_1(\omega)^2 + I_2(\omega)^2 + I_3(\omega)^2]^{1/2}. \quad (64)$$

A set of curves in figure 1 shows, for different critical damping rates $\xi$ of the fuzzy, the mean real part of the impedance for a constant unit modal density in $\omega$ and small fluctuation parameters
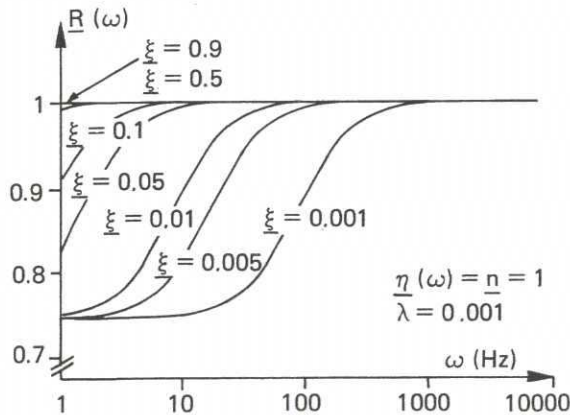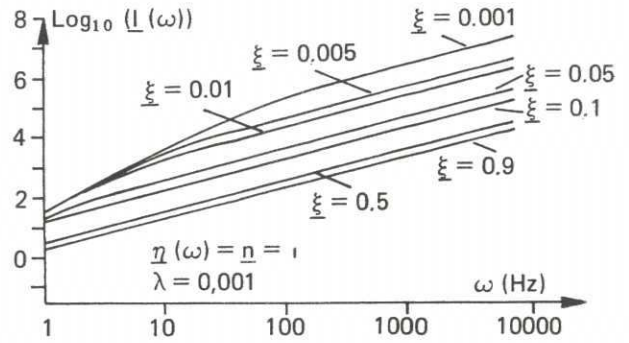


Fig. 2

($\lambda = 0.001$). The positive character of the $\omega \mapsto R(\omega)$ function shows that, for the computation parameters chosen, the real part of the impedance is of the body type. Also, as $\omega$ increases, we tend toward the situation where all of the equivalent mass of fuzzy is set into vibration. The associative computation of the mean imaginary part of the impedance (Fig. 2) shows a linear increase in the function $\omega \mapsto I(\omega)$ for a given $\xi$. The decrease in the level of the curves as $\xi$ increases provides a way of verifying that the apparent dissipation brought in by the fuzzy actually does involve kinetic and potential energy transfer from the main structure to the fuzzy.

Figures 3 and 4 graph the functions $\omega \mapsto R(\omega)$ and $\omega \mapsto I(\omega)$ for fixed $\lambda = 0.001$ and $\xi = 0.005$, for different functions $\omega \mapsto n(\omega)$. Figure 4 is interesting
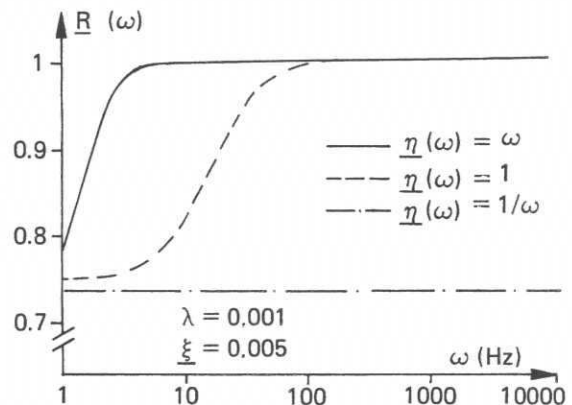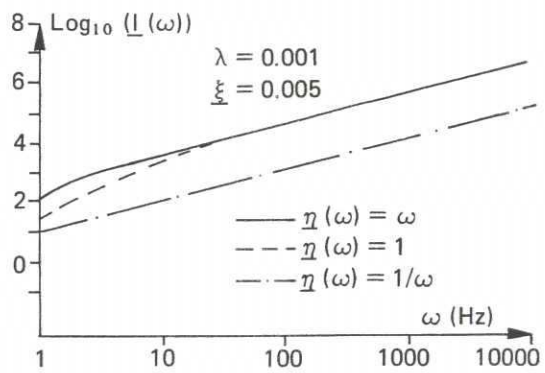


Fig. 3



Fig. 1



Fig. 4

on the practical level because it illustrates the relative character of the modal density of the fuzzy from a frequency point of view: compare the solid and dashed curves corresponding to the values $\underline{n}(\omega) = \omega$ and $\underline{n}(\omega) = \text{Const.} = 1$, respectively. For high values of $\omega$ the first curve can be considered as the limit curve, with respect to the second, of the function $\omega \mapsto I(\omega)$ as $n(\omega) \to +\infty$. Yet we observe that the two curves come together for the same values of $\omega$. This means that, for the imaginary part of the probabilistic impedance, a given numerical value of the mean modal density of the fuzzy is meaningful only relative to a given frequency domain. Here, for example, a constant mean modal frequency of $1 \, \text{s} \cdot \text{rad}^{-1}$ corresponds to an infinite apparent modal density beyond 100 Hz.

Let us note that other computations show that this result does not depend either on $\lambda$ or on $\xi$.

Figures 5 and 6 show the standard deviation $\omega \mapsto \sigma_R(\omega)$ and $\omega \mapsto \sigma_I(\omega)$ for $\underline{n}(\omega) = \text{Const.} = 1$ and $\underline{\xi} = 0.005$, for different values of $\lambda$.
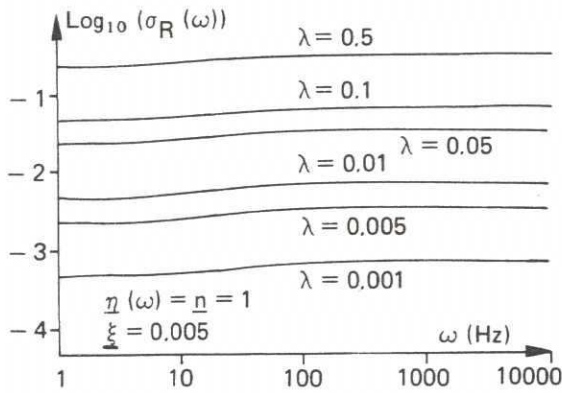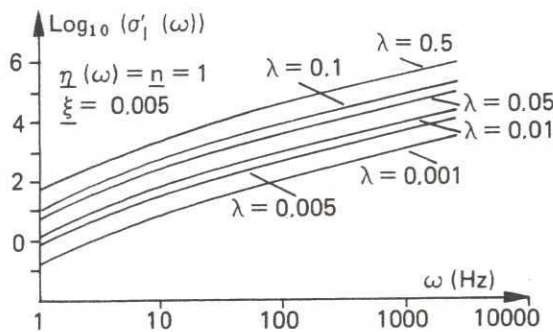


Fig. 5



Fig. 6

## V. — VALIDATION OF FUZZY THEORY BY NUMERICAL SIMULATION

### V,1. — SIMULATION OBJECTIVES

Part I of this paper presented the theoretical elements of the probabilistic approach to modeling structural fuzzy, and the principle of solving the equation of the linear vibrations of structure with its fuzzy. In particular, we showed that there exists a $\lambda_F > 0$ such that, for $\|\lambda(\omega)\|_\infty < \lambda_F$, the random solution exists, is unique and is written $\mathscr{P}$ a. s. for $d\omega$ almost all $\omega$ in $B_n$:

$$\hat{\mathbf{U}}_n(\omega, \lambda) = \hat{U}_n^{(0)}(\omega, \lambda)$$
$$+ \sum_{k=1}^{+\infty} \left\{ \sum_{l_1=1}^{L} \cdots \sum_{l_k=1}^{L} X_{l_1} \cdots X_{l_k} \hat{U}_{l_1 \ldots l_k, n}^{(k)}(\omega, \lambda) \right\}. \quad (65)$$

We recall that if $L$ fuzzy laws are used in the model, the function $\omega \mapsto \lambda(\omega)$ has values in $(]0, 1[)^{3L}$ and can be used to check the dispersion of the mechanical parameters of the fuzzy laws.

Yet from a practical point of view, this function can take on large values in norm. This will happen, for example, if we use finite elements of fuzzy to model a part of the structure where the mechanical characteristics are poorly known. For the method to be effective, the constant $\lambda_F$ used to define (65) must be able to take on large enough values to cover the majority of cases in practice.

But the problem does not stop there. It is quite evident that, from a numerical point of view, the solution will be sought in the form:

$$\hat{\mathbf{U}}_n(\omega, \lambda) = \hat{U}_n^{(0)}(\omega, \lambda)$$
$$+ \sum_{k=1}^{\mathscr{K}} \left\{ \sum_{l_1=1}^{L} \cdots \sum_{l_k=1}^{L} X_{l_1} \cdots X_{l_k} \hat{U}_{l_1 \ldots l_k, n}^{(k)}(\omega, \lambda) \right\} \quad (66)$$

where $\mathscr{K}$ is a finite positive integer. Yet the volume of calculations needed to construct (66) is directly proportional to $\dfrac{L(L^{\mathscr{K}} - 1)}{L - 1}$, where $L$ is the total number of independent random variables used to construct the constitutive equations for the entire fuzzy of the structure. A compromise thus has to be found in the order of expansion $\mathscr{K}$, between the precision of the approximation (66) and the computation cost, both of which must remain reasonable and capable of producing the expected results.

The numerical simulation thus has the triple objective:

— first to validate the computer developments;

— then to determine the domain of values $\lambda_F$ for which relation (65) [or (66)] is verified;

— finally to determine the order $\mathscr{K}$ of the expansion (66) which can achieve the correct results for the previous values of $\lambda_F$.

In fact, the second point is going to be very different because we are going to show that the simulation

method used forces us to consider high values of $\lambda_F$ at the start, which will lead us to determine the minimum upper bound to be placed on the $\lambda_F$ values to make them compatible with (65). As for the third point, we will limit ourselves to the first-order expansion by showing that it leads to entirely correct results for the various structures studied.

## V,2. — PRINCIPLE OF THE SIMULATION

For a given master structure and band of analysis $B$, the numerical simulation involves four stages:

(a) The structure is identified mechanically over $B$, i.e. we calculate the forced response of the structure to an excitation $F$ (deterministic in our case) such that $F \in H_B(\mathbb{R}, \mathbb{C}^N)$, by finite elements using the MF method;

(b) A structural complexification is modeled on the main structure with mechanical disturbance systems having random mechanical and geometric characteristics, with these characteristics being determined according to given laws of probability;

(c) For $Q$ independent drawings of all of the above characteristics, we compute as in (a) the forced response of this complexified structure and then determine the statistical quantities over the $Q$ drawings by estimating first and second order moments of the mechanical observations of the system;

(d) Finally, we compute the response of the structure with the structural fuzzy theory, for mechanical parameters of the fuzzy that are equivalent in law (which implies equivalency of mean and of standard deviation) to the $Q$ previous drawings and the results are compared with those of the statistic from stage (c).

## V,3. — CHOICE OF DISTURBER SYSTEMS

We have chosen the simple damped linear oscillator as an elementary mechanical model for representing the structural complexification introduced by the fuzzy.

We have made this choice for two reasons:

— These are discrete systems whose various mechanical parameters are well known and whose simple and inexpensive modeling allow us to reproduce them in large numbers on a given master structure;

— They correspond to the subjacent deterministic mechanical model of the fuzzy law and thereby assure us, which is imperative for the simulation, that they model a fuzzy that clearly belongs to the class defined by the law used.

Each oscillator is modelled by a bar element (a single degree of freedom) having zero mass and a linear law of visco-elastic behavior with instantaneous memory [7], to which a point mass $m_0$ is attached. Its random characteristics are its spatial location in the master structure, the mass $m_0$, the eigenfrequency $\omega_0$ of the associated conservative system and the damping $\xi_0$ (or the imaginary part of its complex tensile-compressive modulus $E_0$, which comes down to the same thing).

## V,4. — MECHANICAL HYPOTHESIS CONCERNING THE SIMULATED FUZZY

We introduce the following hypotheses for all of the structures analyzed. They in no way detract from the generality of the findings but do simplify the computations considerably:

(a) The fuzzy is homogeneous or locally homogeneous over the surface $\Sigma$ of the master structure;

(b) It is orthotropic and acts in only one direction, so we can limit the number of degrees of freedom of the models for the simulations (because in the fuzzy theory the number of degrees of freedom is not increased);

(c) the fuzzy cutoff frequency is zero, so that for a given MF band $B_n \left( B = \bigcup_{n=1}^{M_B} B_n \right)$, the fuzzy vibrates throughout the entire band;

(d) The mechanical parameters of the fuzzy are constant with respect to $\omega$.

## V,5. — APPROXIMATION OF THE FUZZY BY DISCRETE SYSTEMS

In this section we consider the problem of modeling a homogeneous fuzzy over an area $S$ of $\mathbb{R}^3$ of nonzero finite measure $\Delta S$, with a finite number $M$ of oscillators distributed randomly over $S$ in a frequency band of width $\Delta\omega$.

### (a) Spatial aspect

We recall that, for the fuzzy constitutive equation we have constructed, the fuzzy must be seen as a continuous system. In other words, if $m$ represents a generic point of $S$, the probabilistic fuzzy impedance operator applied to $S$ is a continuous operator with respect to $m$. It thus appears that the model of a homogeneous fuzzy over a surface $S$ of nonzero measure requires an infinite number of oscillators or, equivalently, that a fuzzy simulated by a finite number $M$ of discrete systems cannot be perfectly homogeneous over $S$.

Yet there exists an optimum model that consists of partitioning $S$ into a finite number of subsurfaces $K_S$

of constant measure $\Delta S_e$: $S = \bigcup_{k=1}^{K_S} S_k$ with, for all $k = 1, \ldots, K_S$, $\Delta S_k = \Delta S_e = \Delta S / K_S$. By taking the same laws of probability on each $S_k$ to determine the random parameters of the oscillators, we model a homogeneous fuzzy "relatively to the scale of surface element $\Delta S_e$". In particular, the number of oscillators contained in each $S_k$ over the band $\Delta\omega$ will be constant and denoted by $M_s$.

### (b) Frequency aspect

Another loss of homogeneity comes from the dispersion specific to the random drawer for determining the $M_S$ eigenfrequencies of the oscillators in a surface element $S_k$. To reduce this dispersion we have to carry out a heterodyning during the selection of the eigenfrequencies, which will be all the more effective as $M_S$ is large.

This finally leads us to determine the values of $K_S$ and $M_S$ that are maximum together, which should verify $K_S \cdot M_S = M$. Trivially, we get:

$$K_S = M_S = M^{1/2}. \tag{67}$$

Notes: (1) The results are identical when $S$ is a curve of $\mathbb{R}^3$ (as in a linear fuzzy).

(2) The procedure used defines the mean modal density of the fuzzy from the number of oscillators used for modeling it. In fact, it is of course possible to proceed in the reverse direction, in which case $M_S$ is given and we get

$$M = M_S^2, \tag{68}$$

a relation which is more licit than (67) since $K_S$, $M_S$ and $M \in \mathbb{N}$.


## V,6. — SELECTION OF THE RANDOM CHARACTERISTICS OF THE OSCILLATORS

From now on we assume the quantities $\Delta S_e$ and $M$ (and thus $K_S$ and $M_S$) to be known for a given structure and MF band. For fixed $k$ in $\{1, 2, \ldots, K_S\}$, a point $m$ of $S_k$ is represented by its local curvilinear coordinates $(\mu, \eta)$, where

$$\mu \in [-\mu_k; \mu_k] \quad \text{and} \quad \eta \in [-\eta_k; \eta_k].$$

We let

$$B_n = \left[ \Omega_n - \frac{\Delta\omega}{2}; \Omega_n + \frac{\Delta\omega}{2} \right].$$

The oscillator characteristics over the band $B_n$ are drawn as follows:

(a) For each $S_k$ ($k = 1, 2, \ldots, K_S$), we determine $M_S$ realizations of the random vector variable $r = (\mu, \eta)$ at values in $[-\mu_k; \mu_k] \times [-\eta_k; \eta_k]$ according to the law of probability:

$$dP_r(x, y) = \frac{1}{4\mu_k \eta_k} \mathbf{1}_{[-\mu_k; \mu_k]}(x)$$
$$\times \mathbf{1}_{[-\eta_k; \eta_k]}(y) \, dx \, dy \tag{69}$$

the values $\{r_i\}_i$, $i = 1, \ldots, M$, corresponding to the spatial locations on $S_k$ of the oscillators contained in the surface element $S_k$ for the band $B_n$.

(b) Still for each $S_k$ ($k = 1, 2, \ldots, M$), we draw the eigenfrequencies of the $M_s$ oscillators at random. To do this, we adopt a value of the modal density fluctuation parameter $\lambda_3$ and partition the band $B_n$ into $M_S$ disjunct sub-bands $B_{nj}$ of width $\delta\omega = \frac{\Delta\omega}{M_S}$, centered at frequencies $\omega_j = \Omega_n - \frac{\Delta\omega}{2} + (j-1)\delta\omega$ [$\lambda_3$ and $\Delta\omega$ are constants according to hypothesis 5.4 (d)].

For each $j$ ($j = 1, 2, \ldots, M_S$), we determine a realization $\omega_{0j}$ of the random variable $\omega_0$ at values in $[\Omega_1; \Omega_2]$ of the probability law coming from the theory of construction of the fuzzy law (see Part I):

$$dP_{\omega_0}(\tilde{\omega}, \omega_j) = h(\tilde{\omega}, \omega_j) \mathbf{1}_{\Omega_1; \Omega_2]}(\tilde{\omega}) \, d\tilde{\omega} \tag{70}$$

with

$$h(\tilde{\omega}, \omega_j) = \begin{cases} \dfrac{n_0}{} & \text{si } \tilde{\omega} \in [\omega_j - \delta_1; \omega_j + \delta_1] \\[2mm] \dfrac{1}{16\lambda_3 n_0 (\omega_j - \tilde{\omega})^2} - n_0 \dfrac{(\lambda_3 - 1)^2}{4\lambda_3} \\[2mm] \text{si} \\[1mm] \omega \in [\Omega_1; \omega_j - \delta_1] \cup [\omega_j + \delta_1; \Omega_2] \end{cases} \tag{71}$$

where

$$\delta_1 = \frac{1}{2n_0(1+\lambda_3)}, \quad \Omega_1 = \omega_j - \frac{1}{2n_0(1-\lambda_3)}, \quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} \tag{72}$$
$$\Omega_2 = \omega_j + \frac{1}{2n_0(1+\lambda_3)}$$

$n_0$ being the known mean modal density of the fuzzy.

This way we get values $\{\omega_{0j}\}_j$, $j = 1, 2, \ldots, M_S$ for each $S_k$, $M_S$, and the doublets $\{r_i, \omega_{0i}\}_i$, $1 = 1, 2, \ldots, M_S$ are assigned at random to the $M_S$ oscillators contained in $S_k$.

(c) The drawing of the masses and critical damping rates of the oscillators is simpler. For the two parameters we adopt the mean values $m_0$ (resp. $\xi_0$), the parameters $\lambda_1$ (resp. $\lambda_2$) and we draw $M$ realizations $\{m_{0i}\}_{i=1, \ldots, M}$ and $\{\xi_{0j}\}_{j=1, \ldots, M}$ of the random variables

$$m_0 = \underline{m}_0(1 \times X_{m_0}) \tag{73}$$
$$\xi_0 = \underline{\xi}_0(1 + X_{\xi_0}), \tag{74}$$

where $X_{m0}$ (resp. $X_{\xi 0}$) is a centered random variable of uniform law having the support $[-\lambda_1; \lambda_1]$ (resp. $[-\lambda_2; \lambda_2]$). These $M$ realizations are then assigned randomly to the $M$ oscillators of the model.

## V,7. — MECHANICAL PARAMETERS OF THE EQUIVALENT FUZZY

According to the comments in section V,5, a fuzzy modeled by a finite set of discrete systems over a surface area $S$ is homogeneous over $S$ only relative to the scale of surface element $\Delta S_e$. Under these conditions, we must associate parameters relative to the $\Delta S_e$ scale for the simulation fuzzy with the mechanical parameters defined at a given point (from a spatial point of view) of the fuzzy constitutive equation.

*Mean modal density $\underline{n}_f$*

For a surface element of area $\Delta S_e$ we have, on the average, one oscillator throughout the frequency interval of width $\Delta\omega$, whence

$$\underline{n}_f = \frac{M_S}{\Delta\omega}. \tag{75}$$

*Mean mass $\underline{m}_f$ per unit length or area*

For a surface element of area $\Delta S_e$, the equivalent mean resonant mass of fuzzy associated with the above mean modal density is that of an oscillator

$$\underline{m}_f = \underline{m}_0 \frac{M_S}{\Delta S}. \tag{76}$$

*Fluctuation parameters*

The definition of a surface element of homogeneity only allows us to model a homogeneous fuzzy by averaging over $S$. It is in effect clear that the relative positions of the oscillators inside the surface elements $S_k$ may vary widely as $k$ describes $\{1, 2, \ldots, K_S\}$ and thus introduce a local dispersion. This dispersion is given by the following result:

For the chosen model of a homogeneous fuzzy over a surface $S$ by a finite number $M$ of discrete systems such that $M = M_S^2$, the real loss of homogeneity causes a specific dispersion which is expressed by:

$$\lambda_h = \frac{\sqrt{3}}{M_S^{1/2}} \left(1 - \frac{1}{M_S}\right)^{1/2}. \tag{77}$$

Let $S = \bigcup_{k=1}^{K_S} S_k$ be a finite partition of the surface $S$ into $K_S$ surface elements $S_k$ of area $\Delta S_e$. For each $k$ ($k = 1, 2, \ldots, K_S$), we make $M_S$ independent draw-

ings of points of $S_k$, with each drawing following the same law of uniform probability.

Let $N_{i,q}^k$ be the random discrete variable representing the number of points selected for selection $q$ in the surface sub-element $\delta S_k^i$ of $S_k$ having the area $\delta S = \dfrac{\Delta S_e}{M_S}$.

Since only one point is determined at each drawing, $N_{i,q}^k$ can take on only the values 0 or 1. Then, since the law of probability of the selection is uniform, we have:

$$P(N_{i,q}^k = 1) = \frac{1}{M_S} \tag{78}$$

$$P(N_{i,q}^k = 0) = 1 - \frac{1}{M_S}. \tag{79}$$

So that for fixed $i$, $k$ and $q$ we get:

$$\mathscr{E}(N_{i,q}^k) = \frac{1}{M_S} \tag{80}$$

$$\mathscr{E}\{(N_{i,q}^k)^2\} = \frac{1}{M_S}. \tag{81}$$

Now let the random variable $\mathbb{N}_i^k = \sum_{q=1}^{M_S} N_{i,q}^k$, which represents the number of points contained in $\delta S_k^i$ for fixed $i$ and $k$ after $M_S$ drawings.

According to (80) and (81) we get:

$$\mathscr{E}\{(\mathbb{N}_i^k)^2\} - \mathscr{E}(\mathbb{N}_i^k)^2 = 1 - \frac{1}{M_S}. \tag{82}$$

Let us also define the random variable $\mathbb{Z}_i = \dfrac{1}{M_S} \sum_{k=1}^{M_S} \mathbb{N}_i^k$. It is the variance of this random variable which give us the dispersion we are looking for, since it measures the fluctuations of the number of points contained in $\delta S_k^i$ for fixed $i$ as we scan the $K_S$ elements $S_k$ ($k = 1, 2, \ldots, K_S$).

A simple calculation gives:

$$\sigma_{\mathbb{Z}_i}^2 = \frac{1}{M_S}\left(1 - \frac{1}{M_S}\right). \tag{83}$$

As $\mathscr{E}(\mathbb{Z}_i) = 1$, the variation index $\lambda_h$ defined by

$$\lambda_h = \sqrt{3}\,\frac{\sigma_{\mathbb{Z}_i}}{\mathscr{E}(\mathbb{Z}_i)} \tag{84}$$

yields (77).

*Note 1:* The dispersion parameter $\lambda_h$ affects only the equivalent mass and modal density variables of the fuzzy constitutive equation because the critical damping of the fuzzy is an intrinsic parameter that is unaffected by problems of homogeneity.

*Note* 2: $\lambda_h$ is a parameter specific to the simulation method used and has to be added to the dispersion parameters $\lambda_1$ and $\lambda_2$ that are specific to the fuzzy constitutive equation. In fact, for the values of $M$, $\lambda_1$ and $\lambda_2$ used in applications, we shall see that $\lambda_1 \ll \lambda_k$ and $\lambda_2 \ll \lambda_h$, which will bring us to study wide-dispersion fuzzies.

## VI. – APPLICATIONS

In the following computations, the structure is in a vacuum and is referenced to an orthonormal cartesian system $(x, y, z)$. Also, the findings presented correspond to an expansion of the first-order random solution.

### VI,1. – BEAM WITH HOMOGENEOUS FUZZY

The master structure is a cantilever beam running along the $x$ axis, clamped at $x = 0$, having length $L = 2$ and free at $x = L$, with constant cross-sectional area $S_e = 1 \times 10^{-4}$. It is excited in bending mode in the $(x, y)$ plane by a unit point force applied at $x = L$. The band studied is 0-1,0000 Hz, partitioned into 10 MF bands 100 Hz wide.

The beam is homogeneous with a mass density $\rho = 31,400$, Young's modulus $E = 2.1 \times 10^{11}$, Poisson coefficient $\nu = 0.3$, bending inertia $I = 1 \times 10^{-8}$ and structural damping level $\xi = 0.003$.

The fuzzy is homogeneous, orthotropic (nonzero in the $y$ direction) and is modeled by 2,890 oscillators in the band analyzed. On each narrow MF band, the mean parameters of the oscillators are $n_0 = 2.89 \text{ Hz}^{-1}$, $\xi_0 = 0.002$ and $m = 0.006$ (or a fuzzy-master structure mass ratio of the order of 3%). The dispersion parameters are $\lambda_1 = \lambda_2 = \lambda_3 = 0.001$ and $\lambda_h = 0.4$.

The structure is modeled by straight two-node beam elements and the fuzzy by two-node line elements.

The mechanical observations of the system are the acceleration energies in each 5 Hz band and the averaged boundary condition impedances in each 5 Hz band. For the simulation, there are ten drawings for the energy computation and five for the impedance computation.

Figures 7 to 10 summarize all of the energy computations (energy expressed in decibels) for the observation points $x = L$ and $x = 1.4$.
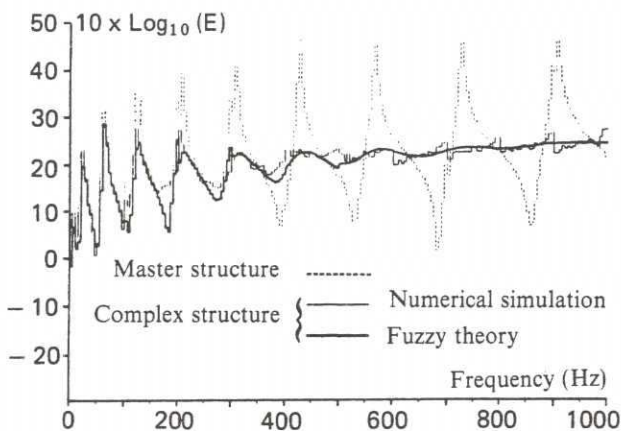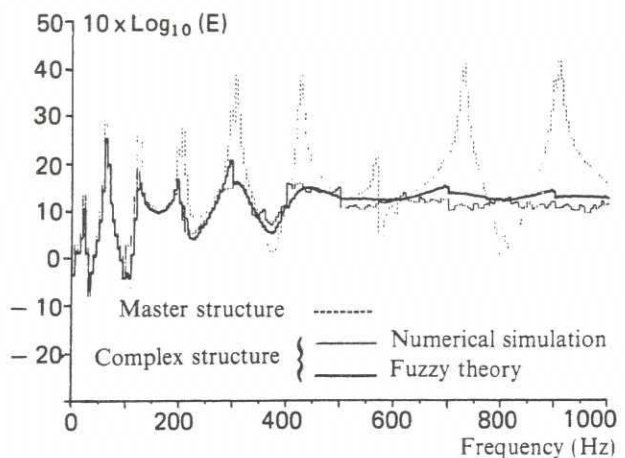
Fig. 7. – Mean energy (at point $x = 2$).
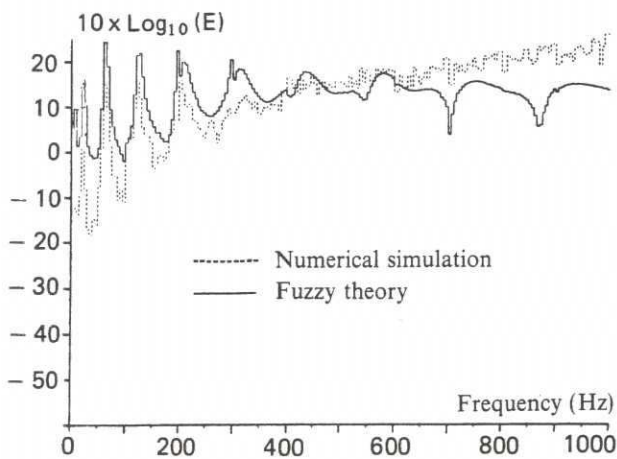
Fig. 9. – Energy mean (at point $x = 1.4$).

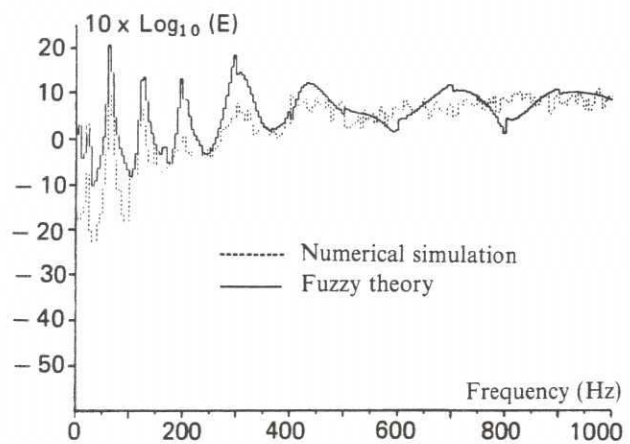Fig. 8. – Standard deviation of the energy (at $x = 2$).

Fig. 10. – Standard deviation of the energy (at $x = 1.4$).

We note that, in the band we have chosen to analyze, the response of the master structure is typically modal; but despite this and the weak mass disturbance introduced by the fuzzy, the smoothing of the resonance peaks is very large.

The agreement between the simulation and the theory is excellent, in spite of a slight drift of the standard deviation between 700 and 1,000 Hz. Yet for the energy type observations the significant quantities are the mean parts, which are r m s values considering the fluctuating part of the random response [relation (58)].

So the results show us that, for this structure and for a high fuzzy dispersion parameter ($\lambda_h = 40\%$), the expansion of the first order solution seems to be sufficient. Moreover, a second order computation shows that the exact gain in accuracy (less than 1 dB) does not justify the effort needed to compute it.

Figures 11 to 14 illustrate a few results of direct and cross boundary condition impedances. The scale of the coordinate axis is linear and each of the curves is normalized with respect to the maximum of the quantity observed throughout the band analyzed.
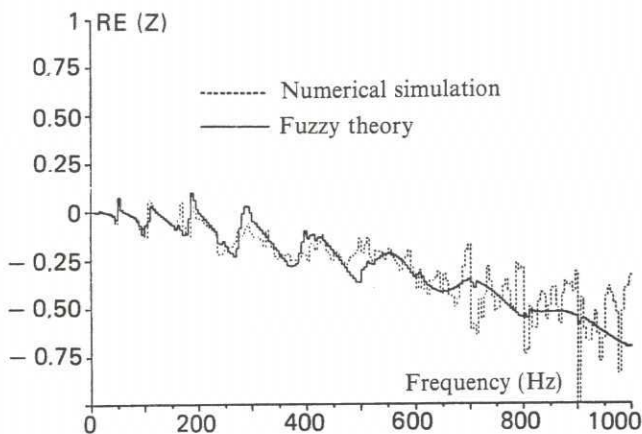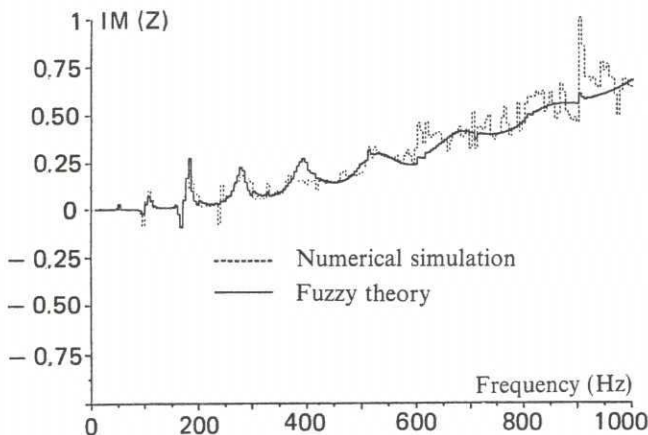


Fig. 11. – Self-impedance. Real part.



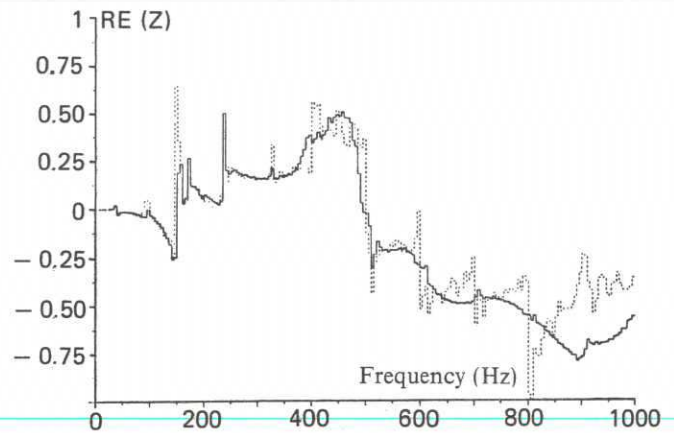Fig. 12. – Self-impedance. Imaginary part.
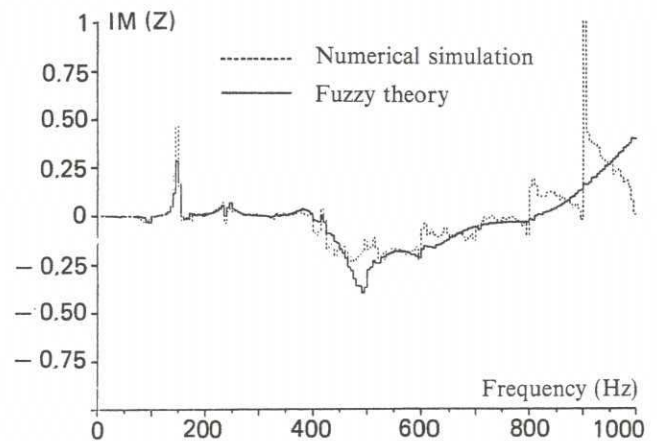


Fig. 13. – Cross impedance. Real part.



Fig. 14. – Cross impedance. Imaginary part.

## VI,2. – BEAM WITH LOCALLY HOMOGENEOUS FUZZY

The master structure, the boundary conditions and the excitation forces are all the same as in section VI,1. The fuzzy is orthotropic in the $y$ direction, locally homogeneous in the two disjunct zones defined by $0.5 < x < 0.8$ and $1.3 < x < 1.7$. The fuzzy constitutive equations in each zone are different, but globally-
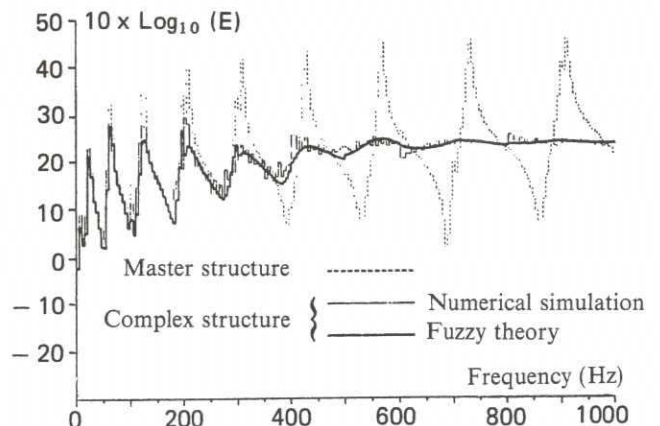


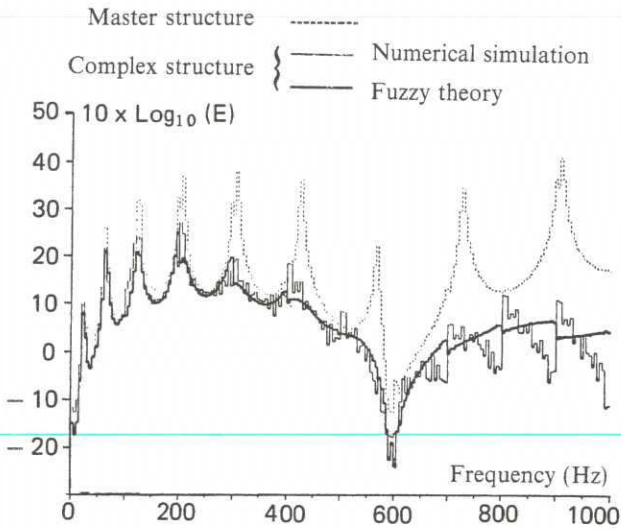Fig. 15. – Mean of the energy (at point $x = 2$).

Fig. 16. – Mean energy (at point $x = 0.33$).

—with respect to the beam—the mean parameters of the fuzzy are equivalent to those in section VI,1. However, the dispersion parameters $\lambda_h$ in each zone are 55 and 44% here and the number of drawings for the simulation is five.

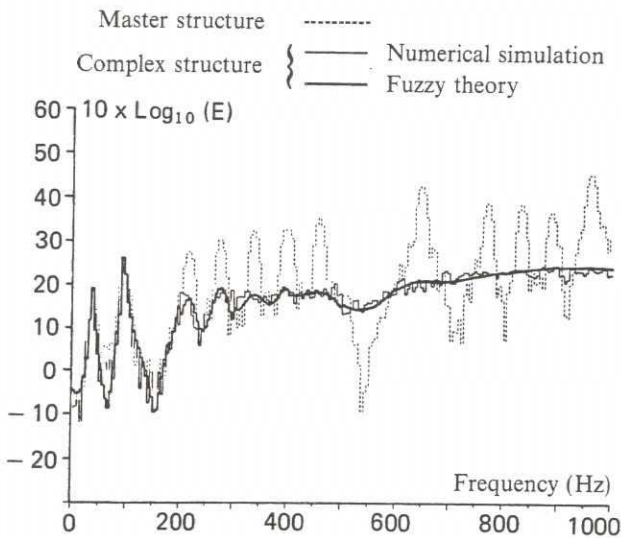Figures 15 and 16 show the mean parts of the acceleration energy, in decibels, for each 5 Hz, at the excitation point and in the vicinity of the clamping point. We can note that there is practically no difference between figure 15 and the section VI,1, although there is no fuzzy at the level of the excitation point.

## VI,3. – PLATE WITH HOMOGENEOUS FUZZY

The master structure here is a plate centered at (0, 0, 0), of length $L = 1$, width $l = 0.5$ and thickness $e = 0.003$ and having its mean plane in $z = 0$. It is homogeneous and isotropic and has a mass density $\rho = 7,850$, Young's modulus $E = 2.1 \times 10^{11}$, Poisson coefficient $\nu = 0.3$ and structural damping level $\xi = 0.003$. As before, the 0-1,000 Hz band analyzed is broken down into ten narrow MF bands, each 100 Hz wide. We are concerned with the bending vibrations of the plate as it rests on its edges and is excited by a unit point force applied along the $z$ axis at its center.

The fuzzy is orthotropic, homogeneous and acts only in the $z$ direction. It is modeled in each MF band by 576 $z$ axis oscillators with mean parameters $\underline{m}_0 = 0.000\,56$, $\underline{n}_0 = 5.6$ Hz$^{-1}$ and $\underline{\xi}_0 = 0.003$. The dispersion parameters are $\lambda_1 = \lambda_2 = \lambda_3 = 0.001$ and $\lambda_h = 0.34$.

Fig. 17. – Mean energy. Point $(x, y) = (0, 0)$.
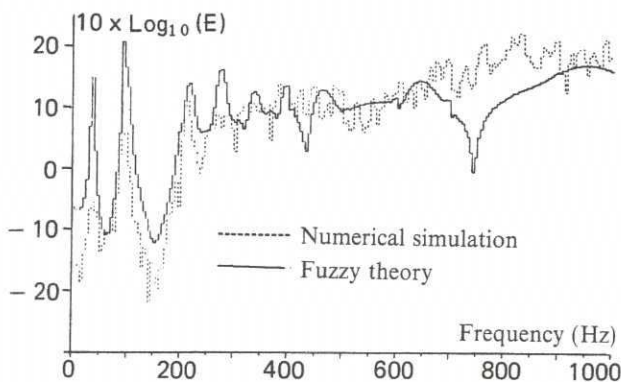
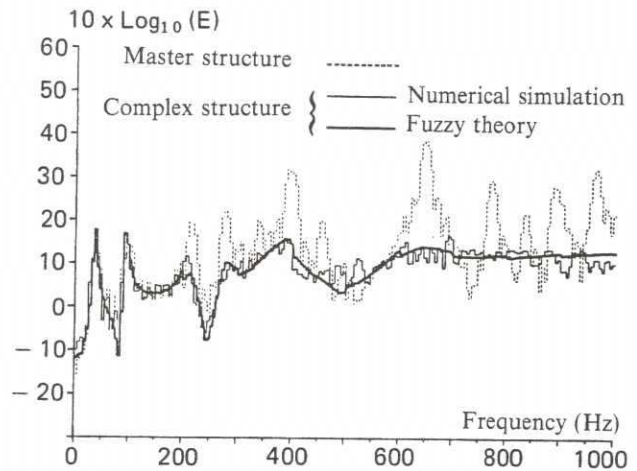Fig. 18. – Standard deviation of the energy. Point $(x, y) = (0, 0)$.

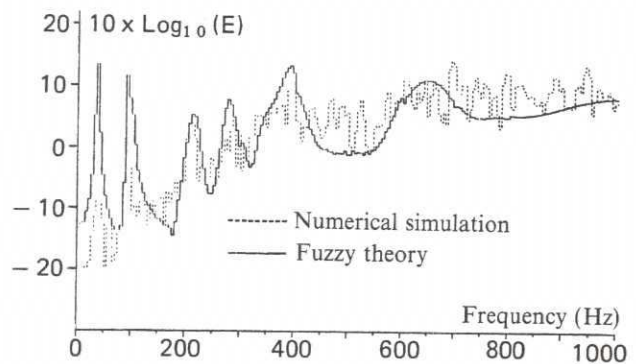Fig. 19. – Mean of the energy. Point $(x, y) = \left(\dfrac{L}{8}, \dfrac{l}{8}\right)$.

Fig. 20. – Standard deviation of the energy. Point $(x, y) = \left(\dfrac{L}{8}, \dfrac{l}{8}\right)$.

The finite elements of the master structure are three-node plates and the finite elements of the fuzzy are three-node surfaces. The number of drawings for the statistical computation is five and the quantities computed are the acceleration energies in each 5 Hz band, represented in decibels in figures 17 to 20.

Here again, the response is rather modal throughout the band analyzed. Aside from the first two modes, the smoothing of the resonance peaks is large, with the response tending even toward a quasi-static response at the excitation point (Fig. 17). The agreement between the simulation and the fuzzy theory is excellent and confirms the findings for the beam, for a two dimensional medium.

## VI,4. – SHELL WITH HOMOGENEOUS FUZZY

The master structure here is a circular cylinder. It is open-ended, centered at $(0, 0, 0)$, with the $z$ axis as its axis of revolution. Its length is $L = 2.52$, thickness $e = 0.045$ and mean radius $R = 0.619$. Its ends are swivel-mounted in two rings, nondeformable in their planes. The 0-1,300 Hz band analyzed is partitioned into 13 narrow MF bands and the unit point force is applied at $(0, R, 0)$ along the $y$ axis.

The master structure is homogeneous and isotropic. It has a mass density $\rho = 1,920$, Young's modulus $E = 4.5 \times 10^{10}$, Poisson coefficient $\nu = 0.3$ and structural damping level $\xi = 0.003$.

The fuzzy is homogeneous, orthotropic and acts only radially. It is modeled in each MF band by 256 oscillators with mean parameters $m_0 = 0.1$, $n_0 = 2.56$ Hz$^{-1}$, and $\xi_0 = 0.003$. The dispersion parameters are $\lambda_1 = \lambda_2 = \lambda_3 = 0.001$ and $\lambda_h = 0.4$.

The structure is modeled in eight-node shell elements and the fuzzy in eight-node surface elements. The vertical plane of symmetry at $x = 0$ was used for all of the computations because the symmetric problem of the plate shows that a homogeneous fuzzy reflects the symmetries on the average.
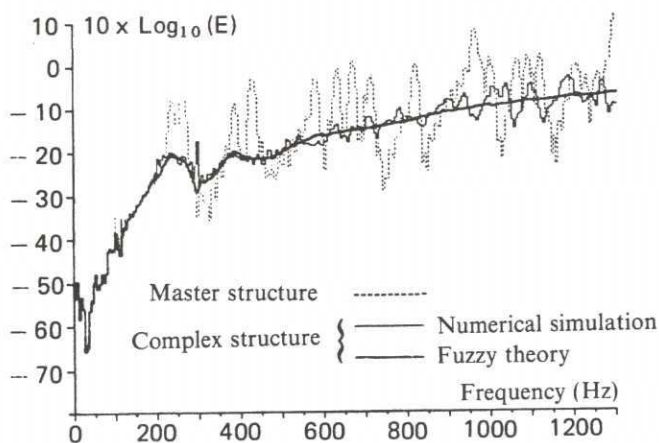


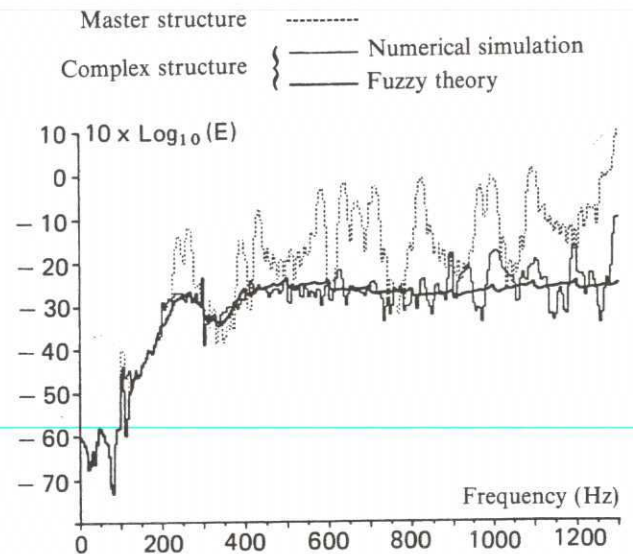Fig. 21. – Mean energy. Point $(0, R, 0)$.



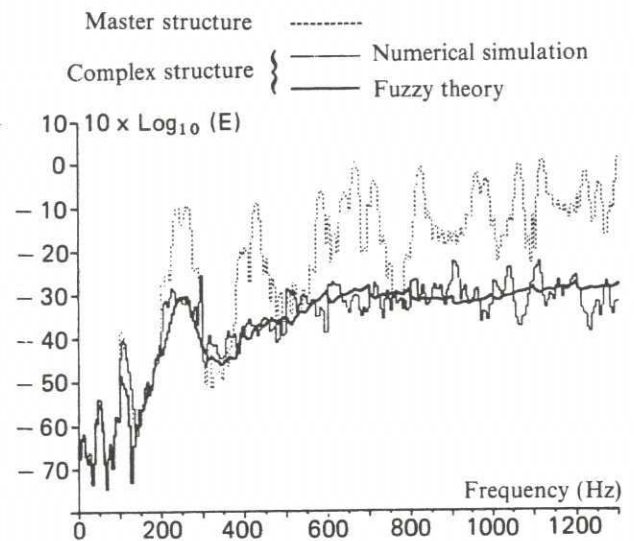Fig. 22. – Mean energy. Point $\left(0, R, \dfrac{L}{5}\right)$.



Fig. 23. – Mean energy. Point $(R, 0, 0)$.

The number of drawings in the statistical computation is five, and the mechanical observations are the normal acceleration energies in each 5 Hz band.

Figures 21 to 23 summarize the computations for a few points of observation. The theory and simulation again compare very well here, despite the small number of selections. From a mechanical point of view we can note a major modification of the structure dynamic for the points where the dynamic is controlled by the energy propagations (Fig 22 and 23).

## VII . – CONCLUSIONS

In this second part of our publication we have presented the numerical and computer developments

of the probabilistic model of the fuzzy in complex structures.

The validity of the fuzzy constitutive equation was analyzed by numerical simulation on a few standard structures, and this has brought out two essential points:

— The theoretical findings seem to be valid for fuzzy dispersion parameters going at least up to 50%, which should make it possible to cover a sufficient number of practical cases;

— For these dispersion parameters, the expansion of the first-order random solution is sufficient and thus allows reasonable computation costs, conforming with the approach to the problem.

The problem still remains open to further developments, though. Other probabilistic fuzzy constitutive equations can be constructed from different elementary deterministic mechanical models, e.g. to take into account the spatial memory of the fuzzy. It may also be thought to construct other laws by inverse identification on the basis of statistical data from experiments on real structures.

The tool presented here should allow parametric studies, at lesser cost, of mechanical phenomena such as the energy propagation in the structure, the impedance couplings, truncation problems, etc.

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