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► **To cite this version:**

Aurel Spătaru. Only the first term of some series counts. *Statistics and Probability Letters*, 2011, 81 (10), pp.1547. 10.1016/j.spl.2011.05.012 . hal-00770345

**HAL Id: hal-00770345**

**<https://hal.science/hal-00770345>**

Submitted on 5 Jan 2013

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## Accepted Manuscript

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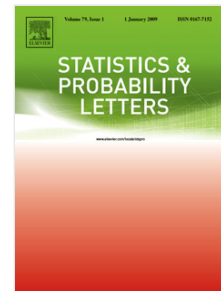
PII: S0167-7152(11)00187-8  
DOI: [10.1016/j.spl.2011.05.012](https://doi.org/10.1016/j.spl.2011.05.012)  
Reference: STAPRO 6009

To appear in: *Statistics and Probability Letters*

Received date: 28 November 2010

Revised date: 25 May 2011

Accepted date: 25 May 2011



Please cite this article as: Spătaru, A., Only the first term of some series counts. *Statistics and Probability Letters* (2011), doi:10.1016/j.spl.2011.05.012

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# Only the first term of some series counts

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## Abstract

Let  $X, X_1, X_2, \dots$  be i.i.d. random variables, and set  $S_n = X_1 + \dots + X_n$ . We prove that for three important distributions of  $X$ , namely normal, exponential and geometric, series of the type  $\sum_{n \geq 1} a_n P(|S_n| \geq xb_n)$  or  $\sum_{n \geq 1} a_n P(S_n \geq xb_n)$  behave like their first term as  $x \rightarrow \infty$ .

**Keywords:** Tail probabilities of sums of i.i.d. random variables, normal distribution, exponential distribution, geometric distribution.

## 1. Introduction

Let  $X, X_1, X_2, \dots$  be i.i.d. random variables with partial sums  $S_n = X_1 + \dots + X_n, n \geq 1$ . Several authors considered series of the type

$$f(x) = \sum_{n \geq 1} a_n P(|S_n| \geq xb_n), \quad x > 0, \quad (1.1)$$

or, in case  $X \geq 0$ ,

$$f(x) = \sum_{n \geq 1} a_n P(S_n \geq xb_n), \quad x > 0, \quad (1.2)$$

where  $a_n, b_n > 0$ . If  $\sum_{n \geq 1} a_n = \infty$ , the primary focus was to obtain necessary and sufficient moment conditions for the convergence of these series, i.e. to find a threshold  $\alpha$  such that  $f(x) = \infty$  for  $x < \alpha$ , while  $f(x) < \infty$  for  $x > \alpha$ . Then it was sensible to look for a normalizing

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elementary function  $g(x)$ ,  $x > \alpha$ , such that the ratio  $f(x)/g(x)$  have a nontrivial limit as  $x \searrow \alpha$ . The first result in this direction was due to Heyde (1975) who proved that

$$\lim_{x \searrow 0} x^2 \sum_{n \geq 1} P(S_n \geq xn) = EX^2$$

whenever  $EX^2 < \infty$  and  $EX = 0$ . (Hsu and Robbins (1947) and Erdős (1949,1950) showed that  $\sum_{n \geq 1} P(S_n \geq xn) < \infty$ ,  $x > 0$ , if and only if  $EX^2 < \infty$  and  $EX = 0$ .) References to the rich literature on this so-called precise asymptotics problem are given in Spătaru (2010).

The purpose of this paper is to deal with a dual problem motivated also by insurance applications. Namely, if  $f(x_0) < \infty$  for some  $x_0 > 0$ , then  $\lim_{x \rightarrow \infty} f(x) = 0$  by dominated convergence. Then it is interesting to determine a normalizing simple function  $g(x)$ ,  $x \geq x_0$ , such that  $f(x)/g(x)$  have a nondegenerate limit as  $x \rightarrow \infty$ . It turns out that in three important cases presented below this limit is 1, and  $g(x)$  equals the first term of the series. This phenomenon might be more general. Actually, if  $EX^2 < \infty$  and  $EX = 0$ , Pruss (1997) proved that there are absolute constants  $C_1, C_2 > 0$  such that

$$C_1 x^{-2} E[X^2 I\{|X| \geq x\}] \leq \sum_{n \geq 1} P(|S_n| \geq xn) \leq C_2 x^{-2} E[X^2 I\{|X| \geq x\}], \quad x > 0. \quad (1.3)$$

Since  $P(|S_1| \geq x) = P(|X| \geq x) \leq x^{-2} E[X^2 I\{|X| \geq x\}]$ ,  $x > 0$ , (1.3) may suggest that  $\sum_{n \geq 1} P(|S_n| \geq xn) \sim P(|S_1| \geq x)$  as  $x \rightarrow \infty$  whenever the series is positive for all  $x > 0$ .

The three cases refers to the normal distribution and series (1.1), where the assumption that  $X_1, X_2, \dots$  have a common distribution is dropped, and to the exponential and geometric distributions and series (1.2).

## 2. Normal distribution

For  $n \geq 1$ , we assume that  $X_n$  has a normal distribution with zero mean and  $\sigma_n^2$  variance, and set  $B_n^2 = \sigma_1^2 + \dots + \sigma_n^2$ .

**Lemma 1.** Let  $\Phi$  denote the standard normal distribution function, and put  $\Psi(x) = 1 - \Phi(x) + \Phi(-x)$ ,  $x \geq 0$ . For  $c > 1$ , the following assertions hold:

- (i) the function  $\Psi(cx)/\Psi(x)$ ,  $x \geq 0$ , is decreasing;
- (ii)  $\lim_{x \rightarrow \infty} \Psi(cx)/\Psi(x) = 0$ .

**Proof.** To verify (i), it suffices to show that the function

$$R(x) = \left( \int_{cx}^{\infty} e^{-y^2/2} dy \right) / \int_x^{\infty} e^{-y^2/2} dy, \quad x \geq 0,$$

is decreasing. For  $0 \leq x \leq y$ , we have

$$\frac{x^2}{2}(c^2 - 1) \leq \frac{y^2}{2}(c^2 - 1) \implies -\frac{x^2}{2} - \frac{c^2 y^2}{2} \leq -\frac{y^2}{2} - \frac{c^2 x^2}{2} \implies e^{-x^2/2} e^{-c^2 y^2/2} \leq e^{-y^2/2} e^{-c^2 x^2/2}.$$

Therefore, for  $x \geq 0$ ,

$$\begin{aligned} ce^{-x^2/2} \int_x^{\infty} e^{-c^2 y^2/2} dy &\leq ce^{-c^2 x^2/2} \int_x^{\infty} e^{-y^2/2} dy \implies e^{-x^2/2} \int_x^{\infty} e^{-y^2/2} dy \leq ce^{-c^2 x^2/2} \int_x^{\infty} e^{-y^2/2} dy \\ &\implies R'(x) \leq 0. \end{aligned}$$

Next, as  $\Psi(x) \sim \frac{2}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}$  as  $x \rightarrow \infty$  (see, e.g., Corollary 2, p. 49 of Chow and Teicher (1978)), we obtain

$$\frac{\Psi(cx)}{\Psi(x)} \sim \left( \frac{e^{-c^2 x^2/2}}{cx} \right) \frac{x}{e^{-x^2/2}} = \frac{1}{c} e^{-(c^2-1)x^2/2} \rightarrow 0 \text{ as } x \rightarrow \infty. \quad \square$$

**Theorem 1.** Suppose that  $\sum_{n \geq 1} a_n P(|S_n| \geq x_0 b_n)$  converges for some  $x_0 > 0$ , where  $b_n/b_1 > B_n/B_1$ ,  $n \geq 2$ . Then

$$\sum_{n \geq 1} a_n P(|S_n| \geq x b_n) \sim a_1 P(|S_1| \geq x b_1) = a_1 \Psi(x b_1 / \sigma_1) \text{ as } x \rightarrow \infty. \quad (2.1)$$

**Proof.** Write

$$\sum_{n \geq 1} a_n P(|S_n| \geq x b_n) = a_1 P(|S_1| \geq x b_1) + \sum_{n \geq 2} a_n P(|S_n| \geq x b_n), \quad x > 0. \quad (2.2)$$

For  $x \geq x_0$ , applying Lemma 1.i with  $c = \frac{b_n}{B_n} \cdot \frac{B_1}{b_1}$ ,  $n \geq 2$ , we get

$$\begin{aligned} \sum_{n \geq 2} \frac{a_n P(|S_n| \geq x b_n)}{a_1 P(|S_1| \geq x b_1)} &= \sum_{n \geq 2} \frac{a_n \Psi(x \frac{b_n}{B_n})}{a_1 \Psi(x \frac{b_1}{B_1})} = \sum_{n \geq 2} \frac{a_n \Psi(x \frac{b_1}{B_1} \cdot \frac{b_n}{B_n} \cdot \frac{B_1}{b_1})}{a_1 \Psi(x \frac{b_1}{B_1})} \\ &\leq \sum_{n \geq 2} \frac{a_n \Psi(x_0 \frac{b_1}{B_1} \cdot \frac{b_n}{B_n} \cdot \frac{B_1}{b_1})}{a_1 \Psi(x_0 \frac{b_1}{B_1})} = \frac{1}{a_1 \Psi(x_0 \frac{b_1}{B_1})} \sum_{n \geq 2} a_n P(|S_n| \geq x_0 b_n) < \infty. \end{aligned}$$

Consequently, by dominated convergence and Lemma 1.ii, we have

$$\lim_{x \rightarrow \infty} \sum_{n \geq 2} \frac{a_n P(|S_n| \geq x b_n)}{a_1 P(|S_1| \geq x b_1)} = \sum_{n \geq 2} \frac{a_n}{a_1} \lim_{x \rightarrow \infty} \frac{\Psi(x b_1 b_n / b_1 \sqrt{n})}{\Psi(x b_1)} = 0. \quad (2.3)$$

Now (2.1) follows from (2.2) and (2.3).  $\square$

### 3. Exponential distribution

In this section we assume that  $X$  has an exponential distribution with rate parameter  $\lambda$ .

We need the following lemma.

**Lemma 2.** For  $n \geq 1$ , define

$$h_n(x) = e^{-x b_n} \sum_{k=0}^{n-1} \frac{(x b_n)^k}{k!}, \quad x \geq 0.$$

where  $b_n > 0$  and  $b_n - b_1 \geq C^{-1}(n-1)$  for some  $C > 0$ . Then, for  $n \geq 2$ , we have:

- (i) the function  $h_n(x)/h_1(x)$ ,  $x \geq 0$ , is decreasing for  $x > C$ ;
- (ii)  $\lim_{x \rightarrow \infty} h_n(x)/h_1(x) = 0$ .

**Proof.** For  $n \geq 2$  and  $1 \leq k \leq n-1$ , the derivative of the function  $e^{-x(b_n-b_1)}x^k$ ,  $x \geq 0$ , is negative whenever  $x > k/(b_n - b_1)$ . This entails that the derivative of the function  $h_n(x)/h_1(x)$ ,  $x \geq 0$ , is negative whenever  $x > C \geq (n-1)/(b_n - b_1)$ . Thus (i) holds, and (ii) is obvious.  $\square$

**Theorem 2.** Assume that  $b_n$ ,  $n \geq 1$ , are as in Lemma 2 and  $\sum_{n \geq 1} a_n P(S_n \geq x_0 b_n) < \infty$  for some  $x_0 > 0$ . Then

$$\sum_{n \geq 1} a_n P(S_n \geq x b_n) \sim a_1 P(S_1 \geq x b_1) = a_1 e^{-\lambda b_1 x} \text{ as } x \rightarrow \infty. \quad (3.1)$$

**Proof.** Without loss of generality, suppose that  $\lambda = 1$ . Then  $S_n$  has an Erlang distribution with shape parameter  $n$  and rate parameter 1, and so  $P(S_n \geq x) = e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}$ ,  $n \geq 1$ ,  $x \geq 0$ . We have

$$\sum_{n \geq 1} a_n P(S_n \geq xb_n) = a_1 P(S_1 \geq xb_1) + \sum_{n \geq 2} a_n P(S_n \geq xb_n), \quad x > 0. \quad (3.2)$$

For  $x \geq x_0 \vee C$ , on account of Lemma 2.i, we obtain

$$\begin{aligned} \sum_{n \geq 2} \frac{a_n P(S_n \geq xb_n)}{a_1 P(S_1 \geq xb_1)} &= \sum_{n \geq 2} \frac{a_n h_n(x)}{a_1 h_1(x)} \leq \sum_{n \geq 2} \frac{a_n h_n(x_0 \vee C)}{a_1 h_1(x_0 \vee C)} \\ &= \frac{1}{a_1 h_1(x_0 \vee C)} \sum_{n \geq 2} a_n P(S_n \geq (x_0 \vee C)b_n) < \infty. \end{aligned}$$

Therefore, by dominated convergence and Lemma 2.ii, we get

$$\lim_{x \rightarrow \infty} \sum_{n \geq 2} \frac{a_n P(S_n \geq xb_n)}{a_1 P(S_1 \geq xb_1)} = \sum_{n \geq 2} \frac{a_n}{a_1} \lim_{x \rightarrow \infty} \frac{h_n(x)}{h_1(x)} = 0. \quad (3.3)$$

By (3.2) and (3.3), (3.1) obtains.  $\square$

#### 4. Geometric distribution

Now we assume that  $X$  is geometrically distributed with parameter  $p$  and taking values in the set  $\{0, 1, \dots\}$ . Then  $S_n$  follows a negative binomial distribution with parameters  $n$  and  $p$  expressed in terms of the regularized incomplete beta function  $I_p$ , namely we have

$$P(S_n \geq k) = I_p(k, n) = \sum_{j=k}^{k+n-1} \binom{k+n-1}{j} p^j q^{k+n-1-j}, \quad n, k \geq 1, \quad (4.1)$$

where  $q = 1 - p$ .

**Lemma 3.** Let  $0 < p < 1$ . For  $n \geq 2$ , the following assertions hold:

- (i) the sequence  $\{I_p(mn, n)/I_p(m, 1) : m \geq -1/\log p\}$  is decreasing;
- (ii)  $\lim_{m \rightarrow \infty} I_p(mn, n)/I_p(m, 1) = 0$ .

**Proof.** By (10), we get  $I_p(m, 1) = P(S_1 \geq m) = p^m$ ,  $m \geq 1$ . For  $n \geq 2$  and  $m \geq 1$ , on account of (10), we have

$$\begin{aligned} I_p(mn, n) &= \binom{mn+n-1}{n-1} p^{mn} q^{n-1} + \binom{mn+n-1}{n-2} p^{mn+1} q^{n-2} \\ &\quad + \dots + \binom{mn+n-1}{1} p^{mn+n-2} q + p^{mn+n-1} \\ &= \frac{(mn+n-1)(mn+n-2)\dots(mn+1)}{(n-1)!} p^{mn} q^{n-1} \\ &\quad + \frac{(mn+n-1)(mn+n-2)\dots(mn+2)}{(n-2)!} p^{mn+1} q^{n-2} \\ &\quad + \dots + (mn+n-1) p^{mn+n-2} q + p^{mn+n-1}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \frac{I_p(mn, n)}{I_p(m, 1)} &= \frac{(mn+n-1)(mn+n-2)\dots(mn+1)}{(n-1)!} p^{m(n-1)} q^{n-1} \\ &\quad + \frac{(mn+n-1)(mn+n-2)\dots(mn+2)}{(n-2)!} p^{m(n-1)+1} q^{n-2} \\ &\quad + \dots + (mn+n-1) p^{m(n-1)+n-2} q + p^{m(n-1)+n-1} \tag{4.2} \\ &= \frac{(mp^m n + (n-1)p^m)(mp^m n + (n-2)p^m)\dots(mp^m n + p^m)}{(n-1)!} q^{n-1} \\ &\quad + \frac{(mp^m n + (n-1)p^m)(mp^m n + (n-2)p^m)\dots(mp^m n + 2p^m)}{(n-2)!} p^{m+1} q^{n-2} \\ &\quad + \dots + (mp^m n + (n-1)p^m) p^{(m+1)(n-2)} q + p^{(m+1)(n-1)}. \end{aligned}$$

As the function  $xp^x$ ,  $x \geq 1$ , is decreasing for  $x \geq -1/\log p$ , (4.2) shows that (i) holds. Since

$$\lim_{m \rightarrow \infty} mp^m = \lim_{m \rightarrow \infty} p^m = \lim_{m \rightarrow \infty} p^{(m+1)l} = 0, \text{ (ii) follows as well. } \square$$

**Theorem 3.** Suppose that  $\sum_{n \geq 1} a_n P(S_n \geq m_0 n) < \infty$  for some  $m_0 \geq 1$ . Then

$$\sum_{n \geq 1} a_n P(S_n \geq mn) \sim a_1 P(S_1 \geq m) = a_1 p^m \text{ as } m \rightarrow \infty. \tag{4.3}$$



**Proof.** Notice that  $\lim_{m \rightarrow \infty} \sum_{n \geq 1} a_n P(S_n \geq mn) = 0$  by dominated convergence. Write

$$\sum_{n \geq 1} a_n P(S_n \geq mn) = a_1 P(S_1 \geq m) + \sum_{n \geq 2} a_n P(S_n \geq mn), \quad m \geq 1. \quad (4.4)$$

Let  $m_1 \geq -1/\log p$ . Then, for  $m \geq m_0 \vee m_1$ , in view of Lemma 3.i, we have

$$\begin{aligned} \sum_{n \geq 2} \frac{a_n P(S_n \geq mn)}{a_1 P(S_1 \geq m)} &= \sum_{n \geq 2} \frac{a_n I_p(mn, n)}{a_1 I_p(m, 1)} \leq \sum_{n \geq 2} \frac{a_n I_p((m_0 \vee m_1)n, n)}{a_1 I_p(m_0 \vee m_1, 1)} \\ &= \frac{1}{a_1 I_p(m_0 \vee m_1, 1)} \sum_{n \geq 2} a_n P(S_n \geq (m_0 \vee m_1)n) < \infty. \end{aligned}$$

Hence, by dominated convergence and Lemma 3.ii, we get

$$\lim_{m \rightarrow \infty} \sum_{n \geq 2} \frac{a_n P(S_n \geq mn)}{a_1 P(S_1 \geq m)} = \sum_{n \geq 2} \frac{a_n}{a_1} \lim_{m \rightarrow \infty} \frac{I_p(mn, n)}{I_p(m, 1)} = 0. \quad (4.5)$$

Thus (4.3) follows from (4.4) and (4.5).  $\square$

### Acknowledgment

I am grateful to Gheorghiuță Zbăganu for discussing motivating this paper, and to the referee for pointing out some misprints in the original version of the manuscript.

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