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The purpose of the present paper is a contribution to the study of steady-state p.d.f. for mechanical systems of higher dimension, using the FKP equation method. The article contains three parts. In part I we recall results about Ito stochastic differential equations and FKP equation and we set the problem. In part II we present a result concerning an exact steady-state p.d.f. for a class of nonlinear dynamical systems of higher dimension. In part III we develop a numerical method to solve the FKP equation in a reasonable higher dimension. We present a numerical example in a 12 dimensions case.

## INTRODUCTION

We know that one of the basic tools for the treatment of diffusive Markov processes is the Kolmogrov's forward equation, also called the Fokker-Planck equation (FKP equation). For mechanical applications this tool allowed the study of the response of nonlinear oscillators to stochastic excitation.

Recently T. K. Caughey[9] drew a review of the state of the art in the field of the exact steady-state probability density function (p.d.f.) for one-dimensional nonlinear oscillator to gaussian white noise excitation.

If the excitation is not a white noise, but a 'coloured' process, the stochastic averaging method[46,51,74,76,77] can be used. For one-dimensional nonlinear oscillator, many results have been obtained in the last decade[32,55-58,74].

If the non-white excitation is a 'coloured' stationary physically realizable gaussian process, a Markov realization of this process can be built to obtain the FKP equation[40,42-44] instead of the averaging stochastic method. But in this case, the dimension of the FKP equation is greater than 2 for a one-dimensional nonlinear oscillator, and no exact steady-state p.d.f. can be obtained. A numerical method must be used to solve the FKP equation.

For nonlinear dynamical systems of higher dimension excited by a gaussian white noise, the situation is different and very little is known. The exact steady-state p.d.f. has been obtained for some specific nonlinear oscillator of higher dimension[10,11,13]. In the general case, no exact solution can be obtained and numerical methods must be performed. Unfortunately the numerical solution of the FKP equation in higher dimension is very difficult to build. It is for this reason that the FKP method is not used in this situation and an approximate method has been developed. For instance, the stochastic linearization method allows the building of an approximation of some problems of higher dimension[5,12,31,47,67,71,72,81]. Nevertheless, the most general approximate method is in the present time, the direct numerical simulation of the nonlinear stochastic differential equations[4,23,62-66,71,73].

The purpose of the present paper is a contribution to the study of steady-state p.d.f. for mechanical systems of higher dimension, using the FKP equation method.

This article contains three parts: (1) In part I we recall results about Ito stochastic differential equations and FKP equation and we set the problem. (2) In part II we shall present a result concerning an exact steady-state p.d.f. for a class of nonlinear dynamical systems of higher dimension. This result contains the result in Ref 11. (3) In part III we shall develop a numerical method to solve the FKP equation in a reasonable higher dimension. We shall present a numerical example in a 12 dimensions case.

## PART I: PREREQUISITIES FROM THE THEORY OF NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

The aim of this first part is to recall certain notions and to state the results that we shall use in the following parts of the present paper.

### *Notations and setting the problem*

Let $x=(x_1,\ldots,x_n)$ be a $\mathbb{R}^n$ vector. Throughout this paper, we identify the $x$ vector with the $(n\times 1)$ column matrix of its $x_j$ components. The Euclidian space $\mathbb{R}^n$ is equipped with the usual scalar product $\langle x,y\rangle = \sum_{j=1}^n x_j y_j$, $x$ and $y$ in $\mathbb{R}^n$ and the associated norm $\|x\| = \langle x,x\rangle^{1/2}$. We denote by $\mathrm{Mat}_{\mathbb{R}}(n,n)$ the set of all the $(n\times n)$ real matrices, we take $|a| = (\sum_{j,k=1}^n a_{jk}^2)^{1/2}$ for $a\in \mathrm{Mat}_{\mathbb{R}}(n,n)$, and $a^T$ denotes the transpose of $a$. Let $W(t)=(W_1(t),\ldots,W_n(t))$ be the $\mathbb{R}^n$-valued normalized Wiener process on $\mathbb{R}^+=[0,+\infty[$. Its $\mathrm{Mat}_{\mathbb{R}}(n,n)$-valued covariance

function is $C_W(t,t')=\min(t,t')I$ where $I$ denotes the unit $(n\times n)$ matrix.

Let $X_0=(X_{0,1},\ldots,X_{0,n})$ be a $\mathbb{R}^n$-valued second order random variable which is independent of the Wiener process $W$. We denote by $\rho_{X_0}(y)$ the probability density function on $\mathbb{R}^n$ of the random variable $X_0$. We consider the Ito stochastic differential equation on $\mathbb{R}^n$:

$$dX(t)=b(X(t),t)\,dt+a(X(t),t)\,dW(t),\quad t>0 \tag{1}$$

with the random initial condition:

$$X(0)=X_0 \text{ a.s.} \tag{2}$$

where $,t\longrightarrow b(x,t)=(b_j(x,t))_j$ and $x,t\longrightarrow a(x,t)=(a_{jk}(x,t))_{jk}$ are two mappings defined on $\mathbb{R}^n\times\mathbb{R}^+$ with values respectively in $\mathbb{R}^n$ and $\mathrm{Mat}_{\mathbb{R}}(n,n)$. Concerning the existence and uniqueness of the diffusion process $X(t)$ solution of problems (1)–(2) in the uniform Lipschitz case, we refer to Refs 21, 22 and 28. But this case is rarely encountered and in the following parts other tools are necessary. On the other hand, in what follows we are concerned with the stationary solution of the initial value problem (1)–(2). So it is necessary that all coefficients $b_j(x,t)=b_j(x)$ and $a_{jk}(x,t)=a_{jk}(x)$ are independent of time $t$. Throughput this paper we restrict the development to this case, i.e. we consider only stochastic differential equations of the time homogeneous markovian type.

*Diffusive process and FKP equation*

Suppose we give continuous functions $x\longrightarrow b(x)$: $\mathbb{R}^n\longrightarrow\mathbb{R}^n$ and $x\longrightarrow a(x)$: $\mathbb{R}^n\longrightarrow\mathrm{Mat}_{\mathbb{R}}(n,n)$ with the following properties:

*(P1)* – For all fixed positive real number $R>0$ and for all $x$ and $y$ with $\|x\|\leqslant R$, $\|y\|\leqslant R$, there is a positive constant $K_R$ such that:

$$\begin{cases}\|b(x)-b(y)\|^2+|a(x)-a(y)|^2\leqslant K_R\|x-y\|^2\\ \|b(x)\|^2+|a(x)|^2\leqslant K_R(1+\|x\|^2)\end{cases} \tag{2}$$

*(P2)* – Let $L$ be the following differential operator on $\mathbb{R}^n$ such that for any real function $x\longrightarrow u(x)$ defined on $\mathbb{R}^n$ we have:

$$(Lu)(x)=\sum_{j=1}^{n} b_j(x)\frac{\partial u(x)}{\partial x_j}+\frac{1}{2}\sum_{j,k=1}^{n}\sigma_{jk}(x)\frac{\partial^2 u(x)}{\partial x_j\,\partial x_k} \tag{4}$$

where $\sigma(x)=a(x)a(x)^T\in\mathrm{Mat}_{\mathbb{R}}(n,n)$. We suppose that there exists a Liapounov function $x,t\longrightarrow V_1(x,t)$ defined on $\mathbb{R}^n\times\mathbb{R}^+$ with values in $\mathbb{R}^+$, with continuous second derivative, which satisfies the two conditions:

$$\exists\lambda>0:\ \forall(x,t)\in\mathbb{R}^n\times\mathbb{R}^+,\ \frac{\partial}{\partial t}V_1(x,t)+(LV_1)(x,t)\leqslant\lambda V_1(x,t) \tag{5}$$

$$Inf_{\|x\|>r,t>0}\ V_1(x,t)\longrightarrow+\infty\quad\text{as } r\longrightarrow+\infty \tag{6}$$

From all the above assumptions, we have the following results[24,44]:

*(R1)* – The initial value problem:

$$dX(t)=b(X)\,dt+a(X)\,dW(t),\quad t>0 \tag{7}$$

$$X(0)=X_0 \text{ a.s.} \tag{8}$$

has a unique solution $X$ which is a $\mathbb{R}^n$-valued diffusive Markov process defined a.s. for all $t\geqslant 0$.

*(R2)* – For every $t$ in $\mathbb{R}^-$ the random variable $X(t)$ has a probability density function $\rho(t,x)$ which satisfies the FKP equation on $\mathbb{R}^n$:

$$\frac{\partial}{\partial t}\rho+\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(b_j(x)\rho)-\frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial^2}{\partial x_j\,\partial x_k}(\sigma_{jk}(x)\rho)=0 \tag{9}$$

for $t>0$, with the initial condition:

$$\rho(t,x)\longrightarrow\rho_{X_0}(x)\quad\text{as } t\downarrow 0 \tag{10}$$

*Existence of an asymptotic stationary solution and steady-state FKP equation*

We suppose that in addition of (P1) and (P2), there exists another real positive valued function $x\longrightarrow V_2(x)$ defined on $\mathbb{R}^n$, with a continuous second derivative and such that:

$$\sup_{\|x\|>R}LV_2(x)=-C_R,\quad C_R\longrightarrow+\infty\quad\text{as } R\longrightarrow+\infty \tag{11}$$

with these assumptions we have the following results[24,28,44].

*(R3)* – The solution $X$ of problem (7)–(8) tends in probability for $t\longrightarrow+\infty$, to a $\mathbb{R}^n$-valued stationary process $X_S(t)$ and we have:

$$\rho_S(x)=\lim_{t\to+\infty}\rho(t,x) \tag{12}$$

where $\rho_S(x)$ is the p.d.f. on $\mathbb{R}^n$ of the random variable $X_S(t)$ for any fixed $t$. The steady-state p.d.f. $\rho_S$ satisfies the steady-state FKP equation associated with (9):

$$\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(b_j(x)\rho_S(x))-\frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial^2}{\partial x_j\,\partial x_k}(\sigma_{jk}(x)\rho_S(x))=0 \tag{13}$$

with the normalization condition:

$$\int_{\mathbb{R}^n}\rho_S(x)\,dx=1 \tag{14}$$

## PART II: EXACT STEADY-STATE p.d.f. FOR A CLASS OF NONLINEAR MECHANICAL SYSTEM OF HIGHER DIMENSION

In this part we establish the exact stead-state p.d.f. for a class of nonlinear dynamical system described by canonical equations.

*Canonical equations of a class of dynamical systems*

We consider a nonlinear dynamical system of dimension $m\geqslant 1$ described in the canonical form.

Let $Q(t)=(Q_1(t),\ldots,Q_m(t))$ be the generalized coordinates and $\dot{Q}(t)=dQ(t)/dt$ the generalized velocity.

We denote by $P(t)=(P_1(t), \ldots, P_m(t))$ the generalized momentum canonically conjugated from $Q(t)$. The variables $P$ and $Q$ are the canonical variables. We denote by $p=(p_1, \ldots, p_m)$ and $q=(q_1, \ldots, q_m)$ the variables associated with the $P(t)$ and $Q(t)$ stochastic processes. Let $p,q \to h(p,q)$ be a real function defined on $\mathbb{R}^m \times \mathbb{R}^m$. For $z \in \{p,q\}$ we shall denote by $\partial_z h$ and $\partial_z^2 h$ respectively the $\mathbb{R}^m$-vector and the $(m \times m)$ real matrix such that:

$$(\partial_z h)_j = \frac{\partial h}{\partial z_j}, \qquad [\partial_z^2 h]_{jk} = \frac{\partial^2 h}{\partial z_j \partial z_k}$$

The nonlinear dynamical system is subjected to a $\mathbb{R}^m$-valued normalized gaussian white noise denoted by $\dot{W}_2(t)=(\dot{W}_{2,1}(t), \ldots, \dot{W}_{2,m}(t))$. It is described by the following generalized nonlinear stochastic differential equation on $\mathbb{R}^n=\mathbb{R}^m \times \mathbb{R}^m$ with $n=2m$:

$$\begin{cases} \dot{Q} = \partial_p H(Q,P), & t>0 \\ \dot{P} = -\partial_q H(Q,P) + F(Q,P,\dot{W}_2) \end{cases} \tag{15}$$

with the random initial condition:

$$Q(0)=Q_0, \qquad P(0)=P_0 \text{ a.s.} \tag{16}$$

where:

(i) The derivatives $\dot{Q}$ and $\dot{P}$ with respect to $t$ of the stochastic processes $Q(t)$ and $P(t)$, should be understood in the sense of the theory of generalized stochastic processes[44].

(ii) The Hamiltonian function of the corresponding associated conservative system (i.e., $F \equiv 0$) is denoted by $H$. We shall assume that (1) $q,p \to H(q,p)$ is a function on $\mathbb{R}^m \times \mathbb{R}^m$ with values in $[0, +\infty[$, independent of time $t$; (2) function $H$ has a continuous second derivative, and for all $p$ and $q$ in $\mathbb{R}^m$, the matrix $\partial_p^2 H \in \mathrm{Mat}_{\mathbb{R}}(m,m)$ is positive-definite: (3) for all $p$ and $q$ in $\mathbb{R}^m$, we have the following inequalities:

$$0 < \langle [\partial_p^2 H] z, z \rangle \leqslant C_T \|z\|^2, \quad \forall z \in \mathbb{R}^m, \quad \|z\| > 0 \tag{17}$$

where $C_T$ is a positive real constant which does not depend on $p$ and $q$. In addition we shall assume that:

$$\mathrm{Inf}_{\|q\|^2 + \|p\|^2 > R^2} H(q,p) \to +\infty \quad \text{if } R \to +\infty \tag{18}$$

(iii) The initial condition $(Q_0, P_0)$ is a given $\mathbb{R}^m \times \mathbb{R}^m$-valued second order random variables and has a probability density function $\rho_{Q_0,P_0}(q,p)$.

(iv) The non-conservative force $F=(F_1, \ldots, F_m)$ is expressed in terms of the canonical variables. We consider here the restricted class of systems such that

$$F(Q,P,\dot{W}_2) = -f(H)G\dot{Q} + g(H)S\dot{W}_2 \tag{19}$$

with $\dot{Q}=\partial_p H$ (first equation (15)), and where $S$ and $G$ are two constant real $(m \times m)$ matrices such that $G=SS^T$ and $|S|>0$. Hence denoting by $N$ the rank of $S$ we have $1 \leqslant N \leqslant m$, and $G$ is a positive matrix, but not necessarily a positive definite matrix. Functions $f$ and $g$ are continuous on $[0, +\infty[$ with values in $]0, +\infty[$, and $g$ is differentiable. In addition, there exists $r_0>0$ and real constants $\mu>0$, $C_0>0$, $C_1>0$, $C_2>0$, $-\infty < \alpha_2 \leqslant 1$, $\alpha_1 > \alpha_2 - 1$, $\alpha_0 < \alpha_1 - \alpha_2 + 1$ such that:

$$\forall r \geqslant r_0, \quad f(r) \geqslant C_1 r^{\alpha_1} \tag{20}$$

$$\forall r \geqslant r_0, \quad C_0 \exp(-\mu r^{\alpha_0}) \leqslant g^2(r) \leqslant C_2 r^{\alpha_2} \tag{21}$$

(v) In order to apply the results (R1) to (R3) of part I, we introduce the following notations:

$$x = \begin{pmatrix} q \\ p \end{pmatrix} \in \mathbb{R}^n; \quad X(t) = \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix}; \quad X_0 = \begin{pmatrix} Q_0 \\ P_0 \end{pmatrix} \tag{22}$$

$$b(x) = \begin{pmatrix} \partial_p H \\ -\partial_q H - f(H) G \partial_p H \end{pmatrix} \in \mathbb{R}^n \tag{23}$$

$$a(x) = \left(\begin{array}{c|c} 0_{(m \times m)} & 0_{(m \times m)} \\ 0_{(m \times m)} & g(H)S \end{array}\right) \in \mathrm{Mat}_{\mathbb{R}}(n,n) \tag{24}$$

Hence the initial value problem (15–16) can be written as:

$$\begin{cases} \dot{X} = b(X) + a(X)\dot{W}, & t>0 \\ X(0) = X_0 \text{ a.s.} \end{cases} \tag{25}$$

with $\dot{W}=(\dot{W}_1, \dot{W}_2)$ a $\mathbb{R}^n$-valued normalized gaussian white noise. We know (see Ref. 44) that the generalized stochastic problem (25) is equivalent to the following Ito stochastic differential equation with initial value:

$$\begin{cases} \mathrm{d}X(t) = b(X(t))\,\mathrm{d}t + a(X(t))\,\mathrm{d}W(t), & t>0 \\ X(0) = X_0 \text{ a.s.} \end{cases} \tag{26}$$

Finally, we shall suppose functions $f$, $g$ and $H$ to be such that the mappings $b$ and $a$ on $\mathbb{R}^n$, defined by relations (23)–(24) verify the inequalities (3) and that the random variable $X_0$ is independent of $\{W(t), t \geqslant 0\}$.

## EXISTENCE AND UNIQUENESS OF A REGULAR SOLUTION AND FKP EQUATION

Let $u$ and $v$ be two real functions defined on $\mathbb{R}^m \times \mathbb{R}^m$ with a continuous derivative. We introduce the Poisson bracket notation:

$$[u,v] = \sum_{j=1}^{m} \left( \frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j} \right) = \langle \partial_q u, \partial_p v \rangle - \langle \partial_p u, \partial_q v \rangle \tag{27}$$

Let $L$ be the differential operator on $\mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^n$ which is defined by (4) and which is associated with the $a$ and $b$ mappings given by (23) and (24). Let $q,p \to V(q,p)$ be a real function on $\mathbb{R}^m \times \mathbb{R}^m$ with a continuous second derivative. Hence we have:

$$LV = [V,H] - f(H)\langle S^T \partial_p H, S^T \partial_p V \rangle + \frac{1}{2} g^2(H) \sum_{k=1}^{m} \langle (\partial_p^2 V) S^k, S^k \rangle \tag{28}$$

with $S^k \in \mathbb{R}^m$ such that $S_j^k = S_{jk}$. In the particular case where $V=\varphi(H)$, with $r \to \varphi(r)$ a real function on $\mathbb{R}$, we

obtain:

$$LV=\left(\frac{1}{2}g^2(H)\varphi''(H)-f(H)\varphi'(H)\right)\|S^T\,\partial_p H\|^2$$
$$+\frac{1}{2}g^2(H)\varphi'(H)\sum_{k=1}^{m}\langle(\partial_p^2 H)S^k,S^k\rangle \qquad (29)$$

with $\varphi'(r)=\mathrm{d}/\mathrm{d}r\,\varphi(r)$. We observe that in this case we have used the Poisson bracket property: $[\varphi(H),H]=\varphi'(H)[H,H]=0$ and the usual differential calculus: $\partial_p V=\varphi'(H)\,\partial_p H$, $\partial_p^2 V=\varphi''(H)\,\partial_p H(\partial_p H)^T+\varphi'(H)\,\partial_p^2 H$.

Let us consider the Liapounov function $V_1(q,p)=H(q,p)+\gamma$ with $\gamma$ a positive real number. The conditions (6) is satisfied in view of (18). Note that $\partial V_1/\partial t=0$. Using the relation (29) we obtain:

$$LV_1=-f(H)\|S^T\,\partial_p H\|^2+\frac{1}{2}g^2(H)\sum_{k=1}^{m}\langle(\partial_p^2 H)S^k,S^k\rangle$$

It is clear that for all fixed positive real number $R$, condition (5) is satisfied for all $p$ and $q$ such that $\|p\|^2+\|q\|^2\leqslant R$. Hence we can limit the proof of the condition (5) to the case $H\geqslant H_0>r_0$ with $H_0>1$ and where $r_0$ is the positive real number introduced in (21). The inequalities (17) yield

$$0<\sum_{k=1}^{m}\langle(\partial_p^2 H)S^k,S^k\rangle\leqslant C_T|S|^2 \qquad (30)$$

and (21) gives $g^2(H)\leqslant C_2 H$ because $\alpha_2\leqslant 1$ and $H\geqslant H_0>1$. Hence we have for $H\geqslant H_0$:

$$LV_1\leqslant -f(H)\|S^T\,\partial_p H\|^2+\frac{1}{2}C_2C_T|S|^2H$$

and condition (5) is satisified with $\lambda=\frac{1}{2}C_2C_T|S|^2>0$ because $\gamma>0$ and $-f(H)\|S^T{}_p H\|^2\leqslant 0$. So, we have proved the following result in using (R1) and (R2) of part I:

**Result 1:** The nonlinear stochastic problem (15) subjected to all conditions defined in (ii), (iii), (iv) and (v), has a unique regular solution. For every $t>0$, the random variable $X(t)$ has a probability density function $\rho(t,q,p)$ which satisfies the FKP equation on $\mathbb{R}^m\times\mathbb{R}^m$:

$$\frac{\partial\rho}{\partial t}+[\rho,H]-\mathrm{div}_p J(\rho)=0,\quad t>0 \qquad (31)$$

with the initial condition:

$$\rho(t,q,p)\longrightarrow\rho_{Q_0,P_0}(q,p)\quad\text{as } t\downarrow 0 \qquad (32)$$

where $\mathrm{div}_p J=\sum_{j=1}^{m}\partial/\partial p_j J_j$ and where $J(\rho)$ is the $\mathbb{R}^m$-vector:

$$J(\rho)=\rho(f(H)+g(H)g'(H))G\,\partial_p H+\frac{1}{2}g^2(H)G\,\partial_p\rho \qquad (33)$$

*Existence and construction of the exact steady-state p.d.f*

Asymptotic stationary solution of system (15)–(16) does not exist with the previous assumption. We must change the assumption on $S$ introduced in (iv).

**Result 2:** With the assumptions of the result 1, if the rank of $S$ is equal to $m$ (hence $G$ is a positive definite matrix), the process $X(t)$ tends in probability for $t\longrightarrow+\infty$ to a $\mathbb{R}^m\times\mathbb{R}^m$-valued stationary process $X_S(t)=\{Q_S(t),P_S(t)\}$. For every $t$, the steady-state p.d.f. $\rho_S(q,p)$ of the random variable $X_S(t)$ satisfies the steady-state FKP equation on $\mathbb{R}^m\times\mathbb{R}^m$:

$$[\rho_S,H]-\mathrm{div}_p J(\rho_S)=0 \qquad (34)$$

with the normalization condition:

$$\int_{\mathbb{R}^m\times\mathbb{R}^m}\rho_S(q,p)\,\mathrm{d}q\,\mathrm{d}p=1 \qquad (35)$$

where $J$ is given by (33). Problem (34)–(35) has an exact unique solution which is written as:

$$\rho_S(q,p)=C_N\frac{1}{g^2(H(q,p))}\exp\left(-2\int_0^{H(q,p)}g^{-2}(r)f(r)\,\mathrm{d}r\right) \qquad (36)$$

where $C_N$ is a positive real constant defined by condition (35).

**Proof:**

(a) Existence of $X_S(t)$-process is obtained applying result (R3) of part I. The proof of this point is lengthy because several cases must be considered. Hence we shall only give some elements. First of all one can prove that the assumptions on $H$ allow to consider only the case $\|q\|^2+\|p\|^2>R^2$ with $\|p\|>0$. Hence $\|S^T\,\partial_p H\|>0$ because rank $S=m$ and $\|\partial_p H\|>0$ in this case. Next we can exhibit a $V_2$-function the condition (11) to be verified. For instance the case $\alpha_2<\alpha_1$ is easy to perform. The choice $V_2=H$ yields as $R\longrightarrow+\infty$,

$$LV_2\leqslant H\left(-C_1H^{\alpha_1-\alpha_2}\|S^T\,\partial_p H\|^2+\frac{1}{2}C_2C_T|S|^2\right)\longrightarrow-\infty$$

(b) The application of (13) and (14) yields (34) and (35) taking into account result 1. Exact steady-state p.d.f. is built by searching it in the form $\rho_S(q,p)=\varphi(H)$. Hence $[\rho_S,H]=0$, $J(\rho_S)=\psi G\,\partial_p H$ with $\psi\equiv(f+gg')\varphi+\frac{1}{2}g^2\varphi'$ and equation (34) is verified when $\varphi$ is solution of equation $\psi=0$. Note that $\rho_S$ is integrable on $\mathbb{R}^m\times\mathbb{R}^m$ because we have (18) and as $H\longrightarrow+\infty$,

$$\rho_S(q,p)\leqslant\frac{C_N}{C_0}\exp\left(-H^{\gamma}\left(\frac{C_1}{C_2}-\mu H^{\alpha_0-\gamma}\right)\right)\sim\frac{C_N}{C_0}\exp(-H^{\gamma})$$

with $\gamma=\alpha_1-\alpha_2+1>0$ and $\alpha_0-\gamma<0$.

**Remark:**

Conditions (ii) yield that equation $\partial_p H(q,p)=y$ can be solved locally in $q$ for $y$ in $\mathbb{R}^m$. Let $p=h(q,y)\in\mathbb{R}^m$ be the solution. If we assume that $h$ can be built for all $y$ in $\mathbb{R}^m$ and that $y\longrightarrow h(q,y)$ is a continuously differentiable $\mathbb{R}^m$-valued defined on $\mathbb{R}^m$, hence the steady-state p.d.f. $\rho_{Q,\dot{Q}}(q,y)$ of the random variable $\{Q(t),\dot{Q}(t)\}$, with respect

to $dq\,dy$ is given by:

$$\rho_{Q,\dot{Q}}(q,y)=\rho_S(q,h(q,y))|\det[\hat{c}_y h(q,y)]| \qquad (37)$$

where $[\hat{c}_y h]_{jk}=\partial h_j/\partial y_k$ is the Jacobian matrix.

**Example 1**

Let $M(q)$ be a symmetric real $(m\times m)$ matrix such that for all $q$ in $\mathbb{R}^m$, $M(q)$ is positive definite and $q\to[M(q)]^{-1}$ is a $C^2$-function bounded on $\mathbb{R}^m$. Let $U(q)$ be a $C^2$-potential function on $\mathbb{R}^m$. Hence:

$$H(q,p)=\frac{1}{2}\langle M(q)^{-1}p,p\rangle+U(q)$$

is the Hamiltonian function associated with the nonlinear dynamical system on $\mathbb{R}^m$:

$$\frac{d}{dt}(M(Q(t))\dot{Q}(t))+f(H)G\dot{Q}(t)+\partial_q U(Q(H))=g(H)S\dot{W}_2$$

The generalized momentum is $P(t)=M(Q(t))\dot{Q}(t)$. When rank $S=m$, the exact steady-state p.d.f. $\rho_{Q,\dot{Q}}(q,y)$ is given by (37) with $h(q,y)=M(q)y$ and $[\hat{\partial}_y h]=M(q)$.

**Example 2**

One can generalize example 1 considering a more general form of the kinetic energy of the dynamical systems:

$$T(q,\dot{q})=\frac{1}{2}\langle M(q)\dot{q},\dot{q}\rangle+\langle B(q),\dot{q}\rangle+C(q)$$

such that $T(q,\dot{q})\geqslant 0$ for all $q$ and $\dot{q}$ in $\mathbb{R}^m$, $M(q)$ as in example 1, $B(q)\in\mathbb{R}^m$ and $C(q)\in\mathbb{R}$.

In this case the generalized momentum is given by $p=\partial_{\dot{q}}T$, i.e. by $p=M(q)\dot{q}+B(q)$, and the Hamiltonian function defined by

$$H=\langle p,\dot{q}\rangle-\mathcal{L}(q,\dot{q})$$

$$\mathcal{L}(q,\dot{q})=T(q,\dot{q})-U(q)$$

is written as:

$$H(q,p)=\frac{1}{2}\langle M^{-1}(q)(p-B(q)),p-B(q)\rangle$$
$$+2\langle M^{-1}(q)(p-B(q)),B(q)\rangle+C(q)+U(q)$$

For this general conservative part, the exact steady-state p.d.f. is given by (36) if rank $S=m$.

## PART III: NUMERICAL RESOLUTION OF THE STEADY-STATE FKP EQUATION IN REASONABLE HIGHER DIMENSION

We consider the initial value problem (7) and we assume that a unique regular solution exists and tends for $t\to+\infty$ to a stationary process $X_S(t)$, i.e., the result (R1) to (R3) hold. Here we are going to construct a numerical approximation of the steady-state FKP equation (13) with condition (14). This approximation is based on Galerkin's method, taking for the base functions the normalized Hermite polynomials in several variables.

The main difficulty of this approach is the convergence rate for a problem of reasonable higher dimension. So, we propose a method to accelerate the convergence rate. We mean by 'reasonable higher dimension', a FKP equation on $\mathbb{R}^n$ with $n$ of the order 10 to 20.

In the following we keep all the notations introduced in the part I.

*Notations in multi-indices*

We consider multi-indices $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n$, $\beta=(\beta_1,\ldots,\beta_n)\in\mathbb{N}^n$ and denote $|\alpha|=\sum_{j=1}^n\alpha_j$, $|\beta|=\sum_{j=1}^n\beta_j$. The notation $|\alpha|=0$ means $\alpha_1=\alpha_2=\ldots=\alpha_n=0$. We let: $\alpha!=\alpha_1!\times\ldots\times\alpha_n!$; $\delta_{\alpha,\beta}=\delta_{\alpha_1\beta_1}\times\ldots\times\delta_{\alpha_n\beta_n}$ where $\delta_{\alpha_j\beta_j}$ is the usual Kronecker symbol; $x^\alpha=x_1^{\alpha_1}\times\ldots\times x_n^{\alpha_n}$. Finally, $(\alpha)^j$ designates the multi-index $(\alpha)^j=(\alpha_1,\ldots,\alpha_{j-1},\alpha_{j+1},\ldots,\alpha_n)$.

*Hilbert space $L^2(\mathbb{R}^n,\nu_n)$*

Let $\nu_n(dx)$ be the canonical normal measure on $\mathbb{R}^n$. We denote its density with respect to $dx$ by $f_n(x)$. Thus

$$\nu_n(dx)=f_n(x)\,dx;\quad f_n(x)=(2\pi)^{-n/2}\exp\left(-\frac{1}{2}\|x\|^2\right) \qquad (38)$$

Let $L^2(\mathbb{R}^n,\nu_n)$ be the Hilbert space of the real functions defined ($\nu_n$ − a.e) on $\mathbb{R}^n$ with integrable square with respect to $\nu_n$. This space is equipped with the scalar product:

$$\ll\varphi,\psi\gg=\int_{\mathbb{R}^n}\varphi(x)\psi(x)f_n(x)\,dx$$

*Normalized Hermite polynomials on $\mathbb{R}$*

The normalized Hermite polynomial $H_n(x)$ on $\mathbb{R}$ of index $m\in\mathbb{N}$ is defined by

$$H_m(x)f_1(x)=(-1)^m\left(\frac{d}{dx}\right)^m f_1(x),\quad x\in\mathbb{R} \qquad (39)$$

and its generating function is written as

$$H_m(x)=\int_{\mathbb{R}}(x+iy)^m f_1(y)\,dy;\quad i=\sqrt{-1} \qquad (40)$$

The polynomials can be built with the recurrenece:

$$H_0(x)=1$$
$$H_{m+1}(x)=xH_m(x)-\frac{d}{dx}H_m(x) \qquad (41)$$

and have the following properties:

The functions $h_m(x)=(m!)^{-1/2}H_m(x)$, $m\in\mathbb{N}$, form an orthonormal basis in $L^2(\mathbb{R},\nu_1)$. The derivative of $H_m(x)$ with respect to $x$ equals $d/dx\,H_m(x)=mH_{m-1}(x)$. The Fourier transform (FT) of $H_m$ with respect to the measure $\nu_1(dx)$ is:

$$\int_{\mathbb{R}}e^{-iux}H_m(x)f(x)\,dx=(-iu)^m\exp\left(-\frac{1}{2}u^2\right) \qquad (42)$$

*Normalized Hermite polynomials on $\mathbb{R}^n$*

The normalized Hermite polynomial $H_\alpha(x)$ on $\mathbb{R}^n$ of

multi-index $\alpha \in \mathbb{N}^n$ is defined as:

$$H_\alpha(x) = H_{\alpha_1}(x_1) \times \ldots \times H_{\alpha_n}(x_n) \tag{43}$$

with $H_{\alpha_j}(x_j)$ the normalized Hermite polynomials on $\mathbb{R}$. Its generating function is written as:

$$H_\alpha(x) = \int_{\mathbb{R}^n} (x+iy)^\alpha f_n(y)\,dy \tag{44}$$

with $(x+iy)\alpha = (x_1+iy_1)^{\alpha_1} \times \ldots \times (x_n+iy_n)^{\alpha_n}$. We list the following properties of the multi-index polynomials

(a) The functions $h_\alpha$ on $\mathbb{R}^n$ of multi-index $\alpha \in \mathbb{N}^n$ such that:

$$h_\alpha(x) = (\sqrt{\alpha!})^{-1} H_\alpha(x) \tag{45}$$

form an orthonormal basis in $L^2(\mathbb{R}^n, \nu^n)$. Thus we have

$$\ll h_\alpha, h_\beta \gg = \delta_{\alpha\beta} \tag{46}$$

(b) The partial derivative of $h_\alpha(x)$ with respect to $x_j$ is written as

$$\frac{\partial}{\partial x_j} h_\alpha(x) = \sqrt{\alpha_j}\, h_{(\alpha)'}(x) \tag{47}$$

(c) The FT of $H_\alpha$ with respect to $\nu_n$ is:

$$\int_{\mathbb{R}^n} e^{-i\langle u,x\rangle} H_\alpha(x) f_n(x)\,dx = (-iu)^\alpha \exp\left(-\frac{1}{2}\|u\|^2\right)$$

for all $u$ in $\mathbb{R}^n$.

(d) In addition for all $j \in \{1, \ldots, n\}$ and for all multi-indices $\alpha$ and $\beta$ in $\mathbb{N}^n$, we let:

$$I_j^{\alpha,\beta} = \int_{\mathbb{R}^n} x_j h_\alpha(x) h_\beta(x) f_n(x)\,dx \tag{48}$$

and we have the following result:

$$I_j^{\alpha,\beta} = \sqrt{\alpha_j}\,\delta_{(\alpha)',\beta} + \sqrt{\beta_j}\,\delta_{\alpha,(\beta)'} \tag{49}$$

*Hermite-polynomial expansion of a probability density function*

Let $\tilde{\rho}_S(x)$ be a probability density function on $\mathbb{R}^n$. Thus $\tilde{\rho}_S(x) \geqslant 0$ and condition (14) is verified, i.e., $\tilde{\rho}_S \in L^1(\mathbb{R}^n)$. We can always rewrite $\tilde{\rho}_S(x)\,dx = q(x)\nu_n(dx)$ where

$$q(x) = \tilde{\rho}_S(x) f_n(x)^{-1} \tag{50}$$

is a $\nu_n$-integrable function on $\mathbb{R}^n$. We say that $q$ is the density of $\tilde{\rho}_S(x)\,dx$ with respect to $\nu_n$.

(i) In the particular case where $q \in L^2(\mathbb{R}^n, \nu_n)$, we can write the expansion in multi-index Hermite-polynomials of this density:

$$q(x) = \sum_{|\alpha|=0}^{+\infty} q_\alpha h_\alpha(x) \tag{51}$$

with $h_\alpha$ defined by (45). This series converges in $L^2(\mathbb{R}^n, \nu_n)$ and for any multi-index $\alpha \in \mathbb{N}^n$, the real number $q_\alpha$ is given by:

$$q_\alpha = \ll q, h_\alpha \gg = \int_{\mathbb{R}^n} \tilde{\rho}_S(x) h_\alpha(x)\,dx \tag{52}$$

As $h_\alpha(x) = 1$ for $|\alpha| = 0$, condition (14) yields:

$$q_\alpha = 1 \qquad \text{for } |\alpha| = 0 \tag{53}$$

Let $(\mathcal{F}\tilde{\rho}_S)(u)$, $u \in \mathbb{R}^n$, the FT of $\tilde{\rho}_S(x)$ such that:

$$(\mathcal{F}\tilde{\rho}_S)(u) = \int_{\mathbb{R}^n} e^{-i\langle u,x\rangle} \rho_S(x)\,dx \tag{54}$$

Hence $\mathcal{F}\tilde{\rho}_S$ admits the expansion:

$$(\mathcal{F}\tilde{\rho}_S)(u) = \exp\left(-\frac{1}{2}\|u\|^2\right) \sum_{|\alpha|=0}^{+\infty} q_\alpha (\sqrt{\alpha!})^{-1} (-iu)^\alpha \tag{55}$$

The expansion (55) of $(\mathcal{F}\tilde{\rho}_S)(u)$ allows to compute easily the moments in function of $q_\alpha$.

(ii) If $q$ does not belong to $L^2(\mathbb{R}^n, \nu_n)$, but $\tilde{\rho}_S$ os such that the real numbers $q_\alpha$ are finite for all $\alpha$, hence we may also consider the expansion (51) and study the convergence of this series in a space that is larger than $L^2(\mathbb{R}^n, \nu_n)$.

*Acceleration of the convergence rate*

In a first step we are considering the following initial linear stochastic problem on $\mathbb{R}^n$:

$$\begin{cases} dX^L(t) = BX^L(t) + A\,dW(t), & t > 0 \\ X^L(0) = X_0 \text{ a.s.} \end{cases} \tag{56}$$

where $A$ and $B$ are two real $(n \times n)$ matrices, the $B$-matrix being such that all the eigenvalues of $B$ have strictly negative real part (in this case $B$ is said to be asymptotically stable), and $W(t)$ is the $\mathbb{R}^n$-valued normalized Wiener process on $\mathbb{R}^n$.

We know that the problem (56) has a unique solution $X^L(t)$ which is a $\mathbb{R}^n$-valued centred second order gaussian diffusion process and that $X^L(t)$ tends in probability for $t \longrightarrow +\infty$ to a $\mathbb{R}^n$-valued stationary process $X_S^L(t)$. For every $t$ in $\mathbb{R}$, $X_S^L(t)$ is a centred second order gaussian random variable and its p.d.f. $\rho_S^L(x)$ on $\mathbb{R}^n$, which satisfies the steady-state FKP equation (13) with $b(x) = Bx$ and $a(x) = A$, is written as

$$\rho_S^L(x) = (2\pi)^{-n/2} (\det C_{X_S^L})^{-1/2} \exp\left(-\frac{1}{2}\langle C_{X_S^L}^{-1} x, x\rangle\right) \tag{57}$$

where $C_{X_S^L} = E(X_S^L(t) X_S^L(t)^T) \in \mathrm{Mat}_{\mathbb{R}}(n,n)$ is the covariance matrix of the random variable $X_S^L(t)$. This covariance matrix can be easily computed.

In general, $C_{X_S^L}$ is very different of the unit matrix $I$ and we know that expansion (50)–(51) of $\rho_S^L$ (given by (57)) requires in order to converge a great many terms in the series (51) because the convergence rate is low in this case. However $C_{X_S^L}$ is a positive definite matrix and we can build the following factorization:

$$C_{X_S^L} = UU^T \tag{58}$$

with $U \in \mathrm{Mat}_{\mathbb{R}}(n,n)$. Let $\tilde{X}^L(t)$ be the $\mathbb{R}^n$-valued stochastic process on $\mathbb{R}^+$ such that

$$X^L(t) = U\tilde{X}^L(t) \tag{59}$$

Hence problem (56) can be written as:

$$\begin{cases} \mathrm{d}\tilde{X}^L(t) = \tilde{B}\tilde{X}^L(t) + \tilde{A}\,\mathrm{d}W(t), \quad t > 0 \\ \tilde{X}(0) = \tilde{X}_0 \end{cases} \tag{60}$$

with $\tilde{B} = U^{-1}BU$, $\tilde{A} = U^{-1}A$ and $\tilde{X}_0 = U^{-1}X_0$. The asymptotic stationary solution $\tilde{X}_S^L(t)$ of problem (60) is a centred diffusion gaussian process. For every time $t$ in $\mathbb{R}$, the covariance matrix $C_{\tilde{X}_S^L}$ of the random variable $\tilde{X}_S^L(t)$ is equal to the unit matrix $I$. In this case the expansion of the steady-stare p.d.f. $\tilde{\rho}_S^L(x)$ of $\tilde{X}_S^L$ requires only one term to obtain an exact convergence. The convergence rate is infinity.

These results show that the Hermite-polynomial expansion of the steady-state p.d.f. $\rho_S$ for the nonlinear problem (7) will be generally slowly convergent.

In view of accelerating the convergence, we are proposing following method.

(1) The above $A$ and $B$ matrices (with $B$ asymptotically stable) is calculated in minimizing the distance of the $X_S(t)$-stationary solution of nonlinear problem (7) from the $X_S^L(t)$-stationary solution of linear problem (56). For instance we can choose the distance $\mathrm{dist}(X_S(t), X_S^L(t)) = E(\|X_S(t) - X_S^L(t)\|^2)$ and apply the stochastic linearization method. Finally when $A$ and $B$ matrices are known, the factorization (58) of $C_{X_S^L}$ can be performed and the $U$ matrix is known.

(2) The $A$ and $B$ matrices being known, nonlinear problem (7) is rewritten as:

$$\begin{cases} \mathrm{d}X(t) = (BX(t) + b^{NL}(X(t)))\,\mathrm{d}t + (A + a^{NL}(X(t)))\,\mathrm{d}W(t) \\ X(0) = x_0 \text{ a.s.} \end{cases} \tag{61}$$

where $x \to b^{NL}(x)$ and $x \to a^{NL}(x)$ are two mapping on $\mathbb{R}^n$ respectively with values in $\mathbb{R}^n$ and $\mathrm{Mat}_{\mathbb{R}}(n,n)$ such that:

$$b^{NL}(x) = b(x) - Bx; \quad a^{NL}(x) = a(x) - A \tag{62}$$

(3) Let $\tilde{X}(t)$ be the $\mathbb{R}^n$-valued stochastic process such that:

$$X(t) = U\tilde{X}(t) \tag{63}$$

In performing this transformation in (7) we obtain:

$$\begin{cases} \mathrm{d}\tilde{X}(t) = \tilde{b}(\tilde{X}(t))\,\mathrm{d}t + \tilde{a}(\tilde{X}(t))\,\mathrm{d}W(t) \\ X(0) = X_0 \end{cases} \tag{64}$$

with

$$\tilde{b}(x) = \tilde{B}x + \tilde{b}^{NL}(x) \in \mathbb{R}^n \tag{65}$$

$$\tilde{a}(x) = \tilde{A} + \tilde{a}^{NL}(x) \in \mathrm{Mat}_{\mathbb{R}}(n,n) \tag{66}$$

where $\tilde{B}$, $\tilde{A}$ and $\tilde{X}_0$ have been previously defined and where $\tilde{b}^{NL}(x) = U^{-1}b^{NL}(Ux) \in \mathbb{R}^n$, $\tilde{a}^{NL}(x) = U^{-1}a^{NL}(x) \in \mathrm{Mat}_{\mathbb{R}}(n,n)$.

(4) Due to the existence of an asymptotic stationary solution of problem (7), the new problem (64) in $\tilde{X}$ has also an asymptotic stationary solution denoted by $\tilde{X}_S(t)$. For every $t$ in $\mathbb{R}$, the steady-state p.d.f. $\tilde{\rho}_S(x)$ of the random variable $\tilde{X}_S(t)$ satisfies the steady-state FKP equation (13), i.e.,

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(\tilde{b}_j(x)\tilde{\rho}_S(x)) - \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2}{\partial x_j\,\partial x_k}(\tilde{\sigma}_{jk}(x)\tilde{\rho}_S(x)) = 0 \tag{67}$$

with

$$\tilde{\sigma}(x) = \tilde{\sigma}^L + \tilde{\sigma}^{NL}(x) \tag{68}$$

$$\tilde{\sigma}^L = \tilde{A}\tilde{A}^T \tag{69}$$

$$\tilde{\sigma}^{NL}(x) = \tilde{A}\tilde{a}^{NL}(x)^T + \tilde{a}^{NL}(x)\tilde{A}^T + \tilde{a}^{NL}\tilde{a}^{NL}(x)^T \tag{70}$$

and the condition:

$$\int_{\mathbb{R}^n} \tilde{\rho}_S(x)\,\mathrm{d}x = 1 \tag{71}$$

Note that the p.d.f. $\rho_S(x)$ is such that $\rho_S(x) = \tilde{\rho}_S(U^{-1}x)|\det U|^{-1}$.

(5) A numerical approximation of problem (67)–(71) is built by using Galerkin's method based on the Hermite-polynomial expansion (50)–(51) of $\tilde{\rho}_S$ (see the next section). But it is clear now, that the convergence rate of the series (51) is better because the nonlinear problem (64) appears as a perturbation of the linear problem (60), and for this last one, the expansion is exactly converged with one term. Note that, the smaller the distance between nonlinear problem and associated linear problem is, the larger the convergence velocity of the $\tilde{\rho}_S$-expansion is.

(6) Convergence of the Hermite-polynomial expansion built in the next section, can be studied by using results (i) and (ii) of the previous section. This mathematical aspect will not be discussed here.

*rth order approximation to steady-state FKP equation*

Let $r > 0$ be an integer and $R$ the number of all multi-indices $\alpha$ such that the length $|\alpha| \in (0, 1, \ldots, r)$. The $r$th order approximation of $\tilde{\rho}_S$, denoted $\tilde{\rho}_{S,r}$, is given by

$$\tilde{\rho}_{S,r}(x) = \sum_{|\alpha|=0}^{r} q_\alpha h_\alpha(x) f_n(x) \tag{72}$$

Substitution of (72) on (67), taking the scalar product of $L^2(\mathbb{R}^n, \mathrm{d}x)$ with $h_\beta$ for any $\beta$ of length $|\beta| \in (0, 1, \ldots, r)$ and integrating by parts we conclude that the vector $q \in \mathbb{R}^R$ of coordinates $q_\alpha$ is solution of the homogeneous linear equation

$$\mathscr{K} q = 0 \tag{73}$$

with $\mathscr{K} = \mathscr{B} + \mathscr{Q} \in \mathrm{Mat}_{\mathbb{R}}(R,R)$ and where $\mathscr{B}$ and $\mathscr{Q}$ are two real $(R \times R)$ matrices with entries:

$$\mathscr{B}_{\beta\alpha} = -\sum_{j=1}^{n} \int_{\mathbb{R}^n} \tilde{b}_j(x)\left(\frac{\partial}{\partial x_j} h_\beta(x)\right) f_n(x)\,\mathrm{d}x \tag{74}$$

$$\mathscr{Q}_{\beta\alpha} = \frac{1}{2}\sum_{j,k=1}^{n} \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_k} h_\beta(x)\right) \frac{\partial}{\partial x_j}(\tilde{\sigma}_{jk}(x) h_\alpha(x) f_n(x))\,\mathrm{d}x \tag{75}$$

We know that condition (71) implies (53). Thus we can introduce the block decomposition of (73).

$$\begin{pmatrix} \mathscr{K}_{11} & \mathscr{K}_{12} \\ \mathscr{K}_{21} & \mathscr{K}_{22} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{q} \end{pmatrix} = 0$$

with $\mathscr{K}_{11} \in \mathbb{R}$, $\mathscr{K}_{12} \in \mathrm{Mat}_{\mathbb{R}}(1, R-1)$, $\mathscr{K}_{21} \in \mathrm{Mat}_{\mathbb{R}}(R-1, 1)$ and $\mathscr{K}_{22} \in \mathrm{Mat}_{\mathbb{R}}(R-1, R-1)$ and where $\mathbf{q} \in \mathbb{R}^{R-1}$ is the vector coordinates $q_\alpha$, $\alpha$: $|\alpha| \in (1, \ldots, r)$. Finally we obtain

$$\tilde{p}_{s,r}(x) = \left(1 + \sum_{|\alpha|=1}^{r} q_\alpha h_\alpha(x)\right) f_n(x) \tag{76}$$

where $\mathbf{q}$ is solution of the linear system on $\mathbb{R}^{R-1}$.

$$\mathscr{K}_{22}\mathbf{q} = -\mathscr{K}_{21} \tag{77}$$

*Computation of the drift coefficients*

Using (65) and (74), property (47) and definition (48), we can write $\mathscr{B}$ as:

$$\mathscr{B}_{\beta\alpha} = \mathscr{B}^{L}_{\beta\alpha} + \mathscr{B}^{NL}_{\beta\alpha} \tag{78}$$

with

$$\mathscr{B}^{L}_{\beta\alpha} = -\sum_{j,k=1}^{n} \sqrt{\beta_j}\, \tilde{B}_{jk} I_k^{\alpha,(\beta)} \tag{79}$$

$$\mathscr{B}^{NL}_{\beta\alpha} = -\sum_{j=1}^{n} \sqrt{\beta_j} \int_{\mathbb{R}^n} \tilde{b}^{NL}_j(x) h_\alpha(x) h_{(\beta)}(x) f_n(x)\, dx \tag{80}$$

(i) Computation of the terms $\mathscr{B}^{L}_{\beta\alpha}$ can be easily and exactly performed using the result (49).
(ii) The terms $\mathscr{B}^{NL}_{\beta\alpha}$ depends on the nonlinear part $\tilde{b}^{NL}(x)$ of the drift vector.

In general no exact calculation can be performed and a numerical method must be used to compute the integrals on $\mathbb{R}^n$. If the dimension $n \leqslant n_0$ with $n_0$ of the order of 10, a 3 to 5 points gauss method with the weight $\exp(-\frac{1}{2}x_j^2)$ can be used for each coordinates. When $n_0 < n < n_1$ with $n_1$ of the order of 20 a Monte Carlo numerical simulation can be used to compute the integrals on $\mathbb{R}^n$. If $n > n_1$ it is not reasonable, in general, to compute these integrals and the proposed method must be abandoned.

*Computation of the diffusion coefficients*

As the drift coefficients, using (68, (75), (47), notation (48) and the orthogonal property (46), coefficients $\mathscr{D}_{\beta\alpha}$ can be written as:

$$\mathscr{D}_{\beta\alpha} = \mathscr{D}^{L}_{\beta\alpha} + \mathscr{D}^{NL}_{\beta\alpha} \tag{81}$$

with

$$\mathscr{D}^{L}_{\beta\alpha} = \frac{1}{2} \sum_{j,k=1}^{n} \sqrt{\beta_k}\, \tilde{\sigma}^{L}_{jk} (\sqrt{\alpha}\, \delta_{(\alpha),(\beta)} - I_j^{\alpha,(\beta)}) \tag{82}$$

$$\mathscr{D}^{NL}_{\beta\alpha} = \frac{1}{2} \sum_{j,k=1}^{n} \sqrt{\beta_k} \int_{\mathbb{R}^n} h_{(\beta)}(x) \frac{\partial}{\partial x_j} (\tilde{\sigma}^{NL}_{jk} h_\alpha(x) f_n(x))\, dx \tag{83}$$

(i) Computation of the coefficients $\mathscr{D}^{L}_{\beta\alpha}$ can be exactly performed using the result (49).
(ii) The coefficients $\mathscr{D}^{NL}_{\beta\alpha}$, which depend on the nonlinear part $\tilde{\sigma}^{NL}$ of the diffusion matrix, are computed as $\mathscr{B}^{NL}_{\beta\alpha}$.

*Numerical example for a FKP equation on* $\mathbb{R}^{12}$

A general computer code, based on the previous method, has been developed to analyse the stationary dynamic stochastic response of a jacket-type offshore structure subjected to wave effects for a given short term sea state.

To realize this code many complementary algebraic calculation and numerical test have been performed: (1) to reduce the dimension of the initial problem; (2) to establish the markovianization of the coloured gaussian input process; (3) to build the pseudo-linear problem by using stochastic linearization method: (4) to optimize the choice of the value of $r$th order approximation; (5) to select the multi-indices which are neglected in the series-expansion (72) and to guarantee the preservation of moment properties; (6) to adapt the calculation of integrals on $\mathbb{R}^n$ of the nonlinear part of the drift coefficients in the case of hydrodynamic nonlinearities; (7) to calculate moments and statistics on sample paths such as upcrossings, downcrossings, absolute maxima on an interval and so on.

It is clear that we cannot develop in detail all these points. However, presenting the numerical example, we shall give some commentaries.

(A) So, the present example concerns a jacket-type offshore structure. For a given short-term sea state, the free surface elevation $\eta$ is modelled by a time-stationary centred second order 'coloured' gaussian random field. This stochastic field is fully described by the unilateral wave spectrum denoted by $S_\eta(\omega)$ and written as

$$S_\eta(\omega) = \bar{H}^2_{1/3} \bar{T}_\eta \lambda\left(\frac{2\pi}{\omega \bar{T}_\eta}\right), \quad \omega > 0$$

where $\lambda$ is a given positive real function on $\mathbb{R}^+$, $\bar{H}_{1/3}$ is the significant height and $\bar{T}_\eta$ is the apparent mean period (see Ref. 44).

The structure is assumed to have a linear behaviour and nonlinearities in the present problem are caused by hydrodynamic viscous drag forces and by the fluid-structure interactions due to motions of the structure as a result of its deformations in time. The present structure is a tapered vertical line-like structure, made with a pipe having circular section, attached to the sea bed and free at the top. The water depth is 30 metres and total height of structure is 40 metres. Denoting by $D$ and $S$ the exterior diameter and section of the pipe, by $C_D$ and $C_m$ the hydrodynamic drag and added mass coefficients relative to the direction perpendicular to the axis of the pipe, we have presently $DC_D = 0.57$ (m) and $SC_m = 0.46$ (m$^2$). Hydrodynamic forces are described by the generalized Morrison formula in terms of the velocity field $V$ and acceleration field $A$ of the incident flow, and of the hydrodynamic coefficients $DC_D$ and $SC_m$. The random modelling of the wave leads us to use a first order kinematics of the incident flow. Thus the stochastic field $(V, A)$ with $\mathbb{R}^3 \times \mathbb{R}^3$-values is also time-stationary and gaussian field (see Ref. 44). The structure is modelled with 30 beam finite elements and we are studying the plane movements. The total number of degrees of freedom is

$N=90$. Thus the discretized system is governed by the following nonlinear stochastic differential on $\mathbb{R}^N$:

$$(\mathcal{M}+\mathcal{M}_0)\ddot{Y}(t)+\mathcal{C}\dot{Y}(t)+\mathcal{K}Y(t)=\mathcal{M}_1\mathbb{A}(t)+F_D(\mathbb{V}(t),\dot{Y}(t)) \tag{84}$$

where:

(1) $Y(t)=(Y_1(t),\ldots,Y_N(t))$ is the vector of the unknown nodal displacements;
(2) $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ are respectively the structural mass, damping and stiffness $(N\times N)$ real symmetric and positive definite matrices;
(3) $\mathcal{M}_0$ is the hydrodynamic added mass $(N\times N)$ real symmetric and positive matrix;
(4) $\mathbb{V}(t)=(\mathbb{V}_1(t),\ldots,\mathbb{V}_M(t)$ and $\mathbb{A}(t)=(\mathbb{A}_1(t),\ldots,\mathbb{A}_M(t)$ are the vectors of the values of the $V$ and $A$ discretized stochastic fields along the wetted elements of the pipe;
(5) $\mathcal{M}_1$ is a $(N\times M)$ real matrix. The term $\mathcal{M}_1\mathbb{A}(t)$ represents the equivalent forces applied to the wetted nodes of the finite element meshes and is produced by the hydrodynamic inertia forces distributed along the pipe.
(6) $v,y\longrightarrow F_D(v,y)$ is a $\mathbb{R}^N$-valued function defined on $\mathbb{R}^M\times\mathbb{R}^N$ and represents the equivalent forces applied to the wetted nodes of the finite element meshes. The term $F_D(\mathbb{V}(t),\dot{Y}(t)$ is produced by the nonlinear hydrodynamic drag forces distributed along the pipe.

(B) Mechanical point of view allows to reduce the dimension $N=90$ of this problem considering the classical generalized eigenvalue problem:

$$\mathcal{K}\psi^i=\omega_i^2(\mathcal{M}+\mathcal{M}_0)\psi^i,\qquad 0\leqslant\omega_1^2\leqslant\omega_2^2\leqslant\ldots$$

which is associated with the homogeneous conservative system. In the present example the first three eigenfrequences are $\omega_1=2.18$, $\omega_2=12.3$ and $\omega_3=35.4$ rad/s. For $\omega>6$ rad/s we know that the energy in the wave spectrum $S_\eta(\omega)$ is negligible. Thus we can represent $Y(t)$ with the first $p$th normal vibration modes $\psi^i\in\mathbb{R}^N$ and write:

$$Y(t)=\sum_{i=1}^{p}Z_i(t)\psi^i,\qquad p\leqslant N \tag{85}$$

In view of the previous values of $\omega_i$, we have selected $p=2$. Projection of differential equation (84) in the subspace of $\mathbb{R}^N$ generated by $(\psi^i,\ldots,\psi^p)$, gives the reduced form of the nonlinear stochastic differential equation on $\mathbb{R}^p$:

$$\mathcal{M}_p\ddot{Z}(t)+\mathcal{C}_p\dot{Z}(t)+\mathcal{K}_pZ(t)=\xi^{(1)}(t)+F_{DR}(\xi^{(2)}(t),\dot{Z}(t)) \tag{86}$$

with $Z(t)=(Z_1(t),\ldots,Z_p(t))$ and where:

(1) $\mathcal{M}_p$, $\mathcal{C}_p$ and $\mathcal{K}_p$ are $(p\times p)$ real symmetric positive definite matrices, $\mathcal{M}_p$ and $\mathcal{K}_p$ being diagonal matrices.
(2) $\xi^{(1)}(t)$ and $\xi^{(2)}(t)$ are two dependent $\mathbb{R}^p$-valued stochastic processes such that for all $i\in(1,\ldots,p)$:

$$\begin{aligned}\xi_i^{(1)}(t)&=\langle\psi^i,\mathcal{M}_1\mathbb{A}(t)\rangle\\ \xi_i^{(2)}(t)&=\sum_{j=1}^{p}R_{ij}^{-1}\langle G^T\mathbb{V}(t),\psi^j\rangle\end{aligned} \tag{87}$$

with $R\in\mathrm{Mat}_{\mathbb{R}}(p,p)$ and $G\in\mathrm{Mat}_{\mathbb{R}}(M,N)$ two matrices which result from the finite element formulation.
(3) $u,z\longrightarrow F_{DR}(u,z)$ is a $\mathbb{R}^p$-valued nonlinear function defined on $\mathbb{R}^p\times\mathbb{R}^p$ and which is deduced from the $F_D$-function.

(C) Let $\xi(t)=(\xi^{(1)}(t),\xi^{(2)}(t))$ be the $\mathbb{R}^p\times\mathbb{R}^p=\mathbb{R}^{2p}$-valued stochastic process which is à centred second order stationary gaussian process because the mapping $\eta\longrightarrow\xi$ is a stable linear filter. Classical spectral analysis technique allows the calculation of the $\mathrm{Mat}_{\mathbb{R}}(2p,2p)$-valued spectral density function $S_\xi(\omega)$ expressed in terms of the frequency response function of this filter and the spectrum $S_\eta(\omega)$. We can verify that $\xi(t)$ is a physically realizable process. So we built an approximate Markov realization of this process using the method developed in Refs 43, 44 and 70. The model which has been developed, is a markovianization with dimension $4p$. Thus, $Q_1$ and $Q_2$ matrices are calculated such that

$$\begin{cases}\xi(t)\simeq Q_1\zeta(t)\\ \mathrm{d}\zeta(t)=Q_2\zeta(t)\,\mathrm{d}t+\mathrm{d}W_2(t)\end{cases} \tag{88}$$

where $\zeta(t)$ is a $\mathbb{R}^{4p}$-valued diffusive stochastic process, $W_2(t)$ is the $\mathbb{R}^{4p}$-valued normalized Wiener process, $Q_1$ is a $(2p\times 4p)$ real matrix and $Q_2$ is an asymptotically stable $(4p\times 4p)$ real matrix.

(D) Putting $X(t)=(Z(t),\dot{Z}(t),\zeta(t))$ the $\mathbb{R}^p\times\mathbb{R}^p\times\mathbb{R}^{4p}=\mathbb{R}^{6p}$-valued stochastic process, equations (86) and (88) can be written in the form (7) with $n=6p$, i.e.:

$$\mathrm{d}X(t)=b(X,t))\,\mathrm{d}t+a\,\mathrm{d}W(t) \tag{89}$$

with $a\in\mathrm{Mat}_{\mathbb{R}}(n,n)$ a constant real matrix.

(E) Stochastic linearization technique is used to build the matrices $A$ and $B$ introduced in the section 'Acceleration of the convergence rate'. In fact this method is applied directly on equation (86) and the second order linearized differential equation which is obtained, is transformed in the canonical form (56).

(F) Finally the previously numerical resolution of the steady-state FKP equation with acceleration of the convergence rate is applied.

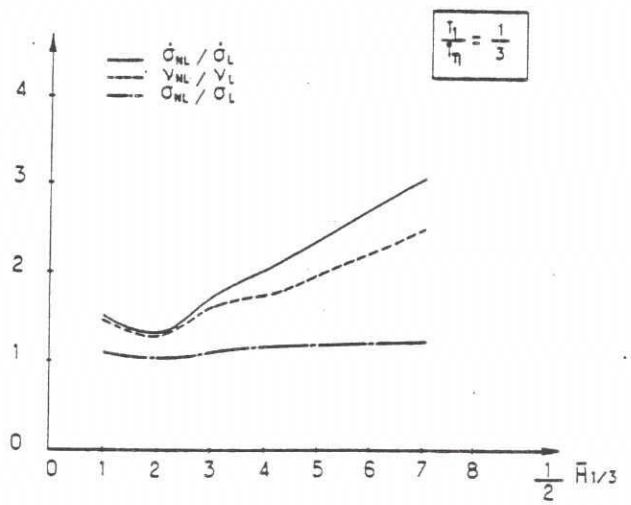


*Fig. 1. Evolution of $\dot{\sigma}_{NL}/\dot{\sigma}_L$, $\sigma_{NL}/\sigma_L$ and $\nu_{NL}/\nu_L$ in function of $\frac{1}{2}\bar{H}_{1/3}$*

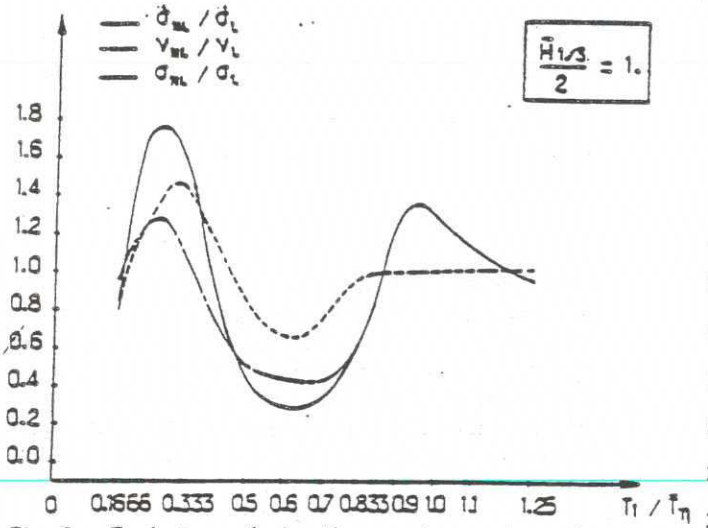


*Fig. 2. Evolution of $\dot\sigma_{NL}/\dot\sigma_L$, $\sigma_{NL}/\sigma_L$ and $\nu_{NL}/\nu_L$ in function of $T_1/\bar T_\eta$*

For the treated example, we have $p=2$ and the steady-state FKP equation has dimension $n=6p=12$. Integral in relation (80) is computed with five gauss points and the retained value of the $r$th order approximation is $r=8$.

(G) Let $Y_{top}(t)$ be the horizontal translation at the top of the structure in the sea-wave direction. It is a component of the $Y(t)$-process which is solution of stochastic equation (84). We put $\sigma_{NL}=(E(Y_{top}(t)^2))^{1/2}$, $\dot\sigma_{NL}=(E(\dot Y_{top}(t)^2))^{1/2}$ and $\nu_{NL}=1/2\pi\ \dot\sigma_{NL}/\sigma_{NL}$. Let $\sigma_L$, $\dot\sigma_L$ and $\nu_L$ be the same variables corresponding to the process $Y^L_{top}(t)$ obtained by stochastic linearization. Figs 1 and 2 show the evolution of $\sigma_{NL}/\sigma_L$, $\dot\sigma_{NL}/\dot\sigma_L$ and $\nu_{NL}/\nu_L$ in function of a parametric investigation expressed in terms of $1/2\,\bar H_{1/3}$ and $T_1/\bar T_\eta$ with $T_1=2\pi\omega_1^{-1}$. These figures show the difference obtained in this case, between the stochastic linearized solution and the nonlinear solution built by numerical resolution of the steady-state FKP equation.

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