

NEW APPROACH
TO SMALL TRANSONIC PERTURBATIONS
FINITE ELEMENT
NUMERICAL SOLVING METHOD
PART 1: NUMERICAL DEVELOPMENTS

by

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ABSTRACT

In part 1, we describe a time-implicit finite element numerical solving method using dual physical/entropic variables for unsteady conservative nonlinear hyperbolic systems. This method can be applied in particular to the Euler equations.

We then show how this formalism can be applied to the unsteady 3D small transonic perturbation equations. The result is a new form of the small transonic perturbation equations in physical variables, accepting vorticity solutions and allowing body interaction to be taken into account, which is not the case for potential small transonic perturbation equations in ADI scheme on structured meshes.

The numerical applications and validations are described in Part II.

Keywords (NASA thesaurus): Fluid dynamics — Numerical methods — Finite elements method — Hyperbolic systems — Nonlinear systems — Euler equations — Small transonic perturbation flow.

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I. — NUMERICAL DEVELOPMENTS

I.1. — INTRODUCTION

In recent years, much research has been done on conservative nonlinear hyperbolic systems in the framework of the Euler equations. The main numerical methods (the list of references below is not comprehensive) include:

(1) Methods based on structured meshes [7, 13, 14, 27, 28, 29, 30, 36, 37, 38, 39, 40, 59, 61]. For instance, **Lerat** and **Sides** [36-39, 59] use time-implicit finite volume schemes and structured meshes and the schemes are globally second-order accurate.

(2) Finite element methods (unstructured mesh) [1, 2, 3, 6, 16, 17, 18, 20-26, 41, 42, 43, 52, 53, 54, 55, 60]. For instance **Angrand**, **Dervieux** *et al.* as well as **Mortchewicz** [48] substitute a formulation by finite elements for **Jameson's** space discretization of finite volumes [29]. **Hughes** [18] uses a formulation by finite elements with a time-implicit scheme and a representation using entropic variables to solve symmetrical systems.

All the techniques mentioned above require introducing an artificial viscosity which is explicitly added to the Euler equations. The numerical discretization schemes introduce a dissipation which is weak for the schemes that are second-order accurate.

Recently, **Lerat** and **Sides** [40] suggested an implicit scheme for transonic computations, without artificial viscosity, using a mesh structured with ADI method and which has a weak internal dissipation.

At CERT/DERI, **Mazet**, **Kalfon** and **Bourdel** [44, 45, 46, 47, 8, 31] studied an approach based on the principle of minimizing entropy generation under constraint of the Euler equations. This approach allows a time-implicit method to be used without introducing artificial viscosity. The variational formulation leads to a finite element method with unstructured mesh. The use of the entropic variables makes it possible to solve symmetrical systems. This technique, which has new aspects, was analyzed numerically by **Mortchewicz** and **Angelini** [49]. The analysis showed that problems remained in this formulation, in particular problems of nonconvergence to a steady state in certain cases.

The final objective of the present work is to develop a numerical method for 3D unsteady transonic flows allowing the use of unstructured meshes.

After a brief review of certain properties of conservative nonlinear hyperbolic systems, a numerical solving method based on a dual formulation with physical/entropic variables, of the type suggested by **Mazet** [44-47] or **Hughes** [18], is discussed. We directly construct the formulation which can also be obtained by a Lagrangian which differs from the one introduced by **Mazet** [44-47].

Space discretization is based on a finite element method applied to the weak formulation. The time scheme is implicit and, each time step, a single symmetrical linear system is solved. Solving can be carried out according to an iterative algorithm which obviates effective assembly of the elementary matrices of finite elements. The method is robust and, for instance, a steady transonic problem can be initialized "abruptly".

For the construction of steady solutions and, a fortiori for unsteady solutions, the dynamics of the physical problem are preserved as the time step is not local.

The method described can be applied to the Euler equations. We describe below how the method is applied to the isentropic Euler small transonic perturbation equations, in order to obtain a code which is intermediate, from the standpoint of numerical costs, between a 3D unsteady small transonic perturbation potential code in ADI scheme on a structured mesh [50, 51] and an implicit 3D unsteady Euler code with an unstructured mesh. These equations have rotational solutions, and the irrotational solutions verify the STP potential equations. The solving method allows the inverse problems to be solved and cases with interactions of lifting surfaces and bodies to be studied, contrary to the possibilities currently available with the potential STP codes in ADI schemes.

However, we will show that the complete STP equations lead to algebraic developments and therefore numerical costs that are similar to those applying to the isentropic Euler equations. In order to satisfy the objective of obtaining an intermediate code, even if its field of application is more restricted, we limited ourselves to the case of the usual simplified form of the small transonic perturbation equations.

The numerical applications are described in Part II of this article.

I.2. — CONSERVATIVE NONLINEAR HYPERBOLIC SYSTEMS AND ENTROPIC VARIABLES

In this section, we recall the results we will be using below on the conservative hyperbolic systems and the properties of the associated entropic variables. The reader is also referred to [12, 19] for general developments on the role of entropy in conservative hyperbolic systems, to [20-26] for general considerations and numerical considerations, to [8, 44-47] for the mathematical developments related to the formulation in entropic variables with applications to the Euler equations and to [10, 11] for the mathematical criteria of hyperbolicity.

I.2.1. — HYPERBOLIC EQUATION CONSIDERED

Let \mathbb{R}^m , m integer ≥ 1 , be the Euclidean vector space equipped with the usual scalar product:

$$\langle W, W' \rangle = \sum_{j=1}^m W_j W'_j, \quad \text{where } W = (W_1, \dots, W_m),$$

$W' = (W'_1, \dots, W'_m)$ are elements of \mathbb{R}^m (W is the physical field). Let d be another integer ≥ 1 which will be the dimension of the physical space ($d=2$ for the 2D case and 3 for the 3D case). We will denote the generic point of \mathbb{R}^d as $x = (x_1, \dots, x_d)$ and the scalar product as $\ll x, x' \gg = \sum_{k=1}^d x_k x'_k$. Below, we will generally use the Einstein convention for index summation.

We consider the following general conservative equation:

$$K \frac{\partial W}{\partial t} + \frac{\partial}{\partial x_k} f^k(W) = 0 \quad (1)$$

where:

(a) K is a given positive real constant originating from a dimensionless constant (reduced frequency) and $t \in \mathbb{R}$ is the dimensionless time.

(b) $(t, x) \mapsto W(t, x) = (W_1(t, x), \dots, W_m(t, x))$ is the physical field of $\mathbb{R} \times \mathbb{R}^d$ in \mathbb{R}^m .

(c) For $k \in \{1, \dots, d\}$, m mappings:

$$W \mapsto f^k(W) = (f_1^k(W), \dots, f_m^k(W)), \quad (2)$$

continuously differentiable from \mathbb{R}^m into \mathbb{R}^m .

(d) Equation (1) is assumed hyperbolic in a part \mathcal{C} of \mathbb{R}^m where \mathcal{C} is a convex open set of \mathbb{R}^m with dimension m .

(e) For any $k \in \{1, \dots, d\}$, we denote the Jacobian matrix of $f^k(W)$ at point W of \mathbb{R}^m as $J^k(W) \in \text{Mat}_{\mathbb{R}}(m, m)$:

$$[J^k(W)]_{ij} = \frac{\partial}{\partial W_j} f_i^k(W), \quad i \text{ et } j \in \{1, \dots, m\}. \quad (3)$$

(f) For any $k \in \{1, \dots, d\}$, we can always write:

$$f^k(W) = J^k(W) W + g^k(W), \quad (4)$$

where functions $W \mapsto g^k(W): \mathbb{R}^m \rightarrow \mathbb{R}^m$ are defined by equalities (4). In the case of the Euler equations, functions f^k are homogeneous in the first degree and functions g^k are then all identically zero. In the case of small transonic perturbations, this is not the case and functions g^k are not all identically zero.

(g) It is recalled that any hyperbolic equation in a conservative form automatically verifies the Rankine-Hugoniot conditions through the discontinuities of field W .

I.2.2. — HYPERBOLICITY CRITERION

Considering (3), everywhere where W is differentiable in x and $\frac{\partial W}{\partial t}$ is taken in the sense of the distributions, equation (1) is written:

$$K \frac{\partial W}{\partial t} + J^k(W) \frac{\partial W}{\partial x_k} = 0. \quad (5)$$

Equation (5) and therefore (1) are hyperbolic in \mathcal{C} [11] if, $\forall W$ in \mathcal{C} and whatever the reals ξ_1, \dots, ξ_d , the equation in ξ_0 :

$$\det \left(K \xi_0 I + \sum_{k=1}^d \xi_k J^k(W) \right) = 0,$$

where $I \in \text{Mat}_{\mathbb{R}}(m, m)$ is the unit matrix with m real roots. Setting $\mu = -K \xi_0$ and:

$$D = \sum_{k=1}^d \xi_k J^k(W) \in \text{Mat}_{\mathbb{R}}(m, m),$$

this equation is written $\det(D - \mu I) = 0$. The m eigenvalues of D must therefore be real.

I,2.3. — ENTROPIC VECTOR ASSOCIATED WITH THE HYPERBOLIC EQUATION

I,2.3.1. — Definition of an entropy

An entropy is defined as a function $W \mapsto S(W)$ of \mathbb{R}^m into \mathbb{R} , strictly convex on \mathcal{E} , and that we will assume continuously differentiable twice in \mathcal{E} .

Under these conditions, in each point W of \mathcal{E} , the Hessian matrix $S''(W) \in \text{Mat}_{\mathbb{R}}(m, m)$ of $S(W)$, such that:

$$[S''(W)]_{ij} = \frac{\partial^2}{\partial W_i \partial W_j} S(W), \quad i \text{ and } j \in \{1, \dots, m\} \quad (6)$$

is a positive-definite symmetrical matrix. Considering the hypotheses on \mathcal{E} and on the differentiability of S , if this property of $S''(W)$ is true in any point W of \mathcal{E} , then S is strictly convex on \mathcal{E} .

In the case of a homogeneous entropy with degree $\alpha > 1$, i. e. $S(\lambda W) = \lambda^\alpha S(W)$, we have $\langle \nabla_W S(W), W \rangle = \alpha S(W)$. Then $\nabla_W S(W)$ is homogeneous with degree $(\alpha - 1) > 0$ and $S''(W)W = (\alpha - 1)\nabla_W S(W)$. This last relation shows that in this case, $\nabla_W S(W) = \tilde{H}(W)W$, where $\tilde{H}(W) = (\alpha - 1)^{-1}S''(W)$ is a positive-definite real symmetrical matrix for any $W \in \mathcal{E}$. For instance, for the Euler equations, such homogeneous entropies exist [49].

In the present case, we will apply these developments to the small transonic perturbation equations. We will show that there exists only one possible entropy and that it is not homogeneous. Therefore, we cannot introduce a hypothesis of homogeneity on the entropy. However, we will introduce a weaker hypothesis which will be verified.

Additional hypothesis: it is assumed that entropy S is such that:

$$\nabla_W S(W) = \tilde{H}(W)W \quad (7)$$

where for any W in \mathcal{E} , $\tilde{H}(W) \in \text{Mat}_{\mathbb{R}}(m, m)$ is a positive-definite real symmetrical matrix.

Remark: the hypothesis introduced does not imply that $\nabla_W S$ is homogeneous and therefore that $\tilde{H}(W)$ is proportional to $S''(W)$.

I,2.3.2. — Definition of the entropic vector associated with the hyperbolic system

If there exist d functions $W \mapsto S^k(W)$ continuously differentiable from \mathbb{R}^m into \mathbb{R} for $k \in \{1, \dots, d\}$, such that:

$$J^k(W)^T \nabla_W S(W) = \nabla_W S^k(W), \quad k \in \{1, \dots, d\}, \quad (8)$$

where T designates the matrix transpose and where $\nabla_W S(W)$ is the gradient of $S(W)$ with respect to W , then the vector $(S(W), S^1(W), \dots, S^d(W))$ of \mathbb{R}^{d+1} is called the entropic vector at point W , associated with the hyperbolic equation (1) and it can be said that (1) has the entropy $S(W)$.

I,2.3.3. — Property of the entropic vector

Everywhere where W is regular, the entropic vector defined in Section I,2.3.2 verifies the following so-called entropy equation:

$$K \frac{\partial}{\partial t} S(W) + \frac{\partial}{\partial x_k} S^k(W) = 0. \quad (9)$$

In effect, taking the scalar product in \mathbb{R}^m of the two members of (5) with $\nabla_W S(W)$ yields, by means of a transposition:

$$K \left\langle \frac{\partial W}{\partial t}, \nabla_W S(W) \right\rangle + \left\langle \frac{\partial W}{\partial x_k}, J^k(W)^T \nabla_W S(W) \right\rangle = 0.$$

Application of equations (8) directly gives the entropy equation (9).

I,2.3.4. — Legendre transform and entropic variables

Since $W \mapsto S(W)$ is a strictly convex function on \mathcal{E} , the Legendre transform of function S is the strictly convex function of $U \mapsto F(U)$ on the convex open set \mathcal{E}^* with the dimension m , defined by:

$$F(U) = \sup_W \{ \langle U, W \rangle - S(W) \}. \quad (10)$$

Function $U \mapsto F(U)$ is called the conjugate function of S of the so-called dual variable $U = (U_1, \dots, U_m)$. Equation (10) expresses the fact that for any values of variables U and W , we have:

$$S(W) + F(U) - \langle U, W \rangle \geq 0. \quad (11)$$

Point $W(U)$ giving the sup of the second member of (10) is unique because of the strict convexity of S . Since function $W \mapsto S(W)$ is differentiable in \mathcal{E} by hypothesis (Sec. I,2.3.1), the subdifferential contains a single subgradient U which is the gradient of $S(W)$. We therefore have:

$$U = \nabla_W S(W). \quad (12)$$

Below, variable U defined by (12) will be called "entropic variable".

Under these conditions, the conjugate function $F(U)$ defined by (10) is given by:

$$F(U) = \langle U, W \rangle - S(W), \quad (13)$$

where U and W are related by equation (12). This Legendre transform is involutive and the conjugate function of $F(U)$ is $S(W)$. The dual relation of (12) is directly obtained from (13) and is written:

$$W = \nabla_U F(U). \quad (14)$$

Since F is strictly convex in U on \mathcal{C}^* , the Hessian matrix $F''(U) \in \text{Mat}_{\mathbb{R}}(m, m)$ of $F(U)$ in point U of \mathcal{C}^* such that:

$$[F''(U)]_{ij} = \frac{\partial^2}{\partial U_i \partial U_j} F(U), \quad i \text{ and } j \in \{1, \dots, n\} \quad (15)$$

is a positive-definite symmetrical matrix.

A. Relation between W and U : Equation (12) is used to establish the transformation giving U knowing W . It is a bijective mapping $W \mapsto U(W) = \nabla_W S(W)$ of \mathcal{C} into \mathcal{C}^* . The reciprocal mapping is the dual relation (14): $U \mapsto W(U) = \nabla_U F(U)$ of \mathcal{C}^* into \mathcal{C} .

B. Algebraic properties: We will need the following algebraic properties:

$$\frac{\partial}{\partial U_j} W_i(U) = [F''(U)]_{ij} \quad (16)$$

$$\frac{\partial}{\partial W_j} U_i(W) = [S''(W)]_{ij}. \quad (17)$$

These two relations are inferred trivially from (12) and (14). If U and W are related by (12) or (14), we have:

$$S''(W) F''(U) = I. \quad (18)$$

In effect:

$$dU_i = \frac{\partial U_i}{\partial W_j} dW_j = \frac{\partial U_i}{\partial W_j} \frac{\partial W_j}{\partial U_k} dU_k$$

giving (18), considering (16) and (17). Since $S''(W)$ and $F''(U)$ are two positive-definite matrices, they are both invertible and we have:

$$F''(U) = S''(W)^{-1}; \quad S''(W) = F''(U)^{-1}. \quad (19)$$

Finally, the following is inferred immediately from (16) and the first equality (19):

$$\frac{\partial W}{\partial x_k} = \frac{\partial W(U)}{\partial U_j} \frac{\partial U_j}{\partial x_k} = S''(W)^{-1} \frac{\partial U}{\partial x_k}. \quad (20)$$

I,2.3.5. — Expression of the hyperbolic equation in entropic variables

By analogy with (12) and (13), we define d functions

$U \mapsto F^k(U)$ of \mathbb{R}^m into \mathbb{R} for $k \in \{1, \dots, d\}$, such that:

$$F^k(U) = \langle U, f^k(W) \rangle - S^k(W). \quad (21)$$

It should be noted that since functions S^k are not convex, transformations (21) are not Legendre transforms.

We have the relation:

$$f^k(W) = \nabla_U F^k(U), \quad k \in \{1, \dots, d\}. \quad (22)$$

In effect, differentiating (21) yields:

$$\begin{aligned} dF^k(U) &= \langle dU, f^k(W) \rangle + \langle U, df^k(W) \rangle - dS^k(W) \\ &= \langle dU, f^k(W) \rangle + \langle U, J^k(W) dW \rangle \\ &\quad - \langle \nabla_W S^k(W), dW \rangle. \end{aligned}$$

Using equation (8) and equation (12) yields:

$$dF^k(U) = \langle dU, f^k(W) \rangle,$$

which proves (22).

In equations (22), W must be considered as the function $U \mapsto W(U)$ of \mathcal{C}^* into \mathcal{C} defined in Section I,2.3.4.A.

Let $F^{k''}(U) \in \text{Mat}_{\mathbb{R}}(m, m)$ be the symmetrical Hessian matrices of $F^k(U)$ such that:

$$[F^{k''}(U)]_{ij} = \frac{\partial^2}{\partial U_i \partial U_j} F^k(U), \quad i \text{ and } j \in \{1, \dots, m\} \quad (23)$$

Substituting (14) and equations (22) into equation (1) yields:

$$K \frac{\partial}{\partial t} \nabla_U F(U) + \frac{\partial}{\partial x_k} \nabla_U F^k(U) = 0$$

giving the expression of the hyperbolic equation in the entropic variable U on \mathcal{C}^* :

$$KF''(U) \frac{\partial U}{\partial t} + F^{k''}(U) \frac{\partial U}{\partial x_k} = 0. \quad (24)$$

It should be noted that expressed as an entropic variable, the hyperbolic operator is symmetrical since $F''(U)$ is a positive-definite symmetrical matrix on \mathcal{C}^* (Sec. I,2.3.4) and matrices $F^{k''}(U)$ are symmetrical for $k \in \{1, \dots, d\}$. Finally, we have the following algebraic property that we will use:

For any k in $\{1, \dots, d\}$ we have:

$$J^k(W) F''(U) = F''(U) J^k(W)^T, \quad (25)$$

or again, using (19)

$$J^k(W) S''(W)^{-1} = S''(W)^{-1} J^k(W)^T. \quad (26)$$

In effect, for $1 \in \{1, \dots, m\}$, equation (22) gives:

$$\frac{\partial}{\partial U_j} f_i^k(W(U)) = \frac{\partial^2}{\partial U_i \partial U_j} F^k(U) = [F^{k''}(U)]_{ij}.$$

In addition, using (2) and (16):

$$\begin{aligned} \frac{\partial}{\partial U_j} f_i^k(W(U)) &= \frac{\partial}{\partial W_i} f_i^k(W) \frac{\partial}{\partial U_j} W_i(U) \\ &= [J^k(W)]_{ii} [F''(U)]_{ij}. \end{aligned}$$

From this we infer the matrix equation:

$$J^k(W) F''(U) = F^{k''}(U), \quad k \in \{1, \dots, d\}. \quad (27)$$

Since matrices $F^{k''}(U)$ are symmetrical, we infer (25).

I,2.3.6. — Existence of an associated entropic vector and hyperbolicity

From the above results it can be inferred that any conservative system of type (1) which has an associated entropic vector is hyperbolic.

In effect, if equation (1) has an entropic vector for W in \mathcal{C} , then, whatever W in \mathcal{C} , matrix $S''(W)$ is positive-definite. Equation (19) then shows that matrix $F''(U)$ is positive-definite for any U in \mathcal{C}^* and equation (24) is hyperbolic on \mathcal{C}^* if, whatever U in \mathcal{C}^* and whatever the reals ξ_1, \dots, ξ_d , the equation in ξ_0 :

$$\det(\xi_0 K F''(U) + \sum_{k=1}^d \xi_k F^{k''}(U)) = 0$$

has m real solutions. This is true since for any U in \mathcal{C}^* , $F''(U)$ is positive-definite and matrix $\sum_{k=1}^d \xi_k F^{k''}(U)$ is real and symmetrical.

I,3. — NUMERICAL SOLVING METHOD

I,3.1. — TIME DISCRETIZATION

Let Δt be the time step. The values at time $t = (n+1)\Delta t$ (or $t = n\Delta t$) are denoted W , U , etc. (or W^n , U^n , etc.).

We use the following numerical scheme. Equations (1) and (5) time-discretized are written as follows:

$$\frac{K}{\Delta t} (W - W^n) + \frac{\partial}{\partial x_k} f^k(W) = 0 \quad (28)$$

$$\frac{K}{\Delta t} (W - W^n) + J^k(W) \frac{\partial W}{\partial x_k} = 0. \quad (29)$$

I,3.2. — CONSTRUCTION OF THE SYSTEM OF EQUATIONS IN VARIABLES W AND U

We will construct the system of equations in W and U . Substituting (20) into (29) yields, everywhere where U is regular:

$$\frac{K}{\Delta t} (W - W^n) + J^k(W) S''(W)^{-1} \frac{\partial U}{\partial x_k} = 0.$$

This equation can also be written using the commutation equation (26):

$$\frac{K}{\Delta t} (W - W^n) + S''(W)^{-1} J^k(W)^T \frac{\partial U}{\partial x_k} = 0.$$

Since $S''(W)$ is positive-definite for any W in \mathcal{C} , the two left members can be multiplied, giving:

$$\frac{K}{\Delta t} S''(W) (W - W^n) + J^k(W)^T \frac{\partial U}{\partial x_k} = 0. \quad (30)$$

The two basic equations are now (30) and (28) in which variables W and U are considered independent. However, the unique physical solution must verify equation (12) considering the construction. From the standpoint of numerical solving, we could, for instance, impose equation (12) as a constraint by means of a Lagrange multiplier. This approach was not however used, since it considerably complicates the numerical formulation. We therefore preferred to use a numerical type technique which is designed to give the operators of the mixed problem good properties of symmetry, positivity and contraction. Obviously, we must make sure that the numerical solution obtained effectively verifies equation (12). As equations (12) and (14) are mutually dual, we penalize equation (30) with (12) and equation (28) with (14).

Let μ be a positive real depending on x such that (Sec. I,3.3.4, Remark 4):

$$\forall x, \quad \mu(x) \in]0, 2[. \quad (31)$$

The system of equations in mixed variables W and U that we consider is then written:

$$\begin{aligned} \frac{K}{\Delta t} S''(W) (W - W^n) + J^k(W)^T \frac{\partial U}{\partial x_k} \\ + \frac{K}{\Delta t} \left(1 - \frac{\mu}{2}\right) (\nabla_W S(W) - U) = 0 \\ - \frac{K}{\Delta t} (W - W^n) - \frac{\partial}{\partial x_k} f^k(W) \\ + \frac{K}{\Delta t} \frac{\mu}{2} (W - \nabla_U F(U)) = 0. \end{aligned} \quad (32)$$

(a) Let $\tilde{H}(W)$ be given by (7) and let us define matrix $\hat{H}(W) \in \text{Mat}_{\mathbb{R}}(m, m)$ by the equality:

$$\left(2 - \frac{\mu}{2}\right) \hat{H}(W) = S''(W) + \left(1 - \frac{\mu}{2}\right) \tilde{H}(W). \quad (33)$$

Since, for any W in \mathcal{C} , matrices $S''(W)$ and $\tilde{H}(W)$ are positive-definite real symmetrical matrices and since $1 - \mu/2 > 0$, matrix $\hat{H}(W)$ is positive-definite, real and symmetrical for any W in \mathcal{C} .

(b) Considering hypothesis (7), equation (12) is written:

$$U = \nabla_w S(W) = \tilde{H}(W) W. \quad (34)$$

For any W in \mathcal{C} , $\tilde{H}(W)^{-1}$ exists and therefore $W = \tilde{H}(W)^{-1} U$. By comparison with equation (14), we infer that:

$$\nabla_U F(U) = \tilde{H}(W)^{-1} U. \quad (35)$$

In this last equation, we must consider W as function $W(U)$ of \mathcal{C}^* in \mathcal{C} .

(c) Using (33) and (35) and reorganizing the terms of system (32) gives the final form of the system of equations of the mixed formulation:

$$\begin{aligned} & \frac{K}{\Delta t} \left(2 - \frac{\mu}{2}\right) \hat{H}(W) W \\ & + \left[-\frac{K}{\Delta t} \left(1 - \frac{\mu}{2}\right) U + J^k(W)^T \frac{\partial U}{\partial x_k} \right] \\ & = \frac{K}{\Delta t} S''(W) W^n \quad (36) \end{aligned}$$

$$\begin{aligned} & \left[-\frac{K}{\Delta t} \left(1 - \frac{\mu}{2}\right) W - \frac{\partial}{\partial x_k} f^k(W) \right] \\ & - \frac{K}{\Delta t} \frac{\mu}{2} \tilde{H}(W)^{-1} U = -\frac{K}{\Delta t} W^n. \quad (37) \end{aligned}$$

I.3.3. — NUMERICAL SOLVING OF THE MIXED FORMULATION BY THE FINITE ELEMENT METHOD

I.3.3.1. — Computation domain and hypotheses

In the reference configuration, the computation domain is the connected bounded open set Ω of \mathbb{R}^n with boundary $\partial\Omega$ (generally not connected).

We will denote the unit normal to $\partial\Omega$, external to Ω , as $N = (N_1, \dots, N_d)$.

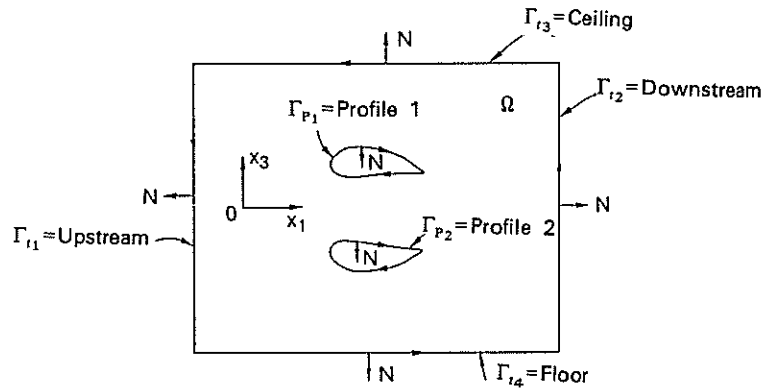


Diagram 1. — Example of a 2D computation domain in the reference configuration.

Diagram 1 illustrates a 2D configuration of a reference computation domain.

At time $t = (n+1) \Delta t$, the domain is denoted Ω_t and its boundary, whose unit normal external to Ω_t is $N^t = (N_1^t, \dots, N_d^t)$, is denoted $\partial\Omega_t$.

Let $\eta = (\eta_1, \dots, \eta_d)$ be the point of $\partial\Omega_t$ which is the transform in the movement of point $x = (x_1, \dots, x_d)$ of the reference configuration $\partial\Omega$ of the boundary. We write:

$$\eta = x + h(x, t) \quad (38)$$

where $h(x, t) = (h_1(x, t), \dots, h_d(x, t))$ is a given function defined on $\partial\Omega \times \mathbb{R}$ with values in \mathbb{R}^d .

At time t , the boundary $\partial\Omega_t$ is written:

$$\partial\Omega_t = \Gamma_f \cup \Gamma_{p,t} \quad (39)$$

where:

Γ_f is the union of the fixed boundaries. Therefore $\Gamma_f \in \partial\Omega$ and $h(x, t) = 0, \forall x \in \Gamma_f$ and $\forall t$.

$\Gamma_{p,t}$ is the union of the moving boundaries.

Below, small movements of the moving boundaries $\Gamma_{p,t}$ around the reference configuration are assumed. We therefore linearize the movements of $\Gamma_{p,t}$ around the corresponding part Γ_p of the reference boundary.

(a) *Volume integral*: considering the above hypothesis, the volume integrals are written:

$$\int_{\Omega_t} \simeq \int_{\Omega}. \quad (40)$$

(b) *Edge integral*: in the weak formulation, the flux integral on $\partial\Omega_t$ of a vector field $X = (X_1, \dots, X_d)$ will be involved:

$$\mathcal{J} = \int_{\partial\Omega_t} \ll N^t, X \gg d\sigma^t \quad (41)$$

where $d\sigma'$ is the surface measure borne by $\partial\Omega_r$. Considering (39), it is written:

$$\mathcal{J} = \mathcal{J}_f + \mathcal{J}_p \quad (42)$$

$$\mathcal{J}_f = \int_{\Gamma_f} \ll X(x), d\sigma(x) \gg \quad (43)$$

$$\mathcal{J}_p = \int_{\Gamma_{p,t}} \ll X(\eta), d\sigma'(\eta) \gg$$

where $d\sigma = (N_1 d\sigma, \dots, N_d d\sigma)$ and $d\sigma' = (N'_1 d\sigma', \dots, N'_d d\sigma')$ are oriented area elements relative to Γ_f and $\Gamma_{p,t}$ respectively. Considering the movement linearization hypothesis, the edge integral \mathcal{J}_p can be written:

$$\mathcal{J}_p \simeq \int_{\Gamma_p} \ll X(x), d\sigma'(\eta(x)) \gg. \quad (44)$$

I.3.3.2. — Weak formulation of the mixed problem

The objective is to numerically solve the equations by the finite element method [4, 5, 9, 56, 62] to allow the use of unstructured meshes. It is therefore necessary to first establish the weak formulation of system (36)-(37) which directly allows us to build the volume operators and the boundary operators. It is then necessary to specify how the boundary conditions are introduced in the boundary operators for each type of problem. We will go into this aspect in Section I.4.

First we formally establish the weak formulation. Then we choose the approximation spaces.

Let Ψ and Φ be two functions defined on $\Omega \simeq \Omega$ with sufficiently regular values in \mathbb{R}^m .

(a) Taking the scalar product in \mathbb{R}^m of the two members of (36) with Ψ and summing on Ω_r , considering (40), yields:

$$\begin{aligned} & \int_{\Omega} \frac{K}{\Delta t} \left(2 - \frac{\mu}{2} \right) \langle \hat{H}(W) W, \Psi \rangle dx \\ & + \int_{\Omega} \left[-\frac{K}{\Delta t} \left(1 - \frac{\mu}{2} \right) \langle U, \Psi \rangle \right. \\ & \quad \left. + \langle J^k(W)^T \frac{\partial U}{\partial x_k}, \Psi \rangle \right] dx \\ & = \int_{\Omega} \frac{K}{\Delta t} \langle S''(W) W^n, \Psi \rangle dx. \quad (45) \end{aligned}$$

(b) Similarly, we take the scalar product in \mathbb{R}^m of the two members of (37) with Φ , we sum on Ω_r , we apply Green's theorem to the volume term in $\langle \frac{\partial}{\partial x_k} f^k(W), \Phi \rangle$, we use (4) and (40)-(44) and

obtain:

$$\begin{aligned} & \int_{\Omega} \left[-\frac{K}{\Delta t} \left(1 - \frac{\mu}{2} \right) \langle W, \Phi \rangle + \langle J^k(W) W, \frac{\partial \Phi}{\partial x_k} \rangle \right] dx \\ & - \int_{\Omega} \frac{K}{\Delta t} \frac{\mu}{2} \langle \hat{H}(W)^{-1} U, \Phi \rangle dx \\ & - \left[\int_{\Gamma_f} \ll X(W, \Phi), d\sigma \gg + \int_{\Gamma_p} \ll X(W, \Phi), d\sigma' \gg \right] \\ & = - \int_{\Omega} \langle g^k(W), \frac{\partial \Phi}{\partial x_k} \rangle dx \\ & - \int_{\Omega} \frac{K}{\Delta t} \langle W^n, \Phi \rangle dx \quad (46) \end{aligned}$$

where we set

$$X(W, \Phi) = (X_1(W, \Phi), \dots, X_d(W, \Phi)) \in \mathbb{R}^d$$

with, for $k \in \{1, \dots, d\}$:

$$X_k(W, \Phi) = \langle f^k(W), \Phi \rangle. \quad (47)$$

Remark: the equations we established can also be inferred from the following Lagrangian at time t :

$$\begin{aligned} L_t(W, U) = & \frac{K}{\Delta t} \int_{\Omega} \langle \nabla_W S(W), W - W^n \rangle dx \\ & - \frac{K}{\Delta t} \int_{\Omega} \langle W - W^n, U \rangle dx \\ & + \int_{\Omega} \langle f^k(W), \frac{\partial U}{\partial x_k} \rangle dx - \int_{\partial\Omega} F^k(U) N'_k d\sigma' \\ & + \frac{K}{\Delta t} \frac{1}{2} \int_{\Omega} \mu (\langle U, W \rangle - S(W) - F(U)) dx. \end{aligned}$$

This Lagrangian is similar to that introduced in [8, 44-47]. We added the last term, whose sign is negative, considering (11).

I.3.3.3. — Construction of the approximation spaces

Let \mathcal{W} and \mathcal{U} be the spatial approximation spaces for fields W and U respectively. These spaces, with a finite dimension, are constructed by the finite element method from the same "triangulation" of domain $\bar{\Omega}$.

A. Choice of triangulation

Domain $\bar{\Omega}$ is meshed by finite elements with three nodes (triangles) in 2D ($d=2$) and with four nodes (tetrahedrons) in 3D ($d=3$) (see Diagram 2).

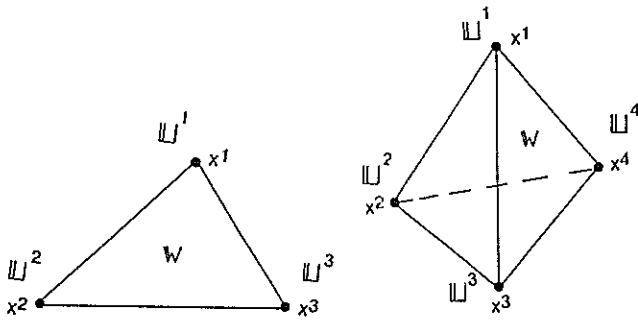


Diagram 2. — Finite volume elements.

We will call these elements volume elements (even in 2D). The total number of volume elements is M_v and the total number of nodes is N_0 . We have:

$$\bar{\Omega} \simeq \bigcup_{j=1}^{M_v} T_j. \quad (48)$$

To simplify these notations, we will drop index J of the elements when no confusion is possible. The local numbers of the nodes of any volume element of domain T are denoted l , where $l \in \{1, \dots, d+1\}$ and $x^l = (x_1^l, \dots, x_d^l)$ are the coordinates of node l . We also need finite boundary elements. They are defined topologically by the trace of the volume mesh on $\partial\Omega$. The boundary elements are therefore elements with two nodes (line segment common to one side of a triangle) in 2D and elements with three nodes (triangle common to a tetrahedron face) in 3D. The total number of boundary elements is denoted M_b . We have:

$$\partial\Omega \simeq \bigcup_{I=1}^{M_b} T_{b,I}. \quad (49)$$

As above, we will drop index I and we will denote the local numbers of the nodes of any boundary element of domain T_b as $l \in \{1, \dots, d\}$ and the coordinates of node l as $x^l = (x_1^l, \dots, x_d^l)$.

The nodes of the boundary elements are obviously common to the nodes of the volume elements.

B. Choice of the approximation space \mathcal{W}

For \mathcal{W} we take all the step functions defined (almost everywhere) on Ω , with values in \mathbb{R}^m , such that each function W of \mathcal{W} is constant on each finite volume element. Let T be the domain of any finite volume element. We then have:

$$\forall x \in T, \quad W(x) = W \in \mathbb{R}^m \quad (50)$$

where $W = (W_1, \dots, W_m)$ is the constant value of field W on the finite volume element considered. We

made this choice for the following reasons:

- Discretized field W is local to each finite volume element. Variables W are independent from one element to another. This allows us to locally eliminate field W , which will be done algebraically. The implicit field will therefore concern only discretized field U .

- The algebra is considerably simplified.

- The functional regularity is sufficient as can be seen on (45)-(46).

However, the functions of \mathcal{W} do not have a natural trace on boundary $\partial\Omega$. We can construct one by lifting. From a numerical standpoint, the lifting can be local or global. Global lifting is "accurate" but too costly from a numerical standpoint. We therefore excluded it. Local lifting is not costly, but it is not sufficiently "accurate" to be able to be used in the boundary operator.

Under these conditions, we decided to express the boundary term of (46) by means of the entropic variables U , since the approximation space \mathcal{U} will have a trace (continuous functions). The boundary term of (46) will therefore be written:

$$\mathcal{J}(U, \Phi) = - \left[\int_{\Gamma_f} \ll X(U, \Phi), d\sigma \gg + \int_{\Gamma_p} \ll X(U, \Phi), d\sigma' \gg \right], \quad (51)$$

where we set

$$X(U, \Phi) = (X_1(U, \Phi), \dots, X_d(U, \Phi)),$$

where for any $k \in \{1, \dots, d\}$

$$X_k(U, \Phi) = \langle \hat{f}^k(U), \Phi \rangle \in \mathbb{R} \quad (52)$$

$$\hat{f}^k(U) = f^k(W(U)) \in \mathbb{R}^m \quad (53)$$

where $U \mapsto W(U): \mathcal{G}^* \rightarrow \mathcal{G}$ is the function defined in Section I,2.3.4A.

C. Choice of approximation space \mathcal{U}

For \mathcal{U} , we take all the continuous functions U of $\bar{\Omega}$ into \mathbb{R}^m , such that the restriction of U to any finite volume element is a first-degree polynomial in variables x_1, \dots, x_d , with coefficients in \mathbb{R}^m . The functions of \mathcal{U} therefore have a trace on boundary $\partial\Omega$.

C1. Finite volume elements: For field U , we are therefore in the conventional framework of linear finite elements with three nodes (2D case) or four nodes (3D case).

Let $x \mapsto \varphi_l(x): T \rightarrow \mathbb{R}$, $l \in \{1, \dots, d+1\}$ be the $d+1$ interpolation functions for a finite volume ele-

ment of domain T . They are such that:

$$\varphi_l(x^j) = \delta_{lj}, \quad l \text{ and } j \in \{1, \dots, d+1\} \quad (54)$$

and we have:

$$\forall x \in T, \quad U(x) = \sum_{l=1}^{d+1} \varphi_l(x) \mathbb{U}^l \quad (55)$$

where \mathbb{U}^l is the value of field U at node x^l :

$$\mathbb{U}^l = U(x^l) \in \mathbb{R}^m, \quad l \in \{1, \dots, d+1\}. \quad (56)$$

C2. Finite boundary elements: Here again, we are in the conventional framework of linear finite elements with two or three nodes. Let $x \mapsto \varphi'_l(x): T_b \rightarrow \mathbb{R}$, $l \in \{1, \dots, d\}$ be the d interpolation functions for a finite element of domain T_b . They are such that:

$$\varphi'_l(x^j) = \delta_{lj}, \quad l \text{ and } j \in \{1, \dots, d\} \quad (57)$$

and we have:

$$\forall x \in T_b, \quad U(x) = \sum_{l=1}^d \varphi'_l(x) \mathbb{U}^l. \quad (58)$$

I,3.3.4. — Construction of the elementary matrices of a finite volume element

Let T be the domain of any fixed finite volume element, $L = d+1$ be the number of nodes, $\mathbb{W} \in \mathbb{R}^m$ be the constant value of field W on T and $\mathbb{U}^1, \dots, \mathbb{U}^L$ be the values of field U at the L nodes. Let μ be the value, taken as constant, of $x \mapsto \mu(x)$ on T . The value of μ can change from one element to another. We introduce the following real constants:

$$\begin{aligned} CS &= \int_T dx; & CS_0 &= \frac{K}{\Delta t} CS; \\ CS_1 &= CS_0 \left(1 - \frac{\mu}{2}\right) \end{aligned} \quad (59)$$

and for l and $l' \in \{1, \dots, L\}$ and $k \in \{1, \dots, d\}$:

$$\begin{aligned} \alpha_l &= \frac{1}{CS} \int_T \varphi_l(x) dx; & \beta_{lk} &= \frac{1}{CS} \int_T \frac{\partial}{\partial x_k} \varphi_l(x) dx; \\ \gamma_{ll'} &= \frac{1}{CS} \int_T \varphi_l(x) \varphi_{l'}(x) dx \end{aligned} \quad (60)$$

For l and $l' \in \{1, \dots, L\}$, we define matrices $G_n^l(\mathbb{W}) \in \text{Mat}_{\mathbb{R}}(m, 1)$, $E^l(\mathbb{W}) \in \text{Mat}_{\mathbb{R}}(m, m)$ and $A^{ll'}(\mathbb{W}) \in \text{Mat}_{\mathbb{R}}(m, m)$ such that:

$$G_n^l(\mathbb{W}) = -CS \sum_{k=1}^d \beta_{lk} g^k(\mathbb{W}) - CS_0 \alpha_l \mathbb{W}^n \quad (61)$$

$$E^l(\mathbb{W}) = -\alpha_l CS_1 I + CS \sum_{k=1}^d \beta_{lk} J^k(\mathbb{W}) \quad (62)$$

$$A^{ll'}(\mathbb{W}) = \frac{1}{2} \mu CS_0 \gamma_{ll'} \tilde{H}(\mathbb{W})^{-1} \quad (63)$$

where $I \in \text{Mat}_{\mathbb{R}}(m, m)$ is the unit matrix. We define the following matrices by blocks: $\mathbb{U} \in \text{Mat}_{\mathbb{R}}(Lm, 1)$, $G_n(\mathbb{W}) \in \text{Mat}_{\mathbb{R}}(Lm, 1)$, $E(\mathbb{W}) \in \text{Mat}_{\mathbb{R}}(Lm, m)$, and $A(\mathbb{W}) \in \text{Mat}_{\mathbb{R}}(Lm, Lm)$, such that:

$$\left. \begin{aligned} \mathbb{U} &= \begin{bmatrix} \mathbb{U}^1 \\ \vdots \\ \mathbb{U}^L \end{bmatrix}; & G_n(\mathbb{W}) &= \begin{bmatrix} G_n^1(\mathbb{W}) \\ \vdots \\ G_n^L(\mathbb{W}) \end{bmatrix}; \\ E(\mathbb{W}) &= \begin{bmatrix} E^1(\mathbb{W}) \\ \vdots \\ E^L(\mathbb{W}) \end{bmatrix} \end{aligned} \right\} \quad (64)$$

$$A(\mathbb{W}) = \begin{bmatrix} A^{11}(\mathbb{W}) & \cdot & \cdot & \dots & A^{1L}(\mathbb{W}) \\ \cdot & A^{22}(\mathbb{W}) & \cdot & \dots & \cdot \\ \cdot & \cdot & A^{33}(\mathbb{W}) & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ A^{L1}(\mathbb{W}) & \cdot & \cdot & \dots & A^{LL}(\mathbb{W}) \end{bmatrix}.$$

Then, based on (45) and (46) and on (50) and (55), the equations relative to the finite volume elements of domain T are written:

$$\begin{aligned} \left(2 - \frac{\mu}{2}\right) CS_0 \tilde{H}(\mathbb{W}) \mathbb{W} + E(\mathbb{W})^T \mathbb{U} \\ = CS_0 S''(\mathbb{W}) \mathbb{W}^n \quad (65) \\ E(\mathbb{W}) \mathbb{W} - A(\mathbb{W}) \mathbb{U} = G_n(\mathbb{W}). \quad (66) \end{aligned}$$

Matrices $\tilde{H}(\mathbb{W})$, $S''(\mathbb{W})$, $A(\mathbb{W})$ are symmetrical. Variable \mathbb{U} is not local to the element treated, but \mathbb{W} is. Since, according to Section I,3.2 a, $\tilde{H}(\mathbb{W})$ is invertible for any \mathbb{W} in \mathcal{C} and $\left(2 - \frac{\mu}{2}\right) CS_0 > 0$, we can infer the following from equation (65):

$$\begin{aligned} \mathbb{W} &= \left(2 - \frac{\mu}{2}\right)^{-1} \hat{\mathbb{W}}^n \\ &\quad - \left(1 - \frac{\mu}{2}\right) \left(2 - \frac{\mu}{2}\right)^{-1} \hat{\mathbb{H}}(\mathbb{W})^{-1} E(\mathbb{W})^T \mathbb{U} \end{aligned} \quad (67)$$

where we set:

$$\hat{\mathbb{W}}^n = \tilde{H}(\mathbb{W})^{-1} S''(\mathbb{W}) \mathbb{W}^n \quad (68)$$

$$\hat{\mathbb{H}}(\mathbb{W})^{-1} = (CS_1)^{-1} \tilde{H}(\mathbb{W})^{-1}. \quad (69)$$

Substituting (67) in (66) and changing all the signs gives the following matrix equation:

$$P(\mathbb{W}) \mathbb{U} = \hat{G}^n(\mathbb{W}) \quad (70)$$

where $P(W) \in \text{Mat}_{\mathbb{R}}(Lm, Lm)$ is the matrix of the finite volume element which is written:

$$P(W) = \left(1 - \frac{\mu}{2}\right) \left(2 - \frac{\mu}{2}\right)^{-1} E(W) \hat{H}(W)^{-1} E(W)^T + A(W) \quad (71)$$

and $\hat{G}^n(W) \in \text{Mat}_{\mathbb{R}}(Lm, 1)$ is the "second member" column matrix of the finite volume element, written:

$$\hat{G}^n(W) = \left(2 - \frac{\mu}{2}\right)^{-1} E(W) \hat{W}^n - G_n(W). \quad (72)$$

Remarks:

(1) Matrix $P(W)$ is positive-definite, real and symmetrical for any W in \mathcal{E} .

(2) Equality (70) is symbolic. The entropic variables U are not local to the finite volume element. The system is obtained by assembling elementary matrices $P(W)$ and "second member" matrices $\hat{G}^n(W)$ on the set of finite volume elements.

(3) As we changed all the signs of (70), we will also change the signs of the elementary matrices for the finite boundary elements.

(4) To establish a law $x \mapsto \mu(x)$, we analyzed the associated linear problem.

The system of mixed equations formally has structure (65)-(66). We show that the operator is contracting if $\mu \in]0, 2[$. In order to obtain the same conditioning on the set of finite volume elements of a given mesh, we constructed the following law. The finite volume element of domain T with "area" CS has the following value for μ :

$$\mu = \mu_1 - \left(\frac{CS_{\min}}{CS}\right)^{1/d} (\mu_1 - \mu_0) \quad (73)$$

where μ_0 and μ_1 are two fixed constants independent of the mesh such that:

$$0 < \mu_0 < \mu_1 < 2, \quad (74)$$

and where CS_{\min} is the smallest value of CS on the set of finite volume elements of a given mesh:

$$CS_{\min} = \min_{J \in \{1, \dots, M_v\}} \{CS\} \quad (75)$$

It can be noted that on the smallest finite element $\mu = \mu_0$ and on the largest $\mu \sim \mu_1$ insofar as $CS_{\max} \gg CS_{\min}$, which is the case in practice.

I.3.3.5. — Construction of the elementary matrix for the finite boundary elements

A. Case of a boundary element on a moving boundary

Let T_b be the domain of a finite boundary element belonging to Γ_p in the reference configuration. The

number of nodes is d and the values of field U in these nodes are U^1, \dots, U^d with $U^i \in \mathbb{R}^m$.

Considering the remark of Section I.3.3.4, the remark of Section I.3.3.2 and (51), the elementary matrix sought is that associated with mapping $-\mathcal{J}_p(U, \Phi)$ on $\mathcal{U} \times \mathcal{U}$:

$$U, \Phi \mapsto -\mathcal{J}_p(U, \Phi) = \int_{T_b} \ll X(U, \Phi), d\sigma^t \gg \quad (76)$$

where X is defined by (52) and (53) and where, for any $x \in T_b$:

$$\left. \begin{aligned} U(x) &= \sum_{i=1}^d \varphi_i'(x) U^i \\ \Phi(x) &= \sum_{i'=1}^d \varphi_{i'}'(x) \Phi^{i'} \end{aligned} \right\} \quad (77)$$

which are equations (58). Under these conditions, the elementary matrix $B_p(U) \in \text{Mat}_{\mathbb{R}}(md, md)$ of the finite boundary element is defined by the equality:

$$\Phi^T B_p(U) U = \int_{T_b} \ll X(U, \Phi), d\sigma^t \gg \quad (78)$$

where U and Φ are the following matrices of $\text{Mat}_{\mathbb{R}}(md, 1)$, defined by blocks:

$$U = \begin{bmatrix} U^1 \\ \vdots \\ U^d \end{bmatrix}; \quad \Phi = \begin{bmatrix} \Phi^1 \\ \vdots \\ \Phi^d \end{bmatrix} \quad (79)$$

B. Case of a boundary element on a fixed boundary

The computations are identical to those for point A above. It is sufficient to replace $d\sigma^t$ by $d\sigma$. The elementary matrix $B_f(U) \in \text{Mat}_{\mathbb{R}}(md, md)$ is therefore defined by the following equality:

$$\Phi^T B_f(U) U = \int_{T_b} \ll X(U, \Phi), d\sigma \gg. \quad (80)$$

C. Remarks

(1) Since matrix $B_p(U)$ is constructed by (78) it is sufficient to set $h=0$ to obtain the expression for $B_f(U)$, considering (38).

(2) Those boundary conditions which do not depend on the Dirichlet conditions must be developed in (78) and (80).

(3) In most of the physical cases encountered (Euler equations, STP equations, etc.), matrices $B_f(U)$ and $B_p(U)$ are symmetrical, but not necessarily positive-definite. We will therefore assume them to be symmetrical.

In addition, the boundary conditions are generally in a subspace of the trace and the boundary operators are then defined in this subspace. We will therefore assume these matrices to be positive-definite below.

(4) In 2D ($d=2$), the integrand of the second member of (78) has the 1-differential form:

$$\ll X(U, \Phi), d\sigma' \gg = X_1(U, \Phi) d\eta_3 - X_3(U, \Phi) d\eta_1 \quad (81)$$

where η is given by (38) and the variations in t can be ignored (t fixed). We therefore have:

$$d\eta_k = dx_k + \frac{\partial}{\partial x_1} h_k dx_1 + \frac{\partial}{\partial x_3} h_k dx_3, \quad k \in \{1, 3\}. \quad (82)$$

For the integral of (80), we simply have (setting $h \equiv 0$):

$$\ll X(U, \Phi), d\sigma \gg = X_1(U, \Phi) dx_3 - X_3(U, \Phi) dx_1. \quad (83)$$

(5) In 3D ($d=3$), the integrand of the second member of (78) has the 2-differential form:

$$\ll X(U, \Phi), d\sigma' \gg = X_1(U, \Phi) d\eta_2 \wedge d\eta_3 + X_2(U, \Phi) d\eta_3 \wedge d\eta_1 + X_3(U, \Phi) d\eta_1 \wedge d\eta_2 \quad (84)$$

where η is given by (38) and, as above (t fixed):

$$d\eta_k = dx_k + \frac{\partial}{\partial x_1} h_k dx_1 + \frac{\partial}{\partial x_2} h_k dx_2 + \frac{\partial}{\partial x_3} h_k dx_3, \quad k \in \{1, 2, 3\}. \quad (85)$$

We can express the second member of (84) and we keep only the linear terms, the terms of type $\frac{\partial}{\partial x_2} h_3 \frac{\partial}{\partial x_1} h_2 dx_1 \wedge dx_2$ being dropped. For integral (80), by setting $h \equiv 0$, we simply have:

$$\ll X(U, \Phi), d\sigma \gg = X_1(U, \Phi) dx_2 \wedge dx_3 + X_2(U, \Phi) dx_3 \wedge dx_1 + X_3(U, \Phi) dx_1 \wedge dx_2. \quad (86)$$

(6) The development of the elementary boundary matrices is specific to each problem.

It should be stressed that the linearization of the boundary conditions occurs only for the unsteady part.

I.3.3.6. — Solving Method

The assembly of the elementary volume and boundary matrices leads to the following matrix equation:

$$\mathcal{P}(w, u_b)u = \mathcal{F}^n(w, u_b, (n+1)\Delta t) \quad (87)$$

where:

$$w = (W^1, \dots, W^{M_v}) \in (\mathbb{R}^m)^{M_v};$$

$$u = (U^1, \dots, U^{N_0}) \in (\mathbb{R}^m)^{N_0} \simeq \mathbb{R}^v$$

where $v = m \times N_0$ is the total number of degrees of freedom of the entropic variables, u_b are the u DOFs involved in the boundary elements, $\mathcal{F}^n(w, u_b, (n+1)\Delta t) \in \mathbb{R}^v$ is the second assembled member and $\mathcal{P}(w, u_b) \in \text{Mat}_{\mathbb{R}}(v, v)$ is the assembled matrix. Any Dirichlet conditions on u due to the boundary conditions must be added to (87).

Matrix $\mathcal{P}(w, u_b)$ is positive-definite symmetrical. Equation (87) is nonlinear in u and w . It can be solved by a complete or incomplete Newton's method.

Actually, in the framework of the numerical analysis conducted for the small transonic perturbation equations, we obtained an excellent solution by approximating the system by:

$$\mathcal{P}(w^n, u_b^n)u = \mathcal{F}^n(w^n, u_b^n, (n+1)\Delta t), \quad (88)$$

which leads to solving a linear system. In the unsteady case, this scheme introduces a negligible time error if the time step is not too large. Then, knowing u at time $(n+1)\Delta t$, we computed w at the same time by equations (67) and (68), with, for each volume element,

$$\left. \begin{aligned} W &= \left(2 - \frac{\mu}{2}\right)^{-1} \hat{W}^{n,n} \\ &- \left(1 - \frac{\mu}{2}\right) \left(2 - \frac{\mu}{2}\right)^{-1} \hat{H}(W^n)^{-1} E(W^n)^T U \\ \hat{W}^{n,n} &= \hat{H}(W^n)^{-1} S''(W^n) W^n. \end{aligned} \right\} \quad (89)$$

We used a preconditioned conjugate gradient method to solve (88). Since this method is iterative, matrices \mathcal{P} and \mathcal{F}^n are not effectively assembled. There is therefore no real problem of memory space, since only the elementary matrices of the finite elements must be stored. In addition, the Dirichlet conditions are directly taken into account algorithmically in the conjugate gradient.

I.4. — APPLICATIONS TO SMALL TRANSONIC PERTURBATIONS (STP)

I.4.1. — REVIEW OF THE 3D SMALL TRANSONIC PERTURBATION EQUATION

We consider a compressible, irrotational 3D flow ($d=3$) of an inviscid fluid varying isentropically. The system of Cartesian coordinates x_1, x_2, x_3 is such that x_1 is directed according to the infinite upstream velocity of the flow (see diagram 3).

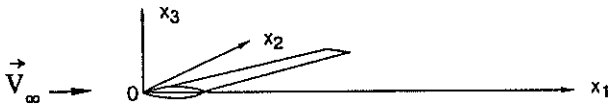


Diagram 3. — Coordinate systems.

The associated 2D problems ($d=2$) will be in plane $0x_1x_3$. We consider the dimensionless problem. We denote as M (instead of M_∞ to simplify the notations) the infinite upstream Mach number and as $\hat{V}=(\hat{V}_1, \hat{V}_2, \hat{V}_3)$, $\hat{\rho}$ and \hat{p} the dimensionless velocity, density and pressure physical quantities. Let $\hat{\phi}$ be the total velocity potential: $\hat{V}=\text{grad } \hat{\phi}$. The isentropic pressure coefficient is such that:

$$-\frac{C_p}{2} = \frac{1-\hat{\rho}^\gamma}{\gamma M^2} \quad (90)$$

where γ is the adiabatic index ($\gamma=7/5=1.4$ for a diatomic gas). The mass conservation equation is written:

$$K \frac{\partial \hat{\rho}}{\partial t} + \text{div}(\hat{\rho} \text{grad } \hat{\phi}) = 0 \quad (91)$$

and Bernoulli's theorem gives:

$$K \frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2} (\|\text{grad } \hat{\phi}\|^2 - 1) + \frac{\hat{\rho}^{\gamma-1} - 1}{(\gamma-1) M^2} = 0. \quad (92)$$

For small transonic perturbations, we introduce the perturbation quantities ϕ, ρ and $V \doteq (V_1, V_2, V_3) = \text{grad } \phi$ such that:

$$\left. \begin{aligned} \hat{\phi} &= x_1 + \phi; & \hat{V}_1 &= 1 + V_1 \\ \hat{\rho} &= 1 + \rho; & \hat{V}_2 &= V_2 \\ \hat{p} &= 1 + p; & \hat{V}_3 &= V_3. \end{aligned} \right\} \quad (93)$$

By setting:

$$\mu = K \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x_1} + \frac{1}{2} \|\text{grad } \phi\|^2 \quad (94)$$

equation (92) gives:

$$\hat{\rho} = [1 - M^2 (\gamma-1) \mu]^{1/(\gamma-1)}. \quad (95)$$

Expanding $\mu \mapsto \hat{\rho}(\mu)$ around 0, substituting the expansion in (91) and preserving only the terms of the same order directly yields the so-called complete 3D small transonic perturbation equation which governs the perturbation potential:

$$\begin{aligned} & -K^2 M^2 \frac{\partial^2 \phi}{\partial t^2} - 2 K M^2 \frac{\partial^2 \phi}{\partial x_1 \partial t} \\ & + \frac{\partial}{\partial x_1} \left[\left(1 - M^2 - \frac{\lambda}{2} \frac{\partial \phi}{\partial x_1} \right) \frac{\partial \phi}{\partial x_1} \right. \\ & \quad \left. - \frac{M^2}{2} \left\{ \left(\frac{\partial \phi}{\partial x_2} \right)^2 + \left(\frac{\partial \phi}{\partial x_3} \right)^2 \right\} \right] \end{aligned}$$

$$\begin{aligned} & + \frac{\partial}{\partial x_2} \left[\frac{\partial \phi}{\partial x_2} \left(1 - M^2 \frac{\partial \phi}{\partial x_1} \right) \right] \\ & + \frac{\partial}{\partial x_3} \left[\frac{\partial \phi}{\partial x_3} \left(1 - M^2 \frac{\partial \phi}{\partial x_1} \right) \right] = 0 \quad (96) \end{aligned}$$

where:

$$\lambda = M^2 [3 - M^2 (2 - \gamma)]. \quad (97)$$

For many applications, the contribution of the non-linear terms in $\partial/\partial x_2$ and $\partial/\partial x_3$ is negligible and the so-called simplified 3D small transonic perturbation equation is then written [51]:

$$\begin{aligned} & -K^2 M^2 \frac{\partial^2 \phi}{\partial t^2} - 2 K M^2 \frac{\partial^2 \phi}{\partial x_1 \partial t} \\ & + \frac{\partial}{\partial x_1} \left[\left(1 - M^2 - \frac{\lambda}{2} \frac{\partial \phi}{\partial x_1} \right) \frac{\partial \phi}{\partial x_1} \right] \\ & + \frac{\partial}{\partial x_2} \frac{\partial \phi}{\partial x_2} + \frac{\partial}{\partial x_3} \frac{\partial \phi}{\partial x_3} = 0. \end{aligned} \quad (98)$$

I.4.2. — EXPRESSION OF THE COMPLETE EQUATION AS A CONSERVATIVE EQUATION

It will be shown that (96) is equivalent to an equation of type (1). We set $W = (W_1, W_2, W_3, W_4) \in \mathbb{R}^4$ ($m=4$ in 3D), with:

$$W_1 = -M^2 \left[K \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x_1} \right] \quad (99)$$

$$W_{k+1} = V_k = \frac{\partial \phi}{\partial x_k}, \quad k \in \{1, 2, 3\} \quad (100)$$

$$\begin{aligned} & K \frac{\partial}{\partial t} W_1 + \frac{\partial}{\partial x_1} \left[W_1 + W_2 - \frac{\lambda}{2} W_2^2 - \frac{1}{2} M^2 (W_3^2 + W_4^2) \right] \\ & + \frac{\partial}{\partial x_2} [W_3 (1 - M^2 W_2)] \\ & + \frac{\partial}{\partial x_3} [W_4 (1 - M^2 W_2)] = 0. \end{aligned} \quad (101)$$

Differentiating the two members of (99) with respect to x_k , $k \in \{1, 2, 3\}$, yields the three following equations, considering (100):

$$K \frac{\partial}{\partial t} W_{k+1} + \frac{\partial}{\partial x_1} W_{k+1} + M^{-2} \frac{\partial}{\partial x_k} W_1 = 0. \quad (102)$$

The four equations (101) and (102) are therefore written in form (1):

$$K \frac{\partial}{\partial t} W + \sum_{k=1}^3 \frac{\partial}{\partial x_k} f^k(W) = 0 \quad (103)$$

where:

$$\begin{aligned} f^1(W) &= \begin{bmatrix} W_1 + W_2 - \frac{\lambda}{2} W_2^2 - \frac{1}{2} M^2 (W_3^2 + W_4^2) \\ W_2 + M^{-2} W_1 \\ W_3 \\ W_4 \end{bmatrix}; \\ f^2(W) &= \begin{bmatrix} W_3 (1 - M^2 W_2) \\ 0 \\ M^{-2} W_1 \\ 0 \end{bmatrix}; \\ f^3(W) &= \begin{bmatrix} W_4 (1 - M^2 W_2) \\ 0 \\ 0 \\ M^{-2} W_1 \end{bmatrix} \end{aligned} \quad (104)$$

This conservative form accepts the appropriate Rankine-Hugoniot equations.

I,4.3. — ENTROPY AND ENTROPIC VECTOR ASSOCIATED WITH THE COMPLETE EQUATION

The entropic vector associated with problem (103)-(104) is written:

$$S(W) = \frac{1}{2} W_1^2 + \frac{1}{2} M^2 (W_2^2 + W_3^2 + W_4^2) - \frac{\lambda}{6} M^2 W_2^3 - \frac{1}{2} M^4 W_2 (W_3^2 + W_4^2) \quad (105)$$

$$\begin{aligned} S^1(W) &= S(W) \\ &+ W_1 \left[W_2 - \frac{\lambda}{2} W_2^2 - \frac{1}{2} M^2 (W_3^2 + W_4^2) \right] \left. \begin{aligned} S^2(W) &= W_1 W_3 (1 - M^2 W_2) \\ S^3(W) &= W_1 W_4 (1 - M^2 W_2) \end{aligned} \right\} \quad (106) \end{aligned}$$

The Hessian matrix of $S(W)$ defined by (6) is written:

$$S''(W) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & M^2 (1 - \lambda W_2) & -M^4 W_3 & -M^4 W_4 \\ 0 & -M^4 W_3 & M^2 (1 - M^2 W_2) & 0 \\ 0 & -M^4 W_4 & 0 & M^2 (1 - M^2 W_2) \end{bmatrix}. \quad (107)$$

Cone \mathcal{C} is determined by stating that $S''(W)$ is positive-definite. We demonstrated that entropy (105) is the one and only possible entropy for problem (103)-(104) except for the linear terms and constants. It is not homogeneous. The additional hypothesis introduced on S in Section I,2.3 is verified and matrix

$\tilde{H}(W)$ defined by (7) is written:

$$\tilde{H}(W) = S'' \left(\frac{1}{2} W \right). \quad (108)$$

Matrix $\hat{H}(W)$ defined by (33) is then written:

$$\hat{H}(W) = \left(2 - \frac{\mu}{2} \right)^{-1} \times \left[S''(W) + \left(1 - \frac{\mu}{2} \right) S'' \left(\frac{1}{2} W \right) \right]. \quad (109)$$

It can thus be seen that matrices $\hat{H}(W)$, $\tilde{H}(W)$ and $S''(W)$ are not diagonal.

The complete equation (96) leads to an algebra which, from a numerical standpoint, generates a volume of computations roughly equal to that of the Euler equations, isentropic or not. There is no real gain with respect to the Euler equations and it is therefore preferable to use the Euler equations. We therefore consider below the simplified 3D small transonic perturbation equation (98).

I,4.4. — CONSERVATIVE EQUATION ASSOCIATED WITH THE SIMPLIFIED EQUATION

We use equation (98) instead of (96). We set (99) and (100). Equation (101) is modified, equations (102) are unchanged. We obtain the conservative form (103) with:

$$\begin{aligned} f^1(W) &= \begin{bmatrix} W_1 + W_2 - \frac{\lambda}{2} W_2^2 \\ W_2 + M^{-2} W_1 \\ W_3 \\ W_4 \end{bmatrix}; \\ f^2(W) &= \begin{bmatrix} W_3 \\ 0 \\ M^{-2} W_1 \\ 0 \end{bmatrix}; \quad f^3(W) = \begin{bmatrix} W_4 \\ 0 \\ 0 \\ M^{-2} W_1 \end{bmatrix}. \end{aligned} \quad (110)$$

In this case, the Jacobian matrices $J^k(W)$ are very simple and functions $g^k(W)$ defined by (4) are written:

$$g^1(W) = \begin{bmatrix} \frac{\lambda}{2} W_2^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad g^2(W) = g^3(W) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (111)$$

Remark 1: The mass conservation equation (91) is written:

$$K \frac{\partial \hat{p}}{\partial t} + \frac{\partial}{\partial x_1} \hat{p} \hat{V}_1 + \frac{\partial}{\partial x_2} \hat{p} \hat{V}_2 + \frac{\partial}{\partial x_3} \hat{p} \hat{V}_3 = 0. \quad (112)$$

The first equation (103) with (110) can also be written:

$$K \frac{\partial}{\partial t} (1 + W_1) + \frac{\partial}{\partial x_1} \left[1 + W_1 + W_2 - \frac{\lambda}{2} W_2^2 \right] + \frac{\partial}{\partial x_2} W_3 + \frac{\partial}{\partial x_3} W_4 = 0. \quad (113)$$

By identification of (112) and (113) we infer the relations between physical variables $\hat{\rho}$, \hat{V} and variable W :

$$\left. \begin{aligned} \hat{\rho} &= 1 + W_1 \\ \hat{\rho} \hat{V}_1 &= 1 + W_1 + W_2 - \frac{\lambda}{2} W_2^2 \\ \hat{\rho} \hat{V}_2 &= W_3 \\ \hat{\rho} \hat{V}_3 &= W_4 \end{aligned} \right\} \quad (114)$$

Considering the second equation (93) and the first equation (114), we infer that W is the density perturbation variable:

$$W_1 = \rho. \quad (115)$$

Remark 2: The pressure coefficient C_p given by (90) is written:

$$C_p = 2(\gamma M^2)^{-1} [(1 + W_1)^\gamma - 1].$$

We can obviously compute C_p by this equation, but to remain consistent with the small transonic perturbation formulation, we write $(1 + W_1)^\gamma \sim 1 + \gamma W_1$ and obtain:

$$C_p = 2 M^{-2} W_1. \quad (116)$$

Remark 3: We denote the local Mach number as M_{loc} to distinguish it from the infinite Mach number denoted M .

It is written:

$$M_{loc}^2 = \left[\hat{\rho} \left(1 + \frac{1}{2} \gamma M^2 C_p \right) \right]^{-1} M^2 \|\hat{\rho} \hat{V}\|^2; \\ \|\hat{\rho} \hat{V}\|^2 = \sum_{k=1}^3 (\hat{\rho} \hat{V}_k)^2.$$

Using (90) yields:

$$M_{loc}^2 = M^2 \|\hat{\rho} \hat{V}\|^2 \hat{\rho}^{-(\gamma+1)}.$$

The use of the approximation of Remark 2 then gives the formula:

$$M_{loc} = M \|\hat{\rho} \hat{V}\| [1 + (\gamma + 1) W_1]^{-1/2}. \quad (117)$$

Remark 4: To obtain the linear problem, it is sufficient to set $\lambda = 0$. Thus, the formulation also allows the subsonic and supersonic linearized problem to be solved.

It should also be noted that (103)/(104) or (110) has rotational solutions.

I,4.5. — ENTROPY AND ENTROPIC VECTOR ASSOCIATED WITH THE SIMPLIFIED EQUATION

The entropic vector associated with problem (103)/(110) is written:

$$S(W) = \frac{1}{2} W_1^2 + \frac{1}{2} M^2 (W_2^2 + W_3^2 + W_4^2) - \frac{\lambda}{6} M^2 W_2^3. \quad (118)$$

$$\left. \begin{aligned} S^1(W) &= S(W) + W_1 \left(W_2 - \frac{\lambda}{2} W_2^2 \right) \\ S^2(W) &= W_1 W_3 \\ S^3(W) &= W_1 W_4 \end{aligned} \right\} \quad (119)$$

The Hessian matrix $S''(W)$ of $S(W)$ is diagonal and is written:

$$S''(W) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & M^2(1 - \lambda W_2) & 0 & 0 \\ 0 & 0 & M^2 & 0 \\ 0 & 0 & 0 & M^2 \end{bmatrix} \quad (120)$$

Considering the results of Sections I,2.3.2 and I,2.3.6, the system is hyperbolic if matrix $S''(W)$ is positive-definite, i.e. if $1 - \lambda W_2 > 0$. Accordingly, the cone $\mathcal{C} \subset \mathbb{R}^4$ is defined by:

$$\mathcal{C} = \left\{ W \in \mathbb{R}^4; W_2 < \frac{1}{\lambda} \right\}. \quad (121)$$

Here again, entropy (118) is the one and only entropy for problem (103)/(110). It is not homogeneous and matrix $\tilde{H}(W)$ defined by (7) is diagonal and is written $\tilde{H}(W) = S''\left(\frac{1}{2} W\right)$, like (108). It is effectively positive-definite and symmetrical for any W in \mathcal{C} , since $1 - \lambda W_2 > 0 \Rightarrow 1 - \frac{\lambda}{2} W_2 > 0$.

Remark: The eigenvalues of the Jacobian $J^1(W)$ are written $\mu = 1$ (double) and $\mu = 1 \pm M^{-1} \sqrt{1 - \lambda W_2}$. A single eigenvalue may vanish for the critical value W_2^* of W_2 , written:

$$W_2^* = \lambda^{-1} (1 - M^2). \quad (122)$$

The expression of the critical C_p is then:

$$C_p^* = -2\lambda^{-1} (1 - M^2). \quad (123)$$

I,4.6. — ENTROPIC VARIABLES FOR THE SIMPLIFIED EQUATION

Let $U = (U_1, U_2, U_3, U_4) \in \mathbb{R}^4$ be the entropic variable.

(a) Equation $W \mapsto U(W) = \nabla_W S(W) = \hat{H}(W) W$ (see I,2.3.4 A) defined on \mathcal{C} is written, considering (118):

$$\left. \begin{aligned} U_1 &= W_1 \\ U_2 &= M^2 \left(1 - \frac{\lambda}{2} W_2 \right) W_2 \\ U_3 &= M^2 W_3 \\ U_4 &= M^2 W_4 \end{aligned} \right\} \quad (124)$$

(b) Equation $U \mapsto W(U) = \nabla_U F(U)$ (see I,2.3.4 A) defined on \mathcal{C}^* is written:

$$\left. \begin{aligned} W_1 &= U_1 \\ W_2 &= 2 U_2 \left[M^2 \left(1 + \sqrt{1 - \frac{2\lambda}{M^2} U_2} \right) \right]^{-1} \\ W_3 &= M^{-2} U_3 \\ W_4 &= M^{-2} U_4 \end{aligned} \right\} \quad (125)$$

where \mathcal{C}^* is the subspace of \mathbb{R}^4 such that:

$$\mathcal{C}^* = \{ U \in \mathbb{R}^4; U_2 < (2\lambda)^{-1} M^2 \}. \quad (126)$$

I,4.7. — ALGEBRAIC COMPLEMENTS FOR THE SIMPLIFIED EQUATION

(a) Matrix $\hat{H}(W)$ defined by (33) is diagonal and is written:

$$\hat{H}(W) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & M^2(1 - \hat{\lambda} W_2) & 0 & 0 \\ 0 & 0 & M^2 & 0 \\ 0 & 0 & 0 & M^2 \end{bmatrix}; \quad \hat{\lambda} = \lambda \left(\frac{3 - \mu/2}{4 - \mu} \right), \quad (127)$$

(b) It should be noted that expressions (61)-(63) are extremely simple, that matrix $P(W)$ defined by (71) can be calculated algebraically with no difficulty, since $\hat{H}(W)^{-1}$ is diagonal, and finally that $\hat{W}^{n,n}$ defined in (89) is expressed simply:

$$\hat{W}^{n,n} = \begin{bmatrix} W_1^n \\ \Lambda^n W_2^n \\ W_3^n \\ W_4^n \end{bmatrix}; \quad \Lambda^n = (1 - \lambda W_2^n)(1 - \hat{\lambda} W_2^n)^{-1}. \quad (128)$$

(c) Functions $\hat{f}^k(U)$ defined for $k \in \{1, 2, 3\}$ are written:

$$\left. \begin{aligned} \hat{f}^1(U) &= M^{-2} \begin{bmatrix} M^2 U_1 + U_2 \\ U_1 + 2 U_2 [1 + \sqrt{1 - 2\lambda M^{-2} U_2}]^{-1} \\ U_3 \\ U_4 \end{bmatrix} \\ \hat{f}^2(U) &= M^{-2} \begin{bmatrix} U_3 \\ 0 \\ U_1 \\ 0 \end{bmatrix}; \quad \hat{f}^3(U) = M^{-2} \begin{bmatrix} U_4 \\ 0 \\ 0 \\ U_1 \end{bmatrix} \end{aligned} \right\} \quad (129)$$

I,4.8. — MATRIX OF THE EDGE ELEMENTS AND BOUNDARY CONDITIONS IN THE 2D CASE

To simplify the presentation, we will develop only the 2D case for airfoils.

The boundaries are oriented as shown in Diagram 1.

I,4.8.1. — Matrix of an edge element on a moving boundary

Let $x^1 = (x_1^1, x_3^1)$ and $x^2 = (x_1^2, x_3^2)$ be the coordinates of the two nodes of the element. Node 1 is the originating node and node 2 is the end node and 12 is pointed according to the orientation of boundary Γ_p . The parameterizing of the finite element is defined by:

$$\left. \begin{aligned} [0, 1] \ni r \mapsto x(r) &= (x_1(r), x_3(r)) \in \mathbb{R}^2 \\ x_1 &= (1-r)x_1^1 + rx_1^2, \quad x_3 = (1-r)x_3^1 + rx_3^2 \end{aligned} \right\} \quad (130)$$

and, since $\varphi_1'(x(r)) = 1-r$ and $\varphi_2'(x(r)) = r$, we have:

$$\Phi = (1-r)\Phi^1 + r\Phi^2; \quad U = (1-r)U^1 + rU^2 \quad (131)$$

$$\left. \begin{aligned} \hat{h}_k(r, t) &= h_k(x(r), t); \\ \eta_k(r) &= x_k(r) + \hat{h}_k(r, t); \quad k \in \{1, 3\}. \end{aligned} \right\} \quad (132)$$

Using (78), (81), (52), (53), and (129), the elementary matrix $B_p(U) \in \text{Mat}_{\mathbb{R}}(6, 6)$ is symmetrical and is written by blocks:

$$B_p(U) = \begin{bmatrix} \frac{1}{3} b_h(U) & \frac{1}{6} b_h(U) \\ \frac{1}{6} b_h(U) & \frac{1}{3} b_h(U) \end{bmatrix} \quad (133)$$

where $b_h(\mathbb{U}) \in \text{Mat}_{\mathbb{R}}(3, 3)$ is the symmetrical matrix:

$$b_h(\mathbb{U}) = \begin{bmatrix} P_0 & M^{-2} P_0 & -P_1 \\ M^{-2} P_0 & M^{-2} \beta_{12}(\mathbb{U}) P_0 & 0 \\ -P_1 & 0 & M^{-2} P_0 \end{bmatrix} \quad (134)$$

in which we set:

$$\left. \begin{aligned} P_0 &= x_3^2 - x_3^1 + \hat{h}_3(1, t) - \hat{h}_3(0, t) \\ P_1 &= M^{-2} [x_1^2 - x_1^1 + \hat{h}_1(1, t) - \hat{h}_1(0, t)] \\ \beta_{12}(\mathbb{U}) &= 2 [1 + \sqrt{1 - M^{-2} \lambda (\mathbb{U}_2^1 + \mathbb{U}_2^2)}]^{-1} \end{aligned} \right\} \quad (135)$$

I.4.8.2. — *Matrix of an edge element on a fixed boundary*

For a boundary element on Γ_f , it is sufficient to set $h \equiv 0$. We then obtain:

$$B_f(\mathbb{U}) = \begin{bmatrix} \frac{1}{3} b_0(\mathbb{U}) & \frac{1}{6} b_0(\mathbb{U}) \\ \frac{1}{6} b_0(\mathbb{U}) & \frac{1}{3} b_0(\mathbb{U}) \end{bmatrix} \quad (136)$$

where $b_0(\mathbb{U})$ is given by (134), having set $\hat{h}_1 = \hat{h}_3 \equiv 0$ in (135).

I.4.8.3. — *Boundary conditions and modifications of the finite edge element matrices*

A. *Infinite boundaries*

As the meshes are unstructured, we chose a rectangle for the infinite boundaries. The upstream and downstream boundaries are line segments parallel to axis x_3 . The "ceiling" and the "floor" are line segments parallel to axis x_1 . These boundaries are fixed. The supersonic case is conventional and does not raise any problems. We will discuss only the subsonic case ($M < 1$) which requires "adjustments".

Subsonic upstream: We write:

$$M^{-1} W_1 + W_2 = 0; \quad W_4 = 0, \quad (137)$$

We use (124) and we approximate (137) linearly ($\lambda = 0$), yielding:

$$U_1 + M^{-1} U_2 = 0; \quad U_4 = 0. \quad (138)$$

In each node x^i , we write two Dirichlet conditions determined from (137):

$$\mathbb{U}_2 = -M \mathbb{U}_1^n, \quad \mathbb{U}_4 = 0. \quad (139)$$

Considering (138), the first line of $b_0(\mathbb{U})$ is written:

$$\begin{aligned} b_0(\mathbb{U}) \mathbb{U}^i &= P_0 \mathbb{U}_1^i + M^{-2} P_0 \mathbb{U}_2^i - P_1 \mathbb{U}_4^i \\ &= P_0 (1 - M^{-1}) \mathbb{U}_1^i. \end{aligned}$$

Considering the two Dirichlet conditions, matrix $b_0(\mathbb{U})$ can then be written:

$$b_0(\mathbb{U}) = \begin{bmatrix} (x_3^2 - x_3^1)(1 - M^{-1}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (140)$$

Since $M < 1$, $1 - M^{-1} < 0$, but on the upstream $x_3^2 - x_3^1 < 0$. Therefore, matrices $B_0(\mathbb{U})$ are nonnegative on the subsonic upstream.

Subsonic downstream: We write $W_1 = 0$. Therefore $U_1 = 0$ according to (124). Therefore, in each node x^i of the downstream, we write a Dirichlet condition:

$$\mathbb{U}_1^i = 0. \quad (141)$$

In matrix $b_0(\mathbb{U})$, we drop the nonlinear term ($\lambda = 0$ as on the upstream).

Considering the Dirichlet condition, matrix $b_0(\mathbb{U})$ is then written:

$$b_0(\mathbb{U}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M^{-2}(x_3^2 - x_3^1) & 0 \\ 0 & 0 & M^{-2}(x_3^2 - x_3^1) \end{bmatrix}. \quad (142)$$

But on the downstream, $x_3^2 - x_3^1 > 0$. Therefore matrices $B_0(\mathbb{U})$ are nonnegative on the subsonic downstream.

Subsonic ceiling and floor: Here we consider only the case of profiles and not that of cascades. If these two boundaries are not physical walls, we can write the nonreflection conditions using the Riemann invariants. In this case, these two boundaries of the computation domain are placed far from the aerodynamic body(ies) and it is equivalent to consider them as physical walls. Below, we therefore consider only physical walls and, in each node x^i of the ceiling and floor (parallel to x_1) we write, considering (124), a Dirichlet condition:

$$\mathbb{U}_4^i = 0. \quad (143)$$

On the ceiling and floor, we have $P_0 = x_3^2 - x_3^1 = 0$, and, considering the Dirichlet condition, matrix $b_0(\mathbb{U})$ is zero. Matrices $B_0(\mathbb{U})$ are therefore nonnegative on the subsonic floor and ceiling.

B. Profile boundaries

In 2D, equation (112) on $\Gamma_{p,t}$ is written:

$$K \frac{\partial \hat{p}}{\partial t} + \frac{\partial}{\partial \eta_1} (\hat{p} \hat{V}_1) + \frac{\partial}{\partial \eta_3} (\hat{p} \hat{V}_3) = 0. \quad (144)$$

We consider the associated 2-differential form:

$$K \hat{p} d\eta_1 \wedge d\eta_3 + (\hat{p} \hat{V}_1) d\eta_3 \wedge dt + (\hat{p} \hat{V}_3) dt \wedge d\eta_1 = 0. \quad (145)$$

We consider the linearization hypothesis for the movements and the reference boundary Γ_p is assumed parametrized in x_1 . Therefore, for $k \in \{1, 3\}$:

$$\left. \begin{aligned} \eta_k &= x_k + h_k(x_1, t) \\ d\eta_k &= dx_k + \frac{\partial h_k}{\partial x_1} dx_1 + \frac{\partial h_k}{\partial t} dt \end{aligned} \right\} \quad (146)$$

Then, using (114), (124) and (125), we obtain the 1-differential form on Γ_p which must be zero. It is the boundary condition:

$$\begin{aligned} M^{-2} \left(1 + \frac{\partial h_1}{\partial x_1} \right) \left[(1 + U_1) K M^2 \frac{\partial h_3}{\partial t} - U_4 \right] dx_1 \\ + \left[1 + U_1 + M^{-2} U_2 - (1 + U_1) K \frac{\partial h_1}{\partial t} \right] \\ \times \left[dx_3 + \frac{\partial h_3}{\partial x_1} dx_1 \right] = 0. \end{aligned} \quad (147)$$

On a finite boundary element Γ_p , we obtain, based on (147), using (130) and (132) and taking notations (133) and (134):

$$\left. \begin{aligned} [B_p(\mathbb{U}) \mathbb{U}]_1 + \mathcal{S}(\mathbb{U})_1 &= 0 \\ [B_p(\mathbb{U}) \mathbb{U}]_4 + \mathcal{S}(\mathbb{U})_4 &= 0 \end{aligned} \right\} \quad (148)$$

where:

$$\left. \begin{aligned} \mathcal{S}(\mathbb{U})_1 &= \frac{1}{2} P_4 + (P_4 - P_0) \left(\frac{1}{3} \mathbb{U}_1^1 + \frac{1}{6} \mathbb{U}_1^2 \right) \\ \mathcal{S}(\mathbb{U})_4 &= \frac{1}{2} P_4 + (P_4 - P_0) \left(\frac{1}{6} \mathbb{U}_1^1 + \frac{1}{3} \mathbb{U}_1^2 \right) \end{aligned} \right\} \quad (149)$$

and:

$$\left. \begin{aligned} P_4 &= P_0 - P_0 P_3 + P_1 P_2 \\ P_2 &= M^2 K \frac{\partial}{\partial t} \frac{1}{2} [h_3(x_1^1, t) + h_3(x_1^2, t)] \\ P_3 &= K \frac{\partial}{\partial t} \frac{1}{2} [h_1(x_1^1, t) + h_1(x_1^2, t)] \end{aligned} \right\} \quad (150)$$

For 2D profiles, we can divide each member of (147) by $M^{-2} \left(1 + \frac{\partial h_1}{\partial x_1} \right) dx_1$.

This yields an equation imposed as Dirichlet condition in each node x^l of the profile boundary:

$$\mathbb{U}_4^l = g(t, \mathbb{U}_1^{l,n}, \mathbb{U}_2^{l,n}). \quad (151)$$

Based on (133), using (148) and the fact that there is a Dirichlet condition (151), the elementary matrix $B_p(\mathbb{U})$ is given by (133) with $b_h(\mathbb{U})$ which is written:

$$b_h(\mathbb{U}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M^{-2} \beta_{12}(\mathbb{U}) P_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (152)$$

The second member of the finite boundary element is the column matrix $\mathcal{S}(\mathbb{U}) \in \text{Mat}_{\mathbb{R}}(6, 1)$ with $\mathcal{S}(\mathbb{U})_1$ and $\mathcal{S}(\mathbb{U})_4$ given by (149) and:

$$\left. \begin{aligned} \mathcal{S}(\mathbb{U})_2 &= -M^{-2} P_0 \left(\frac{1}{3} \mathbb{U}_1^1 + \frac{1}{6} \mathbb{U}_1^2 \right) \\ \mathcal{S}(\mathbb{U})_5 &= -M^{-2} P_0 \left(\frac{1}{6} \mathbb{U}_1^1 + \frac{1}{3} \mathbb{U}_1^2 \right) \end{aligned} \right\} \quad (153)$$

1.5. — CONCLUSIONS

We described a numerical method for solving the conservative hyperbolic equations by an implicit finite element method without artificial viscosity. It is applied to a new form of the small transonic perturbation equations. The numerical applications and the validation of the approach are given in Part II of this article.

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