Numerical simulation of homogeneous and inhomogeneous Gaussian stochastic vector fields

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NUMERICAL SIMULATION OF HOMOGENEOUS AND INHOMOGENEOUS GAUSSIAN STOCHASTIC VECTOR FIELDS

by

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ABSTRACT

The first part of this paper presents a numerical simulation method for homogeneous and inhomogeneous stochastic Gaussian vector fields. The convergence properties of the method are studied in detail as these are the only criteria for guaranteeing the quality of the simulated paths and hence the representativity of the simulation compared with the probabilistic model. Numerical simulation formulae are given explicitly in this first part.

In the second part, an example is given of a velocity field simulation of atmospheric turbulence, as modeled by an isotropic homogeneous Gaussian field and by an inhomogeneous Gaussian field.

I. – INTRODUCTION

Numerical simulation methods are efficient for constructing an approximated solution of certain nonlinear stochastic dynamic problems. In fact, they are presently the only methods which can be used to construct a solution for numerous large dimension linear stochastic problems for which there are no other practical methods [55, 56]. For instance, it is desired to solve the equation:

$$\dot{Y}(t)=f(X(t), Y(t)), t>0, \qquad Y(0)=y_0$$

where f is a nonlinear mapping and $X(t)$ is a stationary stochastic process defined on a probabilistic space $(A, \mathcal{A}, P)$. The simulation method then consists of constructing sample paths $t \mapsto X(t, a_i)$, $i=1, N$, $a_i \in A$ to solve the equations

$$\dot{Y}(t, a_i)=f(X(t, a_i), Y(t, a_i)), t>0, \qquad i=1, N,$$

then, from these solutions, constructing estimators of the random values of the response of the nonlinear system.

Such an approach is used to predict the response of an aircraft to continuous atmospheric turbulence by a numerical model in presence of nonlinearities in the coupled system: aerodynamics-structure-servocontrol (control surface rate saturation nonlinearity, structural nonlinearities, aerodynamic nonlinearity in the transonic state, etc.) [45]. For this type of study, it is therefore necessary to numerically simulate stochastic fields. For the example just mentioned, the field to be simulated is that of continuous atmospheric turbulence.

Although numerical simulation methods are generally conceptually simple, they are relatively complicated to implement. In effect, the mathematical properties of the stochastic fields to be simulated must mandatorily be verified, since they are the only way of guaranteeing the "quality" of the simulated paths and therefore the representativity of the simulation with respect to the probabilistic model. Furthermore, whereas the choice of method has little impact on the cost of simulation in the case of processes (random values indexed by a scalar parameter), in the case of fields (values indexed by a vector parameter, for instance one coordinate for time and three coordinates for space), the choice of an unsuitable method can lead to prohibitive numerical costs. There are many methods for numerically simulating stochastic processes and fields with real or vector values, stationary or not, homogeneous or not, Gaussian or not. Synoptic data on the main methods can be found, for instance, in [55].

The first part of this paper presents a numerical simulation method for homogeneous and inhomogeneous Gaussian stochastic vector fields, which case occurs very often in practice. Certain bases of the method described are not new. For instance, a description and applications can be found in [51, 52, 53]. However, for the reasons mentioned, this paper gives a complete, relatively detailed description for the general case of vector fields, including the properties which appear essential to us for controlling the quality of numerical simulation and, finally, directly usable, the numerical procedure being presented integrally. The case of stationary and nonstationary scalar Gaussian processes is a special case of the results described.

The second part of the paper is devoted to an application in the case of an isotropic homogeneous, then nonhomogeneous field. The application was chosen in the framework of aircraft response to atmospheric turbulence. For this type of application, the Gaussian assumption of the turbulent field is sufficient [44, 69], and it is first attempted to restore only the spectral tensor of the simulated field. This quantity is sufficiently significant to analyze the nonlinear responses of mechanical systems. Under these conditions, the atmospheric turbulence models described herein do not necessarily verify all the constraints related to fluid mechanics.

II. – NUMERICAL SIMULATION OF HOMOGENEOUS AND NONHOMOGENEOUS GAUSSIAN VECTOR FIELDS

II,1. – NOTATIONS

Let m be a positive integer. The Euclidean space $\mathbb{R}^m$ referenced to its canonical base is equipped with the usual scalar product $\langle x, y \rangle = \sum_{j=1}^{m} x_j y_j$

and the associated norm $\|x\| = \langle x, x \rangle^{1/2}$ where

$$x=(x_1, \ldots, x_m), y=(y_1, \ldots, y_m).$$

For x and $y \in \mathbb{C}^m$, we note

$$\langle x, \bar{y} \rangle = \sum_{j=1}^{m} x_j \bar{y}_j \qquad \text{and} \qquad \|x\| = \langle x, \bar{x} \rangle^{1/2}$$

where $\bar{y}$ designates the conjugate of y. Let $\mathbb{K}\mathbb{R}$ or $\mathbb{C}$, and n and m be two positive integers. We will identify the vector space of linear mappings of $\mathbb{K}^n$ into $\mathbb{K}^m$ with the space of matrices $\mathrm{Mat}_{\mathbb{K}}(m, n)$ with m rows and n columns whose elements are in $\mathbb{K}$. We therefore have $\mathbb{K}^n \sim \mathrm{Mat}_{\mathbb{K}}(n, 1)$. Let $A \in \mathrm{Mat}_{\mathbb{C}}(n, m)$. We denote the transpose matrix of A as A^T, the

conjugate matrix as $\bar{A}$ and the adjoint matrix as $A^*=\bar{A}^T$. We will denote as $\mathrm{tr}\,A=\sum_{j=1}^{m} A_{jj}$ the trace of A and will use the matrix norm $\|A\|=\sum_{j=1}^{n}\sum_{i=1}^{m}|A_{ji}|$.

II,1.1. — *Field indexing parameters and sets*

In the discussion below, d will be an integer $\geqq 1$. We will denote as $t=(t_1, \ldots, t_d)$ the generic point of $\mathbb{R}^d$ which will be the indexing parameter for the fields considered. We will also use the notations $t'=(t'_1, \ldots, t'_d)$ and $u=(u_1, \ldots, u_d)$ in place of t when we need several points.

Let $T_1, \ldots, T_d$ be d positive real numbers and $\bar{T}$ be the subset of $\mathbb{R}^d$ (a closed, bounded, rectangular subset) such that:

$$\bar{T}=[0, T_1]\times[0, T_2]\times\ldots\times[0, T_d] \tag{1}$$

with measure

$$|T|=T_1\times\ldots\times T_d=\int_{\bar{T}} dt \tag{2}$$

where dt is the Lebesgue measure on $\mathbb{R}^d$. We set:

$$T_{\inf}=\mathrm{Inf}\{T_1, \ldots, T_d\}. \tag{3}$$

The Greek letter α denotes an integer multiple index with dimension d reserved for t:

$$\alpha=(\alpha_1, \ldots, \alpha_d)\in\mathbb{N}^d, \qquad \alpha_j\in\mathbb{N}, \tag{4}$$

For α fixed, t_α denotes the point of $\mathbb{R}^d$

$$t_\alpha=(t_{1,\,\alpha_1}, \ldots, t_{d,\,\alpha_d}). \tag{5}$$

where $t_{j,\,\alpha_j}\in\mathbb{R}$ is the value of the j-th coordinate t_j of t.

II,1.2. — *Field parameters and spectral domain*

Let $\Omega_1, \ldots, \Omega_d$ be d positive real fixed numbers. We denote as $\bar{\Omega}$ the subset of $\mathbb{R}^d$

$$\bar{\Omega}=[-\Omega_1, \Omega_1]\times\ldots\times[-\Omega_d, \Omega_d] \tag{6}$$

defined as the product of the subsets $[-\Omega_j, \Omega_j]$ of $\mathbb{R}$ called its edges. The compact part $\bar{\Omega}$ of $\mathbb{R}^d$ is the support of the matricial spectral measures considered and will be called the "spectral domain".

Let $\omega=(\omega_1, \ldots, \omega_d)$ be the generic point of $\bar{\Omega}\subset\mathbb{R}^d$. When we need another point, we will write $\omega'=(\omega'_1, \ldots, \omega'_d)$. As above, we denote as

$$|\Omega|=\Omega_1\times\ldots\times\Omega_d=\int_{\bar{\Omega}} d\omega \tag{7}$$

the measure of $\bar{\Omega}$ where $d\omega$ is the Lebesgue measure on $\mathbb{R}^d$.

The Greek letter β denotes an integer multiple index with dimension d reserved for ω:

$$\beta=(\beta_1, \ldots, \beta_d)\in\mathbb{N}^d, \ \beta_j\in\mathbb{N}. \tag{8}$$

For β fixed, ω_β denotes the point of $\bar{\Omega}$:

$$\omega_\beta=(\omega_{1,\,\beta_1}, \ldots, \omega_{d,\,\beta_d}) \tag{9}$$

where $\omega_{j,\,\beta_j}$ is the value of the j-th coordinate ω_j of ω.

II,1.3. — *Subdivision of spectral domain* $\bar{\Omega}$

The establishment of numerical simulation formulas for the fields requires introducing a subdivision of spectral domain $\bar{\Omega}$. We could take any subdivision, but in this case, we would obtain a formulation which would not be suitable for the multidimensional Fast Fourier Transform (FFT) algorithms and the numerical cost of simulation would be high. In order to be able to use the FFT, we must use a subdivision with a constant step for each coordinate although possibly different from one coordinate to another.

Let $M_1, \ldots, M_d$ be d positive fixed integers and $N_1=2M_1, \ldots, N_d=2M_d$ be the d associated positive even integers. We set:

$$\begin{aligned} N&=N_1\times\ldots\times N_d\\ N_{\inf}&=\mathrm{Inf}\{N_1, \ldots, N_d\}. \end{aligned} \tag{10}$$

In the discussion below, we will denote as B_N the subset of $\mathbb{N}^d$ such that (cartesian product of subsets of $\mathbb{N}$)

$$B_N=\prod_{j=1}^{d}\{0, 1, 2, \ldots, N_j-1\} \tag{12}$$

Therefore if

$$\beta\in B_N, \ \beta_j\in\{0, 1, \ldots, N_j-1\}, \ \forall j\in\{1, \ldots, d\}.$$

For any $j\in\{1, \ldots, d\}$, we consider the subdivision of the edge $[-\Omega_j, \Omega_j]$ with a constant step Δ_j such that:

$$\Delta_j=2\Omega_j N_j^{-1}=\Omega_j M_j^{-1} \tag{13}$$

whose meshes are the subsubsets of $[-\Omega_j, \Omega_j]$ with center:

$$\begin{aligned} &\omega_{j,\,\beta_j}=-\Omega_j+\left(\beta_j+\frac{1}{2}\right)\Delta_j,\\ &\beta_j\in\{0, 1, \ldots, N_j-1\} \end{aligned} \tag{14}$$

We then define the subdivision of $\bar{\Omega}$ as the finite union of subsubsets Q_β of $\bar{\Omega}$ with disjoint insides:

$$\bar{\Omega} = \bigcup_{\beta \in B_N} Q_\beta \tag{15}$$

such that

$$Q_\beta = \prod_{j=1}^{d} [\omega_{j,\beta_j} - \Delta_j/2,\ \omega_{j,\beta_j} + \Delta_j/2] \tag{16}$$

$$|Q_\beta| = \int_{Q_\beta} d\omega = |\Delta| \tag{17}$$

where $|\Delta|$ is such that

$$|\Delta| = \Delta_1 \times \ldots \times \Delta_d. \tag{18}$$

It should be noted that subset Q_β of $\bar{\Omega}$ is centered in point ω_β and that there is no subset centered in the origin of $\bar{\Omega}$, which is a requisite for what follows.

Let $\Delta = (\Delta_1, \ldots, \Delta_d)$ be the vector of $\mathbb{R}^d$ and $\|\Delta\| = \langle \Delta, \Delta \rangle^{1/2}$ its norm (not to be confused with $|\Delta|$ and $\|\Delta\|$).

We then have:

$$N_{\inf} \to +\infty \quad \Leftrightarrow \quad \|\Delta\| \to 0. \tag{19}$$

Finally, for any mapping defined on $\bar{\Omega}$, we will use the following condensed notation

$$\sum_{\beta \in B_N} f(\omega_\beta) = \sum_{\beta_1 = 0}^{N_1 - 1} \ldots \sum_{\beta_d = 0}^{N_d - 1} f(\omega_{1,\beta_1}, \ldots, \omega_{d,\beta_d}) \tag{20}$$

where ω_{j,β_j} is defined by (14).

II,2. – HYPOTHESES ON THE FIELDS CONSIDERED

Below, $(A, \mathscr{A}, P)$ denotes a probabilistic space, E the mathematical expectation, and d and n two integers $\geqq 1$.

We consider a Gaussian, second-order, centered stochastic field $X(t) = (X_1(t), \ldots, X_n(t))$ defined on $(A, \mathscr{A}, P)$, indexed on an open set $\mathscr{O}$ of $\mathbb{R}^d$ with values in $\mathbb{R}^n$, whose autocorrelation function $t, t' \mapsto R(t, t') = E(X(t) X(t')^T) : \mathscr{O} \times \mathscr{O} \to \mathrm{Mat}_{\mathbb{R}}(n, n)$ can be represented:

$$R(t, t') = \int_{\mathbb{R}^d} e^{i\langle \omega,\ t - t' \rangle} Q(t, \omega) S(\omega) Q(t', \omega)^* d\omega \tag{21}$$

with the following hypotheses:

(H1): $\omega \mapsto S(\omega)$ is a function of $\mathbb{R}^d$ in $\mathrm{Mat}_{\mathbb{C}}(n, n)$ with compact support $\bar{\Omega}$ continuous on $\bar{\Omega}$ and such that for any $\omega \in \mathbb{R}^d$, $S(\omega)$ is a positive Hermitian matrix:

$$\left.\begin{array}{l} S(\omega) = S(\omega)^* \\ \langle S(\omega) x, \bar{x} \rangle \geqq 0, \qquad \forall x \in \mathbb{C}^n \end{array}\right\} \tag{22}$$

and such that

$$S(-\omega) = S(\omega). \tag{23}$$

(H2): $t, \omega \mapsto Q(t, \omega)$ is a continuous mapping of $\mathscr{O} \times \mathbb{R}^d$ into $\mathrm{Mat}_{\mathbb{C}}(n, n)$ such that for any $\omega \in \bar{\Omega}$:

$$Q(t, -\omega) = \overline{Q(t, \omega)}. \tag{24}$$

For any compact set $\mathscr{K} \subset \mathscr{O}$, there exists a positive real constant $C_{\mathscr{K}}$ which depends on $\mathscr{K}$ and is independent of ω such that $\forall t$ and $t' \in \mathscr{K}$, $\forall \omega \in \bar{\Omega}$, $\forall p, q \in \{1, \ldots, n\}$

$$\left.\begin{array}{l} \big||Q_{pq}(t, \omega)| - |Q_{pq}(t', \omega)|\big| \leqq C_{\mathscr{K}} \|t - t'\| \\ \big|\psi_{pq}(t, \omega)| - |\psi_{pq}(t', \omega)|\big| \leqq C_{\mathscr{K}} \|t' - t\| \end{array}\right\} \tag{25}$$

where we have set

$$Q_{pq}(t, \omega) = |Q_{pq}(t, \omega)| \exp(i \psi_{pq}(t, \omega)).$$

Comments

(i) Let $X^C(t) = X^R(t) + i X^I(t)$ be a field with values in $\mathbb{C}^m$ and m be an integer $\geqq 1$, where $X^R(t)$ and $X^I(t)$ have values $\mathbb{R}^m$.

We can identify $\mathbb{C}$ to $\mathbb{R}^2$ and $X^C(t)$ to $X(t) = \{X^R(t), X^I(t)\}$ with values $\mathbb{R}^n$, $n = 2m$. Thus the case of fields with values $\mathbb{C}^m$ can be identified with the case $\mathbb{R}^n$.

(ii) By hypothesis, $X(t)$ has values $\mathbb{R}^n$ and therefore $R(t, t') \in \mathrm{Mat}_{\mathbb{R}}(n, n)$. Thus, the imaginary part of integral (21) is identically zero and we can also write:

$$R(t, t') = \mathrm{Re} \int_{\bar{\Omega}} \times e^{i\langle \omega,\ t - t' \rangle} Q(t, \omega) S(\omega) Q(t', \omega)^* d\omega \tag{26}$$

(iii) Hypothesis (22) requires function $\omega \mapsto \mathrm{tr}\, S(\omega)$ on $\mathbb{R}^d$ to have positive real values and be continuous on $\bar{\Omega}$ and therefore means that measure $\mathrm{tr}\, S(\omega)\, d\omega$ is positive-bounded on $\mathbb{R}^d$.

(iv) The hypothesis of continuity of Q on $\mathscr{O} \times \mathbb{R}^d$ is not restrictive. Since S has compact support $\bar{\Omega}$, it is sufficient for $t, \omega \mapsto Q(t, \omega)$ to be continuous on $\mathscr{O} \times \bar{\Omega}$. As the values of function $t, \omega \mapsto Q(t, \omega)$ are not involved in $\mathscr{O} \times (\mathbb{R}^d \setminus \bar{\Omega})$, we can make a continuous extension.

II,3. – PROPERTIES OF THE FIELDS CONSIDERED

II,3.1. – CASE OF A HOMOGENEOUS FIELD

In all of Section II,3.1., we take $\mathcal{O}=\mathbb{R}^d$. As the field is Gaussian and centered, a necessary and sufficient condition for the field *to be homogeneous* (or stationary if $d=1$ and also if $d>1$) is for $R(t, t')=R(t-t')$ and therefore for $Q(t, \omega)=I$, where I is the unit matrix of $\mathrm{Mat}_{\mathbb{R}}(n, n)$. Representation (21) is then written

$$R(t-t')=\mathrm{Ré}\int_{\bar{\Omega}} e^{i\langle \omega,\, t-t'\rangle} S(\omega)\, d\omega \qquad (27)$$

and $S(\omega)$ is the density of the matricial spectral measure of the second-order homogeneous field $X(t)$. It is known that one necessary and sufficient condition to have representation (27) is for the field to be mean-square continuous and for the matricial spectral measure $M(d\omega)$ to accept a density $S(\omega)$ with respect to $d\omega$.

The hypotheses introduced in Section II.2 require the following properties:

(P1) The function $u\mapsto R(u)$ is continuous on $\mathbb{R}^d$ and $\lim_{\|u\|\to+\infty}|R(u)|=0$. It is infinitely differentiable on $\mathbb{R}^d$ and therefore field $X(t)$ is mean-square infinitely differentiable on $\mathbb{R}^d$.

Functions S and S^2 are integrable on $\mathbb{R}^d$.

Therefore, R^2 is integrable on $\mathbb{R}^d$ but R is not integrable on $\mathbb{R}^d$ in the general case.

(P2) Field $X(t)$ has continuous sample paths on $\mathbb{R}^d$, almost surely. This property will be useful for studying statistics on paths of the type $\sup_{t\in\bar{T}}\|X(t)\|$, where T is defined by (1). The proof is obtained by directly applying proposition A1.1 of Section A.1.iv of the appendix, since, $Q(t, \omega)$ being equal to I, (A1.6) and (A1.7) are verified and $M(d\omega)=S(\omega)\,d\omega$ and $\mathrm{supp}\, S=\bar{\Omega}$.

(P3) The hypotheses of Shannon's theorem for the fields are verified. For $j\in\{1, \ldots, d\}$, we set $\tau_j=\pi\Omega_j^{-1}$ as sampling step for coordinate t_j. Let $\alpha=(\alpha_1, \ldots, \alpha_d)\in\mathbb{Z}^d$ be the multiple index of relative integers of dimension d. We set

$$t_\alpha=(\alpha_1\tau_1, \ldots, \alpha_d\tau_d)\in\mathbb{R}^d$$

as sampling points in t, and $t=(t_1, \ldots, t_d)$ is any point of $\mathbb{R}^d$. We then have [55]:

$$X(t)=\sum_{\alpha\in\mathbb{Z}^d} X(t_\alpha)\prod_{j=1}^{d}\frac{\sin[\Omega_j(t_j-\alpha_j\tau_j)]}{\Omega_j(t_j-\alpha_j\tau_j)} \qquad (28)$$

Remark on the hypothesis of the compact support of S: For certain applications, S does not have a compact support. This is the case for instance of the frozen isotropic atmospheric turbulence model. But for physical fields, the total power

$$E(\|X(t)\|^2)=\mathrm{tr}\,R(0)=\int_{\mathbb{R}^d}\mathrm{tr}\,S(\omega)\,d\omega<+\infty \qquad (29)$$

is finite: the field is second-order and S is integrable on $\mathbb{R}^d$. Therefore $\forall\varepsilon>0$, $\exists\bar{\Omega}$ defined by (6) such that:

$$0<\int_{\mathbb{R}^d\ \bar{\Omega}}\mathrm{tr}\,S(\omega)\,d\omega\leqq\varepsilon E(\|X(t)\|^2). \qquad (30)$$

We then conventionally approach S by $1_{\bar{\Omega}}S$, where $1_{\bar{\Omega}}$ is the indicator function of $\bar{\Omega}$ on $\mathbb{R}^d$.

The approximated field thus constructed is mean-square infinitely differentiable on $\mathbb{R}^d$, whereas the initial field may not even have been mean-square differentiable once. Thus, criterion (30) for determination of $\bar{\Omega}$ is suitable if no mean-square differentiation of field $X(t)$ on $\mathbb{R}^d$ occurs in the problem analyzed.

However, if the initial field $X(t)$ has a spectral matrix density S which does not have a compact support and which is such that:

$$\int_{\mathbb{R}^d}\|\omega\|^{2p}\,\mathrm{tr}\,S(\omega)\,d\omega<+\infty \qquad (31)$$

for p integer $\geqq 1$, property related to the existence of mean-square differentials on $\mathbb{R}^d$ of the field, and we need the property:

$$\int_{\mathbb{R}^d}\|\omega\|^{2q}\,\mathrm{tr}\,S(\omega)\,d\omega<+\infty \qquad (32)$$

for $q\leqq p$, then criterion (30) for determination of $\bar{\Omega}$ must mandatorily be replaced by:

$$\int_{\mathbb{R}^d\ \bar{\Omega}}\|\omega\|^{2q}\,\mathrm{tr}\,S(\omega)\,d\omega\leqq\varepsilon E(\|X(t)\|^2). \qquad (33)$$

II,3.2. – *Case of a nonhomogeneous field*

In this case, $\mathcal{O}$ can be any open set of $\mathbb{R}^d$ and $Q(t, \omega)$ depends explicitly on t. Function S is called the density with respect to $d\omega$ of the structural matrix measure and function $t, \omega\mapsto S_X(t, \omega):\mathcal{O}\times\mathbb{R}^d\to\mathrm{Mat}_{\mathbb{C}}(n, n)$ such that:

$$S_X(t, \omega)=Q(t, \omega)\,S(\omega)\,Q(t, \omega)^* \qquad (34)$$

is the instantaneous spectral density. The support of $\omega\mapsto S_X(t, \omega)$ is $\bar{\Omega}$, uniformly in t, and $t, \omega\mapsto S_X(t, \omega)$ is continuous on $\mathcal{O}\times\bar{\Omega}$. The field is therefore effectively of second-order since:

$$E(\|X(t)\|^2)=\mathrm{tr}\,R(t, t)$$
$$=\mathrm{Ré}\int_{\bar{\Omega}}\mathrm{tr}\,S_X(t, \omega)\,d\omega<+\infty, \qquad \forall t\in\mathcal{O} \qquad (35)$$

The hypotheses of Section II.2 imply the following properties:

(P1) field $X(t)$ almost certainly has continuous sample paths on $\mathcal{O}$. The proof is obtained by applying proposition A1.1 of Section A1-iv of the appendix. In effect, considering (25), (A1-6) is satisfied;

(P2) let $\tau=(\tau_1, \ldots, \tau_d)\in\mathbb{R}^d$ with $\tau_j=\pi\Omega_j^{-1}$. It can be said that function t, $\omega\mapsto Q(t, \omega)$ oscillates slowly if, for any t and any

$$\omega\in\bar{\Omega},\ Q(t+\tau, \omega)\sim Q(t, \omega).$$

In this case, the sampling in t is controlled by the support $\bar{\Omega}$ of S, as in the case of homogeneous fields for which we use Shannon's theorem. Below, we will assume that this hypothesis is verified.

II,4. — CONSTRUCTION OF AN APPROACHING SEQUENCE OF THE FIELD

II,4.1. — *Splitting of S*

The numerical simulation will be performed on an approximation of the initial field, which is an element of an approaching sequence converging to the initial field.

According to (22), for any $\omega\in\bar{\Omega}$, $S(\omega)$ is a positive Hermitian matrix. Therefore, there always exists a matrix $H(\omega)\in\mathrm{Mat}_{\mathbb{C}}(n, n)$ such that:

$$S(\omega)=H(\omega)\,H(\omega)^* \tag{36}$$

and function $\omega\mapsto H(\omega)$ is continuous on $\bar{\Omega}$.

Let $r=\mathrm{rank}\,S(\omega)$, for $r\leqq n$ we can always construct $H(\omega)$ by the general linear algebraic method based either on the use of the base of eigenvectors of the Hermitian matrix S or on the LU Gaussian reduction in the supplement of the kernel which it is known how to construct.

In the particular case where $r=n$, $S(\omega)$ is positive-defined and we can use Cholesky's method.

For any $t\in\mathcal{O}$ and $\omega\in\bar{\Omega}$, we define matrix $\mathbb{H}(t, \omega)\in\mathrm{Mat}_{\mathbb{C}}(n, n)$ such that:

$$\mathbb{H}(t, \omega)=Q(t, \omega)\,H(\omega). \tag{37}$$

Function t, $\omega\mapsto\mathbb{H}(t, \omega)$ is continuous on $\mathcal{O}\times\bar{\Omega}$.

Under these conditions, (26) is written:

$$R(t, t')=\mathrm{Ré}\int_{\bar{\Omega}} \times e^{i\langle\omega,\, t-t'\rangle}\,\mathbb{H}(t, \omega)\,\mathbb{H}(t', \omega)^*\,d\omega. \tag{38}$$

If the field is homogeneous, we have $\mathbb{H}(t, \omega)=H(\omega)$ independent of t.

II,4.2. — *Definition of the approaching sequence*

We use all the notations of Section II.1. For N fixed, we define a stochastic field $X^N(t)=(X_1^N(t), \ldots, X_n^N(t))$ on $(A, \mathcal{A}, P)$ indexed on $\mathcal{O}$ with values in $\mathbb{R}^n$ such that for any $p\in\{1, \ldots, n\}$

$$X_p^N(t)=\sqrt{2|\Delta|}\,\mathrm{Ré}\left\{\sum_{q=1}^{n}\sum_{\beta\in B_N}\mathbb{H}_{pq}(t, \omega_\beta)\,Z_{q,\beta} \times\exp(i\Phi_{q,\beta}+i\langle t, \omega_\beta\rangle)\right\}. \tag{39}$$

where

(a) $\{\Phi_{q,\beta}\}_{q,\beta}$ $K=n\times N$ are random variables defined on $(A, \mathcal{A}, P)$, uniform on $[0, 2\pi]$ and mutually independent.

(b) $\{Z_{q,\beta}\}_{q,\beta}$ corresponds to the two following possible choices:

Choice 1: The $Z_{q,\beta}$ are constants equal to 1

$$\forall q\in\{1, \ldots, n\},\quad \forall\beta\in B_N,\quad Z_{q,\beta}=1. \tag{40}$$

Choice 2: The $Z_{q,\beta}$ are random variables defined by

$$Z_{q,\beta}=\sqrt{-\mathrm{Log}\,\Psi_{q,\beta}} \tag{41}$$

where $\{\Psi_{q,\beta}\}$ are K random variables defined on $(A, \mathcal{A}, P)$, uniform on $[0, 1]$ independent of one another and of random variables $\Phi_{q,\beta}$.

Comments: The two choices lead to the same second-order characteristics for field $X^N(t)$. However, the first choice gives a field which is not Gaussian but which will be asymptotically Gaussian for $N_{\mathrm{inf}}\to+\infty$, whereas for the second choice, X^N is a Gaussian field for any fixed N.

II,5. — PROPERTIES OF THE FIELDS OF THE APPROACHING SEQUENCE

II,5.1. — *Second-order properties*

For any fixed N, the stochastic field $X^N(t)$ defined has the following properties regardless of whether (40) or (41) is chosen for $Z_{q,\beta}$:

(i) it is of the second order and, for any $t\in\mathcal{O}$, we have:

$$E(\|X^N(t)\|^2)=|\Delta|\sum_{\beta\in B_N}\mathrm{tr}\,S_X(t, \omega_\beta)<+\infty \tag{42}$$

(ii) it is centered, and its autocorrelation function t,

$$t'\mapsto R^N(t, t')=E(X^N(t)\,X^N(t')^T):\mathcal{O}\times\mathcal{O}\to\mathrm{Mat}_{\mathbb{R}}(n, n)$$

is written:

$$R^N(t,t') = \text{Ré} \int_{\mathbb{R}^d} e^{i\langle \omega,\, t-t'\rangle} \times Q(t,\omega)\, M^N(d\omega)\, Q(t',\omega)^* \quad (43)$$

where $M^N(d\omega)$ is the measure on $\mathbb{R}^d$ with values in $\text{Mat}_{\mathbb{C}}(n,n)$ such that:

$$M^N(d\omega) = |\Delta| \sum_{\beta \in B_N} S(\omega_\beta)\, \delta_{\omega_\beta}(\omega) \quad (44)$$

and

$$\delta_{\omega_\beta} = \bigotimes_{j=1}^{d} \delta_{\omega_{j,\beta_j}} \quad (45)$$

where $\delta_{\omega_{j,\beta_j}}$ is the Dirac measure on $\mathbb{R}$ in point ω_{j,β_j} for coordinate $\omega_j \in \mathbb{R}$ of ω.

(iii) field $X^N(t)$ is mean-square continuous.

(iv) if $\mathcal{O} = \mathbb{R}^d$ and $Q(t,\omega) = I$, field $X^N(t)$ is mean-square homogeneous.

Proof: for any y and y' in $\mathbb{R}$, we have:

$$E(\cos(y + \Phi_{q,\beta})) = 0 \quad (46)$$

$$E(\cos(y+\Phi_{q,\beta})\cos(y'+\Phi_{q',\beta'})) = \delta_{qq'}, \delta_{\beta\beta'}, \frac{1}{2}\cos(y-y') \quad (47)$$

for choice 1 (40)

$$E(Z_{q,\beta}^2) = 1, \quad (48)$$

for choice 2 (41)

$$E(Z_{q,\beta}^2) = \int_0^1 (\sqrt{-\text{Log}\,\psi})\, d\psi = 1. \quad (49)$$

By writing $\mathbb{H}(t,\omega_\beta) = |\mathbb{H}(t,\omega_\beta)| \exp(i\theta_{pq}(t,\omega_\beta))$, relation (39) can be written:

$$X_p^N(t) = \sqrt{2|\Delta|} \sum_{q=1}^{n} \sum_{\beta \in B_N} |\mathbb{H}_{pq}(t,\omega_\beta)|\, Z_{q,\beta} \times \cos(\langle t, \omega_\beta\rangle + \theta_{pq}(t,\omega_\beta) + \Phi_{q,\beta}) \quad (50)$$

considering (46) to (49) and since $\Phi_{q,\beta}$ are independent of $\Psi_{q,\beta}$, it can be seen that $E(X_p^N(t)) = 0$, and therefore that the field is centered and we obtain:

$$\begin{aligned} E(X_p^N(t)\, X_p^N,(t')) &= |\Delta| \sum_{q=1}^{n} \sum_{\beta \in B_N} |\mathbb{H}_{pq}(t,\omega_\beta)\mathbb{H}_{p',q}(t',\omega_\beta)| \\ &\times \cos(\langle t-t', \omega_\beta\rangle + \theta_{pq}(t,\omega_\beta) - \theta_{p',q}(t',\omega_\beta)) \\ &= \text{Ré}\left\{ |\Delta| \sum_{q=1}^{n} \sum_{\beta \in B_N} \mathbb{H}_{pq}(t,\omega_\beta)\, \overline{\mathbb{H}_{p',q}(t',\omega_\beta)} \times \exp(i\langle \omega_\beta, t-t'\rangle) \right\} \end{aligned}$$

Since $Q(t,\omega)\, S(\omega)\, Q(t',\omega)^* = \mathbb{H}(t,\omega)\, \mathbb{H}(t',\omega)^*$, we obtain (43)-(44). We infer (42) from (43)-(44). Since B_N has a finite cardinal, R^N is continuous on $\mathcal{O} \times \mathcal{O}$, proving point (iii).

Finally, since $X^N(t)$ is a centered field and, in case (iv), $R^N(t,t') = R^N(t-t')$ depends only on $t-t'$, the field is mean-square homogeneous, which proves point (iv).

II,5.2. — *Properties in law*

(i) *Choice* 1: if the $Z_{q,\beta}$ are defined by (40), for any fixed N, the stochastic field $X^N(t)$ is not Gaussian.

(ii) *Choice* 2: if the $Z_{q,\beta}$ are defined by (41), then, for any fixed N,

— the stochastic field $X^N(t)$ is a Gaussian field;

— if, in addition, $\mathcal{O} = \mathbb{R}^d$ and $Q(t,\omega) = I$, then field $X^N(t)$ is homogeneous.

Proof: Let us begin by proving point (ii). For any y, we set:

$$U_{q,\beta} = \sqrt{-\log \Psi_{q,\beta}}\, \cos(\Phi_{q,\beta} + y_{q,\beta}). \quad (51)$$

Then, random variables $\{U_{q,\beta}\}_{q,\beta}$ are mutually independent and for any q and β fixed, random variable $U_{q,\beta}$, which is centered and which has a covariance of 1/2 according to (46), (47) and (49), is a Gaussian random variable. Considering (50), we can infer that X^N is a Gaussian field. (It can be demonstrated that $U_{q,\beta}$ is Gaussian by calculating its characteristic function, for instance.) In the case $\mathcal{O} = \mathbb{R}^d$ and $Q(t,\omega) = I$, according to II,5.1.-iv, X^N is mean-square homogeneous. But since it is Gaussian, it can be inferred that it is homogeneous (in law).

The proof of point (i) is inferred from point (ii).

II,5.3. — *Properties of the sample paths*

(i) *Choice* 1: The $Z_{q,\beta}$ are defined by (40). For any integer $m \geqq 1$ and any compact set $\mathcal{K}$ of $\mathcal{O}$, there exists a positive constant $C_{\mathcal{K}}$ independent of N such that $\forall\, t$ and t' in $\mathcal{K}$ and $\forall\, N$

$$\sum_{p=1}^{n} E(|X_p^N(t) - X_p^N(t')|^{2m}) \leqq C_{\mathcal{K}} \|t-t'\|^{2m}, \quad (52)$$

and field $X^N(t)$, which is not Gaussian, almost certainly has continuous sample paths on $\mathcal{O}$.

(ii) *Choice* 2: The $Z_{q,\beta}$ are defined by (41). For any compact set $\mathcal{K}$ of $\mathcal{O}$, there exists a positive constant $C_{\mathcal{K}}$ independent of N such that $\forall\, t$ and t' in $\mathcal{K}$ and $\forall\, N$

$$E(\|X(t) - X(t')\|^2) \leqq C_{\mathcal{K}} \|t-t'\|^2 \quad (53)$$

and field $X^N(t)$ which is Gaussian almost certainly has its continuous sample paths on $\mathcal{O}$.

(iii) *Homogeneous case:* if $\mathcal{O}=\mathbb{R}^d$ and $Q(t,\omega)=I$, and whatever the choice, 1 or 2, the paths $t_j \mapsto X^N(t, a)$, $a \in A$ are periodic functions with period:

$$T_j^0 = 2\pi (1/2\Delta_j)^{-1}, \qquad j \in \{1, \ldots, d\}. \tag{54}$$

By setting $T^0=(T_1^0, \ldots, T_d^0) \in \mathbb{R}^d$, $t \mapsto X^N(t, a)$ is a periodic function on $\mathbb{R}^d$ with period T^0.

Proof: (i) The result is a direct result of proposition A1.2 of Section A1.vi of the appendix for the non-Gaussian case. Hypothesis (A1.11) is verified since t, $\omega \mapsto \mathbb{H}(t, \omega)$ is continuous on $\mathcal{O} \times \bar{\Omega}$ and hypothesis (A1.12) results from (25)-(37).

(ii) The result results from proposition A1.1 of Section A1.iv of the appendix for the Gaussian case, knowing that (A1.5) is given by (43). Equation (53) is none other than (A1.8) of the proof of proposition A1.1.

(iii) The proof is trivial.

II,6. – STUDY OF CONVERGENCE OF THE APPROACHING SEQUENCE

We constructed an approaching sequence X^N of field X.

The numerical simulation will be carried out on an element X^N of this sequence for N fixed. It is therefore necessary to study convergence of X^N to X. We will establish three types of results.

– The first concerns convergence of the second-order values.

– The second is relative to convergence in law (weak convergence) of the sequence of fields X^N to the Gaussian field X.

– Finally, the third result, finer than the above, shows weak convergence on the continuous functions with compact support, used to insure convergence of the statistics on the sample paths.

II,6.1. – *Convergence of second-order quantities*

For any N, field $X^N(t)$ is centered like field $X(t)$.

The results are as follows for second-order quantities (valid for choices 1 and 2). For any t and t' in $\mathcal{O}$:

(i) the sequence $\{Q(t,\omega) M^N(d\omega) Q(t,\omega)^*\}_N$ of measures converges narrowly for $N_{\inf} \to +\infty$ to measure $Q(t,\omega) S(\omega) Q(t',\omega)^* d\omega$

(ii) the sequence $R^N(t, t')$ converges to $R(t, t')$.

(iii) if, for any p and $q \in \{1, \ldots, n\}$, the functions $\omega \mapsto s(\omega) = [Q(t,\omega) S(\omega) Q(t',\omega)^*]_{pq}$ are in $C^r(\bar{\Omega}, \mathbb{C})$, with $r=1$ or 2, then the speeds of convergence are in $N_{\inf}^{-r}$.

Proof: For any t and t' fixed in $\mathcal{O}$ and p, q fixed in $\{1, \ldots, n\}$, the function $s \in C^0(\bar{\Omega}, \mathbb{C})$. We use lemma A2.2 of Section A.2. (ii) of the appendix. Point (i) results directly from it, as do points (ii) and (iii) by taking $s'(\omega) = s(\omega) \exp(i\langle \omega, t-t' \rangle)$.

II,6.2. – *Convergence in law of the approaching sequence*

(i) *Choice* 1: (*a*) the sequence of non-Gaussian stochastic fields X^N converges in law (or weakly) to the Gaussian field X when $N_{\inf} \to +\infty$.

(*b*) if $\mathcal{O}=\mathbb{R}^d$, $Q(t,\omega)=I$, mean-square homogeneous converges in law to the homogeneous Gaussian field X.

(ii) *Choice* 2: (*a*) the sequence of stochastic Gaussian fields X^N converges in law to the Gaussian field X when $N_{\inf} \to +\infty$.

(*b*) if $\mathcal{O}=\mathbb{R}^d$, $Q(t,\omega)=I$, the sequence of homogeneous Gaussian fields X^N converges in law to the homogeneous Gaussian field X.

Proof: (i) the proof of (*a*) is given in Section A.3 of the appendix. The proof of the result is in the central limit theorem. The proof of point (*b*) results from point (*a*) and the fact that a Gaussian field mean-square homogeneous is homogeneous and vice versa.

(ii) The proof is trivial since sequence X^N is Gaussian and centered and $R^N(t, t') \to R(t, t')$ according to II.6.1 (ii).

II,6.3. – *Weak convergence on continuous functions*

Let $\mathcal{K}$ be any compact part of $\mathcal{O}$, for instance subset $\bar{T}$ defined by (1). Let $C^0(\mathcal{K})$ be the space of continuous functions on $\mathcal{K}$ with values $\mathbb{R}$ equipped with the uniform convergence norm:

$$\forall f \in C^0(\mathcal{K}), \quad |||f||| = \sup_{t \in \mathcal{K}} |f(t)| \tag{55}$$

let $p \in \{1, \ldots, n\}$ be fixed. For instance, it might be desired to know whether a random variable $\sup_{t \in \mathcal{K}} |X_p^N(t)|$ converges in law to random variable $\sup_{t \in \mathcal{K}} |X_p(t)|$ or even if the second-order characteristics converge. This type of result cannot be inferred only from the convergence in law of the approaching sequence X^N to X established in Section II.6.2. More is required and it is the following result:

Let g be a continuous function of $C^0(\mathcal{K})$ in $\mathbb{R}$. We then have:

$$\mathcal{L}(g(X^N)) \xrightarrow[N_{\inf} \to +\infty]{\text{tightly}} \mathcal{L}(g(X)). \tag{56}$$

Proof: In Section II.3., it was shown that field $X(t)$ almost certainly had its continuous sample paths

on $\mathcal{O}$. In addition, it was demonstrated in Section II.5.3 that for choices 1 and 2 and for any fixed N, field $X^N(t)$ almost certainly had its continuous sample paths on $\mathcal{O}$.

In addition, according to the results of II.5.2, we know that X^N converges weakly to X.

But, since inequalities (52) (for choice 1) and (53) (for choice 2) are uniform in N, weak convergence is inferred on $C^0(\mathcal{K})$.

Applications: The functional $g(f)=\sup_{t\in\mathcal{K}}|f(t)|$ is the norm $|||f|||$ of f on $C^0(\mathcal{K})$ and it is therefore a continuous functional. It can be inferred that:

$$\mathcal{L}\left(\sup_{t\in\mathcal{K}}|X_p^N(t)|\right)\xrightarrow[N_{\inf}\to+\infty]{\text{tightly}}\mathcal{L}\left(\sup_{t\in\mathcal{K}}|X_p(t)|\right) \quad (57)$$

II,7. — NUMERICAL SIMULATION FORMULAS

II,7.1. — Homogeneous case. Rapid simulation formula by FFT

(i) *General case:* We consider the case $\mathcal{O}=\mathbb{R}^d$, $Q(t,\omega)=I$ and therefore (39) is written as follows for $p\in\{1,\ldots,n\}$ and N fixed:

$$X_p^N(t)=\sqrt{2\Delta}\,\mathrm{Ré}\left\{\sum_{q=1}^{n}\sum_{\beta\in B_N}H_{pq}(\omega_\beta)Z_{q,\beta}\times\exp(i\Phi_{q,\beta}+i\langle t,\omega_\beta\rangle)\right\} \quad (58)$$

where $H(\omega)$ is defined in II.4.1.

According to Shannon's theorem (II.3.1.P3), it is necessary to sample coordinate t_j of t at frequency

$$f_{l,j}=2f_{\max,j}\text{ where }f_{\max,j}=\Omega_j/2\pi.$$

Therefore $f_{l,j}=\tau_j^{-1}$.

More generally, we introduce the parameters $\nu_j=2m_j$, m_j integer $\geqq 1$ and coordinate t_j is sampled at $f_{l,j}=\nu_j f_{\max,j}=\nu_j\Omega_j(2\pi)^{-1}$. The sampling step δ_j is written $\delta_j=f_{l,j}^{-1}$, *i.e.* using (13):

$$\delta_j=2\pi(\hat{N}_j\Delta_j)^{-1},\quad j\in\{1,\ldots,d\} \quad (59)$$

$$\hat{N}_j=1/2\,\nu_j N_j,\qquad \nu_j=2m_j,$$

$$m_j\geqq 1,\quad j\in\{1,\ldots,d\} \quad (60)$$

Let $A_{\hat{N}}$ be the subset of $\mathbb{N}^d$ such that:

$$A_{\hat{N}}=\prod_{j=1}^{d}\{0,1,2,\ldots,\hat{N}_j-1\}. \quad (61)$$

Therefore, if the multiple index

$$\alpha\in A_{\hat{N}},\qquad \alpha_j\in\{0,1,\ldots,\hat{N}_j-1\}$$

for any j in $\{1,\ldots,d\}$.

The sampling points for coordinate t_j are therefore:

$$t_{j,\alpha_j}=\alpha_j\delta_j,\qquad \alpha_j\in\{0,1,\ldots,\hat{N}_j-1\} \quad (62)$$

and considering (5), the set of sampling points for t is:

$$t_\alpha=(t_{1,\alpha_1},\ldots,t_{d,\alpha_d}),\qquad \alpha\in A_{\hat{N}} \quad (63)$$

The set of sampling points for ω is:

$$\omega_\beta=(\omega_{1,\beta_1},\ldots,\omega_{d,\beta_d}),\qquad \beta\in B_N \quad (64)$$

with (12) and (14).

It can be noted that according to (54), the period T_j^0 for coordinate t_j is written $T_j^0=4\pi/\Delta_j$ and since $(\hat{N}_j-1)\delta_j\sim 2\pi/\Delta_j$, it can be seen that choice (62) is correct since a half-period is used. A simple computation for $\alpha\in A_{\hat{N}}$ and $\beta\in B_N$ gives:

$$\langle\omega_\beta,t_\alpha\rangle=-2\pi\sum_{j=1}^{d}\alpha_j(\nu_j^{-1}-(2\hat{N}_j)^{-1})+2\pi\sum_{j=1}^{d}\alpha_j\beta_j\hat{N}_j^{-1}. \quad (65)$$

Since $\hat{N}_j\geqq N_j$, $A_{\hat{N}}\supseteq B_N$, and for $\beta\in A_{\hat{N}}$, we set:

$$x_\beta^{(p)}=0\quad\text{if}\quad\beta\notin B_N$$

$$x_\beta^{(p)}=\sum_{q=1}^{n}H_{pq}(\omega_\beta)Z_{q,\beta}\exp(i\Phi_{q,\beta})\quad\text{if}\quad\beta\in B_N. \quad (66)$$

Then relation (58) is written for any $\alpha\in A_{\hat{N}}$:

$$X_p^N(t_\alpha)=\sqrt{2\Delta}\,\mathrm{Ré}\left\{\tilde{X}_p^N(\alpha)\times\exp\left[-2i\pi\sum_{j=1}^{d}\alpha_j(\nu_j^{-1}-(2\hat{N}_j^{-1}))\right]\right\} \quad (67)$$

where

$$\tilde{X}_p^N(\alpha)=\sum_{\beta\in A_{\hat{N}}}x_\beta^{(p)}\exp\left[\sum_{j=1}^{d}2i\pi\alpha_j\beta_j\hat{N}_j^{-1}\right] \quad (68)$$

where $\tilde{X}_p^N$, given by (68), can be computed by FFT.

(ii) *Particular case: folded formula:* We introduce the following additional hypothesis relative to the symmetry of the field: $\forall j\in\{1,\ldots,d\},\forall p$ and $\forall q\in\{1,\ldots,n\}$. We have:

$$H_{pq}(\omega_1,\ldots,-\omega_j,\ldots,\omega_d)=H_{pq}(\omega_1,\ldots,\omega_j,\ldots,\omega_d). \quad (69)$$

Then, with M_j such that $N_j=2M_j$ we set:

$$\hat{M}_j=\nu_j M_j,\qquad \nu_j=2m_j,\quad m_j\geqq 1$$

$$A_{\hat{M}}=\prod_{j=1}^{d}\{0,1,\ldots,\hat{M}_j-1\}$$

$$B_M=\prod_{j=1}^{d}\{1,2,\ldots,M_j\}$$

$$\delta_j = 2\pi (\hat{M}_j \Delta_j)^{-1}, \qquad t_{j,\alpha_j} = \alpha_j \delta_j$$
$$\Delta_j = \Omega_j M_j^{-1}, \qquad \omega_{j,\beta_j} = \beta_j \Delta_j$$

and, as in (66), for $\beta \in A_{\hat{M}}$:, we set:

$$x_\beta^{(p)} = 0 \quad \text{if} \quad \beta \notin B_M$$

$$x_\beta^{(p)} = \sum_{q=1}^{n} H_{pq}(\omega_\beta) Z_{q,\beta} \exp(i\Phi_{q,\beta}) \quad \text{if} \quad \beta \in B_M.$$

Then for any $\alpha \in A_{\hat{M}}$:, we have:

$$X_p^N(t_\alpha) = \sqrt{2\Delta}\, \text{Ré}\{\tilde{X}_p^N(\alpha)\} \tag{70}$$

$$\tilde{X}_p^N(\alpha) = \sum_{\beta \in A_{\hat{M}}} x_\beta^{(p)} \exp\left[\sum_{j=1}^{d} 2i\pi \alpha_j \beta_j \hat{M}_j^{-1}\right] \tag{71}$$

where $\tilde{X}_p^N$, given by (71) can be calculated by FFT.

(iii) *Comments:* The introduction of parameters $\nu_1, \ldots, \nu_d$ makes it possible to preserve the FFT algorithms, even if a finer solution is desired for coordinate t_j than for coordinate ω_j. For $m_j = 1$, we have $\nu_j = 2$ and $\hat{N}_j = N_j$ which is the conventional case.

II.7.2. — *Nonhomogeneous case*

In this case, we must use (39) with $\mathbb{H}_{pq}(t, \omega)$ depending on t. The same expansion can be made as in the homogeneous case, but in this case, $x_\alpha^{(p)}$ defined by (66) depends on α and (71) can no longer be calculated by FFT. However, if the field remains homogeneous for certain coordinates and nonhomogeneous for others, approach (70)-(71) is useful since it allows part of the computations to be made with partial FFT for the homogeneous coordinates.

III. — ATMOSPHERIC TURBULENCE MODEL AND SIMULATION

Below, the aim is to use a model not to determine the properties of atmospheric turbulence from the standpoint of fluid mechanics but to calculate the response to turbulence of nonlinear mechanical systems such as an aircraft using a turbulence model where only the second-order quantities are significant and in which the Gaussian hypothesis is acceptable, as was stated in the introduction.

III,1. — STATEMENT OF THE PROBLEM

A simplified turbulence model, called cylindrical, is often used [9, 58] to compute the responses of an aircraft. Recently, after measurements made in flight [69], it was observed that this simple model gave results which were farther from the experimental responses than the isotropic turbulence model, in particular for aircraft with a long extension. However, this turbulence model, which gives good results when the aircraft is flying at constant altitude, is no longer suitable at low altitude [0, 300 m] and in particular during landing and takeoff phases, since the loss of isotropy and homogeneity due to the closeness of the ground must be taken into account.

In this section, we illustrate the methods described above by constructing simulations of atmospheric turbulence, modeled first by an isotropic homogeneous field then by a model of an isotropic inhomogeneous field that we will construct.

Below, the three space coordinates (x, y, z) are denoted (t_1, t_2, t_3) and the associated wave vector is denoted $(\omega_1, \omega_2, \omega_3) = \omega$.

III,2. — MODELING OF TURBULENCE BY AN ISOTROPIC FIELD. SIMULATION

Atmospheric turbulence is modeled by a stochastic field $X(t) = (X_1, t_2, t_3),\ X_2(t_1, t_2, t_3),\ X_3(t_1, t_2, t_3))$ indexed on $\mathbb{R}^3$ with values in $\mathbb{R}^3$, Gaussian, homogeneous, independent of physical time (frozen turbulence), second-order, centered whose matricial spectral measure accepts a density given by [2, 25]:

$$S(\omega) = -\frac{f(\|\omega\|)}{4\pi \|\omega\|^2}\left(\frac{\omega\omega^T}{\|\omega\|^2} - I\right) \tag{72}$$

where I is the identity matrix of $\text{Mat}_{\mathbb{R}}(3,3)$,

$$f(\|\omega\|) = \frac{55}{9}\,\frac{\Gamma(5/6)}{\sqrt{\pi}\,\Gamma(1/3)}\,\frac{\sigma^2}{\omega_e}\,\frac{(\|\omega\|/\omega_e)^4}{(1+(\|\omega\|/\omega_e)^2)^{17/6}}$$

$\sigma^2 = E(\|X(t)\|^2)$ and ω_e is a reference wave number.

The density S does not have a compact support. Therefore, we will simulate the field whose spectral matrix density is $\mathbf{1}_{\tilde{\Omega}} S(\omega)$.

We can then directly use equation (58) to construct field simulations. In order to check the results, we construct a spectral matrix density estimator, whose expectation is given by:

$$\hat{S}_{pq}(\omega) = (2\pi)^{-3} \mathcal{N}^{-1} \sum_{k=1}^{\mathcal{N}} \hat{X}_p^{(k)}(\omega)\, \overline{\hat{X}_q^{(k)}(\omega)},\ 1 \leqq p, q \leqq 3$$

where

$$\hat{X}_p^{(k)}(\omega) = (T_1 T_2 T_3)^{-(1/2)} \times \int_0^{T_1}\int_0^{T_2}\int_0^{T_3} X_p^{(k)}(t_1, t_2, t_3) \times e^{-i(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3)}\, dt_1\, dt_2\, dt_3$$

where $X^{(k)}(t_1, t_2, t_3)$, $t_j \in [0\ T_j]$, $j \in \{1, 2, 3\}$, is the result of the kth simulation of field X and $\mathcal{N}$ is the total number of simulations.

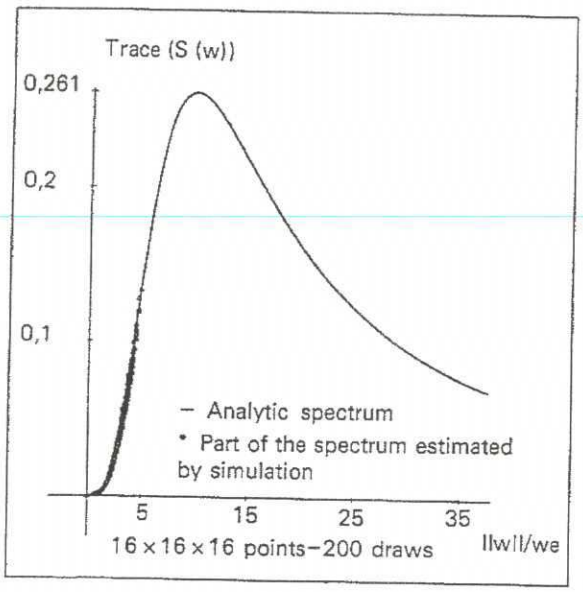

Fig. 1. – Isotropic turbulence.

Figure 1 shows the analytic and estimated power spectral measure of the field, *i. e.* the trace of matrix $S(\omega)$, which is a function of only the wave number.

We used 16 points per coordinate to construct the simulation, with the mean of the estimator calculated on $\mathcal{N} = 200$ simulations, and the spectral domain $\bar{\Omega}$ truncated to obtain a sufficiently fine spectral resolution for the estimator.

III,3. – CONSTRUCTION OF A MODEL FOR ANISOTROPIC TURBULENCE IN THE ATMOSPHERIC BOUNDARY LAYER [0-300 m] AND SIMULATION

The model we propose reconstructs an instantaneous spectral density from the experimental data

$$S_X(t, \omega) = Q(t, \omega)\, S(\omega)\, Q(t, \omega)^*$$

for the turbulent field. Such an approach was described in a simpler framework, using a different simulation method [20, 21]. Having the instantaneous matricial spectral measure, we can write equation (21)

$$R(t, t') = \iiint_{\mathbb{R}^3} e^{i\langle \omega, t-t' \rangle}\, Q(t, \omega)\, S(\omega)\, Q(t', \omega)^*\, d\omega$$

and use the methods described in the first part to simulate the field. We will first establish the mathematical framework used to construct the model.

III,3.1. – *Hypotheses on the turbulent field*

The field of atmospheric turbulence in the boundary layer [0-300 m] is modeled by a Gaussian field indexed on $\mathbb{R}^3$ with values in $\mathbb{R}^3$, centered, mean-square continuous, horizontally homogeneous, *i. e.* for the first two coordinates of point $t = (t_1, t_2, t_3)$ of space. The autocorrelation function therefore verifies the property:

$$R(t, t') = R(t_1 - t'_1, t_2 - t'_2, t_3, t'_3). \tag{73}$$

We denote as $\Phi(\omega_1, t_2 - t'_2, t_3, t'_3) \in \mathrm{Mat}_{\mathbb{C}}(3, 3)$ the one-dimensional spectral matrix density defined by:

$$R(t_1 - t'_1, t_2 - t'_2, t_3, t'_3) = \int_{\mathbb{R}} e^{i\omega_1 (t_1 - t'_1)}\, \Phi(\omega_1, t_2 - t'_2, t_3, t'_3)\, d\omega_1. \tag{74}$$

The transverse spectral matrix densities $\Psi(\omega_1, \omega_2, t_3, t'_3) \in \mathrm{Mat}_{\mathbb{C}}(3, 3)$ can be defined in the same way by:

$$R(t_1 - t'_1, t_2 - t'_2, t_3, t'_3) = \iint_{\mathbb{R}^2} e^{i\omega_1 (t_1 - t'_1) + i\omega_2 (t_2 - t'_2)}\, \Psi(\omega_1, \omega_2, t_3, t'_3)\, d\omega_1\, d\omega_2. \tag{75}$$

Functions Φ and Ψ are then related by the relation

$$\Phi(\omega_1, t_2 - t'_2, t_3, t'_3) = \int_{\mathbb{R}} e^{i\omega_2 (t_2 - t'_2)}\, \Psi(\omega_1, \omega_2, t_3, t'_3)\, d\omega_2. \tag{76}$$

With these spectral densities we associate the transverse correlation matrices ρ and γ with values in $\mathrm{Mat}_{\mathbb{R}}(3, 3)$ and the phase angle matrices α and θ with values in $\mathrm{Mat}_{\mathbb{R}}(3, 3)$ such that for j and $k \in \{1, 2, 3\}$:

$$\rho_{k,j}(\omega_1, t_2 - t'_2, t_3)\, \Phi_{kk}(\omega_1, 0, t_3, t_3)\, \Phi_{jj}(\omega_1, 0, t_3, t_3) = |\Phi_{kj}(\omega_1, t_2 - t'_2, t_3, t_3)|^2 \tag{77}$$

$$\Phi_{kj}(\omega_1, t_2 - t'_2, t_3, t_3) = \exp[i\alpha_{kj}(\omega_1, t_2 - t'_2, t_3)]\, |\Phi_{kj}(\omega_1, t_2 - t'_2, t_3, t_3)|$$

$$\gamma_{kj}(\omega_1, \omega_2, t_3, t'_3)\, \Psi_{kk}(\omega_1, \omega_2, t_3, t_3)\, \Psi_{jj}(\omega_1, \omega_2, t'_3, t'_3) = |\Psi_{kj}(\omega_1, \omega_2, t_3, t'_3)|^2 \tag{78}$$

$$\Psi_{kj}(\omega_1, \omega_2, t_3, t'_3) = \exp(i\,\theta_{kj}(\omega_1, \omega_2, t_3, t'_3)) \left|\Psi_{kj}(\omega_1, \omega_2, t_3, t'_3)\right|.$$

II,3.2. — *Existence of a spectral measure*

We introduce the following hypothesis which we will assume verified below:

Hypothesis H: We assume that the functions $t_3, t'_3 \mapsto \gamma(\omega_1, \omega_2, t_3, t'_3)$ and $t_3, t'_3 \mapsto \theta(\omega_1, \omega_2, t_3, t'_3)$ depend only on $\tau_3 = t_3 - t'_3$. We then denote them as:

$$\tau_3 \mapsto \gamma(\omega_1, \omega_2, \tau_3); \qquad \tau_3 \mapsto \theta(\omega_1, \omega_2, \tau_3) \quad (79)$$

Let Γ be the function with values in $\mathrm{Mat}_{\mathbb{C}}(3,3)$ such that:

$$\Gamma_{kj}(\omega_1, \omega_2, \tau_3) = (\gamma_{kj}(\omega_1, \omega_2, \tau_3))^{1/2} \times \exp(i\,\theta_{kj}(\omega_1, \omega_2, \tau_3)) \quad (80)$$

Then, considering hypothesis H and (79), Γ appears to be the complex coherence function relative to Ψ and for k and $j \in \{1, 2, 3\}$ we have

$$\Psi_{kj}(\omega_1, \omega_2, t_3, t'_3) = \Gamma_{kj}(\omega_1, \omega_2, t_3 - t'_3) \times [\Psi_{kk}(\omega_1, \omega_2, t_3, t_3)\,\Psi_{jj}(\omega_1, \omega_2, t'_3, t'_3)]^{1/2}. \quad (81)$$

If, for any ω_1 and ω_2 in $\mathbb{R}$, functions $\tau_3 \mapsto \Gamma(\omega_1, \omega_2, \tau_3)$ and $\omega_3 \mapsto \hat{\Gamma}(\omega_1, \omega_2, \omega_3)$ are integrable on $\mathbb{R}$, with

$$\hat{\Gamma}(\omega_1, \omega_2, \omega_3) = 1/2\pi \times \int_{\mathbb{R}} e^{-i\omega_3\tau_3}\,\Gamma(\omega_1, \omega_2, \tau_3)\,d\tau_3 \quad (82)$$

then, there exists a function

$$\{t_3, \omega\} \mapsto Q(t_3, \omega): \mathbb{R} \times \mathbb{R}^3 \to \mathrm{Mat}_{\mathbb{R}}(3,3)$$

and a function $\omega \mapsto S(\omega): \mathbb{R}^3 \to \mathrm{Mat}_{\mathbb{C}}(3,3)$, verifying (22) and (23) for any $\omega \in \mathbb{R}^3$ and such that:

$$R(t, t') = \iiint_{\mathbb{R}^3} e^{i\langle \omega,\, t - t'\rangle} \times Q(t_3, \omega)\,S(\omega)\,Q(t'_3, \omega)\,d\omega \quad (83)$$

where $Q(t_3, \omega)$ is the diagonal matrix such that:

$$[Q(t_3, \omega)]_{kj} = \delta_{kj}[\Psi_{kk}(\omega_1, \omega_2, t_3, t_3)]^{1/2} \quad (84)$$

where δ_{kj} is Kronecker's symbol and:

$$S(\omega) = \hat{\Gamma}(\omega_1, \omega_2, \omega_3). \quad (85)$$

In effect, according to (81), we have:

$$\Psi_{kj}(\omega_1, \omega_2, t_3, t'_3) = \Psi_{kk}(\omega_1, \omega_2, t_3, t_3)^{1/2} \times \int_{\mathbb{R}} e^{i\omega_3(t_3 - t'_3)}\,\hat{\Gamma}_{kj}(\omega_1, \omega_2, \omega_3)\,d\omega_3 \; \Psi_{jj}(\omega_1, \omega_2, t'_3, t'_3)^{1/2}.$$

Substituting this last equation into (75) yields (83) to (85). Property (22) is general and (23) results from the fact that the field has real values.

Remark: In the problem stated, the functions which will be known are γ and θ. Matrix $S(\omega)$ is therefore constructed from (85), (82) and (80), *i.e.* for j and $k \in \{1, 2, 3\}$:

$$[S(\omega)]_{kj} = 1/2\pi \int_{\mathbb{R}} e^{i\omega_3\tau_3}\,\gamma_{kj}(\omega_1, \omega_2, \tau_3)^{1/2} \exp(i\,\theta_{kj}(\omega_1, \omega_2, \tau_3))\,d\tau_3 \quad (86)$$

III,3.3. — *Application to low altitude turbulence* [0-300 m]

The available experimental data on low altitude atmospheric turbulence mainly consist of the one-dimensional spectral densities $\Phi_{kk}(\omega_1, 0, t_3)$, [15, 16, 33, 41, 42, 50, 57, 62]. Empirical models [10, 14, 20, 21] are also available to describe the behavior of the transverse correlations $\rho_{kk}(\omega_1, t_2 - t'_2, t_3)$ and phase angles $\alpha_{kk}(\omega_1, t_2 - t'_2, t_3)$, $k \in \{1, 2, 3\}$. More specifically:

$$\rho_{kk}(\omega_1, t_2 - t'_2, t_3) = \exp(-1/2\pi\,A_k|\omega_1|\,|t_2 - t'_2|) \quad (87)$$

$$\alpha_{kk}(\omega_1, t_2 - t'_2, t_3) = -B_k\,\omega_1\,(t_2 - t'_2) \quad (88)$$

where quantities A_k and B_k are functions of altitude t_3 which can be assumed constant at low altitude, *which we will assume below.*

From equations (76) and (77) we infer, using (87) and (88)

$$\Psi_{kk}(\omega_1, \omega_2, t_3, t_3) = A_k|\omega_1|\,[1/4\,A_k^2\,\omega_1^2 + 4\pi^2(\omega_1 B_k + \omega_2)^2]^{-1}\,\Phi_{kk}(\omega_1, 0, t_3\,t_3). \quad (89)$$

Currently, there are no experimental models available for the transverse correlations $\gamma_{kj}(\omega_1, \omega_2, t_3, t'_3)$ and the associated phase angles $\theta_{kj}(\omega_1, \omega_2, t_3, t'_3)$.

By analogy with the above models, we introduced hypothesis H and selected the following form:

$$\gamma_{kj}(\omega_1, \omega_2, t_3 - t'_3) = \exp\left[-\frac{1}{2\pi}C_{kj}|\omega_1 + \omega_2|\,|t_3 - t'_3|\right] \quad (90)$$

$$\theta_{kj}(\omega_1, \omega_2, t_3, t'_3) = -D_{kj}(\omega_1 + \omega_2)(t_3 - t'_3) \quad (91)$$

where C_{kj} and D_{kj} are real constants independent of altitude t_3.

The results of Section II.3.2 apply, hypothesis H being verified and functions $\tau_3 \mapsto \Gamma_{kj}(\omega_1, \omega_2, \tau_3)$ and $\omega_3 \mapsto \hat{\Gamma}_{kj}(\omega_1, \omega_2, \omega_3)$ being integrable functions. In this case, matrix $S(\omega)$ is obtained by calculating (86) and we obtain for k and $j \in \{1, 2, 3\}$

$$[S(\omega)]_{kj} = C_{kj}|\omega_1 + \omega_2|\,[1/4\,C_{kj}^2(\omega_1 + \omega_2)^2 + 4\pi^2(D_{kj}(\omega_1 + \omega_2) + \omega_3)]^{-1}. \quad (92)$$

Matrix $Q(t_3, \omega)$ is given by (84) and (89). The diagonal terms, which are the only nonzero terms, are then written as follows for $k \in \{1, 2, 3\}$

$$Q_{kk}(t_3, \omega) = A_k |\omega_1| [1/4 A_k^2 \omega_1^2 + 4\pi^2 (\omega_1 B_k + \omega_2)^2]^{-1} \Phi_{kk}(\omega_1, 0, t_3, t_3). \quad (93)$$

III,3.4. — *Practical considerations*

(i) It can be seen from equations (93) and (92) that functions $Q(t_3, \omega)$ and $S(\omega)$ are not defined for the zero wave vector $\omega = (0, 0, 0)$. This is due to the fact that the experimental spectra $\Phi_{kk}(\omega_1, 0, t_3)$ are known only for $|\omega_1| \geqq \omega_{1,0} (\sim 10^{-2}$ rad/m).

We will extend $S(\omega)$ and $Q(t_3, \omega)$ by continuity on subset $[-\omega_{1,0}, \omega_{1,0}]$. It is then clear that function Q verifies properties (24) and (25).

However, it can be verified that these functions are integrable in a neighborhood of this point.

(ii) The construction of the model does not depend on the form of the experimental spectra. It can therefore be used for various meteorological situations. It does however depend on the form (87), (88), (90) and (91) of the transverse correlations and phase angles. In addition, it must be verified that the choice of constants A_k, B_k, C_{kj}, and D_{kj} makes matrix $S(\omega)$ positive Hermitian. Assuming that these quantities are independent of the altitude makes the model more particularly suited to the case of turbulence in a neutrally stable meteorological situation caused by strong winds: in effect, these constants vary more rapidly with altitude in the case of convection turbulence.

(iii) *Example of experimental spectra:* Empirical models of one-dimensional spectra $\Phi_{kk}(\omega_1, 0, t_3)$ describing a turbulence in neutral situation near very rough ground (for instance in airports near cities) can be found in [16, 10, 35, 62, 41]. They are written

$$\begin{aligned}
\Phi_{11}(\omega_1, 0, t_3) &= 105/4\pi u_*^2 \; t_3 [.44 + 33/2\pi t_3 |\omega_1|]^{-5/3} \\
\Phi_{22}(\omega_1, 0, t_3) &= 17/4\pi u_*^2 \; t_3 [.38 + 9.5/2\pi t_3 |\omega_1|]^{-5/3} \\
\Phi_{33}(\omega_1, 0, t_3) &= 2/4\pi u_*^2 \; t_3 [.44 + 5.3 (t_3 |\omega_1|/2\pi)^{5/3}]^{-1}
\end{aligned} \quad (94)$$

where u_* is the friction speed.

(iv) Determination of Constants [20, 21, 50]:

As an initial approximation, it can be assumed that the lateral component of turbulence is independent of the other two components:

$$C_{12} = C_{21} = C_{23} = C_{32} = 0.$$

In the above references can be found:

$$C_{11} = 19, \quad C_{22} = 13,$$

$$C_{33} = 13, \quad C_{13} = C_{31} = 16$$

$$D_{11} = 1, \quad D_{22} = 2, \quad D_{33} = 2,$$

$$D_{13} = D_{31} = 1, \quad A_1 = 19.$$

For constants A_k and B_k, for which no experimental measurements are available, we arbitrarily set the following values, for the sole purpose of numerically checking the simulation method:

$$A_k = C_{kk} \quad \text{for } k = 2,3$$

and

$$B_k = D_{kk} \quad \text{for } k = 1, 2, 3.$$

Figure 2 shows an estimate of the one-dimensional spectrum of the second component of anisotropic turbulence for a fixed altitude t_3, compared with the empirical spectrum, $\Phi_{22}(\omega_1, 0, t_3)$.

CONCLUSION

We described the general results obtained on the numerical simulation of Gaussian vector fields. The applications of numerical simulation algorithms proposed for the case of isotropic, homogeneous turbulence and anisotropic inhomogeneous turbulence show the efficiency of the methods developed. Furthermore, we propose a constant spectrum model for turbulence in the 0-300 m atmospheric boundary layer. What now remains to be done is to supply the model with constants characterizing the various transverse correlations and phase angles of turbulence, which requires experimental measurements in certain cases.

APPENDICES

A.1 STOCHASTIC FIELD ALMOST CERTAINLY HAVING CONTINUOUS SAMPLE PATHS

Let d and n be two positive integers $\geqq 1$. Vector spaces $\mathbb{R}^d$ and $\mathbb{R}^n$ are equipped with the Euclidean scalar product noted $\langle \, . \, , \, . \, \rangle$ and the associated norm noted $\| \, . \, \|$. Let $(A, \mathcal{A}, P)$ be a probabilistic space and E be the mathematical expectation.

Let $X(t) = (X_1(t), \ldots, X_n(t))$ be a stochastic field defined on $(A, \mathcal{A}, P)$, indexed on an indeterminate open set $\mathcal{O}$ of $\mathbb{R}^d$ with values in $\mathbb{R}^n$.

Field $X(t)$ almost certainly has continuous sample paths if, for P-almost every $a \in A$, paths $t = (t_1, \ldots, t_d) \mapsto X(t, a)$ are continuous functions of $\mathcal{O}$ in $\mathbb{R}^n$.

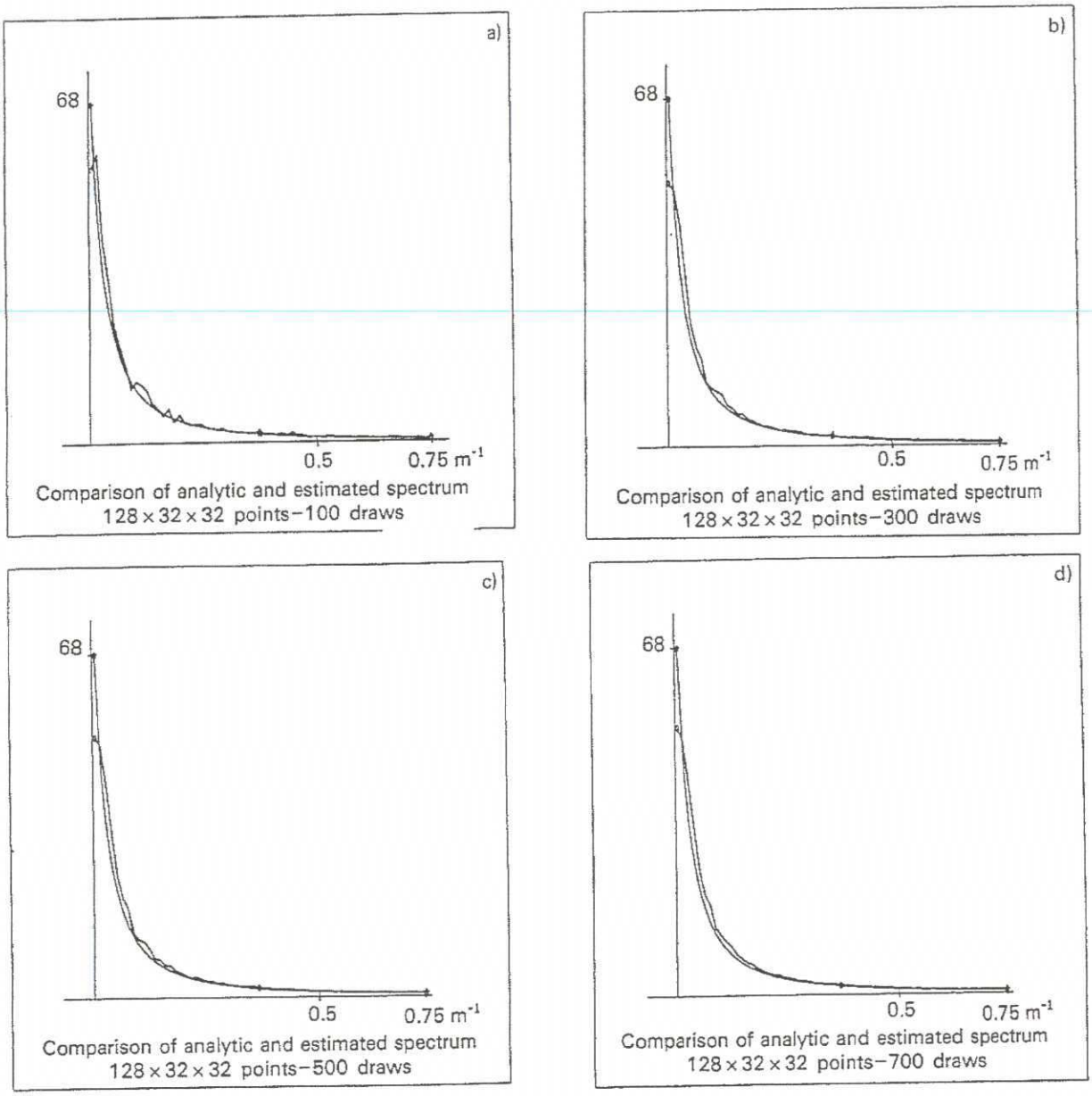

Fig. 2. – Anisotropic turbulence. One-dimensional longitudinal spectrum.

The following result, which is a generalization of Kolmogorov's lemma for the case of fields ($d>1$) gives a very useful sufficient condition.

(i) *Sufficient condition in the general case* [63]: If there exist two positive constants η and λ and, if for any compact set $\mathscr{K}$ of $\mathcal{O}$, there exists a third constant $C_{\mathscr{K}}$ such that:

$$E(\|X(t)-X(t')\|^{\lambda}) \leqq C_{\mathscr{K}}\|t-t'\|^{d+\eta}, \qquad \forall t,t' \in \mathscr{K} \quad (A1.1)$$

then process $X(t)$ almost certainly has its continuous sample paths on $\mathcal{O}$.

(ii) *Remarks:* (1) On $\mathbb{R}^n$, all the norms are equivalent. We can therefore take the norm

$$\|x\| = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}, \qquad 1 \leqq p < +\infty, \quad (A1.2)$$

which gives the Euclidean norm for $p=2$.

Condition (A1.1) can therefore be replaced by the following condition, with $\lambda \geqq 1$:

$$\sum_{q=1}^{n} E(|X_q(t)-X_q(t')|^{\lambda}) \leqq C_{\mathscr{K}}\|t-t'\|^{d+\eta}, \qquad \forall t,t' \in \mathscr{K}. \quad (A1.3)$$

(2) Condition (A1.1) or (A1.3) is sufficient.

It is not necessary, but no hypothesis was introduced on the law of the field. This field can be non-Gaussian, homogeneous or nonhomogeneous. If $X(t)$ is a Gaussian field, we can use (A1.1) or (A1.3), but in this case, it is possible to weaken the condition if the Gaussian character is taken into account. We then obtain the following classical results:

(iii) *Sufficient condition in the Gaussian case* [35]: If $X(t)$ is a second-order Gaussian field for which the mean function $t \mapsto m_x(t) = E(X(t))$ is continuous from $\mathcal{O}$ into $\mathbb{R}^n$, if there exists a positive constant η and if for any compact set $\mathcal{K}$ of $\mathcal{O}$, there exists a second positive constant $C_{\mathcal{K}}$ such that:

$$E(\|X(t)-X(t')\|^2) \leqq C_{\mathcal{K}} |\mathrm{Log}(\|t-t'\|)|^{-(1+\eta)} \quad (A1.4)$$

for any t and t' in $\mathcal{K}$, with $\|t-t'\| \to 0$, then $X(t)$ almost certainly has its continuous sample paths on $\mathcal{O}$.

(iv) *Proposition* A1.1: Let

$$X(t) = (X_1(t), \ldots, X_n(t))$$

be a stochastic field defined on $(A, \mathcal{A}, P)$, indexed on an open set $\mathcal{O}$ of $\mathbb{R}^d$ with values in $\mathbb{R}^n$, Gaussian, second-order, centered, whose autocorrelation function

$$t, t' \mapsto R(t, t') = E(X(t) X(t')^T) : \mathcal{O} \times \mathcal{O} \to \mathrm{Mat}_{\mathbb{R}}(n, n)$$

can be represented:

$$R(t, t') = \mathrm{Ré} \int_{\mathbb{R}^d} \times e^{i\langle \omega, t-t' \rangle} Q(t, \omega) M(d\omega) Q(t', \omega)^* \quad (A1.5)$$

with the following hypotheses:

(H1) $M(d\omega)$ is a measure on $\mathbb{R}^d$ with values in $\mathrm{Mat}_{\mathbb{C}}(n, n)$, such that for any Borel set B of $\mathbb{R}^d$, $M(B) = \int_B M(d\omega)$ is a positive Hermitian matrix and $M(\mathbb{R}^d) < +\infty$.

(H2) $t, \omega \mapsto Q(t, \omega)$ is a continuous mapping of $\mathcal{O} \times \mathbb{R}^d$ into $\mathrm{Mat}_{\mathbb{C}}(n, n)$ such that for any compact set $\mathcal{K}$ of $\mathcal{O}$:

$$\sup_{t \in \mathcal{K}} \int_{\mathbb{R}^d} \|\omega\|^2 \; Q(t, \omega)^2 \; M(d\omega) = C'_{\mathcal{K}} < +\infty \quad (A1.6)$$

and for any t and t' in $\mathcal{K}$

$$\int_{\mathbb{R}^d} |Q(t, \omega) - Q(t', \omega)|^2 \, M(d\omega) \leqq C''_{\mathcal{K}} \|t-t'\|^2 \quad (A1.7)$$

where $C'_{\mathcal{K}}$ and $C''_{\mathcal{K}}$ are positive finite real constants depending on $\mathcal{K}$.

Under these hypotheses, field $X(t)$ almost certainly has continuous sample paths on $\mathcal{O}$.

Proof: we have:

$$E(\|X(t)-X(t')\|^2) = \mathrm{tr}\{R(t, t) + R(t', t') - R(t, t') - R(t', t)\}$$

$$= \mathrm{Ré} \int_{\mathbb{R}^d} \mathrm{tr}\{A(t, t', \omega) M(d\omega) A(t, t', \omega)^*\} \leqq \int_{\mathbb{R}^d} |A(t, t', \omega)|^2 \, M(d\omega)$$

where:

$$A(t, t', \omega) = Q(t, \omega) e^{i\langle \omega, t-t' \rangle} - Q(t', \omega) = [Q(t, \omega) - Q(t', \omega)] e^{i\langle \omega, t-t' \rangle} + (e^{i\langle \omega, t-t' \rangle} - 1) Q(t', \omega)$$

but since

$$|e^{i\langle \omega, u \rangle} - 1|^2 = 2(1 - \cos\langle \omega, u \rangle) \leqq 2\langle \omega, u \rangle^2 \leqq 2\|\omega\|^2 \|u\|^2$$

we have

$$A(t, t', \omega) \leqq Q(t, \omega) - Q(t', \omega) + \sqrt{2} \|t-t'\| \, \|\omega\| \; Q(t', \omega)$$

for a and b real and positive, and considering (A1.6) and (A1.7), we infer that for any t and t' in $\mathcal{K}$:

$$E(\|X(t)-X(t')\|^2) \leqq C_{\mathcal{K}} \|t-t'\|^2 \quad (A1.8)$$

where

$$C_{\mathcal{K}} = 4 C'_{\mathcal{K}} + 2 C''_{\mathcal{K}}$$

for

$$\|u\| \to 0, \qquad \|u\|^2 |\mathrm{Log}(\|u\|)|^2 \to 0 < 1.$$

The application of (A1.4) gives the result.

(v) *Remark:* If, in proposition A1.1, we take $\mathcal{O} = \mathbb{R}^d$ and we assume the field to be homogeneous, then we have (A1.5) where $Q(t, \omega) = I$ is the real unit matrix and M is the matricial spectral measure. Under these conditions, (A1.7) is automatically satisfied and a sufficient condition for $X(t)$ to almost certainly have its continuous sample paths on $\mathbb{R}^d$ is to have (A1.6) which is equivalent to:

$$\int_{\mathbb{R}^d} \|\omega\|^2 \, \mathrm{tr}\, M \, d\omega = C < +\infty \quad (A1.9)$$

considering the properties of the matricial spectral measure.

(vi) *Proposition* A1.2: Let $\bar{\Omega}$ be the compact part of $\mathbb{R}^d$ defined by (6), where N is defined by (10).

B_N is defined by (12), ω_β by (9)-(14), $|\Delta|$ by (18). We use notation (20).

Let $X^N(t)=(X_1^N(t),\ldots,X_n^N(t))$ be the stochastic field defined on $(A, \mathscr{A}, P)$, indexed on any open set $\mathcal{O}$ of $\mathbb{R}^d$ with values in $\mathbb{R}^n$, such that for any $p\in\{1,\ldots,n\}$:

$$X_p^N(t)=\sqrt{2|\Delta|}\,\mathrm{Re}\left\{\sum_{q=1}^{n}\sum_{\beta\in B_N}\mathbb{H}_{pq}(t,\omega_\beta)\times\exp(i\Phi_{q,\beta}+i\langle t,\omega_\beta\rangle)\right\} \quad (A\,1.10)$$

with the following hypotheses:

(H1) the $\Phi_{q,\beta}$ are $K=n\times N$ random variables, uniform on $[0,2\pi]$, mutually independent;

(H2) for any p and q in $\{1,\ldots,n\}$, functions t, $\omega\mapsto\mathbb{H}_{pq}(t,\omega)$ are continuous from $\mathcal{O}\times\Omega$ into $\mathbb{C}$, and for any compact set $\mathscr{K}$ of $\mathcal{O}$:

$$\sup_{t\in\mathscr{K},\,\omega\in\Omega}|\mathbb{H}_{pq}(t,\omega)|\leqq C'_{\mathscr{K}} \quad (A\,1.11)$$

this means that the $\mathbb{H}_{pq}$ are bounded on $\mathscr{K}\times\bar{\Omega}$.

For any t and t' in $\mathscr{K}$, there exists $C''_{\mathscr{K}}>0$ independent of ω such that:

$$\begin{aligned}&\big|\,|\mathbb{H}_{pq}(t,\omega)|-|\mathbb{H}_{pq}(t',\omega)|\,\big|\leqq C''_{\mathscr{K}}\|t-t'\|\\&|\theta_{pq}(t,\omega)-\theta_{pq}(t',\omega)|\leqq C''_{\mathscr{K}}\|t-t'\|\end{aligned} \quad (A\,1.12)$$

where

$$\mathbb{H}_{pq}(t,\omega)=|\mathbb{H}_{pq}(t,\omega)|\exp(i\theta_{pq}(t,\omega)). \quad (A\,1.13)$$

Under these hypotheses, we have:

(1) for any integer $m\geqq 1$, $\exists C_{\mathscr{K}}>0$, such that $\forall t$ and $t'\in\mathscr{K}$ and $\forall N$:

$$\sum_{p=1}^{n}E(|X_p^N(t)-X_p^N(t')|^{2m})\leqq C_{\mathscr{K}}\|t-t'\|^{2m} \quad (A\,1.14)$$

which means that upper bound (A1.14) is uniform in N;

(2) Field $X^N(t)$ almost certainly has its continuous sample paths on $\mathcal{O}$.

Proof: Criterion (A1.3) allows us to initially consider a component $X_p(t)$, $p\in\{1,\ldots,n\}$ of $X(t)$. Considering the expression (A1.13), for any t and t' in $\mathcal{O}$, we have:

$$\begin{aligned}Z_p^N&=X_p^N(t)-X_p^N(t')\\&=\sum_{k=1}^{K}\{A_k\cos(a_k+\Phi_k)+B_k\sin(b_k+\Phi_k)\}\end{aligned}$$

where we introduced the condensed index $k=(q,\beta)$, $k\in\{1,\ldots,K\}$, $K=nN$, and where we set:

$$\begin{aligned}&\Phi_k=\Phi_{q,\beta};\ A_k=\sqrt{2|\Delta|}\,(|\mathbb{H}_{pq}(t,\omega_\beta)|-|\mathbb{H}_{pq}(t',\omega_\beta)|)\\&B_k=-2\sqrt{2|\Delta|}\,|\mathbb{H}_{pq}(t',\omega_\beta)|\sin(1/2[\langle t-t',\omega_\beta\rangle\\&\qquad+\theta_{pq}(t,\omega_\beta)-\theta_{pq}(t',\omega_\beta)])\\&a_k=\langle t,\omega_\beta\rangle+\theta_{pq}(t,\omega_\beta)\\&b_k=1/2[\langle t+t',\omega_\beta\rangle+\theta_{pq}(t,\omega_\beta)+\theta_{pq}(t',\omega_\beta)].\end{aligned}$$

For any $m\geqq 1$, we have:

$$(Z_p^N)^{2m}=\sum_{i_1+\ldots+i_K=2m}\frac{(2m)!}{i_1!\ldots i_K!}\prod_{k=1}^{K}J_k^{i_k} \quad (A\,1.15)$$

where:

$$J_k=A_k\cos(a_k+\Phi_k)+B_k\sin(b_k+\Phi_k).$$

Since random variables $\Phi_1,\ldots,\Phi_K$ are mutually independent, the same is true for $J_1,\ldots,J_K$ and we have commutation $E\prod_k=\prod_k E$. In addition, for $i_k=2r+1$ odd, we have $E(J_k^{i_k})=0$. Sum (A1.15) then has only the contributions for which we simultaneously have $i_1,i_2,\ldots,i_K$ even numbers. This yields:

$$E\{(Z_p^N)^{2m}\}=\sum_{i_1+\ldots+i_K=m}\frac{(2m)!}{(2i_1)!\times\ldots\times(2i_K)!}\times\prod_{k=1}^{K}E(J_k^{2i_k}).$$

We have the following upper bound for any k:

$$E(J_k^{2i_k})\leqq(|A_k|+|B_k|)^{2i_k}.$$

Considering (A1.11) and (A1.12), for any t and t' in $\mathscr{K}$, we infer:

$$E(J_k^{2i_k})\leqq(\sqrt{2\Delta}\,\tilde{C}_{\mathscr{K}}\|t-t'\|)^{2(i_1+\ldots+i_K)},\qquad\forall k.$$

where

$$\tilde{C}_{\mathscr{K}}=C''_{\mathscr{K}}+C'_{\mathscr{K}}(C''_{\mathscr{K}}+\sup_{\omega\in\bar{\Omega}}\|\omega\|)$$

In addition, since $(2i_k)!\geqq i_k!$, we have:

$$E\{(Z_p^N)^{2m}\}\leqq\frac{(2m)!}{m!}\sum_{i_1+\ldots+i_K=m}\times\frac{m!}{i_1!\ldots i_K!}(\sqrt{2|\Delta|}\,\tilde{C}_{\mathscr{K}}\|t-t'\|)^{2(i_1+\ldots+i_K)}$$

as well as the identity:

$$C_N=\sum_{i_1+\ldots+i_K=m}\frac{m!}{i_1!\ldots i_K!}=\left(\sum_{k=1}^{K}1\right)^m=K^m.$$

Therefore, for any t and $t' \in \mathscr{K}$, we obrain:

$$\sum_{p=1}^{n} E\{(Z_{2p}^N)^{2m}\} \leqq C_{\mathscr{K}} \| t-t' \|^{2m}$$

where $C_{\mathscr{K}}$ is the constant independent of N:

$$C_{\mathscr{K}} = (m!)^{-1} \times (2m)! \times 2^m n^{m+1} \times |\Omega|^m \times \bar{C}_{\mathscr{K}}^{2m}$$

since $|\Delta| K = |\Delta| N n = |\Omega| n$, according to (7), (10), (13) and (18), which proves point (1) of the proposition. The proof of point (2) is inferred directly from (A1.14) and (A1.3).

(vii) *Remark:* Since functions $\mathbb{H}_{pq}$ are continuous on $\mathscr{O} \times \bar{\Omega}$, if they are independent of t, then conditions (A1.11) and (A1.12) are automatically satisfied.

A.2. *Convergence Lemmas:* We establish two convergence lemmas which allow us to prove convergence of the second-order quantities of the approaching sequence.

We use all the notations of Section II.1 of the text.

(i) **Speed of Convergence Lemma A2.1:** *Let $\omega \mapsto f(\omega)$ be a function defined on $\bar{\Omega}$ with values in $\mathbb{C}$ continuously differentiable p times where $p=1$ or 2: $f \in C^p(\bar{\Omega}, \mathbb{C})$. Then, there exists a finite positive real constant $C_0 > 0$ such that $\forall N$ we have:*

$$\left| \int_{\bar{\Omega}} f(\omega)\, d\omega - |\Delta| \sum_{\beta \in B_n} f(\omega_\beta) \right| \leqq C_0 \| \Delta \|^p. \quad (A2.1)$$

This means that $|\Delta| \sum_{\beta \in B_N} f(\omega_\beta)$ converges to $\int_{\bar{\Omega}} f(\omega)\, d\omega$ when $N_{\inf} \to +\infty$ [i.e. $\| \Delta \| \to 0$ according to (19) of the text] with the speed of convergence $N_{\inf}^{-p}$ since we have:

$$\| \Delta \|^p = \left[\sum_{j=1}^{d} \frac{(2\Omega_j)^2}{N_j^2} \right]^{p/2} \quad (A2.2)$$

Proof: We begin by demonstrating the result for $p=2$.

Let $(\nabla f)(\omega) = (\partial_1 f(\omega), \ldots, \partial_d f(\omega)) \in \mathbb{C}^d$ be the gradient of f in ω and $h(\omega) \in \mathrm{Mat}_{\mathbb{C}}(d, d)$ be the Hessian matrix of f in ω such that:

$$[h(\omega)]_{ij} = \partial_i \partial_j f(\omega) \quad (A2.3)$$

where ∂_j denotes the partial derivative with respect to the j-th coordinate ω_j of $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$: let $R(\omega, \omega_\beta) \in \mathbb{C}$ such that:

$$R(\omega, \omega_\beta) = f(\omega) - f(\omega_\beta) - \langle (\nabla f)(\omega_\beta), \omega - \omega_\beta \rangle - 1/2 \langle h(\omega_\beta)(\omega - \omega_\beta), \omega - \omega_\beta \rangle \quad (A2.4)$$

According to Taylor's formula, $\forall \varepsilon > 0$, $\exists r > 0$ such that for $\| \omega - \omega_\beta \| \leqq r$

$$|R(\omega - \omega_\beta)| \leqq \varepsilon \| \omega - \omega_\beta \|^2. \quad (A2.5)$$

In addition, [using (15) to (18) of the text] we can write

$$\int_{\bar{\Omega}} f(\omega)\, d\omega = |\Delta| \sum_{\beta \in B_N} f(\omega_\beta) + \sum_{\beta \in B_N} \int_{Q_\beta} (f(\omega) - f(\omega_\beta))\, d\omega \quad (A2.6)$$

Using (A2.4), we infer from (A2.6) that:

$$\mathrm{Int} = \int_{\bar{\Omega}} f(\omega)\, d\omega - |\Delta| \sum_{\beta \in B_N} f(\omega_\beta) \quad (A2.7)$$

is written:

$$\mathrm{Int} = \mathrm{Int}_1 + \mathrm{Int}_2 + \mathrm{Int}_3$$

$$\mathrm{Int}_1 = \sum_{\beta \in B_N} \int_{Q_\beta} R(\omega, \omega_\beta)\, d\omega$$

$$\mathrm{Int}_2 = \sum_{\beta \in B_N} \int_{Q_\beta} \langle (\nabla f)(\omega_\beta), \omega - \omega_\beta \rangle\, d\omega \quad (A2.8)$$

$$\mathrm{Int}_3 = 1/2 \sum_{\beta \in B_N} \int_{Q_\beta} \langle h(\omega_\beta)(\omega - \omega_\beta), (\omega - \omega_\beta) \rangle\, d\omega.$$

We introduce the change of integration variable:

$$\omega = \omega_\beta + \omega' \quad (A2.9)$$

and we set

$$Q_0 = \prod_{j=1}^{d} [-\Delta_j/2, \Delta_j/2]$$
$$R_0(\omega_\beta, \omega') = R(\omega_\beta + \omega', \omega_\beta). \quad (A2.10)$$

Considering (A2.5), $\exists C_1 > 0$.

$$|R_0(\omega_\beta, \omega')| \leqq C_1 \| \omega' \|^2, \quad \forall \beta \in B_N, \quad \forall \omega' \in Q_0. \quad (A2.11)$$

Accordingly:

$$|\mathrm{Int}_1| \leqq C_1 \sum_{\beta \in B_N} \int_{Q_0} \| \omega' \|^2\, d\omega' = N C_1 \int_{Q_0} \| \omega' \|^2\, d\omega$$

yielding:

$$|\mathrm{Int}_1| \leqq C_1 / 12\, 2^d |\Omega| \| \Delta \|^2. \quad (A2.12)$$

In addition, for any N fixed, we have:

$$\mathrm{Int}_2 = \sum_{\beta \in B_N} \int_{Q_0} \langle (\nabla f)(\omega_\beta),\, \omega' \rangle \, d\omega' = 0 \qquad (A\,2.13)$$

since $\int_{Q_0} \omega'_j \, d\omega' = 0, \ \forall\, j \in \{1, \ldots, d\}$.

Similarly, since $\int_{Q_0} \omega'_i \omega'_j \, d\omega' = 0$, for any $i \neq j$, we have:

$$\mathrm{Int}_3 = 1/2 \sum_{\beta \in B_N} \sum_{j=1}^{d} [h(\omega_\beta)]_{jj} \int_{Q_0} (\omega'_j)^2 \, d\omega'$$

$$= 1/24 \, |\Delta| \sum_{j=1}^{d} \Delta_j^2 \sum_{\beta \in B_N} [h(\omega_\beta)]_{jj},$$

Since $\omega \mapsto h(\omega)$ is continuous on compact set $\bar{\Omega}$, $\exists\, C_2 > 0$:

$$|[h(\omega_\beta)]_{jj}| \leqq C_2, \forall\, j \in \{1, \ldots, d\}, \quad \forall\, \beta \in B_N \qquad (A\,21.14)$$

We infer that:

$$|\mathrm{Int}_3| \leqq C_2/24\, 2^d |\Omega| \, \|\Delta\|^2. \qquad (A\,2.15)$$

setting:

$$C_0 = 1/12\, 2^d |\Omega| \sup(C_1,\, C_2/2)$$

and considering (A2.8), (A2.12), (A2.13) and (A2.15), we infer (A2.1) for $p=2$. For $p=1$, the demonstration is the same, limiting Taylor's expansion to the first order.

(ii) **Lemma A2.2 for convergence of a sequence of measures:** *Let $\omega \mapsto s(\omega)$ be a continuous function of $\bar{\Omega}$ into $\mathbb{C}$ and $\mu(d\omega) = s(\omega)\, d\omega$ be the measure on $\bar{\Omega}$ with values in $\mathbb{C}$ defined by density s. We have:*

$$\mu(\bar{\Omega}) = \int_{\bar{\Omega}} \mu(d\omega) = r, \qquad |r| < +\infty. \quad (A\,2.16)$$

Let $\{\mu^N(d\omega)\}_N$ be the sequence of measures on $\bar{\Omega}$ with values $\mathbb{C}$ such that:

$$\mu^N(d\omega) = |\Delta| \sum_{\beta \in B_N} s(\omega_\beta)\, \delta_{\omega_\beta} \qquad (A\,2.17)$$

where δ_{ω_β} is the measure defined by (43). *We have:*

$$\mu^N(\bar{\Omega}) = \int_{\bar{\Omega}} \mu^N(d\omega) = |\Delta| \sum_{\beta \in B_N} s(\omega_\beta) = r^N,$$

$$|r^N| < +\infty. \qquad (A\,2.18)$$

Then:

(1) *The sequence of measures $\{\mu^N\}_N$ converges tightly to measure μ on $\bar{\Omega}$ when $N_{\inf} \to +\infty$ (the tight convergence implies the weak convergence), and therefore the numerical sequence on $\mathbb{C}$ $\{r^N\}_N$ converges to r,*

(2) *If $s \in C^p(\bar{\Omega}, \mathbb{C})$, for any $p=1$ or 2, the speed of convergence is in $N_{\inf}^{-p}$.*

Proof: (1) By definition, the sequence $\{\mu^N\}_N$ tightly converges to μ if $\forall\, \varphi \in C^0(\bar{\Omega}, \mathbb{C})$, the numerical sequence on $\mathbb{C}$:

$$\mu^N(\varphi) = \int_{\bar{\Omega}} \varphi(\omega)\, \mu^N(d\omega) \qquad (A\,2.19)$$

converges to:

$$\mu(\varphi) = \int_{\bar{\Omega}} \varphi(\omega)\, \mu(d\omega) \qquad (A\,2.20)$$

however,

$$\mu(\varphi) = \int_{\bar{\Omega}} \varphi(\omega)\, s(\omega)\, d\omega$$

$$\mu^N(\varphi) = |\Delta| \sum_{\beta \in B_N} \varphi(\omega_\beta)\, s(\omega_\beta)$$

since function $\omega \mapsto f(\omega) = \varphi(\omega)\, s(\omega)$ is continuous from $\bar{\Omega}$ into $\mathbb{C}$, we know that for $N_{\inf} \to +\infty$, we have:

$$|\Delta| \sum_{\beta \in B_N} f(\omega_\beta) \to \int_{\bar{\Omega}} f(\omega)\, d\omega.$$

Taking $\varphi(\omega) = 1$, $\forall\, \omega \in \bar{\Omega}$ yields $r^N \to r$;

(2) The proof is obtained directly by applying the above speed of convergence lemma A2.1.

A.3 CONVERGENCE IN LAW OF THE NON-GAUSSIAN APPROACHING SEQUENCE

We again use all the notations of Section II of the text. First, we give a few brief reminders useful to proving the results.

(i) *Definition of convergence in law of the approaching sequence:* Let $\alpha(1), \ldots, \alpha(L)$, L be the multiple indexes of $\mathbb{N}^d$, where L is an indeterminate positive integer, with $\alpha(l) = (\alpha_1(l), \ldots, \alpha_d(l)) \in \mathbb{N}^d$, $\alpha_i(l) \in \mathbb{N}$.

We then say that the sequence of stochastic fields $\{X^N(t)\}_N$ indexed on $\mathcal{O}$ with values in $\mathbb{R}^N$ converges in law to stochastic field $X(t)$ indexed on $\mathcal{O}$ with values in $\mathbb{R}^N$ for $N_{\inf} \to +\infty$ if the system of marginal laws of field $X^N(t)$ tightly converges to the system of marginal laws of field $X(t)$, *i.e.* if for any $L \in \mathbb{N}$ and any finite family $(t_{\alpha(1)}, \ldots, t_{\alpha(L)})$ of points of $\mathbb{R}^d$, we have:

$$\mathcal{L}(X^N(t_{\alpha(1)}), \ldots, X^N(t_{\alpha(L)})) \xrightarrow[N_{\inf} \to +\infty]{\text{tightly}} \mathcal{L}(X(t_{\alpha(1)}), \ldots, X(t_{\alpha(L)})) \quad (A\,3.1)$$

where $\mathcal{L}(.)$ denotes the law of the random variable in the argument $(.)$.

(ii) *Use of Paul Lévy's theorem:* Let $v_l=(v_{l,1}, \ldots, v_{l,n})\in\mathbb{R}^n$ and $v=(v_1, \ldots, v_L)\in\mathbb{R}^{nL}$. We denote as $v\mapsto\hat{F}_N(v)$ the characteristic function on $\mathbb{R}^{NL}$ with values $\mathbb{C}$ of the random variable $(\mathbb{U}^N=(X^N(t_{\alpha(1)}), \ldots, X^N(t_{\alpha(L)}))$ with values $\mathbb{R}^{nL}$:

$$\hat{F}_N(v)=E\{\exp(i\langle\mathbb{U}^N, v\rangle\} \qquad (A\,3.2)$$

Then, according to Paul Lévy's theorem, if the sequence of characteristic functions $\{\hat{F}_N\}_N$ simply converges to a function $\hat{F}$ defined on $\mathbb{R}^{nL}$ with values $\mathbb{C}$ whose real part Ré $\hat{F}$ is continuous at the origin, then $\hat{F}$ is the characteristic function of a random variable $\mathbb{U}$ with values $\mathbb{R}^n$:

$$\hat{F}(v)=E\{\exp(i\langle\mathbb{U}, v\rangle\} \qquad (A\,3.3)$$

and we have:

$$\mathcal{L}(\mathbb{U}^n) \xrightarrow[N_{\inf}\to+\infty]{\text{tightly}} \mathcal{L}(\mathbb{U}) \qquad (A\,3.4)$$

(iii) *The central limit theorem* [49]: Let $Y_1, \ldots, Y_k$ be K random variables defined on $(A, \mathcal{A}, P)$ with values $\mathbb{R}$, second-order, mutually independent.

For $k\in\{1, \ldots, K\}$, let $m_k=E(Y_k)$ be the mean of Y_k, $\sigma_k^2=E\{(Y_k-m_k)^2\}$ its variance and $P_k(dy_k)$ the probability law in $\mathbb{R}$ of centered randon variable Y_k-m_k. Laws $P_1, \ldots, P_K$ can be different. Let S_K be such that:

$$S_K^2=\sum_{k=1}^{K}\sigma_k^2. \qquad (A\,3.5)$$

Then, if $\forall\varepsilon>0$, we have:

$$\lim_{K\to+\infty} 1/S_K^2\sum_{k=1}^{K}\int_{|y_k|>\varepsilon S_K} y_k^2\,P_k(dy_k)=0 \qquad (A\,3.6)$$

the law of the random variable:

$$1/S_K\sum_{k=1}^{K}(Y_k-m_k) \qquad (A\,3.7)$$

converges tightly for $K\to+\infty$ to the reduced Gaussian law LG (0, 1) on $\mathcal{R}$.

(iv) *Convergence in law of the approaching sequence for choice* 1: The approaching sequence X^N is defined by (39) for choice 1, *i.e.* according to (40) $Z_{q,\beta}=1$, $\forall q$, $\forall\beta$.. Let $K=n\times N$ and $k=(q, \beta)$ be the condensed index. We set:

$$\begin{aligned} A_{k,l_p}&=|\mathbb{H}_{pq}(t_{\alpha(l)}, \omega_\beta)| \\ b_{k,l_p}&=\langle t_{\alpha(l)}, \omega_\beta\rangle+\theta_{pq}(t_{\alpha(l)}, \omega_\beta) \\ \Phi_{q,\beta}&=\Phi_k \end{aligned} \qquad (A\,3.8)$$

$$Y_k=\sqrt{2\Delta}\sum_{l=1}^{L}\sum_{p=1}^{n} v_{l,p}A_{k,l_p}\cos(\Phi_k+b_{k,l_p})$$

then, according to (50) and since $Z_{q,\beta}=1$, we have, with the notations of paragraph (ii) above:

$$\langle\mathbb{U}^N, v\rangle=\sum_{k=1}^{K}Y_k. \qquad (A\,3.9)$$

Since random variables $\{\Phi_k\}_k$ are mutually independent, the same is true for random variables $\{Y_k\}_k$. In addition, for any $k\in\{1, \ldots, K\}$, $m_k=E(Y_k)=0$ according to (46) and, considering (47), a simple computation gives:

$$E(\langle\mathbb{U}^N, v\rangle^2)=\sum_{k=1}^{K}\sigma_k^2=S_K^2 \qquad (A\,3.10)$$

where we used the notations of the central limit theorem.

Developing the left member of (A3.10) yields:

$$S_K^2=\sum_{l=1}^{L}\sum_{l'=1}^{L}\langle R^N(t_{\alpha(l')}, t_{\alpha(l)})v_l, v_{l'}\rangle \qquad (A\,3.11)$$

When $N_{\inf}\to+\infty$, $K\to+\infty$, and according to II.6.1 (ii), $R^N(t, t')\to R(t, t')$. Therefore, by denoting as

$$\mathbb{U}=(X(t_{\alpha(1)}), \ldots, X(t_{\alpha(L)})).$$

the random variable with values $\mathbb{R}^{nL}$, which is such that:

$$\begin{aligned} S_\infty^2&=E(\langle\mathbb{U}, v\rangle^2) \\ &=\sum_{l=1}^{L}\sum_{l'=1}^{L}\langle R(t_{\alpha(l')}, t_{\alpha(l)})v_l, v_{l'}\rangle \end{aligned} \qquad (A\,3.12)$$

we have

$$\lim_{K\to+\infty} S_K^2=S_\infty^2<+\infty. \qquad (A\,3.13)$$

Considering (A3.13), (A3.6) is verified if:

$$\lim_{K\to+\infty}\sum_{k=1}^{K}\mathrm{Int}_k\to 0$$

where

$$\mathrm{Int}_k=\int_{|y_k|>\varepsilon S_K} y_k^2\,P_k(dy_k)$$

considering (A3.8) and by setting:

$$y_k(\varphi)=\sqrt{2|\Delta|}\sum_{l=1}^{L}\sum_{p=1}^{n} v_{l,p}A_{k,l_p}\cos(\varphi+b_{k,l_p})$$

we have:

$$\mathrm{Int}_k=1/2\pi\int_0^{2\pi}1_{\{|y_k(\varphi)|>\varepsilon S_K\}}(y(\varphi))\,y(\varphi)^2\,d\varphi.$$

Inequality $|y_k(\varphi)| > \varepsilon S_K$ is also written:

$$\left| \sum_{l=1}^{L} \sum_{p=1}^{n} v_{l,\,p} A_{k,\,l_p} \cos(\varphi + b_{k,\,l_p}) \right| > \varepsilon / \sqrt{2|\Delta|}\, S_K$$

But we have:

$$|\Delta| = 2^d |\Omega| / N$$

therefore $\forall \varepsilon > 0\, \exists N(\varepsilon)$ such that $\forall N > N(\varepsilon)$ we have: $\varepsilon / \sqrt{2|\Delta|}\, S_K > 2\pi$. Therefore $\forall k$, $\mathrm{Int}_k = 0$, and we can apply the central limit theorem (iii), which, combined with Paul Lévy's theorem, shows that law $\mathscr{L}(\mathbb{U}^N)$ tightly converges to $\mathscr{L}(\mathbb{U})$ where $\mathbb{U}$ is a Gaussian random variable, which shows the tight convergence of the sequence of fields X^N to the Gaussian field X.

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