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NUMERICAL METHODS IN ELASTOACOUSTICS FOR LOW AND MEDIUM FREQUENCY RANGES

by

C. SOIZE, A. DESANTI and J. M. David

ABSTRACT

A few numerical methods are presented for determining the dynamics and sound radiation of structures at low (LF) and medium (MF) frequencies. After discussing the main problems that have to be solved, we introduce mathematical models and formulations for the LF and MF ranges. The second part of the paper is devoted to the numerical methods. At the end, we discuss validation of the methods and computer codes. The conclusions explain the purpose and limits of the numerical methods, as well as future developments.

I. - INTRODUCTION

In this paper, we review the numerical methods developed to predict the dynamics and acoustic radiation of structures in the LF (low frequency) and MF (medium frequency) ranges. The methods described result from the research that has been conducted for several years now by the Structures Department of ONERA on the acoustic stealth of submarines and supported by the DCN (STCAN, CERDAN, Bassin des Carènes test tank) and by DRET Gr6. These methods were also applied more recently to predict internal noise in aircraft structures.

II. - DEFINITION OF THE PROBLEM MECHANICS TO BE SOLVED

To analyze industrial mechanical structures, it is advantageous to have predictive methods that can be used in a sufficiently general framework. In this section, we describe the main constraints and the usual mechanical hypotheses, in order to situate the problem and identify potential difficulties. Not all aspects can be gone into, since the problem is so vast. We assume that the fluids are dense. Therefore, the fluid/structure interactions are large and the simplifications sometimes made for "light" fluids cannot a priori be used. Obviously the methods that we describe apply if the dense fluids are replaced by light fluids, since only the densities change.

II.1. - MECHANICAL CONFIGURATION CONSIDERED

The structure is a 3D elastic body (or a set of bodies) with any geometry, all of whose dimensions are finite, by contrast with certain model problems for which one or more dimensions can be infinite, such as a cylinder with infinite length. This deformable body is placed in a dense fluid (water) or a light fluid (air), said to be external, which occupies an unbounded domain of physical space. This domain can generally be considered infinite or semi-infinite. The body is either still or moving in translation. Only

may be fluids, called internal, that occupy bounded domains (acoustic cavities) and that are dense or light. Here we do not consider systems with dense internal fluids with free surfaces in the presence of a gravity field.

II.2. - MAIN HYPOTHESES APPLIED

The dynamic behavior of the structure, the external fluid and the internal fluids is linear around a static equilibrium configuration taken as reference state. The structural materials are elastic, dissipative, inhomogeneous and often anisotropic (for instance in the case of composites). For elastoacoustic problems, the external fluid and the internal fluids can be considered inviscid and compressible, with small linear irrotational movements. However, to model the problem, it is sometimes necessary to include dissipation terms in the internal fluids, in particular near the walls. Finally, if the external medium is semi-infinite, it is assumed that the boundary condition is such that it can be considered infinite subject to appropriate symmetrization of the coupled fluid-structure system.

II.3. - EXCITATIONS OF THE COUPLED MECHANICAL SYSTEM

The excitations of the coupled system can be acoustic and/or mechanical in nature.

II.3.1. - Acoustic Excitations

The first type of acoustic excitation corresponds to a system of incident waves in the external fluid. These waves are diffracted by the body, inducing an unsteady pressure field on it. By dynamically exciting the elastic body, this pressure field generates an acoustic radiation specific to the structure in the external fluid and a pressure field in the internal fluids. When considering the elastic radiation of the structure, it is an *elastic target problem*. When considering the pressure field induced in the internal fluids, it is a *problem of external/internal acoustic transparency*. The second type of acoustic excitation corresponds to acoustic sources in the internal cavities and the pressure fields within the cavities and/or the radiation of the structure in the external fluid are considered. These are *problems of internal/internal and/or internal/external acoustic transparency*. As regards the mathematical formulations and the numerical analysis, all the cases

described can be considered with mechanical excitations, discussed below.

II,3.2. — Mechanical Excitations

The mechanical forces applied to the structure may be localized in space (concentrated forces) or distributed (field of surface forces, for instance). They generally correspond either to models of known mechanical excitation sources considered external to the coupled system analyzed, making it a *problem of predicting the responses*, or to an arbitrary given used to construct *elastoacoustic transfer functions*. Hydrodynamic excitations are part of the class of mechanical excitations when the hydrodynamic problem can be decoupled from elastoacoustic problem. For instance, as an initial approximation, the turbulent boundary layer wall pressure field due to a flow on a structure can be considered as a mechanical excitation independent of the induced elastic deformation.

II,4. — CHARACTERIZATION OF THE COUPLED FLUID-STRUCTURE SYSTEM RESPONSES

As the behavior of the dynamic coupled fluid-structure system is linear, the system can be completely characterized by *frequency response functions* (or transfer functions). These functions, with functional values, are used to calculate the response of the coupled system to all types of excitations. For elastoacoustics problems, *frequency characterization* is privileged, not only for the excitations, be they deterministic or random, but also and especially for the responses. We are therefore mainly interested in *forced response problems* and we will limit the discussion to this situation. All the equations below are written in the Fourier plane (frequency) and not in the time domain.

For mechanical excitations, we are interested both in the dynamic response of the structure, generally to the acceleration fields, and in the pressure fields in the fluids, used to determine the *noise levels* in the acoustic cavities and the *acoustic radiation* in the near field and far field of the external fluid.

For acoustic excitations, the parameters of interest are obviously the same as above, but in particular: the field scattered in the external fluid for elastic target problems, the field radiated in the external fluid for a problem of internal/external acoustic transpa-

rency and, finally, a pressure field in the internal fluids for a problem of external/internal or internal/internal acoustic transparency.

II,5. — DEFINITION OF THE FREQUENCY RANGES

The frequency range that must be considered in elastoacoustics is very large. A distinction is made between three frequency ranges: *low frequencies (LF)*, *medium frequencies (MF)* and *high frequencies (HF)*, for mechanical reasons (different mechanical behavior, especially for structures), theoretical reasons (theoretical difficulties increasing with frequency for real mechanical systems) and numerical reasons (requirement for intermediate or high power main-frame computer systems). This concept of frequency range is relative and would require a rigorous definition in the framework of the mechanical problem discussed herein, but this would lead to excessively long developments. We therefore assume that this concept is understood, at least intuitively.

It can be said that the LF range is the frequency band in which the response spectra of the coupled fluid/structure system exhibit isolated spectral rays indicating the presence of a modal structure underlying an associated conservative system. This mainly includes the first eigenmodes of the coupled structure/internal fluids system with the added mass effects of the external fluid.

The HF range is the frequency range in which the response spectra of the coupled fluid/structure system do not exhibit any spectral rays or even local disparities. The behavior in this frequency range is very "smooth", indicating that the modal density of the associated conservative system has a uniform frequency that is very high. In this frequency band, the structural boundary conditions play almost no role.

The MF range is the intermediate range where the response spectra exhibit high irregularities indicating that the modal density of the associated conservative system is not constant but exhibits local highs and local lows. In this band, all the structural boundary conditions are involved a priori as well as the real geometry and the real constitutive equation of the materials, exactly as is the case for the LF situation.

Therefore the LF and MF ranges require numerical methods since it is necessary to solve the equations governing the physics of the problem as accurately as

possible. MF methods can no longer be used for the HF range in the current state of our knowledge and today's computers, and the numerical methods are discarded in favor of other predictive methods for the HF range, one of which is Statistical Energy Analysis. Herein, we discuss only the LF and MF ranges for which numerical methods are used.

III. - MATHEMATICAL MODEL AND FORMULATION

Because of length restrictions on this paper, we give only the mathematical formulation directly adapted to the numerical methods we developed, since, for this type of coupled fluid-structure problem, there are several unknown fields and therefore several possibilities for choosing those to be kept and those to be eliminated. Even so, as will be seen, we have to develop two formulations, one for the LF range and the other for the MF range. In addition, the same restriction led us to choose, with regrets, a very symbolic description, but this presentation, even concise, of the mathematical formulation appeared necessary since it is the only way of introducing the numerical methods. As was mentioned in Section II, it is assumed to simplify the discussion that the body is still and that the external medium is infinite instead of semi-infinite.

III.1. - GEOMETRY OF THE COUPLED SYSTEM

The Euclidean space \mathbb{R}^3 is referenced to a cartesian reference system x_1, x_2, x_3 . The generic point is denoted $x = (x_1, x_2, x_3)$. The geometry of the coupled system considered and the notations are defined in Figure 1.

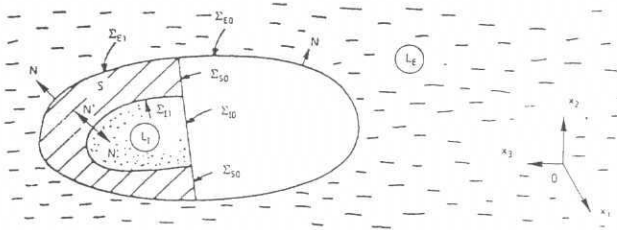


Fig. 1. - Geometry of the coupled fluid-structure system.

The external fluid occupies the open unbounded domain L_E of \mathbb{R}^3 whose boundary is $\Sigma_E = \Sigma_{E_0} \cup \Sigma_{E_1}$. The deformable structure occupies the open bounded domain S of \mathbb{R}^3 with boundary $\Sigma_S = \Sigma_{E_1} \cup \Sigma_{S_0} \cup \Sigma_{I_1}$. The internal fluid (union of the internal fluids) occupies the open bounded domain L_I of \mathbb{R}^3 whose boundary is $\Sigma_I = \Sigma_{I_0} \cup \Sigma_{I_1}$. The unit normal to Σ_E and Σ_I pointing respectively towards

the inside of L_E and L_I is denoted N . Boundaries Σ_{E_0} and Σ_{I_0} are undeformable.

III.2. - EXTERNAL NEUMANN PROBLEM

To formulate the coupled fluid-structure problem described in Section II with the geometry of Figure 1, it is necessary to solve the following external Neumann problem related to the Helmholtz equation:

$$\Delta\Phi + \frac{\omega^2}{c^2}\Phi = 0 \quad \text{in } L_E, \quad (1)$$

$$\frac{\partial\Phi}{\partial N} = v \quad \text{on } \Sigma_E, \quad (2)$$

$$\left. \begin{aligned} \left| \frac{\partial\Phi}{\partial N} - i\frac{\omega}{c}\Phi \right| &= O(r^{-2}), & |\Phi| &= O(r^{-1}), \\ r = \|Ox\| &\rightarrow +\infty, \end{aligned} \right\} (3)$$

where Φ is the velocity potential, c is the speed of sound and ρ is the density of the external fluid, and where v is a normal velocity field defined on Σ_E with values in \mathbb{C} which depends on ω , where $\omega \in \mathbb{R}$ is the angular frequency. The pressure p is given by the equation:

$$p = -i\omega\rho\Phi \quad \text{in } \bar{L}_E. \quad (4)$$

For any ω fixed in \mathbb{R} , problem (1) to (3) has a unique solution [13, 30, 36] and there exist two linear operators $B(\omega/c)$ and $R(x, \omega/c)$ such that:

$$\Phi = B(\omega/c)v \quad \text{in } \Sigma_E, \quad (5)$$

$$\Phi = R(x, \omega/c)v, \quad \forall x \in L_E. \quad (6)$$

The problem of constructing an approximation of finite dimension of the two operators is generally solved by numerical methods (see Sec. IV).

III.3. - CASE OF AN INCIDENT WAVE IN THE EXTERNAL FLUID

The incident wave in L_E is defined by a velocity potential Φ_i with values in \mathbb{C} which verifies equation (1). Boundary Σ_E is subjected to a displacement field \underline{U} whose support is Σ_{E_1} , since Σ_{E_0} has a fixed boundary. The total velocity potential Φ is then written:

$$\Phi = \Phi_i + \Phi_d + \Phi_r,$$

where Φ_d and Φ_r are the diffraction and radiation potentials that are written:

$$\Phi_d = B(\omega/c) \left(-\frac{\partial\Phi_i}{\partial N} \right), \quad \Phi_r = i\omega B(\omega/c)(\underline{U} \cdot N).$$

From this we deduce the expression of the total pressure field on Σ_E :

$$p = p_{id} + p_r, \quad (7)$$

$$p_{id} = -i\omega\rho \left\{ \Phi_i + B(\omega/c) \left(-\frac{\partial\Phi_i}{\partial N} \right) \right\}, \quad (8)$$

$$p_r = \omega^2 \rho B(\omega/c) (\underline{U} \cdot \underline{N}). \quad (9)$$

In any point x of L_E , the total pressure is written:

$$p(x) = p_i(x) + p_{rd}(x), \quad (10)$$

where $p_i(x) = -i\omega\rho\Phi_i(x)$ is the incident pressure and $p_{rd}(x)$ is the scattered pressure, written:

$$p_{rd}(x) = R(x, \omega/c) \left(i\omega \underline{U} \cdot \underline{N} - \frac{\partial\Phi_i}{\partial N} \right). \quad (11)$$

III.4. - EXTERNAL HYDRODYNAMIC FORCES AND COUPLING OPERATOR

Let us assume that there is an incident wave in the external fluid. Let U be the displacement field of the structure at frequency ω defined on S with values in \mathbb{C}^3 . Let \underline{U} be the field defined on Σ_E of support Σ_{E_1} and equal to the trace of U on Σ_{E_1} . Let δU be an arbitrary admissible displacement field of the structure, defined on S and let δU be its trace on Σ_{E_1} . The hydrodynamic force field f_E applied to structure S due to the action of the external fluid and defined by

$$\langle f_E, \delta U \rangle = - \int_{\Sigma_{E_1}} (\delta \underline{U} \cdot \underline{N}) p \, d\sigma$$

is written:

$$f_E = f_{id} + \omega^2 B_E(\omega/c) U, \quad (12)$$

where f_{id} is the hydrodynamic force field defined by:

$$\langle f_{id}, \delta U \rangle = - \int_{\Sigma_{E_1}} (\delta \underline{U} \cdot \underline{N}) p_{id} \, d\sigma, \quad (13)$$

in which p_{id} is given by (8), and where $B_E(\omega/c)$ is the linear external coupling operator that is such that:

$$\begin{aligned} \langle B_E(\omega/c) U, \delta U \rangle \\ = - \int_{\Sigma_{E_1}} \rho (\delta \underline{U} \cdot \underline{N}) B(\omega/c) (\underline{U} \cdot \underline{N}) \, d\sigma. \end{aligned} \quad (14)$$

If there is no incident wave, then f_{id} is zero and there are only the hydrodynamic coupling forces with the external fluid. Finally, it is known that the linear

operator $-\omega^2 B_E(\omega/c)$ can be written:

$$-\omega^2 B_E(\omega/c) = -\omega^2 M_E(\omega/c) + i\omega C_E(\omega/c), \quad (15)$$

where $M_E(\omega/c)$ and $C_E(\omega/c)$ are two real linear operators that are positive-symmetric on the Hilbert spaces adapted to the problem. The first of these operators is the added mass operator and the second is the infinite radiation dissipation operator.

III.5. - FORMULATION ADAPTED TO THE LF RANGE FOR THE COUPLED PROBLEM

For the LF range, it is numerically advantageous to project the equations governing the dynamics of the coupled fluid-structure system onto the truncated basis of the first natural modes of vibration of an associated conservative system with the required spectral properties. Several conditions must be satisfied and the formulation must be specially adapted. First of all, structure S must be able to be assimilated, in the LF range considered, to a medium with a visco-elastic behavior with instantaneous memory, so that the mass and stiffness operators are independent of the frequency. If this model cannot be used as a reasonable approximation, the formulation for the MF range (see Sec. III,6) must be used even for the LF range, since the projection considered does not have any real meaning. Secondly, the presence of internal fluids requires a formulation of the structure-internal fluid coupling that naturally allows identification of an associated conservative system representative as regards the inertia and stiffness terms, for which the spectral problem (eigenvalues and eigenvectors) is correctly stated and can be solved by the most efficient numerical methods. In other words, the discretized spectral problem with finite dimension must have the form:

$$(-\omega^2 [M_0] + [K_0]) X_0 = 0, \quad (16)$$

where $[M_0]$ and $[K_0]$ are two real symmetric matrices with $[M_0]$ positive-definite and $[K_0]$ positive. With these constraints, an integral equation formulation for the internal fluids is not possible, and we therefore preserved the local equations and used a global variational formulation for the coupled problem. To obtain a symmetric problem, we started from the formulation suggested in [32]. The final formulation obtained is summarized below and is developed in [16] for the part concerning the internal fluids and the condensation.

III,5.1. – Model of the Structure

We denote the linear mass, damping and stiffness operators as M_S , C_S and K_S respectively. They are independent of ω considering the hypotheses. They result from the conventional variational formulation of elastodynamic problems and have the usual properties of symmetry and positiveness [9, 13, 20, 23, 35, 43]. To simplify, we assume below that K_S is positive-definite. The structural impedance operators are then written:

$$Z_S(\omega) = -\omega^2 M_S + i\omega C_S + K_S. \quad (17)$$

We denote the element representing the mechanical excitation applied to the structure and depending on the frequency ω as f_S (see Sec. II).

III,5.2. – Model of the Internal Fluids

To simplify the discussion, we assume that for the LF range, the dissipation terms in the internal fluids are negligible compared with the other terms. If this is not the case, these terms can always be added afterwards without any difficulty. The displacement potential Ψ and the pressure field P for the internal fluid (union of the internal fluids) verify the following equations:

$$P = \omega^2 \rho_I \Psi \quad \text{in } \bar{L}_I, \quad (18)$$

$$\Delta \Psi + \frac{1}{\rho_I c_I^2} P = \frac{1}{\rho_I} Q \quad \text{in } L_I, \quad (19)$$

$$\frac{\partial \Psi}{\partial N} = \underline{U}_I \cdot N \quad \text{on } \Sigma_I, \quad (20)$$

where the density ρ_I and the speed of sound c_I are assumed constant in each subdomain of L_I defined by an internal fluid, where Q is an acoustic source term and where \underline{U}_I is the displacement field defined on Σ_I whose support is Σ_{I_1} since boundary Σ_{I_0} is fixed and which is equal to the trace of U on Σ_{I_1} .

III,5.3. – Equation of the Coupled Structure-Internal Fluids-External Fluid System in the LF Domain

The global variational formulation of the coupled problem gives the following equation in operational form:

$$(-\omega^2 M(\omega/c) + i\omega C(\omega/c) + K)X = F \quad (21)$$

where

$$X = \begin{bmatrix} U \\ P \\ \Psi \end{bmatrix}, \quad F = \begin{bmatrix} f_S + f_{id} \\ 0 \\ -\omega^2 q_I \end{bmatrix}, \quad (22)$$

$$M(\omega/c) = \begin{bmatrix} M_S + M_E(\omega/c) & 0 & A \\ 0 & 0 & D \\ A^T & D^T & -K_I \end{bmatrix}, \quad (23)$$

$$C(\omega/c) = \begin{bmatrix} C_S + C_E(\omega/c) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (24)$$

$$K = \begin{bmatrix} K_S & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (25)$$

where A , D , G and q_I are such that if δU , δP and $\delta \Psi$ are arbitrary admissible fields associated with U , P and Ψ respectively, we have:

$$\langle A\Psi, \delta U \rangle = - \int_{\Sigma_{I_1}} \rho_I (\delta \underline{U}_I \cdot N) \Psi \, d\sigma, \quad (26)$$

$$\langle D\Psi, \delta P \rangle = \int_{L_I} \frac{1}{c_I^2} \Psi \, \delta P \, dx, \quad (27)$$

$$\langle GP, \delta P \rangle = \int_{L_I} \frac{1}{\rho_I c_I^2} P \, \delta P \, dx, \quad (28)$$

$$\langle K_I \Psi, \delta \Psi \rangle = \int_{L_I} \rho_I \text{grad } \Psi \cdot \text{grad } \delta \Psi \, dx, \quad (29)$$

$$\langle q_I, \delta \Psi \rangle = \int_{L_I} Q \, \delta \Psi \, dx. \quad (30)$$

The three linear operators $M(\omega/c)$, $C(\omega/c)$ and K are real and symmetric. The conservative problem associated with equation (21) is written:

$$(-\omega^2 M(\omega/c) + K)X = 0, \quad (31)$$

and it is not yet stated satisfactorily for the following reasons:

(1) Operator M depends on ω/c by the external fluid added mass term. The first solution consists of considering the associated incompressible external fluid ($c \rightarrow +\infty$) and we have $M(0)$. This approach entails a nonnegligible extra numerical cost, since it requires first solving an external Neumann problem related to the Laplace equation and which, in any case, does not do away with having to solve the external compressible problem defined in Section III,2. The second solution consists of not taking the added mass effect into account (*i.e.* we set $M_E(\omega/c) = 0$) for constructing the modal projection base sought. We use the second solution since all the applications we conducted show that this method is very efficient, as convergence is rapid in the LF domain considering the number of base vectors chosen.

(2) We assume that the added mass term $M_E(\omega/c)$ momentarily vanishes for construction of the projection base. Then operator (23), independent of ω , is not positive and the numerical problem does not have the required form (16). To overcome this difficulty, it is necessary to eliminate field Ψ from equation (21).

(3) To eliminate Ψ from (21), it is necessary to invert K_I . Let us take $\mathbb{V} = H^1(L_I)/R$, where R is the set of constant fields defined on L_I . Then, since the continuous linear operator K_I from \mathbb{V} into \mathbb{V}' , defined by bilinear form (29) on $\mathbb{V} \times \mathbb{V}$, is injective, it is invertible (whereas it would not be if we did not consider the quotient space), and its inverse is denoted K_I^{-1} . It should be noted that this leads to a few additional numerical difficulties (see Sec. IV) but which cannot be avoided.

With this set of hypotheses, we can then eliminate Ψ from (21), and reintroducing notation (15) yields the formulation for the LF domain governing the coupled structure-internal fluids-external fluid system:

$$(-\omega^2 M_0 + i\omega C_0 + K_0 - \omega^2 B_0(\omega/c)) X_0 = F_0 \quad (32)$$

where:

$$X_0 = \begin{bmatrix} U \\ P \end{bmatrix}, \quad F_0 = \begin{bmatrix} f_S + f_{id} - \omega^2 A K_I^{-1} q_I \\ -\omega^2 D K_I^{-1} q_I \end{bmatrix}, \quad (33)$$

$$M_0 = \begin{bmatrix} M_S + A K_I^{-1} A^T & A K_I^{-1} D^T \\ D K_I^{-1} A^T & D K_I^{-1} D^T \end{bmatrix}, \quad (34)$$

$$\left. \begin{aligned} B_0(\omega/c) &= \begin{bmatrix} B_E(\omega/c) & 0 \\ 0 & 0 \end{bmatrix}, & C_0 &= \begin{bmatrix} C_S & 0 \\ 0 & 0 \end{bmatrix}, \\ K_0 &= \begin{bmatrix} K_S & 0 \\ 0 & G \end{bmatrix}. \end{aligned} \right\} \quad (35)$$

It should be noted that if we can assume all the internal fluids to be incompressible ($c_I \rightarrow +\infty$), then D and G are identically zero and we obtain an independent equation in U (first row of (32)), which considerably simplifies the problem.

III,5.4. — Spectral Problem of the Associated Conservative System

This is the following problem, associated with (32):

$$(-\omega^2 M_0 + K_0) Y = 0. \quad (36)$$

Considering the above hypotheses, the mathematical analysis [13] of problem (16) shows that the spectral problem is correctly stated. The spectrum, which is countable, consists of a series of positive real numbers:

$$0 \leq \omega_1^2 \leq \omega_2^2 \leq \dots,$$

and the associated eigenvectors Y^1, Y^2, \dots , form a basis of the Hilbert space of the problem and are such that:

$$\langle M_0 Y^j, Y^k \rangle = m_j \delta_{jk}, \quad (37)$$

$$\langle K_0 Y^j, Y^k \rangle = m_j \omega_j^2 \delta_{jk}, \quad (37')$$

where the m_j terms are real positive constants related to normalization. Solving problem (36) by the finite element method leads to a numerical problem of type (16). Each (ω_j, Y^j) pair corresponds to an eigenfrequency and an elastoacoustic eigenmode of the conservative coupled structure-internal fluids system in the absence of the external fluid.

III,5.5. — Projected Equation of the Coupled Structure-Internal Fluids-External Fluid System

For any ω in \mathbb{R} , problem (32) has a unique solution whose regularity can be studied [13, 27]. For the LF domain, we construct an approximation X_n of this unique solution X_0 by writing:

$$X_n = \sum_{j=1}^n \chi_j Y^j, \quad (38)$$

where n is the number of eigenvectors chosen and where the χ_j terms with complex values are the new unknowns, called generalized coordinates. Based on equation (32), we then obtain the following matrix equation on \mathbb{C}^n :

$$(-\omega^2 [M_n] + i\omega [C_n] + [K_n] - \omega^2 [B_n(\omega/c)]) \mathbb{X} = F_n, \quad (39)$$

where \mathbb{X} is the column matrix of the χ_j terms, and:

$$\left. \begin{aligned} [M_n]_{jk} &= m_j \delta_{jk}, & [C_n]_{jk} &= \langle C_0 Y^k, Y^j \rangle, \\ [K_n]_{jk} &= m_j \omega_j^2 \delta_{jk}, \end{aligned} \right\} \quad (40)$$

$$[F_n]_j = \langle F_0, Y^j \rangle, \quad (41)$$

$$[B_n(\omega/c)]_{jk} = \langle B_0(\omega/c) Y^k, Y^j \rangle. \quad (42)$$

The main numerical aspects are discussed in Section IV.

III,6. — FORMULATION FOR THE COUPLED PROBLEM ADAPTED TO THE MF RANGE

In the MF frequency range, the conservative system associated with the structure with its internal fluids (without the external fluid) has a modal density with local highs and local lows when the frequency varies in the MF band. This property can moreover be considered as an informal definition of the MF range, as was indicated in Section II,5.

The numerical costs related to calculating a large number of frequencies and eigenmodes of high rank on a system with a very large number of DOFs (degrees of freedom) are high, and all the numerical problems which then arise have not so far been completely solved. In addition, in the MF domain, structural materials such as many composites, for instance, generally have a viscoelastic structure so that the operators for the structure depend significantly on the frequency, preventing a constant approximation. Under these conditions, there is no longer a unique spectral problem representative of the system over a wide MF band. Under these circumstances, projection of an associated conservative system onto a truncated basis of eigenvectors no longer seems suitable. It is therefore unnecessary to obtain a formulation allowing this eigenvector calculation, and the LF formulation of Section II,5 for internal fluids can be dropped for a much simpler method using gyroscopic coupling between the structure and the internal fluids. There is nothing to be changed for the external fluid, since the formulation remains valid. As for the LF range, we do not use an integral equation formulation for the internal fluids since, in the MF domain, we are often led to introduce a dissipation term that is not actually homogeneous in the internal fluids, in particular near the walls, for modeling purposes. Below we summarize the formulation developed for the MF range [42] and which seems quite satisfactory to us.

III.6.1. – Model of the Structure

We denote the linear mass, damping and stiffness operators of the structure that depend on the frequency ω as $M_S(\omega)$, $C_S(\omega)$ and $K_S(\omega)$. For each fixed ω , they have the usual properties of symmetry and positiveness (see Sec. III,5.1). Normally, for viscoelastic materials, only C_S and K_S depend on ω , but not M_S [23]. However, for certain composite structural elements, such as honeycomb sandwich panels, the homogenization techniques associated with kinematic reductions introduce terms that are homogeneous to a mass and that depend on the frequency [19]. This is why we introduce a dependence of M_S on ω in the model. The structural impedance operator is then written:

$$Z_S(\omega) = -\omega^2 M_S(\omega) + i\omega C_S(\omega) + K_S(\omega). \quad (43)$$

Again, as in Section III,5.1, we use the notation f_S for mechanical excitations.

III.6.2. – Model of the Internal Fluids

The formulation is written as “velocity potential”. It is denoted Φ_I and verifies the equations [42]:

$$\operatorname{div}(\{1 + i\omega\lambda(\omega)\} \operatorname{grad} \Phi_I) + \frac{\omega^2}{c_I^2} \Phi_I = \frac{1}{\rho_I} Q \quad \text{in } L_I, \quad (44)$$

$$\{1 + i\omega\lambda(\omega)\} \frac{\partial \Phi_I}{\partial N} = i\omega \underline{U}_I \cdot \underline{N} \quad \text{on } \Sigma_I. \quad (45)$$

The pressure field P is given by the equation:

$$P = -i\omega\rho_I \Phi_I \quad \text{in } \bar{L}_I. \quad (46)$$

The notations ρ_I , c_I , \underline{U}_I and Q have the same meaning as in Section III,5.2. Here we introduced a simple but inhomogeneous dissipation model which has the flexibility required to satisfy the needs of models in the MF domain. Is there is no dissipation, $\lambda(\omega) = 0$ and we obtain an inviscid, compressible, irrotational fluid where Φ_I is a real velocity potential. To model an inhomogeneous dissipation, we write:

$$\lambda(\omega) = \nu(\omega) c_I^{-2}, \quad (47)$$

where ν depends on the space and frequency, has values \mathbb{R}^+ and the dimension of a kinematic viscosity coefficient.

III.6.3. – Equation of the Coupled Structure-Internal Fluids-External Fluid System in the MF Domain

The global variational formulation of the coupled problem gives the following equation in operational form:

$$Z_1(\omega) X_1 = F_1, \quad (48)$$

where $Z_1(\omega)$ is the linear impedance operator:

$$Z_1(\omega) = -\omega^2 M_1(\omega) + i\omega C_1(\omega) + K_1(\omega) - \omega^2 B_0(\omega/c), \quad (49)$$

and where we set:

$$X_1 = \begin{bmatrix} U \\ \Phi_I \end{bmatrix}, \quad F_1 = \begin{bmatrix} f_S + f_{id} \\ q_I \end{bmatrix}, \quad (50)$$

$$\left. \begin{aligned} M_1(\omega) &= \begin{bmatrix} M_S(\omega) & 0 \\ 0 & -M_I \end{bmatrix}, \\ C_1(\omega) &= \begin{bmatrix} C_S(\omega) & -A \\ -A^T & -C_I(\omega) \end{bmatrix}, \\ K_1(\omega) &= \begin{bmatrix} K_S(\omega) & 0 \\ 0 & -K_I \end{bmatrix}, \end{aligned} \right\} \quad (51)$$

where $B_0(\omega/c)$, A , K_I and q_I are the quantities defined in Section III,5.3 and where M_I and $C_I(\omega)$ are the linear operators such that for any admissible field $\delta\Phi_I$ on L_I we have:

$$\langle M_I \Phi_I, \delta\Phi_I \rangle = \int_{L_I} \frac{\rho_I}{c_I^2} \Phi_I \delta\Phi_I dx, \quad (52)$$

$$\langle C_I(\omega) \Phi_I, \delta\Phi_I \rangle = \int_{L_I} \rho_I \lambda(\omega) \text{grad } \Phi_I \cdot \text{grad } \delta\Phi_I dx. \quad (53)$$

For any ω in \mathbb{R} , there is a unique solution of (48) and it is known how to analyze its regularity as a function of the givens [13, 27]. It should be noted that we changed the signs of the second line of equation (48) by blocks so that complex operator $Z_1(\omega)$ was algebraically symmetric, as this property is very useful for the MF numerical method described in Section IV.

III.6.4. – Taking the Structural Complexity Into Account in the MF Domain

In a structure, a distinction can be made between the part accessible to the conventional modeling, called the master structure, and the part inaccessible to this conventional modeling and that we call the structural complexity or fuzzy structure. The fuzzy structure concept is introduced in [39]. It consists of multiple mechanical subsystems connected to the master structure and which generally govern its functionality. Numerical methods are used to construct an approximation of impedance $Z_1(\omega)$ defined by (49). Therefore, in the current state of our knowledge, this can only be done for the master structure coupled with the internal and external fluids. In the LF domain, we partially take the fuzzy structure into account in the mass balances and only the inertial effects of the fuzzy structure are therefore modeled, which is legitimate. In the MF domain, the mechanical subsystems of the fuzzy structure no longer behave like pure masses but are themselves dynamic systems coupled with the master structure. The master structure is therefore subject to a loss by coupling due to the transfer of vibrational energy from the master structure to the fuzzy structure, which induces an apparent dissipation that can cause strong smoothing of the frequency response functions of the master structure. From the standpoint of predictive methods, it is not attempted to identify the fuzzy structure dynamics but to construct unconventional global models of the fuzzy structure to improve the predictions of the dynamic behavior of the master structure. For instance, in [39, 8], we propose a model of the fuzzy structure and its processing that results in introducing an impedance with values in

the random operators $Z_a(\omega)$. Equation (48) is then replaced by:

$$(Z_1(\omega) + Z_a(\omega)) X_1 = F_1. \quad (54)$$

There are then two types of problem. The first concerns construction of a mathematical model of $Z_a(\omega)$ which, in our opinion, must be based on a deterministic mechanical model certain of whose parameters are modeled by random variables for which the probabilistic characteristics must themselves be identified by statistical methods. The second concerns the method used to solve random operator equation (54) assuming the probabilistic model of $Z_a(\omega)$ to be given. This type of research was started in [39, 8] but much still remains to be done considering the difficulties of these aspects.

IV. – NUMERICAL METHODS

The LF and MF numerical methods discussed below are derived directly from the two mathematical formulations described in Section III. These formulations are even an integral part of the method insofar as they were developed for that purpose and with a constant concern for being numerically effective. However, these formulations alone are not sufficient, since two fundamental problems remain. The first concerns effective construction of an approximation of finite rank of all the functional operators introduced. This is conventionally the space discretization phase of the system. The second concerns solving the discretized equations, *i.e.* (36) and (39) for the LF domain, and (48), possibly with variants like (54), for the MF domain. To avoid repetitions, we begin by exposing the part of the methods common to the LF and MF domains then continue with the specific features for each domain.

IV.1. – NUMERICAL METHODS FOR THE EXTERNAL NEUMANN PROBLEM

Here we are concerned with numerical analysis of problem (1)-(3) used to construct the required approximation of the coupling operator $B(\omega/c)$ and the radiation operator $R(x, \omega/c)$ defined in Section III,2. Considering the results of Sections III,3 and III,4, if we are able to construct these approximations, we can directly deduce approximations of (a) the inhomogeneous excitation term f_{id} using (13) and (8) if there is an incident wave, (b) the external coupling operator $B_E(\omega/c)$ using (14), and (c) the radiated pressure in the external fluid using (11) in presence or in the absence of an incident wave, know-

ing the approximation of the displacement field U of the structure. A suitable formulation must be available for this purpose.

IV.1.1. – Additional Information on the Formulation

(a) *Integral equation formulation for the case of a surface Σ_E of any geometry.* If the closed surface Σ_E has an arbitrary geometry, the solution of problem (1)-(3) in an unbounded homogeneous medium can be sought using an integral equation formulation which is well adapted to this situation [13, 30]. There are several possible integral equation formulations. Although problem (1)-(3) always has a unique solution for any ω in \mathbb{R} , certain of these formulations do not supply the solution for a series of real values of ω , called irregular frequencies. The Fredholm alternative obviously works for these values of ω [13]. In practice, for arbitrary geometries, the values of these irregular frequencies are not known before calculation and it is therefore preferable to construct integral formulations which automatically get around this difficulty related to the formulation. Such an approach was developed in the general case by [1] and was validated. This is the approach we chose. Finally, for certain geometric situations where the domain simultaneously involves thin parts and massive parts, for instance a propeller, we developed a formulation based on [1] but which also allows us to replace the upper and lower surfaces of a thin domain by a single mean surface while preserving the interaction terms between the thin parts thus modeled and the massive parts. These aspects are covered in [34].

(b) *Asymptotic formulation for a surface Σ_E with a slender geometry.* The formulation used in (a), which is valid for the general case, induces sufficiently high numerical costs to make it worthwhile to attempt to decrease them if possible. When surface Σ_E is slender, asymptotic methods whose small parameter is related to the slenderness can be used. We then obtain formulations which have much lower numerical costs. We developed and validated such a formulation in [12] for the LF domain and in [6] for the MF domain.

IV.1.2. – Approximation of the Coupling and Radiation Operators

(a) *Case of integral equation formulation.* The finite approximation of the coupling and radiation operators is obtained by constructing the weak formulation associated with the integral equation formulation of Section IV.1(a), then using a finite element

method so that only surface Σ_E has to be meshed. Two versions were developed. The first [1] uses P_0 and Q_0 finite elements, and the second [34], developed recently and which covers the case of multiconnected thin and massive domains, uses P_1 finite elements. We thus construct the matrix of the finite approximation of the coupling operator which is complex, symmetric but not Hermitian, and dense, as well as the corresponding approximation matrix of the radiation operator. These two matrices are functions of ω/c . It should be noted that the numerical cost is due to the requirement for solving a complex linear system with N right-hand sides, where N is equal to the number n of eigenmodes used for the LF method (see Sec. III.5.5) and N is the dimension of the linear system (we therefore invert a matrix) for the MF method. If surface Σ_E has a cyclic geometry, the numerical costs are significantly decreased by taking it into account [34].

(b) *Case of asymptotic formulation.* In this case [6, 12], the coupling operator is directly expressed by an integral operator relative to Σ_E instead of an inverse, as was the case for the general method. To discretize this integral operator, we use a P_0 and Q_0 finite element method and we directly obtain the expression of a matrix of finite approximation of the coupling operator without having to solve a complex linear system with several right-hand sides, contrary to case (a) above; this applies to both the LF and the MF formulation.

IV.2. – NUMERICAL METHODS FOR SPACE DISCRETIZATION OF THE STRUCTURE AND ITS INTERNAL FLUIDS

We use the finite element method to construct finite approximations of the structural impedance operators of equation (17) for the LF model and (43) for the MF model [2, 3, 9, 13, 24, 35, 46]. This is the only method that allows the structures to be discretized in all their variety, both geometrically and as regards the materials and boundary conditions. As for the internal fluids, for the reasons mentioned above, we also use the finite element method. For the LF formulation [16] (or the MF formulation [42]), the elementary matrices for the different types of finite elements (surface and volume) are constructed from operators A , D , G and K_I defined by (26) to (29) (operators $M_I(\omega)$ and $C_I(\omega)$ defined by (52) and (53) respectively).

IV.3. – NUMERICAL SOLVING METHODS FOR THE LF DOMAIN

(a) First we construct the approximation matrices for the operators of the structure and its internal fluids using the method described in Section IV.2. It should be noted that construction of the approximation matrix for operator M_0 defined by (34) requires condensation to eliminate Ψ , and, as was mentioned in Section III.5.3, there is a difficulty related to the inversion of K_I . To construct the quotient space, and so as not to have to modify the formulation of the elementary matrices of the internal fluid finite elements, we chose a Lagrange multiplier technique to impose the constraint. These aspects are developed in [16].

(b) Then we calculate the first n eigenvalues and eigenvectors of a problem of type (16) corresponding to the approximation of the spectral problem (36). The algorithmic method of iterations in subspaces [3] is well suited to this type of problem, since the number of DOFs of the structural model with its internal fluids is much higher than the number n . We can then deduce the matrices defined by equations (40).

(c) We then calculate matrices $[B_n(\omega/c)]$ for values of ω/c covering the LF band considered using the methods described in IV.1. This allows us to construct an approximation of function $\omega \rightarrow [B_n(\omega/c)]$ on the LF band using spline functions.

(d) We solve the complex linear matrix equation (39) on the LF band considered. The frequency resolution chosen can be high since the dimension n of the system is very small and the numerical cost of the solution is therefore negligible.

(e) We then have all the information we need to finish solving the problem. It should be noted that the total numerical cost is due only to the spectral problem (b) and the construction of function $\omega \rightarrow [B_n(\omega/c)]$ of point (c) in the general case, except for slender surfaces for which the numerical cost of (c) is negligible compared with the cost of point (b).

IV.4. – NUMERICAL SOLVING METHODS FOR THE MF DOMAIN

The methods of Section IV.1 and IV.2 are used to construct a numerical model of the coupled structure-internal fluids-external fluid problem in the MF domain, *i.e.* obtain the matrix equation associated with (48) on \mathbb{C}^m , where m is the total number of DOFs of the model. It is written:

$$[Z_1(\omega)][X_1] = [F_1], \quad (55)$$

where

$$[Z_1(\omega)] = -\omega^2 [\tilde{M}_1(\omega)] + i\omega [C_1(\omega)] + [K_1(\omega)], \quad (56)$$

$$[\tilde{M}_1(\omega)] = [M_1(\omega)] + [B_0(\omega/c)]. \quad (57)$$

For any ω in \mathbb{R} , matrix $[\tilde{M}_1(\omega)]$ is complex symmetric, matrices $[C_1(\omega)]$ and $[K_1(\omega)]$ are real symmetric and the complex symmetric matrix $[Z_1(\omega)]$ is invertible.

For the MF domain, it is generally necessary to identify the frequency response of the coupled system over a wide frequency band, with a frequency resolution that must remain relatively high, *i.e.* equation (55) has to be solved for a large number of values of ω . In addition, to capture dynamic phenomena in the MF band, it is necessary to refine the meshes by comparison with a model suited only to the LF band, which means that the number m of DOFs of a numerical MF model is high. These two constraints mean that with today's computers, it is not realistic to consider solving the complex matrix system (55) with dimension m for all the values of ω that must be considered. We feel that the direct approach is currently prohibitive and must be replaced by a more efficient indirect method. We therefore developed an MF method [37, 38, 41, 42], the main points of which are recalled below.

IV.4.1. – MF Solving Method

The problem is to solve linear system (55) over a large frequency band with m high. The MF method consists of writing the wide MF band as a union of narrow bands \mathbb{B}_v and solving the problem on each narrow band using a mixed method with two time scales. The short time scale associated with the center frequency of the narrow band is solved analytically in the Fourier domain without introducing numerical errors, and the long time scale associated with the width of the narrow band \mathbb{B}_v is solved by a numerical method in the time domain.

(a) *Narrow Band MF Signal.* Let

$$\mathbb{B}_v = [\Omega_v - \Delta\omega/2, \Omega_v + \Delta\omega/2]$$

be a narrow MF band with center frequency $\Omega_v > 0$ and bandwidth $\Delta\omega$ such that $0 < \Delta\omega < 2\Omega_v$. The band $[-\Delta\omega/2, \Delta\omega/2]$ is denoted \mathbb{B}_0 . Let τ_L be the long time scale and τ_c be the short time scale such that:

$$\tau_L = \frac{2\pi}{\Delta\omega}, \quad \tau_c = \frac{2\pi}{\Omega_v}. \quad (58)$$

By definition, \mathbb{B}_v is a narrow MF band if:

$$\frac{\tau_L}{\tau_C} = \frac{\Delta\omega}{\Omega_v} \ll 1. \quad (59)$$

Let $H_m(\mathbb{B}_v) = \{F_v \in L^2(\mathbb{R}, \mathbb{C}^m) \mid \text{supp } \hat{F}_v = \mathbb{B}_v\}$ be the Hilbert space of functions $t \mapsto F_v(t): \mathbb{R} \rightarrow \mathbb{C}^m$ with integrable square:

$$\|F_v\|^2 = \int_{\mathbb{R}} \|F_v(t)\|^2 dt < +\infty,$$

and therefore the Fourier transform (FT) of $\omega \mapsto \hat{F}_v(\omega): \mathbb{R} \rightarrow \mathbb{C}^m$ which has an integrable square and is such that:

$$\hat{F}_v(\omega) = \int_{\mathbb{R}} e^{-i\omega t} F_v(t) dt,$$

$$\|F_v\|^2 = \frac{1}{2\pi} \|\hat{F}_v\|^2,$$

has a compact support \mathbb{B}_v , i. e. that:

$$\hat{F}_v(\omega) = 0, \quad \forall \omega \notin \mathbb{B}_v.$$

With any $F_v \in H_m(\mathbb{B}_v)$, we can associate a signal $F_0 \in H_m(\mathbb{B}_0)$ such that:

$$F_0(t) = F_v(t) \exp(-i\Omega_v t). \quad (60)$$

Accordingly, for any ω' in \mathbb{B}_0 , we have:

$$\hat{F}_0(\omega') = \hat{F}_v(\omega' + \Omega_v),$$

which shows that F_0 is a low frequency signal on \mathbb{B}_0 which has only one time scale, the long scale τ_L .

(b) *Frequency Approximation of the Operators.* The frequency response function of the coupled system is identified in the MF domain by solving problems of the following type. Calculate the solution of equation

$$[Z_1(\omega)] \hat{X}_v(\omega) = \hat{F}_v(\omega), \quad (61)$$

where $[Z_1(\omega)]$ is given by (56) and where $F_v \in H_m(\mathbb{B}_v)$. Then, the hypotheses on function $\omega \mapsto [Z_1(\omega)]$ imply that the solution of (61) is such that $X_v \in H_m(\mathbb{B}_v)$. We associate the following equation with (61)

$$(-\omega^2 [M_v] + i\omega [C_v] + [K_v]) \hat{Y}_v(\omega) = \hat{F}_v(\omega), \quad (62)$$

where

$$\left. \begin{aligned} [M_v] &= [\tilde{M}_1(\Omega_v)], & [C_v] &= [C_1(\Omega_v)], \\ [K_v] &= [K_1(\Omega_v)]. \end{aligned} \right\} \quad (63)$$

Then, if functions $\omega \mapsto [\tilde{M}_1(\omega)]$, $[C_1(\omega)]$ and $[K_1(\omega)]$ are continuous on \mathbb{R} , for $\varepsilon > 0$ fixed as small as desi-

red, there is a bandwidth $\Delta\omega$ such that we have:

$$\|\hat{X}_v - \hat{Y}_v\| < \varepsilon.$$

We can therefore replace problem (61) by problem (62)-(63). Signal Y_v whose FT is \hat{Y}_v belongs to $H_m(\mathbb{B}_v)$ and verifies the following differential equation:

$$[M_v] \ddot{Y}_v(t) + [C_v] \dot{Y}_v(t) + [K_v] Y_v(t) = F_v(t). \quad (64)$$

Since Y_v and F_v both belong to $H_m(\mathbb{B}_v)$, there are the two time scales, τ_L and τ_C .

(c) *Analytic Processing of the Short Time Scale.* By making the translation $\omega = \omega' + \Omega_v$ with $\omega' \in \mathbb{B}_0$ and $\omega \in \mathbb{B}_v$, we obtain for any $\omega' \in \mathbb{B}_0$:

$$(-\omega'^2 [M_v] + i\omega' [D_v] + [Z_v]) \hat{Y}_0(\omega') = \hat{F}_0(\omega'), \quad (65)$$

where

$$\begin{aligned} [D_v] &= [C_v] + 2i\Omega_v [M_v], \\ [Z_v] &= -\Omega_v^2 [M_v] + i\Omega_v [C_v] + [K_v], \end{aligned}$$

and where Y_0 and F_0 , which are such that:

$$\begin{aligned} \hat{F}_0(\omega') &= \hat{F}_v(\omega' + \Omega_v), & F_0(t) &= e^{-i\Omega_v t} F_v(t), \\ \hat{Y}_0(\omega') &= \hat{Y}_v(\omega' + \Omega_v), & Y_v(t) &= e^{i\Omega_v t} F_0(t), \end{aligned}$$

are in $H_m(\mathbb{B}_0)$.

(d) *Numerical Processing of the Long Time Scale.* The low frequency signal $Y_0 \in H_m(\mathbb{B}_0)$ which is a solution of (65) and is associated with the mean frequency signal $Y_v \in H_m(\mathbb{B}_v)$, verifies the following low frequency time-difference equation:

$$[M_v] \dot{Y}_0(t) + [D_v] Y_0(t) + [Z_v] Y_0(t) = F_0(t). \quad (66)$$

This LF equation is solved by the Newmark scheme [3] which is an unconditionally stable step numerical integration method. We thus introduce a usual approximation error at this solving level but it should be noted that the error concerns only slow variations of the signal and conventionally remains small.

(e) *Explicit Expression of the MF Solution from the Solution of the Associated LF Problem.* Introducing the conventional family of functions of the Shannon theorem:

$$\Phi_k(t) = \frac{\sin(\pi(t/\tau_L - k))}{\sqrt{\tau_L} \pi(t/\tau_L - k)}, \quad k \in \mathbb{Z},$$

the MF signal $Y_v \in H_m(\mathbb{B}_v)$ is written:

$$Y_v(t) = \sqrt{\tau_L} e^{i\Omega_v t} \sum_{k \in \mathbb{Z}} Y_0(k \tau_L) \Phi_k(t). \quad (67)$$

Its Fourier transform with compact support \mathbb{B}_v is given explicitly by the equation:

$$\hat{Y}_v(\omega) = \tau_L \mathbf{1}_{\mathbb{B}_v}(\omega) \sum_{k \in \mathbb{Z}} Y_0(k \tau_L) e^{-ik \tau_L (\omega - \Omega_v)}. \quad (68)$$

Equation (68) is used to calculate the FT \hat{Y}_v of the MF solution Y_v from the sampled LF time solution $\{Y_0(k \tau_L), k \in \mathbb{Z}\}$ alone.

IV,4.2. – Case of Stationary Random Excitations

Certain acoustic or mechanical excitations are of the stationary random type and the stationary dynamic response of the coupled system is investigated. Let $[T_1(\omega)] = [Z_1(\omega)]^{-1}$ be the complex symmetric frequency response matrix ($m \times m$). The spectral density function $\omega \mapsto [S_{X_1}(\omega)]$, $\omega \mapsto [S_{X_1}(\omega)]$ from \mathbb{R} into $\text{Mat}_{\mathbb{C}}(m, m)$ is then given by the equation:

$$[S_{X_1}(\omega)] = [T_1(\omega)][S_{F_1}(\omega)][T_1(\omega)]^*, \quad (69)$$

where $[S_{F_1}(\omega)]$ is the matrix spectral density of the stationary excitation process. Direct computation of (69) over a broad MF band has much higher numerical costs than in the deterministic case for reasons explained at the beginning of Section IV,4 and also because with this form, it is necessary to completely invert the impedance matrix (56) for each value of ω , then to calculate the product of three complex ($m \times m$) matrices. It is therefore highly desirable to have a method that gets around this difficulty. This is what we have done [40, 42] by reducing the initial problem to solving a small number of deterministic MF problems using the MF method of Section IV,4.1.

IV,4.3. – Taking Mechanical Subsystems Identified by Their Boundary Impedance into Account

It is often useful to be able to introduce in a finite element model a subsystem described by a boundary impedance resulting from measurements or already given by the numerical model. In the general case, a boundary impedance cannot be interpreted as the impedance of a dynamic system governed by a second-order differential equation with constant coefficients due to the elimination of internal degrees of freedom of the subsystem considered to the benefit of its boundary degrees of freedom. It is obviously always possible to couple such a boundary impedance with the impedance of the finite element model in the frequency domain, but it is then necessary to calculate the solution for each frequency and the MF method

described in Section IV,4.1 giving acceptable numerical costs can no longer be used. To overcome this difficulty, we therefore developed [7] a method that allows the MF method to be used even in this case, by introducing a representation by differential equations and hidden variables.

IV,4.4. – Taking Random Operators into Account

This situation is encountered, for instance, when modeling the structural complexity as was indicated in Section III,6.4. For any ω belonging to a wide MF frequency band, it is then necessary to solve the following matrix equation associated with (54):

$$([Z_1(\omega)] + [Z_a(\omega)]) X_1 = F_1, \quad (70)$$

where $[Z_1(\omega)]$ is a complex deterministic matrix and $[Z_a(\omega)]$ is a random matrix. Using a method [39, 8] based on the Neumann series expansion of the random operator, we were able to simplify the numerical solution of this problem to solving a limited number of deterministic MF problems by the numerical MF method described in Section IV,4.1.

IV,5. – FREQUENCY COUPLING METHOD IN THE MF DOMAIN

For many reasons, generally related to construction of the model and use of a large model, a dynamic system is often considered as the assembly of N dynamic subsystems (substructuring). In addition, to study dynamic systems in the MF domain, the Frequency Response Function (FRF) of each subsystem can be determined by a model or by experimental identification. With this FRF then available for a wide MF frequency band for each subsystem, it is then attempted to predict the dynamic behavior, in the frequency domain, of the system resulting from the assembly of the N subsystems. The problem thus stated is not as simple as it looks for the following main reasons: (1) if the FRF is given by measurements, noise is present; (2) at a given frequency, the matrix FRF on the boundary of a subsystem is not necessarily invertible for mechanical reasons or due to the presence of noise, which prevents calculating the boundary impedance at this frequency and therefore conventionally assembling this subsystem; (3) assuming that it was possible to assemble all the subsystems at a given frequency, the matrix may very well not be invertible; (4) in presence of noise, the approach can no longer remain deterministic, and we

then want to predict the upper and lower envelope of the frequency responses of the assembled system, associated with a mean frequency behavior of this assembled system and a confidence interval; (5) even assuming there is no noise on each FRF, the complexity of certain models and the model uncertainties mean that average information is systematically sought with the associated envelopes, as under point (4) above.

We developed [18] a direct frequency coupling method and a mean frequency coupling method corresponding to the deterministic approach and to the statistical approach respectively, as was just mentioned. Since the second method is based on the first one and considering the importance of this type of approach, we describe the direct frequency coupling method that we developed and that overcomes all the numerical problems of inversion, while continuing to privilege global assembly to the utmost using the state variables of each subsystem.

IV,5.1. – *Givens*

Our givens are N subsystems (SS) denoted SS_I , $I \in \{1, \dots, N\}$. Each SS_I is described by an FRF $\omega \mapsto [H^I(\omega)]$, defined on a wide frequency band with values in $\text{Mat}_{\mathbb{C}}(n_I + m_I, n_I + m_I)$, accepting splitting by blocks:

$$\begin{bmatrix} V_1^I(\omega) \\ V_2^I(\omega) \end{bmatrix} = \begin{bmatrix} H_{11}^I(\omega) & H_{12}^I(\omega) \\ H_{21}^I(\omega) & H_{22}^I(\omega) \end{bmatrix} \begin{bmatrix} F_1^I(\omega) \\ F_2^I(\omega) \end{bmatrix}, \quad (71)$$

where $V_1^I(\omega) \in \mathbb{C}^{n_I}$ are the n_I internal degrees of freedom (DOF), $V_2^I(\omega) \in \mathbb{C}^{m_I}$ are the m_I boundary DOFs, $F_1^I(\omega) \in \mathbb{C}^{n_I}$ are the known excitation terms applied to the internal DOFs and $F_2^I(\omega) \in \mathbb{C}^{m_I}$ are the boundary terms which are either givens when the FRF of this subsystem SS_I , considered isolated from the other SSs, is identified, or unknowns which cannot be known until the assembled system has been solved, since it is coupled with the other SSs. Generally, the components of $V_1^I(\omega)$ represent displacements, velocities, accelerations or pressures, the components of $V_2^I(\omega)$ represent displacements, the components of $F_1^I(\omega)$ and $F_2^I(\omega)$ represent forces. Finally, by hypothesis, the complex matrix $[H^I(\omega)]$ is assumed to be symmetric.

The N subsystems are assembled by their boundary DOFs by introducing q master DOFs $U(\omega) = (U_1(\omega), \dots, U_q(\omega)) \in \mathbb{C}^q$. The assembly equation is introduced by the data of N matrices: $[P_I] \in \text{Mat}_{\mathbb{R}}(n_I, q)$, verifying the compatibility condition: $\sum_{I=1}^N [P_I]^T [P_I]$ is a positive-definite matrix, and

such that:

$$V_2^I(\omega) = [P_I] U(\omega). \quad (72)$$

The given excitations of the assembled system are forces $F_1^I(\omega) \in \mathbb{C}^{n_I}$, $I \in \{1, \dots, N\}$ and forces applied to the q master DOFs

$$Q(\omega) = (Q_1(\omega), \dots, Q_q(\omega)) \in \mathbb{C}^q.$$

IV,5.2. – *Subsystem Coupling Method*

Subsystem SS_I is said to be *regular* at fixed frequency ω if matrix $[H_{22}^I(\omega)]$ is invertible; otherwise, it is said to be *irregular*. By reordering the indexing at frequency ω , we denote the indices of the regular SSs as $I \in \{1, \dots, N_0(\omega)\}$ and the indices of the irregular SSs as $J \in \{N_0(\omega) + 1, \dots, N\}$. The number $N_0(\omega)$ depends on ω . It can be zero (all SSs are irregular) or equal to N (all SSs are regular). To test the numerical invertibility of matrix $[H_{22}^I(\omega)]$, we condition it and then we use a Gauss elimination algorithm with double pivoting (rows and columns).

(a) *Solving of Regular Subsystems.* For each regular subsystem SS_I , $I \in \{1, \dots, N_0(\omega)\}$, we calculate the boundary impedance matrix:

$$[D^I(\omega)] = [H_{22}^I(\omega)]^{-1} \in \text{Mat}_{\mathbb{C}}(m_I, m_I), \quad (73)$$

and the boundary forces due to the forces applied to the internal DOFs:

$$E^I(\omega) = [D^I(\omega)] [H_{21}^I(\omega)] F_1^I(\omega) \in \mathbb{C}^{m_I}. \quad (74)$$

(b) *Solving of Irregular Subsystems.* Relative to all the irregular subsystems SS_J , $J \in \{N_0(\omega) + 1, \dots, N\}$, we introduce the vector $F_2(\omega) \in \mathbb{C}^{r(\omega)}$ of unknown boundary forces $F_2^J(\omega)$, $J \in \{N_0(\omega) + 1, \dots, N\}$, such that:

$$F_2(\omega) = \begin{bmatrix} F_2^{N_0(\omega)+1}(\omega) \\ \vdots \\ F_2^N(\omega) \end{bmatrix}. \quad (75)$$

Then, for any

$$J \in \{N_0(\omega) + 1, \dots, N\},$$

we can construct matrices $[\Pi_J] \in \text{Mat}_{\mathbb{R}}(n_J, r(\omega))$ such that:

$$F_2^J(\omega) = [\Pi_J] F_2(\omega). \quad (76)$$

(c) *Assembly of the N Subsystems.*

It is then shown [18] that at frequency ω , the

unknown vectors $U(\omega)$ and $F_2(\omega)$ are the solution of the following matrix equation:

$$[\mathbf{A}(\omega)]\mathbf{X}(\omega) = \mathbf{R}(\omega)$$

expressed by blocks:

$$\begin{bmatrix} Z_{UU}(\omega) & Z_{UF}(\omega) \\ Z_{UF}^T(\omega) & Z_{FF}(\omega) \end{bmatrix} \begin{bmatrix} U(\omega) \\ F_2(\omega) \end{bmatrix} = \begin{bmatrix} \mathbb{G}_U(\omega) \\ \mathbb{G}_F(\omega) \end{bmatrix}, \quad (78)$$

where:

$$\begin{aligned} [Z_{UU}(\omega)] &= \sum_{I=1}^{N_0(\omega)} [P_I]^T [D^I(\omega)] [P_I], \\ [Z_{FF}(\omega)] &= - \sum_{J=N_0(\omega)+1}^N [\Pi_J]^T [H_{22}^J(\omega)] [\Pi_J], \\ [Z_{UF}(\omega)] &= \sum_{J=N_0(\omega)+1}^N [P_J]^T [\Pi_J], \\ \mathbb{G}_U(\omega) &= \mathbb{Q}(\omega) + \sum_{I=1}^{N_0(\omega)} [P_I]^T E^I(\omega), \\ \mathbb{G}_F(\omega) &= \sum_{J=N_0(\omega)+1}^N [\Pi_J]^T [H_{21}^J(\omega)] [F_1^J(\omega)]. \end{aligned}$$

(d) *Solution.* To solve linear system (78) or (77) in condensed form at each frequency ω , we begin by numerically calculating the rank $k(\omega)$ of the complex symmetric matrix $[\mathbf{A}(\omega)] \in \text{Mat}_{\mathbb{C}}(r(\omega)+q, r(\omega)+q)$ using an algorithm based on the Gauss elimination with double pivoting and applied to the conditioned matrix. If $k(\omega) = r(\omega) + q$, then (77) has a unique solution, constructed by solving the linear system by Gauss elimination with double pivoting. If $k(\omega) < r(\omega) + q$, it is checked whether $\mathbf{R}(\omega)$ belongs to the image of the linear operator $\mathbf{A}(\omega)$ whose matrix is $[\mathbf{A}(\omega)]$ on the canonical basis. If it doesn't, there is no solution and the system of external forces applied is not compatible with the connections, which should not occur if the problem is correctly stated. If it does, there are solutions. This situation corresponds to the case where there is a surplus of connections between subsystems for this frequency. We then construct the solution which is in $(\text{Ker } \mathbf{A}(\omega))^\perp$. This amounts to automatically releasing the surplus links at this frequency ω .

V. – SOFTWARE AND APPLICATIONS

V.1. – VALIDATION OF THE METHODS AND SOFTWARE

We developed software codes based on the methods described in Sections III and IV, to make computa-

tions on a variety of models. The central codes are SYDYLI [14] for the mean frequency coupling of dynamic subsystems described by frequency response functions and ADINA-ONERA [15] for hydroelastoacoustic analyses of coupled structure-external fluid-internal fluids systems, covering the axisymmetric and 3D cases and allowing analysis in the LF and MF domains. These codes can be implemented on computers of intermediate power and were optimized for CRAY XMP and CRAY 2 vector computers.

The methods and software were validated as they were developed by comparing them with explicit solutions of model mechanical problems and by comparing them experimentally with more representative mechanical systems for which there is no explicit solution. For instance, we used [25, 26, 31, 33] as measurement data base, and there are many validations of the methods and software in the literature [1, 6, 7, 8, 12, 16, 18, 19, 34, 37, 40, 41, 42].

V.2. – APPLICATIONS

We describe two applications with experimental comparisons. One is relative to the LF domain and the other to the MF domain.

V.2.1. – Application in the LF Domain

The mechanical system is a 3D inhomogeneous structure made of composite materials (the geometric form is illustrated in Figure 2) submerged in a dense fluid (water) occupying a domain of space that can be considered unbounded. One of the objectives is to calculate the values of the first LF resonance frequencies of the structure in the fluid and the associated far-field radiation. For this purpose, we used the C2 3D LF hydroelastoacoustic software system [17] based on the formulation described in Sections III,2 to III,5 and the numerical methods described in Sections IV,1 to IV,3. The hydrodynamic operators for coupling with the external fluid and far-field radiation are calculated by the general integral equation method and made numerical by the finite element method (*see* Secs. IV,1.1 (a) and IV,1.2 (a)). Meshing of surface \sum_E by P_0 and Q_0 finite elements is illustrated in Figure 2.

The mesh includes 6,400 elements. The structural model constructed by the finite element method leads to a model with 21,000 DOFs. The measurements used to make the comparisons with calculations were taken from [31].

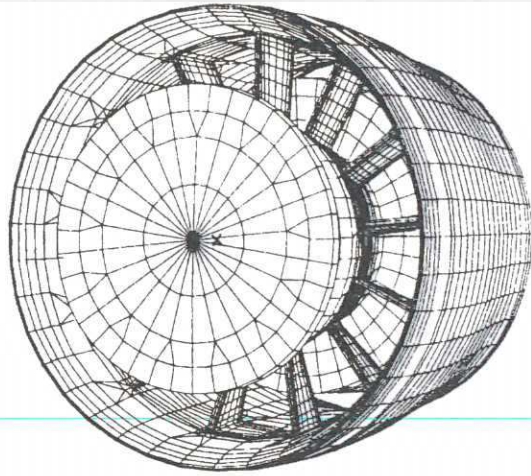


Fig. 2. — Mesh for hydrodynamic calculations.

Table I illustrates the numerical results obtained and shows the computation/measurement comparison expressed as a percentage on the values of the first resonance frequencies of the structure in water and the associated values of the pressures radiated in the far field. The LF hydroelastoacoustic analysis was made in 100 frequency points based on a representation of the structure by its first 20 vibration modes (modes of the associated conservative structure and in a vacuum) on a CRAY 2. The complete analysis required a total of five CPU hours and a memory space of 128 Mwords to 192 Mwords depending on the stage of the analysis.

TABLE I

Computation/measurement comparison of the resonance frequencies of the coupled system and its far-field radiation

$100 \times \frac{(\text{Measurement}-\text{Computation})}{\text{Measurement}} \%$		
Resonance types	Frequency (Hz)	Far-field radiation (in dB)
First bending	1.2 %	-2.0 %
Second bending	1.1 %	2.0 %
Torsion	-2.3 %	2.4 %
First ovalization	-5.1 %	-0.9 %

V.2.2. — Application in the MF Domain

The mechanical system is a submerged submarine for which the dynamic response and far-field radiation are to be calculated for a medium frequency band $ka \in [2., 9]$ (reduced frequency) and for excitations applied to certain of its internal structures. This analysis was made in three phases. Phase 1 concerned development of a model of the submarine by sub-

systems. Phase 2 involved calculation of the frequency response functions (FRF) of the different subsystems (considered isolated). Phase 3 consisted of frequency coupling of the subsystems in the MF domain for hydroelastoacoustic analysis of the complete submarine.

Phase 1: Model Development

Mechanically, the system considered was initially divided into 19 subsystems: the *pressure hull* subsystem SS_c and 18 subsystems SS_i consisting of the internal submarine structures. Subsystem SS_c is a structure consisting of the plating (cylindrical shell with a variable section), all the frames (circumferential stiffeners), the internal bulkheads and the unbounded external fluid coupled to it. As this three-dimensional (3D) subsystem SS_c is axisymmetric from a geometric and a mechanical standpoint, it is subdivided into 24 subsystems corresponding to the symmetric parts $SS_{s,n}$ and the antisymmetric parts $SS_{as,n}$ of the 12 circumferential indices $n=0, 1, 2, \dots, 11$ of the Fourier series expansion with respect to the polar angle θ of the cylindrical coordinates (r, θ, z) , where the z coordinate coincides with the axis of revolution. Subsystem SS_c is therefore synthesized in 3D from the 24 axisymmetric subsystems:

$$SS_c = \bigcup_{n=0}^{11} \{SS_{s,n} \oplus SS_{as,n}\},$$

since the 18 internal structures SS_i are purely 3D, they can only be coupled in 3D. For this type of approach, see [41, 42] for instance.

(a) Modeling and Condensing of a Subsystem $SS_{s,n}$ or $SS_{as,n}$. For each fixed circumferential index n , only the generatrix plane is meshed by axisymmetric finite elements (FEs). For the generatrix plane alone, the model includes 4,500 structural DOFs (axisymmetric hull FEs) and 200 axisymmetric FEs for calculating the radiation and external fluid coupling operators. To use the frequency coupling method described in Section IV,5, each subsystem of type $SS_{s,n}$ or $SS_{as,n}$ is condensed with 64 axisymmetric boundary DOFs and 16 internal pressure DOFs in the external fluid, also axisymmetric.

(b) Condensing of Synthesized 3D Subsystem SS_c . As explained above, subsystem SS_c is 3-D and is obtained by 3-D synthesis of the axisymmetric subsystems. The condensed synthesized model SS_c has 400 3-D boundary DOFs and 90 internal 3-D pressure DOFs in the external fluid.

(c) Modelling and Condensing of Each Subsystem SS_i . Among the 18 subsystems SS_i , 16 subsystems $SS_{ij}, j \in \{1, \dots, 16\}$ model tubes and are processed

by an analytic elastodynamic circular cylindrical thin shell theory. Each subsystem SS_{ij} is condensed with 20 3-D boundary DOFs for coupling with the synthesized 3-D proof hull SS_c . The last two internal structures, SS_{iv} and SS_{ib} , which are 3-D and have an arbitrary geometry, are modeled by the finite element method. The model of the first structure has 4,000 DOFs and is condensed with 16 boundary DOFs for coupling with synthesized 3-D pressure hull SS_c . The second has 850 DOFs and is condensed with 60 boundary DOFs for coupling. Both have three internal DOFs.

Phase 2: Calculation of the FRF of Each Subsystem

The frequency resolution chosen for calculation of all the FRFs corresponds to a division into 1,000 reduced frequency points of the band $ka \in [2., 9.]$ which is processed by the MF method of Section IV,4 as four narrow bands.

Calculation of the FRFs for each subsystem $SS_{s,n}$ and $SS_{as,n}$ $n=0, 1, 2, \dots, 11$ was carried out with the C5 axisymmetric MF hydroelastoacoustic software system [17] in which the main code is ADINA-ONERA and which is based on the formulation

described in Sections III,2, III,3 and III,6 and on the numerical methods described in Sections IV,1, IV,2 and IV,4.

The hydrodynamic external fluid coupling and far-field radiation operators are calculated with the asymptotic formulation adapted to slender geometries and analyzed numerically by the finite element method (see Secs. IV,1 (b) and IV,1.2 (b)).

Calculation of the FRFs for each of subsystems SS_{iv} and SS_{ib} which are 3-D structures without coupled fluid must be carried out with the ADINA-ONERA software based on the formulation described in Section III,6 and the numerical MF method described in Sections IV,2 and IV,4.

Finally, the FRF calculations for subsystems SS_{ij} , $j \in \{1, \dots, 16\}$ were carried out using specially developed code based on direct numerical analysis of an analytic theory using an explicitly known modal representation.

Phase 3: Frequency Coupling of the Subsystems Described by their FRF

Frequency coupling of the above subsystems in the MF domain to make the hydroelastoacoustic analysis of the submarine was carried out using the SYDYLI code [14] based on the method described in Section IV,5. To decrease the very large volume of the data base, frequency coupling was carried out only for one frequency point out of five, *i.e.* for 200 reduced frequency points in the band $ka \in [2., 9.]$. This third phase of the analysis was actually carried out in two steps. First, we made a 3-D synthesis of subsystem SS_c , then we coupled SS_c with all the internal structures: SS_{iv} , SS_{ij} and SS_{ib} , $j \in \{1, \dots, 16\}$.

Results Obtained and Comparisons with Experiments

The entire analysis required 20 CPU hours on a CRAY 2 and 16 CPU hours on a CRAY XMP 416. The results obtained [15] are illustrated in Figures 3 and 4, showing the comparisons between measurements made by the CERDAN of the DCN with the predictions of the model. Figure 3 shows the far-field radiated pressure level in a fixed direction of space for a localized excitation on an internal DOF of subsystem SS_{ib} . Also for a localized excitation on an internal DOF of subsystem SS_{iv} , Figure 4 shows the far-field radiated pressure level on a fixed direction of space for the experimental test and in two neighboring directions surrounding the direction of the test for the numerical model. A relatively marked directivity effect can be seen.

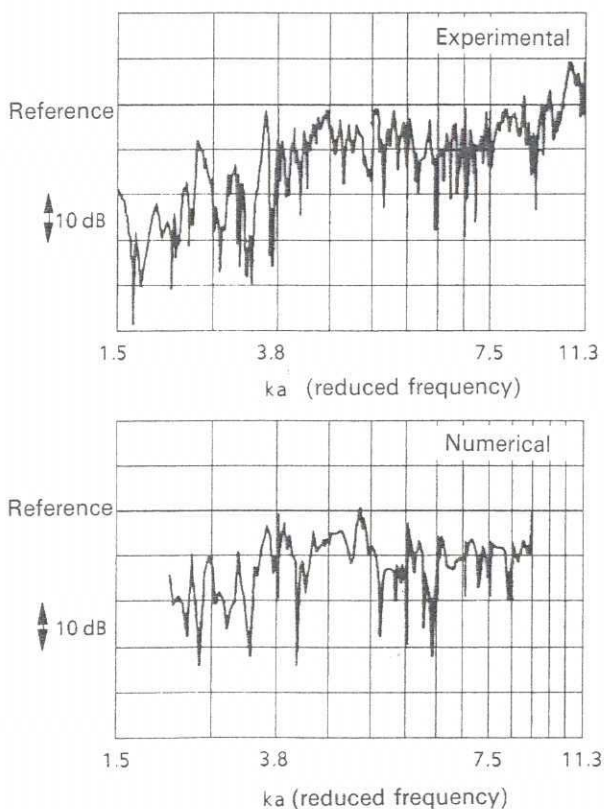


Fig. 3. — Experimental test/numerical model comparison on the pressure level radiated for excitation on SS_{ib} .

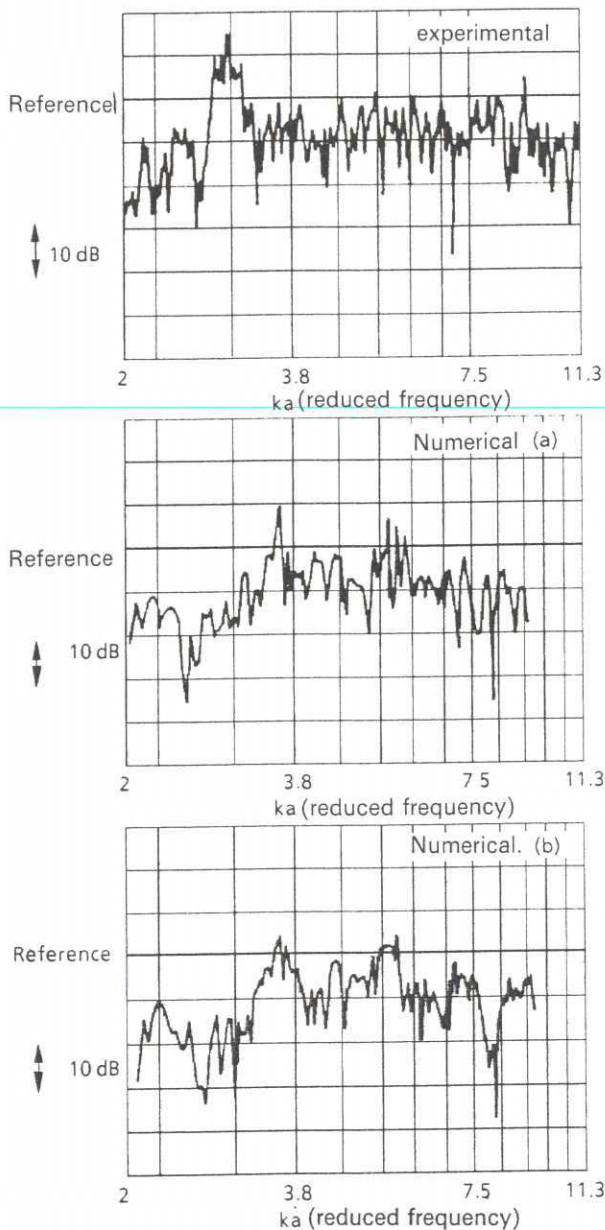


Fig. 4. — Comparison between experimental test and numerical model (a) and (b) on the radiated pressure field for an excitation on SSiv.

VI. — CONCLUSIONS

We described the bases of numerical methods used in codes with an industrial vocation and which are used to make predictions concerning the dynamics and acoustic radiation of structures in the LF and MF ranges. The current limitations of the numerical methods for coupled fluid-structure problems that we discussed are mainly due to the structures and are amplified by the presence of fluids when dense. For

instance, the HF domain cannot be analyzed by the numerical methods described. The limitations are due to several factors, the main of which are as follows in our opinion.

(1) The first factor is not related to the numerical methods themselves but to the level of knowledge concerning structural mechanics: (a) dynamics and model of the materials, certain structural elements such as composites and finally boundary conditions; (b) problem of modeling the fuzzy structure which arises except in the LF domain and more generally, the representativity of the models.

(2) The second is related to the problems in implementing the numerical models, mainly for construction of finite element meshes and supply with mechanical data for structures. These difficulties are substantially increased in the MF domain with respect to the LF domain and are one of the reasons of the current failure in the HF domain.

(3) Finally, the last factor is directly related to computer facilities, power, memory space and flexibility of use of scientific computers. The mathematical formulations and numerical methods developed to best solve the problems raised depend on the computer facilities available at a given time. Considering how fast computers are evolving, the methods should, and it seems to us, be constantly improved. This is true for instance of (a) numerical methods for solving the external Neumann problem related to the Helmholtz equation, based on integral equation formulations, as regards management of memory space and numerical costs of construction and solving with a priori dense matrices; (b) general problems of the finite element method for structures, for which iterative solving methods without actual assembly can be considered.

Finally, to conclude, seems to us that now that numerical methods are available for LF and MF elastoacoustics, it would be desirable to develop uncertainty models on the data to improve the reliability of the predictions, introduce optimization tools, develop model readjustment methods not only in the LF domain but especially in the MF domain and finally, begin to reconsider all the methods to take into account the probable evolution toward massively parallel computers.

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