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# HYBRID NUMERICAL METHOD FOR SOLVING THE HARMONIC MAXWELL EQUATIONS: II. CONSTRUCTION OF THE NUMERICAL APPROXIMATIONS

by

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## ABSTRACT

Part II of this paper deals with the numerical aspects of the mathematical formulation, presented in Part I, for the solution of Maxwell's harmonic equations. Linear isoparametric finite elements are used for external integral and internal differential operators. Three-node and four-node finite elements are used for the surface and volume parts, respectively. Robust numerical approximations were constructed for the integral operators.

*Keywords (NASA thesaurus): Electromagnetic wave – Scatter propagation – Numerical method – Finite element method – Integral equations.*

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## I. – INTRODUCTION

The present second part of our three-part paper is devoted to the numerical approximations of the various operators used in the formulation developed in the first Part, for solving Maxwell's harmonic equations. All of the notation remains the same. Details can be found in [1].

## II. – DISCRETIZATION OPTIONS

For the reasons explained in section II, 2 of Part I, it is important to minimize the numerical costs of constructing elementary matrices of the various finite elements used in the formulation. So we chose finite elements of the lowest possible degree that are compatible with the minimum required regularity, namely isoparametric finite elements  $P_1$ , which are triangular for the surface elements and tetrahedrons for the volume elements, for the integral operators and for the differential surface and volume operators.

For the integral operators, we have constructed numerical approximations that are good for any geometrical configuration.

## III. – ADAPTATION OF THE INTEGRAL FORMULATION TO A NUMERICAL PROCESS

The integral formulation operators for the external problem, defined by formulae (61) of Part I, are adapted by the finite element method.

### III.1. – FINITE ELEMENTS ON THE EXTERNAL SURFACE

#### III.1.1. – Grid Generation for the External Surface

The surface  $\partial\Omega_i = \Gamma_c \cup \Gamma_d$  is meshed with triangles. These must, of course, comply with the usual compatibility conditions for finite elements, which we will assume to hold from now on. We will also assume that the polyhedral surface defined by the grid is the true surface  $\partial\Omega_i$  and will use the same notation for each, to simplify.

#### III.1.2. – Triangular Finite Element $P_1$ and Approximation Space

We cannot use  $P_0$  elements because they result in an approximation space that does not have the regularity required for the integral operators of the problem at hand.  $P_1$  elements will do, however, and the approximation space for all operators on  $\partial\Omega_i$  will be the space of affine functions by triangle, and continuous on the surface  $\partial\Omega_i$ .

We will use  $T$  to denote an arbitrary finite element of the grid  $\partial\Omega_i = \Gamma_c \cup \Gamma_d$ . This is a triangle with apexes  $M_1, M_2$  and  $M_3$ . The coordinates of the node  $M_j$  will be denoted  $(x_j, y_j, z_j)$  in the cartesian frame of reference introduced in section III,1 of Part I. The notation  $T$  will designate the element, but also the surface of  $\mathbb{R}^3$  defined by the triangle  $(M_1, M_2, M_3)$ . To simplify, we will use  $M_j$  to denote the vector  $OM_j$  of  $\mathbb{R}^3$  of components  $(x_j, y_j, z_j)$ . The area of the surface element  $T$  (area of surface  $T$ ) is:

$$|T| = \int_T ds = \frac{1}{2} \|(M_2 - M_1) \times (M_3 - M_1)\| \quad (1)$$

in which  $\|V\|$  designates the euclidian norm of  $V \in \mathbb{R}^3$ .

#### III.1.3. – Element Size

The fixed wave number  $k$  is associated with the wavelength  $\Lambda = \frac{2\pi}{k}$ . By hypothesis, the grid of  $\partial\Omega_i$  is generated such that  $\forall T, \sqrt{|T|} \ll \Lambda$ . This hypothesis will be of constant use in developing the numerical process.

#### III.1.4. – Normal and Orientation

Let  $n_T$  be the unit normal to the planar surface  $T$  such that:

$$\begin{aligned} n_T &= \frac{(M_2 - M_1) \times (M_3 - M_1)}{\|(M_2 - M_1) \times (M_3 - M_1)\|} \\ &= \frac{1}{2|T|} (M_2 - M_1) \times (M_3 - M_1) \quad (2) \end{aligned}$$

We define the element  $W_T = n_T |T|$  of  $\mathbb{R}^3$  which, considering (1), is written:

$$W_T = n_T |T| = \frac{1}{2} (M_2 - M_1) \times (M_3 - M_1) \in \mathbb{R}^3 \quad (3)$$

The surface  $\partial\Omega_i$  is oriented so that the unit normal is directed outward at  $\partial\Omega_i$ . We assume from here on that the triplet  $M_1, M_2, M_3$  is ordered such that the normal  $n_T$  defined by (2) is outward at  $\partial\Omega_i$ . This ordered triplet  $M_1, M_2, M_3$  defines the incidence of the element  $T$  and we deduce from this the sense of

the curvilinear abscissa on the closest edge  $\partial T$  of  $T$ :

$$\partial T = M_1 M_2 + M_2 M_3 + M_3 M_1 \quad (4)$$

### III.2. - $P_1$ INTERPOLATION FORMULAE

We will use  $(x, y, z)$  from now on to designate the cartesian coordinates of a given point  $M$  on  $T$ , where we will use the barycentric coordinates  $\lambda_1, \lambda_2, \lambda_3$  for parametering the surface:

$$M = \lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3 \in T \quad (5-1)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \quad 0 \leq \lambda_i \leq 1, \quad \forall i \in \{1, 2, 3\} \quad (5-2)$$

Let  $A_T$  be a function on  $T$  with values in  $\mathbb{K}^3$ , and  $A_T^1, A_T^2, A_T^3$  the values of  $A_T$  in  $M_1, M_2, M_3$ :

$$A_T^i = A_T(M_i) \in \mathbb{K}^3, \quad i \in \{1, 2, 3\} \quad (6)$$

We then have:

$$A_T = \lambda_1 A_T^1 + \lambda_2 A_T^2 + \lambda_3 A_T^3 \quad (7)$$

We will use the same conventions for expressing a function  $\varphi_T$  continuous on  $T$  with values in  $\mathbb{K}^3$ .

#### III.2.1. - Integration Formula

If  $\alpha_1, \alpha_2, \alpha_3$  are three positive or zero integers, we have the formula:

$$\int_T \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3} ds = 2 |T| \frac{\alpha_1! \alpha_2! \alpha_3!}{(\alpha_1 + \alpha_2 + \alpha_3 + 2)!} \quad (8)$$

#### III.2.2. - Barycentric Averages

We will use  $\underline{M}_T$  to denote the point  $T$  that is the barycenter of the triangle  $M_1, M_2, M_3$ :

$$\underline{M}_T = \frac{1}{|T|} \int_T M ds = \frac{1}{3} (M_1 + M_2 + M_3) \quad (9)$$

and we will use  $\underline{A}_T \in \mathbb{K}^3$  to denote the barycentric average of  $A_T$  on  $T$ :

$$\underline{A}_T = \frac{1}{|T|} \int_T A_T ds = \frac{1}{3} (A_T^1 + A_T^2 + A_T^3) \quad (10)$$

We then get the following relations:

$$\int_T A_T ds = |T| \underline{A}_T \quad (11-1)$$

$$\int_T A_T \times n ds = -W_T^0 \underline{A}_T, \quad W_T^0 = (W_T \times) \quad (11-2)$$

$$\int_T A_T \cdot n ds = \underline{A}_T \cdot W_T \quad (11-3)$$

#### III.2.3. - Calculation of the Integral over $T$ of Differential Form $d(A_T dM)$

Considering the orientation of the edge  $\partial T$ , we get from Stokes' theorem:

$$\int_T d(A_T \cdot dM) = A_T^1 \cdot V_T^1 + A_T^2 \cdot V_T^2 + A_T^3 \cdot V_T^3 \quad (12)$$

with  $V_T^i \in \mathbb{R}^3$  for  $i \in \{1, 2, 3\}$ , and:

$$V_T^1 = \frac{1}{2} (M_2 - M_3) \quad (13-1)$$

$$V_T^2 = \frac{1}{2} (M_3 - M_1) \quad (13-2)$$

$$V_T^3 = \frac{1}{2} (M_1 - M_2) \quad (13-3)$$

### III.3. - FORMULAE FOR THE KERNEL OF THE EXTERNAL PROBLEM

To state the expressions for the integral operators explicitly in numerical form, we will need information concerning the gradient of the kernel  $N(r)$  defined by (34) in Part I, the solid angle, the normal derivative of the kernel and lastly its average value.

#### III.3.1. - Kernel Gradient

Once all of the calculations are made, this is expressed:

$$\text{grad}' N = g(r) \frac{1}{r^3} (M' - M) \quad (14-1)$$

$$g(r) = -(1 + jkr) e^{-jkr} \quad (14-2)$$

#### III.3.2. - Solid Angle

Let  $M$  be a fixed point of  $\mathbb{R}^3$ . And let  $\omega'_M$  be the solid-angle differential form, which is of class  $C^\infty$  and is closed in the complement of  $M$ :  $d' \omega'_M = 0$ , expressed [17]:

$$\omega'_M = \frac{1}{r^3} ((x' - x) dy' \wedge dz' + (y' - y) dz' \wedge dx' + (z' - z) dx' \wedge dy') \quad (15-1)$$

or, using the definitions of section III.4.2 in Part I:

$$\omega'_M = \frac{1}{r^3} (M' - M) \cdot n' ds' \quad (15-2)$$

Moreover, we have:

$$\int_{\partial \Omega_i} \omega'_M = \begin{cases} 4\pi & \text{if } M \in \Omega_i \\ 0 & \text{if } M \in \Omega_c = \mathbb{R}^3 \setminus \Omega_i \end{cases} \quad (15-3)$$

After the computations, we get the following formulae that can be used directly in the numerical process.

For  $T \neq T'$ :

$$\Omega_{TT'} = \int_{T'} \omega'_{M_T} = Sg \{ (M_{T'} - M_T) \cdot n_{T'} \} \times \left( 2\pi - \sum_{i=1}^3 \cos^{-1}(\alpha'_i \cdot \alpha'_{i+1}) \right) \quad (16-1)$$

$$\alpha'_i = \frac{\rho'_i \times \rho'_{i+1}}{\|\rho'_i \times \rho'_{i+1}\|} \quad \text{if } \|\rho'_i \times \rho'_{i+1}\| \neq 0 \quad \left. \vphantom{\alpha'_i} \right\} (16-2)$$

$$\text{et } \alpha'_i = 0 \quad \text{if } \|\rho'_i \times \rho'_{i+1}\| = 0$$

$$\rho'_i = M'_i - M_T \in \mathbb{R}^3$$

$$Sg(a) = \frac{a}{|a|} \quad \text{for } a \in \mathbb{R}^* \quad \left. \vphantom{Sg(a)} \right\} (16-3)$$

For  $T = T'$ :

$$\Omega_{T'T} = \int_{T'} \omega'_{M_T} = 0 \quad (16-4)$$

### III.3.3. — Integral of the Normal Derivative of the Kernel

Using (14) and (15-2), we get

$$\frac{d' N}{dn'} ds' = g(r) \omega'_{M_T} \quad (17)$$

and deduce the following approximation from it using (16). For any fixed  $T$  and any fixed  $T'$  (with  $T \neq T'$  or  $T = T'$ ) and for any fixed  $M = M_T \in T$ :

$$\int_{T'} \frac{d' N}{dn'} ds' = \int_{T'} g(M_T, M') \omega'_{M_T} \simeq g(M_T, M_{T'}) \Omega'_{TT'} \quad (18-1)$$

$$g(M_T, M_{T'}) = g(r(M_T, M_{T'})) \quad (18-2)$$

### III.3.4. — Average Value of the Kernel

For any  $T$  and  $T'$  with  $T \neq T'$  or  $T = T'$ , this average value is defined by:

$$\mathcal{N}_{TT'} = \frac{1}{|T||T'|} \int_T \int_{T'} N ds ds' \quad (19)$$

The approximation of (19) that we have constructed not only preserves the symmetry but can also be counted on when elements exist face to face at a distance between barycenters that is smaller than the characteristic dimension of the two elements. The

approximation is written:

$$\mathcal{N}_{TT'} \simeq \mathbb{N}_{TT'} + \frac{1}{2} (J_{TT'} + J_{T'T}) \quad (20-1)$$

$$\mathbb{N}_{TT'} \simeq -jk \quad (20-2)$$

$$\mathbb{N}_{TT'} = \frac{e^{-jk r(M_T, M_{T'})} - 1}{r(M_T, M_{T'})} \quad (20-3)$$

$$J_{TT'} = \frac{1}{|T|} \int_T \frac{ds}{r}, \quad r = \|M_{T'} - M\| \quad (20-4)$$

For  $T \neq T'$ , the integral of (20-4) is computed numerically by a Gauss method. For  $T = T'$ , we get an expression for (20-4) that is expressed using the log function (see [1]).

### III.4. — $\tilde{P}$ OPERATOR

This operator is defined by (61-1) of Part I as:

$$\langle \tilde{P} A, \varphi \rangle = -\frac{jk}{2} \int_{\partial\Omega_i} \varphi \cdot (n \times A) ds$$

So what we want is to compute, for an arbitrary finite element  $T$  of  $\partial\Omega_i$ :

$$\langle \tilde{P}_{TT} A_T, \varphi_T \rangle = -\frac{jk}{2} \int_T \varphi_T \cdot (n \times A_T) ds \quad (21)$$

Using (3) and (8) we get:

$$\langle \tilde{P}_{TT} A_T, \varphi_T \rangle = \sum_{i=1}^3 F_T^i(A_T) \cdot \varphi_T^i \quad (22-1)$$

$$F_T^i(A_T) = -\frac{jk}{24} \sum_{l=1}^3 (1 + \delta_{il}) W_T^0 A_T^l \in \mathbb{C}^3 \quad (22-2)$$

$$W_T^0 = (W_T \times) \quad (22-3)$$

with  $\delta_{il} = 0$  if  $i \neq l$  and  $\delta_{il} = 1$  if  $i = l$ .

### III.5. — $\tilde{I}$ OPERATOR

The operators  $\tilde{I}_{cc}$  and  $\tilde{I}_{dd}$  defined by (61-2) and (61-3) of Part I are the restrictions of the operator  $\tilde{I}$  to  $\Gamma_c$  and  $\Gamma_d$ , such that:

$$\langle \tilde{I} A, \varphi \rangle = \int_{\partial\Omega_i} \varphi \cdot A ds$$

For a given finite element  $T$  of  $\partial\Omega_i$ , the object is therefore to compute

$$\langle \tilde{I}_{TT} A_T, \varphi_T \rangle = \int_T \varphi_T \cdot A_T ds \quad (23)$$

Using (8), we get:

$$\langle \tilde{I}_{TT} A_T, \varphi_T \rangle = \sum_{i=1}^3 F_T^i(A_T) \cdot \varphi_T^i \quad (24-1)$$

$$F_T^i(A_T) = \frac{|T|}{12} \sum_{i=1}^3 (1 + \delta_{ii}) A_T^i \in \mathbb{C}^3 \quad (24-2)$$

### III.6. - $\tilde{Q}$ OPERATOR

The  $\tilde{Q}$  operator is defined by (61-4) in Part I:

$$\langle \tilde{Q} A, \varphi \rangle = \frac{jk}{4\pi} \int_{\partial\Omega_i} ds \int_{\partial\Omega_i} (\varphi \times n) \cdot [(n' \times A') \times \text{grad}' N] ds' \quad (25)$$

Let  $T$  be the "receiver" finite element and  $\varphi_T$  the continuous function defined on  $T$  with values in  $\mathbb{R}^3$  such that:

$$\varphi_T = \lambda_1 \varphi_T^1 + \lambda_2 \varphi_T^2 + \lambda_3 \varphi_T^3, \quad \varphi_T^i = \varphi_T(M_i) \quad (25)$$

Let  $T'$  be the "transmitter" finite element and  $A_{T'}$  the continuous function defined on  $T'$  with values in  $\mathbb{C}^3$  such that

$$A_{T'} = \lambda_1 A_{T'}^1 + \lambda_2 A_{T'}^2 + \lambda_3 A_{T'}^3, \quad A_{T'}^i = A_{T'}(M_i) \quad (26)$$

What we compute is:

$$\langle \tilde{Q}_{TT'} A_{T'}, \varphi_T \rangle = \frac{jk}{4\pi} \int_T ds \int_{T'} (\varphi_T \times n_T) \cdot [(n_{T'} \times A_{T'}) \times \text{grad}' N] ds' \quad (27)$$

#### III, 6.1. - Expression of $\tilde{Q}_{TT}$

Using the properties of the mixed product and the relation (14-1), and considering the fact that  $(M' - M) \cdot n_T = 0$ , for  $\forall M$  and  $M' \in T$ , we get:

$$\langle \tilde{Q}_{TT} A_T, \varphi_T \rangle = 0 \quad (28)$$

#### III,6.2. - Expression of $\tilde{Q}_{TT'}$

For  $T \neq T'$ , the  $\tilde{Q}$  operator extracts the tangential part of  $\varphi$  and  $A$ . So we must preserve the expressions  $\varphi_T \times n_T$  and  $n_{T'} \times A_{T'}$  when adapting to the numerical process. Moreover, we know that the  $\tilde{Q}$  operator is algebraically symmetrical (see section VII, 4 in Part I). As this symmetry is hidden, it must be preserved when adapting to the numerical process. We will therefore have to write it explicitly. Expression (27) is not easy to adapt for numerical purposes because the expression to be integrated in the second member is not in differential form. Moreover, if  $T$  and  $T'$  are two elements that are face to face ( $n_{T'} = -n_T$ ) and the distance  $\|M_T - M_{T'}\|$  is small compared with  $\sqrt{T}$  or  $\sqrt{T'}$ , then  $r^{-1}$  is very large and the computation would be very poorly conditioned. So we must transform the second member of

(27) to bring out differential forms that will be easily adaptable to numerical computation, and in particular the solid angle differential form, to arrive at a numerical formula that is stable in all cases (*i.e.* the problem of face-to-face elements near each other as we have just mentioned). After a somewhat lengthy set of calculations that can be found in detail in [1], we get the new form we want for (27) for  $T \neq T'$ , which is written:

$$\langle \tilde{Q}_{TT'} A_{T'}, \varphi_T \rangle = \frac{jk}{8\pi} \int_T \int_{T'} \{ g(r) \omega'_M(U \cdot n_T) ds - g(r) \omega_M(U \cdot n_{T'}) ds' + (n_{T'} \times U) \cdot d'(N dM') ds - (n_T \times U) \cdot d(N dM) ds' \} \quad (29-1)$$

$$U = (\varphi_T \times n_T) \times (n_{T'} \times A_{T'}) \quad (29-2)$$

We can then construct the numerical approximation of (29) using the theorem of the average along the way, and we get [1]:

$$\langle \tilde{Q}_{TT'} A_{T'}, \varphi_T \rangle \simeq \sum_{i=1}^3 F_{TT'}^i(A_{T'}) \cdot \varphi_T^i \quad (30-1)$$

with  $i \in \{1, 2, 3\}$  such that:

$$F_{TT'}^i(A_{T'}) = \frac{jk}{24\pi} \{ g(M_T, M_{T'}) [\Omega_{T'T}(n_T^0)^2 W_T^0 - \Omega_{TT'} W_T^0 (n_T^0)^2] A_T + W_T^0 [(n_T^0 \cdot \nabla_{TT'}) \times (n_T^0 \cdot A_T)] - n_T^0 [(n_T^0 \cdot \nabla_{T'T}) \times (W_T^0 \cdot A_T)] \} \quad (30-2)$$

with  $g(M_T, M_{T'})$  defined by (18-2),  $W_T^0$  by (11-2),  $\Omega_{TT'}$  by (16) and

$$n_T^0 = (n_T \times) \quad (30-3)$$

$$(n_T^0)^2 = n_T \otimes n_T - I \quad (30-4)$$

$$\nabla_{T'T} = \sum_{i=1}^3 N(M_i, M_{T'}) V_T^i \quad (30-5)$$

in which  $V_T^i$  is defined by (13).

### III,7. - $\tilde{B}$ OPERATOR

The  $\tilde{B}$  operator is defined by (61-5) in Part I:

$$\langle \tilde{B} A, \varphi \rangle = - \frac{j}{4\pi} \int_{\partial\Omega_i} \int_{\partial\Omega_i} N d(\varphi \cdot dM) d'(A' \cdot dM')$$

Returning to the notation of (25) and (26), we compute:

$$\langle \tilde{B}_{TT'} A_{T'}, \varphi_T \rangle = - \frac{j}{4\pi} \int_T \int_{T'} N d(\varphi_T \cdot dM) d'(A_{T'} \cdot dM') \quad (31)$$

It is observed that, in the framework of the  $P_1$  elements,  $d(\varphi_T, dM)$  is a "constant" on  $T$ , as  $d'(A_{T'}^i, dM')$  is on  $T'$ . So we apply the theorem of the average, which is in this case exact:

$$\langle \tilde{B}_{TT'} A_{T'}, \varphi_T \rangle = \sum_{i=1}^3 F_{TT'}^i(A_{T'}) \cdot \varphi_T^i \quad (32-1)$$

with  $F_{TT'}^i(A_{T'}) \in \mathbb{C}^3$  for  $i \in \{1, 2, 3\}$  such that:

$$F_{TT'}^i(A_{T'}) = -\frac{j}{4\pi} \mathcal{N}_{TT'} V_T^i \left( \sum_{l=1}^3 A_{T'}^l \cdot V_{T'}^l \right) \quad (32-2)$$

in which  $\mathcal{N}_{TT'}$  is computed by (20) and  $V_T^i$  by (13).

### III,8. - $\tilde{S}$ OPERATOR

This  $\tilde{S}$  operator is defined by (61-6) of Part I:

$$\langle \tilde{S}A, \varphi \rangle = -\frac{jk^2}{4\pi} \int_{\partial\Omega_i} ds \int_{\partial\Omega_i} N ds' (\varphi \times n) \cdot (A' \times n')$$

Using the notation of (25) and (26), we compute:

$$\begin{aligned} \langle \tilde{S}_{TT'} A_{T'}, \varphi_T \rangle \\ = -\frac{jk^2}{4\pi} \int_T ds \int_{T'} N ds' (\varphi_T \times n_T) \cdot (A_{T'} \times n_{T'}) \end{aligned} \quad (33)$$

Using the theorem of the average and (11-2), we construct the following numerical approximation:

$$\langle \tilde{S}_{TT'} A_{T'}, \varphi_T \rangle \simeq \sum_{i=1}^3 F_{TT'}^i(A_{T'}) \cdot \varphi_T^i \quad (34-1)$$

with  $F_{TT'}^i(A_{T'}) \in \mathbb{C}^3$  for  $i \in \{1, 2, 3\}$  such that:

$$F_{TT'}^i(A_{T'}) = \frac{jk^2}{12\pi} \mathcal{N}_{TT'} W_T^0 W_{T'}^0 A_{T'} \quad (34-2)$$

in which the various terms of (34-2) are given by (10) (11-2) and (20).

### III,9. - OPERATORS FOR COMPUTING THE SCATTERED ELECTROMAGNETIC FIELD

Here we must adapt the operators  $R_1(M)$ ,  $R_2(M)$  and  $R_3(M)$  defined by (78) in Part I. Let  $M$  be a fixed-point in  $\Omega_e$ ,  $T'$  the "transmitter" finite element,  $A_{T'}$  (resp.  $B_{T'}$ ) the continuous function defined on  $T'$  with values in  $\mathbb{C}^3$  such that:

$$\left. \begin{aligned} A_{T'} &= \lambda_1 A_{T'}^1 + \lambda_2 A_{T'}^2 + \lambda_3 A_{T'}^3, \\ A_{T'}^i &= A_{T'}(M_i) \end{aligned} \right\} \quad (35-1)$$

$$\left. \begin{aligned} B_{T'} &= \lambda_1 B_{T'}^1 + \lambda_2 B_{T'}^2 + \lambda_3 B_{T'}^3, \\ B_{T'}^i &= B_{T'}(M_i) \end{aligned} \right\} \quad (35-2)$$

We then have:

$$R_{1,T'}(M) A_{T'} = \int_{T'} A_{T'}' \frac{d'N}{dn'} ds'$$

$$R_{2,T'}(M) A_{T'} = \int_{T'} A_{T'}' \times d'(N dM')$$

$$R_{3,T'}(M) B_{T'} = jk \int_{T'} N B_{T'}' ds'$$

Using the theorem of the average, (17) and (18), we get the following numerical approximation:

$$R_{1,T'}(M) A_{T'} \simeq g(M, M_{T'}) \Omega_{MT'} \underline{A}_{T'} \quad (36-1)$$

$$R_{2,T'}(M) A_{T'} \simeq \underline{A}_{T'} \times \mathbb{V}_{MT'} \quad (36-2)$$

$$R_{3,T'}(M) B_{T'} \simeq jk |T'| N(M, M_{T'}) \underline{B}_{T'} \quad (36-3)$$

with  $g(M, M_{T'})$  given by (18-2),  $\Omega_{MT'}$  by (16) by replacing the point  $M_T$  by the point  $M$ , and:

$$\mathbb{V}_{MT'} = \sum_{i=1}^3 N(M_i', M) V_{T'}^i \quad (36-4)$$

with  $V_{T'}^i$  given by (13) and  $\underline{A}_{T'}$  and  $\underline{B}_{T'}$  by:

$$\underline{A}_{T'} = \frac{1}{|T'|} \int_{T'} A_{T'} ds' \quad (37-1)$$

$$\underline{B}_{T'} = \frac{1}{|T'|} \int_{T'} B_{T'} ds' \quad (37-2)$$

It will be noted that the relations (37) can be expressed using (10), but we will also need integral expressions because we will not directly know the value of certain fields at the grid nodes of  $\partial\Omega_i$ .

### III,10. - COMPUTATION OF AVERAGE VALUES OF THE FIELDS OVER THE EXTERNAL SURFACE

Relations (36) use average values  $\underline{A}_{T'}$  and  $\underline{B}_{T'}$  of  $A_{T'}$  and  $B_{T'}$  on  $T'$ . Relations (79) of Part I show that the fields in question are  $H_e$  and  $E_e$  on  $\partial\Omega_i$ , the expressions for which are given by relations (75) and (76) in Part I.

We spell out the computation of these average fields in the following sections.

III,10.1. — *Expression of the Average Fields on the External Surface for Computing the Scattered Magnetic Field*

Using (75) and (76) from Part I, and taking  $A = H_e$  and  $B = n \times E_e$ , we get directly:

$$T' \in \Gamma_c: \underline{A}_{T'} = (\underline{H}_{e,c})_{T'} \\ = (I - n_{T'} \otimes n_{T'}) (\underline{H}_{i,c})_{T'} - n_{T'} \times (\underline{H}_{r,0})_{T'} \quad (38-1)$$

$$\underline{B}_{T'} = n_{T'} \times (\underline{E}_{e,c})_{T'} = 0 \quad (38-2)$$

$$T' \in \Gamma_d: \underline{A}_{T'} = (\underline{H}_{e,d})_{T'} \\ = (I + (n_{T'} \otimes n_{T'}) (\mu_{T'}^* - I)) (\underline{H}_{bd})_{T'} \quad (38-3)$$

$$\underline{B}_{T'} = n_{T'} \times (\underline{E}_{e,d})_{T'} = n_{T'} \times (\underline{E}_{i,d})_{T'} + (\underline{E}_{r,0})_{T'} \quad (38-4)$$

III,10.2. — *Expression of the Average Fields on the External Surface for Computing the Scattered Electric Field*

We therefore have  $A = E_e$  and  $B = n \times H_e$  and we use (75) and (76) from Part I to get:

$$T' \in \Gamma_c: \underline{A}_{T'} = (\underline{E}_{e,c})_{T'} \\ = \frac{1}{jk|T'|} n_{T'} \sum_{i=1}^3 V_{T'}^i \cdot \{ (\underline{H}_{i,c})_{T'}^i - n_{T'} \times (\underline{H}_{r,0})_{T'}^i \} \quad (39-1)$$

$$\underline{B}_{T'} = n_{T'} \times (\underline{H}_{e,c})_{T'} = n_{T'} \times (\underline{H}_{i,c})_{T'} + (\underline{H}_{r,0})_{T'} \quad (39-2)$$

$$\underline{A}_{T'} = \frac{1}{jk|T'|} n_{T'} \sum_{i=1}^3 V_{T'}^i \cdot (\underline{H}_{bd})_{T'}^i \\ - (I - n_{T'} \otimes n_{T'}) (\underline{E}_{i,d})_{T'} - n_{T'} \times (\underline{E}_{r,0})_{T'} \quad (39-3)$$

$$\underline{B}_{T'} = n_{T'} \times (\underline{H}_{e,d})_{T'} = n_{T'} \times (\underline{H}_{bd})_{T'} \quad (39-4)$$

III,11. —  $m_c$  AND  $m_d$  OPERATORS

The  $m_c$  and  $m_d$  operators are defined by (62) in Part I. The operator  $m$  can be considered such that:

$$\langle m A, \delta \lambda \rangle = \int_{\partial \Omega_i} \delta \lambda A \cdot n \, ds$$

and for a given finite element  $T$  of  $\partial \Omega_i$ , we then compute:

$$\langle m_{TT} A_T, (\delta \lambda)_T \rangle = \int_T (\delta \lambda)_T (A_T \cdot n_T) \, ds. \quad (40)$$

We obtain the following expressions adapted for numerical computations:

$$\langle m_{TT} A_T, (\delta \lambda)_T \rangle = \sum_{i=1}^3 F_T^i(A_T) \cdot (\delta \lambda)_T^i \quad (41-1)$$

in which  $F_T^i(A_T) \in \mathbb{C}$  for  $i \in \{1, 2, 3\}$  such that:

$$F_T^i(A_T) = \sum_{i=1}^3 \left( \frac{1 + \delta_{ii}}{12} \right) (A_T^i \cdot W_T) \quad (41-2)$$

with  $W_T$  given by (3).

III,12. — OPERATORS FOR COMPUTING THE FAR FIELD

Here we want to adapt the operators of equations (82) from Part I, to compute the reflected electromagnetic far field  $\mathcal{E}_r$  and  $\mathcal{H}_r$ , asymptotically. The calculations are spelled out in [18]. Only  $\mathcal{E}_r$  is computed since  $\mathcal{H}_r$  is deduced immediately from it. According to (82-1) of Part I, we need only to put the following integral in numerical form:

$$\mathbb{I}_{T'}(u) = \int_{T'} \mathcal{N}_u A' \, ds' \quad (42)$$

in which  $\mathcal{N}_u$  is defined by (80) in Part I and  $A'$  is written:

$$A' = u \times (n' \times E'_c) - [I - u \otimes u] (H'_c \times n'). \quad (43)$$

The theorem of the average is not used here, but we compute the integral of (42) numerically using a seven-point Gauss method. The fields  $n' \times E'_c$  and  $H'_c \times n'$  are given by the relations (14-2), (75-2) and (16-2), (76-3) of Part I, and are computed at the Gaussian integration points using the interpolation  $P_1$ .

IV. — NUMERICAL FORM OF DIFFERENTIAL OPERATORS FOR THE INTERIOR PROBLEM

These operators, introduced in section IV of Part I, will be adapted to the numerical process by the finite element method. The problem is therefore to construct elementary matrices.

IV,1. — VOLUME AND SURFACE FINITE ELEMENTS FOR THE INTERNAL BOUNDARIES

IV,1.1. — *Meshing of Internal Interfaces*

The external interface  $\partial \Omega_i$  has been meshed with triangles. The internal interfaces are of two types:  $\Gamma_{cd}$  and  $\Gamma_{mn}$  (see Part I). All of these internal interfaces are meshed with triangles exactly like the  $\partial \Omega_i$  grid. We assume that the usual compatibility is ensured between the grid of  $\partial \Omega_i$  and that of the  $\Gamma_{cd}$  and



$\Gamma_{mn}$  surfaces (matching of edges with the various intersections of these surfaces).

IV.1.2. — *Grid of Dielectric Volume Domains*

Each dielectric domain  $\Omega_n$  has a boundary  $\partial\Omega_n$  (see Part I) which is now meshed with triangles, since it is the union of the surfaces  $\partial\Omega_n \cap (\Gamma_c \cup \Gamma_d)$ ,  $\partial\Omega_n \cap \Gamma_{cd}$  and  $\partial\Omega_n \cap \Gamma_{mn}$ , which are all meshed.

Each domain  $\Omega_n$  is then filled with tetrahedra and the volume grid of  $\Omega_n$  on  $\partial\Omega_n$  coincides with the existing grid of  $\partial\Omega_n$ . This ensures compatibility, and the surface grid of  $\partial\Omega_n$  consists of the triangular faces of the tetrahedra.

So there is only a series of nodes on the surfaces  $\partial\Omega_n \cap (\Gamma_c \cup \Gamma_d)$  and  $\partial\Omega_n \cap \Gamma_{cd}$ , and there will be two series of geometrically matching nodes (the triangulation nodes of  $\Gamma_{mn}$  are doubled up) overlaying each other on the  $\Gamma_{mn} = \partial\Omega_n \cap \partial\Omega_n$  surfaces, in order to construct the  $T_{\Gamma_{cd}}$  operator defined by (28-4) in Part I.

*Notation Convention for all of Section IV:*

We have to develop the calculation of the elementary matrices for the internal operators linked to each  $\Omega_n$  domain. To simplify the notation, we drop the  $n$  superscript whenever there is no possible confusion, writing  $\Omega$ ,  $H$ ,  $\delta H$ ,  $\mu^*$ ,  $\underline{\varepsilon}^*$ ,  $\delta\lambda_{\Omega}$ ,  $\delta\lambda_{cd}$ , instead of  $\Omega_n$ ,  $H^n$ ,  $\delta H^n$ ,  $\underline{\mu}^{*n}$ ,  $\underline{\varepsilon}^{*n}$ ,  $\delta\lambda_{cd,n}$ .

On the other hand, to deal with the  $T_{\Gamma_{mn}}$  operator for  $\Gamma_{mn}$ , we have to keep the  $n$  and  $n'$  subscripts.

IV.1.3. —  *$P_1$  Tetrahedral Finite Element and Approximation Space*

We take  $P_1$  elements. The approximation space for the internal operators on  $\Omega$  will be the space of affine functions by tetrahedron, and continuous in  $\Omega$  (it is each  $\Omega_n$  here, as the  $H$  field is discontinuous through the  $\Gamma_{mn}$  boundaries). This choice is compatible with the approximation space for the operators of the external problem on  $\partial\Omega$ , since the tetrahedral elements  $P_1$  yield triangular  $P_1$  elements for each of their faces.

We will use  $V$  to denote any finite element in the grid of  $\Omega$ . This is a tetrahedron whose apexes will be referred to as  $M_1$ ,  $M_2$ ,  $M_3$  and  $M_4$ . The notation  $V$  will designate the element, but also the part of  $\mathbb{R}^3$  defined by the tetrahedron  $(M_1, M_2, M_3, M_4)$ .

The volume element  $V$  will be denoted:

$$|V| = \int_V dx = \frac{1}{6} |det| \quad (44-1)$$

$$det = [M_2 - M_1, M_3 - M_1, M_4 - M_1] \quad (44-2)$$

in which  $[\dots, \dots, \dots]$  always designates the mixed product.

We introduce the following condensed notation:

$$\left. \begin{aligned} N^1 &= \frac{1}{det} (M_3 - M_4) \times (M_2 - M_4) \\ N^2 &= \frac{1}{det} (M_3 - M_1) \times (M_4 - M_1) \\ N^3 &= \frac{1}{det} (M_1 - M_2) \times (M_4 - M_2) \\ N^4 &= \frac{1}{det} (M_1 - M_3) \times (M_2 - M_3) \end{aligned} \right\} \quad (45)$$

For  $j \in \{1, 2, 3, 4\}$ ,  $N^j \in \mathbb{R}^3$ . The vector  $N^j$  is normal to the face opposite the node  $j$ .

IV.2. —  *$P_1$  INTERPOLATION FORMULAE*

On  $V$  we use the barycentric coordinates  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  to parameter  $V$ :

$$M = \lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3 + \lambda_4 M_4 \in T \quad (46-1)$$

$$\left. \begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= 1, \\ 0 \leq \lambda_i \leq 1, \quad \forall i \in \{1, 2, 3, 4\} \end{aligned} \right\} \quad (46-2)$$

Let  $\varphi$  be a continuous function on  $V$  with values in  $\mathbb{K}^3$ , and  $\varphi^1, \varphi^2, \varphi^3, \varphi^4$  be the values of  $\varphi$  at  $M_1, M_2, M_3$  and  $M_4$ :

$$\varphi^i = \varphi(M_i) \in \mathbb{K}^3, \quad i \in \{1, 2, 3, 4\}. \quad (47)$$

We then have:

$$\varphi = \lambda_1 \varphi^1 + \lambda_2 \varphi^2 + \lambda_3 \varphi^3 + \lambda_4 \varphi^4 \quad (48)$$

As before, we have the following integration formula. If  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4$  are four positive or zero integers:

$$\int_V \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3} \lambda_4^{\alpha_4} ds = 6 |V| \frac{\alpha_1! \alpha_2! \alpha_3! \alpha_4!}{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 3)!} \quad (49)$$

IV.2.1. — *Computation of Gradient. Divergence and Rotational Operators*

Let  $\varphi$  be a function of the type (48) on  $V$  with values  $\mathbb{K}$ . The gradient of  $\varphi$  is written using (45):

$$\nabla \varphi = \sum_{i=1}^4 \varphi^i \cdot N^i. \quad (50-1)$$

Let  $\Psi$  be a function on  $V$  with values  $\mathbb{K}^3$  of the type (48). The divergence and rotational of  $\Psi$  is written

using (45):

$$\operatorname{div} \Psi = \sum_{i=1}^4 \Psi^i \cdot N^i \quad (50-2)$$

$$\operatorname{rot} \Psi = \sum_{i=1}^4 N^i \times \Psi^i. \quad (50-3)$$

#### IV.3. – NUMERICAL EXPRESSION OF THE $\mathbb{M}^n$

For a dielectric domain  $\Omega_n$ , the  $\mathbb{M}^n$  operator is defined by (21) of Part I as:

$$\begin{aligned} \ll \mathbb{M}^n H^n, \delta H \gg &= jk^2 \int_{\Omega_n} (\underline{\mu}^* H) \cdot \delta H \, dx \\ &- j \int_{\Omega_n} ([\underline{\varepsilon}^*]^{-1} \operatorname{rot} H) \cdot \operatorname{rot} \delta H \, dx \end{aligned}$$

in which  $\underline{\mu}^*$  and  $\underline{\varepsilon}^*$  are the complex symmetrical ( $3 \times 3$ ) matrices defined by (4) and (12) in Part I, and are continuous on  $\Omega_n$ . Let  $V$  be a finite element of  $\Omega_n$ , and  $H_V$  and  $\delta H_V$  be the functions defined on  $V$  with values in  $\mathbb{C}^3$ , of the type (48). What we then have to state is:

$$\begin{aligned} \ll \mathbb{M}_V^n H_V^n, \delta H_V \gg &= jk^2 \int_V (\underline{\mu}^* H_V) \cdot \delta H_V \, dx \\ &- j \int_V ([\underline{\varepsilon}^*]^{-1} \operatorname{rot} H_V) \cdot \operatorname{rot} \delta H_V \, dx \quad (51) \end{aligned}$$

Using (49), (50-3), we deduce the expression we are looking for:

$$\ll \mathbb{M}_V^n H_V^n, \delta H_V \gg = \sum_{l=1}^4 F_V^l(H_V) \cdot \delta H_V^l \quad (52-1)$$

With (49) for  $l \in \{1, 2, 3, 4\}$  such that:

$$\begin{aligned} F_V^l(H_V) &= j |V| \sum_{i=1}^4 \left\{ k^2 \left( \frac{1 + \delta_{il}}{20} \right) [\underline{\mu}_V^*] \right. \\ &\quad \left. + [N^{l0}] [\underline{\varepsilon}_V^*]^{-1} [N^{l0}] \right\} H_V^i \quad (52-2) \end{aligned}$$

$$[N^{l0}] = (N^l \times) \quad (52-3)$$

$$[\underline{\mu}_V^*] = [\underline{\mu}_V^* (\underline{M}_V)]; \quad [\underline{\varepsilon}_V^*]^{-1} = [\underline{\varepsilon}_V^* (\underline{M}_V)]^{-1} \quad (52-4)$$

in which  $\underline{M}_V$  designates the barycenter of the element  $V$ .

#### IV.4. – NUMERICAL EXPRESSION OF THE $\operatorname{Div}^n$ OPERATOR

For a dielectric domain  $\Omega_n$ , the  $\operatorname{Div}^n$  operator is defined by (28-2) of Part I as:

$$\ll \operatorname{Div}^n H, \delta \lambda_\Omega \gg = \int_{\Omega_n} \delta \lambda_\Omega \operatorname{div} (\underline{\mu}^* H) \, dx.$$

Let  $V$  be a finite element of  $\Omega_n$ , and let  $H_V$  and  $\delta \lambda_V$  be the continuous functions defined on  $V$  with values in  $\mathbb{C}^3$  and  $\mathbb{C}$ , of the type (48). What we calculate is:

$$\ll \operatorname{Div}_V^n H_V, \delta \lambda_V \gg = \int_V \delta \lambda_V \operatorname{div} (\underline{\mu}_V^* H_V) \, dx \quad (53)$$

Using (50-2), we get:

$$\ll \operatorname{Div}_V^n H_V^n, \delta \lambda_V \gg = \sum_{l=1}^4 F_V^l(H_V) \cdot \delta \lambda_V^l \quad (54-1)$$

with, for  $l \in \{1, 2, 3, 4\}$  and  $F_V^l(H_V) \in \mathbb{C}$ , and independent of  $l$ , such that:

$$F_V^l(H_V) = \frac{1}{4} |V| \sum_{i=1}^4 (\underline{\mu}_V^* H_V^i) \cdot N^i. \quad (54-2)$$

#### IV.5. – NUMERICAL EXPRESSION FOR THE $T_{\Gamma_{cd}}$ , AND $U_{cd}^n$ OPERATORS

##### IV.5.1. – $U_{cd}^n$ Operator

This operator is relative to the surface  $\Gamma_{cd}$ , which is the union of the interfaces between the perfect internal conductor  $\Omega_c$  and the internal dielectric  $\Omega_n$ . It is defined by (28-5) in Part I as:

$$\langle U_{cd}^n H^n, \delta \lambda_{cd} \rangle = \int_{\Gamma_{cd}} \delta \lambda_{cd, n} (\underline{\mu}^{*n} H^n) \cdot n \, ds.$$

We will therefore use  $T$  to denote an arbitrary three-node  $P_1$  finite of  $\Gamma_{cd}$ . This element is oriented in accordance with the convention adopted in Section III.1.4, so that the normal to  $T$ ,  $n_T$ , defined by (2), is directed toward the middle of the conductor  $\Omega_c$ . We then have to compute:

$$\langle U_T H_T, \delta \lambda_T \rangle = \int_T \delta \lambda_T (\underline{\mu}^* H_T) \cdot n_T \, ds \quad (55)$$

with  $H_T$  and  $\delta \lambda_T$  being the continuous functions on  $T$  with values in  $\mathbb{C}^3$  and  $\mathbb{C}$ , respectively, of type (7). Using (8), we get:

$$\langle U_T H_T, \delta \lambda_T \rangle = \sum_{l=1}^3 F_T^l(H_T) \cdot \delta \lambda_T^l \quad (56-1)$$

with, for  $l \in \{1, 2, 3\}$  and  $F_T^l(H_T) \in \mathbb{C}$ , such that:

$$F_T^l(H_T) = \sum_{i=1}^3 \left( \frac{1 + \delta_{il}}{12} \right) (\underline{\mu}^* H_T^i) \cdot W_T \quad (56-2)$$

with  $W_T$  defined by (3) and  $\underline{\mu}_T^* = \underline{\mu}^* (\underline{M}_T)$  where  $\underline{M}_T$  is given by (9).

#### IV,5.2. — $T_{\Gamma_{mn}}$ Operator

This operator is relative to the surface  $\Gamma_{mn}$ , which is the interface between  $\Omega_n$  and  $\Omega_{n'}$ . It is defined by (28-4) in Part I, *i.e.* for  $(n, n')$  fixed in  $\mathcal{N}_\Gamma$  (see section IV-2 in Part I):

$$\begin{aligned} \langle T_{\Gamma_{mn}} H, \delta\lambda_{\Gamma_{mn}} \rangle &= k \int_{\Gamma_{mn}} (R_{\Gamma_{mn}}^n H^n - R_{\Gamma_{mn}}^{n'} H^{n'}) \cdot \delta\lambda_{\Gamma_{mn}} ds \\ R_{\Gamma_{mn}}^n &= (n \otimes n) \underline{\mu}^{*n} - (n \times) \\ R_{\Gamma_{mn}}^{n'} &= (n' \otimes n') \underline{\mu}^{*n'} + (n' \times) \end{aligned}$$

The surface  $\Gamma_{mn}$  is oriented as in section III and is meshed by three-node  $P_1$  finite elements. Let  $T$  be an arbitrary element of  $\Gamma_{mn}$ , whose nodes are  $M_1, M_2, M_3$ . We choose this orientation so that the normal  $n_T$  defined by (2) is directed from  $\Omega_n$  toward  $\Omega_{n'}$ . We introduce two  $P_1$  finite elements denoted  $T_n$  and  $T_{n'}$ , such that  $T_n$  coincides with  $T$  and with one face of a tetrahedron of  $\Omega_n$ , and such that  $T_{n'}$  coincides with  $T$  and with one face of a tetrahedron of  $\Omega_{n'}$ . The normal of  $T_n$  (resp.  $T_{n'}$ ) is  $n_{T_n} = n_T$  (resp.  $n_{T_{n'}} = -n_T$ ).

This is equivalent to doubling the grid of  $\Gamma_{mn}$ . Geometrically, the  $T_n$  and  $T_{n'}$  elements coincide, so what we compute for any finite element  $T$  of  $\Gamma_{mn}$  is:

$$\begin{aligned} \langle T_T H_T, \delta\lambda_T \rangle &= k \int_T (R_T^n H_{T_n} - R_T^{n'} H_{T_{n'}}) \cdot \delta\lambda_T ds \quad (57-1) \\ R_T^n &= (n_T \otimes n_T) \underline{\mu}^{*n} - (n_T \times) \quad (57-2) \\ R_T^{n'} &= (n_T \otimes n_T) \underline{\mu}^{*n'} - (n_T \times) \quad (57-3) \end{aligned}$$

with  $\delta\lambda_T, H_{T_n}$  and  $H_{T_{n'}}$  being three continuous functions on  $T$  with values in  $\mathbb{C}^3$ , of type (7). Using (8), we get:

$$\langle T_T H_T, \delta\lambda_T \rangle = \sum_{l=1}^3 F_T^l(H_T) \cdot \delta\lambda_T^l \quad (58-1)$$

with  $F_T^l(H_T) \in \mathbb{C}^3$ , for  $l \in \{1, 2, 3\}$  such that:

$$F_T^l(H_T) = k |T| \sum_{i=1}^3 \left( \frac{1 + \delta_{il}}{12} \right) (R_T^n H_{T_n}^i - R_T^{n'} H_{T_{n'}}^i) \quad (58-2)$$

#### V. — COMPLEMENTARY REMARKS CONCERNING THE TREATMENT OF EDGES AND THEIR INTERSECTIONS

Let us recall that each dielectric domain  $\Omega_n$  is meshed on its own, so the edge  $\partial\Omega_n$  of  $\Omega_n$  might consist of interfaces of the type  $\Gamma_{cd,n}$ ,  $\Gamma_{mn}$  and  $\Gamma_{d,n}$ . We are concerned here with the treatment of edges

consisting of the intersection of the outer surface with a surface separating two internal media ( $\partial\Omega_i \cap \Gamma_{cd,n}$  or  $\partial\Omega_i \cap \Gamma_{mn}$ ).

#### V,1. — $\partial\Omega_i \cap \Gamma_{cd,n}$ TYPE EDGE

We proceed by continuity with the  $\Gamma_{cd,n}$  interface. The nodes of this edge belong to the dielectric and the unknowns are  $H_{bd}^n$  and  $E_{e,\tau}^v$ . But we need to know  $n \times H_{e,c}$  at the nodes of this edge for the integral operators. So we use the transmission condition (16-2) of Part I between  $\Omega_e$  and  $\Omega_n$  concerning the continuity of the magnetic tangential.

#### V,2. — $\partial\Omega_i \cap \Gamma_{mn}$ TYPE EDGE

On an edge like this, the electric and magnetic fields are continuous. That is, we assume that the surface is locally  $C_1$  on this edge so that it admits a tangent plane on the edge, and we use  $n_e$  to denote the unit normal to  $\partial\Omega_i$  at a given node on the edge. The tangent part of  $H_{bd}^n$  and  $H_{bd}^{n'}$  is continuous through  $\Gamma_d: n_e \times H_{bd}^n = n_e \times H_e$  and  $n_e \times H_{bd}^{n'} = n_e \times H_e$ , whence  $n_e \times (H_{bd}^n - H_{bd}^{n'}) = 0$ , making  $H_{bd}^n - H_{bd}^{n'}$  colinear at  $n_e$ .

Moreover, calling  $n$  the normal to  $\Gamma_{mn}$  at this node, we have  $n \times H_{bd}^n = n \times H_{bd}^{n'}$ , whence  $H_{bd}^n - H_{bd}^{n'}$  is colinear at  $n$ .

Yet  $n_e \neq n$  (the tangent planes at  $\Gamma_d$  and  $\Gamma_{mn}$  are assumed not to be identical), which implies  $H_{bd}^n = H_{bd}^{n'}$ .

An analogous line of reasoning applies to the electric field, so we have  $E_{e,\tau}^n = E_{e,\tau}^{n'}$ , in particular.

#### V,3. — INTERSECTION OF SEVERAL EDGES ON THE OUTER SURFACE

The procedure of V,1 and V,2 applies here too. The only difference stems from the fact that there may be more than two nodes matching at this point of the edge intersection.

#### VI. — CONCLUSION

We have given explicit expressions for the numerical form of the various terms in the formulation developed in Part I. These expressions minimize the numerical cost, considering the constraints we adopted (see section II of Part I) while maintaining the reliability of the approximations for the various possible geometric configurations.

The validation of these approximations is presented in Part III.

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