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# A model and numerical method in the medium frequency range for vibroacoustic predictions using the theory of structural fuzzy

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In linear dynamical analysis of complex mechanical systems, the structural fuzzy is defined as the set of minor subsystems that are connected to the master structure but are not accessible by classical modeling. The notion of master structure is presently extended to others elements such as an external dense compressible fluid strongly coupled with the primary structure. For the low-frequency (LF) dynamical analysis, the modeling of the structural fuzzy is commonly made with a system of masses. If the LF modeling of the structural fuzzy is applied in the medium-frequency (MF) domain, there are some large differences between calculations and experiment. It is therefore necessary to take into account internal degrees of freedom of the structural fuzzy. A global probabilistic modeling of the structural fuzzy is proposed to improve the calculated estimates of the MF vibrations into the master structure and of the far field radiated by itself in or out of context of the acoustic scattering. This paper reviews the author's previous work and introduces the type II probabilistic constitutive law. In this paper: (1) a probabilistic modeling of the structural fuzzy is presented, (2) two probabilistic constitutive laws of the structural fuzzy are constructed, (3) the modeling in the MF range for vibroacoustic predictions using theory of structural fuzzy is developed, (4) the fuzzy solution in the MF range is studied, and (5) numerical simulations on standard structures and on a submerged complex industrial structure are shown.

## INTRODUCTION

This paper deals with vibroacoustic predictions in the medium-frequency (MF) range. Typically we are interested in acoustic field radiated from a submerged elastic structure locally submitted to internal dynamical loads, or in acoustic scattering of an incident wave from a fluid-loaded elastic structure. In this MF domain, the modeling of the "secondary" mechanical subsystems attached to the "primary" structure must be introduced to improve the predictions. In this way, we have introduced the concept of the structural fuzzy,<sup>1</sup> we have developed this theory<sup>1-3</sup> with an adapted numerical analysis,<sup>4,5</sup> which is based on the MF method,<sup>6,7</sup> and we have presented some applications for the validation.<sup>5,3</sup> The MF method<sup>6</sup> that we have developed to solve forced vibration,<sup>6,7</sup> radiation problems,<sup>7,8</sup> and scattering problems,<sup>9,10</sup> has been recently used by Vasudevan<sup>11,12</sup> to solve radiation and scattering problems. In the same time the theory of structural fuzzy has been revisited by Sparrow,<sup>13</sup> who has initialized new research in the area of acoustic scattering from elastic structures with attached structural fuzzy.<sup>14</sup>

We present in this paper a survey of the theory of the structural fuzzy for structural modeling in the MF range, theory that we will present in context of sound radiation and acoustic scattering problems, including in these developments a new probabilistic fuzzy constitutive law with spatial memory and in presenting recent numerical simulation of the far field radiated in water by a stiffened shell cylinder with and without structural fuzzy.

## I. GENERAL CONCEPTS OF MODELING USING THEORY OF STRUCTURAL FUZZY

### A. Concept of master structure and structural fuzzy

In the context of vibroacoustic prediction problems, we will use the term *master structure* to designate the part of the mechanical system that is accessible to conventional modeling, i.e., the system whose mechanical properties, geometry, boundary conditions, and excitations are known with sufficient accuracy and whose necessary modeling requires implementation and leads to an analytical approach or to a numerical approach at a cost that remains reasonable. Presently, this concept of master structure extends to other elements in addition to those involved in the primary structural stiffness. Where there is strong interaction between the primary structure and dense external fluid (submerged structure), and, or, dense or light internal fluids, it is necessary to globally model the primary structure and the fluids in order to analyze the dynamics. The system that can be modeled in this way is the *master structure*. The complement to the master structure with respect to the complete mechanical system is designated as the *structural fuzzy*. It is by definition the part that is not accessible to conventional modeling because details are unknown or are imprecisely known. The structural fuzzy consists of the secondary mechanical systems "attached" to the primary structure.

Typically, this concept can be used for the predictions of the sound radiation from an elastic structure under dynamical loads, or for the predictions of an incident acoustic

wave scattered from a fluid-loaded elastic structure, where there is a lack of knowledge about internal subsystems such as internal geometry, boundary conditions, internal degrees of freedom (DOF), and about specific properties such as mass, damping, and stiffness.

## B. Physical role of the structural fuzzy

In the following we discuss modeling of weakly damped structures, and the terminology *frequency response function* means, if no more is precised, the linear relation in the frequency domain between (1) forces applied to the structure and (2) structural displacements or far field radiated by the structure.

In the low-frequency (LF) domain, modeling generally consists of taking as master structure the primary structure, the fluids, the solid masses, etc., which are conventionally modeled and a system equivalent to the structural fuzzy, modeled globally by pure masses. The predicted LF response functions which exhibit resonance peaks due to the modes of vibration are generally well correlated to experimental results.<sup>15,16</sup>

In the MF domain, the conclusions are the same for the particular case of a "pure" structure, i.e., where there is no structural fuzzy in the above sense. Then, the comparisons between the predicted MF frequency response functions and experimental results are satisfactory.<sup>7,8,10,16,17</sup> *A contrario*, the problem is radically different for the structures in the MF domain. Vibration experiments give smooth morphology for the MF frequency response functions, but traditional predictions based on an equivalent pure mass modeling of the structural fuzzy, do not. The rates of dissipation that would correspond to such smoothing are much too high to allow this phenomenon to be explained by mechanical damping alone. The apparent dissipation occurring on the master structure is due to the energy transfer to the structural fuzzy, which includes mechanical systems attached to the master structure entering in vibration in the MF domain. It is therefore the internal dynamical DOF of the fuzzy that are involved, and a pure mass model can in no case account for this phenomena. These observations mean that in order to predict the MF vibrations in the master structure of a mechanical system, we must model the dynamical effects that the structural fuzzy has on the master structure.

## C. Method for accounting for the structural fuzzy

Let us note that we are only to model the effects of the structural fuzzy on the master structure through the common boundary. We therefore propose a probabilistic approach so that the conventional model (adapted to the MF domain) of the master structure, plus the probabilistic model of the effects of the structural fuzzy qualitatively and quantitatively restore the average behavior of the master structure of the mechanical system. Consequently, the concept of structural fuzzy is introduced to account for missing DOF in structural vibration problems with the following constraint. We want to construct a modeling that allows us to predict, not only the modulus of the direct and

crossed frequency response functions of the master structure in the MF domain, but also the associated average phases. For instance the SEA method,<sup>18</sup> which is well adapted for the high-frequency (HF) domain, does not give such an information on the phases.

The state equation, which governs the linear dynamics of the master structure (formulated with velocity field for primary structure), is expressed conventionally as an impedance term in the frequency domain. Therefore, the dynamical effects of the structural fuzzy on the master structure must be modeled by introducing a boundary impedance. As the model of the fuzzy is probabilistic, the effects of the fuzzy on the master structure will be modeled by setting a *probabilistic boundary impedance of the structural fuzzy*. Therefore, the number of DOF in the master structure model is not increased by taking into account the effects of the fuzzy.

## D. Concept of probabilistic constitutive law of the structural fuzzy

The probabilistic boundary impedance of the structural fuzzy is not intrinsic, since it depends on the local geometry of the common boundaries on which the fuzzy is "attached" and the DOF of the primary structure velocity field on the same boundaries. We therefore introduce the concept of *probabilistic constitutive law of the structural fuzzy* which can be assimilated to an impedance. The constitutive laws are used to construct the *boundary random impedance operator of the structural fuzzy*. A basic problem is the construction or identification of the fuzzy constitutive laws according to the nature of the structural fuzzies, i.e., the classes of fuzzy. We will give in Sec. III the construction of two constitutive laws.

## II. PROBABILISTIC MODELING OF THE STRUCTURAL FUZZY

We summarize hereinafter the main ideas concerning the probabilistic modeling of the structural fuzzy that we have developed with mathematical details,<sup>1,2</sup> with the view of helping the reading of the next sections.

Let  $\Gamma$  be the common boundary between the structural fuzzy and the master structure. Boundary  $\Gamma$  is assumed to be known in the deterministic sense. We will denote by  $S$  and  $S_I$  the parts of  $\Gamma$  such that  $S_I = \{m_1, \dots, m_I\}$  is a discrete set consisting of  $I$  points  $m_i \in \Gamma$  and such that  $S$  is a continuous set of generic points  $m$  of  $\Gamma$ . The part of the fuzzy relative to part  $S_I$ , (resp.  $S$ ) of  $\Gamma$  will be said to be *fuzzy with locally discrete boundary*, (resp. *fuzzy with continuous boundary*), because the connections of fuzzy to the master structure are discrete, (resp. continuous).

We introduce the following hypotheses concerning the structural fuzzy. Each mechanical subsystem of the fuzzy is assumed to have a linear behavior, is weakly dissipative, and the associated conservative mechanical subsystem is a linear dynamical system whose spectrum of natural vibration frequencies is discrete. In all the following we will denote by  $\omega$  the angular frequency and the problem will be formulated in the frequency domain using the Fourier

transform. To simplify the description, we will assume that the coupling state variable of the master structure with the structural fuzzy is a displacement field  $m \rightarrow \hat{\mathbf{u}}(m, \omega)$  defined on  $\Gamma$  with values in  $\mathbb{C}^3$ , or equivalently the velocity vector field  $m \rightarrow \hat{\mathbf{v}}(m, \omega) = i\omega \hat{\mathbf{u}}(m, \omega)$ , with  $i = \sqrt{-1}$  and where we are using  $\exp(i\omega t)$  time convention. The dual variable is then a force density vector field  $m \rightarrow \hat{\mathbf{f}}(m, \omega)$  defined on  $\Gamma$  with values in  $\mathbb{C}^3$ . Actually, we could reason on any other pair of variables.

### A. Fuzzy with continuous boundary

Let  $Z(m, m', \omega)$  be the probabilistic boundary impedance of the structural fuzzy relative to  $S$ , such that:

$$\hat{\mathbf{f}}(m, \omega) = \int_S Z(m, m', \omega) \hat{\mathbf{v}}(m', \omega) ds(m'), \quad (1)$$

where  $ds(m)$  denotes the positive measure on  $S$ . It is assumed that for any  $\omega$  in  $\mathbb{R}$  and for any  $m$  and  $m'$  in  $S$ :

(1)  $Z(m, m', \omega)$  is a second-order random variable defined on a probabilistic space  $(\mathcal{A}, \mathcal{F}, \mathcal{P})$  with values in the  $3 \times 3$  complex matrices relative to the canonical basis, where  $\mathcal{P}$  is the probability measure on  $(\mathcal{A}, \mathcal{F})$ .

(2) There is the symmetry property  $Z(m, m', \omega) = Z(m', m, \omega)^T$ , where the right exponent  $T$  denotes the transposition of matrix.

(3) Finally, denoting by  $\mathbb{X}$  the space of functions  $m \rightarrow \mathbf{w}(m)$  defined on  $S$  with values in  $\mathbb{C}^3$  with integrable square for  $ds$ , equipped with the scalar product

$$\langle \langle \mathbf{w}, \mathbf{w}' \rangle \rangle = \int_S \langle \mathbf{w}(m), \overline{\mathbf{w}'(m)} \rangle ds(m),$$

where

$$\langle \mathbf{w}(m), \overline{\mathbf{w}'(m)} \rangle = \sum_{j=1}^3 w_j(m) \overline{w'_j(m)}$$

and where an overbar means the complex conjugate, we will assume that for any function  $\hat{\mathbf{v}}(\omega) \in \mathbb{X}$ , the integral of the right-hand side of Eq. (1) is defined  $\mathcal{P}$ -almost surely.

With these assumptions, the boundary random impedance operator  $Z(\omega)$  of the structural fuzzy with continuous boundary  $S$ , defined for any  $\hat{\mathbf{v}}(\omega)$  and  $\hat{\mathbf{w}}(\omega)$  in  $\mathbb{X}$  by:

$$\begin{aligned} \langle \langle Z(\omega) \hat{\mathbf{v}}(\omega), \hat{\mathbf{w}}(\omega) \rangle \rangle &= \int_S \int_S \langle Z(m, m', \omega) \hat{\mathbf{v}}(m', \omega) \\ &\quad \times \overline{\hat{\mathbf{w}}(m, \omega)} \rangle ds(m) ds(m'), \end{aligned} \quad (2)$$

is  $\mathcal{P}$ -almost surely linear and continuous from  $\mathbb{X}$  to  $\mathbb{X}$ . With the operator notations, Eq. (1) can be written as:

$$\hat{\mathbf{f}}(\omega) = Z(\omega) \hat{\mathbf{v}}(\omega) = i\omega Z(\omega) \hat{\mathbf{u}}(\omega). \quad (3)$$

Let  $\mathbf{w} = (w_1, w_2, w_3)$  be the components of  $\mathbf{w} \in \mathbb{X}$  relative to the canonical basis. For any point  $m$  of  $S$ , we associate a local orthonormal basis related to the boundary  $S$ , and note  $\Phi(m)$  as the orthogonal  $3 \times 3$  real matrix of the linear operator for transition from canonical basis to this local basis, such that  $\Phi(m)^{-1} = \Phi(m)^T$ . For any  $\hat{\mathbf{v}}(\omega)$  in  $\mathbb{X}$  we associate  $\hat{\mathbf{v}}_0(\omega)$  in  $\mathbb{X}$  such that  $\hat{\mathbf{v}}(m, \omega) = \Phi(m) \hat{\mathbf{v}}_0(m, \omega)$ . Therefore, we deduce from Eq. (2) that for all  $\hat{\mathbf{v}}(\omega)$  and

$\hat{\mathbf{w}}(\omega)$  in  $\mathbb{X}$ , we have  $\langle \langle Z(\omega) \hat{\mathbf{v}}(\omega), \hat{\mathbf{w}}(\omega) \rangle \rangle = \langle \langle Z_0(m, m', \omega) \hat{\mathbf{v}}_0(\omega), \hat{\mathbf{w}}_0(\omega) \rangle \rangle$ , where  $Z_0(m, m', \omega)$  is the  $3 \times 3$  complex matrix

$$Z_0(m, m', \omega) = \Phi(m)^T Z(m, m', \omega) \Phi(m'), \quad (4)$$

which is called the *probabilistic constitutive law* of the structural fuzzy relative to  $S$ .

*Orthotropic fuzzy.* The structural fuzzy with continuous boundary is said to be orthotropic on  $S$  for a frequency band  $B_n$  if there exists  $m \rightarrow \Phi(m)$ , i.e., a family of local basis on  $S$  such that the matrix  $Z_0(m, m', \omega)$  is diagonal for any  $\omega$  in  $B_n$  and for any  $m$  and  $m'$  in  $S$ . We then have  $[Z_0(m, m', \omega)]_{jk} = \delta_{jk} z_j(m, m', \omega)$ . The probabilistic constitutive law for the orthotropic fuzzy is therefore the given of the family of second-order random variables with values in  $\mathbb{C}$ :  $\{z_j(m, m', \omega), j \in \{1, 2, 3\}, \omega \in B_n, m \in S, m' \in S\}$ .

*Isotropic fuzzy.* The above orthotropic fuzzy is said to be isotropic if  $z_1(m, m', \omega) = z_2(m, m', \omega) = z_3(m, m', \omega) = z(m, m', \omega)$ . Then, the probabilistic constitutive law of isotropic fuzzy is such that  $Z_0(m, m', \omega) = z(m, m', \omega) I_d$ , where  $I_d$  is the identity matrix, and where  $\{z(m, m', \omega), \omega \in B_n, m \in S, m' \in S\}$  is a given family of second-order random variables with values in  $\mathbb{C}$ .

*Anisotropic fuzzy.* The structural fuzzy with continuous boundary is said to be anisotropic on  $S$  for a frequency band  $B_n$  if it is neither orthotropic nor isotropic. In this case, the probabilistic constitutive law is a family of second-order random variables  $Z(m, m', \omega)$  with values in the  $3 \times 3$  complex matrices.

*Homogeneous fuzzy.* An anisotropic (resp. orthotropic, isotropic) fuzzy on  $S$  for  $B_n$  is said to be homogeneous if for all  $m$  and  $m'$  in  $S$  we have  $Z(m, m', \omega) = Z(\omega)$  (resp.  $z_j(m, m', \omega) = z_j(\omega)$ ,  $z(m, m', \omega) = z(\omega)$ ). It can be noted that in the orthotropic or isotropic case, the homogeneity on  $S$  does not necessarily mean that  $Z(m, m', \omega)$  is independent of  $m$  and  $m'$ , contrary to the anisotropic case.

*Locally homogeneous fuzzy.* A fuzzy is said to be locally homogeneous on  $S$  for band  $B_n$  if there is a finite partition of  $S$ :  $S = \cup_j S_j$  such that the fuzzy is homogeneous on each part  $S_j$  of  $S$ .

### B. Fuzzy with locally discrete boundary

Let  $J = 3I$  and let  $Z_J(\omega)$  be the  $J \times J$  complex matrix of the random impedance operator of the structural fuzzy with locally discrete boundary  $S_I$ . It is assumed that  $Z_J(\omega)$  is a second-order random variable defined on the probabilistic space  $(\mathcal{A}, \mathcal{F}, \mathcal{P})$  with values in the  $J \times J$  complex matrices and such that  $Z_J(\omega) = Z_J(\omega)^T$ . Let  $\hat{\mathbf{U}}_J(\omega) = (\hat{\mathbf{u}}(m_1, \omega), \dots, \hat{\mathbf{u}}(m_I, \omega))$  with values in  $\mathbb{C}^J$  the displacement vector relative to  $S_I$  and  $\hat{\mathbf{V}}_J(\omega) = i\omega \hat{\mathbf{U}}_J(\omega)$  the velocity vector. Let  $\hat{\mathbf{F}}_J(\omega) = (\hat{\mathbf{f}}_1(\omega), \dots, \hat{\mathbf{f}}_I(\omega))$  with values in  $\mathbb{C}^J$  the dual variable of  $\hat{\mathbf{U}}_J(\omega)$ , where the  $\hat{\mathbf{f}}_j(\omega)$  have values in  $\mathbb{C}^3$ . Then the equation relating the velocity vector  $\hat{\mathbf{V}}_J(\omega)$  to the force vector  $\hat{\mathbf{F}}_J(\omega)$  relative to  $S_I$  is written:

$$\hat{\mathbf{F}}_J(\omega) = Z_J(\omega) \hat{\mathbf{V}}_J(\omega) = i\omega Z_J(\omega) \hat{\mathbf{U}}_J(\omega). \quad (5)$$

### III. PROBABILISTIC CONSTITUTIVE LAW OF THE STRUCTURAL FUZZY

In this section, (a) we recall the results concerning the probabilistic constitutive law of the structural fuzzy with continuous boundary<sup>1,2</sup> which can be used for an orthotropic or isotropic locally homogeneous fuzzy. This law that we will call in the following as the type I probabilistic constitutive law, does not allow us to introduce an effect of a spatial memory into the structural fuzzy, this hypothesis being consistent with the assumption of homogeneousness. (b) We present an unpublished probabilistic constitutive law of the structural fuzzy with continuous boundary, referenced in the following as the type II probabilistic constitutive law, which can be used for an orthotropic or isotropic locally uniform fuzzy with a spatial memory effect inside the structural fuzzy.

Note that, consistently with the results of Sec. II A, these two laws are defined by one quantity,  $z$  in the isotropic case, or by three quantities,  $z_1, z_2, z_3$  in the orthotropic case. Actually, we will construct laws with scalar values of type  $z$ . These laws will depend on mechanical parameters. We will therefore be able to use it for the orthotropic case by assigning different values to the parameters for each local direction 1, 2, and 3.

#### A. Type I probabilistic constitutive law

The use of the concept of locally homogeneous fuzzy allows the mechanical parameters of the type I probabilistic constitutive law to be locally independent as regards the space variables. Consequently, this law is defined by the scalar valued impedance  $z(\omega)$  which does not depend on the  $m$  space variable. The most elementary mechanical model verifying the mechanical hypotheses introduced in Sec. II is that of a simple linear oscillator excited by its support. We have used this as underlying deterministic basis for constructing the probabilistic constitutive law. This obviously does not mean that the law obtained is only suitable for simulating behavior of a structural fuzzy consisting of simple oscillators. On the contrary, this law is capable of representing the dynamical behavior of the complex mechanical systems comprising the fuzzy. This is due to the fact that the parameters of the simple oscillator are modeled by random variables and the probabilistic law obtained generates a random family of oscillators. The locally homogeneous fuzzy hypothesis (see the end of Sec. II A) means that all the points of the boundary  $S_j \subset S = \cup_j S_j$  are equivalent and in this case, the coupling force at a point  $m$  of  $S_j$ , due to the actions of the structural fuzzy on the master structure, only depends on the velocity of the master structure at this same point  $m$ . Consequently with this law, the effect of the structural fuzzy on the master structure is modeled by an infinity of DOF without coupling between these DOF, and there is no spatial memory effect inside the structural fuzzy.

The model of this law can be summarized as the following.<sup>1,2</sup> The mechanical parameters of the law, which depend implicitly on  $j$ , are:

(1) A deterministic cutoff frequency denoted by  $\Omega_c$

such that for  $\omega < \Omega_c$ , the structural fuzzy behaves like a pure mass as seen by  $S_j$ .

(2) An equivalent mass of the fuzzy per unit measure of the boundary  $S_j$ , described by a random real positive valued function  $\omega \rightarrow \mu(\omega)$  defined for  $\omega > 0$ .

(3) A rate of internal dissipation of the structural fuzzy described by a random function  $\omega \rightarrow \xi(\omega)$  defined for  $\omega \geq \Omega_c$  with values in  $]0, 1[$ .

(4) A modal density of the fuzzy described by a random real positive-valued function  $\omega \rightarrow n(\omega)$  defined for  $\omega \geq \Omega_c$ . By definition,  $n(\omega)\Delta\omega$  is the number of natural frequencies of the structural fuzzy in the vicinity  $\Delta\omega$  of  $\omega$ . The distance between two natural frequencies of the fuzzy in the vicinity of  $\omega$  is therefore  $\eta(\omega) = 1/n(\omega)$ . Obviously, for  $\omega$  fixed, this frequency does not necessarily coincide with a natural frequency  $\omega_p(\omega)$  of the fuzzy. It is a probabilistic model of the random variable  $\omega_p(\omega)$  that allow us to introduce the probability for  $\omega_p(\omega) \in (\omega, \omega + d\omega)$ .

The algebraic expression of the impedance  $z(\omega)$  in any point  $m \in S_j$ , deduced from the simple oscillator formulation, is such that:

$$i\omega z(\omega) = -\omega^2 R(\omega) + i\omega I(\omega), \quad (6)$$

where for  $\omega \in ]0, \Omega_c[$ ,  $R$  and  $I$  are such that:

$$R(\omega) = \mu(\omega), \quad I(\omega) = 0, \quad (7)$$

and where for  $\omega \geq \Omega_c$ ,  $R$  and  $I$  are written as:

$$R(\omega) = \frac{\mu(\omega)v^2(\omega)(v^2(\omega) - 1 + 4\xi^2(\omega))}{(v^2(\omega) - 1)^2 + 4v^2(\omega)\xi^2(\omega)}, \quad (8)$$

$$I(\omega) = \frac{2\omega\mu(\omega)v(\omega)\xi(\omega)}{(v^2(\omega) - 1)^2 + 4v^2(\omega)\xi^2(\omega)}, \quad (9)$$

where we have noted:

$$v(\omega) = \omega_p(\omega)/\omega. \quad (10)$$

For each  $\omega$  fixed, the mechanical parameters are modeled by mutually independent random variables such that:

$$\mu(\omega) = \underline{\mu}(\omega)(1 + Y_1(\omega)), \quad (11)$$

$$\xi(\omega) = \underline{\xi}(\omega)(1 + Y_2(\omega)), \quad (12)$$

$$n(\omega) = \underline{n}(\omega)(1 + Y_3(\omega)), \quad (13)$$

where  $\underline{\mu}(\omega)$ ,  $\underline{\xi}(\omega)$ , and  $\underline{n}(\omega)$  are the mean values, where the  $\mathbb{R}^d$ -valued random variable  $\mathbf{Y}(\omega) = \{Y_1(\omega), \dots, Y_d(\omega)\}$  is such that for  $j \in \{1, \dots, d\}$ ,  $Y_j(\omega) = 3^{-1/2}\lambda_j(\omega)X_j$ , with  $d=3$  and where  $X_1, \dots, X_d$  are mutually independent uniform normalized random variables (i.e., centered with unit variance) defined on the probabilistic space  $(\mathcal{A}, \mathcal{F}, \mathcal{P})$ . The control of the dispersion of these mechanical parameters is introduced by the function  $\omega \rightarrow \lambda(\omega) = (\lambda_1(\omega), \dots, \lambda_d(\omega))$  defined for  $\omega > 0$  with values in  $(]0, 1[)^d$ . These data define the probability density function  $p_{\mathbf{Y}(\omega)}(\mathbf{y}; \omega)$  of the random variable  $\mathbf{Y}(\omega)$ , with  $\mathbf{y} = (y_1, \dots, y_d)$ .

By construction we assume that the conditional probability density function of the random variable  $\omega_p(\omega)$  knowing that  $Y_3(\omega) = y_3$  is written as:

$$P_{\omega_p(\omega)|Y_3(\omega)}(\tilde{\omega};\omega|y_3) = [2\eta(\omega,y_3)]^{-1} \mathbf{1}_{[\omega-\eta(\omega,y_3),\omega+\eta(\omega,y_3)]}(\tilde{\omega}), \quad (14)$$

where  $2\eta(\omega,y_3) = [\underline{n}(\omega)(1+y_3)]^{-1}$  and where  $\mathbf{1}_B(\tilde{\omega}) = 1$  if  $\tilde{\omega} \in B$  and  $=0$  if  $\tilde{\omega} \notin B$ . Note that it is natural to introduce Eq. (14) because, since the distance between two natural frequencies of the fuzzy in the vicinity of  $\omega$  is  $2\eta(\omega,y_3)$ , knowing that  $Y_3(\omega) = y_3$ , the probability of presence of a natural frequency on an interval with measure  $2\eta(\omega,y_3)$  is equal to 1. Furthermore, if the modal density  $\underline{n}(\omega) \rightarrow +\infty$ , the conditional probability must approach the Dirac measure at point  $\omega$ , since the probability of presence of a natural frequency  $\omega_p(\omega) = \omega$  then approaches 1. The model proposed has these properties.

Finally, we deduce the probability density function of the  $\mathbb{R}^{d+1}$ -valued random variable  $(Y(\omega), \omega_p(\omega))$  which is written as:

$$P_{Y(\omega),\omega_p(\omega)}(y,\tilde{\omega};\omega) = P_{Y(\omega)}(y;\omega) \times P_{\omega_p(\omega)|Y_3(\omega)}(\tilde{\omega};\omega|y_3). \quad (15)$$

For  $\Omega_c, \underline{\mu}, \underline{\xi}, \underline{n}$ , and  $\lambda$  given, the probabilistic constitutive law of type I is described by the family of second-order random variables  $\{z(\omega;\lambda), \omega > 0\}$ , defined on  $(\mathcal{A}, \mathcal{F}, \mathcal{P})$ , with values in  $\mathbb{C}$ , and which is completely defined (1) by the mapping  $(Y(\omega), \omega_p(\omega)) \rightarrow z(\omega;\lambda)$  which is deduced from Eqs. (6) to (13), and (2) by the known probability density function defined by Eq. (15). Consequently, we can calculate the probability law of the  $\mathbb{C}$ -valued random variable  $z(\omega;\lambda)$ . In particular, denoting by  $E$  the mathematical expectation, the mean  $\underline{z}(\omega;\lambda) = E\{z(\omega;\lambda)\}$  is explicitly calculated<sup>1,2</sup> using Eq. (15), and is written as

$$i\omega \underline{z}(\omega;\lambda) = -\omega^2 \underline{R}(\omega;\lambda) + i\omega \underline{I}(\omega;\lambda), \quad (16)$$

where  $\underline{R}(\omega;\lambda) = E\{R(\omega;\lambda)\}$  and  $\underline{I}(\omega;\lambda) = E\{I(\omega;\lambda)\}$ . If we want to keep the exact nonlinear mapping of the random fluctuation  $z(\omega;\lambda) - \underline{z}(\omega;\lambda)$  that we have defined above, the global probabilistic analysis of the vibration, radiation, and scattering problems of the master structure in presence of the structural fuzzy must be solved in using Monte Carlo simulation method, due to the presence of nonlinearities in the mapping. In this case the numerical cost is too much expensive. For this reason we have preferred to develop a perturbation method joined to a second-order analysis (see Sec. V), which requires to construct a linear dependency of the random fluctuation  $i\omega(z(\omega;\lambda) - \underline{z}(\omega;\lambda))$  in function of the random variable  $\mathbf{X} = (X_1, X_2, X_3)$ . Consequently we have written the following approximation  $\tilde{z}(\omega;\lambda)$  of  $z(\omega;\lambda)$ :

$$i\omega \tilde{z}(\omega;\lambda) = i\omega \underline{z}(\omega;\lambda) + \sum_{j=1}^d X_j (-\omega^2 R_j(\omega;\lambda) + i\omega I_j(\omega;\lambda)), \quad (17)$$

where  $i\omega \underline{z}(\omega;\lambda)$  is given by Eq. (16), and where the  $\mathbb{R}$ -valued functions  $\omega \rightarrow R_j(\omega;\lambda)$  and  $\omega \rightarrow I_j(\omega;\lambda)$  for  $j \in \{1, \dots, d\}$  have been calculated to minimize  $E\{|i\omega z(\omega;\lambda) - i\omega \tilde{z}(\omega;\lambda)|^2\}$ . For any  $\omega$  fixed and for any  $j \in \{1, \dots, d\}$  we have proved<sup>1,2</sup> that:

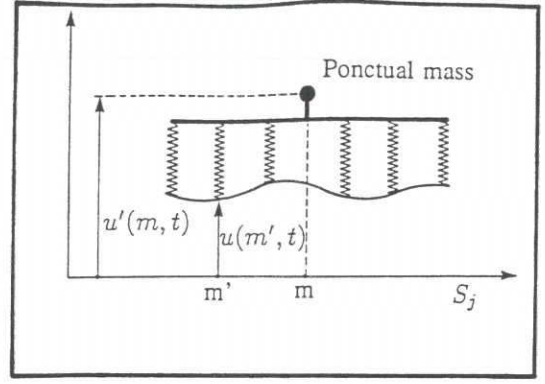


FIG. 1. Underlying deterministic mechanical schema for type II law.

$$R_j(\omega;\lambda) \rightarrow 0, \quad I_j(\omega;\lambda) \rightarrow 0, \quad \text{if } \|\lambda(\omega)\|_\infty \rightarrow 0. \quad (18)$$

The reader will be able to find all the explicit expressions of  $\underline{R}, \underline{I}, R_j$ , and  $I_j$  for  $j \in \{1, 2, 3\}$  in Refs. 1 and 2, but also in the next Sec. III B in taking inside the corresponding formula of the type II law, the special values  $\underline{\alpha}(\omega) = 1$  and  $\lambda_4(\omega) = 0$  for which the type II law gives in this case the type I law.

## B. Type II probabilistic constitutive law

The type II probabilistic constitutive law models a locally orthotropic or isotropic fuzzy with spatial memory effect inside the structural fuzzy, i.e., there is a finite partition of  $S: S = \cup_j S_j$ , such that on each part  $S_j$  of the boundary, the constitutive law can be described by a scalar-valued impedance  $z(m, m', \omega)$  which depends *a priori* on the space variables  $m$  and  $m'$ . In addition we will assume that the spatial memory effect is uniform and isotropic on  $S_j$ , which is to say that  $z(m, m', \omega) = z(\|m - m'\|, \omega)$  on the boundary  $S_j$ . To explain the origin of this probabilistic constitutive law, we now develop the underlying deterministic bases. Let us note that these hypotheses allow us to consider the space variable with only one dimension, and to replace for a short while and for this construction, the boundary  $S_j$  by the real line  $\mathbb{R}$ . Therefore we have  $m \in \mathbb{R}$  and  $ds(m) = dm$ . At any point  $m$  of  $\mathbb{R}$ , we associate a ponctual mass  $\mu$  with absolute displacement  $u'(m, t)$ , and we consider a density of simple oscillators (see Fig. 1). Each of these oscillators has its support connected to a point  $m'$  of  $\mathbb{R}$ , with absolute displacement  $u(m', t)$  and we will denote by  $k_\epsilon(m, m')$  and  $c_\epsilon(m, m')$  the density of stiffness and damping such that:

$$k_\epsilon(m, m') = \mu \omega^2 g_\epsilon(m - m'),$$

$$c_\epsilon(m, m') = 2\xi \mu \omega g_\epsilon(m - m'), \quad (19)$$

where  $g_\epsilon$  is an even positive-valued function defined on  $\mathbb{R}$ , with an integral over  $\mathbb{R}$  equal to 1 and which is such that the measure  $g_\epsilon(m) dm$  on  $\mathbb{R}$  tends towards the Dirac measure  $\delta_0(m)$  on  $\mathbb{R}$  at the point  $0 \in \mathbb{R}$  if  $\epsilon \rightarrow 0$ . The parameter  $\epsilon$  will be directly correlated to the effective length of the isotropic spatial memory around the point  $m \in S_j$ . Let us return to the present construction where  $S_j$  is replaced by

$\mathbb{R}$ . Let  $f_\epsilon(m, t)$  be the real-valued force density field applied to the boundary and equal to the opposite of the reactions of the oscillators. Then, the equations in the time domain are:

$$\begin{aligned} & \mu \frac{\partial^2}{\partial t^2} u'(m, t) + \int_{\mathbb{R}} c_\epsilon(m, m') \frac{\partial}{\partial t} \{u'(m, t) - u(m', t)\} dm' \\ & + \int_{\mathbb{R}} k_\epsilon(m, m') (u'(m, t) - u(m', t)) dm' = 0, \quad (20) \\ f_\epsilon(m, t) = & - \int_{\mathbb{R}} c_\epsilon(m, m') \frac{\partial}{\partial t} \{u'(m', t) - u(m, t)\} dm' \\ & - \int_{\mathbb{R}} k_\epsilon(m, m') (u'(m', t) - u(m, t)) dm' = 0. \end{aligned} \quad (21)$$

Taking the Fourier transform with respect to  $t$  of Eqs. (20) and (21), using the properties of the function  $g_\epsilon$ , eliminating the field  $\hat{u}'(m', \omega)$  between the two transformed equations and introducing the velocity  $\hat{v}(m', \omega) = i\omega \hat{u}(m', \omega)$ , we obtain the relation:

$$\hat{f}_\epsilon(m, \omega) = \int_{\mathbb{R}} z_\epsilon(m - m', \omega) \hat{v}(m', \omega) dm', \quad (22)$$

which is the convolution product  $z_\epsilon(\cdot, \omega) * \hat{v}(\cdot, \omega)$ , and where  $z_\epsilon(m - m', \omega)$  is such that:

$$\begin{aligned} i\omega z_\epsilon(m - m', \omega) dm' = & a(\omega) \delta_0(m' - m) - a(\omega)^2 b(\omega)^{-1} \\ & \times h_\epsilon(m - m') dm', \end{aligned} \quad (23)$$

$$a(\omega) = 2i\omega \xi \mu \omega_p + \mu \omega_p^2, \quad (24)$$

$$b(\omega) = -\omega^2 \mu + 2i\omega \xi \mu \omega_p + \mu \omega_p^2, \quad (25)$$

$$h_\epsilon = g_\epsilon * g_\epsilon. \quad (26)$$

Let us note that if  $\epsilon \rightarrow 0$ , the impedance  $z_\epsilon(m - m', \omega) dm' \rightarrow z(\omega) \delta_0(m' - m)$ , with  $z(\omega)$  defined by Eq. (6) and, by Eqs. (8)–(10), i.e., if the length of spatial memory tends towards zero, then the underlying deterministic mechanical system is the simple linear oscillator excited by its support and used in the modeling of the type I law.

From a point of view of the modeling, the law that would be based on Eq. (23) would not be completely satisfying because, due to its nonlocal nature, it would induce an increase in the numerical difficulties for solving the coupling with the master structure. Our final objective is then to construct an equivalent local law by introducing an equivalent coupling factor that we will denote by  $\alpha(\omega)$ . Therefore Eq. (22) is replaced by

$$\hat{f}(m, \omega) = z_{\text{equ}}(\omega) \hat{v}(m, \omega), \quad (27)$$

where the local impedance  $z_{\text{equ}}(\omega)$ , which does not depend on  $m$  for all  $m \in S_j$ , is such that

$$i\omega z_{\text{equ}}(\omega) = a(\omega) - a(\omega)^2 b(\omega)^{-1} \alpha(\omega), \quad (28)$$

with  $a(\omega)$  and  $b(\omega)$  given by Eqs. (24) and (25). We see that the type I law is deduced from Eq. (27) in taking

$\alpha(\omega) = 1$ , this value corresponding to  $\epsilon \rightarrow 0$  (no spatial memory effect). On the other hand, if such an equivalent local modeling exists,  $\alpha(\omega)$  can be determined to minimize the difference of mechanical power:

$$\Delta \mathcal{P}(\omega) = \langle \langle \hat{f}_\epsilon(\omega) - \hat{f}(\omega), \hat{v}(\omega) \rangle \rangle, \quad (29)$$

where  $\hat{f}_\epsilon(m, \omega)$  and  $\hat{f}(m, \omega)$  are given by Eq. (1) with  $Z(m, m', \omega)$  constructed, respectively, in using  $z_\epsilon$  defined by Eq. (23) and  $z_{\text{equ}}$  defined by Eq. (28). *A priori*,  $\alpha(\omega)$  is a complex-valued random variable and must be determined statistically. It is not self-evident that the impedance defined by Eq. (28) can model correctly a structural fuzzy with spatial memory. We have checked that it is possible by studying the example of a straight beam, having a finite length  $L$ , excited in bending mode, weakly damped, with a structural fuzzy given by Eqs. (22)–(26) with

$$g_\epsilon(m) = \epsilon^{-2} (\epsilon - |m|) \mathbf{1}_{[-\epsilon, \epsilon]}(m), \quad (30)$$

where  $c_\epsilon = 2\epsilon/L$  represents the effective length of the spatial memory, and where  $\mu$ ,  $\xi$ , and  $\omega_p$  are modeled by the random variables defined by Eqs. (11)–(15). For each value of  $\omega$  we have calculated by Monte Carlo simulation an estimation of the statistical mean of the random variable  $\alpha(\omega)$  which minimizes Eq. (29). We have shown in detail<sup>19</sup> that for the small values  $c_\epsilon \leq 0.04$ , we could replace  $\alpha(\omega)$  by the statistical mean of its modulus  $|\alpha(\omega)|$ , because first the mean of the phase is close to zero and second the root-mean square of the random variable  $|\alpha(\omega)|$  is much smaller than its mean. For  $c_\epsilon \geq 0.05$ , the above hypotheses are not so well satisfied, but in this case we have proved<sup>19</sup> that the structural fuzzy has practically no effect on the dynamical response of the beam and consequently, we can keep these hypotheses for the construction of the type II law, for any value of the parameter  $c_\epsilon$ .

Finally, the type II probabilistic constitutive law is constructed by starting from the above deterministic modeling defined by Eq. (28), where for each  $\omega$  fixed, the parameters  $\mu(\omega)$ ,  $\xi(\omega)$ , and  $n(\omega)$  are the random variables which are modeled by Eqs. (11)–(13), and where the equivalent coupling factor  $\alpha(\omega)$  is the real positive valued random variable:

$$\alpha(\omega) = \underline{\alpha}(\omega) (1 + Y_4(\omega)), \quad (31)$$

where  $\underline{\alpha}(\omega)$  is the mean value that is a real function with values in  $[0, 1]$ , and where  $Y_4(\omega)$  is the random variable defined in Sec. III A. The probabilistic modeling of the parameters is exactly the same as that for the type I law and consequently Eqs. (14)–(18) hold in taking in the formula the value  $d=4$ . The approximation of the impedance  $z_{\text{equ}}$  is always denoted by  $\tilde{z}(\omega; \lambda)$ , is constructed as indicated in Sec. III A, and is given by Eq. (17) with  $\underline{z}(\omega; \lambda) = E\{z_{\text{equ}}(\omega; \lambda)\}$  given by Eq. (16), where  $\underline{R}$ ,  $\underline{I}$ ,  $\underline{R}_j$ , and  $\underline{I}_j$  for  $j \in \{1, \dots, 4\}$  are explicitly given by the following equations which are proved in Ref. 19.

For  $\omega \in ]0, \Omega_c[$ , i.e., below the cutoff frequency, we have

$$\underline{R}(\omega; \lambda) = \underline{\mu}(\omega), \quad (32)$$

$$\underline{I}(\omega;\lambda)=0, \quad (33)$$

$$R_1(\omega;\lambda)=3^{-1/2}\underline{\mu}(\omega)\lambda_1(\omega), \quad (34)$$

$$R_j(\omega;\lambda)=0, \quad \text{for } j \in \{2,3,4\}, \quad (35)$$

$$I_j(\omega;\lambda)=0, \quad \text{for } j \in \{1,2,3,4\}. \quad (36)$$

For  $\omega \in [\Omega_c, +\infty[$ , i.e., above the cutoff frequency, we have

$$\underline{R}(\omega;\lambda)=\beta_0^R(\omega)\rho^R(\omega;\lambda), \quad (37)$$

$$R_1(\omega;\lambda)=\beta_1^R(\omega;\lambda)\rho^R(\omega;\lambda), \quad (38)$$

$$R_2(\omega;\lambda)=\gamma_2^R(\omega;\lambda)\underline{\alpha}(\omega)\underline{J}_5(\omega;\lambda), \quad (39)$$

$$R_3(\omega;\lambda)=\gamma_3^R(\omega;\lambda)\underline{\alpha}(\omega)\underline{J}_4(\omega;\lambda), \quad (40)$$

$$R_4(\omega;\lambda)=\beta_4^R(\omega;\lambda)\underline{\alpha}(\omega)[\kappa_0(\omega;\lambda)+\underline{J}_3(\omega;\lambda)], \quad (41)$$

$$\underline{I}(\omega;\lambda)=\beta_0^I(\omega)\rho^I(\omega;\lambda), \quad (42)$$

$$I_1(\omega;\lambda)=\beta_1^I(\omega;\lambda)\rho^I(\omega;\lambda), \quad (43)$$

$$I_2(\omega;\lambda)=\gamma_2^I(\omega;\lambda)[3^{-1}\kappa_1(\omega)\lambda_2^2(\omega)(1-\underline{\alpha}(\omega)) \\ +\underline{\alpha}(\omega)\underline{J}_2(\omega;\lambda)], \quad (44)$$

$$I_3(\omega;\lambda)=\gamma_3^I(\omega;\lambda)\underline{\alpha}(\omega)\underline{J}_1(\omega;\lambda), \quad (45)$$

$$I_4(\omega;\lambda)=\beta_4^I(\omega;\lambda)\underline{\alpha}(\omega)[\underline{J}_0(\omega;\lambda)-\kappa_1(\omega)], \quad (46)$$

with

$$\beta_0^R(\omega)=\omega\underline{\mu}(\omega)\underline{n}(\omega),$$

$$\beta_0^I(\omega)=2^{-1}\pi\omega^2\underline{\mu}(\omega)\underline{n}(\omega),$$

$$\beta_j^R(\omega;\lambda)=3^{-1/2}\lambda_j(\omega)\beta_0^R(\omega), \quad \text{for } j=1,4,$$

$$\beta_j^I(\omega;\lambda)=3^{-1/2}\lambda_j(\omega)\beta_0^I(\omega), \quad \text{for } j=1,4,$$

$$\gamma_j^R(\omega;\lambda)=3^{1/2}\lambda_j(\omega)^{-1}\beta_0^R(\omega), \quad \text{for } j=2,3,$$

$$\gamma_j^I(\omega;\lambda)=3^{1/2}\lambda_j(\omega)^{-1}\beta_0^I(\omega), \quad \text{for } j=2,3,$$

$$\rho^R(\omega;\lambda)=\kappa_0(\omega;\lambda)(\underline{\alpha}(\omega)-1)+\underline{\alpha}(\omega)\underline{J}_3(\omega;\lambda),$$

$$\rho^I(\omega;\lambda)=\kappa_1(\omega)(1-\underline{\alpha}(\omega))+\underline{\alpha}(\omega)\underline{J}_0(\omega;\lambda),$$

$$\kappa_0(\omega;\lambda)=\frac{1}{\omega\underline{n}(\omega)}+\frac{1}{12\omega^3\underline{n}^3(\omega)(1-\lambda_3^2(\omega))},$$

$$\kappa_1(\omega)=4\underline{\xi}(\omega)[\pi\omega\underline{n}(\omega)]^{-1},$$

and where for  $k \in \{0,1,2,3,4,5\}$ , we have

$$\underline{J}_k(\omega;\lambda)=\frac{1}{4}\int_{-1}^1 dy_2 \int_{-1}^1 dy_3 \\ \times J_k(\omega,\underline{\xi}(\omega)[1+\lambda_2(\omega)y_2],\lambda_3(\omega)y_3),$$

with

$$J_0(\omega,x,y)=\pi^{-1}(1-x^2)^{-1/2}(1+y)[\arctan Y_+(\omega,x,y) \\ -\arctan Y_-(\omega,x,y)].$$

$$J_1(\omega,x,y)=yJ_0(\omega,x,y),$$

$$J_2(\omega,x,y)=\underline{\xi}(\omega)^{-1}[x-\underline{\xi}(\omega)]J_0(\omega,x,y),$$

$$J_3(\omega,x,y)=\frac{1}{\omega\underline{n}(\omega)}-\frac{(1+y)}{4(1-x^2)^{1/2}}\ln\left(\frac{N(\omega,x,y)}{D(\omega,x,y)}\right),$$

$$J_4(\omega,x,y)=yJ_3(\omega,x,y),$$

$$J_5(\omega,x,y)=\underline{\xi}(\omega)^{-1}[x-\underline{\xi}(\omega)]J_3(\omega,x,y),$$

and

$$Y_{\pm}(\omega,x,y)=x^{-1}(1-x^2)^{-1/2}[\theta_{\pm}(\omega,y)+x^2],$$

$$\theta_{\pm}(\omega,y)=[(1\pm\tau(\omega,y))^2-1]/2,$$

$$\tau(\omega,y)=[2\omega\underline{n}(\omega)(1+y)]^{-1},$$

$$N(\omega,x,y)=[U_+(\omega,y)+W_+(\omega,x,y)] \\ \times [U_-(\omega,y)-W_-(\omega,x,y)],$$

$$D(\omega,x,y)=[U_+(\omega,y)-W_+(\omega,x,y)] \\ \times [U_-(\omega,y)+W_-(\omega,x,y)],$$

$$U_{\pm}(\omega,y)=2[\theta_{\pm}(\omega,y)+1],$$

$$W_{\pm}(\omega,x,y)=2(1-x^2)^{1/2}[1\pm\tau(\omega,y)].$$

If we take the special values  $\underline{\alpha}(\omega)=1$  and  $\lambda_4(\omega)=0$  into Eqs. (32)–(46), we get the explicit formula for the type I law developed in Sec. III A. Let us note that the integrals  $\underline{J}_k(\omega;\lambda)$  are computed by a numerical method.

#### IV. MODELING IN THE MF RANGE FOR VIBROACOUSTIC PREDICTIONS USING STRUCTURAL FUZZY

This section deals with the modeling and the mathematical formulation in the medium-frequency range for vibroacoustic predictions using the theory of structural fuzzy. The formulation developed is specially adapted to numerical methods for complex geometry and to analytical analysis for simple geometry.

The physical space  $\mathbb{R}^3$  is referred to a Cartesian reference system  $Ox_1x_2x_3$  and we will denote by  $\mathbf{x}=(x_1,x_2,x_3)$  the generic point  $m$  of  $\mathbb{R}^3$ . The geometry of the coupled system and the notations are defined in Fig. 2. The external fluid occupies the open unbounded domain  $D_E$  of  $\mathbb{R}^3$  with boundary  $\Sigma_E$ . The primary structure is an elastic continuum that occupies the open-bounded domain  $D_S$  of  $\mathbb{R}^3$  with boundary  $\partial D_S=\Sigma_E\cup\Sigma_S\cup\Sigma_I\cup\Gamma$ . The part  $\Gamma$  is the common boundary between the structural fuzzy and the primary structure. The internal fluid occupies an open bounded domain  $D_I$  of  $\mathbb{R}^3$  with boundary  $\Sigma_I$ . We denote as  $\mathbf{n}$  the unit normal to  $\partial D_S$  external to  $D_S$  and we denote as  $\mathbf{n}'=-\mathbf{n}$  the unit normal to  $\Sigma_I$  external to  $D_I$ .

#### A. Boundary integral equation formulation for the exterior Neumann problem

To formulate the coupling between the primary structure  $D_S$  and the external unviscous compressible dense fluid  $D_E$ , it is necessary to solve the following harmonic exterior Neumann problem for Helmholtz equation:



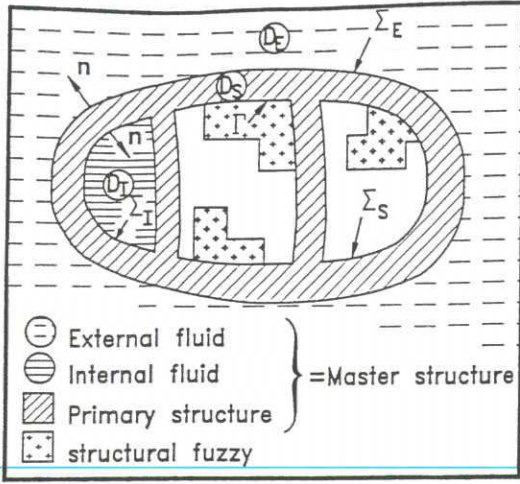


FIG. 2. Geometry of the coupled system

$$\nabla^2 \Phi + (\omega/c)^2 \Phi = 0 \quad \text{in } D_E, \quad (47)$$

$$\frac{\partial \Phi}{\partial \mathbf{n}} = \hat{v} \quad \text{on } \Sigma_E, \quad (48)$$

$$\left| \frac{\partial \Phi}{\partial \mathbf{n}} - \frac{i\Phi\omega}{c} \right| = O(r^{-2}), \quad |\Phi| = O(r^{-1}), \quad r = \|\mathbf{x}\| \rightarrow +\infty, \quad (49)$$

where  $\Phi$  is the harmonic velocity potential,  $c$  and  $\rho$  are the acoustic fluid velocity and the density of the external fluid, and  $\hat{v}$  is a normal velocity field defined on  $\Sigma_E$  with values in  $\mathbb{C}$  which depends on  $\omega$ , where  $\omega \in \mathbb{R}$  is the angular frequency. The harmonic pressure field is such that

$$\hat{p} = -i\omega\rho\Phi \quad \text{in } \bar{D}_E = D_E \cup \Sigma_E. \quad (50)$$

For all  $\omega$  fixed in  $\mathbb{R}$ , the problem defined by Eqs. (47)–(49) has a unique solution<sup>25,26</sup> and there is two linear operators  $\mathbf{B}(\omega/c)$  and  $\mathcal{R}(\mathbf{x}, \omega/c)$  such that

$$\Phi = \mathbf{B}(\omega/c)\hat{v} \quad \text{on } \Sigma_E, \quad (51)$$

$$\Phi(\mathbf{x}) = \mathcal{R}(\mathbf{x}, \omega/c)\hat{v}, \quad \forall \mathbf{x} \in D_E. \quad (52)$$

For some simple shapes, these operators can explicitly be constructed.<sup>20–24</sup> For the general case in the MF range, numerical methods based on boundary integral equations are necessary. Several boundary integral equation formulations related to  $\Sigma_E$  can be used to construct the operators  $\mathbf{B}(\omega/c)$  and  $\mathcal{R}(\mathbf{x}, \omega/c)$ , but we know<sup>25,26</sup> that Eqs. (47)–(49) are always a unique solution for all real  $\omega$ . However, some boundary integral equation formulations do not yield the solution for a set of discrete frequencies, that we will call the “irregular frequencies.” Of course, for this irregular frequencies we know that the Fredholm’s alternative is verified.<sup>25</sup> In practice, for any general geometries of  $\Sigma_E$ , the values of these irregular frequencies are unknown before performing any calculations. Consequently, for the general case, we used the following boundary integral equation<sup>27</sup> that eliminates automatically the numerical difficulties induced by the irregular frequencies. Let  $G(\mathbf{x}, \mathbf{x}')$  be the fundamental solution of the Helmholtz equation in

$\mathbb{R}^3$ , which is written as  $G(\mathbf{x}, \mathbf{x}') = -(4\pi\|\mathbf{x} - \mathbf{x}'\|)^{-1} \exp(-i\|\mathbf{x} - \mathbf{x}'\|\omega/c)$ . The variational formulation yields the following operatorial boundary integral equation:<sup>27</sup>

$$\begin{bmatrix} 0 \\ \Phi \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_3' \\ \mathbf{A}_3 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \Psi \\ \hat{v} \end{bmatrix}, \quad (53)$$

where  $\mathbf{A}_1$ ,  $\mathbf{A}_2$ , and  $\mathbf{A}_3$  are the linear operators such that, if  $\delta\Psi$  and  $\delta\hat{v}$  are any test functions belonging to the admissible class of functions associated, respectively, with  $\Psi$  and  $\hat{v}$ , we have

$$\langle \langle \mathbf{A}_1 \Psi, \delta\Psi \rangle \rangle = \int_{\Sigma_E} \int_{\Sigma_E} \left( \frac{d}{dn} \frac{d'G}{dn'} \right) \Psi' \delta\Psi ds' ds, \quad (54)$$

$$\langle \langle \mathbf{A}_2 \hat{v}, \delta\hat{v} \rangle \rangle = - \int_{\Sigma_E} \int_{\Sigma_E} G \hat{v}' \delta\hat{v} ds' ds, \quad (55)$$

$$\langle \langle \mathbf{A}_3 \Psi, \delta\hat{v} \rangle \rangle = \frac{1}{2} \int_{\Sigma_E} \Psi \delta\hat{v} ds - \int_{\Sigma_E} \int_{\Sigma_E} \frac{d'G}{dn'} \Psi' \delta\hat{v} ds' ds, \quad (56)$$

where  $ds$  is the surface measure borne by  $\Sigma_E$ , where the “prime” is related to the point  $\mathbf{x}'$  and where  $\mathbf{A}_3'$  denotes the algebraic transposition of the operator  $\mathbf{A}_3$ . Using differential geometry calculus and Stokes’ theorem, Eq. (54) can be written as:<sup>28</sup>

$$\begin{aligned} \langle \langle \mathbf{A}_1 \Psi, \delta\Psi \rangle \rangle &= - \int_{\Sigma_E} \int_{\Sigma_E} G \langle d(\delta\Psi d\mathbf{x}), d'(\Psi' d\mathbf{x}') \rangle \\ &\quad + \frac{\omega^2}{c^2} \int_{\Sigma_E} \int_{\Sigma_E} G \langle \mathbf{n}, \mathbf{n}' \rangle \Psi' \delta\Psi ds' ds, \end{aligned} \quad (57)$$

where  $d$  and  $d'$  denote the exterior derivative operator with respect to  $\mathbf{x}$  and  $\mathbf{x}'$ . The complex-valued boundary integral operator defined by Eq. (53) is algebraically symmetric but not Hermitian. For all fixed  $\omega \in \mathbb{R}$  and for all sufficiently smooth field  $\hat{v}$  defined on  $\Sigma_E$ , Eq. (53) gives the unique solution  $\Phi$  of the exterior Neumann problem and yields the construction of the operator  $\mathbf{B}(\omega/c)$  defined by Eq. (51). Let us note that the right-hand side of Eq. (57) allows us to use the space of all the continuous functions on  $\Sigma_E$  for constructing the numerical approximation by the finite element method. Finally, the radiating operator  $\mathcal{R}(\mathbf{x}, \omega/c)$  defined by Eq. (52) for all  $\mathbf{x}$  in  $D_E$  is written as:

$$\mathcal{R}(\mathbf{x}, \omega/c) = \mathcal{R}_1(\mathbf{x}, \omega/c) + \mathcal{R}_2(\mathbf{x}, \omega/c)\mathbf{B}(\omega/c), \quad (58)$$

where  $\mathcal{R}_1(\mathbf{x}, \omega/c)$  and  $\mathcal{R}_2(\mathbf{x}, \omega/c)$  are the two linear operators such that:

$$\mathcal{R}_1\left(\mathbf{x}, \frac{\omega}{c}\right)\hat{v} = \int_{\Sigma_E} G \hat{v}' ds', \quad (59)$$

$$\mathcal{R}_2\left(\mathbf{x}, \frac{\omega}{c}\right)\Phi = - \int_{\Sigma_E} \frac{d'G}{dn'} \Phi' ds'. \quad (60)$$

Where  $\Sigma_E$  is a slender surface of revolution, an asymptotic method was developed<sup>29</sup> and it allows a decrease in the numerical costs.

## B. Actions due to acoustic scattering from the structure

Let us consider the mechanical system under the action of an incident acoustic wave in  $D_E$  and any applied forces. Let  $\mathbf{x} \rightarrow \hat{\mathbf{U}}(\mathbf{x}, \omega)$  be the total displacement field defined on  $D_S$  with values in  $\mathbb{C}^3$  of the primary elastic structure at frequency  $\omega$ . Consequently the boundary  $\Sigma_E$  is submitted to the displacement field  $\mathbf{x} \rightarrow \hat{\mathbf{u}}(\mathbf{x}, \omega)$ , which is the trace on  $\Sigma_E$  of the field  $\hat{\mathbf{U}}$ . The incident acoustic wave is defined by a velocity potential  $\Phi_i$  with values in  $\mathbb{C}$ , verifying Eq. (47). Let  $\Phi$  be the total velocity potential verifying Eqs. (47)–(49). The Neumann boundary condition on  $\Sigma_E$  defined by Eq. (48) is such that  $\hat{v} = i\omega \langle \hat{\mathbf{u}}, \mathbf{n} \rangle$ . Then, we can write  $\Phi = \Phi_i + \Phi_s$ , where the scattered velocity potential  $\Phi_s$  verifies Eqs. (47)–(49) with the Neumann condition on  $\Sigma_E: \partial \Phi_s / \partial \mathbf{n} = \hat{v} - \partial \Phi_i / \partial \mathbf{n}$ . Therefore, we deduce from Eq. (51) that  $\Phi_s = \mathbf{B}(\omega/c) \{i\omega \langle \hat{\mathbf{u}}, \mathbf{n} \rangle - \partial \Phi_i / \partial \mathbf{n}\}$ . We deduce from Eq. (50) the expression of the total pressure field on  $\Sigma_E$ :

$$\hat{p} = \hat{p}_{id} + \hat{p}_r, \quad (61)$$

$$\hat{p}_{id} = -i\omega \rho \left[ \Phi_i + \mathbf{B} \left( \frac{\omega}{c} \right) \left( -\frac{\partial \Phi_i}{\partial \mathbf{n}} \right) \right], \quad (62)$$

$$\hat{p}_r = \omega^2 \rho \mathbf{B}(\omega/c) (\langle \hat{\mathbf{u}}, \mathbf{n} \rangle). \quad (63)$$

At any point  $\mathbf{x}$  in  $D_E$ , the total pressure is written as:

$$\hat{p}(\mathbf{x}, \omega) = \hat{p}_i(\mathbf{x}, \omega) + \hat{p}_s(\mathbf{x}, \omega), \quad (64)$$

where  $\hat{p}_i(\mathbf{x}, \omega) = -i\omega \rho \Phi_i(\mathbf{x}, \omega)$  is the incident acoustic wave and  $\hat{p}_s(\mathbf{x}, \omega)$  is the scattered pressure which is such that

$$\hat{p}_s(\mathbf{x}, \omega) = \mathcal{R} \left( \mathbf{x}, \frac{\omega}{c} \right) \left[ i\omega \langle \hat{\mathbf{u}}, \mathbf{n} \rangle - \frac{\partial \Phi_i}{\partial \mathbf{n}} \right]. \quad (65)$$

Let  $\delta \mathbf{U}$  be an any  $\mathbb{C}^3$ -valued admissible displacement field defined on  $D_S$  of the primary structure and  $\delta \mathbf{u}$  its trace on  $\Sigma_E$ . Then the  $\mathbb{C}^3$ -valued force field  $\hat{\mathbf{f}}_E$  applied to the primary structure and due to the action of the external fluid is defined by the equation:

$$\langle \langle \hat{\mathbf{f}}_E, \delta \mathbf{U} \rangle \rangle = - \int_{\Sigma_E} \langle \delta \mathbf{u}, \mathbf{n} \rangle \hat{p} \, ds,$$

where  $\hat{p}$  is given by Eqs. (61) to (63). Therefore, we can write

$$\hat{\mathbf{f}}_E = \hat{\mathbf{f}}_{id} + \omega^2 \mathbf{B}_E(\omega/c) \hat{\mathbf{U}}, \quad (66)$$

where  $\hat{\mathbf{f}}_{id}$  is the external force field due to the action of the incident wave on the structure and which is such that

$$\langle \langle \hat{\mathbf{f}}_{id}, \delta \mathbf{U} \rangle \rangle = - \int_{\Sigma_E} \langle \delta \mathbf{u}, \mathbf{n} \rangle \hat{p}_{id} \, ds, \quad (67)$$

with  $\hat{p}_{id}$  given by Eq. (62) and where  $\mathbf{B}_E(\omega/c)$  is the external fluid coupling operator which is such that

$$\left\langle \left\langle \mathbf{B}_E \left( \frac{\omega}{c} \right) \hat{\mathbf{U}}, \delta \mathbf{U} \right\rangle \right\rangle = - \int_{\Sigma_E} \rho \langle \delta \mathbf{u}, \mathbf{n} \rangle \mathbf{B} \left( \frac{\omega}{c} \right) (\langle \hat{\mathbf{u}}, \mathbf{n} \rangle) \, ds. \quad (68)$$

## C. Modeling of the primary structure

The primary structure is an inhomogeneous, anisotropic, linear viscoelastic solid continuum  $D_S$ . Let  $\mathbf{x} \rightarrow \hat{\mathbf{U}}(\mathbf{x}, \omega)$  the  $\mathbb{C}^3$ -valued displacement field of this primary structure defined on  $D_S$ . Let  $\mathbf{M}_S$ ,  $\mathbf{C}_S(\omega)$ , and  $\mathbf{K}_S(\omega)$  be the mass, damping, and stiffness real linear operators, constructed by the variational formulation, and which are positive-definite symmetric.<sup>25,30–34</sup> The dependency on  $\omega$  of  $\mathbf{C}_S$  and  $\mathbf{K}_S$  is due to the presence of viscoelastic materials.<sup>7,25,32,35</sup> Then, the impedance operator  $\mathbf{Z}_S(\omega)$  of the primary structure is such that

$$i\omega \mathbf{Z}_S(\omega) = -\omega^2 \mathbf{M}_S + i\omega \mathbf{C}_S(\omega) + \mathbf{K}_S(\omega). \quad (69)$$

We will denote as  $\hat{\mathbf{f}}_S$  the element that represents the applied mechanical forces on the primary structure and which depends on the frequency  $\omega$ .

## D. Modeling of the internal fluid

The internal fluid  $D_I$  is compressible, with density  $\rho_I$  and acoustic velocity  $c_I$ . It is assumed that there is a source term  $\hat{\mathbf{Q}}_I$  in the internal fluid  $D_I$  and we denote as  $\hat{p}_I$  the pressure field in  $\bar{D}_I = D_I \cup \Sigma_I$ . For the MF modeling, the formulation is written with a velocity potential  $\Phi_I$  which is the solution of the following internal Neumann problem:

$$\text{div}([1 + i\omega \beta_I(\omega)] \nabla \Phi_I) + (\omega/c_I)^2 \Phi_I = \rho_I^{-1} \hat{\mathbf{Q}}_I \quad \text{in } D_I, \quad (70)$$

$$[1 + i\omega \beta_I(\omega)] \frac{\partial \Phi_I}{\partial \mathbf{n}} = i\omega \langle \hat{\mathbf{u}}, \mathbf{n} \rangle \quad \text{on } \Sigma_I, \quad (71)$$

$$\hat{p}_I = -i\omega \rho_I \Phi_I \quad \text{in } \bar{D}_I, \quad (72)$$

where  $\hat{\mathbf{u}}$  is the trace of  $\hat{\mathbf{U}}$  on  $\Sigma_I$ . Coefficient  $\beta_I$  is real and depends on  $\mathbf{x}$  and  $\omega$ . For  $\beta_I \equiv 0$ , we have an usual inviscid compressible fluid (acoustic fluid). For certain models in the MF range, it is advantageous to have an inhomogeneous dissipation term depending on the frequency to simulate loss phenomena in the neighborhood of walls of the acoustic cavity or in certain parts of the fluid domain  $D_I$ . For modeling an inhomogeneous dissipation, we write  $\beta_I(\mathbf{x}, \omega) = \nu(\mathbf{x}, \omega) c_I^{-2}$ , where  $\nu(\mathbf{x}, \omega) \in \mathbb{R}^+$  has the dimension of a kinematic viscosity coefficient.

Let  $\delta \mathbf{U}$  and  $\delta \Phi_I$  be any test functions belonging to the admissible class of functions associated, respectively, with  $\hat{\mathbf{U}}$  and  $\Phi_I$ , and  $\delta \mathbf{u}$  is the trace of  $\delta \mathbf{U}$  on  $\Sigma_I$ . The variational formulation yields the excitation source  $\hat{\mathbf{q}}_I$ , the fluid-structure coupling operator  $\mathbf{H}_I$  such that

$$\langle \langle \hat{\mathbf{q}}_I, \delta \Phi_I \rangle \rangle = \int_{D_I} \hat{\mathbf{Q}}_I \delta \Phi_I \, dx, \quad (73)$$

$$\langle \langle \mathbf{H}_I \Phi_I, \delta \mathbf{U} \rangle \rangle = - \int_{\Sigma_I} \rho_I \langle \delta \mathbf{u}, \mathbf{n} \rangle \Phi_I \, ds, \quad (74)$$

and the impedance operator  $\mathbf{Z}_I(\omega)$  of the internal fluid such that

$$i\omega \mathbf{Z}_I(\omega) = -\omega^2 \mathbf{M}_I + i\omega \mathbf{C}_I(\omega) + \mathbf{K}_I, \quad (75)$$

with  $\mathbf{M}_I$ ,  $\mathbf{C}_I(\omega)$ , and  $\mathbf{K}_I$ , the mass, damping, and stiffness real symmetric positive operators defined by

$$\langle\langle \mathbf{M}_I \Phi_I, \delta \Phi_I \rangle\rangle = \int_{D_I} \rho_I c_I^{-2} \Phi_I \delta \Phi_I dx, \quad (76)$$

$$\langle\langle \mathbf{C}_I(\omega) \Phi_I, \delta \Phi_I \rangle\rangle = \int_{D_I} \rho_I \beta_I(\omega) \langle \nabla \Phi_I, \nabla(\delta \Phi_I) \rangle dx, \quad (77)$$

$$\langle\langle \mathbf{K}_I \Phi_I, \delta \Phi_I \rangle\rangle = \int_{D_I} \rho_I \langle \nabla \Phi_I, \nabla(\delta \Phi_I) \rangle dx. \quad (78)$$

### E. Modeling of the structural fuzzy

The structural fuzzy is applied to the part  $\Gamma$  of boundary  $\partial D_S$ . Using the developments of Secs. II and III, the boundary impedance operator of the structural fuzzy, denoted as  $\mathbf{Z}_\Gamma$  and related on the  $C^3$ -valued velocity field  $i\omega \hat{\mathbf{U}}$  of the primary structure  $D_S$ , is constructed by using the different probabilistic constitutive laws related to the classes of fuzzy of the actual problem. Taking into account the form of the Eqs. (16) and (17),  $\mathbf{Z}_\Gamma$  will depend on  $\omega$  and on the vector-valued dispersion parameter  $\omega \rightarrow \lambda(\omega) = (\lambda_1(\omega), \dots, \lambda_L(\omega))$  with values in  $([0,1]^L \subset \mathbb{R}^L$ , where  $L$  is a given positive integer. For any fixed  $\omega$ ,  $\mathbf{Z}_\Gamma(\omega; \lambda)$  is a complex operator-valued second-order random variable defined on the probabilistic space  $(\mathcal{A}, \mathcal{F}, \mathcal{P})$ , which is algebraically symmetric and which is such that

$$\mathbf{Z}_\Gamma(\omega; \lambda) = \underline{\mathbf{Z}}_\Gamma(\omega; \lambda) + \mathbf{Z}_r(\omega; \lambda), \quad (79)$$

where  $\underline{\mathbf{Z}}_\Gamma(\omega; \lambda) = E\{\mathbf{Z}_\Gamma(\omega; \lambda)\}$  is the mean which is such that

$$i\omega \underline{\mathbf{Z}}_\Gamma(\omega; \lambda) = -\omega^2 \underline{\mathbf{R}}_\Gamma(\omega; \lambda) + i\omega \underline{\mathbf{I}}_\Gamma(\omega; \lambda), \quad (80)$$

and where  $\mathbf{Z}_r(\omega; \lambda)$  is the random fluctuation which is such that

$$i\omega \mathbf{Z}_r(\omega; \lambda) = \sum_{j=1}^L X_j (-\omega^2 \mathbf{R}_{\Gamma,j}(\omega; \lambda) + i\omega \mathbf{I}_{\Gamma,j}(\omega; \lambda)), \quad (81)$$

with  $X_1, \dots, X_L$  some real-valued centered random variables defined on  $(\mathcal{A}, \mathcal{F}, \mathcal{P})$ , mutually independent, with unit variance and uniform probability law. By construction [see Eq. (18)], if  $\|\lambda(\omega)\|_\infty \rightarrow 0$ , then  $\mathbf{Z}_\Gamma(\omega; \lambda) \rightarrow \underline{\mathbf{Z}}_\Gamma(\omega; \lambda)$ . Finally, for any  $\omega$ ,  $\underline{\mathbf{R}}_\Gamma(\omega; \lambda)$  and  $\mathbf{R}_{\Gamma,j}(\omega; \lambda)$  for  $j \in \{1, \dots, L\}$  are real symmetric operators, and,  $\underline{\mathbf{I}}_\Gamma(\omega; \lambda)$  and  $\mathbf{I}_{\Gamma,j}(\omega; \lambda)$  for  $j \in \{1, \dots, L\}$  are real symmetric positive operators.

### F. Probabilistic equation for fluid-structure interaction with structural fuzzy

Using Eqs. (66), (69), (73)–(75), and (79), we obtain by assembling, the equation of the vibrations of the master structure (i.e., primary structure with external and internal fluids) with its structural fuzzy, which is written with the operatorial notation as

$$i\omega [\mathbf{Z}_{\text{mast}}(\omega) + \mathbf{Z}_{\text{fuz}}(\omega; \lambda)] \hat{\mathbf{W}}(\omega) = \hat{\mathbf{F}}(\omega), \quad (82)$$

where the state variable  $\hat{\mathbf{W}}$  and the excitation term  $\hat{\mathbf{F}}$  are written in block form as

$$\hat{\mathbf{W}} = \begin{bmatrix} \hat{\mathbf{U}} \\ \Phi_I \end{bmatrix}, \quad \hat{\mathbf{F}} = \begin{bmatrix} \hat{\mathbf{f}}_S + \hat{\mathbf{f}}_{id} \\ \hat{q}_I \end{bmatrix}, \quad (83)$$

where the impedance operator  $\mathbf{Z}_{\text{mast}}(\omega)$  of the master structure is such that

$$i\omega \mathbf{Z}_{\text{mast}}(\omega) = -\omega^2 \mathbf{M}_1 + i\omega \mathbf{C}_1(\omega) + \mathbf{K}_1(\omega) - \omega^2 \mathbf{B}_0(\omega/c), \quad (84)$$

with  $\mathbf{M}_1$ ,  $\mathbf{C}_1(\omega)$ ,  $\mathbf{K}_1(\omega)$ , and  $\mathbf{B}_0(\omega/c)$  the operators written in block form as

$$\mathbf{M}_1(\omega) = \begin{bmatrix} \mathbf{M}_S & 0 \\ 0 & -\mathbf{M}_I \end{bmatrix}, \quad \mathbf{C}_1(\omega) = \begin{bmatrix} \mathbf{C}_S(\omega) & -\mathbf{H}_I \\ -\mathbf{H}_I^t & -\mathbf{C}_I(\omega) \end{bmatrix}, \quad (85)$$

$$\mathbf{K}_1(\omega) = \begin{bmatrix} \mathbf{K}_S(\omega) & 0 \\ 0 & -\mathbf{K}_I \end{bmatrix}, \quad \mathbf{B}_0(\omega/c) = \begin{bmatrix} \mathbf{B}_E(\omega/c) & 0 \\ 0 & 0 \end{bmatrix}, \quad (86)$$

and where, thanks to Eq. (79), the boundary impedance operator of the structural fuzzy is written as

$$\mathbf{Z}_{\text{fuz}}(\omega; \lambda) = \underline{\mathbf{Z}}_{\text{fuz}}(\omega; \lambda) + \mathbf{Z}_{\text{rand}}(\omega; \lambda), \quad (87)$$

where  $\underline{\mathbf{Z}}_{\text{fuz}}(\omega; \lambda) = E\{\mathbf{Z}_{\text{fuz}}(\omega; \lambda)\}$  is the mean that is written in block form as

$$\underline{\mathbf{Z}}_{\text{fuz}}(\omega; \lambda) = \begin{bmatrix} \underline{\mathbf{Z}}_\Gamma(\omega; \lambda) & 0 \\ 0 & 0 \end{bmatrix}, \quad (88)$$

with  $\underline{\mathbf{Z}}_\Gamma(\omega; \lambda)$  given by Eq. (80), and where  $\mathbf{Z}_{\text{rand}}(\omega; \lambda)$  is the random fluctuation such that

$$i\omega \mathbf{Z}_{\text{rand}}(\omega; \lambda) = \sum_{j=1}^L X_j (-\omega^2 \mathbf{R}_j(\omega; \lambda) + i\omega \mathbf{I}_j(\omega; \lambda)), \quad (89)$$

the operators  $\mathbf{R}_j(\omega; \lambda)$  and  $\mathbf{I}_j(\omega; \lambda)$  for  $j \in \{1, \dots, L\}$  being such that

$$\mathbf{R}_j(\omega; \lambda) = \begin{bmatrix} \mathbf{R}_{\Gamma,j}(\omega; \lambda) & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{I}_j(\omega; \lambda) = \begin{bmatrix} \mathbf{I}_{\Gamma,j}(\omega; \lambda) & 0 \\ 0 & 0 \end{bmatrix}. \quad (90)$$

Let us note that the operator  $i\omega(\mathbf{Z}_{\text{mast}}(\omega) + \mathbf{Z}_{\text{fuz}}(\omega; \lambda))$  is complex symmetric but non-Hermitian.

## V. FUZZY SOLUTION IN THE MF RANGE

To construct an approximation of finite dimension of the operatorial Eqs. (82) and (65), we use the finite element method.<sup>25,30,36–38</sup> Boundaries  $\Sigma_E$ ,  $\Sigma_I$ , and  $\Gamma$  are meshed with finite elements: (1) which are compatible on  $\Sigma_E$  with the finite elements of the primary structure<sup>7,16,17</sup> and with the finite elements of the external fluid coupling operator,<sup>7,16,28</sup> (2) which are compatible on  $\Sigma_I$  with the finite elements of internal fluid,<sup>7,8,16</sup> and (3) which are compatible on  $\Gamma$  with the finite elements of the boundary impedance operator of the structural fuzzy.<sup>5</sup> To simplify the writing in the following, we will keep the operatorial notations before discretizing the operators.

To obtain an efficient numerical method for constructing the fuzzy solution in the MF range of Eqs. (82) and (65), two main problems have to be solved. The first one

concerns the effective construction of the random solution of the probabilistic Eq. (82). The second one is to use an adapted MF analysis that avoids computing the solution at all discrete frequencies by solving directly a linear equation frequency by frequency, because the finite element models that represent a realistic fluid–structure interaction problem with structural fuzzy in a wide medium frequency range, have very large number of degrees of freedom and we want to decrease the numerical cost. This kind of problem is well known and the reader will be able to find some explanations of this situation for instance in Soize<sup>6,7</sup> and Vasudevan.<sup>12</sup>

### A. Construction of the random solution

Let  $\mathbb{B} = [\omega_{\min}, \omega_{\max}] \subset \mathbb{R}^+$  be the MF band of analysis. Considering Eq. (87), we can write Eq. (82) as

$$i\omega [\mathbf{Z}_0(\omega; \lambda) + \mathbf{Z}_{\text{rand}}(\omega; \lambda)] \hat{\mathbf{W}}(\omega) = \hat{\mathbf{F}}(\omega), \quad (91)$$

where  $\mathbf{Z}_0(\omega; \lambda)$  is the deterministic impedance operator,

$$\mathbf{Z}_0(\omega; \lambda) = \mathbf{Z}_{\text{mast}}(\omega) + \mathbf{Z}_{\text{fuz}}(\omega; \lambda), \quad (92)$$

and where  $\mathbf{Z}_{\text{rand}}(\omega; \lambda)$  is the random operator due to the fluctuations of the structural fuzzy and defined by Eq. (89).

We have proved<sup>1</sup> that, with some usual smooth hypotheses that are sufficient for the applications, and if the vector-valued dispersion parameter  $\lambda$  of the structural fuzzy is small enough in norm, i.e., if there exists  $\lambda_0$  sufficiently small such that

$$0 < \sup_{\omega \in \mathbb{B}} \|\lambda(\omega)\|_{\infty} \leq \lambda_0, \quad (93)$$

then we have the following results: (1) Eq. (82) has a unique solution almost surely for any  $\omega \in \mathbb{B}$ ; (2) for all  $\omega \in \mathbb{B}$  the operator  $\mathbf{Z}_0(\omega; \lambda)$  is invertible; (3) setting

$$\mathbf{T}(\omega; \lambda) = -\mathbf{Z}_0(\omega; \lambda)^{-1} \mathbf{Z}_{\text{rand}}(\omega; \lambda), \quad (94)$$

we can write Eq. (91) as

$$i\omega \mathbf{Z}_0(\omega; \lambda) [1 - \mathbf{T}(\omega; \lambda)] \hat{\mathbf{W}}(\omega) = \hat{\mathbf{F}}(\omega), \quad (95)$$

and for any  $\omega \in \mathbb{B}$  we have the Neuman's expansion:<sup>39,40</sup>

$$[1 - \mathbf{T}(\omega; \lambda)]^{-1} = \sum_{k=0}^{+\infty} \mathbf{T}(\omega; \lambda)^k, \quad (96)$$

the series of the right member of Eq. (96) being almost surely convergent.

Therefore, the random solution of Eq. (82) can be written for any  $\omega \in \mathbb{B}$  as

$$\hat{\mathbf{W}}(\omega; \lambda) = \sum_{k=0}^{+\infty} \mathbf{T}(\omega; \lambda)^k [i\omega \mathbf{Z}_0(\omega; \lambda)]^{-1} \hat{\mathbf{F}}(\omega). \quad (97)$$

Substituting Eq. (94) into Eq. (97) and using Eq. (89), we deduce that the random solution given by Eq. (97) admits the expansion:

$$\begin{aligned} \hat{\mathbf{W}}(\omega; \lambda) &= \hat{\mathbf{W}}^{(0)}(\omega; \lambda) \\ &+ \sum_{k=1}^{\mathcal{X}} \left\{ \sum_{j_1, \dots, j_k=1}^L X_{j_1} \cdots X_{j_k} \hat{\mathbf{W}}_{j_1, \dots, j_k}^{(k)}(\omega; \lambda) \right\}, \end{aligned} \quad (98)$$

with  $\mathcal{X} = +\infty$ , where  $\hat{\mathbf{W}}^{(0)}(\omega; \lambda)$  is the solution of the deterministic linear equation:

$$i\omega \mathbf{Z}_0(\omega; \lambda) \hat{\mathbf{W}}^{(0)}(\omega; \lambda) = \hat{\mathbf{F}}(\omega), \quad (99)$$

and where  $\hat{\mathbf{W}}_{j_1, \dots, j_k}^{(k)}(\omega; \lambda)$  for  $k \geq 1$  are the solutions of the following recurrent deterministic linear equations:

$$i\omega \mathbf{Z}_0(\omega; \lambda) \hat{\mathbf{W}}_{j_1, \dots, j_k}^{(k)}(\omega; \lambda) = \hat{\mathbf{F}}_{j_1, \dots, j_k}^{(k)}(\omega; \lambda), \quad (100)$$

$$\begin{aligned} \hat{\mathbf{F}}_{j_1, \dots, j_k}^{(k)}(\omega; \lambda) &= -[-\omega^2 \mathbf{R}_{j_k}(\omega; \lambda) + i\omega \mathbf{I}_{j_k}(\omega; \lambda)] \\ &\times \hat{\mathbf{W}}_{j_1, \dots, j_{k-1}}^{(k-1)}(\omega; \lambda), \end{aligned} \quad (101)$$

with the convention that  $\hat{\mathbf{W}}_{j_1, \dots, j_k}^{(k)}(\omega; \lambda) = \hat{\mathbf{W}}^{(0)}(\omega; \lambda)$  for  $k=0$ . Of course, the recurrence is limited to a finite integer  $\mathcal{X} \geq 1$ , and we will say that we have constructed the solution of order  $\mathcal{X}$ . Full mathematical details about the convergence of the solution in function of  $\lambda_0$  and  $\mathcal{X}$  can be found in Soize.<sup>1</sup> As soon as the deterministic fields  $\hat{\mathbf{W}}_{j_1, \dots, j_k}^{(k)}(\omega; \lambda)$  for  $0 \leq k \leq \mathcal{X}$  are known by calculation for any  $\omega \in \mathbb{B}$  (see Sec. V B), we deduce directly from Eq. (83) and without computation the deterministic fields  $\hat{\mathbf{U}}_{j_1, \dots, j_k}^{(k)}(\omega; \lambda)$  and  $\hat{\Phi}_{I_{j_1, \dots, j_k}}^{(k)}(\omega; \lambda)$  for  $0 \leq k \leq \mathcal{X}$ . Therefore, we deduce the  $\mathcal{X}$ -order expansion of the following factors.

(1) The displacement field  $\hat{\mathbf{U}}(\omega; \lambda)$  of the primary structure  $D_S$  and its trace  $\hat{\mathbf{u}}(\omega; \lambda)$  on  $\Sigma_E$

$$\begin{aligned} \hat{\mathbf{U}}(\omega; \lambda) &= \hat{\mathbf{U}}^{(0)}(\omega; \lambda) \\ &+ \sum_{k=1}^{\mathcal{X}} \left\{ \sum_{j_1, \dots, j_k=1}^L X_{j_1} \cdots X_{j_k} \hat{\mathbf{U}}_{j_1, \dots, j_k}^{(k)}(\omega; \lambda) \right\}, \end{aligned} \quad (102)$$

and, hence, the velocity field  $i\omega \hat{\mathbf{U}}(\omega; \lambda)$  and the acceleration field  $-\omega^2 \hat{\mathbf{U}}(\omega; \lambda)$ :

(2) The pressure field in the internal fluid  $D_I$  by using Eq. (72):

$$\begin{aligned} \hat{p}_I(\omega; \lambda) &= \hat{p}_I^{(0)}(\omega; \lambda) + \sum_{k=1}^{\mathcal{X}} \left\{ \sum_{j_1, \dots, j_k=1}^L X_{j_1} \cdots X_{j_k} \right. \\ &\left. \times \hat{p}_{I_{j_1, \dots, j_k}}^{(k)}(\omega; \lambda) \right\}, \end{aligned} \quad (103)$$

with  $\hat{p}_{I_{j_1, \dots, j_k}}^{(k)}(\omega; \lambda) = -i\omega \rho_I \Phi_{I_{j_1, \dots, j_k}}^{(k)}(\omega; \lambda)$  for  $0 \leq k \leq \mathcal{X}$

(3) The scattered pressure in the external fluid at any point  $\mathbf{x} \in D_E$  due to the incident acoustic wave and to the other actions, by using Eqs. (65) and (102):

$$\hat{p}_s(\mathbf{x}, \omega; \lambda) = \hat{p}_s^{(0)}(\mathbf{x}, \omega; \lambda) + \sum_{k=1}^{\mathcal{K}} \left\{ \sum_{j_1, \dots, j_k=1}^L X_{j_1} \cdots X_{j_k} \hat{p}_{s_{j_1, \dots, j_k}}^{(k)}(\mathbf{x}, \omega; \lambda) \right\},$$

$$\hat{p}_s^{(0)}(\mathbf{x}, \omega; \lambda) = \mathcal{R} \left( \mathbf{x}, \frac{\omega}{c} \right) \left\{ i\omega \langle \hat{\mathbf{u}}^{(0)}(\omega; \lambda), \mathbf{n} \rangle - \frac{\partial \Phi_i}{\partial \mathbf{n}} \right\},$$

$$\hat{p}_{s_{j_1, \dots, j_k}}^{(k)}(\mathbf{x}, \omega; \lambda) = \mathcal{R}(\mathbf{x}, \omega/c) \{ i\omega \langle \hat{\mathbf{u}}_{j_1, \dots, j_k}^{(k)}(\omega; \lambda), \mathbf{n} \rangle \}, \quad (104)$$

for  $1 \leq k \leq \mathcal{K}$ . Let us note that Eq. (104) yields also the radiated pressure due only to the actions of the applied force  $\hat{\mathbf{f}}_S$  on the primary structure and of the source term  $\hat{\mathbf{Q}}_I$  in the internal fluid if the incident acoustic wave vanishes, i.e., if  $\Phi_i \equiv 0$ . In this last case,  $\partial \Phi_i / \partial \mathbf{n} = 0$  in Eq. (104), and  $\hat{\mathbf{f}}_{id} = 0$  in the Eq. (83) which defines  $\hat{\mathbf{F}}$ .

We see that Eqs. (102)–(104) are of the same type and consequently, any scalar observation of the mechanical system (for instance a coordinate of the acceleration of the primary structure at a fixed point, the pressure in the internal fluid or the scattered pressure in the external fluid at a fixed point) will be denoted by  $\hat{r}(\omega)$  and will be a  $\mathbb{C}$ -valued random variable that depends on the parameter  $\lambda$  and that admits the  $\mathcal{K}$ -order expansion:

$$\hat{r}(\omega) = \hat{r}^{(0)}(\omega) + \sum_{k=1}^{\mathcal{K}} \left\{ \sum_{j_1, \dots, j_k=1}^L X_{j_1} \cdots X_{j_k} \hat{r}_{j_1, \dots, j_k}^{(k)}(\omega) \right\}. \quad (105)$$

For the applications we are interested in three types of quantities associated with  $\hat{r}(\omega)$ .

(1) The first type concerns the real part, the imaginary part and the square of the modulus of  $\hat{r}(\omega)$ , i.e., the following mappings  $\omega \rightarrow \text{Re}\{\hat{r}(\omega)\}$ ,  $\omega \rightarrow \text{Im}\{\hat{r}(\omega)\}$ ,  $\omega \rightarrow |\hat{r}(\omega)|^2$ .

(2) The second type is related to the average of quantities calculated over a small frequency band of bandwidth  $\delta\omega$ , and defined by

$$\langle \hat{r} \rangle(\omega) = (\delta\omega)^{-1} \int_{\omega - \delta\omega/2}^{\omega + \delta\omega/2} \hat{r}(\tilde{\omega}) d\tilde{\omega}, \quad (106)$$

and concerns, as the first type, the real part, the imaginary part, and the square of the modulus of  $\langle \hat{r} \rangle(\omega)$ , i.e., the following mappings  $\omega \rightarrow \text{Re}\{\langle \hat{r} \rangle(\omega)\}$ , etc.

(3) The third type concerns the energy  $\mathcal{E}_r(\omega)$  of  $r$  in the subbands  $[\omega - \delta\omega/2, \omega + \delta\omega/2]$  defined by the formula:

$$\mathcal{E}_r(\omega) = (2\pi)^{-1} \int_{\omega - \delta\omega/2}^{\omega + \delta\omega/2} |\hat{r}(\tilde{\omega})|^2 d\tilde{\omega}. \quad (107)$$

We can calculate all the desired statistical moments of the random variables of the first, second, or third type, by using Eq. (105) and the usual results concerning the transformation of finite vector-valued random variables.<sup>41,42</sup> In particular we are interested in first two second-order moments, i.e., the mean value and the root-mean square of these random variables. For instance, for the first-order expansion ( $\mathcal{K} = 1$ ), the calculations are algebraically simple but sometimes fastidious and are not reproduced here.<sup>2,5</sup>

## B. Brief remarks about the method of analysis in the MF range

Taking into account Eqs. (99) and (100), we see that the equations that we have to solve are all of the following type:

$$i\omega \mathbf{Z}_0(\omega) \hat{\mathbf{W}}(\omega) = \hat{\mathbf{F}}(\omega), \quad (108)$$

for  $\omega \in \mathbb{B}$ , where the unknown is  $\hat{\mathbf{W}}(\omega)$ . We can write

$$-\omega^2 \mathbf{B}_0(\omega/c) = -\omega^2 \mathbf{M}_0(\omega/c) + i\omega \mathbf{C}_0(\omega/c), \quad (109)$$

where  $\mathbf{M}_0(\omega/c)$  and  $\mathbf{C}_0(\omega/c)$  are some real symmetric operators, and where  $\mathbf{B}_0(\omega/c)$  is the external fluid coupling operator defined by Eqs. (68) and (86). Then, thanks to Eqs. (80), (84), (88), (92), and (109), Eq. (108) can be written as

$$(-\omega^2 \mathbf{M}(\omega) + i\omega \mathbf{C}(\omega) + \mathbf{K}(\omega)) \hat{\mathbf{W}}(\omega) = \hat{\mathbf{F}}(\omega), \quad (110)$$

where  $\mathbf{M}(\omega)$ ,  $\mathbf{C}(\omega)$ , and  $\mathbf{K}(\omega)$  are three real symmetric operators that can be expressed in terms of the previously defined operators:  $\mathbf{M}_S$ ,  $\mathbf{C}_S(\omega)$ , and  $\mathbf{K}_S(\omega)$  (primary structure operators),  $\mathbf{M}_I$ ,  $\mathbf{C}_I(\omega)$ ,  $\mathbf{K}_I$ , and  $\mathbf{H}_I$  (internal fluid operators),  $\mathbf{M}_0(\omega/c)$  and  $\mathbf{C}_0(\omega/c)$  (external fluid operators),  $\mathbf{R}_\Gamma(\omega; \lambda)$  and  $\mathbf{I}_\Gamma(\omega; \lambda)$  (structural fuzzy mean operators). For solving Eq. (110), we use the MF method that we have developed<sup>6</sup> and that has been validated in previous works.<sup>6–12</sup> This method avoids computing the solution  $\hat{\mathbf{W}}(\omega)$  at all discrete frequencies by solving directly the linear Eq. (110) frequency by frequency. It is based on specific time integration and transform techniques and allows us to compute the solution  $\hat{\mathbf{W}}(\omega)$  at all discrete frequencies, but with a numerical cost that is much smaller than the numerical cost induced by the direct frequency per frequency method or by the direct time-integration method. Let us note that this MF method yields the real and the imaginary parts of  $\hat{\mathbf{W}}(\omega)$ , i.e., the modulus and the phase, at all discrete frequencies. Consequently no new approximation or no new average is introduced at this level of the present construction. Full details about this MF method applied to an equation of the type (110) are given in Soize<sup>2,5,7</sup> and a brief review is given in Soize.<sup>16,17</sup>

## VI. NUMERICAL SIMULATIONS AND APPLICATIONS

This section deals with three applications. The first and second one are relatively simple structures in a vacuum using type I and type II probabilistic constitutive laws, and give some elements which validate the theory of the structural fuzzy. The third one concerns a highly complex structure submerged in water for which the radiation problem is studied. All the results presented hereinafter have been performed at ONERA with a general finite element structural code in which the fluid–structure interaction (external and internal fluids), the theory of structural fuzzy, and the MF method for constructing the solution in the MF range have been implemented. The third example has been performed on a CRAY computer. All the results presented correspond to a first-order expansion of the random fluctuation of the solution.

### A. Beam in a vacuum with type I structural fuzzy

(1) The master structure is a homogeneous cantilever beam running along the  $x$  axis, having length  $L=2$  m, clamped at  $x=0$ , free at  $x=L$ , with constant cross-sectional area  $10^{-4}$  m<sup>2</sup>, mass density 31 400 kg/m<sup>3</sup>, Young's modulus  $2.1 \times 10^{11}$  N/m<sup>2</sup>, bending inertia  $10^{-8}$  m<sup>4</sup> and structural damping rate 0.003. The frequency band of analysis studied is  $\mathbb{B}=[0,1000]$  Hz. It is excited in bending mode in the  $(x,y)$  plane by a point force applied at  $x=2$  m, with unit modulus and flat spectrum over all the band  $\mathbb{B}$ . For the MF analysis method,<sup>6,7</sup> the band  $\mathbb{B}$  is partitioned into 10 MF narrow bands. Each MF narrow band has a bandwidth  $\Delta\omega=2\pi \times 100$  rad/s.

(2) The structural fuzzy is applied on all the length of the beam and is modeled by a homogeneous orthotropic fuzzy with continuous boundary, constructed with the type I probabilistic constitutive law. The input fuzzy parameters of this type I law have the following values for each MF narrow band (see Sec. III A):  $\mu=5.196 \times 10^{-3}$  kg/m with dispersion parameter  $\lambda_1=0.4$ ,  $\xi=0.002$  with dispersion parameter  $\lambda_2=0.001$ , and  $\eta=2.756 \times 10^{-2}$  (rad/s)<sup>-1</sup> with dispersion parameter  $\lambda_3=0.4$ .

(3) With the theory of the structural fuzzy, the master structure is modeled by straight two-node beam finite elements, and the type I law orthotropic fuzzy with continuous boundary, by two-node line finite elements.

(4) The solution of reference is constructed by the Monte Carlo simulation method. The structural fuzzy is modeled by  $N$  simple oscillators for each MF narrow band of bandwidth  $\Delta\omega$ , acting in the  $y$  direction, and which are homogeneously and randomly attached along the master structure. The value of  $N$  is such that  $\eta = \sqrt{N}/\Delta\omega$  which yields  $N=300$ . Each oscillator is a damping-truss element with a critical damping rate  $\xi_0=0.002$  and a concentrated mass  $\mu_0=6 \times 10^{-4}$  kg such that  $\mu = \mu_0 \sqrt{N}/L$ . Consequently, we see that on each MF narrow band, the ratio of the mean mass of the structural fuzzy over the mass of the master structure is only 2.86%. The master structure is modeled with 300 straight two-node beam finite elements. All the parameters and the location over the beam of each oscillator are simulated using the probability density function defined by Eq. (15).

(5) Figures 3 and 4 are relative to the  $y$  acceleration of the beam at  $x=1.4$  m. Let us recall that excitation is at  $x=2$  m. Fig. 3(a) and (b) concerns the acceleration energies calculated by Eq. (107) with  $\delta\omega=2\pi \times 5$  rad/s, and Fig. 4(a) and (b) are relative to the real and imaginary parts of the acceleration response, calculated by Eq. (106) with  $\delta\omega=2\pi \times 5$  rad/s. On Fig. 3(a) and (b), we show the mean response and the root-mean square in dB of the master structure with structural fuzzy. The agreement between the response of reference performed by Monte Carlo simulation (thin line) and the response obtained by the theory of structural fuzzy (thick line) is excellent for the mean response and for the root-mean-square response. Figure 4(a) [respectively (b)] shows the mean response (thick line) and the root-mean-square response (thin line) in m/s<sup>2</sup> for the real part (respectively, the imaginary part) of

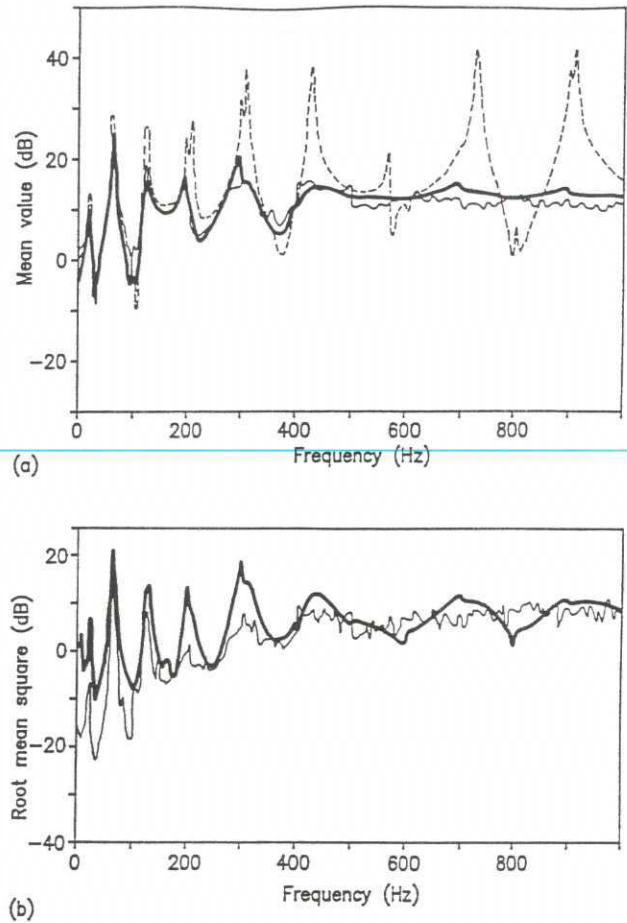


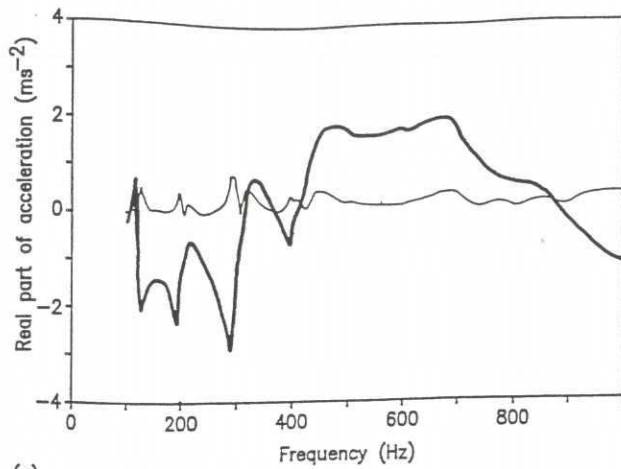
FIG. 3. (a) Mean values and (b) root-mean-square values, in dB, of the transverse acceleration energies over a 5-Hz bandwidth, at  $x=1.4$  m of a cantilever beam in a vacuum excited by a unit transverse load applied at  $x=2$  m, without structural fuzzy [thin broken line on Fig. (a)], and with a type I structural fuzzy performed by Monte Carlo numerical simulation (thin lines) and by the theory of structural fuzzy (thick lines).

the response obtained with the theory of structural fuzzy. Finally, one can see in Fig. 3(a) the response of the master structure without fuzzy (thin broken line), and which is typically modal, but despite this and the weak mass disturbance introduced by the fuzzy, the smoothing of the resonance peaks is very large.

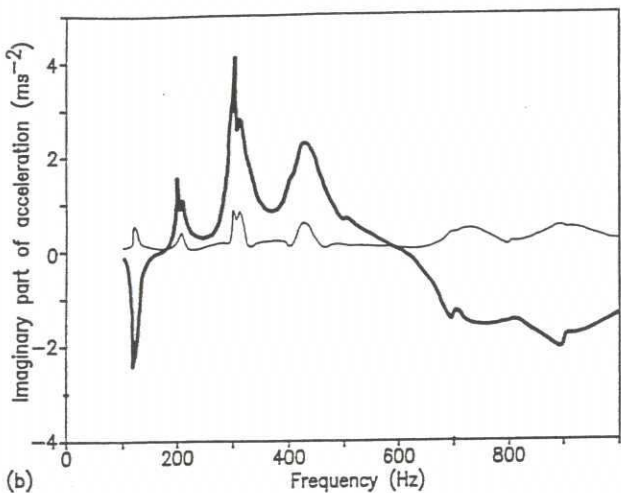
### B. Beam in a vacuum with type II structural fuzzy

(1) The master structure is a homogeneous beam running along the  $x$  axis, having length  $L=2$  m, simply supported at  $x=0$  and at  $x=L$ , with constant cross-sectional area  $10^{-4}$  m<sup>2</sup>, mass density 20 000 kg/m<sup>3</sup>, Young's modulus  $2.1 \times 10^{11}$  N/m<sup>2</sup>, bending inertia  $10^{-12}$  m<sup>4</sup> and structural damping rate 0.003. The frequency band of analysis studied is  $\mathbb{B}=[300, 600]$  Hz. It is excited in bending mode in the  $(x,y)$  plane by a point force applied at  $x=1$  m, with unit modulus and flat spectrum over all the band  $\mathbb{B}$ . For the MF analysis method,<sup>6,7</sup> the band  $\mathbb{B}$  is partitioned into 3 MF narrow bands. Each MF narrow band has a bandwidth  $\Delta\omega=2\pi \times 100$  rad/s.

(2) The structural fuzzy is applied on all the length of the beam and is modeled by a uniform orthotropic fuzzy with spatial memory (its a fuzzy with continuous bound-



(a)



(b)

FIG. 4. (a) Real part and (b) imaginary part, in  $\text{m/s}^2$ , of the transverse acceleration averaged over a 5-Hz bandwidth, at  $x=1.4$  m of a cantilever beam in a vacuum with a type I structural fuzzy, excited by a unit transverse load applied at  $x=2$  m. Mean values (thick lines) and root-mean-square values (thin lines), obtained by the theory of structural fuzzy.

ary) and constructed with the type II probabilistic constitutive law. The input fuzzy parameters of this type II law have the following values for each MF narrow band (see Sec. III B):  $\mu=8 \times 10^{-4}$  kg/m with dispersion parameter  $\lambda_1=0.1$ ,  $\xi=0.003$  with dispersion parameter  $\lambda_2=0.001$ ,  $\eta=2.256 \times 10^{-2}$   $(\text{rad/s})^{-1}$  with dispersion parameter  $\lambda_3=0.1$ , and for the mean value of the equivalent coupling factor  $\alpha$ , the values defined by the graph of Fig. 5, which depends on the narrow frequency band and on the length of spatial memory  $c_\epsilon=2\epsilon/L$  that we have introduced in Sec. III B. The corresponding dispersion parameter is  $\lambda_4=0.001$ .

(3) With the theory of the structural fuzzy, the master structure is modeled by straight two-node beam finite elements, and the type II law orthotropic fuzzy with continuous boundary and spatial memory effect, by two-node line finite elements.

(4) The solution of reference is constructed by the Monte Carlo simulation method. The structural fuzzy is modeled for each MF narrow band of bandwidth  $\Delta\omega$ , by  $N$  oscillators following the discrete model<sup>19</sup> associated with

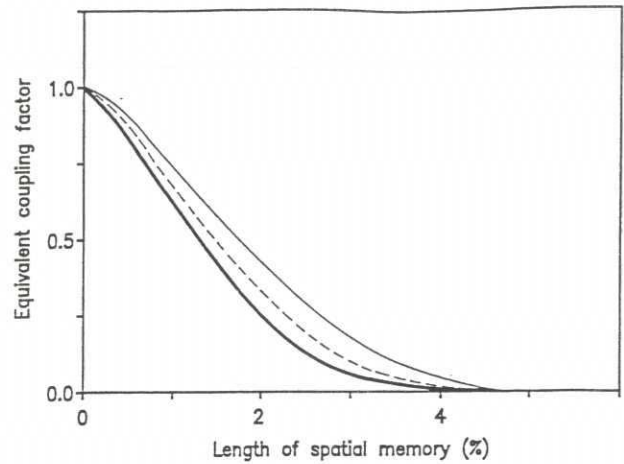


FIG. 5. Mean value of the equivalent factor  $\alpha$  of the type II probabilistic fuzzy constitutive law for a simply supported beam of length  $L=2$  m in a vacuum, as a function of the fuzzy spatial memory length  $c_\epsilon=2\epsilon/L$  expressed in percent, and statistically estimated with the Monte Carlo simulation method. Results for different frequency bands: [300, 400] Hz (thin line), [400, 500] Hz (thin broken line), [500, 600] Hz (thick line).

the continuous model defined by Eqs. (19)–(26), the law of the coupling memory used being defined by Eq. (30). These oscillators with spatial memory act in the  $y$  direction. In fact the equivalent coupling factor  $\alpha$  of the Fig. 5 has been determined as we have explained in Sec. III B. The value of  $N$  is such that<sup>19</sup>  $\underline{n} = \sqrt{N}/\Delta\omega$  which yields  $N=201$ . Each oscillator with spatial memory has a critical damping rate  $\xi_0=0.003$  and a concentrated mass  $\mu_0=1.128 \times 10^{-4}$  kg such that<sup>19</sup>  $\underline{\mu} = \mu_0\sqrt{N}/L$ . Let us note that in Eq. (19), parameters  $\xi_0$  and  $\mu_0$  are denoted respectively as  $\xi$  and  $\mu$ . Consequently, we see that on each MF narrow band, the ratio of the mean mass of the structural fuzzy over the mass of the master structure is only 0.56%, i.e., less than 1%. The master structure is modeled with 200 straight two-node beam finite elements. All the parameters and the location over the beam of each oscillator are simulated using the probability density function defined by Eq. (15) with  $d=4$ .

(5) Figure 6 gives the  $y$  acceleration of the beam at  $x=1.2$  m with structural fuzzy with spatial memory, and concerns the mean value in dB of the square of the modulus of acceleration at discrete frequencies, with a resolution of 10 Hz. The agreement between the responses of reference performed by Monte Carlo simulation for two lengths of spatial memory  $c_\epsilon=1\%$  (thick broken line) and  $c_\epsilon=3\%$  (thin broken line), and the responses obtained by the theory of structural fuzzy  $c_\epsilon=1\%$  (thick line) and  $c_\epsilon=3\%$  (thin line), is satisfying.

### C. Far field radiated by an highly complex submerged structure with structural fuzzy

We are interested in this application to calculate in the MF range the acoustic field radiated by a complex structure with structural fuzzy, immersed in water and excited by mechanical forces applied to the structure.

(1) The primary structure is a slender steel shell cylinder having a large number of transverse stiffeners, a few transverse walls, and several internal three-dimensional

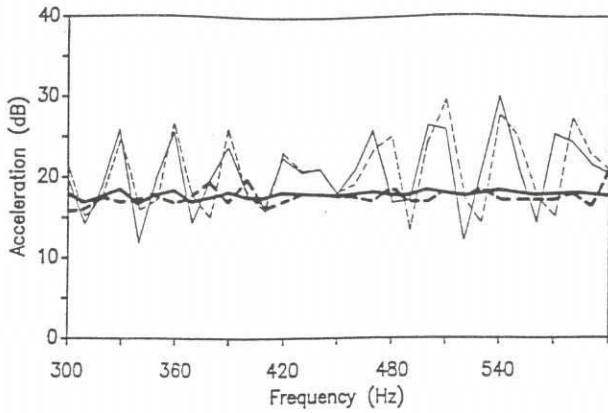


FIG. 6. Mean values in dB at discrete frequencies of the square of the modulus of the transverse acceleration at  $x=1.2$  m of a simply supported beam of length  $L=2$  m, in a vacuum, with a type II structural fuzzy whose  $\underline{g}$  is given by Fig. 5 in function of the fuzzy spatial memory length  $c_\epsilon=2\epsilon/L$ , and excited by a unit transverse load applied at  $x=1$  m. Results performed by Monte Carlo simulation for  $c_\epsilon=1\%$  (broken thick line),  $c_\epsilon=3\%$  (broken thin line), and by the theory of structural fuzzy for  $c_\epsilon=1\%$  (thick line),  $c_\epsilon=3\%$  (thin line).

subsystems. The radius of circular sections of this stiffened cylindrical shell and the shell thickness are variable. This primary structure is globally a three-dimensional structure with an axisymmetric part. The master structure is this primary structure submerged in an unbounded external dense acoustic fluid (water). The medium frequency band of analysis studied is  $\mathbb{B}_r=[0.8, 1.25]$  that we have expressed in terms of the reduced frequency  $\omega_r=\omega/\omega_R$ , where  $\omega_R$  is the ring frequency of the stiffened cylindrical shell. Three kind of external loads denoted as  $M_D$ ,  $F_R$ , and  $F_A$  are applied to the same point  $m_0$  of the cylindrical shell and are respectively a moment about the tangent to the circular section passing by  $m_0$ , a radial force along the normal to the surface of the shell passing by  $m_0$  and an axial force. Each force has a unit modulus and flat spectrum over all the band  $\mathbb{B}_r$ . For the MF analysis method,<sup>6,7</sup> the band  $\mathbb{B}_r$  is partitioned into 4 reduced MF narrow bands. On each MF narrow band the ratio of the mean mass of the structural fuzzy over the mass of the primary structure is only 0.2%, i.e., less than 1%.

(2) The structural fuzzy is applied on all the inner side of the shell and is modeled by an homogeneous orthotropic fuzzy with continuous boundary which acts in the radial direction along the normal to the surface of the shell and which is constructed with the type I probabilistic constitutive law. The input fuzzy parameters of this type I law have the following values for each MF narrow band (see Sec. III A):  $\underline{\mu}=28$  kg/m<sup>2</sup>,  $\underline{\xi}=0.001$ ,  $\underline{n}=10^{-2}$  (rad/s)<sup>-1</sup>, with dispersion parameters  $\underline{\lambda}_1=\underline{\lambda}_2=\underline{\lambda}_3=0.001$ .

(3) With the theory of the structural fuzzy, the master structure (primary structure and external fluid coupling operator), and the structural fuzzy with continuous boundary, are modeled by finite element method. Taking into account that the dispersion parameters of the structural fuzzy are very small, we are going to present only the results concerning the mean of the responses.

(4) Figure 7 presents the levels of mean pressure in

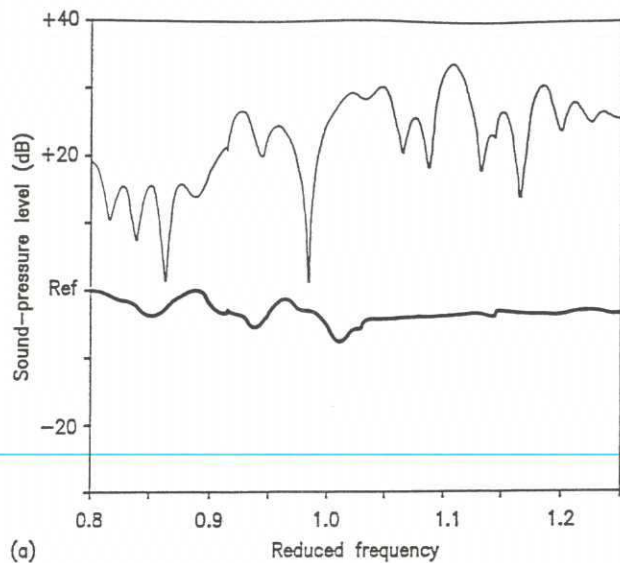
dB, at discrete reduced frequencies, pressure which is radiated in the far radial direction passing by the point  $m_0$ . Figure 7(a), (b), and (c) shows, respectively, the levels for the  $M_D$ ,  $F_R$ , and  $F_A$  forces. On each figure, the thick line represents the response of the master structure with the structural fuzzy, and the thin line the response without structural fuzzy. Figure 7(c) shows that the structural fuzzy has a small effect on the pressure levels because in this case the orthotropic fuzzy acts in the radial direction of the shell while the axial excitation  $F_A$  induces mainly axial dynamical deformations in the shell, and consequently, this structural fuzzy has a small effect on the levels of radiated pressure. However smoothing of the peaks is preserved in the three cases.

## VII. CONCLUSIONS

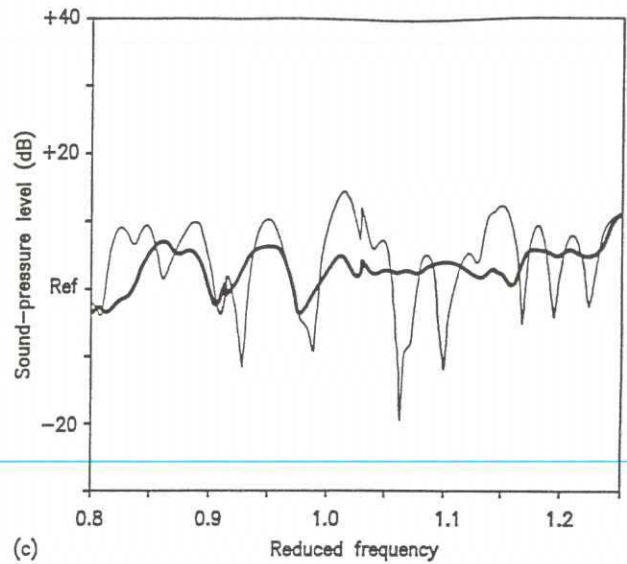
(1) In the area of the physical modeling of highly complex mechanical systems and in the medium frequency range, the theory of structural fuzzy that we have presented in this paper and which is based on a probabilistic approach, is useful: (a) to model globally all the structural details which are known but which are not accessible to conventional modeling while keeping a usual deterministic modeling for the master structure. In this case, structural fuzzy is the result of the natural complexity of the structure or is due to the implementation of a voluntary disorder in the master structure for some dynamical behavior reasons; (b) to model a structure whose some structural details are unknown or are imprecisely known, for instance a lack of knowledge about the presence of subsystems and some uncertain data. In this case, structural fuzzy is not due to the impossibility to model conventionally a large number of known structural details as in the previous point, but is due to the fuzziness in the knowledge of the structure; (c) to simplify the modeling of a structure, the master structure being always conventionally modeled, but some substructures being voluntarily suppressed and replaced by fuzzy substructures in order to include an automatic sensitivity analysis to input parameters of the modeling.

(2) In the area of the construction of the structural fuzzy theory that we have presented: (a) Structural fuzzy is modeled by a boundary random impedance operator and does not increase the number of degrees of freedom of the master structure modeling. (b) This theory allows us to preserve and to predict modulus and phases of master structure responses, i.e., to study the wave propagations in the primary structure and in the internal and external acoustic fluids. Consequently we have a theory for solving in the MF range the forced vibrations, the radiation and the scattering problems, in presence of structural fuzzy. (c) Boundary random impedance operator is constructed with some probabilistic constitutive laws related to the classes of fuzzy. Two probabilistic constitutive laws have been developed from two different elementary deterministic mechanical models. The first one represents a structural fuzzy with no spatial memory effect inside the fuzzy, while the second one has this memory effect. These two laws are easy to use in a modeling and depend only on usual me-

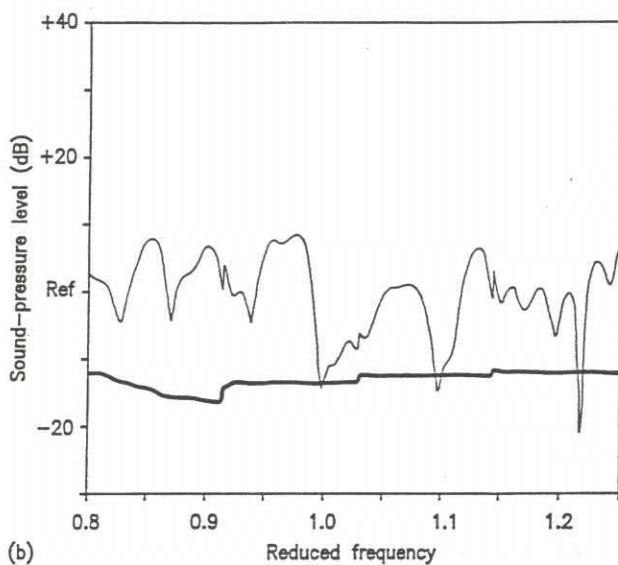




(a)



(c)



(b)

FIG. 7. Mean values of the sound pressure in dB at discrete reduced frequencies of the far field radiated in a radial direction by a complex stiffened cylindrical shell structure submerged in water and excited by (a)  $M_D$ —moment, (b)  $F_R$ —radial force, (c)  $F_A$ —axial force, applied at the same point of the shell. Shown are the results without structural fuzzy (thin lines) and the results with type I radial orthotropic fuzzy performed by the theory of structural fuzzy (thick lines).

chanical scalar parameters, called input fuzzy parameters, and which are the input data of these laws. For each input fuzzy parameter, a mean value and a dispersion around the mean are given in input. The validity of these two constitutive laws was analyzed by numerical simulation on a few standard structures in previous works<sup>1,2,4,5</sup> and in the present paper. It should be noted that we introduce in this paper a new type of fuzzy, characterized by a type II probabilistic constitutive law. (d) An important problem still remains open to further developments and concerns the possibilities to fit the input fuzzy parameters of the two proposed constitutive laws on the basis of statistical data from experiments on real structures or on the basis of perturbed numerical models of mechanical subsystems.<sup>3</sup> (e) Another problem is opened and concerns the choice of the uniform probability density distribution for the input fuzzy parameters.

(3) In the area of the fuzzy solution in the MF range, the method that we have developed avoids the Monte Carlo simulation which would be for this kind of problem much too expensive. This method is such that: (a) the effects of the structural fuzzy on the master structure ap-

pear as a mean impedance of the structural fuzzy plus a random fluctuation around its mean due to the dispersion of the input fuzzy parameters; (b) the mean impedance of the structural fuzzy adds to the impedance of the master structure, and the resulting impedance is exactly treated without introducing any approximation; (c) the random fluctuation part is treated as a perturbation of the already known solution for the master structure with its mean fuzzy. For sufficiently small value of the dispersion parameters, the first order expansion of the random fluctuation of the solution is sufficient. If it is not the case, a higher-order expansion can be used; (d) all the statistical moments of the response can theoretically be calculated. In practice the computer code that we have developed calculates only the mean value of the solution and the variance, i.e., the deviation from the mean, using a first or a second-order expansion. We have verified on beams, plates, and shells in vacuo with structural fuzzy<sup>5</sup> that for a dispersion of the input fuzzy parameters going at least up 50%, first order expansion was reasonably sufficient.

(4) Finally in the area of the applications, we have studied several cases for simple and highly complex struc-

tures with structural fuzzy, in vacuo, or submerged in a dense fluid, and studying in this last case the radiation problems. A contrario a large area is opened concerning applications for acoustic scattering from dense fluid-loaded elastic structure with structural fuzzy. Studies have begun in this way.<sup>13,14</sup>

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