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Quantum Mechanics Revisited

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Abstract

From a general study of the relations between models, meaning the variables with their mathematical properties, and the measures they represent, a new formalism is developed, which covers the scope of Quantum Mechanics. In the paper we prove that the states of a system can be represented in a Hilbert space, that a self-adjoint operator is associated to any observable, that the result of a measure must be the eigen value of the operator and appear with the usual probability. Furthermore an equivalent of the Wigner's theorem holds, which leads to the demonstration of the Schrödinger equation, still valid in the General Relativity context.

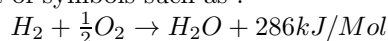
These results are based on general assumptions, which do not involve any hypothesis about determinism, the role of the observer or others usually debated.

The formalism presented sustains the usual "axioms" of Quantum Mechanics, but open new developments, notably by considering localized variables, functions and sections on vector bundles and their jet extensions..

After almost a century the interpretation of quantum mechanics stays a largely open subject. If, for most of the workers who use it everyday, this is not a matter of concern, the unending flow of papers on this topic shows that, for some people at least, this is an issue. Rightly so, because, whatever one's philosophical belief, one cannot feel comfortable with a successful scientific theory which, according to some of the most authorized voices in physics, is beyond our understanding. And a scientist cannot truly be convinced by the usual argument : "It works, so we have to accept it". The capability to provide experimentally verifiable predictions is not the only criterium for a scientific theory. A "black box" in the "cloud" which answers rightly to our questions is not a scientific theory, if we have no knowledge of the basis upon which it has been designed. A scientific theory should provide a set of concepts and a formalism which can be easily and indisputably understood and used by the workers in the field. And this leads to look for a theory which helps us to describe, understand and as far as it is possible, explain, the world we live in.

It would be preposterous to try to refute or simply to interfere in the fierce debate which has involved the greatest scientists of the past century. So, let us

say that I step aside the philosophical debate, even if, for the sake of clarity, I need to say that I side with a realist interpretation of physics, meaning that there is a physical reality which exists independantly of our beliefs or "conscience". My focus is more limited. From a realist point of view, a theory itself can be seen as an object of its own. A physical theory is a construct, in which some phenomena are singled out, are given a representation in some formal system, with the double purpose to explain why and how the real world works, and to make predictions. The validation of a theory comes from a process of rational and objective experimentation in which the predictions are confronted with the measures, taken as they are, but always interpreted in the format of the representation of the theory. So, between the big discourse about physical laws, which identifies atoms, fields, forces,... and the brute collect of data there is an intermediary step in which the concepts and the data are formatted in order to be usable. And this format, even if it is hidden from the view of the large public because its understanding requires specialized knowledge, plays a central role in the acceptance and usage of a theory. This step is what I call a model, based upon some general concepts, but translated in a workable formal system. The most illuminating model is the atomic representation used in chemistry. A set of symbols such as :



tells us almost everything which is useful to understand and work with chemical experiments.

The analysis of formal systems is most advanced in mathematics, where mathematical logic has helped to understand (and correct) the consistency and formalization of the theory of sets or arithmetic.

In physics the formal system relies almost exclusively on mathematics, where physical objects and their properties are represented by mathematical objects with their corresponding properties. And it is clear that progres in physics has been closely related to the advances in mathematics, which provide a larger collection of representation. Classical mechanics could not have been developed without the derivative and integral calculus, General Relativity without the concept of manifolds, and the Standard Model without the support of the representation of groups.

So where do we stand in Quantum Physics ? The question is a bit muddled, as any student discovers in the many "introductory books" on the subject, by a constant mixing of physical experiments and formal systems, where it is not always easy to understand if the formal system validates the experiments or the converse. So he can be told that the position and the momentum of a particle cannot be simultaneously measured because their operators do not commute, but if he asks why it is not so at a macroscopic level, he faces a long explanation which sums up usually to "this is the quantum world", understand the magic kingdom.

Actually, "Quantum Physics" encompasses several theories, with three distinct areas :

- i) The duality matter / wave : the indisputable fact that particles can behave

like fields which propagate, and conversely force fields can behave like pointwise particles. This departure from the classical "picture" (say Newtonian mechanics and Maxwell's electromagnetism) requires a new formalism, that is certainly well rendered by the "quantum mechanics" proper, but goes far beyond. The string, brane, or quantum loop theories illustrate the need for a new model for the world of particles. The "spin" of a particle is a phenomenon which requires similarly a new theoretical foundation (probably linked to the first one).

ii) The "quantum mechanics" (QM) which is presented in all the books on the subject (such as summarized by Weinberg) as a set of "axioms" :

- Physical states of a system are represented by vectors in a Hilbert space, defined up to a complex number (a ray in a projective Hilbert space)

- Observables are represented by hermitian operators

- The only values that can be observed for an operator are one of its eigen values λ_k corresponding to the eigen vector ψ_k

- The probability to observe λ_k if the system is in the state ψ is proportional to $|\langle\psi, \psi_k\rangle|^2$

- If two systems with Hilbert space H_1, H_2 interact, the states of the total system are represented in $H_1 \otimes H_2$

- The Schrödinger equation

iii) The "Quantum theory of fields" (QTF) which is broadly an adapted application of the previous theories to interacting particles, and is summarized in the standard model.

These three aspects are entangled, for historical, practical and pedagogical reasons, but are distinct. Whatever their success as a predictive tool, QM and QTM cannot alone explain the duality matter / field, and the fact that the basic axioms of QM should apply to any system is still a matter of puzzlement.

In this paper I will focus on Quantum Mechanics proper, that is the set of axioms which sustain the models developed for studying the atomic and sub-atomic world. My purpose is to look for a logical and physical basis for these axioms. So I will stay on the most general level of physics, meaning a "system" which could be any object of the study of a physicist.

It is clear that, in its common and usually practiced form, Quantum Mechanics is not fully satisfying for any soul in quest of mathematical correctness. In order to improve the formal side of Quantum Mechanics, most, if not all the studies, have followed the path of an algebraic construct, whether in the general picture (Bratelli, Araki and others) or in the quantum theory of fields (Halvorson and others). In this framework the focus is moved from the Hilbert space to the set of observables, and indeed a system is itself defined through the algebra of its observables. This provides a more comfortable background to develop a mathematical theory, notably with respect to the always sensitive issues of continuity, and many results that are certainly useful. But this approach has a fundamental drawback : it leads further from an understanding of the physical foundations of the theory itself. To tell that a system should be represented by a von Neumann algebra does not explain more than why a state should be represented in a Hilbert space at the beginning. The sophistication of

the mathematical wrapping does not improve the understanding of the foundations of Quantum Mechanics : in both cases the axioms are just that, they are granted. And actually they are more muddled. There is no use to repeat that "the experiments validate the theory" : as long as the theory does not tell us why and how (other than through a philosophical discourse) it does not work at a macroscopic scale, it is not validated, indeed it is invalidated daily.

The approach in this paper is, in some ways, the opposite. We will focus on the interaction between measures and formal representations of a system. And to do so we will stick mainly to the common presentation of Quantum Mechanics, which, besides its formal imperfections (which are not my concern), is closer to the physics as it is done every day. We will just try to understand what lies besides the narrative which begins with something like "Let be a system,..." and later goes on by "let be X,Y, Z the fields,...". As this is the universal presentation of any physical experiment, it should deserve more than a putative glance. And from the study of the relations between experiments and formal physical theory, in the most general context, we will **prove** the following :

- the state of a system can be represented in an affine Hilbert space
- if two systems interact it is possible to represent the states in the tensor product of the Hilbert spaces
- to each observable is associated a self-adjoint operator
- the results which can be obtained through an observable belong to the vector space of its operator, and they appear with a probability $|\langle\psi, \psi_k\rangle|^2$
- an analog to the Wigner's theorem holds for any gauge transformations between observers
- the Schrödinger equation holds in the General Relativity context, and the presence of a universal constant such as \hbar is necessary

Many theorems will be used in this paper. They concern a broad range of mathematical topics. So it is convenient to refer these theorems to a compendium of mathematics, that I have published recently. They are cited as (JCD Th. XXX).

1 DEFINITIONS

It is necessary to introduce some definitions to set the picture in which we will work.

The process of measure, meaning of going from a physical system to a an understanding of the results, comprises several steps.

$$\text{System} \xrightarrow{\text{Measure}} \text{Data} \xrightarrow{\text{Assignment}} \text{Model} \xrightarrow{\text{Interpretation}} \text{Formal theory}$$

1.1 System

Many statements in physics start with the word "system". Its ubiquitousness requires some precisions.

A physical system is a delimited area of the universe, including all the physical objects that it comprises. A physical system changes with the time. We assume that it can be observed at different times (not necessarily continuously), so that its identity is preserved : this is always the same system that is observed (with possibly all the alterations that can occur within). Moreover the measures always occur at some time : its evolution is followed by a sequence of measures. The measures can be related to a phenomenon or to its evolution, but in any case it is always assumed that they are done at a given time.

1.2 Model

To describe the system the physicist uses a model : this is a finite set of quantities - the **variables** - related to identified physical phenomena occurring in the system, which could be measured. So a physical model relies on a choice : it does not include everything, the choice may be relevant or not.

The variables are the crux of the picture. In one hand they define the frame which is used to collect and organize the data, on the other hand they are mathematical objects with a precise definition in some formal theory.

The purpose of the model may be to know the initial data prior to an experiment (in this case no relation is assumed between the variables), or to check the validity of the forecasts of some theory. In this case the variables are supposed to be linked, but the outcome of the measures is taken as it is, meaning that their analysis with regard to the theory (checking that the assumed relations hold) is another phase which is not in the scope of the present paper. Here we focus on the relation between variables and measures and the variables can be considered as independant.

The model of a system may or not include the external actions on the system : this is up to the physicist who defines the system, and if they are included they are variables of the model, subject to possible measurements.

Data, coming from measures, are assigned to each variable. It is assumed that there is no "intermediary variables", meaning a variable the value of which is computed from other variables, as they would be useless in this framework.

If the measures are related to the change of some phenomenon the variables are the rate of change at a given time (the value of their derivative).

The measures are supposed to be possible, in the meaning that they are related to observable phenomena and their outcomes are real figures.

A **configuration** of the system is the set of all the measures that can be possibly obtained with a model, even if all these measures are not usually made.

1.3 Variables

Each variable of the model is described in some formal system, and we assume that it is represented by some mathematical object having precise properties (such as tensoriality, regularity,...). So they are maps which acquire a definite value (meaning real figures) for a configuration of the system. A configuration is a (huge) set of raw data, and a state of the system is the organized set of these values, where the figures have been assigned to their respective mathematical objects.

The number of variables is finite, say N . So we have a family $(\Xi_n)_{n=1}^N$ of variables, attached to a family of procedures $(\varphi_n)_{n=1}^N$, which are designed to assign to the variables the values of the measures of any configuration.

The variables of the model can be sorted according to their range and their domain.

Range : Some variable can take only a finite set of values. Other variables can take continuous values, either as scalars or as components of some tensorial mathematical objects.

Domain : Some variables are related to the whole of the system or to specific objects singled out in the system. Others are localized, meaning that their domain is an uncountable set of points, and so they are represented by functions. In particular this happens whenever :

- a force field is involved : by definition its extension is all over the spatial area of the system, so the possible measures cover the value of the field at each point

- particles (or whatever objects which are deemed localized) are involved, which are either indistinguishable, or can be subjected to a transformation which is deemed significant. Then the measures should include some procedure telling which kind of particle is present at each location

1.4 Assignment

Whenever a variable is a function the assignment process comprises two steps : the collection of the data, and the estimation of the function from these data, usually by some statistical method from a sample of data. However the size of the sample is not limited a priori and the effective value of the variable could be measured in any point. If some precise specification is assumed for the function, say that it depends on a finite number of parameters, these parameters are the variables, but then the specification is deemed pertinent (it is not checked in the model). In the other cases the specification is limited to the appartenance to some family of functions, chosen usually with regards to its regularity (such as continuity). The description of the function by some parameters is then done in another step. The meaning of the representation is that the observed value should be consistent with the choice of the function. If a field is supposed to be constant in the area of the system then its measure shall always give the same result whatever the location of the measure.

In the following we assume that the variables are either discrete variables, taking a finite number of values, or "continuous" variables that can be represented as vectors of a vector space, possibly infinite dimensional. The latter class includes variables (defined for single objects or the system) or functions with values in a vector space.

Functions which take discrete values may enter this picture, if it is assumed (in accordance with the usual experimental process) that a continuous function is first measured, then adjusted to a family of step functions (with constant value on some domain). Then they are considered here as continuously valued functions.

1.5 States

A state of the system is described in the model by a finite set of variables, discrete or continuous, and a finite set of functions which represent the variables whose output is an infinite number of measures. By the assignation process the physicist goes from a configuration (the set of raw measures) to a state (an organized set of mathematical objects which have precise values).

The picture is very similar to any model of statistical mechanics with an infinite number of "degrees of freedom". The difference here is that we do not involve any lagrangian or similar law linking the measures. In quantum mechanics it is usual to introduce a "wave function" $\psi(x)$ to represent a particle. Without entering the debate about the quantity that could be measured, in the model it should appear as a function ψ with the proper mathematical characteristics, so belonging to an infinite dimensional space of functions.

1.6 Example

The system is composed of a particle with its position q in some region Ω and momentum p , spin component s , in a force field F .

The set of configurations M is all the conceivable measures of s (say an integer), q and p (say the components of 3 vectors in some frame), F (the components of a vector at each point of Ω). So M is a (huge) set of figures, a small part of which is effectively known (but any measure could theoretically be done).

The set of states E_0 is a subset of the product of two vector spaces (one for each value of s) such as : $\mathbb{R}^3 \times \mathbb{R}^3 \times C_b(\Omega; \mathbb{R}^3)$ where $C_b(\Omega; \mathbb{R}^3)$ is the space of bounded functions on Ω valued in \mathbb{R}^3 .

It is clear that the association between a measure (made by some complicated procedure φ) and a variable is founded on a formal model, which brings some order in M .

2 HILBERT SPACE

In this first section we will prove that the state of the system can be represented in a Hilbert affine space. The system is studied from a static point of view (there is no evolution involved).

2.1 The Hilbert space of states

We need to distinguish discrete and continuous variables. In a first step we assume that there is no "discrete" variables.

2.1.1 First proposition without discrete variables

Proposition 1 *Whenever the measures on a system involve a function, without any discrete variable, the set of states of the system can be embedded as an open connected subset H_0 of an infinite dimensional, separable, real Hilbert space H , defined uniquely up to isomorphism, and there is an open convex subset, containing 0 , such that its vectors represent states of the system.*

Proof.

i) Let us denote M the set of possible configurations of the system corresponding to continuous variables. Each configuration is described in the set of maps $(\varphi_n)_{n=1}^N$ whose values are scalars or functions, corresponding to a state of the system. Without loss of generality we can assume that whenever a variable is a function, this function belongs to a Banach vector space : the space of bounded, or continuous, or with compact support functions, the latter happening as the geometrical area covered by the system is bounded. So the set E_0 of the states of the system is some open subset of a Banach vector space E .

The procedures $(\varphi_n)_{n=1}^N$ used to do the measures are assumed to cover all the possible configurations, possibly by combining several procedures addressing specific ranges of values. And when different procedures are used on the same subset M_0 of configurations, it is assumed that there is an unambiguous way (a calibration) to convert the measures done with a procedure n_1 into the measures done with a procedure n_2 on the same subset M_0 .

So the set M of configurations has the structure of a manifold modeled on the Banach vector space E . Whenever a function is involved, the manifold M is infinite dimensional, and if not it is finite dimensional.

iii) Because of the imprecision of the measures, the physicist introduces some kind of granularity in the set M . It can be done by several methods, but they amount to the definition of steps of tolerance. Around each result of a measure a collection of neighborhoods is defined, with the purpose to assess the proximity of states. This collection is usually finite, but it suffices that it is countable. This collection generates, by union and finite intersection, open subsets and defines a topology which is second countable (JCD Def.551). Therefore M is separable (JCD Th.555). It is reasonable to assume that this topology is also regular, meaning that for any point p of M , closed subset C of M , there are

open subsets O, O' such that : $p \in O, C \subset O', O \cap O' = \emptyset$ (JCD Def.556). Being separable and regular M is also metrizable (JCD Th.585). Because the number of charts $(\varphi_n)_{n=1}^N$ is finite E is also separable.

iv) From there the Henderson theorem states that, if M is infinite dimensional, it can be embedded as an open subset H_0 of an infinite dimensional separable real Hilbert space H , defined uniquely up to isomorphism. There is a countable, locally finite, simplicial complex K such that M is homeomorphic to $[K] \times H$. Moreover this structure is smooth and the set $H - H_0$ is isomorphic to H , ∂H_0 is homeomorphic to H_0 and $\overline{H_0}$ (Henderson, JCD Th.1324). So $H - H_0$ is connected and its complement $\overline{H_0}$ is also connected. If M is finite dimensional, say N , it can be embedded in \mathbb{R}^N which is a finite dimensional Hilbert space. And we come back to the classical picture of statistical mechanics with a finite dimensional configuration space.

v) Let us denote $\langle \rangle$ the scalar product on H (this is a bilinear symmetric positive definite form) and $\| \cdot \|$ the associated norm. The map : $H_0 \rightarrow \mathbb{R} :: \langle u, u \rangle$ is bounded from below and continuous, so it has a minimum u_0 in H_0 that we call a "ground state". By translation of H_0 with u_0 we can assume that 0 belongs to H_0 . There is a largest convex subset of H which contains H_0 , defined as the intersection of all the convex subset contained in H_0 . Its interior is an open convex subset C . It is not empty : because 0 belongs to H_0 which is open in H , there is an open ball $B_0 = (0, r)$ contained in H_0 . ■

2.1.2 Complex structure

Proposition 2 *The set H can be endowed with the structure of a complex Hilbert space*

The Hilbert space H has the structure of a real vector space. Any infinite dimensional real vector space admits, on the same set, a complex structure (JCD Th.313) $J \in \mathcal{L}(H; H) : J^2 = -Id_H$ which, combined with the scalar product, can define a hermitian, definite positive sesquilinear form $\gamma(u, v)$ on H , making it a complex Hilbert space.

Proof.

H has a countable hilbertian basis $(\varepsilon_\alpha)_{\alpha \in \mathbb{N}}$ because it is separable.

Define :

$$J(\varepsilon_{2\alpha}) = \varepsilon_{2\alpha+1}; J(\varepsilon_{2\alpha+1}) = -\varepsilon_{2\alpha}$$

$$\forall \psi \in H : i\psi = J(\psi)$$

$$\text{So : } i(\varepsilon_{2\alpha}) = \varepsilon_{2\alpha+1}; i(\varepsilon_{2\alpha+1}) = -\varepsilon_{2\alpha}$$

The bases $\varepsilon_{2\alpha}$ or $\varepsilon_{2\alpha+1}$ are complex bases of H :

$$\psi = \sum_{\alpha} \psi^{2\alpha} \varepsilon_{2\alpha} + \psi^{2\alpha+1} \varepsilon_{2\alpha+1} = \sum_{\alpha} (\psi^{2\alpha} - i\psi^{2\alpha+1}) \varepsilon_{2\alpha} = \sum_{\alpha} (-i\psi^{2\alpha} + \psi^{2\alpha+1}) \varepsilon_{2\alpha+1}$$

$$\|\psi\|^2 = \sum_{\alpha} |\psi^{2\alpha} - i\psi^{2\alpha+1}|^2 = \sum_{\alpha} |\psi^{2\alpha}|^2 + |\psi^{2\alpha+1}|^2 + i \left(-\overline{\psi^{2\alpha}} \psi^{2\alpha+1} + \psi^{2\alpha} \overline{\psi^{2\alpha+1}} \right)$$

$$= \sum_{\alpha} |\psi^{2\alpha}|^2 + |\psi^{2\alpha+1}|^2 + i (-\psi^{2\alpha} \psi^{2\alpha+1} + \psi^{2\alpha} \psi^{2\alpha+1})$$

Thus $\varepsilon_{2\alpha}$ is a hilbertian complex basis

H has a structure of complex vector space that we denote $H_{\mathbb{C}}$

The map : $T : H \rightarrow H_{\mathbb{C}} : T(\psi) = \sum_{\alpha} (\psi^{2\alpha} - i\psi^{2\alpha+1}) \varepsilon_{2\alpha}$ is linear and continuous

The map : $\bar{T} : H \rightarrow H_{\mathbb{C}} : \bar{T}(\psi) = \sum_{\alpha} (\psi^{2\alpha} + i\psi^{2\alpha+1}) \varepsilon_{2\alpha}$ is antilinear and continuous

Define : $\gamma(\psi, \psi') = g(\bar{T}(\psi), T(\psi'))$

γ is sesquilinear

$$\gamma(\psi, \psi') = g\left(\sum_{\alpha} (\psi^{2\alpha} + i\psi^{2\alpha+1}) \varepsilon_{2\alpha}, \sum_{\alpha} (\psi'^{2\alpha} - i\psi'^{2\alpha+1}) \varepsilon_{2\alpha}\right)$$

$$= \sum_{\alpha} (\psi^{2\alpha} + i\psi^{2\alpha+1}) (\psi'^{2\alpha} - i\psi'^{2\alpha+1})$$

$$= \sum_{\alpha} \psi^{2\alpha} \psi'^{2\alpha} + \psi^{2\alpha+1} \psi'^{2\alpha+1} + i(\psi^{2\alpha+1} \psi'^{2\alpha} - \psi^{2\alpha} \psi'^{2\alpha+1})$$

$$\gamma(\psi, \psi) = 0 \Rightarrow g(\psi, \psi) = 0 \Rightarrow \psi = 0$$

Thus γ is definite positive ■

This does not impact the measures which are always supposed to be real. The complex structure is more a convenience, for mathematical purposes, than a physical requirement.

2.1.3 Second proposition including discrete variables

Proposition 3 *Whenever the measures on a system involve a function, and discrete variables taking d values, the set of states of the system can be embedded as d open connected subsets H_{\varkappa} of an affine space, modelled on an infinite dimensional, separable, real Hilbert space H , defined uniquely up to isomorphism.*

Proof.

i) Now we assume that there are n discrete variables $(D_k)_{k=1}^n$ and d_k possible values for each. Denoting these values by consecutive integers : $1, 2, \dots, d_k$ the possible configurations of the system are any combination $\{i_1, i_2, \dots, i_n\}$, $i_k \in \{1, 2, \dots, d_k\}$ that is a total of $d = d_1 \times d_2 \times \dots \times d_n$ different states. So we can consider a single discrete variable D taking the values $\varkappa = 1, 2, \dots, d$

ii) It will be convenient to represent D as a vector in \mathbb{C}^d each value of D being represented as a vector of the canonical basis of \mathbb{C}^d .

iii) By definition the system is in one of the configurations given by the value \varkappa of the variable D , and the value of the other measures, which are common. Within each of these discrete configurations the previous result holds : the continuous variables are represented by a vector ψ belonging to an open, connected, subset H_{\varkappa} of the same Hilbert space H .

As above we translate H_{\varkappa} such that the minimum of $\langle u, u \rangle$ is 0.

By the Zorn lemma we can pick up any d vectors $(v_k)_{k=1}^d$ orthonormal to represent the values of D by a vector $\psi_d \in H$.

iv) We can identify a state of the system as a couple (ψ_d, ψ) of vectors of H . It is convenient to define on the set (ψ_d, ψ) the structure of affine space \widehat{H} modelled on the Hilbert space H :

Define the map : $\rightarrow : (H \times H) \times (H \times H) \rightarrow H :: \overrightarrow{(\psi_d, \psi), (\psi'_d, \psi')} = \psi' - \psi$

It meets the required properties :

$$\overrightarrow{(u_1, \psi_1), (u_2, \psi_2)} + \overrightarrow{(u_2, \psi_2), (u_3, \psi_3)} + \overrightarrow{(u_3, \psi_3), (u_1, \psi_1)} = \psi_2 - \psi_1 + \psi_3 - \psi_2 + \psi_1 - \psi_3 = 0$$

For (u, ψ) fixed, the map $\tau : H \rightarrow \widehat{H} :: \tau(v) = (u, \psi + v)$ is a bijection.

With this structure the ground states corresponding to $D = \varkappa$ are represented by a point G_\varkappa with first coordinate $\psi_d = v_\varkappa$ and the set of states $(v_\varkappa, H_\varkappa)$ is an open subset \widehat{H}_\varkappa in the hyperplane $(G_\varkappa, H_\varkappa)$. We have a collection of d open, connected, subsets \widehat{H}_\varkappa of the affine space \widehat{H} modelled on the Hilbert space H . ■

It is clear that this construct is quite artificial, and that many parameters are our choice. However this is a convenient representation, notably to define "subrepresentations" (related to a part of the discrete variables) and the scalar product of the vectors of two ground states $\langle \psi_d, \psi'_d \rangle$. And it fits well with the representation used for the continuous variables. Moreover we keep the important property that \widehat{H}_\varkappa is an open subset of the affine Hilbert space.

2.1.4 Structures involved and notations

For each non discrete measure Ξ_n the results belong to an open subset E_0 of a Banach vector space E_n and $E = \bigoplus_{n=1}^N E_n$.

An atlas of the manifold M is given by an open cover $(O_a)_{a \in A}$ of M , and a collection of maps $(\Phi_a)_{a \in A} : \Phi_a : O_a \rightarrow E$ where $\Phi_a = (\varphi_{an})$ is a set of measuring maps. The sets $E_a = \Phi_a(O_a)$ are open subsets of the Banach vector space E .

The embedding of M is the diffeomorphism $\iota : M \rightarrow H_0 \subset H$. So the map $\pi_a : O_a \rightarrow H :: \psi = \pi_a(x_a) = \iota \circ \Phi_a^{-1}(x_a)$ is a diffeomorphism.

$$\begin{array}{c} M \xrightarrow{\Phi} E_0 \\ \downarrow \\ \downarrow \iota \\ \downarrow \\ H_0 \end{array}$$

The manifold M has a the structure of of a smooth Hilbert manifold, so there is an atlas such that the maps Φ_a, ι are smooth.

Another set of procedures $(\varphi'_p)_{p=1}^P$ induces on M_\varkappa an atlas $(E, (O'_b, \Phi'_b)_{b \in B})$ which is compatible with the atlas $(E, (O_a, \Phi_a)_{a \in A})$, and thus defines the same structure of manifold, if and only if $\forall (a, b) \in A \times B, O_a \cap O'_b \neq \emptyset$ the map $\Phi'_b \circ \Phi_a^{-1}$ is a diffeomorphism on $E'_b \cap E_a$. Then $H_\varkappa, H'_\varkappa$ are diffeomorphic, and H, H' can be identified.

Because ι is a diffeomorphism, a measure Ξ_n can be seen as a map $\Xi_n : H_0 \rightarrow E_n$ (with the awareness that Ξ_n is formally defined by different, compatible, charts, on an open cover). So in the following a chart X will be seen as a map $X : H_0 \rightarrow E$ and the subset \widehat{H}_0 of the affine Hilbert space \widehat{H} will be called the set of states of the system. When a variable is a function, the

value $\Xi_n(\psi)$ is a function belonging to the family of functions E_n with domain in some set.

The association $H_0 \rightarrow E$ is not unique : any chart X defines a different association. The same point in H can be read as different vectors x of E , depending on the chart used. Conversely a set of measures x of E can be associated to different vectors ψ of H , depending on the chart. In the following we will use the latter point of view.

A complete set of measures is comprised of X (valued in E) and D (valued in $\{D\}=1,2,\dots,d$). The set of states has the structure of a subset \widehat{E}_0 of a Banach affine space \widehat{E} modelled on the Banach vector space E . The set of open subsets $(\widehat{H}_\varkappa)_{\varkappa=1,\dots,d}$ in \widehat{H} with $\widehat{H}_\varkappa = (v_\varkappa, \widehat{H}_\varkappa)$ is an open subset of \widehat{H} denoted \widehat{H}_0 .

2.1.5 Comments

1. Each subset \widehat{H}_\varkappa is connected, but not necessarily convex. The usual operations of a vector space are available in H but it can happen that their result lays out of H_\varkappa .

2. The topological considerations are crucial in the proof, and the separability of the model is a direct consequence of the granularity of the measurement procedures.

4. The space H has no physical content or meaning, it is only a part of the formalism which is used. The evolution with the time of the vector ψ representing the state of the system, which is seen further below, has nothing to do with any kind of propagation.

2.2 Linear charts

The charts $X : H_0 \rightarrow E_0$ do exist and are defined by the practical association of data and states of the system, but usually are not formalized by the observer. So in order to be able to use H we need some formal way to associate vectors of E and vectors of H . As the choice of compatible charts does not matter, it is handy to find linear charts. We prove now the following:

Proposition 4 *For any chart $X : H_0 \rightarrow E_0$ and basis $(e_i)_{i \in I}$ there is an inner product on E such that :*

- *there is a unique orthonormal basis $(\tilde{e}_i)_{i \in I}$ on E , and a unique hilbertian basis $(\tilde{\varepsilon}_i)_{i \in I}$ of H with $X(\tilde{\varepsilon}_i) = \tilde{e}_i$*
- *the map $\Upsilon : E \rightarrow H :: \Upsilon(\sum_{i \in I} \tilde{x}^i \tilde{e}_i) = \sum_{i \in I} \tilde{x}^i \tilde{\varepsilon}_i$ is a linear isometry*
- *any vector $\psi \in H$ can be written : $\psi = \sum_{i \in I} \langle \phi_i, \psi \rangle_H \varepsilon_i$ with $(\phi_i)_{i \in I}$ uniquely defined vectors of H*

For each value of the discrete variable there is such a linear map Υ_\varkappa and the collection (Υ_\varkappa) is an affine atlas for \widehat{H}_0

We will use the following result (see Neeb p.60,61 and JCD Def.1155 for more) :

If F is a Hilbert space of functions : $F \subset \mathbb{C}^E$ with domain any set and valued in \mathbb{C} or \mathbb{R} such that the evaluation maps : $\forall x \in E : \widehat{F}_x : F \rightarrow \mathbb{C} :: \widehat{F}_x(f) = f(x)$ are continuous, then $\widehat{F}_x \in F'$ (the dual of F) and has an associated vector : $F_x \in F : \langle F_x, f \rangle = \widehat{F}_x(f) = f(x)$ and the function : $K : E \times E \rightarrow \mathbb{C} :: K(x, y) = \langle F_x, F_y \rangle = \widehat{F}_x(F_y) = F_y(x)$ is a positive kernel, in the meaning that for any finite subsets $(x_p, y_p)_{p=1}^n$ the $n \times n$ matrix $[K(x_p, y_p)]$ is hermitian, positive semi-definite and $|K(x, y)| \leq K(x, x)K(y, y)$

Moreover if K is a positive kernel on E and X is any map : $X : H \rightarrow E$ on any set H , then $X^*K(\psi, \psi') = K(X(\psi), X(\psi'))$ is still a positive kernel.

Proof.

At first we consider the continuous variables, for a fixed value of the discrete variables.

i) The Hilbert space H has a positive definite kernel : $P : H \times H \rightarrow \mathbb{C} :: P(\psi_1, \psi_2) = \langle \psi_1, \psi_2 \rangle$. Thus, for any chart X , the function : $K : M_{\mathcal{X}} \times M_{\mathcal{X}} \rightarrow \mathbb{C} :: K(X(\psi_1), X(\psi_2)) = P(\psi_1, \psi_2) = \langle \psi_1, \psi_2 \rangle$ is positive definite on M , and we have a triple (M, X^{-1}, H) . Its canonical realization is the Hilbert space H_K of functions $K_x : M \rightarrow \mathbb{C} :: K_x(y) = K(x, y)$ where $x, y \in M$ which has for scalar product : $\langle K_x, K_y \rangle_{H_K} = K(x, y) = \langle X^{-1}(x), X^{-1}(y) \rangle_H$

So : $\forall \psi_x, \psi_y \in H : K_{X(\psi_x)}(X(\psi_y)) = K(X(\psi_x), X(\psi_y)) = \langle \psi_x, \psi_y \rangle = \langle K_{X(\psi_x)}, K_{X(\psi_y)} \rangle_{H_K}$

ii) Let $(e_i)_{i \in I}$ be any usual basis of the vector space E (so only a finite number of components are non null), and assuming that the vectors $e_i \in M$, denote $\varepsilon_i = X^{-1}(e_i) \in H$. K is definite positive and hermitian, so $K_{e_i}(e_j) = \langle \varepsilon_i, \varepsilon_j \rangle = K_{ij}$ defines a positive definite hermitian sesquilinear form on E :

$$\langle x, y \rangle_E = \sum_{i,j \in I} K_{ij} \bar{x}^i y_j = \sum_{i,j \in I} \bar{x}^i y^j K(e_i, e_j) = \left\langle \sum_{i \in I} x^i \varepsilon_i, \sum_{j \in I} y^j \varepsilon_j \right\rangle_H$$

The vectors ε_i are linearly independant, because the determinant $\det [\langle \varepsilon_i, \varepsilon_j \rangle]_{i,j \in J}$ is non null for any finite subset J of I .

Let $H_1 = \overline{Span(\varepsilon_i)_{i \in I}}$ be the closure of the vector subspace of H generated by the family $(\varepsilon_i)_{i \in I}$. This is a closed vector subspace of H , thus a Hilbert space.

By the Ehrhardt-Schmidt procedure it is always possible to define an orthonormal basis $(\tilde{\varepsilon}_i)_{i \in I}$ of H_1 with $\tilde{\varepsilon}_i = \tilde{L}\varepsilon_i \in H_1$ defined up to a unitary transformation $U : U^*U = Id : \langle U\tilde{\varepsilon}_i, U\tilde{\varepsilon}_j \rangle = \langle \tilde{\varepsilon}_i, U^*U\tilde{\varepsilon}_j \rangle = \langle \tilde{\varepsilon}_i, \tilde{\varepsilon}_j \rangle$. The subset H_0 contains a convex subset C of 0 , thus we can assume that $\tilde{\varepsilon}_i \in H_0$ and there is $\tilde{e}_i = X(\tilde{\varepsilon}_i)$ which is an orthonormal basis $\tilde{e}_i = Le_i$ of E . A vector $X \in E$ has for coordinates in this new basis : $X = \sum_{i \in I} x^i e_i = \sum_{i \in I} \tilde{x}^i \tilde{e}_i$ where only a finite number of components \tilde{x}^i is non null.

We have the following diagram :

$$e_i \xrightarrow{X^{-1}} \varepsilon_i \xrightarrow{\tilde{L}} \tilde{\varepsilon}_i \xrightarrow{X} \tilde{e}_i \xrightarrow{L^{-1}} e_i$$

$$X^{-1} \circ \tilde{L} \circ X = L$$

L, \tilde{L} are related :

$$\langle \tilde{e}_i, \tilde{e}_j \rangle_E = \delta_{ij} = ([L]^* [K] [L])_j^i$$

$$\langle \tilde{\varepsilon}_i, \tilde{\varepsilon}_j \rangle_H = \delta_{ij} = \left([\tilde{L}]^* [K] [\tilde{L}] \right)_j^i$$

So there is a unitary map U such that : $[\tilde{L}] = [L][U]$:

$$\begin{aligned} [U] &= [L]^{-1} [\tilde{L}] \Rightarrow [U]^* [U] = [\tilde{L}]^* [L^{-1}]^* [L]^{-1} [\tilde{L}] \\ &= [\tilde{L}]^* [L^{-1}]^* ([L]^* [K] [L]) [L]^{-1} [\tilde{L}] = [\tilde{L}]^* [K] [\tilde{L}] = I \end{aligned}$$

and it is possible to choose $(\tilde{\varepsilon}_i)_{i \in I}$ such that $[\tilde{L}] = [L]$.

Then $[K] = [L]^* [L]^{-1} \Leftrightarrow [K]^{-1} = [L] [L]^*$

iii) The map : $\hat{\pi} : \ell^2(I) \rightarrow H_1 :: \hat{\pi}_I(y) = \sum_{i \in I} y^i \tilde{\varepsilon}_i$ is an isomorphism of Hilbert spaces and $\forall \psi \in H_1 : \psi = \sum_{i \in I} \langle \tilde{\varepsilon}_i, \psi \rangle_H \tilde{\varepsilon}_i$ (JCD Th.1084).

Define the map : $\pi : E \rightarrow \mathbb{R}_0^I :: \pi(\sum_{i \in I} \tilde{x}^i \tilde{e}_i) = (\tilde{x}^i)_{i \in I}$ where \mathbb{R}_0^I is the subset of the set of maps $I \rightarrow \mathbb{R}$ such that only a finite number of components is non zero. It is bijective.

$\mathbb{R}_0^I \subset \ell^2(I)$ so the map : $\Upsilon = \hat{\pi} \circ \pi : E \rightarrow H_1 :: \Upsilon(\sum_{i \in I} \tilde{x}^i \tilde{e}_i) = \sum_{i \in I} \tilde{x}^i \tilde{\varepsilon}_i$ is well defined. It is linear, injective, continuous with norm 1 with respect to the norm induced on E by K . It preserves the scalar product :

$$\begin{aligned} \langle \sum_{i \in I} \tilde{x}^i \tilde{e}_i, \sum_{i \in I} \tilde{y}^i \tilde{e}_i \rangle_E &= \sum_{ij \in I} \tilde{x}^i \tilde{y}^j \langle \tilde{e}_i, \tilde{e}_j \rangle_E \\ &= \sum_{ij \in I} \tilde{x}^i \tilde{y}^j \langle \tilde{\varepsilon}_i, \tilde{\varepsilon}_j \rangle_H = \langle \Upsilon(\sum_{i \in I} \tilde{x}^i \tilde{e}_i), \Upsilon(\sum_{i \in I} \tilde{y}^i \tilde{e}_i) \rangle_H \end{aligned}$$

Thus $\forall X \in E : \tilde{x}^i = \langle \tilde{\varepsilon}_i, \Upsilon(x) \rangle_H$

Moreover $\Upsilon(e_j) = \varepsilon_j$:

$$\begin{aligned} \Upsilon(e_j) &= \Upsilon\left(\sum_{i \in I} (L^{-1})_j^i \tilde{e}_i\right) = \sum_{i \in I} (L^{-1})_j^i \tilde{\varepsilon}_i = \sum_{ik \in I} (L^{-1})_j^i \tilde{L}_i^k \varepsilon_k = \\ &= \sum_{k \in I} \left([\tilde{L}] [L^{-1}]\right)_j^k \varepsilon_k = \varepsilon_j \end{aligned}$$

iv) Let p be the orthogonal projection of H on H_1 : $\|\psi - p(\psi)\| = \min_{u \in H_1} \|\psi - u\|$

Then : $\psi = p(\psi) + o(\psi)$ where $o(\psi) \in H_1^\perp$ is the orthogonal complement of H_1 :

$$\begin{aligned} \langle \tilde{\varepsilon}_i, p(\psi) + o(\psi) \rangle &= \left\langle \tilde{\varepsilon}_i, \sum_{j \in I} \langle \tilde{\varepsilon}_j, \psi \rangle_H \tilde{\varepsilon}_j + o(\psi) \right\rangle \\ &= \sum_{j \in I} \langle \tilde{\varepsilon}_j, \psi \rangle_H \langle \tilde{\varepsilon}_i, \tilde{\varepsilon}_j \rangle + \langle \tilde{\varepsilon}_i, o(\psi) \rangle = \langle \tilde{\varepsilon}_i, \psi \rangle_H + \langle \tilde{\varepsilon}_i, o(\psi) \rangle = \langle \tilde{\varepsilon}_i, \psi \rangle_H \end{aligned}$$

There is a convex subset C of H containing 0 which is contained in H_0 . There is $r > 0$ such that : $\forall \psi \in H : \|\psi\| < r \Rightarrow \psi \in H_0$ and as $\|\psi\|^2 = \|p(\psi)\|^2 + \|o(\psi)\|^2$ we have $p(\psi), o(\psi) \in H_0$. Then :

$$\forall i \in I : \langle \tilde{\varepsilon}_i, o(\psi) \rangle_H = K(\tilde{e}_i, X(o(\psi))) = 0 \Rightarrow X(o(\psi)) = 0 \Rightarrow o(\psi) = 0$$

Thus H_1 is dense in H and as it is closed $H = H_1$, the map Υ is an isometry and $\forall \psi \in H : \psi = p(\psi) \in H_1$. The basis $(\tilde{\varepsilon}_i)_{i \in I}$ is a hilbertian basis of H . Any vector $\psi \in H$ can be written : $\psi = \sum_{i \in I} \langle \tilde{\varepsilon}_i, \psi \rangle_H \tilde{\varepsilon}_i$ and $\sum_{i \in I} |\langle \tilde{\varepsilon}_i, \psi \rangle_H|^2 < \infty$

v) For any $j \in I$:

$$\begin{aligned} \langle \tilde{\varepsilon}_i, \psi \rangle_H \tilde{\varepsilon}_i &= \langle L(\varepsilon_i), \psi \rangle_H L(\varepsilon_i) = \sum_{j,k \in I} ([L] [L]^*)_k^j \langle \varepsilon_j, \psi \rangle_H \varepsilon_k \\ &= \sum_{j,k \in I} \left([K]^{-1}\right)_k^j \langle \varepsilon_i, \psi \rangle_H \varepsilon_k = \sum_{j,k \in I} \left\langle \left([K]^{-1}\right)_j^k \varepsilon_j, \psi \right\rangle_H \varepsilon_k \end{aligned}$$

with $[K]^{-1} = [L][L]^*$ and $[K]^* = [K]$

Let us denote : $\phi_i = \sum_{j \in I} ([K]^{-1})_j^i \varepsilon_j$ then any vector $\psi \in H$ can be written : $\psi = \sum_{i \in I} \langle \tilde{\varepsilon}_i, \psi \rangle_H \tilde{\varepsilon}_i = \sum_{i \in I} \langle \phi_i, \psi \rangle_H \varepsilon_i$ and the series is absolutely convergent.

So $\forall x \in E : \Upsilon \left(\sum_{i \in I} x^i e_i \right) = \sum_{i \in I} x^i \varepsilon_i$ with $x^i = \langle \phi_i, \psi \rangle_H$.

vi) The chart $\Upsilon^{-1} : H \rightarrow E$ is compatible with X because $X \circ \Upsilon : E \rightarrow E :: X \circ \Upsilon \left(\sum_{i \in I} \tilde{x}^i \tilde{e}_i \right) = X \left(\sum_{i \in I} \tilde{x}^i \tilde{\varepsilon}_i \right) = X \left(\sum_{i \in I} x^i \varepsilon_i \right)$ is a diffeomorphism.

vii) This procedure holds for any subset H_{\varkappa} . D is, by construction, a bijective map with the ground states $D(G_{\varkappa})$ thus we have a collection $(\Upsilon_{\varkappa})_{\varkappa=1}^d$ of affine maps for \hat{H}_0 . ■

2.2.1 Additional results

We have additional results which are used in the following.

1. For any vector of E : $x = \sum_{i \in I} x^i e_i$ where only a finite number of components \tilde{x}^i is non null. So the series $x = \sum_{i \in I} x^i e_i$ is absolutely convergent in the Banach space E, and because Υ is linear the series $\Upsilon \left(\sum_{i \in I} x^i e_i \right) = \sum_{i \in I} x^i \varepsilon_i$ is absolutely convergent in H. Conversely for any summable series $\sum_{i \in I} x^i \varepsilon_i$ in H, that is :

$$\exists \psi \in H : \forall \varepsilon > 0, \exists J \subset I, \text{card}(J) < \infty, \forall K \subset J : \left\| \sum_{i \in K} x^i \varepsilon_i - \psi \right\| < \varepsilon$$

the series $\sum_{i \in I} x^i e_i$ is summable in E, because $\|\Upsilon\| = 1$ and as I is countable, both series are absolutely convergent in E (JCD Th.954)

2. We have the relations : $\langle \phi_j, \varepsilon_k \rangle = \delta_{jk}$ because :

$$\begin{aligned} \langle \phi_j, \varepsilon_k \rangle &= \left\langle \sum_{i \in I} ([K]^{-1})_i^j \varepsilon_i, \varepsilon_k \right\rangle = \sum_{i \in I} ([K]^{-1})_i^j \langle \varepsilon_i, \varepsilon_k \rangle \\ &= \sum_{i \in I} ([K]^{-1})_i^j [K]_k^i = \delta_{jk} \end{aligned}$$

2.2.2 Comments

These linear charts are crucial in the following of the paper. Some comments are useful.

1. The structure of manifold M of the set of "raw data" is hidden to the observer. The chart $X : H \rightarrow E$ does exist formally, but is not known to the observer. As for any manifold, any chart can be used to identify a point of M. The choice of a linear chart sums up to associate to a set of measures $x = \sum_{i \in I} x^i e_i$ a state : $\psi = \Upsilon(x) = \sum_{i \in I} x^i \varepsilon_i$. The map Υ is uniquely defined by the chart X and the basis $(e_i)_{i \in I}$ and ψ is uniquely defined by $x^i = \langle \phi_i, \psi \rangle$. Of course the states $\psi_1 = X^{-1}(x)$ and $\psi_2 = \Upsilon(x)$ are not the same (except for $x = e_i$), but Υ respects the structure of manifold, through the associations $\varepsilon_i = X^{-1}(e_i)$ and $\tilde{\varepsilon}_i = \sum_{j \in I} [L]_i^j \varepsilon_j$ which are characteristic of M.

2. The linear chart is built through an association between components of vectors in coordinated bases in E and H. Once a basis has been decided in E,

the key values are the components x^i , which appear both in the measurement process and the identification of the states in H .

The relations which occur in a change of basis are seen in the next section.

3. Using the linear chart, to each vector v_k is associated a vector $v_k = \Upsilon^{-1}(v_k) \in E$ thus the condition $\psi_d \in (v_k)_{k=1}^d$ reads $X_d = \Upsilon^{-1}(\psi_d) \in (v_k)_{k=1}^d$

2.3 Product of systems

2.3.1 Decomposition of the Hilbert space

Proposition 5 *If the model is comprised of N continuous variables $(\Xi_k)_{k=1}^N$, each belonging to a Banach vector space E_k , then the real Hilbert space H of states of the system is the Hilbert sum of N Hilbert space $H = \bigoplus_{k=1}^N H_k$ and any vector ψ representing a state of the system is uniquely the sum of N vectors ψ_k , each image of the value of one variable Ξ_k in the state ψ*

Proof.

By definition $E = \prod_{k=1}^N E_k$. The set $E_k^0 = \{0, \dots, E_k, \dots, 0\} \subset E$ is a vector subspace of E . A basis of E_k^0 is a subfamily $(e_i)_{i \in I_k}$ of a basis $(e_i)_{i \in I}$ of E . E_k^0 has for image by the continuous linear map Υ a closed vector subspace H_k of H . Any vector x of E reads : $x \in \prod_{k=1}^N E_k : x = \sum_{k=1}^N \sum_{i \in I_k} x^i e_i$ and it has for image by $\Upsilon : \psi = \Upsilon(x) = \sum_{k=1}^N \sum_{i \in I_k} x^i \varepsilon_i = \sum_{k=1}^N \psi_k$ with $\psi_k \in H_k$. This decomposition of $\Upsilon(x)$ is unique.

Conversely, the family $(e_i)_{i \in I_k}$ has for image by Υ the set $(\varepsilon_i)_{i \in I_k}$ which are linearly independant vectors of H_k . It is always possible to build an orthonormal basis $(\tilde{\varepsilon}_i)_{i \in I_k}$ from these vectors as done previously. H_k is a closed subspace of H , so it is a Hilbert space. The map : $\hat{\pi}_k : \ell^2(I_k) \rightarrow H_k :: \hat{\pi}_k(x) = \sum_{i \in I_k} x^i \tilde{\varepsilon}_i$ is an isomorphism of Hilbert spaces and : $\forall \psi \in H_k : \psi = \sum_{i \in I_k} \langle \tilde{\varepsilon}_i, \psi \rangle_H \tilde{\varepsilon}_i$.

The vector subspaces H_k are orthogonal : Υ is an isometry, and

$$\forall \psi_k \in H_k, \psi_l \in H_l, k \neq l : \langle \psi_k, \psi_l \rangle_H = \langle \Upsilon^{-1}(\psi_k), \Upsilon^{-1}(\psi_l) \rangle_E = 0$$

Any vector $\psi \in H$ reads : $\psi = \sum_{k=1}^N \pi_k(\psi)$ with the orthogonal projection $\pi_k : H \rightarrow H_k :: \pi_k(\psi) = \sum_{i \in I_k} \langle \tilde{\varepsilon}_i, \psi \rangle_H \tilde{\varepsilon}_i$ so H is the Hilbert sum of the H_k . ■

As a consequence the definite positive kernel of (E, Υ) decomposes as :

$$K((\Xi_1, \dots, \Xi_N), (\Xi'_1, \dots, \Xi'_N)) = \sum_{k=1}^N K_k(\Xi_k, \Xi'_k) = \sum_{k=1}^N \langle \Upsilon(\Xi_k), \Upsilon(\Xi'_k) \rangle_H$$

This decomposition comes handy when we have to "translate" relations between variables into relations between vector states, notably if they are linear. But it requires that we keep the real Hilbert space structure.

2.3.2 Interacting systems

Position of the problem

In the general assumptions above the system is not necessarily isolated. If

there is an action of the "outside" onto the system, this action must be accounted for among the variables : it is subject to measures and has no special status.

If there are two systems S_1, S_2 which interact with each other, then one can consider the two systems together, that is the product $S_1 \times S_2$ of the models. We keep all the variables as they were, the manifold of the configuration is the product of the manifolds, we have a new Hilbert space, which can be identified to the Hilbert sum of the previous spaces, according to the result above.

However to account for the interactions, in each model of S_1, S_2 we need to introduce the variables Z_1, Z_2 which account for the external action. It seems logical to drop these variables, and to consider a model of the interacting systems that we denote S_{1+2} by keeping only the variables which are specific to each system. We have the following diagram :

$$\begin{array}{ccc}
 \lrcorner & S_1 & \ulcorner \\
 X_1 & & Z_1 \\
 E_1 & \times & E_{Z1} \\
 \\
 H_1 & \oplus & H_{Z1} \\
 \psi_1 & & \psi_{Z1} \\
 \\
 & & \lrcorner & S_2 & \ulcorner \\
 & & X_2 & & Z_2 \\
 & & E_2 & \times & E_{Z2} \\
 \\
 & & H_2 & \oplus & H_{Z2} \\
 & & \psi_2 & & \psi_{Z2} \\
 \\
 & & \lrcorner & S_{1+2} & \ulcorner \\
 & & X_1 & & X_2 \\
 & & E_1 & \times & E_2 \\
 \\
 & & H'_1 & \oplus & H'_2 \\
 & & \psi'_1 & & \psi'_2
 \end{array}$$

The vector spaces E_1, E_2 for the definition of the variables X_1, X_2 specific to each system do not change. In each case the vector space to consider is the product : $E_1 \times E_{Z1}, E_2 \times E_{Z2}, E_1 \times E_2$.

Using a linear map it is possible to split the vectors representing the states :

$$\begin{aligned}
 H_1 &= \Upsilon_1(E_1 \times \{0\}), H_{Z1} = \Upsilon_1(\{0\} \times E_{Z1}), \\
 H_2 &= \Upsilon_2(E_2 \times \{0\}), H_{Z2} = \Upsilon_2(\{0\} \times E_{Z2}), \\
 H'_1 &= \Upsilon_{1+2}(E_1 \times \{0\}), H'_2 = \Upsilon_{1+2}(\{0\} \times E_2)
 \end{aligned}$$

This model is just the projection of $S_1 \times S_2$ on $E_1 \times E_2$. This always can be done, however it is clear that we miss some important features, meaning the interactions. So we could change the model in order to regain the information that we lost.

Proposition

Let us denote S'_{1+2} this new model. Its variable will be denoted Y , valued in a Banach vector space E' with basis $(e'_i)_{i \in I'}$. There will be another Hilbert space H' , and a linear map $\Upsilon' : E' \rightarrow H'$ similarly defined. As we have the choice of the model, we will impose some properties to Y and E' .

a) The variables X_1, X_2 can be deduced from the value of Y : there must be a bilinear bijective map : $\Phi : E_1 \times E_2 \rightarrow E'$

b) Φ must be such that whenever the systems S_1, S_2 are in the states ψ_1, ψ_2 then S'_{1+2} is in the state ψ' and

$$\Upsilon'^{-1}(\psi') = \Phi(\Upsilon_1^{-1}(\psi_1), \Upsilon_2^{-1}(\psi_2))$$

c) The positive kernel is a defining feature of the models, so we want a positive kernel K' of (E', Υ') such that :

$$\forall X_1, X'_1 \in E_1, \forall X_2, X'_2 \in E_2 :$$

$$K'(\Phi(X_1, X_2), \Phi(X'_1, X'_2)) = K_1(X_1, X'_1) \times K_2(X_2, X'_2)$$

We will prove the following :

Proposition 6 *Whenever two systems S_1, S_2 interact, there is a model S'_{1+2} , encompassing the two systems and meeting the conditions a,b,c above. It is obtained by taking the tensor product of the variables specific to S_1, S_2 . Then the Hilbert space of S'_{1+2} is the tensorial product of the Hilbert spaces associated to each system.*

Proof.

The map : $\varphi : H_1 \times H_2 \rightarrow H' :: \varphi(\psi_1, \psi_2) = \Phi(\Upsilon_1^{-1}(\psi_1), \Upsilon_2^{-1}(\psi_2))$ is bilinear. So, by the universal property of the tensorial product, there is a unique map $\widehat{\varphi} : H_1 \otimes H_2 \rightarrow H'$ such that : $\varphi = \widehat{\varphi} \circ \iota$ where $\iota : H_1 \times H_2 \rightarrow H_1 \otimes H_2$ is the tensorial product.

The condition iii) reads :

$$\begin{aligned} & \langle \Upsilon_1(X_1), \Upsilon_1(X'_1) \rangle_{H_1} \times \langle \Upsilon_2(X_2), \Upsilon_2(X'_2) \rangle_{H_2} \\ &= \langle (\Upsilon' \circ \Phi(\Upsilon_1(X_1), \Upsilon_2(X_2)), \Upsilon' \circ \Phi(\Upsilon_1(X'_1), \Upsilon_2(X'_2))) \rangle_{H'} \\ & \langle \psi_1, \psi'_1 \rangle_{H_1} \times \langle \psi_2, \psi'_2 \rangle_{H_2} = \langle \varphi(\psi_1, \psi_2), \varphi(\psi'_1, \psi'_2) \rangle_{H'} = \langle \widehat{\varphi}(\psi_1 \otimes \psi_2), \widehat{\varphi}(\psi'_1 \otimes \psi'_2) \rangle_{H'} \end{aligned}$$

The scalar products on H_1, H_2 extend in a scalar product on $H_1 \otimes H_2$,

endowing the latter with the structure of a Hilbert space with :

$$\langle (\psi_1 \otimes \psi_2), (\psi'_1 \otimes \psi'_2) \rangle_{H_1 \otimes H_2} = \langle \psi_1, \psi'_1 \rangle_{H_1} \langle \psi_2, \psi'_2 \rangle_{H_2}$$

and then the reproducing kernel is the product of the reproducing kernels (JCD Th.1163).

So we must have : $\langle \widehat{\varphi}(\psi_1 \otimes \psi_2), \widehat{\varphi}(\psi'_1 \otimes \psi'_2) \rangle_{H'} = \langle \psi_1 \otimes \psi_2, \psi'_1 \otimes \psi'_2 \rangle_{H_1 \otimes H_2}$ and $\widehat{\varphi}$ must be an isometry : $H_1 \otimes H_2 \rightarrow H'$

So the simplest solution is to take $H' = H_1 \otimes H_2$ and then $E' = E_1 \otimes E_2$. However this solution is not unique, and indeed makes sense only if the systems are defined by similar variables.

This proposition extends obviously for any discrete variable, thus it holds for any model. ■

3 OPERATORS

3.1 Main results

The vector space E is infinite dimensional, so an effective physical measure is a finite set of figures. And, to be consistent with the model, we define a

primary observable as any finite set of coordinates for a point representing a state in the affine space \widehat{E}_0 . And we distinguish continuous and discrete primary observables.

3.1.1 Continuous primary observables

At first we consider the case without any discrete variable. A primary observable Y is a map : $Y : E \rightarrow E :: Y(x) = \sum_{j \in J} x^j e_j$ where J is a finite subset of I expressed in some basis $(e_i)_{i \in I}$ of E .

Main result

Proposition 7 *To any physical continuous primary observable Y on the system is associated uniquely a hermitian operator \widehat{Y} on $H : \widehat{Y} = \Upsilon \circ Y \circ \Upsilon^{-1}$ with a linear chart Υ of H , such that the measure of Y , if the system is in the state $\psi \in H_0$, is $Y(x) = \sum_{j \in J} \langle \phi_j, \psi \rangle e_j$. Moreover \widehat{Y} is a compact, Hilbert-Schmidt operator and $\|\widehat{Y}_J\| = 1$.*

We use the notations and definitions of the previous section.

Proof.

i) Given any chart X and basis $(e_i)_{i \in I}$ of E we can define a basis $(\varepsilon_i)_{i \in I}$ of H , and the bijective, linear, maps :

$\pi : E \rightarrow \mathbb{R}_0^I :: \pi(\sum_{i \in I} x^i e_i) = (x^i)_{i \in I}$ where \mathbb{R}_0^I is the subset of the set of maps $I \rightarrow \mathbb{R}$ such that only a finite number of components is non zero.

$\widehat{\pi} : H \rightarrow \ell^2(I) :: \widehat{\pi}(\sum_{i \in I} \psi^i \varepsilon_i) = (\psi^i)_{i \in I}$ and $\psi^i = \langle \phi^i, \psi \rangle$

The linear chart $\Upsilon = \widehat{\pi}^{-1} \circ \pi :: E \rightarrow H$ is such that to the configuration X is associated the state ψ with $x^i = \langle \phi^i, \psi \rangle = \psi^i$

ii) A primary continuous observable is the selection of a finite number of components of $I : Y_J(X) = \sum_{j \in J} x^j e_j$ where J is a finite subset of I .

Let us denote the map : $\tau_J : \mathbb{R}_0^I \rightarrow \mathbb{R}_0^I :: \tau_J((x^i)_{i \in I}) = (y^i)_{i \in I}$ where $y^i = x^i$ if $i \in J$ and $y^i = 0$ otherwise.

So : $Y_J = \pi^{-1} \circ \tau_J \circ \pi$

iii) To the observable Y_J on E we associate the operator \widehat{Y}_J on H :

$\widehat{Y}_J = \widehat{\pi}^{-1} \circ \tau_J \circ \widehat{\pi} :: H \rightarrow H :: \widehat{Y}_J(\sum_{i \in I} \psi^i \varepsilon_i) = \sum_{i \in J} \psi^i \varepsilon_i$

which reads :

$\widehat{Y}_J = (\widehat{\pi}^{-1} \circ \pi) \circ (\pi^{-1} \circ \tau_J \circ \pi) \circ (\pi^{-1} \circ \widehat{\pi}) = \Upsilon \circ Y_J \circ \Upsilon^{-1}$

$$\begin{array}{ccc} E & \xrightarrow{Y} & E_J \\ \downarrow \Upsilon & & \downarrow \Upsilon \\ H & \xrightarrow{\widehat{Y}_J} & H_J \end{array}$$

where $E_J = \text{Span}(e_i)_{i \in J}$, $H_J = \Upsilon(E_J)$

So : $\widehat{Y}_J(\psi) = \sum_{j \in J} \langle \phi_j, \psi \rangle \varepsilon_j$ with $\phi_j = \sum_{i \in I} ([K]^{-1})_j^i \varepsilon_i$ and $x^i = \langle \phi^i, \psi \rangle = \psi^i$

This is a linear, continuous, operator. \widehat{Y}_J depends uniquely of the basis $(\varepsilon_i)_{i \in J}$ chosen to define Y_J .

Y_J is a projection $Y_J^2 = Y_J$ and \widehat{Y}_J is also a projection : $\widehat{Y}_J^2 = \Upsilon \circ Y_J \circ \Upsilon^{-1} \circ \Upsilon \circ Y_J \circ \Upsilon^{-1} = \widehat{Y}_J$ on the vector subspace H_J .

Thus :

$$\widehat{Y}_J^2(\psi) = \sum_{j \in J} \langle \phi_j, \sum_{i \in J} \langle \phi_i, \psi \rangle \varepsilon_i \rangle \varepsilon_j = \sum_{i, j \in J} \langle \phi_i, \psi \rangle \langle \phi_j, \varepsilon_i \rangle \varepsilon_j = \sum_{j \in J} \langle \phi_j, \psi \rangle \varepsilon_j$$

$$\text{ii) Its adjoint } \widehat{Y}_J^* \text{ is such that : } \forall \psi, \psi' \in H : \langle \widehat{Y}_J \psi, \psi' \rangle = \langle \psi, \widehat{Y}_J^* \psi' \rangle$$

$$\text{A short computation gives : } \widehat{Y}_J^*(\psi) = \sum_{j \in J} \langle \varepsilon_j, \psi \rangle \phi_j$$

$$\text{Thus } \widehat{Y}_J \text{ is hermitian iff } \forall \psi \in H : \sum_{j \in J} \langle \varepsilon_j, \psi \rangle \phi_j = \sum_{j \in J} \langle \phi_j, \psi \rangle \varepsilon_j$$

Expressed with respect to the independant vectors ε_i with $[K'] = [K]^{-1}$ and $[K] = [K]^*$

$$\widehat{Y}_J(\psi) = \sum_{j \in J} \sum_{i \in I} \overline{[K']_j^i} \langle \varepsilon_i, \psi \rangle \varepsilon_j = \sum_{j \in J} \sum_{i \in I} [K']_i^j \langle \varepsilon_i, \psi \rangle \varepsilon_j$$

$$\widehat{Y}_J^*(\psi) = \sum_{j \in J} \sum_{i \in I} [K']_j^i \langle \varepsilon_j, \psi \rangle \varepsilon_i$$

If we take two vectors $\psi = \varepsilon_p, \psi' = \varepsilon_q$:

$$\widehat{Y}_J(\varepsilon_p) = \sum_{j \in J} \left(\sum_{i \in I} [K']_i^j [K]_p^i \right) \varepsilon_j = \sum_{j \in J} \delta_j^p \varepsilon_j$$

$$\widehat{Y}_J^*(\varepsilon_p) = \sum_{j \in J} \left(\sum_{i \in I} [K']_j^i [K]_p^i \right) \varepsilon_i$$

$$\langle \widehat{Y}_J \varepsilon_p, \varepsilon_q \rangle = \sum_{j \in J} \delta_j^p \langle \varepsilon_j, \varepsilon_q \rangle = \sum_{j \in J} \delta_j^p [K]_q^j$$

$$\langle \varepsilon_p, \widehat{Y}_J^* \varepsilon_q \rangle = \sum_{j \in J} \left(\sum_{i \in I} [K']_j^i [K]_p^i \right) \langle \varepsilon_p, \varepsilon_i \rangle$$

$$= \sum_{j \in J} \left(\sum_{i \in I} [K]_i^p [K']_j^i \right) [K]_p^j = \sum_{j \in J} \delta_j^p [K]_q^j$$

Thus \widehat{Y}_J is hermitian on H .

iii) Moreover \widehat{Y}_J has a finite rank because Y_J has a finite rank, thus \widehat{Y}_J is a compact operator.

\widehat{Y}_J is a Hilbert-Schmidt operator if J is a finite subset : take the Hilbertian basis $\tilde{\varepsilon}_i$ in H :

$$\sum_{i \in I} \left\| \widehat{Y}_J(\tilde{\varepsilon}_i) \right\|^2 = \sum_{i, j \in J} |\langle \phi_j, \tilde{\varepsilon}_i \rangle|^2 \|\varepsilon_j\|^2 = \sum_{j \in J} \|\phi_j\|^2 \|\varepsilon_j\|^2 < \infty$$

Because \widehat{Y}_J is a projection, it has two eigen values : 1 and 0. The eigen space corresponding to 1 is the vector subspace H_J generated by $(\varepsilon_i)_{i \in J}$. Because it is a compact hermitian operator the eigen spaces corresponding to the eigen value 0 is orthogonal to H_J . This is the orthogonal complement to the vector subspace ϕ_J generated by $(\phi_i)_{i \in J}$ thus $\phi_J = H_J$ and \widehat{Y}_J is the orthogonal projection on H_J and $\left\| \widehat{Y}_J \right\| = 1$.

\widehat{Y}_J is a trace class operator if J is a finite subset, with trace $\dim H_J$

$$\sum_{i \in I} \langle \widehat{Y}(\tilde{\varepsilon}_i), \tilde{\varepsilon}_i \rangle = \sum_{i, j \in J} \overline{\langle \phi_j, \tilde{\varepsilon}_i \rangle} \langle \varepsilon_j, \tilde{\varepsilon}_i \rangle = \sum_{i, j \in J} \phi_j^i \varepsilon_j^i$$

$$= \sum_{j \in J} \langle \phi_j, \varepsilon_j \rangle = \sum_{j \in J} \sum_{k \in I} \overline{[K']_j^k} \langle \varepsilon_k, \varepsilon_j \rangle$$

$$= \sum_{j \in J} \sum_{k \in I} [K']_k^j [K]_j^k = \sum_{j \in J} \delta_{jj} = \dim H_J$$

iv) Any vector of H can be written $\psi = \sum_{i \in I} \psi^i \varepsilon_i$ and the vectors $\varepsilon_i \in H_0$. Thus the image $\widehat{Y}_J(\psi) = \sum_{j \in J} \psi^j \varepsilon_j \in H_J \subset H_0$ if $\psi \in H_0$ and the operators are well defined as maps : $H_0 \rightarrow H_J$. ■

Further properties of continuous observables

1. As any vector $\psi \in H$ can be written : $\psi = \sum_{i \in I} \langle \phi_i, \psi \rangle \varepsilon_i$ the series : $\widehat{Y}_J(\psi) = \sum_{i \in J} \langle \phi_i, \psi \rangle \varepsilon_i$ is still convergent for any subset J of I , even infinite and we denote H_J the closure of $\text{Span}(\varepsilon_j, j \in J)$. In the following we will precise when J is finite.

2. Any primary observable can be written : $\widehat{Y}_J(\psi) = \sum_{j \in J} \phi^j(\psi) \varepsilon_j$ where $\phi^j \in E'$ is the 1-form associated to ϕ_j such that $\phi^j(\psi) \in \mathbb{R}$

3. As $\varepsilon_j = [L^{-1}]_j^i \tilde{\varepsilon}_i$, $\phi_j = \sum_{i \in I} ([K]^{-1})_j^i \varepsilon_i = \sum_{i \in I} ([K]^{-1})_j^i [L^{-1}]_i^k \tilde{\varepsilon}_k = \sum_{k \in I} ([L]^*)_j^k \tilde{\varepsilon}_k$ with $[K]^{-1} = [L][L]^*$

$$\widehat{Y}_J(\psi) = \sum_{j \in J, kl \in I} \langle ([L]^*)_j^k \tilde{\varepsilon}_k, \psi \rangle [L^{-1}]_j^l \tilde{\varepsilon}_l = \sum_{j \in J, kl \in I} ([L])_k^j [L^{-1}]_j^l \langle \tilde{\varepsilon}_k, \psi \rangle \tilde{\varepsilon}_l$$

$$\text{So : } \widehat{Y}_J(\psi) = \sum_{kl \in I} A^{kl} \langle \tilde{\varepsilon}_k, \psi \rangle \tilde{\varepsilon}_l \text{ where } A^{kl} = \sum_{j \in J} [L^{-1}]_j^l [L]_k^j$$

$$\left\| \widehat{Y}_J(\psi) \right\|^2 = \sum_{kl \in I} |A^{kl}|^2 |\tilde{\psi}^k|^2 \leq \sum_{k \in I} |\tilde{\psi}^k|^2$$

3.1.2 Discrete primary observables

Main result

Proposition 8 *To any primary observable D_J on the discrete variables is associated a continuous, hermitian operator \widehat{D}_J in the Hilbert space H wich reads : $\widehat{D}_J(\psi_d) = \sum_{m=1}^d \langle \vartheta_m, \psi_d \rangle v_m$ with $(v_k)_{k=1}^d$ an orthonormal family of vectors of H and $(\vartheta_k)_{k=1}^d$ vectors of H which are linearly independant iff the observable involves all the discrete variables. $\widehat{D}_J(\psi_d)$ corresponds to a unique point in the affine space \widehat{H} belonging to \widehat{H}_0 iff all the discrete variables are involved.*

Proof.

i) Using the notations of the first section, we have n discrete variables $(D_k)_{k=1}^n$ taking their values in $\{1, 2, \dots, d_k\}$ which have been synthetized in the single discrete variable D taking its values in $\{1, 2, \dots, d\}$ and represented by one the vectors of a basis $(e_{\varkappa})_{\varkappa=1}^d$ of \mathbb{C}^d . The value $\vec{D} = e_{\varkappa}$ corresponds to some combination (i_1, i_2, \dots, i_n) , $i_k \in \{1, \dots, d_k\}$ of each the D_k . A reduced observable D_J is a set of discrete variables $(D_k)_{k \in J}$. It takes its values in a subset $(i_{\alpha_1}, i_{\alpha_2}, \dots, i_{\alpha_n})$, $\alpha_k \in \{1, \dots, n\}$ of the values of D , values which are labelled $1, 2, \dots, s$. To a given value \varkappa_J of D_J correspond several values of D .

Let us denote : $A_i^j = 1$ if $D = i \Rightarrow D_J = j$ and $A_i^j = 0$ otherwise. The $s \times d$ matrix $[A]$ has rank s (the states measured by D_J are all distinct) and has exactly one 1 in each column. The measure of D_J can be represented as

a projection of the same vector \vec{D} on a smaller orthogonal basis $(\kappa_k)_{k=1}^s$ of \mathbb{C}^d built from $(e_\varkappa)_{\varkappa=1}^d$:

$$l=1\dots s : \kappa_j = \sum_{\varkappa=1}^d [A]_{\varkappa}^j e_\varkappa \text{ and } [A][A]^t = I_{s \times s}$$

Similarly in H the vector ψ_d representing a state of the system is projected on a smaller basis $(\varrho_k)_{k=1}^s$, built from $(v_k)_{k=1}^d : l=1\dots s : \varrho_j = \sum_{k=1}^d [A]_k^j v_k$

ii) The operator becomes : $\hat{D}_J : H \rightarrow H :: \hat{D}_J(\psi_d) = \sum_{j=1}^s \langle \varrho_j, \psi_d \rangle \varrho_j$. It reads :

$$\begin{aligned} \hat{D}_J(\psi_d) &= \sum_{j=1}^s \langle \varrho_j, \psi_d \rangle \varrho_j = \sum_{j=1}^s \left\langle \sum_{k=1}^d [A]_k^j v_k, \psi_d \right\rangle \left(\sum_{l=1}^d [A]_l^j v_l \right) \\ &= \sum_{j=1}^s \sum_{k,l=1}^d [A]_k^j [A]_l^j \langle v_k, \psi_d \rangle v_l \\ &= \sum_{k,l=1}^d \left([A]^t [A] \right)_k^l \langle v_k, \psi_d \rangle v_l = \sum_{l=1}^d \langle \vartheta_l, \psi_d \rangle v_l \end{aligned}$$

$$\hat{D}_J(\psi_d) = \sum_{j=1}^d \langle \vartheta_j, \psi_d \rangle v_j \text{ with } \vartheta_j = \sum_{k=1}^d \left([A]^t [A] \right)_k^j v_k$$

By permutation of the columns the matrix $[A]$ can be put in the form $[A_J]$ of exactly d/s copies of the unit $s \times s$ matrix, which can be written : $[A_J] = [P][A] \Leftrightarrow [A] = [P]^t [A_J]$ where $[P]$ is a permutation matrix. So $[A]^t [A] = \left([A_J]^t [P] \right) [P]^t [A_J] = [A_J]^t [A_J]$ is a $d \times d$ matrix, of rank s , made of copies of the unit $s \times s$ matrix I and its main diagonal has 1 for elements. So the vectors $(\vartheta_j)_{j=1}^d$ are actually d/s copies of the collection of s vectors ϑ_j .

$$\begin{aligned} \text{They are orthonormal : } \langle \vartheta_i, \vartheta_j \rangle &= \left\langle \sum_{k=1}^d \left([A]^t [A] \right)_k^i v_k, \sum_{k=1}^d \left([A]^t [A] \right)_k^j v_k \right\rangle \\ &= \sum_{k=1}^d \left([A]^t [A] \right)_k^i \left([A]^t [A] \right)_k^j = \left([A]^t [A] [A]^t [A] \right)_j^i = \left([A]^t [A] \right)_j^i = \delta_{ij} \end{aligned}$$

The value for $\psi_d = v_\varkappa$ is :

$\hat{D}_J(\psi_d) = \sum_{j=1}^s \langle \varrho_j, v_\varkappa \rangle \varrho_j = \sum_{j=1}^s [A]_{\varkappa}^j \varrho_j$ and only one coefficient $[A]_{\varkappa}^j$ is non zero.

iii) \hat{D}_J is linear, thus continuous and hermitian.

Indeed its adjoint $\left(\hat{D}_J \right)^*$ is such that :

$$\left\langle \hat{D}_J(\psi_d), \psi'_d \right\rangle = \left\langle \psi_d, \left(\hat{D}_J \right)^* \psi'_d \right\rangle$$

that is : $\left[\left(\hat{D}_J \right)^* \right] = \left[\hat{D}_J \right]^* = \left([A]^t [A] \right)^* = \left([A]^t [A] \right)^t = [A]^t [A]$ because $[A]$ is real.

It is a projection on the vector subspace spanned by $(\varrho_k)_{k=1}^s$

$$\hat{D}_J(\varrho_k) = \sum_{j=1}^s \langle \varrho_j, \varrho_k \rangle \varrho_j = \varrho_k$$

$$\hat{D}_J^2(\psi_d) = \hat{D}_J \left(\sum_{j=1}^s \langle \varrho_j, \psi_d \rangle \varrho_j \right) = \sum_{j=1}^s \langle \varrho_j, \psi_d \rangle \hat{D}_J(\varrho_j) = \hat{D}_J(\psi_d) \quad \blacksquare$$

Example

with $D_1 = \{1, 2\}$, $D_2 = \{1, 2, 3\}$ restricted to D_2

$v_1 \leftrightarrow (1, 1)$, $v_2 \leftrightarrow (1, 2)$, $v_3 \leftrightarrow (1, 3)$, $v_4 \leftrightarrow (2, 1)$, $v_5 \leftrightarrow (2, 2)$, $v_6 \leftrightarrow (2, 3)$,

$\varrho_1 = (v_1 + v_4)$, $\varrho_2 = (v_2 + v_5)$, $\varrho_3 = (v_3 + v_6)$

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\
[A]^t [A] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\
[A] [A]^t &= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Further properties of the discrete operators

1. As a point in the set of states is defined by a couple of vectors (ψ_d, ψ) a primary observable $(D_J, Y_{J'})$ gives a couple of maps $(\widehat{D}_J, \widehat{Y}_{J'})$ on HxH and the image of a point in the affine space \widehat{H} is a point in \widehat{H} .

2. The operator is a projection, its eigen values are (0,1) and the vector subspace spanned by $(\varrho_k)_{k=1}^s$ corresponds to the eigen value 1.

3. The operators associated to different discrete observables commute :

$$\begin{aligned}
\widehat{D}_J \circ \widehat{D}_{J'} (\psi_d) &= \sum_{m=1}^d \left\langle \vartheta_m, \sum_{p=1}^d \langle \vartheta'_p, \psi_d \rangle v_p \right\rangle v_m \\
&= \sum_{m=1}^d \sum_{p=1}^d \langle \vartheta'_p, \psi_d \rangle \langle \vartheta_m, v_p \rangle v_m \\
&= \sum_{m=1}^d \sum_{p=1}^d \left\langle \sum_{l=1}^d \left([A']^t [A'] \right)_l^p v_l, \psi_d \right\rangle \left\langle \sum_{k=1}^d \left([A]^t [A] \right)_k^m v_k, v_p \right\rangle v_m \\
&= \sum_{m=1}^d \sum_{p=1}^d \sum_{l=1}^d \left([A']^t [A'] \right)_l^p \sum_{k=1}^d \left([A]^t [A] \right)_k^m \langle v_l, \psi_d \rangle \langle v_k, v_p \rangle v_m \\
&= \sum_{m=1}^d \sum_{k,l=1}^d \left([A]^t [A] \right)_k^m \left([A']^t [A'] \right)_l^k \langle v_l, \psi_d \rangle v_m \\
&= \sum_{l,m=1}^d \left([A_J]^t [A_J] [A'_{J'}]^t [A'_{J'}] \right)_l^m \langle v_l, \psi_d \rangle v_m
\end{aligned}$$

The product of matrices comprised of unit submatrices commute :

$$[A_J]^t [A_J] [A'_{J'}]^t [A'_{J'}] = [A'_{J'}]^t [A'_{J'}] [A_J]^t [A_J]$$

$$\text{So } \widehat{D}_{J_1} \circ \widehat{D}_{J_2} = \widehat{D}_{J_2} \circ \widehat{D}_{J_1}$$

However the result is not necessarily an operator of the kind \widehat{D}_J for some J.

3.2 Secondary continuous observables

From primary continuous observables, which are the just basic measures that can be done on the system, we can define secondary observables, which are variables defined through the combination of primary observables.

3.2.1 Algebra of primary continuous observables

The simplest combination of primary observables is through an algebra, meaning by linear combination and composition of primary observables.

Proposition 9 *The operators associated to the projection of each vector e_i of a basis of E are a set of orthogonal operators, and are the generators of a commutative C^* -subalgebra A of $\mathcal{L}(H; H)$ and a Hilbert space.*

Proof.

i) For each $j \in I$ we have the primary operator : $\widehat{Y}_j(\psi) = \phi^j(\psi) \varepsilon_j = \langle \phi_j, \psi \rangle \varepsilon_j$ with ϕ^j the 1-form associated to ϕ_j

$$\widehat{Y}_j = \sum_{kl \in I} A^{kl} \widehat{\varepsilon}^k \otimes \widehat{\varepsilon}_l \text{ where } A^{kl} = [L^{-1}]_j^l [L]_k^j$$

This is the projection of ψ on the vector ε_j of the (non hilbertian) basis $(\varepsilon_j)_{j \in I}$.

The primary operators $(\widehat{Y}_j)_{j \in I}$ are mutually orthogonal :

$$\widehat{Y}_j \circ \widehat{Y}_k(\psi) = \langle \phi_k, \psi \rangle \langle \phi_j, \varepsilon_k \rangle \varepsilon_j = \langle \phi_k, \psi \rangle \sum_{i \in I} [K^{-1}]_i^j [K]_k^i \varepsilon_j = \delta_{jk} \widehat{Y}_j(\psi)$$

Any primary observable can be written :

$$\widehat{Y}_J(\psi) = \sum_{j \in J} \widehat{Y}_j(\psi) \text{ and } \sum_{j \in I} \widehat{Y}_j(\psi) = \psi$$

$$\widehat{Y}_{J_1} \circ \widehat{Y}_{J_2} = \widehat{Y}_{J_1 \cap J_2} = \widehat{Y}_{J_2} \circ \widehat{Y}_{J_1}$$

$$\widehat{Y}_{J_1 \cup J_2} = \widehat{Y}_{J_1} + \widehat{Y}_{J_2} - \widehat{Y}_{J_1 \cap J_2} = \widehat{Y}_{J_1} + \widehat{Y}_{J_2} - \widehat{Y}_{J_1} \circ \widehat{Y}_{J_2}$$

Notice that $\langle \widehat{Y}_{j_1} \psi, \widehat{Y}_{j_2} \psi \rangle \neq 0$ usually :

$$\langle \widehat{Y}_{j_1} \psi, \widehat{Y}_{j_2} \psi \rangle = \langle \psi^{j_1} \varepsilon_{j_1}, \psi^{j_2} \varepsilon_{j_2} \rangle = [K]_{j_2}^{j_1} \overline{\psi^{j_1}} \psi^{j_2}$$

ii) Denote $\widehat{Y}_0 = I$ where I is the identity on H and $I_0 = I \cup \{0\}$

For any family $(u^i)_{i \in I_0} \in \mathbb{C}^{I_0}$, $\max_{i \in I_0} |u^i| < \infty$ the endomorphism : $Y = \sum_{i \in I_0} u^i \widehat{Y}_i$ is well defined and $\|Y\| = \sup_{i \in I_0} |u^i|$

Define A as the set of endomorphisms $A = \left\{ Y = \sum_{i \in I_0} u^i \widehat{Y}_i, (u^i)_{i \in I_0} \in \ell^2(I_0) \right\}$

A has the structure of a commutative C^* -subalgebra of $\mathcal{L}(H; H)$ with composition as internal operation :

$$\left(\sum_{i \in I_0} u^i \widehat{Y}_i \right) \circ \left(\sum_{i \in I_0} v^i \widehat{Y}_i \right) = \sum_{i \in I_0} u^i v^i \widehat{Y}_i$$

A has the structure of a Hilbert space with the scalar product : $\langle Y_1, Y_2 \rangle_A = \sum_{i \in I_0} \overline{u_1^i} u_2^i$ and $(\widehat{Y}_i)_{i \in I_0}$ is a Hilbertian basis. ■

Moreover :

1. The adjoint of $Y = \sum_{i \in I} u^i \widehat{Y}_i$ is $Y^* = \sum_{i \in I} \overline{u^i} \widehat{Y}_i$ and the elements of A are normal : $Y \circ Y^* = \sum_{i \in I_0} |u^i|^2 \widehat{Y}_i$

Any element of A is compact, because it is the limit of the sequence of finite rank maps \widehat{Y}_i

It is Hilbert-Schmidt because the space of Hilbert-Schmidt operators is a Hilbert space.

The eigen values and eigen vectors of $Y = \sum_{i \in I} u^i \widehat{Y}_i$ are $(u^i, \varepsilon_i)_{i \in I}$

The eigen values and eigen vectors of $(Y \circ Y^*)^{1/2} = \sum_{i \in I} |u^i| \widehat{Y}_i$ are $(|u^i|, \varepsilon_i)_{i \in I}$.

If they are summable $\sum_{i \in I} |u^i| < \infty$ then Y is trace-class (JCD Th.1115).

2. Each vector $\psi \in H$ defines the continuous linear functional on A :

$$\varphi_\psi : A \rightarrow \mathbb{C} :: \varphi_\psi(Y) = \langle Y\psi, \psi \rangle = \left\langle \sum_{i \in I} u^i \widehat{Y}_i \left(\sum_{k \in I} \psi^k \varepsilon_k \right), \sum_{j \in I} \psi^j \varepsilon_j \right\rangle =$$

$$\sum_{i, j \in I} \bar{u}^i \bar{\psi}^i [K]_j^i \psi^j = [u\psi]^* [K] [\psi]$$

φ_ψ is hermitian ($\varphi_\psi(Y^*) = \overline{\varphi_\psi(Y)}$) and positive ($\varphi_\psi(Y \circ Y^*) \geq 0$).

Such linear functionals are called "states" in the usual algebraic formulation of QM.

3. To the operator $\widehat{Y} = \sum_{i \in I_0} u^i \widehat{Y}_i$ where $(u^i)_{i \in I_0} \in \mathbb{R}_0^{I_0}$ (only a finite number of coefficients is non zero), is associated the observable on E : $Y = \sum_{i \in I_0} u^i Y_i$ which is valued in a finite dimensional subspace of E.

3.2.2 Secondary continuous observables on H

We can go further, but we need two general lemma.

Lemma 10 *There is a bijective correspondance between the projections on a Hilbert space H, meaning the operators*

$$P \in \mathcal{L}(H; H) : P^2 = P, P = P^*$$

and the closed vector subspaces H_P of H. And P is the orthogonal projection on H_P

Proof.

i) If P is a projection, it has the eigen values 0,1 with eigen vector spaces H_0, H_1 . They are closed as preimage of 0 by the continuous maps : $P\psi = 0, (P - Id)\psi = 0$

$$\text{Thus : } H = H_0 \oplus H_1$$

$$\text{Take } \psi \in H : \psi = \psi_0 + \psi_1$$

$$\langle P(\psi_0 + i\psi_1), \psi_0 + \psi_1 \rangle = \langle i\psi_1, \psi_0 + \psi_1 \rangle = -i \langle \psi_1, \psi_0 \rangle - i \langle \psi_1, \psi_1 \rangle$$

$$\langle \psi_0 + i\psi_1, P(\psi_0 + \psi_1) \rangle = \langle \psi_0 + i\psi_1, \psi_1 \rangle = \langle \psi_0, \psi_1 \rangle - i \langle \psi_1, \psi_1 \rangle$$

$$P = P^* \Rightarrow -i \langle \psi_1, \psi_0 \rangle = \langle \psi_0, \psi_1 \rangle$$

$$\langle P(\psi_0 - i\psi_1), \psi_0 + \psi_1 \rangle = \langle -i\psi_1, \psi_0 + \psi_1 \rangle = i \langle \psi_1, \psi_0 \rangle + i \langle \psi_1, \psi_1 \rangle$$

$$\langle \psi_0 - i\psi_1, P(\psi_0 + \psi_1) \rangle = \langle \psi_0 - i\psi_1, \psi_1 \rangle = \langle \psi_0, \psi_1 \rangle + i \langle \psi_1, \psi_1 \rangle$$

$$P = P^* \Rightarrow i \langle \psi_1, \psi_0 \rangle = \langle \psi_0, \psi_1 \rangle$$

$$\langle \psi_0, \psi_1 \rangle = 0 \text{ so } H_0, H_1 \text{ are orthogonal}$$

P has norm 1 thus $\forall u \in H_1 : \|P(\psi - u)\| \leq \|\psi - u\| \Leftrightarrow \|\psi_1 - u\| \leq \|\psi - u\|$ and $\min_{u \in H_1} \|\psi - u\| = \|\psi_1 - u\|$

So P is the orthogonal projection on H_1 and is necessarily unique.

ii) Conversely any orthogonal projection P on a closed vector space meets the properties : continuity, and $P^2 = P, P = P^*$ ■

Lemma 11 For any measurable space (F, S) with σ -algebra S , there is a bijective correspondance between the spectral measures P on the separable Hilbert space H and the maps $f : S \rightarrow H$ with the following properties :

- $f(s)$ is a closed vector subspace of H
- $f(F) = H$
- $\forall s, s' \in S : s \cap s' = \emptyset \Rightarrow f(s) \cap f(s') = \{0\}$

Proof.

i) A spectral measure is a map $P : S \rightarrow \mathcal{L}(H; H)$ such that :

a) $\forall s \in S : P(s)^2 = P(s), P(s) = P^*(s)$

b) $P(F) = Id$

c) $\forall \psi \in H$ the map $\mu : S \rightarrow \mathbb{R} :: \mu(s) = \langle P(s)\psi, \psi \rangle = \|P(s)\psi\|^2$ is a measure on F

ii) With a map f define $P(s)$ as the unique orthogonal projection on $f(s)$. It meets the properties a and b. Let us show that the map μ is countably additive.

Take a countable family $(s_\alpha)_{\alpha \in A}$ of disjointed elements of S . Then $(f(s_\alpha))_{\alpha \in A}$ is a countable family of Hilbert vector subspaces of H . The Hilbert sum $\bigoplus_{\alpha \in A} f(s_\alpha)$ is a Hilbert space H_A , vector subspace of H , which can be identified to $f(\bigcup_{\alpha \in A} s_\alpha)$ and the subspaces $f(s_\alpha)$ are orthogonal. Take any Hilbert basis $(\varepsilon_{\alpha i})_{i \in I_\alpha}$ of $f(s_\alpha)$ then its union is a Hilbert basis of H_A and

$$\forall \psi \in H_A : \sum_{\alpha \in A} \sum_{i \in I_\alpha} |\psi^{\alpha i}|^2 = \sum_{\alpha \in A} \|P(s_\alpha)\psi\|^2 = \|P(\bigcup_{\alpha \in A} s_\alpha)\psi\|^2 < \infty$$

iii) Conversely if P is a spectral measure, using the previous lemma for each $s \in S$ the projection $P(s)$ defines a unique closed vector space H_s of H and $P(s)$ is the orthogonal projection on H_s .

For ψ fixed, because $\mu(s) = \|P(s)\psi\|^2$ is a measure on F , it is countably additive. Take $s, s' \in S : s \cap s' = \emptyset$ then

$$\|P(s \cup s')\psi\|^2 = \|P(s)\psi\|^2 + \|P(s')\psi\|^2$$

$$\text{For any } \psi \in H_{s \cup s'} : \|P(s \cup s')\psi\|^2 = \|\psi\|^2 = \|P(s)\psi\|^2 + \|P(s')\psi\|^2$$

With any Hilbert basis $(\varepsilon_i)_{i \in I}$ of $H_s, (\varepsilon'_i)_{i \in I'}$ of $H_{s'}, \psi \in H_{s \cup s'} : \|\psi\|^2 = \sum_{i \in I} |\psi^i|^2 + \sum_{j \in I'} |\psi'^j|^2$ so $(\varepsilon_i)_{i \in I} \oplus (\varepsilon'_i)_{i \in I'}$ is a Hilbert basis of $H_{s \cup s'}$ and $H_{s \cup s'} = H_s \oplus H_{s'}$ ■

We can now implement these results to the Hilbert space H of the system.

Proposition 12 Any projection $P \neq 0$ on H is of the form $P = \widehat{Y}_J$ for some subset (finite or infinite) J of I

Proof.

Let $P \in \mathcal{L}(H; H) : P^2 = P, P = P^*$ then there is a unique closed vector subspace H_1 such that $P(H) = H_1$ and $H = H_1 \oplus H_0$

$$\text{Define } J = \{i \in I : \varepsilon_i \in H_1\} \Rightarrow \forall i \in J^c : P(\varepsilon_i) = 0 \Leftrightarrow \varepsilon_i \in H_0$$

Any vector $\psi \in H$ can be written :

$\psi = \sum_{i \in I} \langle \phi_i, \psi \rangle_H \varepsilon_i \Rightarrow P(\psi) = \sum_{i \in J} \langle \phi_i, \psi \rangle_H \varepsilon_i = \widehat{Y}_J(\psi)$ ■
 Notice that H_1 can be infinite dimensional.

Proposition 13 *For any measured space (F, S) with σ -algebra S , there is a bijective correspondance between the spectral measures P on the Hilbert space H and the maps $\chi : S \rightarrow 2^I$ such that $\chi(F) = I$ and $\forall s, s' \in S : s \cap s' = \emptyset \Rightarrow \chi(s) \cap \chi(s') = \emptyset$. The spectral measure is then $P(s) = \widehat{Y}_{\chi(s)}$*

Proof.

i) Let P be a spectral measure. Then there is a map $f : S \rightarrow H$ such that $f(s)$ is a closed vector subspace of H
 $f(F) = H$
 $\forall s, s' \in S : s \cap s' = \emptyset \Rightarrow f(s) \cap f(s') = \{0\}$
 For s fixed, $P(s)$ is a projection so $\exists \chi(s) \subset I : P(s) = \widehat{Y}_{\chi(s)}$ and $f(s) = \widehat{Y}_{\chi(s)}(H)$
 $P(F) = Id = \widehat{Y}_I \Leftrightarrow \chi(F) = I$
 $\forall s, s' \in S : s \cap s' = \emptyset \Rightarrow$
 $f(s) \cap f(s') = \{0\} \Leftrightarrow \widehat{Y}_{\chi(s)}(H) \cap \widehat{Y}_{\chi(s')}(H) = \{0\} \Leftrightarrow \chi(s) \cap \chi(s') = \emptyset$
 ii) Conversely, to any map $\chi : S \rightarrow 2^I$ let us associate the map :
 $f(s) = \widehat{Y}_{\chi(s)}(H)$
 This is a closed vector subspace of H , $f(F) = \widehat{Y}_I(H) = H$,
 $\forall s, s' \in S : s \cap s' = \emptyset \Rightarrow f(s) \cap f(s') = \widehat{Y}_{\chi(s)}(H) \cap \widehat{Y}_{\chi(s')}(H) = \{0\}$ ■

As a consequence for any fixed $\psi \in H, \|\psi\| = 1$, the function $\mu : S \rightarrow \mathbb{R} ::$
 $\mu(s) = \langle \widehat{Y}_{\chi(s)}\psi, \psi \rangle = \left\| \widehat{Y}_{\chi(s)}\psi \right\|^2$ is a probability law on (F, S) . Which implies
 that : $\forall s, s' \in S : s \cap s' = \emptyset : \left\| \widehat{Y}_{\chi(s+s')}\psi \right\|^2 = \left\| \widehat{Y}_{\chi(s)}\psi \right\|^2 + \left\| \widehat{Y}_{\chi(s')}\psi \right\|^2$

From there we have the following results :

Theorem 14 *For any measured space (F, S) with σ -algebra S , any map $\chi : S \rightarrow 2^I$ and bounded measurable function $f : F \rightarrow \mathbb{R}$, the spectral integral $:\int_F f(\xi) \widehat{Y}_{\chi(\xi)}$ defines a continuous operator on H . Moreover for any map χ this procedure gives a representation of the C^* algebra of the bounded measurable functions $C_b(F; \mathbb{R})$ and its image is a C^* subalgebra of $\mathcal{L}(H; H)$.*

This is an application of standard theorems on spectral measures (JCD Th 1192, 1197):

The spectral integral is such that there is an operator denoted $\varphi = \int_F f(\xi) \widehat{Y}_{\chi(\xi)}$ with :

$$\forall \psi, \psi' \in H : \langle \varphi(\psi), \psi' \rangle = \int_F f(\xi) \langle \widehat{Y}_{\chi(\xi)}(\psi), \psi' \rangle$$

To this operator one can associate the secondary observable on E :

$$\begin{aligned}\Phi : E \rightarrow E &:: \Phi = \Upsilon^{-1} \circ \varphi \circ \Upsilon \\ \Phi(x) &= \int_F f(\xi) Y_{\chi(\xi)}(x) = \int_F f(\xi) \left(\sum_{j \in \chi(\xi)} x^j e_j \right) \in \mathcal{L}(E; E)\end{aligned}$$

Theorem 15 *For any continuous normal operator φ on H , there is a map $\chi : Sp(f) \rightarrow 2^I$ such that $\varphi = \int_{Sp(f)} s \widehat{Y}_{\chi(s)}$ where $Sp(f)$ is the spectrum of f . If φ has finite range then it belongs to the algebra A (see previous subsection).*

Proof.

i) The first part is the direct application of a classic of spectral analysis (JCD Th.1197). φ has a spectral resolution and so there is a map χ with the properties above.

ii) If φ has a finite range it is necessarily compact, and $\chi(s)$ must be a finite subset of I . By the Riesz theorem (JCD Th.1106) the spectrum of φ is either finite or is a countable sequence converging to 0, contained in disc of radius $\|\varphi\|$ and is identical to the set $(\lambda_n)_{n \in \mathbb{N}}$ of its eigen values (JCD Th.973). For each eigen value λ (except possibly 0), the eigen spaces H_λ are orthogonal for distinct eigen values.

Thus :

as φ has finite range the set $(\lambda_n)_{n=1}^N$ of its distinct eigen values is finite

$$J_n = \chi(\lambda_n) \subset I$$

$$H_n = \widehat{Y}_{\chi(\lambda_n)}(H) = H_{J_n}$$

$$\text{for } n \neq n' : J_n \neq J_{n'}$$

For each distinct eigen value λ_n let $(\psi_{nm})_{m=1}^{N_n}$ be an orthonormal basis of H_n . Then φ reads :

$$\varphi(\psi) = \sum_{n=1}^N \lambda_n \sum_{m=1}^{N_n} \langle \psi_{nm}, \psi \rangle \psi_{nm} \text{ and also : } \varphi = \sum_{n=1}^N \lambda_n \widehat{Y}_{J_n} \text{ where } J_n = \chi(\lambda_n).$$

So $\varphi \in A$ and for each n the eigen vectors ψ_{nm} are a linear combination of the $(\varepsilon_j)_{j \in J_n}$

To φ one can associate the observable : $\Phi : E \rightarrow E :: \Phi = \Upsilon^{-1} \circ \varphi \circ \Upsilon$

$$\Phi(x) = \sum_{n=1}^N \lambda_n Y_{\chi(\lambda_n)}(x)$$

As seen above, the operator is self adjoint iff the eigen values are real, that is if Φ takes real values.

The probability law on the set of eigen values reads, for a vector $\psi \in H$, $\|\psi\| = 1$ fixed :

$$\mu_n = \langle \widehat{Y}_{J_n}(\psi), \psi \rangle = \left\| \widehat{Y}_{J_n} \psi \right\|^2 = \sum_{m=1}^{N_n} |\langle \psi_{nm}, \psi \rangle|^2 \blacksquare$$

3.2.3 Product of observables

1. Secondary observables such as $\Phi(x) = \sum_{n=1}^N \lambda_n Y_{\chi(\lambda_n)}(x)$ or $\int_F f(\xi) Y_{\chi(\xi)}$ can be formally defined, but we need to precise how their value can be measured. This can be conceived in two ways :

- either one proceeds at first to the measure of the primary observables and then to the computation according to the previous formulas

- or, because the result is still a vector of E , their value is measured through another primary observable. This procedure is then modeled as the composition $: Y_J \circ \Phi$ which translates as $\widehat{Y}_J \circ \varphi$ for some finite subset J of I .

The second options seems the more logical, and anyway the only practical whenever the secondary observable is defined through a continuous spectrum.

The product gives :

$\widehat{Y}_J \circ \varphi = \sum_{n=1}^N \lambda_n \widehat{Y}_{\chi(\lambda_n) \cap J}$ in the first case and the subsets $\chi(\lambda_n) \cap J$ are disjointed

$\widehat{Y}_J \circ \varphi = \int_F f(\xi) Y_{\chi(\xi) \cap J}$ in the second case

So, in both cases, it sums up actually to restrict the scope of $\chi(\lambda_n), \chi(\xi)$ to finite subsets.

In the first case the finite collection of finite disjointed sets $(\chi(\lambda_n))_{n=1}^N$ is a finite subset J_φ of I . To make sense the measure by Y_J should encompass only the indices in J_φ and this is possible. Then the result is just φ .

2. The product of two secondary observables has no clear meaning in this picture, and it is not necessary to define the observables on the system. So it does not play a significant role in the following.

3.2.4 Scalar functions

In the previous cases the secondary observables are continuous variables which are either scalars or tensors defined over the whole of the system, or localized maps. In particular $\Phi(x)$ can be the Fourier transform of a map on E .

One can also consider continuous maps $: \lambda : E \rightarrow \mathbb{R}$ meaning belonging to the dual E' of E . They induce a map in $H : \widehat{\lambda} = \lambda \circ \Upsilon^{-1} \in H'$ which is represented by a vector $\phi \in H : \widehat{\lambda}(\psi) = \langle \phi, \psi \rangle = \lambda \circ \Upsilon^{-1}(\psi)$.

As a special case one can consider the vectors $(v_k)_{k=1}^N$ of H , which are an arbitrary family of orthonormal vectors. They can always be chosen such that $: \langle v_k, \psi \rangle = \lambda_k \circ \Upsilon^{-1}(\psi) = \lambda_k(X(\psi))$ which can be useful if the variables are subject to a constraint expressed by an integral.

3.3 Probability

3.3.1 Preliminaries

The formalism developped so far is strictly determinist : to any set of measures (a configuration) corresponds a unique point in the Hilbert affine space. However there is a natural physical probabilist interpretation of some results. But it is necessary at first to make the distinction between formal probability and physical probability.

A probability law is, in mathematics, a positive measure P on some measurable space (F,S) such that $P(S)=1$. So, for any spectral measure P on a measurable space (F,S) and vector $\psi \in H$, the map $: \mu : S \rightarrow \mathbb{R} :: \|P(s)\psi\|^2$ is formally a probability law, meaning that it is positive, countably additive and

$\mu(F) = 1$. But these properties are not in any way related to a physical event, which could or not occur and could be reasonably governed by a probability.

Probability in physics may appear either because the model introduces random variables (the physical phenomenon being determinist or not) or through the process of measures itself. In our picture the second case is explicit. The precision of the measures is involved in two ways :

- by the uncertainty of any measure, which is controlled by a protocol in order to validate the results

- by the fact that we estimate functions from a finite set of data

In our picture the knowledge of the state represented by (ψ_d, ψ) can be seen as the purpose of the measures. If we knew the values of all the variables, we would know precisely the state. But these variables are functions, defined by infinitely many parameters, which are the components $(x_i)_{i \in I}$. Moreover, usually the basis $(e_i)_{i \in I}$ and the indices i are not known by the physicist : we know that they exist from their mathematical definition, but the functions are estimated by statistical methods from one batch of finite measures. For instance the simplest way to estimate a function is to use a linear interpolation from a sample of points. To do so, one does not need to know the basis, but actually the procedure is legitimate because this basis does exist. Because the estimations use controled statistical methods, the uncertainty on the estimates is known. It depends on the nature of the variables which are measured, and on the number of measures which are done.

3.3.2 Main result

Proposition 16 *When the measure of the continuous variables is done according to a precision protocol, any measure of a state $\psi \in H$ of the system has the probability $\nu(J)$ of belonging to $\hat{Y}_J(H)$. ν is a physical probability law on the measurable space (I, \mathcal{I}^I) and if $\|\psi\| = 1$ then $\|\hat{Y}_J(\psi)\|^2 = \nu(J) = \Pr(\psi \in H_J)$*

Proof.

i) To estimate the configuration $X = \sum_{i \in I} y^i e_i$ one uses a batch of data which can be summarized as the primary observable : $Y_J = \sum_{i \in J} y^i e_i$ for a finite set J of I . Let us denote E_J the vector subspace generated by $(e_i)_{i \in J}$ and $\sigma(E_J)$ its Borel σ -algebra. The uncertainty on the measure of X is : $\sigma_J(Y) = \sum_{i \in J^c} y^i e_i$ which can take any value in E_{J^c} . Because the estimation is done using controlled statistical methods, some rules are implemented to quantify the uncertainty of the measure. We assume that the protocol used is such that :

a) For any finite subset J of I , there is a positive, finite, measure μ_J on the Borel σ -algebra of E_{J^c} such that the probability $\Pr(X - Y_J(X) \in \varpi_J)$ is $\mu_J(\varpi_J)$, with $\mu_J(E_{J^c}) = 1$.

b) This measure does not depend of the order of the indices in the collection J

c) For any finite subsets J, K of I , $\varpi_J \in \sigma(E_{J^c})$:

$$\mu_{J \cup K}(\varpi_{J^c} \cup E_{K^c}) = \mu_J(\varpi_{J^c})$$

Then by the Kolomogorov theorem (JCD Th.816) the collection of measures μ_J can be uniquely extended to a measure μ on E such that $\mu(\varpi_J) = \mu_J(\varpi_J)$. And $\mu(E) = 1$.

ii) This measure has for image in the Hilbert space H by the linear map Υ a finite, positive, measure $\hat{\mu}$ on the Borel σ -algebra σ_H of H such that the probability that $o(\psi) = \psi - \hat{Y}_J(\psi) \in \varpi$ is given by $\hat{\mu}(\varpi)$. And $\hat{\mu}(H) = 1$.

Let us denote $H_J = \hat{Y}_J(H)$ and $\nu(J) = 1 - \hat{\mu}(H_J)$. The set 2^I is a σ -algebra of I and $\nu(J)$ is a finite, positive measure on $(I, 2^I)$.

The physicist can see (meaning measure) only states belonging to the vector subspaces H_J , so one can interpret the previous probability the other way around. Whatever the state of the system, the probability that the result of the measure belongs to H_J is $\nu(J)$. The probability increases with J and is 1 if $J=I$, but the measure is not necessarily absolutely continuous, so the probability for $J=j$ is not necessarily null. As one can access only to the states belonging to H_J , for J finite, for all that matters, $\nu(J)$ can be interpreted as the probability that ψ belongs to H_J .

iii) Take any fixed vector $\psi \in H$, $\|\psi\| = 1$ then : $0 \leq \left\| \hat{Y}_J(\psi) \right\|^2 \leq 1$. It is 0 if $\psi \in H_{J^c}$ and 1 if $\psi \in H_J$.

$\left\| \hat{Y}_J(\psi) \right\|^2$ can be seen as a random variable, the value of which is the product of the value of $\|\psi\|^2$ if $\psi \in H_J$ by the probability that $\psi \in H_J$

Thus : $\left\| \hat{Y}_J(\psi) \right\|^2 = \nu(J)$ and one can write : $\left\| \hat{Y}_J(\psi) \right\|^2 = \nu(J) = \Pr(\psi \in H_J)$ ■

Comments

1. Let us take a single continuous variable x , which takes its values in \mathbb{R} . It is clear that any physical measure will at best give a rational number $Y(x) \in \mathbb{Q}$ up to some scale. There are only countably many rational numbers for unaccountably many real scalars. So the probability to get $Y(x) \in \mathbb{Q}$ should be zero. The simple fact of the measure gives an incommensurable weight to rational numbers, implying that each of them has some small, but non null, probability to appear. In this case I can be assimilated to \mathbb{Q} , the subsets J are any finite collection of rational numbers.

2. The experimental proof of some theoretical statements can be deceptive. If a physicist states "the ratio of the circumference of a circle to the length of the diameter is a rational number", he certainly can provide very accurate measures to sustain its statement. But it stays false.

3. Measures on infinite dimensional vector spaces are a delicate topic, but there are sensible solutions, which meet the necessary condition for a probability law : they can be finite (see Gill).

One can now extend this result to spectral measures and secondary observables.

3.3.3 Measure of an observable

Proposition 17 For any spectral measure P on a measurable space (F, S) , valued in $\mathcal{L}(H; H)$ the induced formal probability law : $\mu(s) = \|P(s)(\psi)\|^2$ can be interpreted as $\mu(s)$ is the probability that the physical measure of $\psi \in H, \|\psi\| = 1$ belongs to $P(s)(H)$.

Proof.

Let P be a spectral measure on a measurable space (F, S) , valued in $\mathcal{L}(H; H)$. We know from the previous results that there is a map : $\chi : S \rightarrow 2^I$ such that $\chi(F) = I, \forall s, s' \in S, s \cap s' = \emptyset \Rightarrow \chi(s) \cap \chi(s') = \emptyset$ and $P(s) = \widehat{Y}_{\chi(s)}$.

For any $\psi \in H, \|\psi\| = 1$ the formal probability law is : $\mu(s) = \|P(s)(\psi)\|^2 = \left\| \widehat{Y}_{\chi(s)}(\psi) \right\|^2 = \Pr(\psi \in H_{\chi(s)})$

So it is legitimate to write : $\mu(s) = \nu(\chi(s)) = \Pr(\psi \in H_{\chi(s)})$ and this has a physical meaning. ■

Proposition 18 The measure of any secondary observable is necessarily equal to $\lambda_n \psi_n$ where λ_n is one of the eigen-values of the associated operator φ and ψ_n a vector of the eigen space. The probability that the eigen-value λ_n is picked up when the system is in the state $\psi \in H, \|\psi\| = 1$ is equal to $\sum_{m=1}^{N_n} |\langle \psi_{nm}, \psi \rangle|^2$ where ψ_{nm} are an orthonormal basis of the eigen-vector subspace associated to λ_n

Proof.

Let Φ be a secondary observable. According to our general assumptions and previous results, the associated operator is a compact operator which reads (see notations above) :

$\varphi(\psi) = \sum_{n=1}^N \lambda_n \sum_{m=1}^{N_n} \langle \psi_{nm}, \psi \rangle \psi_{nm}$ and also : $\varphi = \sum_{n=1}^N \lambda_n \widehat{Y}_{J_n}$ where the subsets J_n are finite and disjointed

The induced probability law on the set of eigen values reads, for a vector $\psi \in H, \|\psi\| = 1$ fixed :

$$\nu_n = \left\langle \widehat{Y}_{J_n}(\psi), \psi \right\rangle = \left\| \widehat{Y}_{J_n} \psi \right\|^2 = \sum_{m=1}^{N_n} |\langle \psi_{nm}, \psi \rangle|^2$$

This probability law can be interpreted as the probability that the actual measure of φ provides a result $\varphi(\psi)$ in the vector subspace H_n which is the eigen space associated to λ_n , and the probability that this result shows if $\|\psi\| = 1$ is

$$\left\| \widehat{Y}_{J_n} \psi \right\|^2 = \sum_{m=1}^{N_n} |\langle \psi_{nm}, \psi \rangle|^2$$

Then the result of the measure of φ is :

$\varphi(\psi) = \sum_{n=1}^N \lambda_n \widehat{Y}_{\chi(\lambda_n)}(\psi) = \sum_{n=1}^N \lambda_n \sum_{j \in \chi(\lambda_n)} \langle \phi_j, \psi \rangle \varepsilon_j = \lambda_n \psi_n$ where $\psi_n \in H_n$ ■

3.3.4 Discrete observables

In our definition of discrete observables we have not assumed anything like a topology on the set of measures. And indeed the distinction between the configurations described by D should not be blurred. Thus, in the very general framework that has been adopted here, we cannot go further.

4 THE WIGNER' S THEOREM

4.1 Principles

4.1.1 The observer

The measures are done by an observer. Without engaging the issue of the interaction observer / physical system, the main characteristic of the observer is that he or she has "free will", meaning that this is the operator who implements the experiment (deciding when to proceed to some actions), who chooses the units of measures, and anything which is relevant to these measures such as the frames for localized data. What we sum up by saying that the operator has the "freedom of gauge". And this freedom is not encumbered by what happens in the physical system.

So the question arises of what happens when two different observers of the same system, using similar models (the variables have the same mathematical properties) and similar procedures, compare their data and their estimation of the state of the system.

4.1.2 Symmetries

This question is linked to the symmetries. As it is an ubiquitous word in physics, for the sake of clarity, it needs some precisions. When the definition of a variable is linked to some frame (mathematically and in the measurement procedure), any change of frame should entail a change of the measured value, according to precise, known, rules depending of its mathematical specification : this has nothing to do with some "jump" of a state, only to do with the conversion of data. And the validity of the specification can be checked through the comparison of the measures done by two observers. In this case we say that the measures are equivariant : the figures change according to the same mathematical rules as the corresponding variables.

However it can happen that two observers, using different frames, get the same figures. Usually this phenomenon is met for some classes of observers, or gauge transformations. In this case we say that the system shows a symmetry, which is characterized by the class of gauge transformations for which it is seen. For instance a solid body has an axial symmetry if observers rotating around a precise axis get the same measures. A symmetry can be dealt with in two

ways. Either it is assumed in the model (from some theoretical assumptions or from accounting for some particularities of the physical system) and then the variables are defined accordingly. Or the specifications of the variables are more general, and the symmetry is then a result of the experiment, to be used latter in some theory. But in both cases it can be checked through the measures.

So equivariance and symmetries are two faces of the same feature. They involve the mathematical properties of the variable which has been chosen to represent a physical phenomenon. So far the unique property required for the continuous variable is that they can be represented in a vector space (E). For equivariance it is necessary to use an additional information : the way a variable behave under a gauge transformation, which is initiated by the observer. And equivariance arises if and only if there is some preexistent rule between the measures, thus these rules can be considered as a part of the model.

In this section we will consider equivariance : two observers measure the configuration of the system, using the same model, meaning the same variables with identical mathematical properties, similar measurement procedures but with different frames. For the same state of the system they get two different set of measures X, X' , which are related by some mathematical law $X' = U(X)$ which is defined by the mathematical rules that relate the frames. So U is assumed to exist and the issue is to see what happens at the level of the Hilbert space. This map U is directly related to the model, meaning the mathematical description of the variables, but it can also be checked and measured, by proceeding to experiments using different frames.

In our picture the relation between the measures and the vector in the Hilbert space relies on the basis which is used, so it can be considered from two, equivalent, points of view : these measures correspond to the same state, represented in two different bases, or they correspond to two different states, represented in the same basis. We will adopt the second point of view.

4.2 Theorem

Proposition 19 *If two observers observe the same system, using the same model, their respective space of representation is the same affine Hilbert space, they assign to the same configuration two states which are related by a unitary map in $H \times H$:*

$$\psi'_d = U_d(\psi_d), \psi' = U(\psi)$$

The operators $\widehat{D}_J, \widehat{Y}_J, \widehat{D}'_J, \widehat{Y}'_J$ associated to the same primary observables D_J, Y_J, D'_J, Y'_J and acting on the vectors $(\psi_d, \psi), (\psi'_d, \psi')$ are related by unitary linear maps $\widehat{U}_d, \widehat{U} \in \mathcal{L}(H; H)$: $\widehat{D}'_J = U_d \circ \widehat{D}_J \circ U_d^, \widehat{Y}'_J = \widehat{U} \circ \widehat{Y}_J \circ \widehat{U}^*$*

Continuous variables

Proof.

i) A change of frame for continuous variables may happen in two, non exclusive, cases :

- a variable Ξ is tensorial and its components are described in a basis, and the change of gauge is a change of basis. So it impacts the basis $(e_i)_{i \in I}$ of E itself : a vector V is measured in two different bases as : $V = \sum v^k e_k = \sum v'^k e'_k$. By reporting the components v'^k on the base e_k we get a different vector : $V' = \sum v'^k e_k$ which is related to V by a linear map : $V' = U(V)$. Knowing the rules for the change of basis one can say that V and V' give equivalent descriptions of the vector.

- a variable is a localized map $\Xi(\xi)$ from \mathbb{R}^m to a vector space F and belongs to some space of maps $E \subset C(\mathbb{R}^m; F)$. The change of gauge is applied to the parameters $\xi \in \mathbb{R}^m$ used for the localization. So we assume that there is a map : $\xi' = u(\xi)$. Then the change of gauge impacts the arguments ξ of the map, but the maps Ξ, Ξ' can still be represented in the same space E of maps and there is no change to the basis $(e_i)_{i \in I}$ of E. And we assume that there is some map U such that : $\Xi' = U(\Xi) : \forall \xi : \Xi'(\xi') = \Xi'(u(\xi)) = \Xi(\xi) \Leftrightarrow \Xi' = \Xi \circ u^{-1} = U(\Xi)$

$$\begin{array}{ccccccc}
 & & & \text{Id} & & & \\
 & & & \rightarrow & \rightarrow & \rightarrow & \\
 & \text{F} & & & & & \text{F} \\
 & \uparrow & & & & & \uparrow \\
 \Xi & \uparrow & & & & & \uparrow & \Xi' \\
 & \uparrow & & \text{u} & & & \uparrow & \\
 & \mathbb{R}^m & \rightarrow & \rightarrow & \rightarrow & & \mathbb{R}^m &
 \end{array}$$

ii) We assume that the two observers use the same model, with the same properties of the variables, represented in a vector space E. So the manifold and the Hilbert space are the same. They observe the same physical system at a given time. From their measures and the estimation process they get two vectors $\Xi = \sum_{i \in I} x^i e_i, \Xi' = \sum_{i \in I} x'^i e_i$ expressed in the same basis $(e_i)_{i \in I}$ of E. And there is some mathematical law to convert the vector $\Xi' = U(\Xi)$, not necessarily linear.

On the other hand each vector Ξ represents a state ψ through some chart X : $\Xi = X(\psi)$

We say that the two vectors are equivalent if they represent the same physical state of the system :

$$\Xi' = X(\psi) = U(\Xi) = U(X(\psi))$$

Then, for any states ψ_1, ψ_2 :

$$\langle X(\psi_1), X(\psi_2) \rangle_E = \langle U(X(\psi_1)), U(X(\psi_2)) \rangle_E = \langle \psi_1, \psi_2 \rangle_H$$

iii) If we choose the linear map Υ associated to a basis $(e_i)_{i \in I}$ of E, we associate the vector $\psi = \Upsilon(X)$ to X and $\psi' = \Upsilon(U(X))$ to the configurations X and U(X).

Define the map : $\hat{U} : H \rightarrow H :: \hat{U} = \Upsilon \circ U \circ \Upsilon^{-1}$ so $\hat{U}(\Upsilon(X)) = \Upsilon(U(X))$

$$\left\langle \hat{U}(\Upsilon(X_1)), \hat{U}(\Upsilon(X_2)) \right\rangle_H = \langle \Upsilon(U(X_1)), \Upsilon(U(X_2)) \rangle_H$$

$$= \langle U(X_1), U(X_2) \rangle_E = \langle X_1, X_2 \rangle_E$$

U is defined for any vector of E, so for the orthogonal basis $(\tilde{e}_i)_{i \in I}$ of E.

$$\text{Define : } \hat{U}(\Upsilon(\tilde{e}_i)) = \Upsilon(U(\tilde{e}_i)) = \hat{U}(\tilde{e}_i) = \tilde{e}'_i$$

The set of vectors $(\tilde{e}'_i)_{i \in I}$ is a hilbertian basis of H:

$$\left\langle \widehat{U}(\Upsilon(\tilde{e}_i)), \widehat{U}(\Upsilon(\tilde{e}_j)) \right\rangle_H = \langle U(X_1), U(X_2) \rangle_E = \langle \tilde{e}_i, \tilde{e}_j \rangle_E = \delta_{ij} = \langle \tilde{\varepsilon}'_i, \tilde{\varepsilon}'_j \rangle_H$$

So we can write :

$$\forall \psi \in H : \psi = \sum_{i \in I} \psi^i \tilde{\varepsilon}_i, \widehat{U}(\psi) = \sum_{i \in I} \psi^i \tilde{\varepsilon}'_i$$

$$\text{and : } \psi^i = \langle \varepsilon_i, \psi \rangle = \left\langle \widehat{U}(\varepsilon_i), \widehat{U}(\psi) \right\rangle = \left\langle \varepsilon'_i, \sum_{j \in I} \psi^j \tilde{\varepsilon}'_j \right\rangle = \psi^i$$

$$\text{Thus the map } \widehat{U} \text{ reads : } \widehat{U} : H \rightarrow H :: \widehat{U} \left(\sum_{i \in I} \psi^i \tilde{\varepsilon}_i \right) = \sum_{i \in I} \psi^i \tilde{\varepsilon}'_i$$

It is linear, continuous and unitary : $\left\langle \widehat{U}(\psi_1), \widehat{U}(\psi_2) \right\rangle = \langle \psi_1, \psi_2 \rangle$ and \widehat{U} is invertible

$$\text{Expressed in the Hilbert basis } \tilde{\varepsilon}_i \text{ its matrix is : } \widehat{U}(\tilde{\varepsilon}_i) = \tilde{\varepsilon}'_i = \sum_{j \in I} \left[\widehat{U} \right]_i^j \tilde{\varepsilon}_j$$

iv) Conversely the map $U = \Upsilon^{-1} \circ \widehat{U} \circ \Upsilon$ between the vectors $X, X' = U(X)$ is necessarily linear and unitary with respect to the scalar product on E. If we denote $\tilde{\varepsilon}'_i = \Upsilon^{-1}(\tilde{\varepsilon}'_i) = \sum_{j \in I} \left[\widehat{U} \right]_i^j \tilde{\varepsilon}_j$ one can see that the change of gauge is indeed a change of basis. It reads in the orthonormal basis $(\tilde{e}_i)_{i \in I}$:

$$X' = U(X) = \Upsilon^{-1} \circ \widehat{U} \circ \Upsilon \left(\sum_{i \in I} \tilde{x}^i \tilde{e}_i \right) = \sum_{i \in I} \tilde{x}^i \tilde{e}'_i = \sum_{i \in I} \tilde{x}^i \tilde{e}_i$$

$$\text{and in the basis } (e_i)_{i \in I} \text{ with } e'_i = U(e_i) = \sum_{j \in I} [U]_i^j e_j$$

$$X' = U(X) = \sum_{i \in I} x^i e'_i = \sum_{i \in I} x^i e_i$$

We have :

$$x^i = \sum_{k \in I} [U]_k^i x^k$$

$$\langle e'_i, e'_j \rangle_E = \langle e_i, e_j \rangle_E = [K]_j^i$$

iv) Primary observables are defined by the choice of a finite subset J of I. To make sense, the comparison between measures done by the two observers must involve the same primary observable.

To the observable $Y_J(X) = \sum_{j \in J} x^j e_j$ corresponds the vector $U(Y_J(X)) = \sum_{j \in J} x^j e'_j = \sum_{j \in J} x'^j e_j$ so the operator Y'_J acting on X' is :

$$Y'_J(X') = U \circ Y_J(X) = U \circ Y_J \circ U^{-1}(X')$$

Similarly the associated operator on H is : $\widehat{Y}'_J = \widehat{U} \circ \widehat{Y}_J \circ \widehat{U}^*$ acting on $\psi' = U(\psi)$

$$\widehat{Y}_J = \Upsilon \circ Y_J \circ \Upsilon^{-1}$$

$$\widehat{Y}'_J = \Upsilon \circ U \circ Y_J \circ U^{-1} \circ \Upsilon^{-1} = \Upsilon \circ U \circ \Upsilon^{-1} \circ \Upsilon \circ Y_J \circ \Upsilon^{-1} \circ \Upsilon \circ U^{-1} \circ \Upsilon^{-1}$$

$$= \widehat{U} \circ \widehat{Y}_J \circ \widehat{U}^*$$

And the value which is measured is :

$$\widehat{Y}_J(\psi) = \sum_{j \in J} \langle \phi_j, \psi \rangle \varepsilon_j \text{ with } \langle \phi_j, \psi \rangle = x^j$$

$$\widehat{Y}'_J(\psi') = \sum_{j \in J} \langle \phi_j, \widehat{U}^*(\psi') \rangle \widehat{U}(\varepsilon_j) = \sum_{j \in J} \langle \phi_j, \widehat{U} \widehat{U}^*(\psi') \rangle \varepsilon_j = \sum_{j \in J} \langle \phi_j, \psi' \rangle \varepsilon_j = \sum_{j \in J} \langle \phi_j, \psi \rangle \varepsilon'_j$$

$$\Upsilon^{-1} \left(\widehat{Y}'_J(\psi') \right) = \sum_{j \in J} \langle \phi_j, \psi' \rangle e_j = \sum_{j \in J} \langle \phi_j, \psi \rangle e'_j = \sum_{j \in J} x^j e'_j = \sum_{j \in J} x'^j e_j$$

$$\text{So : } x'^j = \langle \phi_j, \psi' \rangle \blacksquare$$

Discrete variables

Proof.

i) The models are identical, the Hilbert affine space \widehat{H} is the same. As $\langle Uu, Uu \rangle = \langle u, u \rangle$ the ground states are the same. The vectors $(v_k)_{k=1}^d$ are

arbitrary, but the only thing which matters here is their labelling. The second observer can change the labelling of the states by a permutation of the d points. A permutation is an element of the symmetric group $\mathfrak{S}(d)$, which is represented in the vector space H by a unitary isomorphism. So there is a unitary map U_d such that : $v'_i = \sum_{j=1}^d [U_d]_i^j v_j$ with the $d \times d$ unitary matrix $[U_d]$. A given configuration of the system is represented by the same vector in H and, reported in the same basis with the same labelling, by two vectors ψ_d, ψ'_d which are such that : $\psi'_d = U_d(\psi_d)$.

ii) If the observers measure a primary discrete variable, the comparison makes sense only if the two observers measure the same primary observable. The only difference which can occur is related to the labelling of the states which are measured. So they use a primary variable with s positions, and vectors : $(\varrho_j)_{j=1}^s, (\varrho'_j)_{j=1}^s$ which are related by the same permutation matrix $[U_d]$: $\varrho'_j = U_d(\varrho_j)$

The operators are :

$$\begin{aligned}\widehat{D}_J(\psi_d) &= \sum_{j=1}^s \langle \varrho_j, \psi_d \rangle \varrho_j \\ \widehat{D}'_J(\psi'_d) &= \sum_{j=1}^s \langle \varrho'_j, \psi'_d \rangle \varrho'_j = \sum_{j=1}^s \langle U_d(\varrho_j), \psi'_d \rangle U_d(\varrho_j) \\ &= U_d \left(\sum_{j=1}^s \langle U_d(\varrho_j), \psi'_d \rangle \varrho_j \right) = U_d \left(\sum_{j=1}^s \langle \varrho_j, U_d^*(\psi'_d) \rangle \varrho_j \right)\end{aligned}$$

Which reads :

$$\begin{aligned}\widehat{D}'_J(\psi'_d) &= U \left(\widehat{D}_J(U_d^*(\psi'_d)) \right) \\ \widehat{D}'_J &= U_d \circ \widehat{D}_J \circ U_d^* \quad \blacksquare\end{aligned}$$

Comments

1. The observers may measure some of the variables only. The result holds in this case for these variables, with the obvious adjustment on U . The key condition is always that the physical measures must be related.

2. The primary observables \widehat{Y}_J commute, so, for the same value of \widehat{U} , the operators \widehat{Y}'_J commute. But this is not usually the case for $\widehat{Y}_J, \widehat{Y}'_J$.

4.3 Application to groups of transformations

4.3.1 General principle

1. The most important application of the Wigner's theorem is when the measures by the two observers are deduced from each other by the action of a group G .

Then there is a map : $U : G \rightarrow \mathcal{L}(E; E) :: X_g(\psi) = U(g)(X_1(\psi))$ where 1 is the unit in G , $X_g(\psi), X_1(\psi)$ are the measures done by two observers whose frames are deduced by a transformation labeled by G . U is a unitary linear map, as above. And similarly for discrete variables. If moreover U is such that : $U(g \cdot g') = U(g) \circ U(g'); U(1) = Id$ meaning that (E, U) is a unitary representation of the group G , then we have also a unitary representation (H, \widehat{U}) of the group G .

But it does not entail any consequence for the composition of observables : the observers must choose one of the available frames, which can themselves be the composed of frames (by G) but, once the choice has been made, the measures are done by using a definite frame. There is no clear meaning, in this picture, to the composition of observables such that $\widehat{Y}(g') \circ \widehat{Y}(g)$.

2. If the map U is continuous and G is an abelian, locally compact, topological group, and the map $\widehat{U} : G \rightarrow \mathcal{L}(H; H)$ is continuous, there is a spectral measure P on the Pontryagin's dual of G , meaning the set $\widehat{G} = C_0(G; T)$ of continuous maps τ from G to the set T of complex numbers of module 1, such that $\widehat{U}(g) = P(\tau(g))$. Accounting for previous results, that means that there is a map : $\chi : \sigma_{\widehat{G}} \rightarrow 2^I$ on the σ -algebra of \widehat{G} such that : $P(s) = \widehat{Y}_{\chi(s)}$ and $\widehat{U} = \int_G \exp i\tau(g) \widehat{Y}_{\chi \circ \tau(g)}$. Moreover an operator on H commutes with \widehat{U} if and only if it commutes with each $\widehat{Y}_{\chi(s)}$ (JCD Th.1859).

3. If the map : $\widehat{U} : G \rightarrow \mathcal{L}(H; H)$ is continuous, then it has a derivative and $(H, \widehat{U}'(1))$ is a representation of the Lie algebra T_1G and $\forall \kappa \in T_1G$: $\widehat{U}(\exp \kappa) = \exp \widehat{U}'(1) \kappa$ where the first exponential is taken on T_1G and the second on $\mathcal{L}(H; H)$. Moreover $\widehat{U}'(1) \kappa$ is anti-hermitian : $(\widehat{U}'(1) \kappa)^* = -(\widehat{U}'(1) \kappa)$ and $(H, \widehat{U}'(1))$ is a representation of the universal enveloping algebra of T_1G . (JCD Th.1817, 1822, 1828).

4. Any topological group G endowed with a Haar measure has at least a unitary representation (the left or the right regular representation, acting on the arguments) on a Hilbert space of functions. These representations are usually infinite dimensional, but may be finite dimensional on the spaces of polynomials. They provide common specifications for the Hilbert space H itself in Quantum Physics. Notice that the group of displacements in an affine space has no Haar measure.

4.3.2 One parameter groups of transformation

1. If the gauge transformations U are continuous functions of a real scalar θ such that : $U(\theta + \theta') = U(\theta) \circ U(\theta')$, $U(0) = Id$ then they constitute a one parameter : semi-group if $\theta \geq 0$, group if $\theta \in \mathbb{R}$. As E is a Banach vector space, these groups, which are abelian, have some important properties (JCD p.244, 284).

If $\lim_{\theta \rightarrow 0} \|U(\theta) - Id\| = 0$ then the (semi) group is uniformly continuous and : a semi-group can be extended in a group, there is an infinitesimal generator $S \in \mathcal{L}(E; E)$ such that : $\frac{dU}{d\theta} = S \circ U(\theta) \Leftrightarrow U(\theta) = \exp \theta S$ with the exponential of operators on a Banach space.

The associated operator \widehat{U} in H is also a (semi) group of operators, it is unitary, and there is an operator $\widehat{S} \in \mathcal{L}(H; H)$ such that : $\frac{d\widehat{U}}{d\theta} = -i\widehat{S} \circ \widehat{U}(\theta)$. \widehat{S} is a linear map, defined on some subset D of the Hilbert space H which can be defined as : $D = \left\{ \psi \in H : \sup_{\theta \in \mathbb{R}} \left\| \frac{1}{\theta} (\widehat{U}(\theta) - Id) \psi \right\| < \infty \right\}$ and H is a bounded

operator if $\lim_{\theta \rightarrow 0} \|\widehat{U}(\theta) - Id\| = 0$.

As it is easy to check, by differentiation of $\langle \widehat{U}(\theta)\psi, \widehat{U}(\theta)\psi' \rangle$, the operator $i\widehat{S}$ is self-adjoint.

2. Whenever U depends of some Lie group G , any element κ of the Lie algebra T_1G is the infinitesimal generator of the one parameter group of transformations : $U : \mathbb{R} \rightarrow \mathcal{L}(E; E) :: U(\exp \theta \kappa)$

Then, if the continuity conditions above are met, there is an infinitesimal generator S depending on κ :

$$\frac{d\widehat{U}}{d\theta}|_{\theta=0} = \frac{d\widehat{U}}{dg}|_{g=1} \left(\frac{d \exp \theta \kappa}{d\theta} |_{\theta=0} \right) = -i\widehat{S}_\kappa = \widehat{U}'(1_G) \kappa$$

And conversely each value of κ defines a family of gauge transformations with one degree of freedom.

3. For the vector representing the state :

$$\psi(\theta) = \widehat{U}(\theta)\psi(0) \Rightarrow \frac{d\psi}{d\theta} = -i\widehat{S}(\psi(\theta))$$

For the observables we have similarly :

$$\widehat{Y}_J(\theta) = \widehat{U}(\theta) \circ \widehat{Y}_J(0) \circ \widehat{U}(-\theta)$$

$$\frac{d\widehat{Y}_J}{d\theta} = -i\widehat{S} \circ \widehat{U}(\theta) \circ \widehat{Y}_J(0) \circ \widehat{U}(-\theta) + i\widehat{U}(\theta) \circ \widehat{Y}_J(0) \circ \widehat{S} \circ \widehat{U}(-\theta) = -i\widehat{S} \circ \widehat{Y}_J(\theta) + i\widehat{Y}_J(\theta) \circ \widehat{S} = i \left[\widehat{Y}_J(\theta), \widehat{S} \right]$$

because $\widehat{S}, \widehat{U}(\theta)$ commute.

4. As said previously the map U shall be considered as part of the model, as it is directly related to the definition of the variables. Moreover it is also an observable which can be subject to measures. But, as \widehat{U} is unitary, it cannot be self adjoint or trace class.

The status of S is less obvious. It is computed from other observables, and the validity of the rules upon which they rely can be checked this way. However, using the relations above, it may be possible to identify some variables, with a clear physical meaning, which can be measured, and thus to use them in a model. This is the case for the translations and rotations in space which are seen below.

4.3.3 Symmetries

1. A system is symmetric with respect to a transformation represented by U if, for two observers X, X' such that $X' = U(X)$, the value of the observables are equal : $Y_J(X) = Y'_J(X') \Leftrightarrow \sum_{j \in J} x^j e_j = \sum_{j \in J} x'^j e_j$

Then, because $\langle \phi_j, \psi \rangle = x^j, \langle \phi_j, \psi' \rangle = x'^j$ we have : $\widehat{Y}_J(\psi) = \widehat{Y}'_J(\psi') = \widehat{U} \circ \widehat{Y}_J \circ \widehat{U}^*(\psi')$ and conversely.

2. If the transformation is the continuous representation of some unitary group then the one dimensional symmetries are defined by an element κ of the Lie algebra : $\forall \theta \in \mathbb{R} : \widehat{U}(\exp \theta \kappa)\psi(0) = \psi(0)$. The corresponding states must be eigen vectors of $U(\exp \theta \kappa)$ and they must belong to the kernel of \widehat{S}_κ : $\widehat{U}'(1_G)\kappa(\psi(0)) = 0$.

3. A symmetry can be seen for an observable only. Which means that :

$$\forall \psi \in H, \forall \theta \in \mathbb{R} : \widehat{Y}_J(\theta)(\psi) = \widehat{U}(\theta) \circ \widehat{Y}_J(0) \circ \widehat{U}(\theta)^*(\psi) = \widehat{Y}_J(0)(\psi) \Leftrightarrow \widehat{U}(\theta) \circ \widehat{Y}_J(0) = \widehat{Y}_J(0) \circ \widehat{U}(\theta) \Leftrightarrow [\widehat{Y}_J(0), \widehat{U}(\theta)] = 0$$

4.3.4 The units issue

The observers have the choice of their units. If two observers 1,2 use units for an observable x^j such that $x'^j = kx^j$ then we must have $\langle x_1, x_2 \rangle_E = \langle x'_1, x'_2 \rangle_E = \langle \psi_1, \psi_2 \rangle_H = \langle Ux_1, Ux_2 \rangle_E = |k|^2 \langle x_1, x_2 \rangle_E \Rightarrow k = \pm 1$. It implies that the observables must be dimensionless quantities. This is in agreement with the elementary rule that any formal theory should not depend on the units which are used. But has some important consequences as will be seen.

4.4 Fiber bundle extension

Our picture opens the way to an extension which could be very helpful, as it addresses a common class of models of theoretical physics.

4.4.1 Basic model

1. One large class of models has for variables sections of a vector bundle. As any vector bundle can be associated to some principal bundle, there is no loss of generality to use the following general model :

M is a manifold

$P(M, G, \pi)$ is a principal bundle with base M, fiber group G and projection π , with trivialization $p = \varphi(m, g)$ (we will always assume that the bundle is trivial in the bounded area covered by a system)

(V, r) is a finite dimensional representation of the group G, and $(\varkappa_i)_{i=1}^p$ a basis of V

$F[V, r]$ is the associated vector bundle, with holonomic basis $\varkappa_i(m)$ associated to the couple $(\varphi(m, 1), \varkappa_i) \sim (\varphi(m, g^{-1}), r(g)\varkappa_i)$

G and V can be the direct product of similar structures.

The continuous variables are assumed to be a section X belonging to some vector space $E \subset \mathfrak{X}(F)$ of sections on F. The set of sections of a vector bundle has a structure of vector space, with fiberwise operations, which is infinite dimensional, thus it fits well in our picture. It can be the Banach space $\mathfrak{X}_{\infty c}(F)$ of smooth compactly supported sections (as the area covered by the system is bounded, one may assume that the variables are null out of a compact subset of M). Notice that a basis $(e_i)_{i \in I}$ of $\mathfrak{X}(F)$ is a basis of a vector space of maps. Thus $x(m) = \sum_{i \in I} x^i e_i(m)$ where the components x^i are constant, null but for a finite number, and $e_i(m)$ are sections in $\mathfrak{X}(E)$.

2. We can apply the previous results. There is a scalar product on E, an associated Hilbert space H, and each section of $\mathfrak{X}(F)$ is represented by a vector ψ . To each section e_i is associated the vector ε_i . We know of such structures of Hilbert space on set of sections, the most usual being $E=L^2(M, \mu, F)$ with

a Radon measure μ on M (JCD Th.2171). But we have proven that there is necessarily such a structure for (almost) any physical model.

3. Any primary observable can be written as $Y_j(X) = x^j e_j$ so is associated to the dual map e^j . And similarly on H the observable is associated to the vector ϕ_j which is the vector associated to the 1-form : $\varepsilon^j : H \rightarrow \mathbb{C} :: \varepsilon^j(\psi) = \langle \phi_j, \psi \rangle$

The theory of distributions (or generalized functions) can be extended to sections of vector bundles (JCD p.670). Such a distribution acts on $\mathfrak{X}_{\infty c}(F)$ globally (they are not necessarily defined by operations fiberwise). Any distribution $\lambda \in \mathfrak{X}_{\infty c}(F)'$ defines a scalar continuous observable, to which is associated the operator on H : $\hat{\lambda}(\psi) = \lambda(\Upsilon^{-1}(\psi))$ which belongs to the dual of H , and so there is an associated vector $\phi_\lambda \in H$: $\hat{\lambda}(\psi) = \lambda(\Upsilon^{-1}(\psi)) = \langle \phi_\lambda, \psi \rangle$.

4. As we assume that we stay inside the domain of definition of one chart of P , the variable $Y \in \mathfrak{X}(F)$ is defined by a map : $y : M \rightarrow V$ in a vector space S_V of maps. Similarly a section $Z \in \mathfrak{X}(P)$ of P is defined by a map : $z : M \rightarrow G$ belonging to a space of maps S_P (which, usually, is not a vector space).

5. The definition of secondary continuous observable proceeds along the same line as seen previously.

4.4.2 Discrete variables

1. A discrete variable can be seen as a function D taking discrete values $1, 2, \dots, d$ over M . Thus it can be modelled as a partition of M in d disjointed subsets M_k where $D = k$ and the associated vector as: $\psi_d(m) = \sum_{k=1}^d 1_{M_k}(m) v_k \in H$ with the characteristic function of M_k .

2. This model is necessarily discontinuous, which raises many issues, both physical and mathematical. It could be possible to replace the discrete variable by a function taking its values in \mathbb{R} , the splitting in d segments being then part of the estimation process (similar to the use of filters on the signal $x(m)$). However the structure of vector space would not be formally preserved for such continuous variable.

4.4.3 The r jet extension

1. Many usual models involve derivatives of variables. In our picture each derivative is treated independantly. When the variables are sections of a vector bundle one can use the jet formalism. To any vector bundle F one can associate its r jet extension $J^r F$ which is a vector bundle on the same manifold. A section of $J^r F$ is given by a map : $M \rightarrow (\sum_{i=1}^p Z_{\alpha_1 \dots \alpha_s}^i(m) \varkappa_i(m), \alpha_k = 1 \dots \dim M, s = 0 \dots r)$ where the coefficients $Z_{\alpha_1 \dots \alpha_s}^i$ are symmetric in the lower indices. Notice that they are independant, so $\mathfrak{X}(J^r F) \sim \mathfrak{X}(F)^N$: it is isomorphic to the product of N (depending on the dimensions and r) spaces of sections on F .

2. A system described in $J^r F$ gives rise to a Hilbert space H_r which can be seen as the (finite) direct sum of Hilbert spaces :

$$H_r = \bigoplus_{s=0}^r \bigoplus_{1 \leq \alpha_1 \dots \leq \alpha_s \leq \dim M} H_{\alpha_1 \dots \alpha_s} .$$

If $(e_i)_{i \in I}$ is a basis of $\mathfrak{X}(F)$, a basis of $\mathfrak{X}(J^r F)$ reads

$$(e_i^{\alpha_1 \dots \alpha_s}, 1 \leq \dots \leq \alpha_k \leq \dots \leq \dim M, s = 0 \dots r)_{i \in I}$$

and the associated hilbertian basis of H_r :

$$(\widehat{e}_i^{\alpha_1 \dots \alpha_s}, 1 \leq \dots \leq \alpha_k \leq \dots \leq \dim M, s = 0 \dots r)_{i \in I}$$

A vector of H_r reads :

$$\psi_r = \sum_{s=0}^r \sum_{1 \leq \alpha_1 \dots \alpha_s \leq \dim M} \psi_{\alpha_1 \dots \alpha_r} \text{ where } \psi_{\alpha_1 \dots \alpha_r} = \sum_{i \in I} \psi_{\alpha_1 \dots \alpha_r}^i \widehat{e}_i^{\alpha_1 \dots \alpha_s}$$

3. Each section on \widehat{F} induces, by derivation, a section on $J^r F$. The map, denoted J^r is linear and continuous (but neither injective nor surjective). So there is a linear continuous map : $\widehat{J}^r : H \rightarrow H_r$ and, with obvious notations, we have the following commuting diagram :

$$\begin{array}{ccc} \mathfrak{X}(F) & \xrightarrow{J^r} & \mathfrak{X}(J^r F) \\ \Upsilon \downarrow & & \downarrow \Upsilon_r \\ H & \xrightarrow{\widehat{J}^r} & H_r \end{array}$$

$$\widehat{J}^r \circ \Upsilon = \Upsilon_r \circ J^r \Leftrightarrow \widehat{J}^r = \Upsilon_r \circ J^r \circ \Upsilon^{-1}$$

4. If we consider the derivatives with respect to a specific variable α , their values $J_\alpha^r F$ are a vector subspace of $J^r F$ and the map : $\pi_\alpha^r : J^r F \rightarrow J_\alpha^r F$ is a projection. So, this is a primary observable, and there is an associated operator : $\widehat{\pi}_\alpha^r \in \mathcal{L}(H_r; H_r)$ such that :

$$\widehat{\pi}_\alpha^r = \Upsilon_r \circ \pi_\alpha^r \circ \Upsilon_r^{-1}$$

$$\Upsilon_r (\pi_\alpha^r (J^r X)) = \widehat{\pi}_\alpha^r (\Upsilon_r (J^r X))$$

$\widehat{\pi}_\alpha^r$ is linear and self-adjoint. So $\widehat{\pi}_\alpha^r \circ \widehat{J}^r = \Upsilon_r \circ \pi_\alpha^r \circ J^r \circ \Upsilon^{-1}$ is linear.

5. These considerations extend to any linear differential operator :

$D : \mathfrak{X}(J^r F) \rightarrow \mathfrak{X}(F)$ which gives rise to a self-adjoint operator

$\widehat{D} : H_r \rightarrow H :: \widehat{D} = \Upsilon \circ D \circ \Upsilon_r^{-1}$. In particular if there is a first order linear connection on F (induced by a principal connection on P) with covariant derivative :

$$\nabla Y = \sum_{\alpha=1}^{\dim M} \sum_{i,j=1}^p (\partial_\alpha y^i + \Gamma_{\alpha j}^i y^j) d\xi^\alpha \otimes \kappa_i(m), \text{ with } \kappa_i \text{ a basis of } T_1 G,$$

for any $\alpha \in \{1 \dots \dim M\}$ we can consider the associated operator :

$$\widehat{\nabla}_\alpha : H \rightarrow H :: \widehat{\nabla}_\alpha = \Upsilon \circ \nabla_\alpha \circ \Upsilon^{-1}$$

6. The principle of least action states that for any system there is some real function which is stationary. It is usually specified with a lagrangian, but it can be formulated more generally as a distribution Λ acting on the r jet prolongation of a section representing the system. To such a distribution is associated a vector $\phi_r \in H_r$ and the principle of least action reads : $\left\langle \phi_r, \widehat{J}^r \psi \right\rangle_{H_r} = 0$ which is a linear equation.

7. If the variables of the system are assumed to satisfy some partial differential equations, which is defined as a closed subbundle of $J^r F$, it sums up to take a restriction to a vector subspace of \widehat{E} , and to closed vector subspaces of H, H_r .

4.4.4 Gauge transformations

In this model we have two possible gauge transformations :

- a change of local frame : we go from $p_1 = \varphi(m, 1)$ to $z = \varphi(m, g(m))$ where $g(m)$ can varies with m
- a change of chart on M : M is a finite dimensional manifold, and can be described in any compatible atlas

Change of frame

1. The structure of associated vector bundle brings a natural gauge transformation : the observer of reference uses the frame : $p_1 = \varphi(m, 1)$ and any other observer the frame : $z = \varphi(m, g(m))$ defined by a section $Z \in \mathfrak{X}(P)$ of the principal bundle. So we have two measures of the variable Y and an action of $Z \in \mathfrak{X}(P)$ on $Y \in \mathfrak{X}(F)$ defined fiberwise.

There is an obvious structure of an infinite dimensional topological group on $\mathfrak{X}(P)$ by defining the operations fiberwise. The action reads :

$$U : \mathfrak{X}(P) \times \mathfrak{X}(F) \rightarrow \mathfrak{X}(F) :: U(Z)(X)(m) = (\varphi(m, g^{-1}(m)), r(g(m))x)$$

Clearly the measures on X and $U(Z)(X)$ are equivalent so U is unitary with respect to the scalar product on $\mathfrak{X}(F)$ and there is an associated unitary operator :

$$\widehat{U}(Z) : H \rightarrow H :: \widehat{U}(Z) = \Upsilon \circ U(Z) \circ \Upsilon^{-1} \text{ so } \widehat{U}(Z)(\Upsilon(X)) = \Upsilon(U(Z)(X))$$

It sums up to an action of maps $M \rightarrow G$ on maps $M \rightarrow V$

$$U : S_P \times S_V \rightarrow S_V :: U(Z)(x)(m) = r(g(m))x(m)$$

2. The infinitesimal generators of one parameter group of gauge transformations are given by sections of the associated vector bundle $P[T_1G, Ad]$. Fiberwise the action reads :

$$U(\theta)(X)(m) = (\varphi(m, \exp(-\theta\kappa(m))), r(\exp(\theta\kappa(m)))y) \text{ with } \kappa(m) \in T_1G.$$

The previous results apply and, if the action is continuous, for any section $K \in \mathfrak{X}(P[T_1G, Ad])$ there is a map $S(K) \in \mathcal{L}(\mathfrak{X}(F); \mathfrak{X}(F))$ such that :

$$\frac{dU(\theta)}{d\theta} = S(K) \circ U(\theta) \Leftrightarrow U(\theta) = \exp \theta S(K)$$

and a self-adjoint operator $\widehat{S}(K) \in \mathcal{L}(H; H)$ such that : $\frac{d\widehat{U}(\theta)}{d\theta} = \widehat{S}(K) \circ \widehat{U}(\theta)$

Fiberwise we have : $\frac{dy}{d\theta} = r'(1)\kappa(m)(y)$ with $r'(1)\kappa(m) \in L(V; V)$

A section K is the generator of a "Noether current", so to any such current is associated a self-adjoint operator $\widehat{S}(K) \in \mathcal{L}(H; H)$

3. If the physical phenomenon represented by the section X is symmetric, meaning that : $U(Z)(X) = X$ for some section Z generated by a one parameter group of gauge transformations then :

$$\forall \theta : \widehat{U}(Z(\theta))(\psi) = \psi$$

ψ is an eigen vector of the unitary operator $\widehat{U}(Z(\theta))$

Then it makes sense to introduce this specific section Z as an additional variable in the model, the state of the system is represented by a pair (ψ_Y, ψ_Z) and we have the relation : $\psi_Y \in \ker r'(1)\psi_Z$

Change of chart

The model is purely geometric : m and X do not depend on the chart used for M . As above we can assume that the area covered by the system lies in the same open of M , so we can consider one chart φ, φ' for each atlas, defined on the same space \mathbb{R}^n and the change of charts reads : $\varphi(m) = \xi, \varphi'(m) = \xi'$ with $\xi' = \varphi' \circ \varphi^{-1}(\xi) = u(\xi)$

The sections $\Xi(\xi), \Xi'(\xi') = \Xi(u(\xi))$ can be represented in the same space E of maps, and there is a unitary operator U on E such that $\Xi' = U(\Xi)$. It sums up, by using the same basis $(e_i)_{i \in I}$ of E , which are maps $e_i : M \rightarrow E$ to define Ξ' by coordinates $x'^i = U(x^i)$ and the change of chart is global : x'^i, x^i are constant scalar and U does not depend on m . Indeed a chart is a map defined over a large domain, and the chart itself does not depend on the point of the manifold. The associated operator \hat{U} on the Hilbert space H is unitary, and the change of charts sums up to a global isometry. Moreover the unitary operators U, \hat{U} cannot be easily deduced from u . So, usually, the necessary conditions entailed by a change of chart do not bring much.

5 THE EVOLUTION OF THE SYSTEM

Most physical models involve localized data. The coordinates are measured with respect to some frame, which can be changed according to a group of spatial transformations. The Wigner's theorem deals with this kind of issue. However, in physics spatial and time transformations can be related. So far we have considered the system "at a given time". To study the evolution of the system we need to introduce additional assumptions about the geometry of the universe, meaning the model which relates the spatial coordinates and the time coordinate to label the events which are observed. The three basic models are :

- the galilean model of classical physics : the time and the spatial coordinates are not related, and there is a unique universal time for all the observers
- the model of Special Relativity : the universe is modelled as an affine 4 dimensional space, endowed with a lorentzian metric.
- the model of General Relativity : the universe is a four dimensional manifold endowed with a lorentzian metric.

The model which is chosen defines the "gauge transformations" upon which one goes from one observer to the other, both in space and in time. But the definition of a physical system itself depends also on this choice.

Let us first consider the classical case.

5.1 Galilean geometry

5.1.1 The geometry

We are so used to the common geometry that some of its features seem obvious, but they are not.

In the Galilean picture the key features of the geometry of the universe are the separation of time and space, and the isotropy of space. So it can be modelled as the product of a 3 dimensional affine euclidean space S by \mathbb{R} . S is a manifold, with charts defined by frames comprised of an origin and 3 orthonormal vectors and the holonomic bases are the same at any point. Its tangent vector space, where live all physical quantities described as vectors or tensors, is thus a 3 dimensional euclidean vector space, localized at some point of S . The observer located at m uses an orthonormal basis B which is deduced from the holonomic basis by some rotation $R \in SO(3)$.

Physically the measures of length can be done by surveying. Formally, because S is an affine euclidean space, the isometries belong to the group of displacements, semi-product of the group of translations in \mathbb{R}^3 by the group $SO(3)$. If there is a vector V , expressed in the holonomic basis, located at p , then for an observer A_1 in m_1 with frame B_1 there is a displacement which goes from p to A_1 and brings the vector V in the frame of A_1 . The "position" of p with respect to A_1 is given by the translation $p-m_1$. And for two observers A_1, A_2 there is a displacement to go from the position of p and components of V as read by A_1, A_2 .

All this is a bit pedantic, but these precisions will be useful.

5.1.2 Schrödinger and Heisenberg pictures

Because there is a universal time, the follow up of the system can be conceived from two, equivalent, points of view.

1. In the Schrödinger picture there is only one set of configurations of the system, covering the whole evolution of the system. The time is considered as a parameter for the location of the measures, all variables depend on the time. Thus the time is not an observable (it does not have a precise value over the whole evolution).

A continuous variable X reads : $X = \sum_{i \in I} x^i e_i \in E$ where x^i are constant and E is a space of maps (depending on time).

A state, representing the whole evolution of the system, is a vector in the Hilbert space H .

Similarly observables are maps $Y_J : E \rightarrow E_J$ and their associated operators : $\hat{Y}_J : H \rightarrow H_J$

2. In the Heisenberg picture the system is considered at different times : there is a sequence of states indexed by the time.

A continuous variable X reads at t : $X(t) = \sum_{i \in I'} x^i e'_i \in E'$ where x^i are constant and E' is a space of maps (which do not depend on time but may depend on other parameters).

The state at the time t is a vector $\psi(t)$ in the Hilbert space H' , which is the same for all t as the model does not change.

Similarly observables are maps $Y'_j(t) : E' \rightarrow E'_j$ and their associated operators : $\widehat{Y}'_j(t) : H' \rightarrow H'_j$

The time is an observable (it can be measured), a real scalar t .

Both models are legitimate : they are based on different choice of the variables. But of course the measures are done at precise times. We have similar definitions for discrete variables, however, by nature, their evolution is not continuous.

3. If the system is not isolated and there is some action of the "outside" on the system, this action must be incorporated in the model, described by a variable which is a function subject to measurements.

4. One goes from the Schrödinger picture to the Heisenberg picture by an evaluation map :

$$\mathcal{E}(t) : E \rightarrow E' :: \mathcal{E}(t)(X) = X(t)$$

and similarly in the Hilbert spaces :

$$\widehat{\mathcal{E}}(t) : H \rightarrow H' :: \widehat{\mathcal{E}}(t)(\psi) = \psi(t)$$

These maps are linear and we assume that they are continuous morphisms.

The previous formalism can be simplified. We can consider the vectors of the basis $(e_i)_{i \in I}$ as functions of t , so that :

$$\mathcal{E}(t)(X) = \mathcal{E}(t) \left(\sum_{i \in I} x^i e_i \right) = \sum_{i \in I} x^i \mathcal{E}(t)(e_i) = \sum_{i \in I} x^i e_i(t)$$

The components x^i are constant and $E=E'$

Then $H=H'$ and for each value of t there is an associated basis $(\varepsilon_i(t))_{i \in I}$ of H and a vector representing a state has constant components in H .

5. Conversely one goes from the Heisenberg picture to the Schrödinger picture by estimating functions depending of time by a procedure, such as sampling with different frequencies, and usually it involves statistical methods. This point is seen below.

5.1.3 Time evolution

We will prove the following :

Proposition 20 *The evolution of a system is described by a map : $(\psi_d, \psi) : \mathbb{R} \rightarrow \widehat{H}$ and there are self-adjoint operator $\widehat{U}_d, \widehat{U}$ such that : $\psi_d(t) = \widehat{U}_d(t) \psi_d(0)$, $\psi(t) = \widehat{U}(t) \psi(0)$. Moreover for the continuous variables : $\widehat{U}(\theta) = \exp\left(\frac{1}{i\hbar} t H\right)$ where \hbar is some universal constant and H an anti-hermitian operator.*

We will proceed in two steps, using first the Schrödinger picture, then the Heisenberg picture.

The Schrödinger picture

Proof.

i) The galilean model is particular in that the "time dimension" is modelled as an affine space, isomorphic to \mathbb{R} . Thus observers are related by an affine

transformation : $t_2 = at_1 + \theta$ depending on the choice of the origin of time θ and the unit of time a . And because the observables must be dimensionless the only transformations are the translations.

ii) Let us define a fixed observer with time t , which takes measures on the system and its evolution. He measures some $X_0(\psi) \in E_0$ for a given state ψ of the system. Any other observer using a time $t' = t + \theta$ with a fixed $\theta \in \mathbb{R}$ measures $X_\theta(\psi)$ for the same system, and there is a unitary map : $U(\theta)$ such that : $X_\theta(\psi) = U(\theta) X_0(\psi)$. Which is interpreted as the first observer sees a state ψ_0 , the second observer a state ψ_θ and there is a unitary operator $\widehat{U}(\theta) \in \mathcal{L}(H; H)$ such that : $\psi_\theta = \widehat{U}(\theta) \psi_0$. Moreover :

$$\forall \theta, \theta' \in \mathbb{R} : \psi_{\theta+\theta'} = \widehat{U}(\theta) \left(\widehat{U}(\theta') \psi_0 \right) \text{ and } \widehat{U}(0) = Id$$

We have the same result for the discrete variables D .

iii) So we have a one parameter group on a Hilbert space. If, $\forall \psi \in H$ the map $\mathbb{R} \rightarrow \mathcal{L}(H; H) :: \widehat{U}(\theta) \psi$ is continuous, then \widehat{U} is differentiable with respect to θ and the group has an infinitesimal generator, that we will denote H to follow the custom (there is no risk of confusion with the Hilbert space H), such that : $i \frac{d\widehat{U}}{d\theta} |_{\theta=0}(\psi) = H\psi$ and $\widehat{U}(\theta) = \exp(-i\theta H)$.

$$\text{Which reads : } \psi_\theta = \exp(-i\theta H) \psi_0 \Rightarrow i \frac{d\psi}{d\theta} |_{\theta_0} = H\psi_{\theta_0}.$$

One can assume that the one parameter group is continuous for the continuous variable, however there is no general justification to keep this assumption for the discrete variables. Indeed the continuity in this case is equivalent to the stationarity (they are constant). Notice there is a unitary map at each θ , but this map is not continuously defined.

iv) The operators $\widehat{Y}_J(\theta) = \widehat{U}(\theta) \circ \widehat{Y}_J(0) \circ \widehat{U}(\theta)^*$ associated to the observables are differentiable with respect to θ :

$$\frac{d}{d\theta} \widehat{Y}_J(\theta) = i \left[\widehat{Y}_J(\theta), H \right]$$

The group of gauge transformations is abelian. The operators $\widehat{Y}_J(t + \theta), \widehat{Y}_J(\theta)$ commute. The operators H and \widehat{U} commute, by the properties of the exponential. ■

The Heisenberg picture

Proof.

i) We go from the Schrödinger picture to the Heisenberg picture by the evaluation map :

$$\mathcal{E}(t) : E \rightarrow E :: \mathcal{E}(t)(X) = X(t)$$

The action of the translations in time reads :

$$\mathcal{E}(t)(U(\theta)X) = X(t + \theta) = \mathcal{E}(t + \theta)(X) \Leftrightarrow \mathcal{E}(t) \circ U(\theta) = \mathcal{E}(t + \theta) = \mathcal{E}(\theta) \circ U(t)$$

It reads in components :

$$U(\theta)X = \sum_{i \in I} [U(\theta)]_j^i x^j e_i$$

$$\mathcal{E}(t + \theta)X = \sum_{i \in I} x^j e_i(t + \theta) = \sum_{i \in I} [U(t)]_j^i x^j e_i(\theta)$$

$$\text{With } \theta = 0 : \mathcal{E}(t)X = \sum_{i \in I} x^j e_i(t) = \sum_{i \in I} [U(t)]_j^i x^j e_i(0) = U(t) \mathcal{E}(0)X$$

that is : $\mathcal{E}(t) = U(t) \mathcal{E}(0)$

Similarly the evaluation map $\widehat{\mathcal{E}}(t) = \widehat{U}(t) \circ \widehat{\mathcal{E}}(0)$.

ii) The previous equations read :

$$\psi(t) = \exp(-itH) \psi(0)$$

$$i \frac{d\psi}{dt} \Big|_{t_0} = H\psi(t_0)$$

iii) As said previously the observables should be unitless, as it is obvious in the exponential : $\widehat{U}(t) = \exp(-itH)$. But if t is incorporated as a parameter, we need some constant to absorb the unit. This constant does not depend on the system but only on the units which are used. So we see the necessity of the universal constant \hbar in the previous expressions, and we have the usual relations of Quantum Mechanics:

$$\widehat{U}(t) = \exp\left(\frac{1}{i\hbar}Ht\right)$$

$$\frac{d\widehat{U}}{d\theta} \Big|_{\theta=t} = \frac{1}{i\hbar}H\widehat{U}(t) = \frac{1}{i\hbar}\widehat{U}(t)H$$

$$i\hbar \frac{d\psi}{dt} = H\psi(t)$$

$$\frac{d}{d\theta} \widehat{Y}_J(\theta) \Big|_{\theta=t} = -\frac{1}{i\hbar} [H, \widehat{Y}_J(0)] \quad \blacksquare$$

Stationary states

A state is stationary if it is symmetric with respect to the time t. Thus :

$$\psi(t) = \exp(-itH) \psi(0) = \psi(0) \Leftrightarrow \psi(0) \in \ker H$$

$$\text{An observable } Y_J \text{ is stationay if } \widehat{Y}_J(t) = \widehat{Y}_J(0) \Leftrightarrow [H, \widehat{Y}_J(0)] = 0$$

Probability and evolution of the system

One goes from the Heisenberg picture to the Schrödinger picture by estimating functions depending of time from samples of data at different times by a procedure which involves statistical methods. This procedure introduces uncertainty on the result of the measure, depending notably on the frequency of the sampling, in a way similar to any other measure.

Proposition 21 *In the Schrödinger picture, there is physical probability $\nu(J) = \left\| \widehat{Y}_J(\psi) \right\|^2$ for any state $\|\psi\| = 1$ that the trajectory of the states belong to the vector subspace H_J . And the same probability holds for the associated trajectories as measured by an observer.*

There is a probability $\|P(t)(\psi(t))\|^2$ that $\psi(t) \in P(s)(H)$ where P is the spectral resolution of the operator H .

Proof.

i) The uncertainty comes from the measure at any given time : any measure of $X(t) = \sum_{i \in I} x^i e_i(t)$ belongs to some vector subspace E_J , with J a finite subset of I . It comes also from the fact that the map $X : \mathbb{R} \rightarrow E$ is estimated from a finite sample of measures at different times. If we stay in the Schrödinger picture, both fall under the scope of the previous theorem : any measure of a state $\psi \in H$ of the system has the probability $\nu(J)$ of belonging to $\widehat{Y}_J(H)$. ν is a physical probability law on the measurable space $(I, 2^I)$ and if $\|\psi\| = 1$ then

$\|\widehat{Y}_J(\psi)\|^2 = \nu(J) = \Pr(\psi \in H_J)$. The probability is related to full trajectories, not points on the trajectory.

ii) (H, \widehat{U}) is a unitary representation of the abelian group $(\mathbb{R}, +)$, if the map $\widehat{U} : \mathbb{R} \rightarrow \mathcal{L}(H; H)$ is continuous there is a unique spectral measure P on the Borel σ -algebra $\sigma_{\mathbb{R}}$ of \mathbb{R} such that : $\widehat{U}(t) = \int_{\mathbb{R}} \exp(its) P(s)$ (JCD Th.1859,1851). Thus, using the proposition 17, the induced formal probability law : $\mu(s) = \|P(s)(\psi(t))\|^2$ can be interpreted as $\mu(s)$ is the probability that the physical measure of $\psi(t) \in H', \|\psi(t)\| = 1$ belongs to $P(s)(H)$.

iii) If \widehat{U} is continuous, it is smooth, and the function of $t : \langle \widehat{U}(t)\psi, \psi \rangle = \int_{\mathbb{R}} \exp(its) \langle P(s)\psi, \psi \rangle$ is differentiable with respect to t and : $\langle \frac{d}{dt} \widehat{U}(t) |_{t=t_0} \psi, \psi \rangle = \int_{\mathbb{R}} is \exp(it_0s) \langle P(s)\psi, \psi \rangle$. For $t_0 = 0$: it gives $\langle -iH\psi, \psi \rangle = i \int_{\mathbb{R}} s \langle P(s)\psi, \psi \rangle \Leftrightarrow H = \int_{\mathbb{R}} sP(s)$

So the spectral measure P is the spectral resolution of the operator H , $P(\mathbb{R}) = Id$ and $H = \int_{\mathbb{R}} tP(t)$, so H can be seen as the observable associated to the time t . ■

5.1.4 Spatial transformations

To study spatial transformations one can adopt either the Schrödinger point of view, then the spatial transformations are fixed over the whole evolution of the system, or the Heisenberg point of view, and letting the parameters vary with the time. The second is used to study the movement of a rigid body, and gives the classical relations of kinematic. The first is more interesting in our picture.

Framework

1. The choice of a chart for the manifold S is just the choice of an origin O and an orthonormal basis $(\epsilon_{\alpha})_{\alpha=1}^3$. It is arbitrary and will not play any role in the model. Let us consider a first observer 1, whose frame can be assimilated to $(O, (\epsilon_{\alpha})_{\alpha=1}^3)$ and a second observer 2 which is located at p and uses a basis deduced from $(\epsilon_{\alpha})_{\alpha=1}^3$ by a rotation $R \in SO(3)$.

A displacement is defined as a couple of a translation represented by a vector $\tau \in \mathbb{R}^3$ and a rotation $g \in SO(3)$, with

$$\text{product} : (g, \tau) \times (g', \tau') = (gg', g(\tau') + \tau)$$

$$\text{and inverse} (g, \tau)^{-1} = (g^{-1}, -g^{-1}(\tau))$$

2. The main kind of variables which can be considered are :

- the location of an event (such as the position of a singled out particle) : it is represented by a vector in \mathbb{R}^3 and the relation between the variables as observed by 1 and 2 is : $X_2 = R(X_1) + p$ with the displacement (R, p)

- a vector defined over the system : it is represented by a vector V in \mathbb{R}^3 and the relation between the variables as observed by 1 and 2 is : $V_2 = R(V_1)$ with the rotation R

- a localized vector, represented by a map : $Y : \mathbb{R}^3 \rightarrow F$ where F is some vector space, belonging to a vector space of maps E . The relation between the

variables as observed by 1 and 2 is : $Y_2(\xi_2) = Y_1(\xi_1)$ where $\xi_2 = R(\xi_1) + p$

As we are in the Schrödinger picture each variable is a function of the time t , thus we have an infinite dimensional model and we can implement the previous results. R and p are defined globally, so they are constant, but we can consider different families of observers, each one corresponding to an element of the Lie algebra.

We can distinguish vectors ψ_X, ψ_V, ψ_Y corresponding to each kind of variables, which take value in the vector subspaces H_X, H_V, H_Y image of H by the operators associated to the observables. Notice that, because all the variables are functions, all these subspaces are infinite dimensional.

We have the action of the group of displacements on the system, and there are unitary operators U and \widehat{U} , that we can break down in operators acting on the respective vector subspaces. So we can consider $(H_X, \widehat{U}_X), (H_V, \widehat{U}_V), (H_Y, \widehat{U}_Y)$ as distinct infinite dimensional representations of the group of displacements or the group of rotations.

We get a representation of the group of displacement by the product of a representation of the group of rotation, and a representation of the group of translations, following the rules above. Thus it is useful to consider separately the action of each group on the different kind of variables. We assume that the spatial transformations are continuous, then they are smooth.

Spatial rotations

1. A variable V is a map $V: \mathbb{R} \rightarrow \mathbb{R}^3$ which reads : $V = \sum_{i \in I} v^i e_i$ where v^i are constant and the e_i are a basis of some space of maps. For a given observer 2 the rotation R is constant, so $RV = \sum_{i,j \in I} [U(R)]_j^i v^j e_i$ and in the Hilbert space H we have : $\widehat{U}(R)\psi = \sum_{i,j \in I} [U(R)]_j^i v^j \varepsilon_i$.

2. $SO(3)$ is a compact Lie group, thus any continuous unitary representation is reducible in the direct sum of orthogonal finite dimensional irreducible unitary representations. So there is a family $(H_\alpha)_{\alpha \in A}$ of orthogonal, finite dimensional vector subspaces of H which are invariant by $\widehat{U}(R)$ and such that : $H = \oplus_\alpha H_\alpha$. Each family can be labelled by its character $Tr(\widehat{U}(R))$, these characters are those of the maximum taurus of $SO(3)$, which is given by the diagonal matrices with elements $\exp i\theta$, thus one can label the family by the components of the axis of rotation r of R : $H = \oplus_r H_r$ and on H_r the operator \widehat{U} is $\widehat{U}(\exp \kappa_r)$ where κ_r is the element of $so(3)$ with components κ_r . Because H is separable there can be at most countably many such subspace H_r and values of r .

3. Each vector subspace H_r corresponds to the states which are symmetric by a rotation of axis r . And if the system has such a symmetry, then H (at least its part H_V) is necessarily finite dimensional and V reads : $V = \sum_{i=1}^n v^i e_i$. Then there is a finite probability given by $\|\langle \psi, \varepsilon_i \rangle\|^2$ to observe one of the "modes" of rotations e_i .

Translations

1. A variable X is a map $X: \mathbb{R} \rightarrow \mathbb{R}^3$ which reads : $X = \sum_{i \in I} x^i e_i$ where

x^i are constant and the e_i are a basis of some space E of maps. For a given observer 2 the translation P is constant, so:

$$(PX)(t) = \sum_{i \in I} x^i e_i(t) + p = \sum_{i,j \in I} [U(p)]_j^i x^j e_i(t)$$

and in the Hilbert space H we have : $\widehat{U}(p) \psi = \sum_{i,j \in I} [U(p)]_j^i x^j \varepsilon_i$

2. Translations are not a compact group, but this is an abelian group. Its unitary representations are given by spectral integrals : $\widehat{U}(p) = \int_{\mathbb{R}^3} (\exp i \langle \lambda, p \rangle) P(\lambda)$ where $P(\lambda)$ is a spectral measure on \mathbb{R}^3 and $\langle \lambda, p \rangle$ the common scalar product.

Thus we have :

$$\psi(p) = \widehat{U}(p) \psi(0) \text{ with } \widehat{U}(p) = \int_{\mathbb{R}^3} (\exp i \langle \lambda, p \rangle) P(\lambda) \in \mathcal{L}(H; H)$$

and $P(\lambda) = \widehat{X}_{\chi(\lambda)}$ where $\chi(\lambda)$ belongs to 2^I .

$\|P(\lambda)(\psi)\|^2$ can be interpreted as the probability that the physical measure of $\psi \in H, \|\psi\| = 1$ belongs to $P(\lambda)(H)$. So there is a family of trajectories, each with some probability to be observed, and these trajectories are continuous or not according to the choice of E. If the trajectories are supposed to be smooth, the observed trajectories are smooth.

3. A one parameter group would be : $\widehat{U}(\theta p) = \int_{\mathbb{R}^3} (\exp i \langle \lambda, \theta p \rangle) P(\lambda)$ for some fixed value of p, and a symmetry would imply :

$$\forall \theta : (P(\theta) X)(t) = X(t) + P(\theta) = X(t) \text{ so there is no symmetry if } P \neq 0.$$

Local variable

1. A local variable Y is a map : $Y : S \times T \rightarrow F :: Y(\xi, t)$ valued in some vector space F. It reads : $Y = \sum_{i \in I} y^i e_i$ where y^i are constant and e_i is a basis of some vector space E of maps : $S \times \mathbb{R} \rightarrow F$.

A spatial transformation is $Y_2(\xi_2, t) = Y_1(\xi_1, t)$ where $\xi_2 = R(\xi_1) + P$ and R, P are constant. There is a unitary map U such that :

$$Y_2 = (UY_1) = \sum_{i,j \in I} [U]_j^i y^j e_i$$

and (H, \widehat{U}) is a unitary representation of the group of displacements:

$$\widehat{U}(R, P) \circ \widehat{U}(R, P) = \widehat{U}(RR', R(P') + P)$$

$$\widehat{U}(R, P)^{-1} = \widehat{U}(R^{-1}, -R^{-1}(P)) = \widehat{U}(R, P)^*$$

The group of rotations on one hand, the group of translations on the other hand, are both subgroup of the group of displacements and the restriction of \widehat{U} on each subgroup is a representation.

So there are invariant subspaces H_r by rotation such that : $(H_r, \widehat{U}(\exp \kappa_r, 0))$ is an irreducible representation of SO(3) and a spectral measure $P(\lambda)$ on \mathbb{R}^3 such that $\widehat{U}(1, p) = \int_{\mathbb{R}^3} (\exp i \langle \lambda, p \rangle) P(\lambda)$

$$\widehat{U}(R, P) = \widehat{U}(1, P) \circ \widehat{U}(R, 0) = \int_{\mathbb{R}^3} (\exp i \langle \lambda, p \rangle) P(\lambda) \circ \widehat{U}(R, 0)$$

2. The symmetries are deduced directly by differentiation of the equation $X([\exp \theta \kappa] \xi + \theta p, t) = X(\xi, t)$.

5.2 Relativist Geometry

Relativity introduces a dramatic change in the definition of a system and of physical models. Because time and space coordinates are linked, it is no longer

possible to describe the evolution of a system independantly of its spatial extension. Further more, as we have required that a system stays in a precisely defined, and bounded, area, the definition of the system itself cannot be independant from the observer.

5.2.1 The relativist model

Reminder of the principles of relativist geometry

It is more illuminating, and not much difficult, to deal with the geometry of General Relativity. It is based on four basic assumptions :

i) The universe (meaning the "container" of everything) is a four dimensional manifold

ii) There is a lorentzian metric represented by a symmetric bicovariant tensor g , that we will take with the signature $- + + +$. The existence of such metric splits the vector space tangent at each point according to the sign of $g(u,u)$ (with this signature the time like vectors have $g(u,u)<0$ and the space like vectors $g(u,u)>0$), and the subset of time like vectors into two disconnected components (future and past oriented vectors).

iii) All material bodies travel along a "world line", future oriented, parametrized by their proper time. The "proper time" τ of an observer M (as measured by a clock) is such that $g\left(\frac{dM}{d\tau}, \frac{dM}{d\tau}\right) = -c^2$ with the "speed of light" c . Similarly it is assumed that the field forces propagate, as the electromagnetic field, along trajectories such that $g(u,u)=0$ (the "light cone").

iv) There is a linear metric connection which transports orthonormal frames as orthonormal frames

Because space and time are linked, any physical measure of length is based on electromagnetic signals, which implies additional assumptions about the speed and trajectories of such signals. In Special Relativity the isometries are still affine maps, and so the study relies on the group of displacements in Minkovski space (the "Poincaré group"), in a way similar as the Galilean geometry. In General Relativity this is no longer the case and we must use a more elaborate framework.

Network of frames

The definition of a physical system goes in pair with that of a network of privileged observers. A system is still a delimited region of the universe (so here cosmology is excluded) but, because there is no longer a universal time, the evolution of the system (and indeed the definition of the physical system itself) must be related to a family of observers : at each time the spatial extension of the system must be defined. The method is to build a network of frames, in which the events can be consistently measured. This network is a local "gaussian chart" of the manifold.

The starting point is a connected space-like hypersurface (its normal are time like) $S(0)$. It represents the "present" of an observer at its proper time $t=0$. The choice of this hypersurface is arbitrary and crucial, because it defines completely

the system. The metric induced over $S(0)$ is riemannian, so it is possible to define by classical means (such as the "radar" coordinates of Einstein) a system of coordinates and 3 dimensional orthogonal frames.

Over each point x of $S(0)$ there is a unique unitary, time like, future oriented vector $n(x)$ normal to $S(0)$. This vector is "virtual" for an observer located at x , but it defines the tangent to its own world line and the 4th vector of its orthonormal basis. We assume that there is a linear connection on M , so that it defines geodesics : in a neighborhood of x there is a unique geodesic tangent to $n(x)$. The observer can stay on the geodesic by checking that there is no change in the inertial forces. He can similarly transport the frame from x along this world line. So we can define a family of geodesics $\gamma(x, t)$ tangent to $n(x)$ and a vector field n in the future of $S(0)$, which is the infinitesimal generator of diffeomorphisms which maps $S(0)$ to a hypersurface $S(t)$ for each $t \geq 0$. One can prove that the vector field n is orthonormal to each $S(t)$.

In any region of the universe with no singularity this construct is always possible (but certainly not at a cosmological scale because there is always some point where the geodesics cross or vanish). The system is then defined as the area enclosed in a region Ω generated by some open bounded domain of $S(0)$. Each slice of Ω intersecting $S(t)$ is considered by the observer as the system at the time t . We see that the choice of another hypersurface $S(0)$ defines another system. Geometrically Ω is a 4 dimensional manifold and a trivial fiber bundle with base \mathbb{R} with trivialization : $m = \Phi_n(x, t)$ where Φ_n is the flow of the vector field n and x a point of $S(0)$.

Spatial coordinates in $S(0)$ can be established by any conventional method and transported along the vector field n which are geodesics. Notice that events occurring on $S(t)$ cannot be reported live to the observer, but can be reported with a known delay. A bundle of orthonormal bases is built in each point, in a consistent manner, defining a principal bundle structure on M .

A particle which is located at some point x of $S(0)$ (it enters the system at $t=0$) follows its own world line $\mu(\tau)$. Because M is a fiber bundle, for each point $\mu(\tau)$ of its world line there is a unique time $t=\pi(\mu(\tau))$ (consistent with the time of the observer). The 4 velocity $u=\frac{d\mu}{d\tau}$ of the particle is a future oriented, time like vector, which is projected on the base \mathbb{R} as a positive scalar $\pi'(\mu(\tau)) u = \frac{dt}{d\tau} > 0$. So the map $t(\tau)$ is injective : at any time t the particle is in a unique hypersurface $S(t)$, and a particle which enters the system stays in the system (if Ω is "spatially" large enough).

This construct seems a bit abstract, but it is very similar to the one used in the Global Positioning System (see Ashby) which accounts for General Relativity.

The physical model

The geometry impacts the description of the system, and so the list and properties of variables associated to measures which compose the model, in two ways.

All the variables depend at least of the time, and because time is linked with the space coordinates, they must be some functions of the coordinates of a point m , expressed in some frame. So they are sections of vector bundles (or their jet extensions) with base the 4 dimensional manifold M modelling the geometry of the universe. The continuous variables can be considered as section of a common vector bundle E , belonging to some vector subspace of $\mathfrak{X}(E)$. As such they are subject to a change of chart of M . These vector bundles can be associated to the principal bundle P of orthonormal frames on M , with its metric, and then they are also subject to a change of gauge in P . Or they can be associated to other principal bundles (but still with the base M), and then they are subject to change of gauge in these principal bundles.

The general results seen previously apply. In particular :

- i) To any configuration is associated a state of the system, represented by a unique vector ψ of a Hilbert space H .
- ii) There is an inner product on the space $\mathfrak{X}(E)$ and isometries $\Upsilon : \mathfrak{X}(E) \rightarrow H$.
- iii) Any continuous primary observable Y_J , which would be usually some variable Ξ_k , is associated to an operator on H such that : $Y_J(\psi) = \sum_{j \in J} \langle \phi_j, \psi \rangle e_j$ which implies that the section Y_J is a linear function of ψ .
- iv) Whenever the measures done by two observers, using different frames p over M , are related by a unitary map : $X_2 = UX_1$ then the vectors representing the states are similarly related by $\psi_2 = \widehat{U}\psi_1$

5.2.2 Translation in time

We use the chart defined above. The definition of the system itself is linked to the definition of the network of observers. Any change of the hypersurface $S(0)$ or of the boundary of Ω would change the area covered by the system, and the physical objects which are contained within. When $S(0)$ has been chosen, the only freedoms of gauge which are left are the choice of the orthonormal basis at each point x of $S(0)$ and the translation of time.

We will prove the following :

Proposition 22 *The evolution of a system is described by a map : $\psi : \mathbb{R} \rightarrow H$ and there is a self-adjoint operator H , a universal constant \hbar such that : $\psi(t) = \widehat{U}(t)\psi(0)$ where $\widehat{U}(\theta) = \exp\left(\frac{1}{i\hbar}\theta H\right)$.*

So the results of the Galilean Geometry still hold in the General Relativity picture.

Proof.

Indeed the chart is defined by : $m = \Phi_n(x, t)$ with $\Phi_n(x, 0) = x \in S(0)$.

Let us define the chart : $m = \Phi_n(x, t - \theta)$ with $\Phi_n(x, \theta) = x \in S(0)$. It defines the same manifold. Moreover the flow of n is a one parameter group of diffeomorphism : $\Phi_n(x, t + t') = \Phi_n(\Phi_n(x, t), t')$, $\Phi_n(x, 0) = x$

This change of gauge has no impact on the holonomic bases which can be deduced from the chart (the 4th vector is still n). The variable X is expressed with regard to the coordinates (x,t) .

Let us define the map : $U(\theta) : \mathfrak{X}(E) \rightarrow \mathfrak{X}(E) :: U(\theta)(X)(x,t) = X(x,t - \theta)$

$U(\theta)(X) = X(\theta)$ is the variable for an observer which would used a translated time coordinate. And :

$$U(\theta + \theta') = U(\theta) \circ U(\theta'), U(0) = Id$$

$U(\theta)$ defines a one parameter group of transformations on the Banach vector space $\mathfrak{X}(E)$. It has an infinitesimal generator.

The Wigner's theorem can be applied : U must be unitary and the associated operator \hat{U} on H is also unitary. The previous results for the Galilean geometry still hold. The fundamental relations of Quantum Mechanics are valid in this picture. ■

The state of the system at a given time is associated to the configuration on the hypersurface $S(t)$. Of course this result is valid only for the kind of chart which has been defined, but it seems to be the only one that can be implemented for measurements.

5.2.3 Change of spatial frames

The study of gauge transformations will proceed along the lines given for general fiber bundles. But it is useful to give some additional precisions about the geometry of General Relativity.

Further on the geometric model of the universe

The usual model of General Relativity is a four dimensional manifold M endowed with a lorentzian metric g . Then the connection is taken as the only metric, torsionless connection, which is the Lévy-Civita connection.

However this model is neither the most general, nor always the most convenient (at least outside cosmology) which meets the required criteria :

- a) we need a four dimensional vector bundle F over M (which represents the local Minkovski space time)
- b) endowed with a lorentzian scalar product
- c) and a linear connection on F , which preserves the metric
- d) M must be "time orientable"

As any vector bundle can be considered as an associated vector bundle, one can consider first a principal bundle P over M , with a group G which preserves the lorentzian scalar product. Second a unitary representation (\mathbb{R}^4, r) of G , and third a principal connection on P .

Then F is the associated vector bundle $P[\mathbb{R}^4, r]$, it is endowed with an adequate scalar product, and the metric g on M is deduced as a by-product. Moreover any principal connection preserves the scalar product, and the condition to be torsionfree is optional. Besides, the condition d) is easily translated by requiring that G is restricted to its connected component of the identity. The Einstein equations can then be easily deduced the usual way by a lagrangian

method. The topological obstruction lies on the existence of the principal bundle, in the same terms as the existence of a pseudo-riemannian manifold (M,g) . The other constructions are always possible.

This model is "more geometric", as it emphasizes the role of the frames. It is also more convenient for any gauge theory, and for our purpose here.

In this picture the choice of the group G is more open. The most general group which preserves the Lorentz metric is the Spin group (either $\text{Spin}(3,1)$ or $\text{Spin}(1,3)$, which are isomorphic). Then the only choice for the representation is $(\mathbb{R}^4, \mathbf{Ad})$ where the map :

$\mathbf{Ad} : \text{Spin}((3,1) \times \text{Cl}(3,1)) \rightarrow \text{Cl}(3,1) :: \mathbf{Ad}_s u = s \cdot u \cdot s^{-1}$ is defined through the Clifford product \cdot and gives for $u \in \mathbb{R}^4 : \mathbf{Ad}_s u = h(s)u$ with $h(s)$ the element of $\text{SO}(3,1)$ associated to each s and $-s$.

This model offers additional interesting features (JCD th. 2010, 2101, 2103, 2385):

a) F is a spin bundle : at each point m the pair $(F(m), g(m))$ is endowed with the structure of a Clifford algebra $\text{Cl}(m)$ isomorphic to $\text{Cl}(3,1)$ and for any representation (V, r) of the Clifford algebra $\text{Cl}(3,1)$, there are a vector bundle $E = P[V, r]$ and an action $R(m)$ of $\text{Cl}(m)$ on $E(m)$ which makes of $(E(m), R(m))$ a representation of $\text{Cl}(3,1)$ equivalent to (V, r)

b) the principal connection on P induces a linear connection on E , and the connections on E and F are related (this is a Clifford connection)

c) the principal connection on P induces a Dirac-like differential operator $D : J^1 E \rightarrow E$

Rotations

1. In this picture rotations are represented in the principal bundle, and their action on any tensorial bundle built from F goes through the map \mathbf{Ad} (with essentially the same effects as usual). The application of the Wigner's theorem follows the same lines as indicated previously for vector bundles.

The observer of reference uses the frame : $p_1 = \varphi(m, 1)$ and any other observer the frame : $z = \varphi(m, g(m))$ defined by a section $Z \in \mathfrak{X}(P)$ of the principal bundle.

There is an action of the group $\mathfrak{X}(P)$ of sections over P on the space $\mathfrak{X}(E)$ of sections Y of any associated vector bundle $E = P[V, r]$ over P defined fiberwise:

$$U : \mathfrak{X}(P) \times \mathfrak{X}(E) \rightarrow \mathfrak{X}(E) :: U(Z)(Y)(m) = (\varphi(m, g^{-1}(m)), r(g(m))y)$$

so U is unitary with respect to the scalar product on $\mathfrak{X}(E)$ and there is an associated unitary operator :

$$\widehat{U}(Z) : H \rightarrow H :: \widehat{U}(Z) = \Upsilon \circ U(Z) \circ \Upsilon^{-1} \text{ so } \widehat{U}(Z)(\Upsilon(Y)) = \Upsilon(U(Z)(Y))$$

If we restrict Z to global gauge transformations, meaning that $g(m)$ does not depend on M , then (H, \widehat{U}) is a unitary representation of $\text{Spin}(3,1)$, isomorphic to $\text{SL}(C, 2)$.

2. The infinitesimal generators of one parameter group of gauge transformations are given by sections of the associated vector bundle $P[so(3,1), Ad]$. Fiberwise the action reads : $U(\theta)(Y)(m) = (\varphi(m, \exp(-\theta\kappa(m))), r(\exp(\theta\kappa(m)))y)$ with $\kappa(m) \in so(3,1)$.

The previous results apply and, if the action is continuous, for any section $K \in \mathfrak{X}(P[T_1G, Ad])$ there is a map $S(K) \in \mathcal{L}(\mathfrak{X}(E); \mathfrak{X}(E))$ such that : $\frac{dU(\theta)}{d\theta} = S(K) \circ U(\theta) \Leftrightarrow U(\theta) = \exp \theta S(K)$ and a self-adjoint operator $i\widehat{S}(K) \in \mathcal{L}(H; H)$ such that : $\frac{d\widehat{U}(\theta)}{d\theta} = i\widehat{S}(K) \circ \widehat{U}(\theta)$

Fiberwise we have : $\frac{dy}{d\theta} = r'(1) \kappa(m)(y)$ with $r'(1) \kappa(m) \in L(V; V)$

3. With the vector bundle $F = P[\mathbb{R}^4, \mathbf{Ad}]$ we have the rules for a change of orthonormal frames. In its standard representation, any matrix of $SO(3,1)$ takes the form (JCD p.474) :

$$[A] = \begin{bmatrix} \cosh \sqrt{w^t w} & w^t \frac{\sinh \sqrt{w^t w}}{\sqrt{w^t w}} \\ w \frac{\sinh \sqrt{w^t w}}{\sqrt{w^t w}} & I_3 + \frac{\cosh \sqrt{w^t w} - 1}{w^t w} w w^t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \text{ where } w \text{ is a 3 vector and}$$

R a 3x3 matrix of $SO(3)$

If $(\partial_\alpha(m))_{\alpha=0}^3$ is a holonomic basis of F at m , the vector labelled 0 of the basis of the second observer is oriented as its 4 velocity, which has for components in the basis ∂_α : $c u_0 = \frac{1}{\sqrt{1 - \|\frac{v}{c}\|^2}} \begin{bmatrix} c \\ v \end{bmatrix}$ where v is the spatial speed of the the second observer measured by the first observer.

The first column of $[A]$ is u_0 so : $\cosh \sqrt{w^t w} = \frac{1}{\sqrt{1 - \|w\|^2}}$; $w \frac{\sinh \sqrt{w^t w}}{\sqrt{w^t w}} = w \frac{1}{\sqrt{1 - \|w\|^2}}$ which leads to the classical formula with $w = \frac{v}{\|v\|} \arg \tanh \|\frac{v}{c}\|$:

$$[A] = \begin{bmatrix} I_3 + \left(\frac{1}{\sqrt{1 - \frac{\|v\|^2}{c^2}} - 1 \right) \frac{v v^t}{\|v\|^2} & \frac{\frac{v}{c}}{\sqrt{1 - \frac{\|v\|^2}{c^2}}} \\ \frac{\frac{v^t}{c}}{\sqrt{1 - \frac{\|v\|^2}{c^2}}} & \frac{1}{\sqrt{1 - \frac{\|v\|^2}{c^2}}} \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

Moreover $[A] = (\exp p) \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$ with the 4x4 matrices $p = \begin{bmatrix} 0 & w^t \\ w & 0 \end{bmatrix}$ and $[R] = \exp [j(r)]$ where r is a 3 vector, which are the components of the axis of spatial rotation represented by R . The components of w and r fully define an element of the Lie algebra $\mathfrak{so}(3,1)$.

Notice that the derivative with respect to the time, in the gaussian chart, of any section X of F can be identified with the covariant derivative $\nabla_n X$. As a consequence of our assumptions about the chart, we have : $\nabla_n X = \frac{dx}{dt}$. Thus for the vectors of the basis of the second observer we have the derivatives (expressed

in ∂_α): $\frac{du_0}{dt}, \left(\sum_{j=1}^3 \frac{d[R]_i^j}{dt} \partial_j \right)_{i=1}^3$

4. Now let us take as system an observer who travels along his world line in our gaussian chart. Its location at each time t (for the network) is : $m(t) = \Phi_n(x(t), t)$ and $x(t)$ is followed in some chart on $S(0)$ by 3 coordinates $(\xi^\alpha(t))_{\alpha=1}^3$. Moreover his spatial speed is represented by the vector with components : $\left(\frac{d\xi^\alpha}{dt}(t) \right)_{\alpha=1}^3 \equiv v(t)$

The two vectors $v(t), r(t)$ are measurable and so we can take as variables in the model two maps : $V, R : \mathbb{R} \rightarrow \mathbb{R}^3$. A state of the system (that is two maps

V, R) is identified with a vector ψ of a Hilbert space. Using the linear map Υ this vector can be considered as a couple $(\psi_v, \psi_r) \in H \times H$. So we can apply the Schrödinger equation, and we have the two relations :

$$i\hbar \frac{d\psi_v}{dt} = H\psi_v(t)$$

$$i\hbar \frac{d\psi_r}{dt} = J\psi_r(t)$$

with two anti self adjoint operators H, J .

If the maps V, R belong to the Hilbert space $L^2(\mathbb{R}, d\xi, \mathbb{R}^3)$ this space is isomorphic to H , and (ψ_v, ψ_r) can be seen as maps $\psi(\xi, t)$, meaning as scalar fields over M .

5. Other examples can be developed, using the connection over M and the Clifford algebra structure, but it would be out of the scope of this paper.

6 CORRESPONDANCE WITH QUANTUM MECHANICS

The previous definitions and theorems constitute the framework of a new formal system, which can be developed and extended to many areas of theoretical physics. It can be implemented in a broad range of topics, most probably for all the usual problems which are considered by Quantum Mechanics, so its results shall stand for these problems as well. This is not a matter of philosophical point of view : whenever the conditions listed above are met, and they are quite general and should be easily satisfied, the theorems follow by a mathematical demonstration. The picture that we have drawn shares many common features with Quantum Mechanics, but not all of them. The issue is not here to say if QM is right or wrong, but, from the discrepancies with proven results, see what are the additional assumptions that are implicit in the models used in QM. As all these models are related to the atomic and subatomic world, it can shed a light on what should be the proper adjustments to be done to the representation of classical physics to account for the phenomenon encountered at this scale.

6.1 The axioms of Quantum Mechanics

Quantum mechanics relies on a few general "axioms", which are presented more or less in the same words in any book on the subject :

a) The states of a physical system can be represented by "rays" in a complex Hilbert space H . Rays meaning that two vectors which differ by the product by a complex number of module 1 shall be considered as representing the same state.

b) To any physical measure Φ , called an observable, that can be done on the system, is associated a continuous, linear, self-adjoint operator φ on H .

c) Whenever the associated operator φ is compact, the result of any physical measure is one of the eigen-values λ of φ .

d) After the measure the system is in the state represented by the corresponding eigen vector ψ_λ

e) The probability that the measure is λ is equal to $|\langle \psi_\lambda, \psi \rangle|^2$ (with normalized eigen vectors)

f) If a system is in a state represented by a normalized vector ψ , and an experiment is done to test whether it is in one of the states $(\psi_n)_{n=1}^N$ which constitutes an orthonormal set of vectors, then the probability of finding the system in the state ψ_n is $|\langle \psi_n, \psi \rangle|^2$.

g) When two systems interact, the vectors representing the states belong to the tensorial product of the Hilbert spaces.

There are some complications to these axioms to account for the possibility that the operators could be continuous on a dense subspace of H only. They are not relevant here.

As I said in the introduction, I will not use the more elaborate formalisations of QM using C^* -algebras. They are essentially mathematical developments from the axioms listed above, without any additional hint at their physical significance. Moreover they are based upon a general assumption - that a physical system is defined by the set of its observables - that I do not see as pertinent, at least as far as we do not have a better understanding of the physical meaning of these observables.

6.2 Representation of states by vectors of a Hilbert space

The axioms a and g above are met in our picture, with some differences.

6.2.1 Rays

The demonstration on linear maps shows that in our picture the crucial element is the positive kernel K . If we define $\varepsilon_i = e^{i\theta} X^{-1}(e_i)$ we would still have a linear map, with $\psi = e^{i\theta} \Upsilon(x)$. So, the vector ψ associated to a configuration x can be seen as defined up to a complex number of module 1. However the initial chart X actually defines a bijective correspondance between potential states ψ and possible observations x , and, as the choice of a linear chart is arbitrary, there is no need to be uncumbered by rays. Similarly the Hilbert space H is always defined up to an isometry.

6.2.2 Super-selection rules

In Quantum Mechanics some states of a system cannot be achieved (through a preparation for instance) as a combination of other states, and thus "super-selection rules" are required to sort out these specific states. Here there is a simple explanation : because the set H_0 is not the whole of H it can happen that a linear combination of states is not inside H_0 . The remedy is to enlarge the model to account for other physical phenomena, if it appears that these states have a physical meaning.

6.2.3 Tensorial product of Hilbert spaces

We have seen that it is possible to replace two separate models of interacting systems by a single model using the tensorial product of the variables, and thus of the Hilbert spaces. It is clear that this solution is efficient, but it is not mandatory. In our picture the physicist has always the choice of the model. However any "reasonable" model should be based upon the tensor product.

6.2.4 Local vs global theory

There is a fierce debate about the issue of locality in physics, mainly related to the "entanglement" of states for interacting particles. It should be clear that the formal system that we have built is global : more so, it is its main asset. While most of the physical theories are local, with the tools which have been presented we can deal with variables which are global, and get some strong results without many assumptions regarding the local laws. This is certainly also the case for Quantum Mechanics and one of the reasons for its success. However the classic interpretation of QM knows only local variables, and thus the need to resort to product of systems, and objects as complicated as the Hopf spaces. If any model involving a great number of interacting particles would probably always be a difficult subject, one can hope that the formalism presented here could help. However the key issue stays to find the right model to account for the dual behaviour of particles.

6.2.5 Mixed states

In QM there is a distinction between "pure states", which correspond to actual measures, and "mixed states" which are linear combination of pure states, usually not actually observed. There has been a great effort to give a physical meaning to these mixed states.

In our picture such "mixed states" can appear :

- for discrete observables, which are measured using "reduced observables"
- : they give a point which is outside of the affine spaces \widehat{H}_x
- for tensor products : the only tensors which are actually measured are given by a couple of vectors, and so are separable tensors

However the distinction mixed states / pure states does not play any role. Indeed the set of states is usually not all the affine space \widehat{H}_0 . The choice of variables is up to the physicist, they define the model and any result shall be seen in this picture. If the physicist drops part of his model, or introduces tensors, the model is changed, this does not impact the system itself, only the data which can be obtained or their assignation to a mathematical construct.

6.2.6 Wigner's theorem

In the genuine Wigner's theorem of QM (see Weinberg for a demonstration) the map U can be linear and unitary, or antilinear and antiunitary. But if U depends continuously on a parameter then it must be unitary. In our picture we do not

have the choice. Anyway, by far, the most usual case is the former. Moreover, because the space of states is a projective space in Quantum Mechanics, U is defined up to a scalar of module 1.

6.3 Operators and observables

The comparison with QM is not easy, as there is no clear definition of the observables in QM. Strictly speaking they are scalars quantities (or components of vectors) which represent measures which can be done on the system. More commonly it is assumed that an observable is associated to any self-adjoint operator.

The scope of observables in our picture is much larger, and more precise. We have seen that there is a self-adjoint operator associated to any physical measure (a primary observable). It is continuous and compact. And it is possible to define secondary observables, which are not necessarily self-adjoint, but constitute a C^* -algebra.

So the axioms b,c and e are met.

6.3.1 Operators with continuous spectrum

In our picture an observable shall be physically measurable, so by definition it is compact and it has a spectrum comprised of eigen values (except possibly 0) which are isolated points.

In QM an observable can be non compact, but it still has a spectral resolution: $\varphi = \int_{Sp(\varphi)} sP$ and its spectrum $Sp(\varphi)$ is a compact of \mathbb{R} bounded by $\|\varphi\|$. The eigen vectors are identified with the eigen spaces of $P(s)$ for the eigen value 1, meaning the subspaces $P(s)(H)$. The physical interpretation is that the corresponding variable is a random variable (such as the position of a particle).

In our picture a secondary observable such as $\varphi = \int_{Sp(\varphi)} s\hat{Y}_{\chi(s)}$ with an infinite spectrum has still a physical meaning. The set of indices I has the cardinality of \mathbb{N} thus 2^I has the cardinality of \mathbb{R} , so the definition of the map: $\chi : Sp(\varphi) \rightarrow 2^I$ has still a determinist physical meaning (it can be injective). Actually these cases correspond to variables which are functions, that QM cannot handle, but enter fully in our picture. An example is $H = \int_{\mathbb{R}} sP(s)$. Another is the position of a particle: the probability is not related to the position of the particle at a given time but to its trajectory as a whole. The discrepancy between the value of the variable and its estimate, due to the finite number of measures, is at the heart of the introduction of probability in our picture, whereas Quantum Mechanics puts the probability in the physical reality itself.

6.3.2 Axiom d (state after a measure)

This axioms is usually understood as "when any measure is done afterwards, it gives the result that the system in the state ψ_n ".

In our picture the process of measure has no impact on the state of the system, which is fully determined before and after the measure, and stays the same.

Whenever a primary observable is measured, one always get the same result in the same process.

The product of secondary observables has no clear meaning in our picture. However the product of a secondary and a primary observable gives the same result.

6.3.3 Compatible and commuting observables

In our picture the primary observables and the secondary compact observables cover all the range of measures that can be done simultaneously, in "one batch". and the associated operators commute.

In Quantum Mechanics it is postulated that the observables whose operators do not commute cannot be measured simultaneously.

The difference stems from that, in Quantum Mechanics, the list of possible observables is not given explicitly (and indeed, in the C*-algebra formalism, this is the system itself which is supposed to be defined through the operators), so the only way to define simultaneous measures is through the composition of operators, supposing to represent successive physical measurements. In our picture the variables and their properties are the model, they are listed explicitly and the question of simultaneous measures can be dealt with directly, and the associated operators commute. But the product of observables itself has not a clear meaning (except possibly the assignation of values to some variables from the result of a previous set of measures) and this operation is not necessary to address all the cases.

Any variable can be added to a model, but it is always assumed that the value of these variables can be measured (or estimated through an assignation process). It would be useless to add a variable whose value is supposed to be computed from the value of the others. Thus, if it is added, it means that the purpose is to check the relation from the output of the measures, then this is a variable as the others, simultaneously measurable, and the associated operator commute.

For instance the position and the momentum of an object are both observables and can be measured simultaneously at the macroscopic level and their operators commute in our picture. In Quantum Physics the equivalent variables for a particle are defined either by the translation operator or a Fourier transform, and it is common to say that the position and momentum cannot be simultaneously measured because their operators do not commute. Actually the key issue is that the macroscopic model does not hold any more at some scale, and thus it does not make sense to use the same variables to represent a physical reality which needs other tools. Indeed it is simpler, and has more physical sense, to tell that the position and momentum of a particle which can behave like a wave cannot be defined as it is done for a macroscopic object, than to invoke some axiomatic rule to justify this fact.

6.3.4 Probability

The last axiom left is f : the issue of the "transition probability" from one state to another.

It is sometimes treated as a consequence of the other, by considering the operator : $\varphi(\psi) = \sum_{n=1}^N \langle \psi_n, \psi \rangle \psi_n$. It is compact, self-adjoint, but it has only the eigen values 0 and 1, and the axioms cannot tell why one of the vector ψ_n belonging to the same eigen space, should be preferred. If these states were related to discrete observables, similarly there is no obvious reasoning to sustain this axiom.

Actually there is an interpretation in our picture, at least for continuous variables. The only tests that can be made are if ψ belongs to some vector subspace H_J generated by the finite family $(\varepsilon_j)_{j \in J}$. So the axiom can be reformulated as follows :

Proposition 23 *If the system is in a state $\psi \in H$, $\|\psi\| = 1$ and a first measure of the primary observable has shown that $\psi \in H_J$ then the probability that it belongs to $H_{J'}$ for any subset $J' \subset J$ is $\|\widehat{Y}_{J'}(\psi)\|^2$*

Proof. The probability that $\psi \in H_K$ for any subset $K \subset I$ is $\|\widehat{Y}_K(\psi)\|^2$. The probability that $\psi \in H_{J'}$ knowing that $\psi \in H_J$ is :

$$\Pr(\psi \in H_{J'} | \psi \in H_J) = \frac{\Pr(\psi \in H_{J'} \wedge \psi \in H_J)}{\Pr(\psi \in H_{J'} | \psi \in H_J)} = \frac{\Pr(\psi \in H_{J'})}{\Pr(\psi \in H_{J'} | \psi \in H_J)} = \frac{\|\widehat{Y}_{J'}(\psi)\|^2}{\|\widehat{Y}_J(\psi)\|^2} = \|\widehat{Y}_{J'}(\psi)\|^2 \text{ because } \widehat{Y}_{J'}(\psi) = \psi \text{ and } \|\psi\| = 1 \quad \blacksquare$$

6.4 The scale issue

The fact that Quantum Mechanics does not apply at a macroscopic level is one of its major issues. What can we say about the results which are presented in this paper ?

First it is necessary to remind that these results address physical models, not systems. No assumption has been made about the nature of the system, so they should hold whatever the scale, as far as the model meets the precise conditions specified in the first section.

The critical condition is about the number of "degrees of freedom", which must be infinite. And here this not the complexity of the system, or the number of its components which matters, but the kind of variables that the physicist choose to represent the system. If the model does not involves any map, we go back to the traditional framework of Statistical Mechanics, which offers many tools which are similar. Thus models, dealing with macroscopic phenomena and represented with variables in infinite dimensional vector spaces, for instance problems involving field, would fall in the scope of this paper. However in the

most usual cases the Hilbert spaces appear logically in the course of the methods of functional analysis which are used.

Meanwhile the "Quantum scale" requires this kind of model, because of the duality particules / fields (and anyway the problems considered usually involve force fields). It is clear that the success of QM stems from its efficiency to deal with such issues.

Our theorems about probability holds at the macroscopic scale. But they acquire a real physical significance when one faces the problem of estimating a function from a sample of measures.

At the other end of the scale the present formalism does not stand for models in cosmology. Indeed this topic should require an entirely new set of concepts.

7 CONCLUSION

In this paper I have set up the framework for a new formal calculation, which is general, quite simple to use, and is open to many developments, such as for vector bundles and their jet prolongations. Moreover, because each theorem is proven under clear conditions, it is easy and safe to implement.

It validates the general axioms of Quantum Mechanics, and in many way it is an answer to many questions which have been debated. The framework is more general as in QM, but the methods are close, and so should not be a dramatic change for the workers in the field who would want to use it.

As said in the introduction, no assumption has been done in regard to the determinism, the nature of the world, or other philosophical issues. So they stay as they should : a matter of personal belief.

The Quantum World raises many problems, which are far from solved. I think that the main topic is still the pertinent model to account for the duality particules / fields. It has been muddled for too long by the enigma of the Quantum Mechanics. As we can see axioms cannot replace a clear representation of the subatomic world.

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