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Sequential linearization method for viscous/elastic heterogeneous materials

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Abstract

The paper addresses the problem of suitable approximation of the interaction between phases in heterogeneous materials that exhibit both viscous and elastic properties. A novel approach is proposed in which linearized subproblems for an inhomogeneity-matrix system with viscous or elastic interaction rules are solved sequentially within one incremental step. It is demonstrated that in the case of a self-consistent averaging scheme, an additional accommodation subproblem, besides purely viscous and elastic subproblems, is to be solved in order to estimate the material response satisfactorily. By examples of an isotropic two-phase material it is shown that the proposed approach provides acceptable predictions in comparison with the existing models.

Keywords: micromechanics, viscoelasticity, viscoplasticity, homogenization, self-consistent scheme

1. Introduction

Micromechanical modelling of the behaviour of heterogeneous materials under quasi-static loading employs the scale transition between the micro-scale behaviour of the material constituents and the macroscopic scale on which the averaged properties of the heterogeneous material are represented by a homogenized material model. Typical scale-transition schemes in the mean-field micromechanical models, like self-consistent models, are based on the solution to the auxiliary problem where a single inclusion is embedded in a homogeneous matrix of some different material (Hill, 1965b). Usually, advantage is taken of the Eshelby (1957) solution which provides analytic formulae for uniform strains and stresses within an ellipsoidal inclusion embedded in an infinite homogeneous matrix. In this way, the interactions between inclusions or grains in the aggregate are taken into account not directly but in an approximate overall manner dependent on how the matrix properties are defined. The crucial point is that in order to apply the Eshelby formulae, the matrix must obey a linear constitutive law with a single constant tensor of stiffness moduli or compliances.

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This is the case in particular if the matrix is assumed to be linear elastic (possibly anisotropic). Various averaging schemes which use the concept of an ellipsoidal inhomogeneity in a linearly elastic matrix are available, including extensions to anisotropy, composite or coated inclusions, differential and incremental schemes, etc., cf. (Kröner, 1958; Hashin, 1964; Kneer, 1965; Mori and Tanaka, 1973; McLaughlin, 1977; Christensen and Lo, 1979; Willis, 1981; Walpole, 1981; Weng, 1981; Suquet, 1985; Herve and Zaoui, 1990; Cherkaoui et al., 1994; Nemat-Nasser and Hori, 1999; Broohm et al., 2000). If the actual material behaviour is nonlinear then the usual approach is to introduce some kind of linearization, e.g., by using tangent or secant stiffness moduli for the matrix representing an elastoplastic aggregate, with applications and extensions to elastoplastic polycrystals and composites (Hill, 1965a; Hutchinson, 1970; Berveiller and Zaoui, 1979; Tandon and Weng, 1988; Lipinski et al., 1990). Other types of linearization are used to model rigid-viscoplastic materials (Hutchinson, 1976; Molinari et al., 1987; Cailletaud, 1992; Lebensohn and Tome, 1993; de Botton and Ponte Castañeda, 1995; Ponte Castañeda, 1996; Kiryk and Petryk, 1998; Masson et al., 2000; Bornert et al., 2001).

The linearization becomes more difficult if both elastic and viscous properties of the matrix are to be simultaneously taken into account. For linear viscoelasticity the problem can be solved in an elegant way, at least formally, by using the Stieltjes convolutions and Laplace-type transform (Hashin, 1969; Christensen, 1969; Laws and McLaughlin, 1978; Suquet, 1985), although the inverse transform may be difficult and numerically time-consuming in a general anisotropic case. Extensions of that formalism to cases when elasticity is accompanied by non-linear viscous properties, as in elastic-viscoplastic materials, lead to rather complex formulations (Rougier et al., 1994; Masson and Zaoui, 1999; Pierard and Doghri, 2006). As all mean-field approaches to the mechanics of composites and polycrystals are approximate, even for strictly linear problems, it is reasonable to look for simpler linearization schemes.

Therefore, approximate linearization schemes for elasto-viscoplasticity, cf. (Kouddane et al., 1993; Paquin et al., 1999; Sabar et al., 2002; Molinari, 2002; Lahellec and Suquet, 2007; Mercier and Molinari, 2009; Doghri et al., 2010), are of interest. The common task is to approximate the interaction between the matrix material and inhomogeneities without applying the Laplace (or Carson) transform, inversion of which can be cumbersome in a general case. Differences between some of the existing approaches of that type will be illustrated in next chapters. Our aim is to develop a related but conceptually different approach.

The main idea of this paper, not found in the literature, is to abandon the attempts to incorporate both elastic and viscous properties of the matrix in a *single* computational step of an incremental scale-transition scheme. Rather, we propose to use elastic and viscous properties *sequentially* within one incremental step. Accordingly, not a single linear matrix-inhomogeneity problem but two or more different subproblems of that type, with different matrix-inhomogeneity interaction rules, are to be solved separately. The results are combined together to simulate the actual, simultaneously viscous and elastic response of the system to the applied external strain history. It is not evident that such an approach can work well even for linearly viscoelastic isotropic materials, therefore this question is examined in the present paper in more detail. Step-by-step sequential linearization of a strongly nonlinear viscous/elastic (elasto-viscoplastic) material behaviour

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will be addressed in a separate paper, and only an illustrative example of moderate non-linearity is presented below. The attention is restricted to the geometrically linear (small strain) theory. The predictions of the sequential linearization approach proposed here are compared with other approximations as well as with the exact results obtained by using the Laplace transform technique and the correspondence principle.

2. Linear viscoelastic inhomogeneity problem

2.1. Formulation of the sequential approach

We begin with the formulation of the sequential approach to the classical problem of a linear viscoelastic inhomogeneity[1] in a linear viscoelastic infinite matrix, both taken to be of the Maxwell type. Accordingly, a local strain-rate in phase r is decomposed into elastic and viscous parts as follows

$$\dot{\boldsymbol{\varepsilon}}_r = \dot{\boldsymbol{\varepsilon}}_r^e + \dot{\boldsymbol{\varepsilon}}_r^v \,, \qquad \dot{\boldsymbol{\varepsilon}}_r^e = \mathbf{M}_r^e \cdot \dot{\boldsymbol{\sigma}}_r \,, \qquad \dot{\boldsymbol{\varepsilon}}_r^v = \mathbf{M}_r^v \cdot \boldsymbol{\sigma}_r \,, \tag{1}$$

where $\boldsymbol{\sigma}_r$ is a local stress, $\mathbf{M}_r^e$ and $\mathbf{M}_r^v$ are fixed, diagonally symmetric, positive definite tensors of elastic and viscous compliances, respectively, and a superimposed dot denotes time derivative. The respective stiffness moduli tensors $\mathbf{L}_r^e$ and $\mathbf{L}_r^v$ are defined by $\mathbf{L}_r^e = (\mathbf{M}_r^e)^{-1}$ and $\mathbf{L}_r^v = (\mathbf{M}_r^v)^{-1}$, with the inverses taken with due account for the minor symmetries, i.e. $\mathbf{L}_r^e\mathbf{M}_r^e = \mathbf{I} = \mathbf{L}_r^v\mathbf{M}_r^v$, where $\mathbf{I}$ is the symmetrized fourth-order identity tensor, cf. Appendix.

Let $r = 0$ stands for the (infinite) matrix and $r = i$ for the (ellipsoidal) inhomogeneity. The external strain imposed on the matrix at infinity is denoted by $\boldsymbol{\varepsilon}_0$ and is assumed to vary in a prescribed manner with respect to time t, $\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}_0(t)$, while the associated stress in the matrix at infinity is denoted by $\boldsymbol{\sigma}_0$. For the linear *purely elastic* problem ($\mathbf{M}_r^v \equiv \mathbf{0}$), we have the following equation for the differences between strain-rates and stress-rates in the inhomogeneity and in the matrix at infinity (see Appendix),

$$\dot{\boldsymbol{\varepsilon}}_i^e - \dot{\boldsymbol{\varepsilon}}_0 = -\mathbf{M}_*^e \cdot (\dot{\boldsymbol{\sigma}}_i - \dot{\boldsymbol{\sigma}}_0) \qquad \text{if } \mathbf{M}_r^v \equiv \mathbf{0} \,, \tag{2}$$

where $\mathbf{M}_*^e$ is the inverse of the Hill tensor (95) associated with the matrix moduli $\mathbf{L}_0^e$. The classical solution to (2) is expressed in the form

$$\dot{\boldsymbol{\varepsilon}}_i^e(t) = \mathbf{A}_i^e \cdot \dot{\boldsymbol{\varepsilon}}_0(t) \,, \qquad \mathbf{A}_i^e = (\mathbf{I} + \mathbf{P}^e(\mathbf{L}_i^e - \mathbf{L}_0^e))^{-1} \qquad \text{if } \mathbf{M}_r^v \equiv \mathbf{0} \,, \tag{3}$$

where $\mathbf{P}^e$ is the stress polarisation tensor for the linear elastic problem, defined by Eq. (96) for the elastic matrix moduli tensor $\mathbf{L} = \mathbf{L}_0^e$ and for the shape of the ellipsoidal inhomogeneity expressed by $\mathbf{a} = \mathbf{a}_i$.

On the other hand, for the linear *purely viscous* problem ($\mathbf{M}_r^e \equiv \mathbf{0}$) the counterpart to Eq. (2) is

$$\dot{\boldsymbol{\varepsilon}}_i^v - \dot{\boldsymbol{\varepsilon}}_0 = -\mathbf{M}_*^v \cdot (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_0) \qquad \text{if } \mathbf{M}_r^e \equiv \mathbf{0} \,, \tag{4}$$

[1]Following Eshelby (1957), we employ the term "inhomogeneity" in contrast to the term "inclusion" used to indicate an incompatible eigenstrain without perturbation of compliances.

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with the solution

$$\dot{\boldsymbol{\varepsilon}}_i^v(t) = \mathbf{A}_i^v \cdot \boldsymbol{\varepsilon}_0(t)\,, \qquad \mathbf{A}_i^v = (\mathbf{I} + \mathbf{P}^v(\mathbf{L}_i^v - \mathbf{L}_0^v))^{-1} \qquad \text{if } \mathbf{M}_r^e \equiv \mathbf{0}\,. \tag{5}$$

Here, $\mathbf{P}^v$ is the stress polarisation tensor for the linear viscous problem, defined by Eq. (96) on substituting $\mathbf{L} = \mathbf{L}_0^v$ and $\mathbf{a} = \mathbf{a}_i$.

The question we address now is how to determine $\dot{\boldsymbol{\varepsilon}}_i$, at least approximately, without employing the Laplace transform technique, if none of $\mathbf{M}_r^e$ and $\mathbf{M}_r^v$ vanishes. The key concept here is to consider *sequentially* two linear subproblems within *one* infinitesimal time step:

Subproblem V in which the interaction between the inhomogeneity and matrix is assumed as purely viscous,

Subproblem E in which the interaction between the inhomogeneity and matrix is purely elastic.

The inhomogeneity-matrix interaction is defined by the Hill tensor, cf. equation (94). Accordingly, the inhomogeneity-matrix interaction equations for Subproblems V and E, in analogy to Eqs. (4) and (2), are defined by

$$\dot{\boldsymbol{\varepsilon}}_i^{\mathrm{V}} - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}} = -\mathbf{M}_*^v \cdot (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_0^{\mathrm{V}}) \tag{6}$$

and

$$\dot{\boldsymbol{\varepsilon}}_i^{\mathrm{E}} - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{E}} = -\mathbf{M}_*^e \cdot (\dot{\boldsymbol{\sigma}}_i^{\mathrm{E}} - \dot{\boldsymbol{\sigma}}_0^{\mathrm{E}})\,, \tag{7}$$

respectively. In the most general case, in the above equations only the interaction tensors $\mathbf{M}_*^v$, $\mathbf{M}_*^e$ and local stress $\boldsymbol{\sigma}_i$ is treated as known, and $\boldsymbol{\sigma}_0^{\mathrm{V}}$ and all the rate-variables with a superscript V or E are unknowns.

The basic idea is illustrated in Fig. 1 in one-dimensional case. An infinitesimal increment in strain, $\mathrm{d}\boldsymbol{\varepsilon}_r = \dot{\boldsymbol{\varepsilon}}_r \mathrm{d}t$, in each phase r is split conceptually into two parts corresponding to Subproblems V and E, respectively (Fig. 1a). Subproblem V is thought of as corresponding to a fixed (or sufficiently slowly varying) external stress so that elastic interactions between the inhomogeneity and matrix may be disregarded. In turn, Subproblem E corresponds to a sudden change in the stresses, e.g., at the end of an infinitesimal time interval $(t, t + \mathrm{d}t)$ in Subproblem V (Fig. 1b), so that viscous interactions between the inhomogeneity and matrix may be disregarded. To formulate elastic interactions in the rate form, all instantaneous changes in Subproblem E are referred to the time interval $(t, t + \mathrm{d}t)$ and interpreted as purely elastic rates.

The above decomposition is not complete. The second time derivative (acceleration) of a viscous strain in Subproblem V is nonzero in general, $\ddot{\boldsymbol{\varepsilon}}_r^v \neq \mathbf{0}$. The stress-rate related constitutively to $\ddot{\boldsymbol{\varepsilon}}_r^v$ by Eq. $(1)_3$ reads $\mathbf{L}_r^v \cdot \ddot{\boldsymbol{\varepsilon}}_r^v = \dot{\boldsymbol{\sigma}}_r^{\mathrm{A}}$, say. To accommodate such a stress-rate, an extra elastic strain-rate $\dot{\boldsymbol{\varepsilon}}_r^{\mathrm{A}} = \mathbf{M}_r^{\mathrm{A}} \cdot \dot{\boldsymbol{\sigma}}_r^{\mathrm{A}}$ is required by Eq. $(1)_2$. It is called the elastic *accommodation* strain-rate, to distinguish it from a purely elastic strain-rate in Subproblem E. The case when accommodation rates $\dot{\boldsymbol{\sigma}}_r^{\mathrm{A}}$ and $\dot{\boldsymbol{\varepsilon}}_r^{\mathrm{A}}$ are nonzero is roughly indicated by dashed lines in Fig. 1; a more detailed explanation will be provided in Section 3. The problem is that the local accommodation rates are not known.

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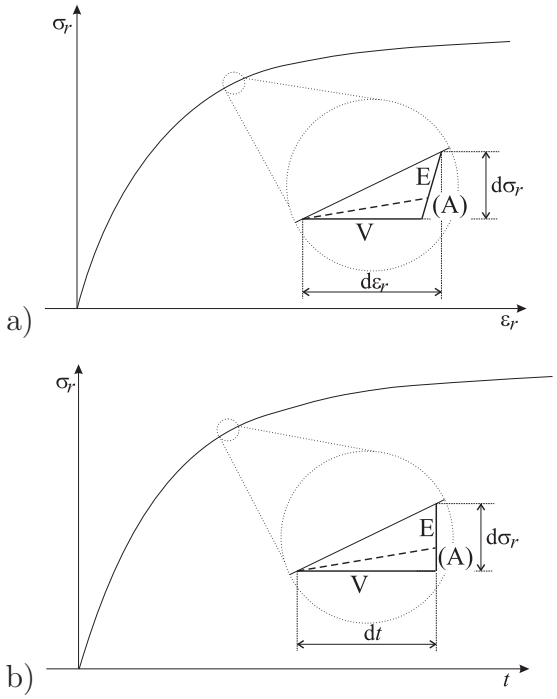

Figure 1: Illustration of the concept of sequential linearization.

There is a number of ways to complement equations (6) and (7) by further assumptions which lead to different variants of approximate models based on the sequential linearization. We impose the requirement that the solution should exhibit certain desired properties indicated at the end of this subsection, in particular, an exact solution should be recovered at least in the most classical, isotropic and incompressible case. It has been found that this requirement can be satisfied by disregarding the local elastic accommodation rates *in the inhomogeneity*, so that

$$\dot{\boldsymbol{\varepsilon}}_i^{\mathrm{V}} = \dot{\boldsymbol{\varepsilon}}_i^{v}\,, \qquad \dot{\boldsymbol{\varepsilon}}_i^{\mathrm{E}} = \dot{\boldsymbol{\varepsilon}}_i^{e}\,, \qquad \dot{\boldsymbol{\sigma}}_i^{\mathrm{E}} = \dot{\boldsymbol{\sigma}}_i\,. \tag{8}$$

It means that not only the interaction rules, but also the *local* strain-rates are separated as purely viscous and purely elastic in Subproblems V and E, respectively. Clearly, this is an approximation in general, to some extent compensated by the freedom in decomposing $\dot{\boldsymbol{\varepsilon}}_0$ into $\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}}$ and $\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{E}}$. Under the assumption (8), $\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}}$ has the meaning of an external strain-rate (at infinity) intended to be compatible in Subproblem V with the viscous strain-rate $\dot{\boldsymbol{\varepsilon}}_i^{v}$ that results from a local uniform stress $\boldsymbol{\sigma}_i$ in an ellipsoidal inhomogeneity. Local elastic accommodation rates are included in Section 3 where the sequential approach to a single inhomogeneity problem is extended to the sequential self-consistent scheme for a viscoelastic multi-phase composite.

Kinematic consistency between the imposed external strain-rate $\dot{\boldsymbol{\varepsilon}}_0$ and those in Subproblems V and E considered jointly is obtained by enforcing the relationship

$$\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}} + \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{E}} = \dot{\boldsymbol{\varepsilon}}_0\,. \tag{9}$$

On substituting relationships (8) and (9) into (6) and (7), the basic equations for

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Subproblems V and E reduce to

$$\dot{\boldsymbol{\varepsilon}}_i^v - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}} = -\mathbf{M}_*^v \cdot (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_0^{\mathrm{V}}) \tag{10}$$

and

$$\dot{\boldsymbol{\varepsilon}}_i^e - \dot{\boldsymbol{\varepsilon}}_0 + \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}} = -\mathbf{M}_*^e \cdot (\dot{\boldsymbol{\sigma}}_i - \dot{\boldsymbol{\sigma}}_0^{\mathrm{E}}) \,, \tag{11}$$

respectively.

Two additional tensor equations are still required. In conjunction with Eqs. (6), (7), (8) and (9), two natural variants are [2]:

- Variant I:

$$\boldsymbol{\sigma}_0^{\mathrm{V}} = \mathbf{L}_0^v \cdot \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}} \,, \qquad \dot{\boldsymbol{\sigma}}_0^{\mathrm{E}} = \mathbf{L}_0^e \cdot \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{E}} \,. \tag{12}$$

- Variant II:

$$\boldsymbol{\sigma}_0^{\mathrm{V}} = \boldsymbol{\sigma}_0 \,, \qquad \dot{\boldsymbol{\sigma}}_0^{\mathrm{E}} = \dot{\boldsymbol{\sigma}}_0 \,. \tag{13}$$

After straightforward rearrangements, Variants I and II of the sequential linearization approach to the viscoelastic inhomogeneity/matrix system reduce to the following sets of equations:

- Variant I:

$$\text{Subproblem V:} \qquad \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}} = (\mathbf{A}_i^v)^{-1} \cdot \dot{\boldsymbol{\varepsilon}}_i^v \,, \tag{14}$$

$$\text{Subproblem E:} \qquad \dot{\boldsymbol{\varepsilon}}_i^e = \mathbf{A}_i^e \cdot (\dot{\boldsymbol{\varepsilon}}_0 - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}}) \,. \tag{15}$$

- Variant II:

$$\text{Subproblem V:} \qquad \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}} = \dot{\boldsymbol{\varepsilon}}_i^v + \mathbf{M}_*^v \cdot (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_0) \,, \tag{16}$$

$$\text{Subproblem E:} \qquad \dot{\boldsymbol{\varepsilon}}_i^e = (\mathbf{I} + \mathbf{M}_*^e \mathbf{L}_i^e)^{-1} \cdot (\dot{\boldsymbol{\varepsilon}}_0 - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}} + \mathbf{M}_*^e \cdot \dot{\boldsymbol{\sigma}}_0) \,. \tag{17}$$

In each Variant, the essence of Subproblem V is to estimate the part $\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}}$ of external strain-rate $\dot{\boldsymbol{\varepsilon}}_0$ that is intended to correspond to $\dot{\boldsymbol{\varepsilon}}_i^e = \mathbf{0}$ in the inhomogeneity in the current stress state, in accord with the assumption (8). In Subproblem E, the difference $(\dot{\boldsymbol{\varepsilon}}_0 - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}})$ is used to determine the elastic strain-rate $\dot{\boldsymbol{\varepsilon}}_i^e$ and the corresponding stress-rate $\dot{\boldsymbol{\sigma}}_i$ in the inhomogeneity. Recall that $\dot{\boldsymbol{\varepsilon}}_i^v = \mathbf{M}_i^v \cdot \boldsymbol{\sigma}_i$ and $\dot{\boldsymbol{\sigma}}_0 = \mathbf{L}_0^e \cdot (\dot{\boldsymbol{\varepsilon}}_0 - \mathbf{M}_0^v \cdot \boldsymbol{\sigma}_0)$ are known in the current stress state.

On summing up equations (10) and (11) and substituting assumptions (13) it is found that Variant II leads to the equation equivalent to that postulated by Molinari (2002). The present derivation is different as it is based on the novel sequential linearization method.

[2] Variant III specified by relations $\mathbf{M}_0^v \cdot \boldsymbol{\sigma}_0^{\mathrm{V}} + \mathbf{M}_0^e \cdot \dot{\boldsymbol{\sigma}}_0^{\mathrm{E}} = \dot{\boldsymbol{\varepsilon}}_0$ and $\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}} = \dot{\boldsymbol{\varepsilon}}_0^v$ (then $\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{E}} = \dot{\boldsymbol{\varepsilon}}_0^e$ which follows from (9) and (1)) has also been tested. This variant, in the special case of a spherical inhomogeneity and matrix being both of an isotropic and incompressible linearly viscoelastic material, leads to an exact solution based on the Laplace transform. However, for compressible materials, the results for Variant III are in a less satisfactory agreement with the exact solution, see Subsection 2.5.

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It is worth noting that in Variant II, after straightforward transformations with the use of Eqs. (6)-(9) and (13), we have the following relationship

$$\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}} - \dot{\boldsymbol{\varepsilon}}_0^{v} = (\mathbf{M}_i^v + \mathbf{M}_*^v) \cdot (\boldsymbol{\sigma}_i - \mathbf{B}_i^v \cdot \boldsymbol{\sigma}_0)\,, \tag{18}$$

where

$$\mathbf{B}_i^v = (\mathbf{M}_i^v + \mathbf{M}_*^v)^{-1}(\mathbf{M}_0^v + \mathbf{M}_*^v) \tag{19}$$

is the stress concentration tensor, cf. Eq. (100), determined for viscous compliances.

The above variants (including Variant III - see the preceding footnote 2) exhibit the following desired properties:

- for the initial stage of the deformation process ($t = 0$ and $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_i = \mathbf{0}$) the problem defined by Eqs (6)-(7) reduces to the purely elastic problem (3),
- for the limit stage of the proportional deformation process ($t \to \infty$ and $\dot{\boldsymbol{\sigma}}_0 = \dot{\boldsymbol{\sigma}}_i = \mathbf{0}$) the problem defined by Eqs (6)-(7) reduces to the purely viscous problem (5),
- in the special case of the spherical inhomogeneity and matrix being both of isotropic and incompressible linear viscoelastic materials, the solution of Eqs (6)-(7) coincides with an exact solution obtained by using the Laplace transform by Hashin (1969), see Subsection 2.5.

2.2. Extension to the Mori-Tanaka scheme for multiphase composites

The above sequential linearization approach is immediately applicable to multiphase composites in conjunction with the averaging scheme of Mori-Tanaka type (Mori and Tanaka, 1973), widely used in the analysis of linear two-phase composite materials. It suffices to define the meaning of strain $\boldsymbol{\varepsilon}_0$ and stress $\boldsymbol{\sigma}_0$ in the above formulae as the corresponding *average* values in the matrix, and impose the conditions

$$\bar{\boldsymbol{\varepsilon}} = \sum_{i=0}^{N} c_i \boldsymbol{\varepsilon}_i\,, \quad \bar{\boldsymbol{\sigma}} = \sum_{i=0}^{N} c_i \boldsymbol{\sigma}_i\,, \quad \sum_{i=0}^{N} c_i = 1, \tag{20}$$

where the quantities with a bar, $\bar{\boldsymbol{\varepsilon}}$ and $\bar{\boldsymbol{\sigma}}$, are the overall variables for the composite, N is the number of different phases r occupying ellipsoidal inhomogeneities, average values in the matrix are denoted by index 0, and c_i denote respective volume fractions in the representative volume element (RVE).

The sequential linearization method formulated above, employed within the Mori-Tanaka scheme, results in the following two interaction equations for subproblems Vi and Ei,

$$\dot{\boldsymbol{\varepsilon}}_i^v - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i} = -\mathbf{M}_*^v \cdot (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_0^{\mathrm{V}i}), \tag{21}$$

$$\dot{\boldsymbol{\varepsilon}}_i^e - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{E}i} = -\mathbf{M}_*^e \cdot (\dot{\boldsymbol{\sigma}}_i - \dot{\boldsymbol{\sigma}}_0^{\mathrm{E}i})\,, \tag{22}$$

with the additional assumption for every variant:

$$\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i} + \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{E}i} = \dot{\boldsymbol{\varepsilon}}_0\,, \tag{23}$$

where $\dot{\boldsymbol{\varepsilon}}_0$ is the average strain-rate in the matrix. As in Section (2.1) above, we have different variants dependent on further specifications, in particular,

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- Variant I:

$$\boldsymbol{\sigma}_0^{\mathrm{V}i} = \mathbf{L}_0^v \cdot \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i}\,, \qquad \dot{\boldsymbol{\sigma}}_0^{\mathrm{E}i} = \mathbf{L}_0^e \cdot \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{E}i}\,, \tag{24}$$

- Variant II:

$$\boldsymbol{\sigma}_0^{\mathrm{V}i} = \boldsymbol{\sigma}_0\,, \qquad \dot{\boldsymbol{\sigma}}_0^{\mathrm{E}i} = \dot{\boldsymbol{\sigma}}_0\,. \tag{25}$$

Jointly with the constitutive laws (1), the equations above are sufficient to determine the elastic strain-rate in each inhomogeneity in terms of the average strain-rate $\dot{\boldsymbol{\varepsilon}}_0$ in the matrix, in full analogy to Eqs. (14)-(17).

The final step is to relate $\dot{\boldsymbol{\varepsilon}}_0$ to the overall strain-rate $\dot{\bar{\boldsymbol{\varepsilon}}}$ for the composite by using the rate form of Eq. $(20)_1$, viz.

$$c_0\dot{\boldsymbol{\varepsilon}}_0 = \dot{\bar{\boldsymbol{\varepsilon}}} - \sum_{i=1}^{N} c_i(\dot{\boldsymbol{\varepsilon}}_i^v + \dot{\boldsymbol{\varepsilon}}_i^e). \tag{26}$$

The specific versions of Eq. (26) for the above two variants are:

- Variant I:

$$\dot{\boldsymbol{\varepsilon}}_0 = \left(c_0\mathbf{I} + \sum_{i=1}^{N} c_i\mathbf{A}_i^e\right)^{-1} \cdot \left(\dot{\bar{\boldsymbol{\varepsilon}}} - \sum_{i=1}^{N} c_i(\dot{\boldsymbol{\varepsilon}}_i^v - \mathbf{A}_i^e \cdot \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i})\right), \tag{27}$$

- Variant II:

$$\dot{\boldsymbol{\varepsilon}}_0 = \left(c_0\mathbf{I} + \sum_{i=1}^{N} c_i\mathbf{A}_i^e\right)^{-1} \cdot \left(\dot{\bar{\boldsymbol{\varepsilon}}} - \sum_{i=1}^{N} c_i\left(\dot{\boldsymbol{\varepsilon}}_i^v - (\mathbf{I} + \mathbf{M}_*^e\mathbf{L}_i^e)^{-1} \cdot (\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i} + \mathbf{M}_*^e\mathbf{L}_0^e \cdot \dot{\boldsymbol{\varepsilon}}_0^v)\right)\right), \tag{28}$$

where in the last equation the following identity has been used:

$$\mathbf{A}_i^e = (\mathbf{I} + \mathbf{M}_*^e\mathbf{L}_i^e)^{-1}(\mathbf{I} + \mathbf{M}_*^e\mathbf{L}_0^e).$$

If $\dot{\bar{\boldsymbol{\varepsilon}}}$ is prescribed then the above equations constitute the full set of equations of the sequential linearization approach to a linear viscoelastic multi-phase composite in conjunction with the Mori-Tanaka averaging scheme.

2.3. Comparison of different approximation schemes

The above two variants of the sequential approach to a viscoelastic inhomogeneity problem are now compared with the Kröner-Weng (Kröner, 1958; Weng, 1981) and Paquin et al. (1999) approaches. For this purpose we use the relationship between a stress-rate in an ellipsoidal inhomogeneity and a remote stress-rate.

In the case of the Kröner-Weng model the interaction is purely elastic, that is

$$\dot{\boldsymbol{\varepsilon}}_i - \dot{\boldsymbol{\varepsilon}}_0 = -\mathbf{M}_*^e \cdot (\dot{\boldsymbol{\sigma}}_i - \dot{\boldsymbol{\sigma}}_0). \tag{29}$$

On substituting the constitutive relationships (1) and rearranging, we obtain the following relationship

$$\dot{\boldsymbol{\sigma}}_i = \mathbf{B}_i^e \cdot \dot{\boldsymbol{\sigma}}_0 - (\mathbf{M}_i^e + \mathbf{M}_*^e)^{-1} \cdot (\dot{\boldsymbol{\varepsilon}}_i^v - \dot{\boldsymbol{\varepsilon}}_0^v)\,, \tag{30}$$

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where

$$\mathbf{B}_i^e = (\mathbf{M}_i^e + \mathbf{M}_*^e)^{-1}(\mathbf{M}_0^e + \mathbf{M}_*^e) \tag{31}$$

is the stress concentration tensor determined for elastic compliances.

The Paquin et al. (1999) approach has been formulated within the self-consistent averaging scheme for heterogeneous materials. The solution for the inhomogeneity problem can be extracted in the limit case when

$$c_i \to 0, \quad \mathbf{A}_0^e \to \mathbf{I}, \quad \mathbf{A}_0^v \to \mathbf{I}. \tag{32}$$

In this case the micro-macro relationship derived by Paquin et al. (1999) reduces to

$$\dot{\boldsymbol{\varepsilon}}_i^e = \dot{\boldsymbol{\varepsilon}}_i - \dot{\boldsymbol{\varepsilon}}_i^v = \mathbf{A}_i^e \cdot \left[\dot{\boldsymbol{\varepsilon}}_0 + (\mathbf{S}^e - \mathbf{S}^v) \cdot \delta\dot{\boldsymbol{\varepsilon}}_i^v - (\mathbf{A}_i^v)^{-1} \cdot \dot{\boldsymbol{\varepsilon}}_i^v\right], \tag{33}$$

where $\mathbf{S}^e$ and $\mathbf{S}^v$ is the Eshelby tensor for the purely elastic and purely viscous matrix, respectively, $\delta\dot{\boldsymbol{\varepsilon}}_i^v = \dot{\boldsymbol{\varepsilon}}_i^v - \mathbf{A}_i^v \cdot \dot{\boldsymbol{\varepsilon}}_0^v$, and $\dot{\boldsymbol{\varepsilon}}_0^v$ is the remote viscous strain-rate. The respective expression for the stress-rate in the inhomogeneity, in terms of the remote stress-rate, is

$$\dot{\boldsymbol{\sigma}}_i = \mathbf{B}_i^e \cdot \dot{\boldsymbol{\sigma}}_0 - (\mathbf{M}_i^e + \mathbf{M}_*^e)^{-1}\mathbf{H}_i^{\mathrm{PQ}} \cdot ((\mathbf{A}_i^v)^{-1} \cdot \dot{\boldsymbol{\varepsilon}}_i^v - \dot{\boldsymbol{\varepsilon}}_0^v), \tag{34}$$

where

$$\mathbf{H}_i^{\mathrm{PQ}} = (\mathbf{I} + \mathbf{M}_*^e\mathbf{L}_0^e)(\mathbf{I} - (\mathbf{S}^e - \mathbf{S}^v)\mathbf{A}_i^v). \tag{35}$$

Note that, similarly to the examined variants of sequential linearization, for the initial and final stages of a proportional deformation process, the rate-solutions for the purely elastic and purely viscous problem, respectively, are recovered. On the contrary, for the final stage of a proportional deformation process for the Kröner-Weng model, a purely viscous solution of the Eshelby problem is not recovered, as one finds $\dot{\boldsymbol{\varepsilon}}_i^v = \dot{\boldsymbol{\varepsilon}}_0^v$ instead.

In case of Variants I and II of the sequential linearization approach, the expression for the stress-rate in the inhomogeneity takes the form

$$\dot{\boldsymbol{\sigma}}_i = \mathbf{B}_i^e \cdot \dot{\boldsymbol{\sigma}}_0 - (\mathbf{M}_i^e + \mathbf{M}_*^e)^{-1}\mathbf{H}^K \cdot ((\mathbf{A}_i^v)^{-1} \cdot \dot{\boldsymbol{\varepsilon}}_i^v - \dot{\boldsymbol{\varepsilon}}_0^v), \tag{36}$$

where[3]

$$\mathbf{H}^{\mathrm{I}} = \mathbf{I} + \mathbf{M}_*^e\mathbf{L}_0^e \quad \text{and} \quad \mathbf{H}^{\mathrm{II}} = \mathbf{I} + \mathbf{M}_*^v\mathbf{L}_0^v. \tag{37}$$

Consider an incompressible isotropic (elastic and viscous) matrix for which the elastic bulk modulus K_0 is infinite and

$$\mathbf{M}_0^{e/v} = \frac{1}{h_0^{e\mathrm{D}/v}}\mathbf{I}_{\mathrm{D}}, \quad \mathbf{I}_{\mathrm{D}} = \mathbf{I} - \mathbf{I}_{\mathrm{P}}, \quad \mathbf{I}_{\mathrm{P}} = \frac{1}{3}\mathbf{1} \otimes \mathbf{1} \tag{38}$$

[3]For Variant III tensor $\mathbf{H}^K$ reads

$$\mathbf{H}^{\mathrm{III}} = (\mathbf{M}_*^e\mathbf{L}_0^e)(\mathbf{M}_*^v\mathbf{L}_0^v)^{-1}(\mathbf{I} + \mathbf{M}_*^v\mathbf{L}_0^v),$$

which for an elastically compressible isotropic material with $\mathbf{M}_r^v = 1/h_r^v\mathbf{I}_D$ and a spherical inhomogeneity can be represented by a single scalar β^{III},

$$\beta^{\mathrm{I}} \leq \beta^{\mathrm{III}} = \frac{5 + 15\eta(\nu)}{3 + 4\eta(\nu)} \leq \frac{15}{7}$$

with $\eta(\nu)$ specified in Eq. (43).

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with $h_0^{\mathrm{eD}} = 2\mu_0$, $h_0^v = 2\eta_0$, where μ_0 and η_0 are the elastic shear modulus and the viscosity of the matrix, respectively; $\mathbf{1}$ is the second-order identity tensor. In this case we have

$$\mathbf{S}^e = \mathbf{S}^v, \quad \mathbf{M}_*^e \mathbf{L}_0^e = \mathbf{M}_*^v \mathbf{L}_0^v \,. \tag{39}$$

Then, all the three variants of the sequential linearization approach and the Paquin et al. model provide the same concentration relationship as the exact solution obtained by Hashin (1969) with use of the Laplace transform and the correspondence principle. This fact was already demonstrated in (Mercier et al., 2005) for the Paquin et al. (1999) and Molinari (2002) approaches, the latter being equivalent to the present Variant II of the sequential linearization approach. This particular exact solution is not recovered using the model developed by Sabar et al. (2002), which is therefore not analyzed in this work in more detail.

In the case of isotropic elastically compressible materials (viscous incompressibility is still assumed, $\mathbf{M}_r^v = 1/h_r^v \mathbf{I}_{\mathrm{D}}$), for which

$$\mathbf{M}_r^e = \frac{1}{h_r^{e\mathrm{P}}} \mathbf{I}_{\mathrm{P}} + \frac{1}{h_r^{e\mathrm{D}}} \mathbf{I}_{\mathrm{D}}, \quad h_r^{e\mathrm{P}} = 3K_r \,, \tag{40}$$

and a spherical inhomogeneity, the fourth order tensor $\mathbf{H}$ in the concentration relationship can be replaced by a single scalar,

$$\dot{\boldsymbol{\sigma}}_i = \mathbf{B}^e \cdot \dot{\boldsymbol{\sigma}}_0 - \beta^K (\mathbf{M}_i^e + \mathbf{M}_*^e)^{-1} \cdot ((\mathbf{A}^v)^{-1} \cdot \dot{\boldsymbol{\varepsilon}}_i^v - \dot{\boldsymbol{\varepsilon}}_0^v), \tag{41}$$

where the coefficient β^K is specified as follows

$$\beta^K = \begin{cases} \frac{5+10\eta(\nu)}{3+4\eta(\nu)} & \text{if} \quad K = \mathrm{I} \\ \frac{5}{3} & \text{if} \quad K = \mathrm{II(M)} \\ \frac{5+10\eta(\nu)\gamma_v}{3+4\eta(\nu)} & \text{if} \quad K = \mathrm{PQ} \end{cases} \tag{42}$$

and (for a non-negative Poisson ratio ν, $0 \le \eta \le 1$)

$$\eta(\nu) = \frac{1-2\nu}{1+\nu}, \quad \gamma_v = \frac{2+\xi_v}{3+\xi_v}, \quad \xi_v = \frac{h_i^v}{h_0^v}, \quad \gamma_v \in \left(\frac{2}{3}, 1\right). \tag{43}$$

The following relation between the coefficients is obtained:

$$\beta^{\mathrm{II}} \le \beta^{\mathrm{PQ}} \le \beta^{\mathrm{I}} \le \frac{10}{7} \,. \tag{44}$$

If $\frac{h_i^v}{h_0^v} \to 0$ (the inhomogeneity behaves as a void in terms of viscous properties) then the Paquin et al. model coincides with Variant II (the Molinari model), $\beta^{\mathrm{PQ}}(\nu, 2/3) = \beta^{\mathrm{II}}$. On the other hand, if $\frac{h_i^v}{h_0^v} \to \infty$ (the inhomogeneity is elastic) the Paquin et al. model coincides with Variant I, $\beta^{\mathrm{PQ}}(\nu, 1) = \beta^{\mathrm{I}}(\nu)$. Dependence of the coefficient β on the Poisson ratio ν for different models and viscosity contrasts is presented in Fig. 2.

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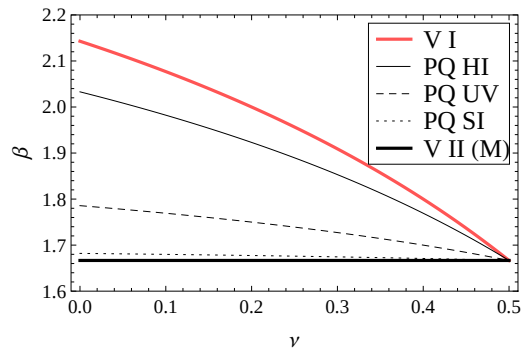

Figure 2: Comparison of coefficients β in the concentration relationship for different variants of the sequential linearization and Paquin et al. (PQ) approaches for isotropic elastically compressible materials and a spherical inhomogeneity; HI (Hard Inclusion) - $h_i^v/h_0^v = 10$, UV (Uniform Viscous properties) - $h_i^v/h_0^v = 1$, SI (Soft Inclusion) - $h_i^v/h_0^v = 0.1$.

2.4. An exact solution in the Laplace transform space

As shown by Hashin (1969) for isotropic materials and by Laws and McLaughlin (1978) in a general anisotropic case, the problem of an ellipsoidal viscoelastic inhomogeneity embedded in a viscoelastic matrix, both described by the Maxwell-type constitutive law (1), can be solved by using the Laplace transform technique and the correspondence principle. Taking the Laplace transform of equations (1) for the inhomogeneity ($r = i$) and for the matrix at infinity ($r = 0$), and assuming the initial conditions $\boldsymbol{\varepsilon}_r|_{t=0} = \mathbf{0}$, $\boldsymbol{\sigma}_r|_{t=0} = \mathbf{0}$, we obtain

$$s\hat{\boldsymbol{\varepsilon}}_r(s) = (s\mathbf{M}_r^e + \mathbf{M}_r^v) \cdot \hat{\boldsymbol{\sigma}}_r(s), \tag{45}$$

denoting $\hat{f}(s) \equiv \int_0^\infty e^{-st} f(t)\, dt$, with s as a complex variable.

We will specify the solution for isotropic materials which are *elastically compressible* and viscously incompressible, of elastic compliance tensors specified by Eq. (40) and viscous compliance tensors as in Eq. (38), and a spherical inhomogeneity. Decompose the strain and stress into hydrostatic and deviatoric parts, denoted by upper indices P and D, respectively. On taking the Laplace transform of equations (1) decomposed in this way, we obtain

$$\hat{\boldsymbol{\sigma}}_r^{\mathrm{P}}(s) = \underbrace{h_r^{\mathrm{P}e}}_{\hat{h}_r^{\mathrm{P}}(s)} \hat{\boldsymbol{\varepsilon}}_r^{\mathrm{P}}(s), \quad \hat{\boldsymbol{\sigma}}_r^{\mathrm{D}}(s) = \underbrace{\frac{s h_r^v}{1 + s\frac{h_r^v}{h_r^{\mathrm{D}e}}}}_{\hat{h}_r^{\mathrm{D}}(s)} \hat{\boldsymbol{\varepsilon}}_r^{\mathrm{D}}(s). \tag{46}$$

Using the correspondence principle and the Eshelby solution for a spherical inclusion, we

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obtain, cf. Hashin (1969),[4]

$$\hat{\boldsymbol{\varepsilon}}_i^{\mathrm{P}}(s) = \hat{A}_i^{\mathrm{P}}(s)\hat{\boldsymbol{\varepsilon}}_0^{\mathrm{P}}(s)\,, \qquad \hat{A}_i^{\mathrm{P}}(s) = \frac{h_0^{\mathrm{P}e} + 2\hat{h}_0^{\mathrm{D}}(s)}{h_i^{\mathrm{P}e} + 2\hat{h}_0^{\mathrm{D}}(s)} \tag{47}$$

and

$$\hat{\boldsymbol{\varepsilon}}_i^{\mathrm{D}}(s) = \hat{A}_i^{\mathrm{D}}(s)\hat{\boldsymbol{\varepsilon}}_0^{\mathrm{D}}(s)\,, \qquad \hat{A}_i^{\mathrm{D}}(s) = \frac{\hat{h}_0^{\mathrm{D}}(s) + \hat{h}_*^{\mathrm{D}}(s)}{\hat{h}_i^{\mathrm{D}}(s) + \hat{h}_*^{\mathrm{D}}(s)}, \tag{48}$$

where

$$\hat{h}_*^{\mathrm{D}}(s) = \hat{h}_0^{\mathrm{D}}(s)\frac{3h_0^{\mathrm{P}e} + 4\hat{h}_0^{\mathrm{D}}(s)}{2(h_0^{\mathrm{P}e} + 3\hat{h}_0^{\mathrm{D}}(s))}. \tag{49}$$

Consider the external strain programme specified as follows

$$\boldsymbol{\varepsilon}_0(t) = \varepsilon_0^{\mathrm{P}}(t)\mathbf{1} + \varepsilon_0^{\mathrm{D}}(t)\mathbf{N}_{\mathrm{D}}, \quad \mathbf{N}_{\mathrm{D}} = \mathrm{const}, \quad \mathrm{tr}\,\mathbf{N}_{\mathrm{D}} = 0, \tag{50}$$

where $\varepsilon_0^{\mathrm{P}}(t), \varepsilon_0^{\mathrm{D}}(t)$ are given scalar functions of time. Thus

$$\hat{\boldsymbol{\varepsilon}}_0(s) = \hat{\varepsilon}_0^{\mathrm{P}}(s)\mathbf{1} + \hat{\varepsilon}_0^{\mathrm{D}}(s)\mathbf{N}_{\mathrm{D}}. \tag{51}$$

From Eqs. (47)-(48) it follows that

$$\hat{\boldsymbol{\varepsilon}}_i^{\mathrm{P}}(s) = \hat{\varepsilon}_i^{\mathrm{P}}(s)\mathbf{1} = \hat{A}_i^{\mathrm{P}}(s)\hat{\varepsilon}_0^{\mathrm{P}}(s)\mathbf{1} \qquad \rightarrow \qquad \hat{\varepsilon}_i^{\mathrm{P}}(s) = \hat{A}_i^{\mathrm{P}}(s)\hat{\varepsilon}_0^{\mathrm{P}}(s), \tag{52}$$

$$\hat{\boldsymbol{\varepsilon}}_i^{\mathrm{D}}(s) = \hat{\varepsilon}_i^{\mathrm{D}}(s)\mathbf{N}_{\mathrm{D}} = \hat{A}_i^{\mathrm{D}}(s)\hat{\varepsilon}_0^{\mathrm{D}}(s)\mathbf{N}_{\mathrm{D}} \qquad \rightarrow \qquad \hat{\varepsilon}_i^{\mathrm{D}}(s) = \hat{A}_i^{\mathrm{D}}(s)\hat{\varepsilon}_0^{\mathrm{D}}(s). \tag{53}$$

Similarly, an 'exact' solution can be derived for the averaged response of a two-phase composite within the Mori-Tanaka-type averaging scheme. Then, the Laplace transform of an *external* strain in Eqs (47)-(48) is replaced by the Laplace transform of the corresponding *average* strain in the matrix. Moreover, the Laplace transform of Eq. $(20)_1$ is used, in the form

$$\hat{\bar{\boldsymbol{\varepsilon}}}(s) = c_i\hat{\boldsymbol{\varepsilon}}_i(s) + (1 - c_i)\hat{\boldsymbol{\varepsilon}}_0(s), \tag{54}$$

where $\hat{\bar{\boldsymbol{\varepsilon}}}(s)$ is prescribed by an equation analogous to Eq. (51). After manipulations outlined above for the spherical inhomogeneity problem, in the case of isotropic materials and compressible elasticity, one finds

$$\hat{\varepsilon}_0^{\mathrm{P/D}}(s) = \frac{\hat{\varepsilon}^{\mathrm{P/D}}(s)}{1 - c_i(1 - \hat{A}_i^{\mathrm{P/D}}(s))}\,, \quad \hat{\varepsilon}_i^{\mathrm{P/D}}(s) = \frac{\hat{A}_i^{\mathrm{P/D}}(s)\hat{\varepsilon}^{\mathrm{P/D}}(s)}{1 - c_i(1 - \hat{A}_i^{\mathrm{P/D}}(s))}. \tag{55}$$

[4]In the case of the corresponding elastic solution there is

$$\mathbf{A}_i = \alpha^{\mathrm{P}}\mathbf{I}_{\mathrm{P}} + \alpha^{\mathrm{D}}\mathbf{I}_{\mathrm{D}} = \frac{h_0^{\mathrm{P}e} + 2h_0^{\mathrm{D}e}}{h_i^{\mathrm{P}e} + 2h_0^{\mathrm{D}e}}\mathbf{I}_{\mathrm{P}} + \frac{h_0^{\mathrm{D}e} + h_*^{\mathrm{D}e}}{h_i^{\mathrm{D}e} + h_*^{\mathrm{D}e}}\mathbf{I}_{\mathrm{D}},$$

where

$$h_*^{\mathrm{D}e} = h_0^{\mathrm{D}e}\frac{3h_0^{\mathrm{P}e} + 4h_0^{\mathrm{D}e}}{2(h_0^{\mathrm{P}e} + 3h_0^{\mathrm{D}e})}.$$

Relations (47) and (48) can be compared with Eqs (6.34)-(6.37) in (Hashin, 1969) using

$$h^{\mathrm{P}} = 3K, \quad h^{\mathrm{D}} = 2\mu, \quad \nu = \frac{h^{\mathrm{P}} - h^{\mathrm{D}}}{2h^{\mathrm{P}} + h^{\mathrm{D}}}.$$

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Table 1: Analysed examples of a spherical inhomogeneity

Case	E_i/E_0	h_i^v/h_m^v
Hard Elastic Inclusion	10	∞
Hard Inclusion	3	10
Soft Inclusion	1/3	1/10
Void	0	~ 0

In the special case of homogeneous elastic properties, exact solutions for an isotropic linear viscoelastic two-phase material have recently been obtained (Ricaud and Masson, 2009) by the finite expansion of relaxation functions into the Prony series, without the need of inversion of the Laplace transform.

2.5. Comparison of results for a tension-compression cycle

Quantitative predictions of the approximate schemes from Section 2.3 are now compared with the exact solution outlined above for a tension-compression cycle under the constraint of constant *overall* volume (not necessarily in individual phases). This is done for the inhomogeneity problem; results for the Mori-Tanaka scheme for a two-phase composite would be analogous. As already mentioned, in the case of *fully incompressible* (both for elastic and viscous deformations) isotropic materials of the inhomogeneity and matrix, an exact solution is obtained. Specifically, the solution obtained with the use of the three variants of the sequential approach, as well as of the Paquin et al. model, for an incompressible spherical inhomogeneity problem coincides with the corresponding exact solution obtained with use of the Laplace transform. Therefore, the analysis is performed below for *elastically compressible* materials. A low value of the Poisson ratio is assumed, equal for the matrix and inhomogeneity, since the difference between approximate schemes is then more pronounced, cf. Fig. 3. Different elastic and viscous properties of the matrix and spherical inhomogeneity (inclusion) have been examined, namely

- a hard elastic inclusion in a viscoelastic matrix,
- a hard/soft viscoelastic inclusion in a viscoelastic matrix,
- a void in viscoelastic matrix.

A *hard* and *soft* inclusion means that its elastic and viscous moduli are, correspondingly, higher or lower than respective moduli of the matrix. Isotropic material properties of the matrix are specified by

$$\nu_{0/i} = 0.05\,, \quad E_0 = 200\,\mathrm{MPa}\,, \quad h_0^v = \frac{2}{3}\,\mathrm{MPa}\times\mathrm{s}. \tag{56}$$

The examined cases of inclusion properties are collected in Table 1.

As a particular case of Eq. (50), the remote strain history $\boldsymbol{\varepsilon}_0(t)$ is assumed as follows

$$\boldsymbol{\varepsilon}_0(t) = (\varepsilon_1(t) - \varepsilon_2(t))\mathbf{N}_\mathrm{D}, \quad \text{where} \quad \begin{cases} \varepsilon_1(t) = 0 & \text{if} \quad t < 0 \\ \varepsilon_1(t) = t & \text{if} \quad t \geq 0 \\ \varepsilon_2(t) = 0 & \text{if} \quad t < T \\ \varepsilon_2(t) = 2(t-T) & \text{if} \quad t \geq T \end{cases} \tag{57}$$

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with

$$\mathbf{N}_{\mathrm{D}} = \frac{1}{2}\left(3\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{1}\right) , \quad [N_{\mathrm{D}ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}, \tag{58}$$

$\mathbf{e}_1$ being the tension-compression direction. The corresponding strain-rate history is

$$\dot{\boldsymbol{\varepsilon}}_0(t) = \dot{\varepsilon}_0^{\mathrm{D}}(t)\mathbf{N}_{\mathrm{D}}, \quad \text{where} \quad \begin{cases} \dot{\varepsilon}_0^{\mathrm{D}}(t) = 0 & \text{if} \quad t < 0 \\ \dot{\varepsilon}_0^{\mathrm{D}}(t) = 1 & \text{if} \quad 0 \le t < T \\ \dot{\varepsilon}_0^{\mathrm{D}}(t) = -1 & \text{if} \quad t \ge T. \end{cases} \quad [1/\mathrm{s}]. \tag{59}$$

The Laplace transform of the strain programme (57) is $\hat{\boldsymbol{\varepsilon}}_0(s) = \hat{\varepsilon}_0(s)\mathbf{N}_{\mathrm{D}}$, with

$$\hat{\varepsilon}_0(s) = \hat{\varepsilon}_1(s) - \hat{\varepsilon}_2(s) = \frac{1}{s^2}\left(1 - 2\exp(-sT)\right). \tag{60}$$

Analytical inversion of the Laplace transform has been performed with the help of *Mathematica* package (Wolfram, 2003).

Conclusions from the analysis are the following (see Fig. 3 and Fig. 4):

- For *hard inclusions*, the approximate schemes give results close to each other, with Variant II (Molinari model) being the softest. As the contrast in viscous properties increases (the asymptotic case is the *hard elastic inclusion*), the exact solution is less stiff in the transient regime than the approximate solutions, therefore Variant II provides the best prediction among the approximate schemes examined. As expected for hard inclusions, predictions of the Paquin model are close to Variant I predictions.

- For *soft inclusions*, the exact solution predicts a 'peak' of stress in the transient regime which is not observed for Variant I and III of the sequential model and can be predicted (although usually less intensive than those in the exact solution) by Variant II and the Paquin model. As expected for soft inclusions, predictions of the Paquin model are close to Variant II predictions.

- *Void* has been modelled as an inhomogeneity with $E_i = 0$ and $h_i^v/h_m^v \to 0$ (a finite value due to numerical reasons), so that $\boldsymbol{\sigma}_i(t) = \mathbf{0}$ as a natural condition for the void. In the case of Variant I, the evolution of the strain-rate in a void coincides with that for the problem of a void in a purely elastic matrix and as such does not agree with an exact solution. Variant II and Paquin et al. model provide good predictions. Similarly to an exact solution, the solution obtained with the use of Variant II is not sensitive to the value of h_i^v/h_m^v ratio, provided $\boldsymbol{\sigma}_i(t) = \mathbf{0}$ is enforced, and in the limit of the proportional straining the purely viscous solution is recovered. This is not the case for the Paquin et al. model for which results depend on h_i^v/h_m^v ratio. In particular, if one assumes $E_i \to 0$ and $h_i^v/h_m^v \to \infty$, predictions of this model tend to the purely elastic solution of the problem for a void and disagree with an exact solution obtained under these conditions.

- The predictions of the Kröner-Weng model (K) are depicted only in Fig. 4. Except of a hard elastic inclusion, for the regimes for which the strain-rate of the matrix

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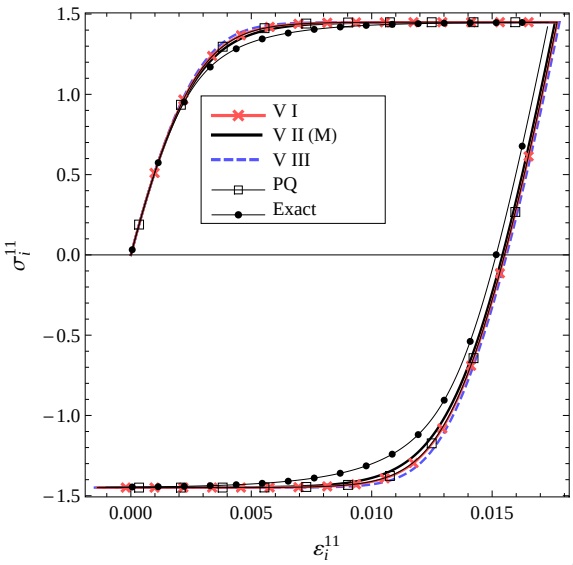
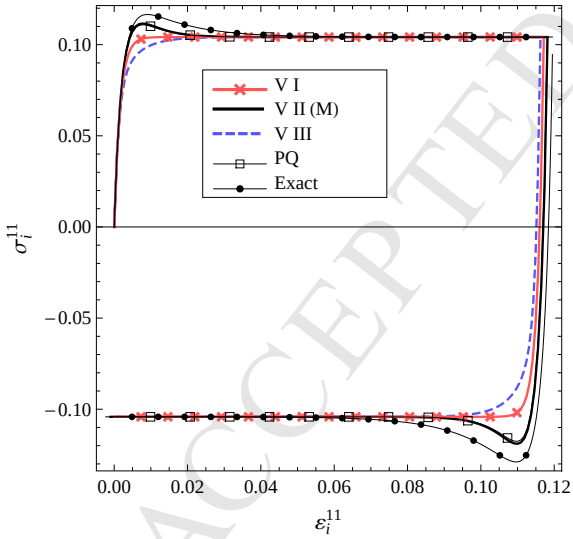
is almost totally viscous, they tend to the Voigt model assumption, $\dot{\varepsilon}_i = \dot{\varepsilon}_0$. The stress-strain loops obtained with use of this model are not presented in Fig. 3 since the stress level is highly overpredicted (hard inclusion) or underpredicted (soft inclusion) as compared to the exact solution in the regimes dominated by a viscous flow.



a)



b)

Figure 3: Comparison of the stress-strain loops predicted by different approaches within a spherical inhomogeneity embedded in an infinite matrix subjected to the tension-compression cycle: a) Hard Inclusion, b) Soft Inclusion.

On the basis of the performed analysis, Variant II of the sequential method has been selected for further developments as it provides the best predictions in comparison to the

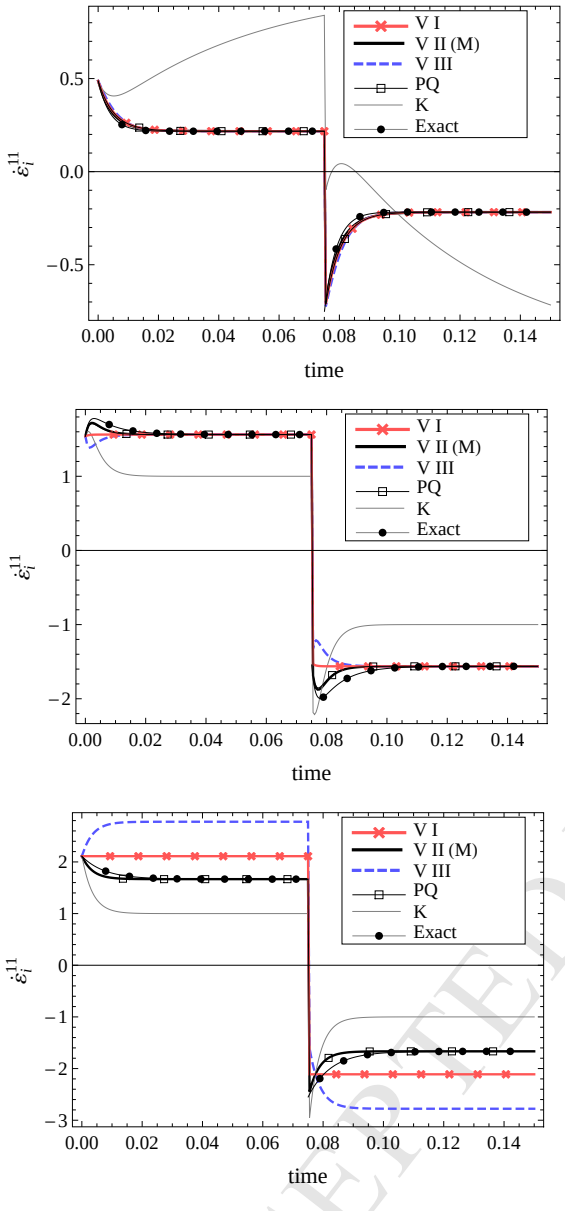


Figure 4: Comparison of strain-rate evolution predicted by different approaches within a spherical inhomogeneity embedded in an infinite matrix subjected to the tension-compression cycle: a) Hard Inclusion, b) Soft Inclusion, c) Void.

exact solution. As indicated above, this variant is equivalent to the Molinari (2002) model in the case of a single inhomogeneity problem.

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3. Self-consistent averaging scheme

3.1. Formulation of the sequential approach

Consider now a heterogeneous material composed of N constituents, of volume fractions c_i, which obey the constitutive relation (1) with elastic compliances $\mathbf{M}_i^e$ and viscous compliances $\mathbf{M}_i^v$, $i = 1, \ldots, N$. The overall strain-rate and stress-rate are now defined by

$$\bar{\boldsymbol{\varepsilon}} = \langle \boldsymbol{\varepsilon}_i \rangle \,, \quad \bar{\boldsymbol{\sigma}} = \langle \boldsymbol{\sigma}_i \rangle \,, \qquad \langle \psi_i \rangle \equiv \sum_{i=1}^{N} c_i \psi_i, \quad \sum_{i=1}^{N} c_i = 1 \,, \tag{61}$$

where $\boldsymbol{\varepsilon}_i$, $\boldsymbol{\sigma}_i$ are themselves averages over individual phases.

In the self-consistent treatment of a multi-phase composite investigated below, fluctuations within each phase are disregarded. The change with respect to the single inhomogeneity problem and the Mori-Tanaka scheme addressed above is that the matrix properties are no longer those of any specific phase and are to be replaced by effective properties of the homogenized composite. Denote by $\bar{\mathbf{M}}^e$ and $\bar{\mathbf{M}}^v$ the fourth-order tensors of effective elastic and viscous compliances of the homogenized composite and by $\bar{\mathbf{L}}^e$ and $\bar{\mathbf{L}}^v$ their inverses, determined for the purely elastic and purely viscous problem, respectively. The latter are determined implicitly from the well-known equations

$$\bar{\mathbf{L}}^e = \langle \mathbf{L}_i^e \mathbf{A}_i^e \rangle, \qquad \mathbf{A}_i^e = (\mathbf{I} + \mathbf{P}^e(\mathbf{L}_i^e - \bar{\mathbf{L}}^e))^{-1}, \tag{62}$$

$$\bar{\mathbf{L}}^v = \langle \mathbf{L}_i^v \mathbf{A}_i^v \rangle, \qquad \mathbf{A}_i^v = (\mathbf{I} + \mathbf{P}^v(\mathbf{L}_i^v - \bar{\mathbf{L}}^v))^{-1}, \tag{63}$$

where the concentration tensors $\mathbf{A}_i^e$ and $\mathbf{A}_i^v$ for phase i, in comparison with Eqs (3) and (5) for a single ellipsoidal inhomogeneity embedded in the matrix of prescribed properties, depend now on the overall properties of the homogenized composite rather than of a given matrix. The remote stress $\boldsymbol{\sigma}_0$ and strain $\boldsymbol{\varepsilon}_0$ at infinity in the inhomogeneity-matrix interaction equation, cf. (94), are equal to the respective overall quantities, $\boldsymbol{\sigma}_0 = \bar{\boldsymbol{\sigma}}$ and $\boldsymbol{\varepsilon}_0 = \bar{\boldsymbol{\varepsilon}}$. In the case of ellipsoidal inhomogeneities of the same shape and orientations, the concentration tensors fulfill the identity $\langle \mathbf{A}_i^{e/v} \rangle = \mathbf{I}$. It is known that the approximations involved in the above equations themselves may not always be realistic, for instance, for a high contrast between phase properties (Christensen, 2005) or in the presence of high anisotropy (Kowalczyk-Gajewska, 2009). Our aim here is, however, not to discuss validity of the self-consistent scheme as such for a linear problem, rather, to examine a more difficult viscous/elastic problem in the case when the self-consistent scheme is appropriate for each linear subproblem separately.

The concept of sequential linearization described in Section 2 is adapted below to the self-consistent averaging scheme. The basic Subproblems V and E are now formulated for each constituent separately and hence consist of Subproblems Vi and Ei, $i = 1, \ldots, N$, respectively. To reduce the number of possible variants, Variant II of the sequential approach from Section 2 is selected as the best from those considered. However, an accommodation Subproblem A, of the origin indicated in Section 2.1, is now also included. Accordingly, we use the following local decompositions

$$\dot{\boldsymbol{\varepsilon}}_i = \dot{\boldsymbol{\varepsilon}}_i^v + \dot{\boldsymbol{\varepsilon}}_i^{\mathrm{A}} + \dot{\boldsymbol{\varepsilon}}_i^{\mathrm{E}} \,, \qquad \dot{\boldsymbol{\sigma}}_i = \dot{\boldsymbol{\sigma}}_i^{\mathrm{A}} + \dot{\boldsymbol{\sigma}}_i^{\mathrm{E}} \tag{64}$$

along with analogous decompositions of the overall response

$$\dot{\boldsymbol{\varepsilon}}_0 = \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}} + \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{A}} + \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{E}}\,, \qquad \dot{\boldsymbol{\sigma}}_0 = \dot{\boldsymbol{\sigma}}_0^{\mathrm{A}} + \dot{\boldsymbol{\sigma}}_0^{\mathrm{E}}\,. \tag{65}$$

In Subproblem Vi we find an external strain-rate at infinity, $\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i}$, intended to be compatible with the viscous strain-rate $\dot{\boldsymbol{\varepsilon}}_i^v$ that results from a local uniform stress $\boldsymbol{\sigma}_i$ in an ellipsoidal inhomogeneity of constituent i. As in Variant II in Section 2, this is done by assuming purely viscous interaction between the inhomogeneity and the homogenized matrix, now of moduli $\bar{\mathbf{L}}^v$ under remote stress $\boldsymbol{\sigma}_0$. Note that those *external* strain-rates $\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i}$ depend on the phase index i and differ from each other in general (and also from *overall* quantities independent of i). They are determined for different phases i from the equation analogous to (10), viz.

$$\dot{\boldsymbol{\varepsilon}}_i^v - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i} = -\bar{\mathbf{M}}_*^v \cdot (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_0)\,, \tag{66}$$

where the Hill tensor inverse $\bar{\mathbf{M}}_*^v$ is associated with the viscous effective moduli tensor $\bar{\mathbf{L}}^v$ for the homogenized composite. Upon multiplying Eq. (66) by c_i and summing up, we obtain $\langle \dot{\boldsymbol{\varepsilon}}_i^v \rangle = \langle \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i} \rangle$ on account of $\boldsymbol{\sigma}_0 = \langle \boldsymbol{\sigma}_i \rangle$.

On the other hand, the homogenized matrix of effective viscous compliance tensor $\bar{\mathbf{M}}^v$ undergoes viscous flow with the strain-rate $\dot{\boldsymbol{\varepsilon}}_0^v = \bar{\mathbf{M}}^v \cdot \boldsymbol{\sigma}_0$ at infinity, while volume averaging for the purely viscous subproblem gives the overall viscous strain-rate $\langle \dot{\boldsymbol{\varepsilon}}_i^v \rangle$. The difference between $\dot{\boldsymbol{\varepsilon}}_0^v$ and $\langle \dot{\boldsymbol{\varepsilon}}_i^v \rangle$ is to be accommodated elastically. The respective internal elastic accommodation rates in phases are distinguished by a superscript A and obey the elastic constitutive relationships $\dot{\boldsymbol{\sigma}}_i^{\mathrm{A}} = \mathbf{L}_i^e \cdot \dot{\boldsymbol{\varepsilon}}_i^{\mathrm{A}}$.

It is possible to retain consistency with Eq. (66) when the accommodation strain-rates are included in Subproblem Vi. For this purpose, each external elastic accommodation strain-rate $\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{A}i}$ is taken to coincide with its internal counterpart $\dot{\boldsymbol{\varepsilon}}_i^{\mathrm{A}}$ for phase i, so that they cancel each other when incorporated into Eq. (66), viz.

$$\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{A}i} = \dot{\boldsymbol{\varepsilon}}_i^{\mathrm{A}} \quad \text{and} \quad (\dot{\boldsymbol{\varepsilon}}_i^v + \dot{\boldsymbol{\varepsilon}}_i^{\mathrm{A}}) - (\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i} + \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{A}i}) = -\bar{\mathbf{M}}_*^v \cdot (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_0)\,. \tag{67}$$

Suppose for a moment that the introduced accommodation strain-rates and stress-rates are known. The purely elastic Subproblem Ei is specified as follows

$$\dot{\boldsymbol{\varepsilon}}_i^{\mathrm{E}} - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{E}i} = -\bar{\mathbf{M}}_*^e \cdot (\dot{\boldsymbol{\sigma}}_i^{\mathrm{E}} - \dot{\boldsymbol{\sigma}}_0^{\mathrm{E}})\,, \tag{68}$$

where the Hill tensor inverse $\bar{\mathbf{M}}_*^e$ is associated with the elastic effective moduli tensor $\bar{\mathbf{L}}^e$ for the homogenized composite. The quantities with a superscript E, involved in Eqs. (64) and (65), represent the differences between total elastic and accommodation terms,

$$\dot{\boldsymbol{\varepsilon}}_i^{\mathrm{E}} = \dot{\boldsymbol{\varepsilon}}_i^e - \dot{\boldsymbol{\varepsilon}}_i^{\mathrm{A}}\,, \qquad \dot{\boldsymbol{\sigma}}_i^{\mathrm{E}} = \dot{\boldsymbol{\sigma}}_i - \dot{\boldsymbol{\sigma}}_i^{\mathrm{A}}\,, \qquad \dot{\boldsymbol{\sigma}}_0^{\mathrm{E}} = \dot{\boldsymbol{\sigma}}_0 - \dot{\boldsymbol{\sigma}}_0^{\mathrm{A}}\,, \tag{69}$$

In analogy to the kinematic consistency condition (9), for each i we assume that

$$\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{E}i} = \dot{\boldsymbol{\varepsilon}}_0 - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{A}i} - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i}\,. \tag{70}$$

It can easily be shown that the requirements

$$\langle \dot{\boldsymbol{\varepsilon}}_i \rangle = \dot{\boldsymbol{\varepsilon}}_0\,, \quad \langle \dot{\boldsymbol{\sigma}}_i \rangle = \dot{\boldsymbol{\sigma}}_0 \tag{71}$$

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together with the equations above lead to

$$\dot{\boldsymbol{\sigma}}_0^{\mathrm{A}} = \left\langle \dot{\boldsymbol{\sigma}}_i^{\mathrm{A}} \right\rangle . \tag{72}$$

One can verify that, if $\dot{\boldsymbol{\sigma}}_i^{\mathrm{A}}$ are known, the set of equations (66)-(71) implies the following equation that relates the local and global responses of the heterogeneous medium:

$$\dot{\boldsymbol{\varepsilon}}_i - \dot{\boldsymbol{\varepsilon}}_0 = -\bar{\mathbf{M}}_*^v \cdot (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_0) - \bar{\mathbf{M}}_*^e \cdot (\dot{\boldsymbol{\sigma}}_i - \dot{\boldsymbol{\sigma}}_0) - \bar{\mathbf{M}}_*^e \cdot (\dot{\boldsymbol{\sigma}}_i^{\mathrm{A}} - \dot{\boldsymbol{\sigma}}_0^{\mathrm{A}}) . \tag{73}$$

It remains, therefore, to estimate the local accommodation stress-rates. There are different possible ways to approximate these quantities.

The simplest way is to neglect the influence of accommodation rates entirely by assuming that

$$\dot{\boldsymbol{\sigma}}_i^{\mathrm{A}} = \dot{\boldsymbol{\sigma}}_0^{\mathrm{A}} . \tag{74}$$

Then, Eq. (73) reduces to that postulated in (Molinari, 2002) for the self-consistent model for elastic-viscoplastic materials, cf. also (Mercier et al., 2005; Mercier and Molinari, 2009).

A more consequent (although still simplified) way is to define the overall elastic accommodation strain-rate and the respective overall accommodation stress-rate as follows

$$\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{A}} = \dot{\boldsymbol{\varepsilon}}_0^v - \langle \dot{\boldsymbol{\varepsilon}}_i^v \rangle = \dot{\boldsymbol{\varepsilon}}_0^v - \langle \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i} \rangle , \qquad \dot{\boldsymbol{\sigma}}_0^{\mathrm{A}} = \bar{\mathbf{L}}^e \cdot \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{A}} . \tag{75}$$

Local accommodation stress-rates can be related to an overall accommodation stress-rate through the viscous concentration tensors $\mathbf{B}_i^v$,

$$\dot{\boldsymbol{\sigma}}_i^{\mathrm{A}} = \mathbf{B}_i^v \cdot \dot{\boldsymbol{\sigma}}_0^{\mathrm{A}} , \tag{76}$$

which is consistent with treating the accommodation stress-rates as constitutively related to *viscous strain accelerations* in the calculation step based on interactions of viscous type, cf. the remark in Section 2.1. Note that Eq. (72) is then satisfied if $\langle \mathbf{B}_i^v \rangle = \mathbf{I}$, which is an identity for ellipsoidal inclusions of equal orientations and aspect ratios. An alternative way, apparently less consequent, is to use elastic concentration tensors instead,

$$\dot{\boldsymbol{\sigma}}_i^{\mathrm{A}} = \mathbf{B}_i^e \cdot \dot{\boldsymbol{\sigma}}_0^{\mathrm{A}} \quad \Longleftrightarrow \quad \dot{\boldsymbol{\varepsilon}}_i^{\mathrm{A}} = \mathbf{A}_i^e \cdot \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{A}}. \tag{77}$$

As an outcome, we obtain three variants of Subproblem A of elastic accommodation in the sequential self-consistent scheme for linearly viscoelastic heterogeneous materials:

- Variant 1 based on Eq. (74),
- Variant 2 based on Eqs (75) and (76),
- Variant 3 based on Eqs (75) and (77).

After straightforward substitutions, the proposed sequential self-consistent scheme for linear viscoelastic heterogeneous materials is summarized in Table 2.

From Eq. (75) for Variants 2 and 3 we obtain

$$\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{A}} = \mathbf{M}_0^v \cdot \boldsymbol{\sigma}_0 - \langle \mathbf{M}_i^v \cdot \boldsymbol{\sigma}_i \rangle. \tag{82}$$

It can easily be verified that

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Table 2: The sequential self-consistent scheme for linear viscoelastic materials

(i) Subproblem V: Determine the external viscous strain-rate $\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i}$ intended to correspond to $\dot{\boldsymbol{\varepsilon}}_i^e = \mathbf{0}$ in the i-th phase in the current stress state, from the equation

$$\dot{\boldsymbol{\varepsilon}}_i^v - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{Vi}} = -\bar{\mathbf{M}}_*^v \cdot (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_0) \qquad \text{with} \quad \dot{\boldsymbol{\varepsilon}}_i^v = \mathbf{M}_i^v \cdot \boldsymbol{\sigma}_i \,. \tag{78}$$

(ii) Subproblem A: Determine the overall elastic accommodation rates of strain and stress from the equations

$$\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{A}} = \bar{\mathbf{M}}^v \cdot \boldsymbol{\sigma}_0 - \langle \dot{\boldsymbol{\varepsilon}}_i^v \rangle, \qquad \dot{\boldsymbol{\sigma}}_0^{\mathrm{A}} = \bar{\mathbf{L}}^e \cdot \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{A}} \tag{79}$$

and define the related internal accommodation rates for each phase by

$$\dot{\boldsymbol{\sigma}}_i^{\mathrm{A}} = \mathbf{B}_i \cdot \dot{\boldsymbol{\sigma}}_0^{\mathrm{A}} \,, \quad \dot{\boldsymbol{\varepsilon}}_i^{\mathrm{A}} = \mathbf{M}_i^e \cdot \dot{\boldsymbol{\sigma}}_i^{\mathrm{A}} \tag{80}$$

using $\mathbf{B}_i = \mathbf{I}$, $\mathbf{B}_i = \mathbf{B}_i^v$ or $\mathbf{B}_i = \mathbf{B}_i^e$ in Variant 1, 2 or 3, respectively.

(iii) Subproblem E: Determine the elastic strain-rate $\dot{\boldsymbol{\varepsilon}}_i^e$ and the corresponding stress-rate $\dot{\boldsymbol{\sigma}}_i$ in the i-th phase, from the equations

$$\dot{\boldsymbol{\varepsilon}}_i^e - \dot{\boldsymbol{\varepsilon}}_0 + \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i} = -\bar{\mathbf{M}}_*^e \cdot (\dot{\boldsymbol{\sigma}}_i - \dot{\boldsymbol{\sigma}}_i^{\mathrm{A}} - \dot{\boldsymbol{\sigma}}_0 + \dot{\boldsymbol{\sigma}}_0^{\mathrm{A}}) \,, \quad \dot{\boldsymbol{\sigma}}_i = \mathbf{L}_i^e \cdot \dot{\boldsymbol{\varepsilon}}_i^e \,, \quad \dot{\boldsymbol{\sigma}}_0 = \langle \dot{\boldsymbol{\sigma}}_i \rangle \,. \tag{81}$$

(iv) Select a time step, update $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}_0$, and continue.

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- when viscous properties are uniform then $\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{A}} = \mathbf{0}$ and both Variants 2 and 3 reduce to Variant 1,
- when elastic properties are uniform then $\dot{\boldsymbol{\sigma}}_i^{\mathrm{A}} = \dot{\boldsymbol{\sigma}}_0^{\mathrm{A}}$ and Variant 3 reduces to Variant 1,
- in the limit when volume fraction of one phase tends to 1, predictions of all three Variants 1, 2 and 3 tend to those of Variant II for an inhomogeneity embedded in an infinite matrix analysed in the preceding Section.

In general, the three variants provide different predictions. They are compared below by an illustrative example.

3.2. Comparison of results

The model is tested for a two-phase incompressible material (data are the same as in (Paquin et al., 1999) and (Mercier and Molinari, 2009)) subjected to a tension-compression cycle under the constraint of constant overall volume. The overall strain-rate is prescribed in the form

$$\dot{\boldsymbol{\varepsilon}}_0(t) = \dot{\varepsilon}(t)\mathbf{N}_{\mathrm{D}}, \quad \text{where} \quad \begin{cases} \dot{\varepsilon}(t) = 0 & \text{if} \quad t < 0 \\ \dot{\varepsilon}(t) = 10^{-4} & \text{if} \quad 0 \le t < T \\ \dot{\varepsilon}(t) = -10^{-4} & \text{if} \quad t \ge T \end{cases} \quad [1/\mathrm{s}]\,, \tag{83}$$

where $\mathbf{N}_{\mathrm{D}}$ is specified by Eq. (58). Both phases are isotropic with compliances of phases specified by Eq. (38) with $h_i^{e\mathrm{D}} = h_i^e$ being a doubled elastic shear modulus and h_i^v a doubled viscosity modulus; a spherical shape of inhomogeneities is assumed. In the first example, the following values of these material parameters are examined:

$$h_1^e = 100\,\mathrm{MPa}, \quad h_2^e = 500\,\mathrm{MPa}, \quad h_1^v = 20\,\mathrm{MPa} \times \mathrm{s}, \quad h_2^v = 2000\,\mathrm{MPa} \times \mathrm{s}\,, \tag{84}$$

for volume fractions of phase 1 equal to $c_1 = 0.1, 0.25, 0.5, 0.75$.

Note that phase 1 as above is less stiff than phase 2 with respect to the elastic and viscous compliances. Other cases have also been examined, namely, uniform viscosity with the material parameters

$$h_1^e = 100\,\mathrm{MPa}, \quad h_2^e = 2000\,\mathrm{MPa}, \quad h_1^v = h_2^v = 2000\,\mathrm{MPa} \times \mathrm{s}\,, \tag{85}$$

and uniform elasticity with

$$h_1^e = h_2^e = 100\,\mathrm{MPa}, \quad h_1^v = 20\,\mathrm{MPa} \times \mathrm{s}, \quad h_2^v = 2000\,\mathrm{MPa} \times \mathrm{s}\,. \tag{86}$$

In both cases $c_1 = 0.5$ is assumed.

In Fig. 5 the calculated overall stress-strain curves are presented for non-uniform viscous and elastic properties and different volume fractions of phase 1. It has been found that differences between the predictions of the analyzed models are best visible in the case of equal volume fractions of phases, therefore in Fig. 6 the evolution of local stresses in phases is shown only for this case. Fig. 7 includes overall stress-strain curves for the cases of uniform elasticity and uniform viscosity. For equal volume fractions of the phases, the

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results obtained with the use of the proposed variants (V1,V2,V3) of the self-consistent model employing the sequential linearization method are compared to the exact[5] solution (R) of Rougier et al. (1993) found with the use of Laplace transform, and to the predictions of the Paquin et al. (1999) model (PQ). In the latter case the concentration relation for the average strain-rate in phase i is of quite a complicated form.

The conclusions from the analysis are the following.

- As it can be seen in Figs 5a and 6, in the case of non-uniform elastic and viscous properties and equal volume fractions of the phases ($c_1 = 0.5$), Variant 2 gives a response similar to that for the Paquin model and clearly softer than that for Variant 1 (equivalent to the Molinari (2002) model). The former two responses are also closer to, although softer than, the exact solution provided by Rougier et al. (1993). Variant 3 is also satisfactory in this case, although the predicted response is stiffer than in the case of the exact solution.

- Very good agreement of Variant 2 with the exact macroscopic solution, in comparison with the remaining variants of the sequential model, is observed for the case of uniform elastic properties (Fig. 7). When the viscous properties are uniform then all three variants collapse into one (thus coincide with the Molinari (2002) model) and provide good predictions. The Paquin model is also successful in both cases.

- When the volume fraction of one phase is noticeably smaller than that of a second phase then the differences between the macroscopic predictions for the examined models are less visible, cf. Fig. 5a-d. If the volume fraction of one phase tends to zero then the problem examined here tends to that of a single inhomogeneity embedded in an infinite matrix, and therefore the predicted responses converge to the exact solution.

The uniform elasticity case is worth being analyzed in more detail. It can be seen that in this case the predictions of Variant 1 and 3 are not satisfactory in comparison with the exact solution. This supports the viewpoint that the reduction of the overall stress level within the accommodation step should be related to viscous non-uniformity, as assumed in Variant 2 of the model. It has been checked that, for the case considered, the accommodation strain-rate defined by Eq. (75) has an opposite direction than the imposed total strain-rate, and within a half of cycle its magnitude increases from zero to some maximum value and then decreases again towards zero in the purely viscous limit. The viscous redistribution of the corresponding accommodation stress-rate between the phases, cf. Eq. (76) and Fig. 8b and d, causes the higher (lower) reduction of the stress level in the stiffer (softer) phase, respectively, than in the case of the remaining variants. It leads to the acceleration of the viscous flow in the softer phase and to a softer overall response of the composite. The evolution of accommodation terms in Variants 2 and 3 in the case of non-uniform viscous and elastic properties is presented in Fig. 8a and c.

On the basis of the analysis performed, Variant 2 of the proposed model is recommended for further use. Variant 1 (equivalent to the Molinari (2002) model), due to its

[5]Of course, exact only within the approximations inherent to the self-consistent scheme itself.

relative simplicity, is also worth considering, especially in the case of a small contrast in viscous properties of the phases. It is worth emphasizing that the softening of the macroscopic response in the transient regime, observed for the recommended Variant 2 in comparison with Variant 1 and making the predictions of the former variant closer to the exact solution, has been achieved here without introducing any additional tuning parameters. The latter technique has been used in (Mercier and Molinari, 2009) in order to improve the Molinari (2002) model predictions in this respect.

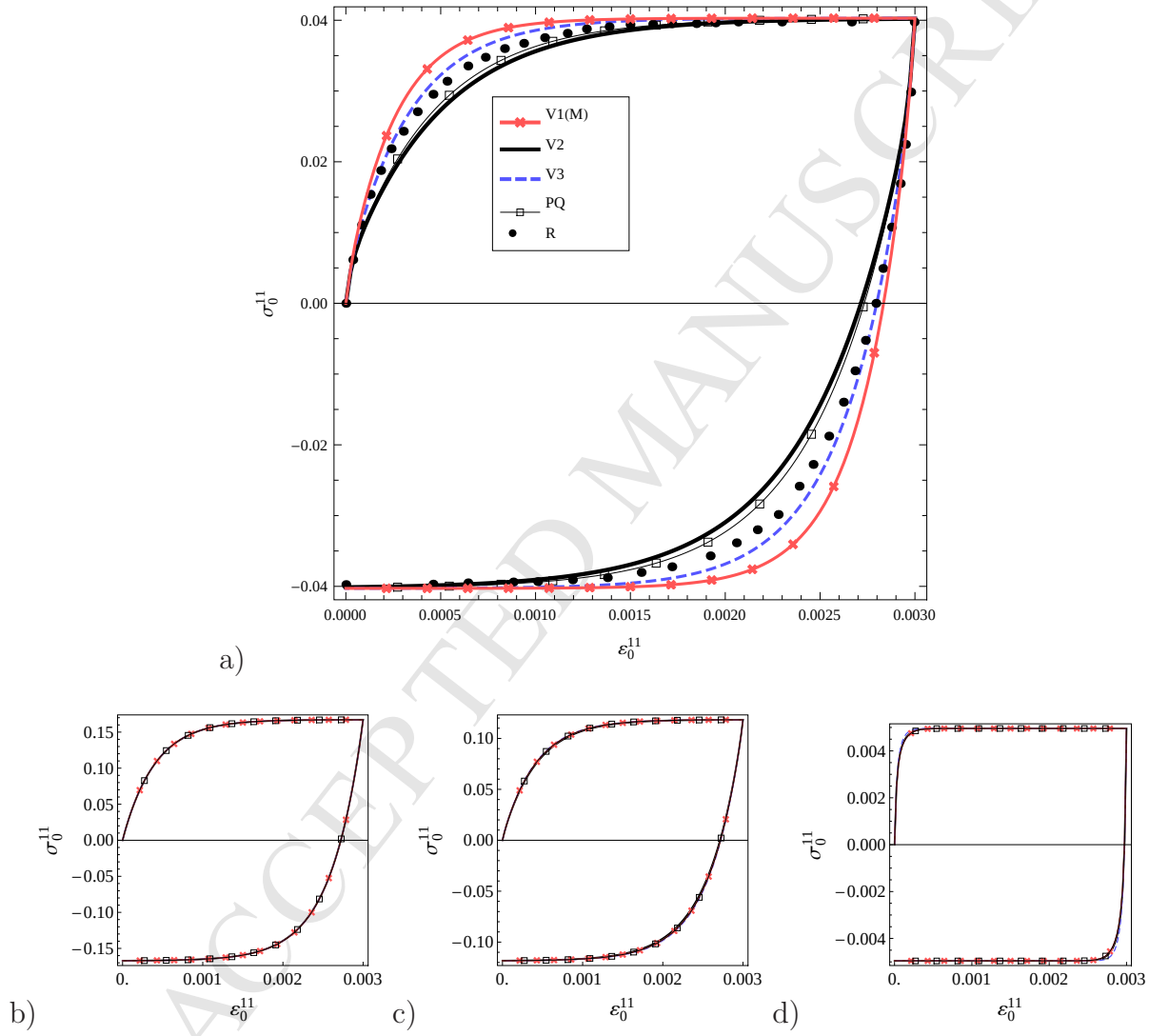

Figure 5: Averaged stress-strain response calculated by different variants of the self-consistent approach for a two-phase material of non-uniform elastic and viscous properties: a) $c_1 = 0.5$, b) $c_1 = 0.1$, c) $c_1 = 0.25$, d) $c_1 = 0.75$.

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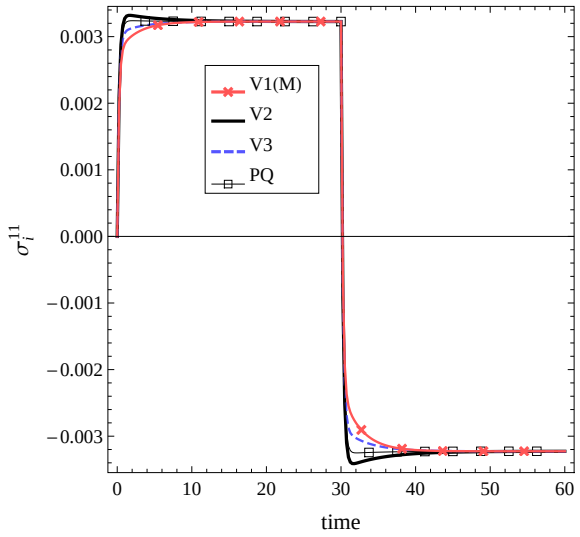
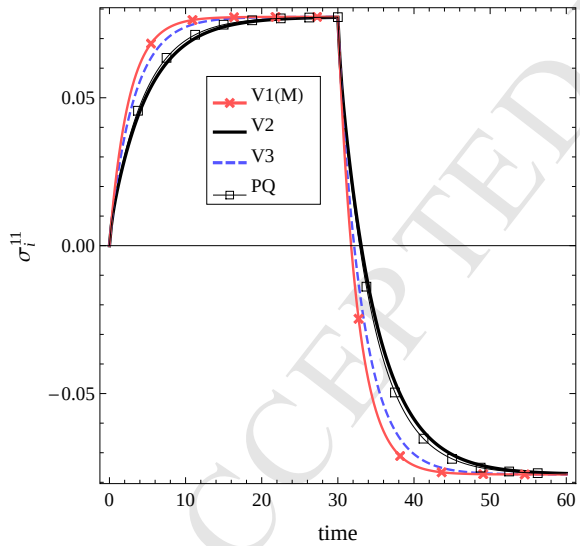


a)



b)

Figure 6: Stress evolution in phase 1 (a) and phase 2 (b) of a two-phase material of non-uniform elastic and viscous properties, corresponding to Fig.5a ($c_1 = 0.5$).

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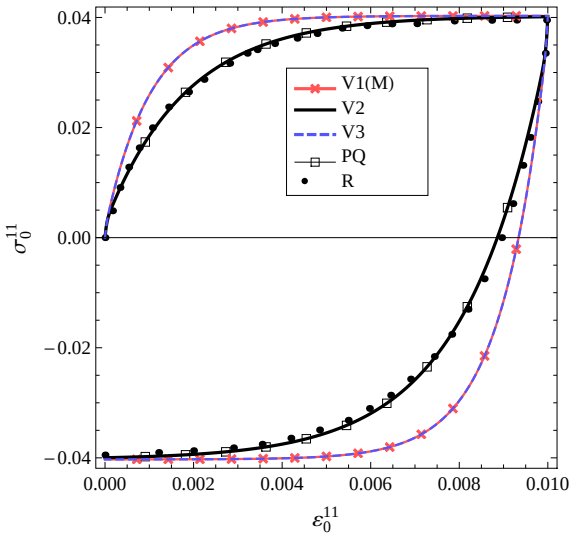
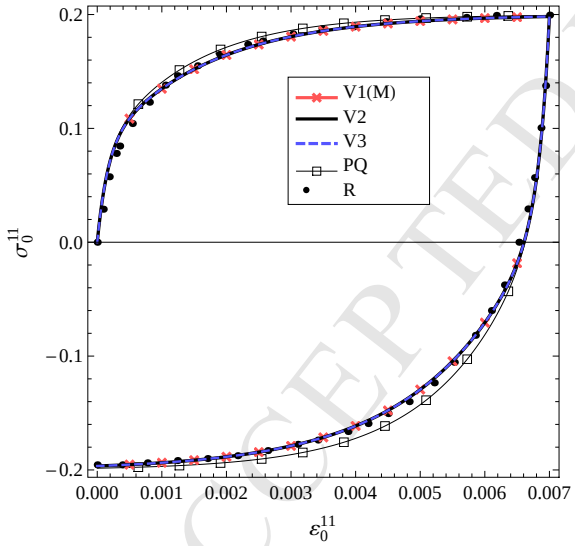


a)



b)

Figure 7: Averaged stress-strain response calculated by different variants of the self-consistent approach for a two-phase material: a) uniform elastic properties (V1=V3), b) uniform viscous properties (V1=V2=V3); $c_1 = 0.5$.

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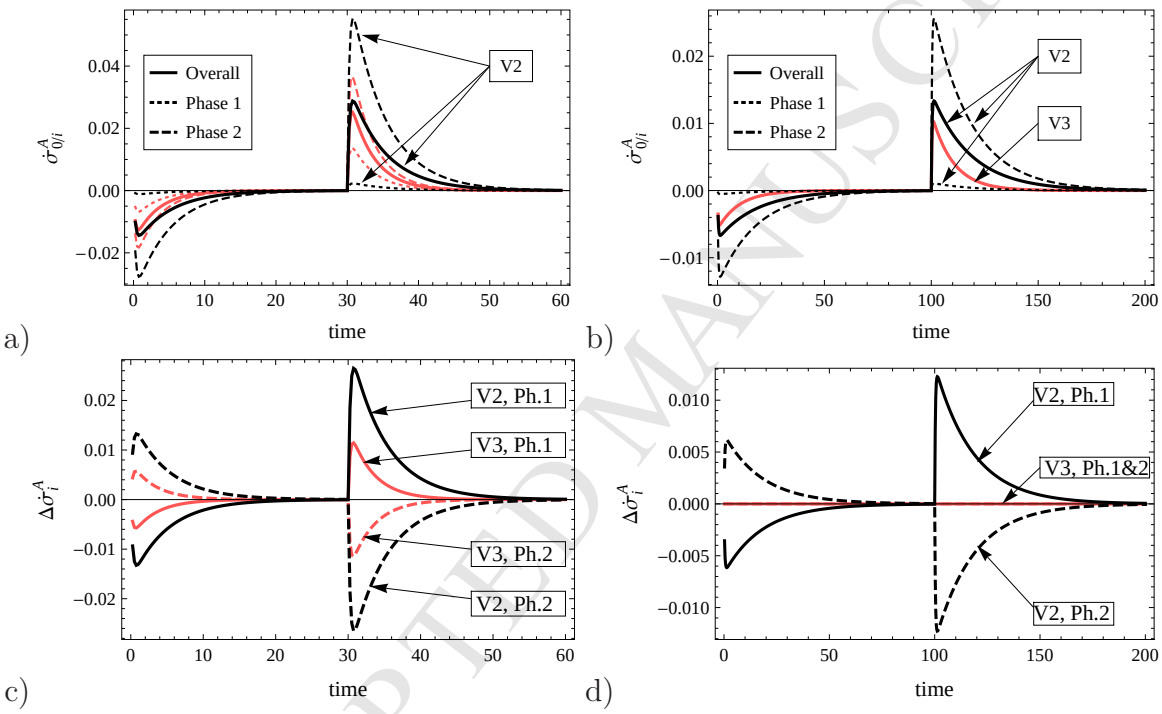

Figure 8: Evolution of local and overall accommodation stress-rates for Variant 2 and Variant 3 of the sequential self-consistent model: a) non-uniform elastic and viscous properties, b) uniform elastic properties, c) difference $\Delta\dot{\sigma}_i^A = \dot{\sigma}_0^A - \dot{\sigma}_i^A$ (identically zero in Variant 1) for non-uniform elastic and viscous properties, d) $\Delta\dot{\sigma}_i^A$ for uniform elastic properties; $c_1 = 0.5$.

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3.3. Extension to non-linear viscosity

The next step is to extend the model to the case of non-linear viscosity,

$$\dot{\boldsymbol{\varepsilon}}_i^v = \mathbf{f}(\boldsymbol{\sigma}_i). \tag{87}$$

That extension is not straightforward. In Subproblem V of the sequential approach, a purely viscous problem is solved by using the Eshelby method. When the constitutive law describing viscous behaviour becomes non-linear, some method of linearization of the constitutive law has to be employed. As it has been mentioned in the introduction, different linearization schemes developed for the rigid-viscoplastic materials can be found in the literature. Three schemes most widely discussed in the literature were proposed by Hutchinson (1976), Molinari et al. (1987) and Masson et al. (2000). We use here the Masson et al. (2000) proposal which seems to provide the most accurate predictions (Lebensohn et al., 2004). In the selected linearization scheme the viscous tangent compliance tensor of a homogenized matrix is determined. Consequently, Subproblem V takes the form

$$\dot{\boldsymbol{\varepsilon}}_i^v - \dot{\boldsymbol{\varepsilon}}_0^{\mathrm{V}i} = -\bar{\mathbf{M}}_*^{v\mathrm{TG}} \cdot (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_0)\,, \tag{88}$$

where

$$\dot{\boldsymbol{\varepsilon}}_i^v = \mathbf{M}_i^{v\mathrm{TG}} \cdot \boldsymbol{\sigma}_i + \dot{\boldsymbol{\varepsilon}}_i^{\mathrm{res}}\,, \quad \mathbf{M}_i^{v\mathrm{TG}} = \frac{\partial \mathbf{f}(\boldsymbol{\sigma}_i)}{\partial \boldsymbol{\sigma}_i}\,, \quad \dot{\boldsymbol{\varepsilon}}_i^{\mathrm{res}} = \mathbf{f}(\boldsymbol{\sigma}_i) - \mathbf{M}_i^{v\mathrm{TG}} \cdot \boldsymbol{\sigma}_i \tag{89}$$

and

$$\bar{\mathbf{M}}^{v\mathrm{TG}} = \langle \mathbf{M}_i^{v\mathrm{TG}} \mathbf{B}_i^{v\mathrm{TG}} \rangle\,, \quad \mathbf{B}_i^{v\mathrm{TG}} = (\mathbf{M}_i^{v\mathrm{TG}} + \bar{\mathbf{M}}_*^{v\mathrm{TG}})^{-1} (\bar{\mathbf{M}}^{v\mathrm{TG}} + \bar{\mathbf{M}}_*^{v\mathrm{TG}})\,. \tag{90}$$

Additionally, following the tangent linearization scheme, it is assumed that

$$\dot{\bar{\boldsymbol{\varepsilon}}}^{\mathrm{res}} = \langle \dot{\boldsymbol{\varepsilon}}_i^{\mathrm{res}} \cdot \mathbf{B}_i^{v\mathrm{TG}} \rangle\,. \tag{91}$$

The Hill tensor inverse $\bar{\mathbf{M}}_*^{v\mathrm{TG}}$ depends on $\bar{\mathbf{M}}^{v\mathrm{TG}}$ and the shape of an inhomogeneity according to the formulae (95) and (96) in the Appendix.

Presence of viscous non-linearity requires also reconsideration of Subproblem A. In the case of Variant 1 we use the formula (74) as previously. Variant 2 requires a definition of the overall accommodation strain-rate and respective accommodation stress-rate. Following the reasoning employed previously, we define

$$\dot{\boldsymbol{\sigma}}_0^{\mathrm{A}} = \bar{\mathbf{L}}^e \cdot (\bar{\mathbf{M}}^{v\mathrm{TG}} \cdot \boldsymbol{\sigma}_0 + \dot{\bar{\boldsymbol{\varepsilon}}}^{\mathrm{res}} - \langle \dot{\boldsymbol{\varepsilon}}_i^v \rangle)\,, \quad \dot{\boldsymbol{\sigma}}_i^{\mathrm{A}} = \mathbf{B}_i^{v\mathrm{TG}} \cdot \dot{\boldsymbol{\sigma}}_0^{\mathrm{A}}\,. \tag{92}$$

Numerical tests for moderate non-linearity[6] of the viscosity law have been performed, that is, for a power-type relationship (87) with an exponent equal to 1.5. The results for a two-phase material in uniaxial tension obtained by using two variants of the sequential self-consistent (SC) scheme are shown in Fig. 9. To enable comparison to other approaches,

[6]The accommodation step in Variant 2 requires further re-consideration in the case of highly non-linear problems. Preliminary calculations have indicated that formula (92) at high non-linearity can introduce too strong reduction of the total instantaneous tangent modulus which can lead to overall softening of the homogenized material. This problem requires further study.

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the constitutive law has been taken the same as that used by Pierard et al. (2007) and Mercier and Molinari (2009); details can be found in these references. A spherical shape of inhomogeneities has been assumed. It can be seen in the Figures that the difference between Variant 2 and Variant 1 (the latter equivalent to the Molinari model) is hardly visible. The reason for that lies in a relatively small contrast in viscous properties of the phases. As already shown in (Mercier and Molinari, 2009), in the examined case the Mori-Tanaka (MT) averaging scheme applied jointly with the Molinari model (here Variant II of the sequential method) is in the excellent agreement with FEM results reported in (Pierard et al., 2007) as concerns the average response of the composite, while the self-consistent scheme slightly overestimates the stress level. In turn, the local stress in the stronger phase is somewhat better predicted by the self-consistent model, especially at the advanced stage of the process.

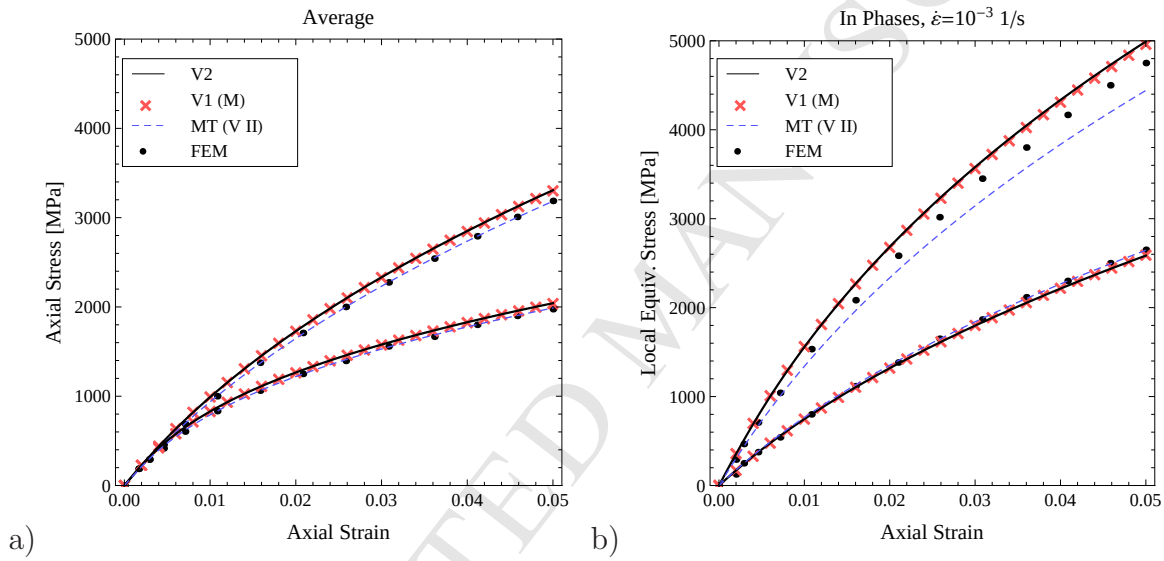

Figure 9: Comparison of the response predicted by the sequential SC and MT models with FEM results taken from (Pierard et al., 2007) for a viscously non-linear two-phase composite in uniaxial tension, $c_i = 0.3$: a) stress-strain curve for the composite at axial strain-rate $\dot{\varepsilon} = 10^{-3}1/s$ (top curves) and $\dot{\varepsilon} = 10^{-4}1/s$ (bottom curves), b) stress-strain curve in phases for $\dot{\varepsilon} = 10^{-3}1/s$ (top curves - inhomogeneity, bottom curves - matrix).

4. Conclusions

In this paper, a new approach has been proposed to predict the behaviour of viscous/elastic heterogeneous materials. Instead of incorporating both elastic and viscous properties of the constituents in a *single* computational step, we propose to use elastic and viscous properties within an applied averaging scheme *sequentially*. The goal is to obtain an efficient homogenization scheme that is able to generate results comparable to the approaches that require much higher computational effort.

First, the model has been evaluated for the classical problem of an ellipsoidal inhomogeneity embedded in an infinite matrix. Predictions of the outlined variants of the

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proposed method have been compared with other models available in the literature, and with the exact analytical solution obtained with use of the Laplace transform and the correspondence principle, in the case of isotropic materials that are viscously and elastically linear and elastically compressible. On the basis of this analysis, the best variant of the sequential decomposition of elastic and viscous responses has been selected for further application into the averaging schemes for heterogeneous materials. It has been found that the variant selected for the inhomogeneity problem is equivalent for that particular problem to the Molinari (2002) model. The coincidence is not automatically extended to the self-consistent treatment of a composite.

The sequential linearization concept can be combined with various averaging schemes. The extension to the Mori-Tanaka averaging scheme for a composite is immediate. The major problem in this paper has been to adapt the sequential linearization approach to the self-consistent averaging scheme for composite materials. In this case, the sequential approach consists of solving *three* subproblems. It has been demonstrated that an additional accommodation step, besides the viscous and elastic subproblems, is needed in order to estimate the material response satisfactorily. By examples of an isotropic two-phase material we have shown that the proposed method provides acceptable predictions in comparison to the existing models.

Acknowledgment

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Appendix

Some well-known formulae for the micro-macro transition in a heterogeneous material are summarized for convenience. Assume an affine (linear with eigenstrains $\boldsymbol{\varepsilon}^{\mathrm{res}}$) relationship between strain and stress tensors,

$$\boldsymbol{\sigma} = \mathbf{L} \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{res}}), \quad \boldsymbol{\varepsilon} = \mathbf{M} \cdot \boldsymbol{\sigma} + \boldsymbol{\varepsilon}^{\mathrm{res}}, \quad \mathbf{LM} = \mathbf{I}, \tag{93}$$

where $\mathbf{L}$ and $\mathbf{M} \equiv \mathbf{L}^{-1}$ are given fourth-order stiffness and compliance tensors, respectively, while $\mathbf{I}$ is the symmetrized fourth-order identity tensor, $I_{ijkl} = (1/2)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. (For a viscous material, strain is to be replaced by strain-rate.) The analytical (Eshelby) solution for an ellipsoidal inclusion embedded in an infinite medium (matrix) of constitutive relations analogous to (93), with uniform stiffness moduli $\mathbf{L}_0$ and uniform residual strain $\boldsymbol{\varepsilon}_0^{\mathrm{res}}$, provides the following fundamental relationship between the stress $\boldsymbol{\sigma}_0$ and strain $\boldsymbol{\varepsilon}_0$ at infinity and their local counterparts,

$$\boldsymbol{\sigma} - \boldsymbol{\sigma}_0 = -\mathbf{L}_* \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0), \tag{94}$$

where $\mathbf{L}_*$ is called the Hill tensor (Hill, 1965b). This fourth-order tensor is calculated as follows

$$\mathbf{L}_* = \mathbf{P}^{-1} - \mathbf{L}_0 \tag{95}$$

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and the polarisation tensor $\mathbf{P}$ is specified by the shape of ellipsoidal inhomogeneity and the matrix properties, namely (Willis, 1981)

$$\mathbf{P} = \frac{1}{4\pi abc} \int_{|\boldsymbol{\nu}|=1} |\mathbf{a}^{-1} \cdot \boldsymbol{\nu}|^{-3} \left(\boldsymbol{\nu} \otimes (\boldsymbol{\nu} \mathbf{L}_0 \boldsymbol{\nu})^{-1} \otimes \boldsymbol{\nu} \right)^{\mathrm{sym}} dS, \tag{96}$$

where $\mathbf{a}$ is the second-order tensor of principal values a, b, c equal to the ellipsoid semi-axes (for a spherical inhomogeneity $a = b = c$ and $\mathbf{a} = a\mathbf{I}$), and the integration is over the surface of the unit sphere $|\boldsymbol{\nu}| = 1$. The Eshelby tensor $\mathbf{S}$ reads

$$\mathbf{S} = \mathbf{P}\mathbf{L}_0 = (\mathbf{I} + \mathbf{M}_0 \mathbf{L}_*)^{-1} = \mathbf{M}_* (\mathbf{M}_0 + \mathbf{M}_*)^{-1},$$

on using the inverses $\mathbf{M}_0 = \mathbf{L}_0^{-1}$, $\mathbf{M}_* = \mathbf{L}_*^{-1}$.

On substituting the constitutive equations for the inhomogeneity and matrix into Eq. (94), the following relationships are derived:

$$\boldsymbol{\sigma} = \mathbf{B} \cdot \boldsymbol{\sigma}_0 + (\mathbf{M} + \mathbf{M}_*)^{-1} \cdot (\boldsymbol{\varepsilon}_0^{\mathrm{res}} - \boldsymbol{\varepsilon}^{\mathrm{res}}), \tag{97}$$

$$\boldsymbol{\varepsilon} = \mathbf{A} \cdot \boldsymbol{\varepsilon}_0 - (\mathbf{L} + \mathbf{L}_*)^{-1} \cdot (\mathbf{L}_0 \cdot \boldsymbol{\varepsilon}_0^{\mathrm{res}} - \mathbf{L} \cdot \boldsymbol{\varepsilon}^{\mathrm{res}}), \tag{98}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the strain and stress concentration tensors, respectively, specified as follows:

$$\mathbf{A} = (\mathbf{L} + \mathbf{L}_*)^{-1} (\mathbf{L}_0 + \mathbf{L}_*) = (\mathbf{I} + \mathbf{P}(\mathbf{L} - \mathbf{L}_0))^{-1}, \tag{99}$$

$$\mathbf{B} = (\mathbf{M} + \mathbf{M}_*)^{-1} (\mathbf{M}_0 + \mathbf{M}_*). \tag{100}$$

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