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A SHARPER ENERGY METHOD FOR THE LOCALIZATION OF THE SUPPORT TO SOME STATIONARY SCHRÖDINGER EQUATIONS WITH A SINGULAR NONLINEARITY

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Abstract

We prove the compactness of the support of the solution of some stationary Schrödinger equations with a singular nonlinear order term. We present here a sharper version of some energy methods previously used in the literature and, in particular, by the authors.

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1 Introduction

Since the beginnings of the eighties of the last century, it is already well-known that the absence of the maximum principle for the case of systems and higher order nonlinear partial differential equations was one of the main motivations of the introduction of suitable energy methods allowing to conclude the compactness of the support of their solutions (see, e.g., the presentation made in the monograph Antontsev, Díaz and Shmarev [1]).

The application of such type of methods to the case of nonlinear Schrödinger equations with a singular zero order term required some important improvements of the method. That was the main object of the author’s papers of Bégout and Díaz [4, 5].

The main goal of this paper is to present a sharper version of the mentioned method potentially able to be applied to many other problems related to this type of Schrödinger equations such as the study of self-similar solutions, case of Neumann boundary conditions, presence of nonlocal terms (such as, for instance, in Hartree-Fock theory in Cazenave [6]), etc. As a matter of fact, the concrete application of this sharper energy method to the concrete case of self-similar solutions of the evolution Schrödinger problem requires many additional arguments justifying the special structure of those solutions, reason why we decided to present it in a separated work (Bégout and Díaz [3]). We send the reader to Bégout and Díaz [3] for a long description of the important role of the compactness of the solution in this context and for many other references related to this qualitative property of the solution.

This paper is organized as follows. Below, we give some notations which will be used throughout this paper. In Section 2, we give the precise “localization” estimates which imply a solution of a partial differential equation to be compactly supported (see Theorems 2.1 and 2.2, and especially estimates (2.1) and (2.3)). In Section 3, we give a tool which permits, from a solution of some partial differential equation, to establish the “localization” estimate (Theorem 3.1). The results of these two sections are proved in Section 4. In Bégout and Díaz [5], localization property is studied for equation

$$-\Delta u + a|u|^{-(1-m)}u + bu = F, \text{ in } \Omega. \tag{1.1}$$

We also study this property here, but with a change of notation. Section 7 helps to understand this new notation (see also Comments 5.1 below for the motivation of this change). Section 5 is devoted to the study of the localization property of the solutions of equation (1.1), in the same spirit as in Bégout

and Díaz [5], but with the homogeneous Neumann boundary condition instead of the homogeneous Dirichlet boundary condition (compare Theorem 5.5 below with Theorem 3.5 in Bégout and Díaz [5]). Finally, Section 6 is concerned by equation (1.1) with the homogeneous Dirichlet boundary condition. We state the same results as in Bégout and Díaz [5], but with the weaker assumption $F \in L^2(\Omega)$ ¹.

Before ending this section, we shall indicate here some of the notations used throughout. We write $i^2 = -1$. We denote by \bar{z} the conjugate of the complex number z , by $\operatorname{Re}(z)$ its real part and by $\operatorname{Im}(z)$ its imaginary part. For $1 \leq p \leq \infty$, p' is the conjugate of p defined by $\frac{1}{p} + \frac{1}{p'} = 1$. Let $j, k \in \mathbb{Z}$ with $j < k$. We then write $\llbracket j, k \rrbracket = [j, k] \cap \mathbb{Z}$. We denote by Γ the boundary of a nonempty subset $\Omega \subseteq \mathbb{R}^N$, $\bar{\Omega}$ its closure, $\Omega^c = \mathbb{R}^N \setminus \Omega$ its complement. Unless if specified, any function lying in a functional space ($L^p(\Omega)$, $W^{m,p}(\Omega)$, etc) is supposed to be a complex-valued function ($L^p(\Omega; \mathbb{C})$, $W^{m,p}(\Omega; \mathbb{C})$, etc). For a Banach space E , we denote by E^* its topological dual and by $\langle \cdot, \cdot \rangle_{E^*, E} \in \mathbb{R}$ the $E^* - E$ duality product. In particular, for any $T \in L^{p'}(\Omega)$ and $\varphi \in L^p(\Omega)$ with $1 \leq p < \infty$, $\langle T, \varphi \rangle_{L^{p'}(\Omega), L^p(\Omega)} = \operatorname{Re} \int_{\Omega} T(x) \overline{\varphi(x)} dx$. As usual, we denote by C auxiliary positive constants, and sometimes, for positive parameters a_1, \dots, a_n , write $C(a_1, \dots, a_n)$ to indicate that the constant C continuously depends only on a_1, \dots, a_n (this convention also holds for constants which are not denoted by “ C ”).

2 From suitable local inequalities to the vanishing of the involved complex functions on some small ball

In this section, we establish some results improving the presentation of some energy methods of Antontsev, Díaz and Shmarev [1] which allow to prove localization properties of solutions of a general class of nonlinear partial differential equations (Sections 5, 6 below and Bégout and Díaz [3]).

Theorem 2.1. *Assume $0 < m < 1$ and let $N \in \mathbb{N}$. Then there exists $C = C(N, m)$ satisfying the following property: let $x_0 \in \mathbb{R}^N$, $\rho_0 > 0$ and $u \in H_{\text{loc}}^1(B(x_0, \rho_0))$. If there exist $L > 0$ and $M > 0$ such that for almost every $\rho \in (0, \rho_0)$,*

$$\|\nabla u\|_{L^2(B(x_0, \rho))}^2 + L \|u\|_{L^{m+1}(B(x_0, \rho))}^{m+1} \leq M \left| \int_{\mathbb{S}(x_0, \rho)} u \overline{\nabla u} \cdot \frac{x - x_0}{|x - x_0|} d\sigma \right|, \quad (2.1)$$

¹in [5], F belongs to $L^{\frac{m+1}{m}}(\Omega) \supseteq L^2(\Omega)$, when $|\Omega| < \infty$ and $m \in (0, 1)$.

then $u|_{B(x_0, \rho_{\max})} \equiv 0$, where

$$\rho_{\max}^\nu = \left(\rho_0^\nu - CM^2 \max \left\{ 1, \frac{1}{L^2} \right\} \max \{ \rho_0^{\nu-1}, 1 \} \right. \\ \left. \times \min_{\tau \in (\frac{m+1}{2}, 1]} \left\{ \frac{E(\rho_0)^{\gamma(\tau)} \max \{ b(\rho_0)^{\mu(\tau)}, b(\rho_0)^{\eta(\tau)} \}}{2\tau - (1+m)} \right\} \right)_+, \quad (2.2)$$

where,

$$E(\rho_0) = \|\nabla u\|_{L^2(B(x_0, \rho_0))}^2, \quad b(\rho_0) = \|u\|_{L^{m+1}(B(x_0, \rho_0))}^{m+1}, \\ k = 2(1+m) + N(1-m), \quad \nu = \frac{k}{m+1} > 2,$$

and where

$$\gamma(\tau) = \frac{2\tau - (1+m)}{k} \in (0, 1), \quad \mu(\tau) = \frac{2(1-\tau)}{k}, \quad \eta(\tau) = \frac{1-m}{1+m} - \gamma(\tau) > 0.$$

for any $\tau \in (\frac{m+1}{2}, 1]$.

Here and in what follows, $r_+ = \max\{0, r\}$ denotes the positive part of the real number r . For $x_0 \in \mathbb{R}^N$ and $r > 0$, $B(x_0, r)$ is the open ball of \mathbb{R}^N of center x_0 and radius r , $\mathbb{S}(x_0, r)$ is its boundary and $\overline{B}(x_0, r)$ is its closure. Finally, σ is the surface measure on a sphere.

Theorem 2.2. *Assume $0 < m < 1$. Let $x_0 \in \mathbb{R}^N$, $\rho_1 > \rho_0 > 0$, $F \in L^2(B(x_0, \rho_1))$ and $u \in H_{\text{loc}}^1(B(x_0, \rho_1))$. If there exist $L > 0$ and $M > 0$ such that for almost every $\rho \in (0, \rho_1)$,*

$$\|\nabla u\|_{L^2(B(x_0, \rho))}^2 + L\|u\|_{L^{m+1}(B(x_0, \rho))}^{m+1} + L\|u\|_{L^2(B(x_0, \rho))}^2 \\ \leq M \left(\left| \int_{\mathbb{S}(x_0, \rho)} u \overline{\nabla u} \cdot \frac{x - x_0}{|x - x_0|} d\sigma \right| + \int_{B(x_0, \rho)} |F(x)u(x)| dx \right), \quad (2.3)$$

then there exist $E_\star > 0$ and $\varepsilon_\star > 0$ satisfying the following property: if $\|\nabla u\|_{L^2(B(x_0, \rho_1))}^2 < E_\star$ and

$$\|F\|_{L^2(B(x_0, \rho))}^2 \leq \varepsilon_\star ((\rho - \rho_0)_+)^p, \quad \forall \rho \in (0, \rho_1), \quad (2.4)$$

where $p = \frac{2(1+m)+N(1-m)}{1-m}$, then $u|_{B(x_0, \rho_0)} \equiv 0$. In other words, with the notation of Theorem 2.1,

$\rho_{\max} = \rho_0$.

Remark 2.3. We may estimate E_\star and ε_\star as

$$E_\star = E_\star \left(\|u\|_{L^{m+1}(B(x_0, \rho_1))}^{-1}, \rho_1, \frac{\rho_0}{\rho_1}, \frac{L}{M}, N, m \right), \\ \varepsilon_\star = \varepsilon_\star \left(\|u\|_{L^{m+1}(B(x_0, \rho_1))}^{-1}, \frac{\rho_0}{\rho_1}, \frac{L}{M}, N, m \right).$$

The dependence on $\frac{1}{\delta}$ means that if δ goes to 0 then E_\star and ε_\star may be very large. Note that $p = \frac{1}{\gamma(1)}$, where γ is the function defined in Theorem 2.1.

Remark 2.4. Note that by Cauchy-Schwarz's inequality, the right-hand side in (2.1) belongs to $L^1_{\text{loc}}([0, \rho_0]; \mathbb{R})$ and so is defined almost everywhere in $(0, \rho_0)$. Consequently, by Hölder's inequality, the right-hand side in (2.3) is defined almost everywhere in $(0, \rho_0)$.

3 A general framework of applications related to the Schrödinger operator

The following result will be applied later to many concrete equations associated to the Schrödinger operator.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a nonempty open subset of \mathbb{R}^N , let $x_0 \in \Omega$, let $\rho_0 > 0$, let $1 \leq p_1, \dots, p_{n_1}, q_1, \dots, q_{n_2} < \infty$, let $F \in L^1_{\text{loc}}(\Omega)$ be such that $F|_{\Omega \cap B(x_0, \rho_0)} \in L^2(\Omega \cap B(x_0, \rho_0))$ and let*

$$f \in C \left(\bigcap_{k=1}^{n_2} L^{q_k}_{\text{loc}}(\Omega); \sum_{j=1}^{n_1} L^{p'_j}_{\text{loc}}(\Omega) \right).$$

Let $u \in H^1_{\text{loc}}(\Omega) \cap L^{p_j}_{\text{loc}}(\Omega) \cap L^{q_k}_{\text{loc}}(\Omega)$, for any $(j, k) \in \llbracket 1, n_1 \rrbracket \times \llbracket 1, n_2 \rrbracket$, be any solution to

$$-\Delta u + f(u) = F, \text{ in } \mathcal{D}'(\Omega). \quad (3.1)$$

If $\rho_0 > \text{dist}(x_0, \Gamma)$ then assume further that

$$f \in C \left(\bigcap_{k=1}^{n_2} L^{q_k}(\Omega); \sum_{j=1}^{n_1} L^{p'_j}(\Omega) \right), \quad u \in H^1_0(\Omega),$$

$$u|_{\Omega \cap B(x_0, \rho_0)} \in L^{p_j}(\Omega \cap B(x_0, \rho_0)) \cap L^{q_k}(\Omega \cap B(x_0, \rho_0)),$$

for any $(j, k) \in \llbracket 1, n_1 \rrbracket \times \llbracket 1, n_2 \rrbracket$. Set for every $\rho \in [0, \rho_0)$,

$$I(\rho) = \left| \int_{\Omega \cap \mathbb{S}(x_0, \rho)} u \overline{\nabla u} \cdot \frac{x - x_0}{|x - x_0|} d\sigma \right|, \quad J(\rho) = \int_{\Omega \cap B(x_0, \rho)} |F(x)u(x)| dx, \quad (3.2)$$

$$w(\rho) = \int_{\Omega \cap \mathbb{S}(x_0, \rho)} u \overline{\nabla u} \cdot \frac{x - x_0}{|x - x_0|} d\sigma, \quad I_{\text{Re}}(\rho) = \text{Re}(w(\rho)), \quad I_{\text{Im}}(\rho) = \text{Im}(w(\rho)). \quad (3.3)$$

Then we have,

$$I, J, I_{\text{Re}}, I_{\text{Im}} \in C([0, \rho_0); \mathbb{R}), \quad (3.4)$$

$$\|\nabla u\|_{L^2(\Omega \cap B(x_0, \rho))}^2 + \text{Re} \left(\int_{\Omega \cap B(x_0, \rho)} f(u) \overline{u} dx \right) = \text{Re} \left(\int_{\Omega \cap B(x_0, \rho)} F(x) \overline{u(x)} dx \right) + I_{\text{Re}}(\rho), \quad (3.5)$$

$$\text{Im} \left(\int_{\Omega \cap B(x_0, \rho)} f(u) \overline{u} dx \right) = \text{Im} \left(\int_{\Omega \cap B(x_0, \rho)} F(x) \overline{u(x)} dx \right) + I_{\text{Im}}(\rho), \quad (3.6)$$

for any $\rho \in [0, \rho_0)$.

Remark 3.2. One easily sees that if $\rho_0 < \text{dist}(x_0, \Gamma)$ then $I, J, I_{\text{Re}}, I_{\text{Im}} \in C([0, \rho_0]; \mathbb{R})$.

Example 3.3. We give some functions f for which Theorem 3.1 applies.

1) Typically, we apply Theorem 3.1 to

$$f(u) = a|u|^{-(1-m)}u + bu + c|x|^2u,$$

with $(a, b, c) \in \mathbb{C}^3$ and $0 < m < 1$. One easily checks that,

$$f \in C\left(L^2_{\text{loc}}(\Omega) \cap L^{m+1}_{\text{loc}}(\Omega); L^2_{\text{loc}}(\Omega) + L^{\frac{m+1}{m}}_{\text{loc}}(\Omega)\right).$$

If in addition, Ω is bounded or if $c = 0$ then one also has,

$$f \in C\left(L^2(\Omega) \cap L^{m+1}(\Omega); L^2(\Omega) + L^{\frac{m+1}{m}}(\Omega)\right).$$

Let $z \in \mathbb{C} \setminus \{0\}$. Since $||z|^{-(1-m)}z| = |z|^m$, it is understood in the above example that $||z|^{-(1-m)}z| = 0$ when $z = 0$.

2) **Hartree-Fock type equations.** Let $V \in L^p(\mathbb{R}^N; \mathbb{R}) + L^\infty(\mathbb{R}^N; \mathbb{R})$, with $\min\{1, \frac{N}{2}\} < p < \infty$ and let $W \in L^q(\mathbb{R}^N; \mathbb{R}) + L^\infty(\mathbb{R}^N; \mathbb{R})$, with $\min\{1, \frac{N}{4}\} < q < \infty$. Set $r = \frac{2p}{p-1}$, $s = \frac{4q}{q-1}$,

$$E = L^2(\mathbb{R}^N) \cap L^4(\mathbb{R}^N) \cap L^r(\mathbb{R}^N) \cap L^s(\mathbb{R}^N),$$

$$f(u) = Vu + (W \star |u|^2)u,$$

for any $u \in H^1(\mathbb{R}^N)$. Then $H^1(\mathbb{R}^N) \hookrightarrow E$ with dense embedding and, by density of $\mathcal{D}(\mathbb{R}^N)$ in spaces $L^m(\mathbb{R}^N)$, for any $m \in [1, \infty)$, we have

$$E^* = L^2(\mathbb{R}^N) + L^{\frac{4}{3}}(\mathbb{R}^N) + L^{r'}(\mathbb{R}^N) + L^{s'}(\mathbb{R}^N),$$

$$f \in C(E; E^*),$$

$$f \in C(H^1(\mathbb{R}^N); H^{-1}(\mathbb{R}^N)).$$

See Cazenave [6] (Proposition 1.1.3, p.3, Proposition 3.2.2, p.58-59, Remark 3.2.3, p.59, Proposition 3.2.9, p.62, Remark 3.2.10, p.63 and Example 3.2.11, p.63).

4 Proofs of the main results

Before proceeding to the proof of Theorems 2.1 and 2.2, we recall the well-known Young's inequality and its particular case.

Lemma 4.1 (Young's inequality). *For any real $x \geq 0$, $y \geq 0$, $\lambda > 1$ and $\varepsilon > 0$, one has*

$$xy \leq \frac{1}{\lambda'} \varepsilon^{\lambda'} x^{\lambda'} + \frac{1}{\lambda} \varepsilon^{-\lambda} y^\lambda, \quad (4.1)$$

and in particular, one has

$$xy \leq \frac{\varepsilon^2}{2} x^2 + \frac{1}{2\varepsilon^2} y^2. \quad (4.2)$$

Proof of Theorems 2.1 and 2.2. We write $\rho_\star = \rho_0$, for the proof of Theorem 2.1 and $\rho_\star = \rho_1$, for the proof of Theorem 2.2. Let us introduce some notations. Let $\rho \in (0, \rho_\star)$. We set

$$\begin{aligned} E(\rho) &= \|\nabla u\|_{L^2(B(x_0, \rho))}^2, & b(\rho) &= \|u\|_{L^{m+1}(B(x_0, \rho))}^{m+1}, & a(\rho) &= \|u\|_{L^2(B(x_0, \rho))}^2, \\ \theta &= \frac{(1+m)+N(1-m)}{k} \in (0, 1), & \ell &= \frac{1}{\theta(1+m)}, & \delta &= \frac{k}{2(1+m)}. \end{aligned}$$

We first assume that $u \in H^1(B(x_0, \rho_\star))$ and we will consider the general case at the end of the proof.

We now proceed with the proof in 5 steps.

Step 1. $E \in W^{1,1}(0, \rho_\star)$, for a.e. $\rho \in (0, \rho_\star)$, $E'(\rho) = \|\nabla u\|_{L^2(\mathbb{S}(x_0, \rho))}^2$ and

$$E(\rho) + b(\rho) \leq \frac{1}{2} \left(K_1(\tau) \rho^{-(\nu-1)} E'(\rho) \right)^{\frac{1}{2}} (E(\rho) + b(\rho))^{\frac{\gamma(\tau)+1}{2}} + (L_1 M)^2 \|F\|_{L^2(B(x_0, \rho))}^2, \quad (4.3)$$

where $K_1(\tau) = CL_1^2 M^2 \max\{\rho_\star^{\nu-1}, 1\} \max\{b(\rho_\star)^{\mu(\tau)}, b(\rho_\star)^{n(\tau)}\}$, $C = C(N, m)$ and $L_1 = \max\{1, \frac{1}{L}\}$.

We have the identity $E(\rho) = \int_0^\rho \left(\int_{\mathbb{S}(x_0, r)} |\nabla u|^2 d\sigma \right) dr$. Since the mapping $r \mapsto \int_{\mathbb{S}(x_0, r)} |\nabla u|^2 d\sigma$ lies in $L^1(0, \rho_\star)$, E is absolutely continuous on $(0, \rho_\star)$. We then get the first part of the claim and it remains to establish (4.3). Let $\rho \in (0, \rho_\star)$. It follows from Cauchy-Schwarz's inequality that

$$\left| \int_{\mathbb{S}(x_0, \rho)} u \overline{\nabla u} \cdot \frac{x - x_0}{|x - x_0|} d\sigma \right| \leq \|\nabla u\|_{L^2(\mathbb{S}(x_0, \rho))} \|u\|_{L^2(\mathbb{S}(x_0, \rho))} = E'(\rho)^{\frac{1}{2}} \|u\|_{L^2(\mathbb{S}(x_0, \rho))}. \quad (4.4)$$

We recall the interpolation-trace inequality (see Corollary 2.1 in Díaz and Véron [7], in which a misprint was present since their δ has to be replaced by $-\delta$):

$$\|u\|_{L^2(\mathbb{S}(x_0, \rho))} \leq C \left(\|\nabla u\|_{L^2(B(x_0, \rho))} + \rho^{-\delta} \|u\|_{L^{m+1}(B(x_0, \rho))} \right)^\theta \|u\|_{L^{m+1}(B(x_0, \rho))}^{1-\theta}, \quad (4.5)$$

where $C = C(N, m)$. Putting together (2.1) (for Theorem 2.1), (2.3) (for Theorem 2.2), (4.4) and (4.5), we obtain,

$$\begin{aligned} &E(\rho) + b(\rho) + \kappa a(\rho) \\ &\leq CL_1 M E'(\rho)^{\frac{1}{2}} \left(E(\rho)^{\frac{1}{2}} + \rho^{-\delta} b(\rho)^{\frac{1}{m+1}} \right)^\theta b(\rho)^{\frac{1-\theta}{m+1}} + L_1 M \int_{B(x_0, \rho)} |F(x)u(x)| dx, \end{aligned} \quad (4.6)$$

where

$$\kappa = \begin{cases} 0, & \text{in the case of Theorem 2.1,} \\ 1, & \text{in the case of Theorem 2.2.} \end{cases}$$

In the case of Theorem 2.2, we apply Young's inequality (4.2) with $x = |F|$, $y = |u|$ and $\varepsilon = \sqrt{L_1 M}$, and we get

$$\int_{B(x_0, \rho)} |F(x)u(x)| dx \leq \frac{L_1 M}{2} \|F\|_{L^2(B(x_0, \rho))}^2 + \frac{1}{2L_1 M} a(\rho), \quad (4.7)$$

for any $\rho \in (0, \rho_*)$. Putting together (4.6) and (4.7), we obtain for both theorems,

$$E(\rho) + b(\rho) \leq C_0 L_1 M E'(\rho)^{\frac{1}{2}} \left(E(\rho)^{\frac{1}{2}} + \rho^{-\delta} b(\rho)^{\frac{1}{m+1}} \right)^\theta b(\rho)^{\frac{1-\theta}{m+1}} + (L_1 M)^2 \|F\|_{L^2(B(x_0, \rho))}^2, \quad (4.8)$$

for almost every $\rho \in (0, \rho_*)$. Let $\tau \in (\frac{m+1}{2}, 1]$ and let $\rho \in (0, \rho_*)$. A straightforward calculation yields

$$\begin{aligned} & \left(E(\rho)^{\frac{1}{2}} + \rho^{-\delta} b(\rho)^{\frac{1}{m+1}} \right) b(\rho)^{\frac{1-\theta}{m+1}} \\ &= E(\rho)^{\frac{1}{2}} b(\rho)^{\frac{1-\theta}{m+1}} + \rho^{-\delta} b(\rho)^{\frac{1}{m+1}} \\ &= E(\rho)^{\frac{1}{2}} b(\rho)^{\tau(1-\theta)\ell} b(\rho)^{(1-\tau)(1-\theta)\ell} + \rho^{-\delta} b(\rho)^{\frac{1}{2} + \tau(1-\theta)\ell} b(\rho)^{\ell - \tau(1-\theta)\ell - \frac{1}{2}} \\ &\leq 2\rho^{-\delta} \max\{\rho_*^\delta, 1\} K_2^2(\tau)^{\frac{1}{2\theta}} (E(\rho) + b(\rho))^{\frac{1}{2} + \tau(1-\theta)\ell}, \end{aligned}$$

where $K_2^2(\tau) = \max\{b(\rho_*)^{\mu(\tau)}, b(\rho_*)^{\eta(\tau)}\}$, since $\frac{\mu(\tau)}{2\theta} = (1-\tau)(1-\theta)\ell$ and $\frac{\eta(\tau)}{2\theta} = \ell - \tau(1-\theta)\ell - \frac{1}{2}$. Hence (4.3) follows from (4.8) and the above estimate with $K_1(\tau) = 16C_0^2 L_1^2 M^2 K_2^2(\tau) \max\{\rho_*^{\nu-1}, 1\}$, since $2\delta\theta = \nu - 1$ and $\theta(\frac{1}{2} + \tau(1-\theta)\ell) = \frac{\gamma(\tau)+1}{2}$.

Step 2. For any $\tau \in (\frac{m+1}{2}, 1]$ and for a.e. $\rho \in (0, \rho_*)$,

$$0 \leq E(\rho)^{1-\gamma(\tau)} \leq K_1(\tau) \rho^{-(\nu-1)} E'(\rho) + (2L_1 M)^{2(1-\gamma(\tau))} \|F\|_{L^2(B(x_0, \rho))}^{2(1-\gamma(\tau))}.$$

Applying Young's inequality (4.1) with $x = \frac{1}{2} (K_1(\tau) \rho^{-(\nu-1)} E'(\rho))^{\frac{1}{2}}$, $y = (E(\rho) + b(\rho))^{\frac{\gamma(\tau)+1}{2}}$, $\lambda = \lambda(\tau) = \frac{2}{\gamma(\tau)+1}$ and $\varepsilon = \varepsilon(\tau) = (\gamma(\tau) + 1)^{\frac{1}{\lambda(\tau)}}$, it follows from Step 1 that,

$$\begin{aligned} & E(\rho) + b(\rho) \\ &\leq \frac{1}{2} \left(K_1(\tau) \rho^{-(\nu-1)} E'(\rho) \right)^{\frac{1}{2}} (E(\rho) + b(\rho))^{\frac{\gamma(\tau)+1}{2}} + (L_1 M)^2 \|F\|_{L^2(B(x_0, \rho))}^2, \\ &\leq \frac{C(\tau)}{2^{\frac{\lambda(\tau)}{\lambda(\tau)-1}}} \left(K_1(\tau) \rho^{-(\nu-1)} E'(\rho) \right)^{\frac{1}{1-\gamma(\tau)}} + \frac{1}{2} (E(\rho) + b(\rho)) + (L_1 M)^2 \|F\|_{L^2(B(x_0, \rho))}^2, \\ &\leq \frac{1}{2} \left(K_1(\tau) \rho^{-(\nu-1)} E'(\rho) \right)^{\frac{1}{1-\gamma(\tau)}} + \frac{1}{2} (E(\rho) + b(\rho)) + (L_1 M)^2 \|F\|_{L^2(B(x_0, \rho))}^2, \end{aligned}$$

since

$$C(\tau) = \frac{\lambda(\tau) - 1}{\lambda(\tau)} \varepsilon(\tau)^{\frac{\lambda(\tau)}{\lambda(\tau)-1}} < \frac{\lambda(\frac{m+1}{2}) - 1}{\lambda(\frac{m+1}{2})} (\gamma(\tau) + 1)^{\frac{1}{\lambda(\tau)-1}} < \frac{1}{2} 2^{\frac{1}{\lambda(\tau)-1}} < \frac{1}{2} 2^{\frac{\lambda(\tau)}{\lambda(\tau)-1}}.$$

We get,

$$E(\rho) + b(\rho) \leq \left(K_1(\tau) \rho^{-(\nu-1)} E'(\rho) \right)^{\frac{1}{1-\gamma(\tau)}} + (2L_1M)^2 \|F\|_{L^2(B(x_0, \rho))}^2.$$

Raising both sides of the above inequality to the power $1 - \gamma(\tau)$ and recalling that $(1 - \gamma(\tau)) \in (0, 1)$, we obtain Step 2.

Step 3. Let $\alpha \in (0, \rho_0]$. If $E(\alpha) = 0$ then necessarily $u|_{B(x_0, \alpha)} \equiv 0$.

Indeed, from our hypothesis, $E' = 0$ on $(0, \alpha)$. Furthermore, $\|F\|_{L^2(B(x_0, \alpha))} = 0$ (from assumption in Theorem 2.1 or (2.4) in Theorem 2.2). It follows from Step 1 and the continuity of b that $b(\alpha) = 0$. Hence Step 3 follows.

Step 4. Proof of Theorem 2.1.

We have $\rho_\star = \rho_0$ and $\|F\|_{L^2(B(x_0, \rho_0))} = 0$. For any $\tau \in (\frac{m+1}{2}, 1]$, set $r(\tau)^\nu = \left(\rho_0^\nu - \nu \frac{K_1(\tau) E(\rho_0)^{\gamma(\tau)}}{\gamma(\tau)} \right)_+$ and let $\rho_{\max} = \max_{\tau \in (\frac{m+1}{2}, 1]} r(\tau)$. Note that definition of ρ_{\max} coincides with (2.2). Let $\tau \in (\frac{m+1}{2}, 1]$. We claim that $E(r(\tau)) = 0$. Otherwise, $E(r(\tau)) > 0$ and so $E > 0$ on $[r(\tau), \rho_0)$. One has from Step 2 (we recall that $\gamma(\tau) - 1 < 0$),

$$\text{for a.e. } \rho \in (r(\tau), \rho_0), \quad K_1(\tau) E'(\rho) E(\rho)^{\gamma(\tau)-1} \geq \rho^{\nu-1}.$$

We integrate this estimate between $r(\tau)$ and ρ_0 . We obtain

$$\nu \frac{K_1(\tau)}{\gamma(\tau)} \left(E(\rho_0)^{\gamma(\tau)} - E(r(\tau))^{\gamma(\tau)} \right) \geq \rho_0^\nu - r^\nu(\tau).$$

By definition of $r(\tau)$, this gives $E(r(\tau)) \leq 0$. A contradiction, hence the claim. In particular, $E(\rho_{\max}) = 0$. It follows from Step 3 that $u|_{B(x_0, \rho_{\max})} \equiv 0$, which is the desired result. It remains to treat the case where $u \in H_{\text{loc}}^1(B(x_0, \rho_0))$. We proceed as follows. Let $n \in \mathbb{N}$, $n > \frac{1}{\rho_0}$. We work on $B(x_0, \rho_0 - \frac{1}{n})$ instead of $B(x_0, \rho_0)$ and apply the above result. Thus $u|_{B(x_0, \rho_{\max}^n)} \equiv 0$, where ρ_{\max}^n is given by (2.2) with $\rho_0 - \frac{1}{n}$ in place of ρ_0 . We then let $n \nearrow \infty$ which leads to the result. This finishes the proof of Theorem 2.1.

Step 5. Proof of Theorem 2.2.

We have $\rho_\star = \rho_1$. Let $\gamma = \gamma(1)$ and set for any $\rho \in [0, \rho_1]$, $G(\rho) = (2L_1M)^{2(1-\gamma)} \|F\|_{L^2(B(x_0, \rho))}^{2(1-\gamma)}$ and $K = K_1(1) \rho_0^{-(\nu-1)}$. Let $E_\star = \left(\frac{\gamma}{2K} (\rho_1 - \rho_0) \right)^{\frac{1}{\gamma}}$ and $\varepsilon_\star = \frac{1}{2^{p'} (2L_1M)^2} \left(\frac{\gamma}{2K} \right)^p$. Note that $p = \frac{1}{\gamma}$. Assume now $E(\rho_1) < E_\star$. Applying Step 2 with $\tau = 1$, one has for a.e. $\rho \in (\rho_0, \rho_1)$,

$$-KE'(\rho) + E(\rho)^{1-\gamma} \leq G(\rho). \quad (4.9)$$

Let define the function H by

$$\forall \rho \in [0, \rho_1], \quad H(\rho) = \left(\frac{\gamma}{2K} (\rho - \rho_0)_+ \right)^{\frac{1}{\gamma}}. \quad (4.10)$$

Then $H(\rho_1) = E_*$, $H \in C^1([0, \rho_1]; \mathbb{R})$ (since $\frac{1}{\gamma} > 2$) and H satisfies

$$\forall \rho \in [0, \rho_1], -KH'(\rho) + \frac{1}{2}H(\rho)^{1-\gamma} = 0, \quad (4.11)$$

$$E(\rho_1) < H(\rho_1). \quad (4.12)$$

Finally and recalling that $\gamma = \frac{1}{p}$, from our hypothesis (2.4) and (4.10), one has

$$\forall \rho \in (0, \rho_1), G(\rho) \leq \frac{1}{2} \left(\frac{\gamma}{2K}(\rho - \rho_0)_+ \right)^{\frac{1-\gamma}{\gamma}} = \frac{1}{2}H(\rho)^{1-\gamma}. \quad (4.13)$$

Putting together (4.9), (4.13) and (4.11), one obtains

$$-KE'(\rho) + E(\rho)^{1-\gamma} \leq -KH'(\rho) + H(\rho)^{1-\gamma}, \text{ for a.e. } \rho \in (\rho_0, \rho_1). \quad (4.14)$$

Now, we claim that for any $\rho \in [\rho_0, \rho_1)$, $E(\rho) \leq H(\rho)$. Indeed, if the claim does not hold, it follows from (4.12) and continuity of E and H that there exist $r \in (\rho_0, \rho_1)$ and $\delta \in (0, r - \rho_0]$ such that

$$E(r) = H(r), \quad (4.15)$$

$$\forall \rho \in (r - \delta, r), E(\rho) > H(\rho). \quad (4.16)$$

It follows from (4.14) and (4.16) that for a.e. $\rho \in (r - \delta, r)$, $H'(\rho) < E'(\rho)$. But, with (4.15), this implies that for any $\rho \in (r - \delta, r)$, $H(\rho) > E(\rho)$, which contradicts (4.16), hence the claim. It follows that $0 \leq E(\rho_0) \leq H(\rho_0) = 0$. We deduce with the help of Step 3 that $u|_{B(x_0, \rho_0)} \equiv 0$, which is the desired result. It remains to treat the case where $u \in H_{\text{loc}}^1(B(x_0, \rho_1))$. We proceed as follows. Assume $E(\rho_1) < E_*$. Then there exists $\varepsilon > 0$ small enough such that $\rho_0 < \rho_1 - \varepsilon$ and $E(\rho_1) < E_*(\varepsilon)$, where $E_*(\varepsilon) = \left(\frac{\gamma}{2K}(\rho_1 - \rho_0 - \varepsilon) \right)^{\frac{1}{\gamma}}$. Since ε_* is a non increasing function of ρ_1 , we do not need to change its definition. Estimates (4.9)–(4.14) holding with $\rho_1 - \varepsilon$ in place of ρ_1 , it follows that $E(\rho_0) = 0$ and we finish with the help of Step 3. This ends the proof of Theorem 2.2. \square

Proof of Theorem 3.1. If $\rho_0 > \text{dist}(x_0, \Gamma)$ then $u \in H_0^1(\Omega)$. So we may extend u by 0 on $\Omega^c \cap B(x_0, \rho_0)$. Denoting \tilde{u} this extension, we have $\tilde{u} \in H_0^1(\Omega \cup B(x_0, \rho_0))$. We first consider the case where $\rho_0 \neq \text{dist}(x_0, \Gamma)$. We deal with $\rho_0 = \text{dist}(x_0, \Gamma)$ at the end of the proof. We first note that $J \in C([0, \rho_0]; \mathbb{R})$ and by Cauchy-Schwarz's inequality, we have

$$\|I\|_{L^1(0, \rho_0)} \leq \|u\|_{H^1(\Omega \cap B(x_0, \rho_0))}^2 < \infty,$$

$$\|J\|_{L^\infty(0, \rho_0)} \leq \|F\|_{L^2(\Omega \cap B(x_0, \rho_0))} \|u\|_{L^2(\Omega \cap B(x_0, \rho_0))} < \infty.$$

We then get, $I, I_{\text{Re}}, I_{\text{Im}} \in L^1((0, \rho_0); \mathbb{R})$, so that $I, J, I_{\text{Re}}, I_{\text{Im}}$ are defined almost everywhere on $(0, \rho_0)$.

It follows from (3.1) that,

$$\langle \nabla u, \nabla \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} + \langle f(u), \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = \langle F, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}, \quad (4.17)$$

for any $\varphi \in \mathcal{D}(\Omega)$. Let $\rho \in (0, \rho_0)$. For any $n \in \mathbb{N}$, $n > \frac{1}{\rho}$, we define the cutoff function $\psi_n \in W^{1,\infty}(\mathbb{R}; \mathbb{R})$ by

$$\forall t \in \mathbb{R}, \psi_n(t) = \begin{cases} 1, & \text{if } |t| \in [0, \rho - \frac{1}{n}], \\ n(\rho - |t|), & \text{if } |t| \in (\rho - \frac{1}{n}, \rho), \\ 0, & \text{if } |t| \in [\rho, \infty), \end{cases}$$

and we set $\tilde{\varphi}_n(x) = \psi_n(|x - x_0|)\tilde{u}(x)$ and $\varphi_n = \tilde{\varphi}_n|_{\Omega}$, for almost every $x \in \Omega \cup B(x_0, \rho_0)$. We easily check that for any $(j, k) \in \llbracket 1, n_1 \rrbracket \times \llbracket 1, n_2 \rrbracket$,

$$\begin{aligned} \varphi_n|_{\Omega \cap B(x_0, \rho_0)} &\in H_0^1(\Omega \cap B(x_0, \rho_0)) \cap L^{p_j}(\Omega \cap B(x_0, \rho_0)) \cap L^{q_k}(\Omega \cap B(x_0, \rho_0)), \\ \tilde{\varphi}_n &\in H_0^1(\Omega \cup B(x_0, \rho_0)) \cap L^{p_j}(\Omega \cup B(x_0, \rho_0)) \cap L^{q_k}(\Omega \cup B(x_0, \rho_0)), \\ \varphi_n &\in H_0^1(\Omega) \cap L^{p_j}(\Omega) \cap L^{q_k}(\Omega). \end{aligned}$$

It follows that there exists $(\varphi_n^m)_{m \in \mathbb{N}} \subset \mathcal{D}(\Omega)$ such that for any $(n, m) \in \mathbb{N}^2$, $\text{supp } \varphi_n^m \subset \Omega \cap B(x_0, \rho_0)$ and

$$\varphi_n^m \xrightarrow[m \rightarrow \infty]{H_0^1(\Omega) \cap L^{p_j}(\Omega) \cap L^{q_k}(\Omega)} \varphi_n,$$

for any $(j, k) \in \llbracket 1, n_1 \rrbracket \times \llbracket 1, n_2 \rrbracket$. Consequently, $\varphi = \varphi_n$ and $\varphi = i\varphi_n$ are admissible test functions in (4.17). We have,

$$\begin{aligned} &\langle \nabla u, \nabla \varphi_n \rangle_{L^2(\Omega), L^2(\Omega)} \\ &= \langle \nabla \tilde{u}, \nabla \tilde{\varphi}_n \rangle_{L^2(\Omega \cup B(0, \rho_0)), L^2(\Omega \cup B(0, \rho_0))} \\ &= \int_{B(x_0, \rho)} \psi_n(|x - x_0|) |\nabla \tilde{u}|^2 dx + \text{Re} \int_{B(x_0, \rho)} \psi_n'(|x - x_0|) \tilde{u} \nabla \tilde{u} \cdot \frac{x - x_0}{|x - x_0|} dx \\ &= \int_{B(x_0, \rho)} \psi_n(|x - x_0|) |\nabla \tilde{u}|^2 dx - n \text{Re} \int_{B(x_0, \rho) \setminus \bar{B}(x_0, \rho - \frac{1}{n})} \tilde{u} \nabla \tilde{u} \cdot \frac{x - x_0}{|x - x_0|} dx \\ &= \int_{B(x_0, \rho)} \psi_n(|x - x_0|) |\nabla \tilde{u}|^2 dx - n \text{Re} \left(\int_{\rho - \frac{1}{n}}^{\rho} \left(\int_{\mathbb{S}(x_0, r)} \tilde{u} \nabla \tilde{u} \cdot \frac{x - x_0}{|x - x_0|} d\sigma \right) dr \right) \\ &= \int_{\Omega \cap B(x_0, \rho)} \psi_n(|x - x_0|) |\nabla u|^2 dx - n \int_{\rho - \frac{1}{n}}^{\rho} I_{\text{Re}}(r) dr, \end{aligned}$$

where we introduced the spherical coordinates (r, σ) at the fifth line. We now let $n \nearrow \infty$. Using the Lebesgue's dominated convergence Theorem and recalling that $I_{\text{Re}} \in L^1((0, \rho_0); \mathbb{R})$, we obtain

$$\lim_{n \rightarrow \infty} \langle \nabla u, \nabla \varphi_n \rangle_{L^2(\Omega), L^2(\Omega)} = \|\nabla u\|_{L^2(\Omega \cap B(x_0, \rho))}^2 - I_{\text{Re}}(\rho). \quad (4.18)$$

Still by the Lebesgue's dominated convergence Theorem, and proceeding as above, we get

$$\lim_{n \rightarrow \infty} \langle \nabla u, i \nabla \varphi_n \rangle_{L^2(\Omega), L^2(\Omega)} = -I_{\text{Im}}(\rho), \quad (4.19)$$

$$\lim_{n \rightarrow \infty} \langle f(u), \varphi_n \rangle_{\sum_{j=1}^{n_1} L^{p'_j}(\Omega), \prod_{j=1}^{n_2} L^{p_j}(\Omega)} = \text{Re} \left(\int_{\Omega \cap B(x_0, \rho)} f(u) \bar{u} dx \right), \quad (4.20)$$

$$\lim_{n \rightarrow \infty} \langle f(u), i \varphi_n \rangle_{\sum_{j=1}^{n_1} L^{p'_j}(\Omega), \prod_{j=1}^{n_2} L^{p_j}(\Omega)} = \text{Im} \left(\int_{\Omega \cap B(x_0, \rho)} f(u) \bar{u} dx \right), \quad (4.21)$$

$$\lim_{n \rightarrow \infty} \langle F, \varphi_n \rangle_{L^2(\Omega), L^2(\Omega)} = \text{Re} \left(\int_{\Omega \cap B(x_0, \rho)} F(x) \overline{u(x)} dx \right), \quad (4.22)$$

$$\lim_{n \rightarrow \infty} \langle F, i \varphi_n \rangle_{L^2(\Omega), L^2(\Omega)} = \text{Im} \left(\int_{\Omega \cap B(x_0, \rho)} F(x) \overline{u(x)} dx \right). \quad (4.23)$$

Choosing $\varphi = \varphi_n$ in (4.17), estimates (4.18), (4.20) and (4.22) allow to pass in the limit as $n \nearrow \infty$ in (4.17). Putting together these estimates, we obtain (3.5). Choosing $\varphi = \varphi_n$ in (4.17), estimates (4.19), (4.21) and (4.23) allow to pass in the limit as $n \nearrow \infty$ in (4.17). Putting together these estimates, we obtain (3.6). We proved that (3.5) and (3.6) hold for almost every $\rho \in (0, \rho_0)$. Since all terms in (3.5) (respectively, in (3.6)) are continuous on $[0, \rho_0)$, except eventually I_{Re} (respectively, I_{Im}), it follows that I_{Re} (respectively, I_{Im}) is continuous and (3.5) (respectively, (3.6)) holds for any $\rho \in [0, \rho_0)$. We then get (3.4). It remains the case $\rho_0 = \text{dist}(x_0, \Gamma)$. It follows from the above proof that (3.4)–(3.6) holds for $\rho_0^n = \rho_0 - \frac{1}{n}$ in place of ρ_0 , for any integer $n > \frac{1}{\rho_0}$. We conclude by letting $n \nearrow \infty$. This finishes the proof. \square

5 Application to the localization property to the case of Neumann boundary conditions

In Bégout and Díaz [5], the authors study the localization property for equation (5.6) below with the homogeneous Dirichlet boundary condition (see, for instance, Theorem 3.5 in Bégout and Díaz [5]). In Theorem 5.5 below, we show that the same property holds with the homogeneous Neumann boundary condition. Note that from Bégout and Díaz [5] to this paper, there was a slight change of notation. See Comments 5.1 below and Section 7 for precision.

Comments 5.1. In the context of the paper of Bégout and Díaz [5], we can establish an existence result with the homogeneous Neumann boundary condition (instead of the homogeneous Dirichlet

condition) and $F \in L^2(\Omega)$ (instead of $F \in L^{\frac{m+1}{m}}(\Omega)$). In Bégout and Díaz [5], we introduced the set,

$$\tilde{\mathbb{A}} = \mathbb{C} \setminus \{z \in \mathbb{C}; \operatorname{Re}(z) = 0 \text{ and } \operatorname{Im}(z) \leq 0\},$$

and assumed that $(\tilde{a}, \tilde{b}) \in \mathbb{C}^2$ satisfies,

$$(\tilde{a}, \tilde{b}) \in \tilde{\mathbb{A}} \times \tilde{\mathbb{A}} \quad \text{and} \quad \begin{cases} \operatorname{Re}(\tilde{a})\operatorname{Re}(\tilde{b}) \geq 0, \\ \text{or} \\ \operatorname{Re}(\tilde{a})\operatorname{Re}(\tilde{b}) < 0 \text{ and } \operatorname{Im}(\tilde{b}) > \frac{\operatorname{Re}(\tilde{b})}{\operatorname{Re}(\tilde{a})}\operatorname{Im}(\tilde{a}), \end{cases} \quad (5.1)$$

with possibly $\tilde{b} = 0$, and we worked with

$$-i\Delta u + \tilde{a}|u|^{-(1-m)}u + \tilde{b}u = \tilde{F}.$$

But here in order to follow a closer notation with most of the works dealing with Schrödinger equations, we do not work any more with this equation but with,

$$-\Delta u + a|u|^{-(1-m)}u + bu = F,$$

and $b \neq 0$. This means that we choose, $\tilde{a} = ia$, $\tilde{b} = ib$ and $\tilde{F} = iF$. Then assumptions on (a, b) are changed by the fact that,

$$\operatorname{Re}(a) = \operatorname{Re}(-i\tilde{a}) = \operatorname{Im}(\tilde{a}), \quad (5.2)$$

$$\operatorname{Im}(b) = \operatorname{Im}(-i\tilde{b}) = -\operatorname{Re}(\tilde{b}). \quad (5.3)$$

It follows that the set $\tilde{\mathbb{A}}$ and (5.1) become,

$$\mathbb{A} = \mathbb{C} \setminus \{z \in \mathbb{C}; \operatorname{Re}(z) \leq 0 \text{ and } \operatorname{Im}(z) = 0\}, \quad (5.4)$$

$$(a, b) \in \mathbb{A} \times \mathbb{A} \quad \text{and} \quad \begin{cases} \operatorname{Im}(a)\operatorname{Im}(b) \geq 0, \\ \text{or} \\ \operatorname{Im}(a)\operatorname{Im}(b) < 0 \text{ and } \operatorname{Re}(b) > \frac{\operatorname{Im}(b)}{\operatorname{Im}(a)}\operatorname{Re}(a). \end{cases} \quad (5.5)$$

Obviously,

$$\left((\tilde{a}, \tilde{b}) \in \tilde{\mathbb{A}} \times \tilde{\mathbb{A}} \text{ satisfies (5.1)} \right) \iff \left((a, b) \in \mathbb{A} \times \mathbb{A} \text{ satisfies (5.5)} \right).$$

Assumptions (5.5) are made to prove the existence and the localization property of solutions to equation

$$-\Delta u + a|u|^{-(1-m)}u + bu = F, \text{ in } L^2(\Omega). \quad (5.6)$$

For uniqueness, the hypotheses are the following (Theorem 2.12 in Bégout and Díaz [2]).

Assumption 5.2 (Uniqueness). Assume that $(a, b) \in \mathbb{C}^2$ satisfies one of the two following conditions.

- 1) $a \neq 0$, $\operatorname{Re}(a) \geq 0$ and $\operatorname{Re}(a\bar{b}) \geq 0$.
- 2) $b \neq 0$, $\operatorname{Re}(b) \geq 0$ and $a = kb$, for some $k \geq 0$.

A geometric interpretation of (5.5) and 1) of Assumption 5.2 is given in Section 7 (as in Section 6 of Bégout and Díaz [5]). Now, we give some results about equation (5.6) when $(a, b) \in \mathbb{A} \times \mathbb{A}$ satisfies (5.5).

Theorem 5.3 (Neumann boundary conditions). *Let Ω be a nonempty bounded open subset of \mathbb{R}^N having a C^1 boundary, let ν be the outward unit normal vector to Γ , let $0 < m < 1$ and let $(a, b) \in \mathbb{A}^2$ satisfies (5.5).*

1. For any $F \in L^2(\Omega)$, there exists at least one solution $u \in H^1(\Omega)$ to

$$\begin{cases} -\Delta u + a|u|^{-(1-m)}u + bu = F, & \text{in } L^2(\Omega), \\ \frac{\partial u}{\partial \nu}|_{\Gamma} = 0. \end{cases} \quad (5.7)$$

Symmetry property. *If furthermore, for any $\mathcal{R} \in SO_N(\mathbb{R})$, $\mathcal{R}\Omega = \Omega$ and if $F \in L^2(\Omega)$ is spherically symmetric then there exists a spherically symmetric solution $u \in H^1(\Omega) \cap H_{\text{loc}}^2(\Omega)$ of (5.7). For $N = 1$, this means that if F is an even (respectively, odd) function then u is also an even (respectively, odd) function.*

2. If furthermore (a, b) satisfies Assumption 5.2 then the solution of (5.7) is unique.
3. Let $u \in H^1(\Omega)$ be any solution to (5.7). Then $u \in H_{\text{loc}}^2(\Omega)$. In addition,

$$\|u\|_{H^1(\Omega)} \leq M\|F\|_{L^2(\Omega)}, \quad (5.8)$$

where $M = M(|a|, |b|)$. Finally, if for some $\alpha \in (0, m]$, $F \in C_{\text{loc}}^{0,\alpha}(\Omega)$ then $u \in C_{\text{loc}}^{2,\alpha}(\Omega)$.

Here and in what follows, $SO_N(\mathbb{R})$ denotes the special orthogonal group of \mathbb{R}^N .

Remark 5.4. One easily checks that if $(a, b) \in \mathbb{A}^2$ satisfies $\operatorname{Re}(a) \geq 0$ and $\operatorname{Re}(a\bar{b}) \geq 0$ then $(a, b) \in \mathbb{A}^2$ verifies (5.5). In this case, uniqueness assumptions imply existence assumptions.

Proof of Theorem 5.3. The result comes from Bégout and Díaz [2]: Theorem 2.9 (existence and symmetry property), Theorem 2.12 (uniqueness), Theorem 2.11 (*a priori* estimate (5.8)) and Theorem 2.15 (local smoothness). The proof of the theorem is achieved. \square

Theorem 5.5. *Let Ω be a nonempty bounded open subset of \mathbb{R}^N having a C^1 boundary, let $0 < m < 1$ and let $(a, b) \in \mathbb{A}^2$ satisfies (5.5). Then there exists $\varepsilon_\star > 0$ such that for any $0 < \varepsilon \leq \varepsilon_\star$, there exists $\delta_0 = \delta_0(\varepsilon, |a|, |b|, N, m) > 0$ satisfying the following property. Let $F \in L^2(\Omega)$ and let $u \in H^1(\Omega)$ be a solution to (5.7). If uniqueness holds for the problem (5.7)², $\text{supp } F$ is a compact set and $\|F\|_{L^2(\Omega)} \leq \delta_0$ then $\text{supp } u \subset K(\varepsilon) \subset \Omega$, where*

$$K(\varepsilon) = \left\{ x \in \mathbb{R}^N; \exists y \in \text{supp } F \text{ such that } |x - y| \leq \varepsilon \right\},$$

which is compact.

The proof relies on the following lemma.

Lemma 5.6. *Let $\Omega \subset \mathbb{R}^N$ be a nonempty open subset of \mathbb{R}^N , let $0 < m < 1$ and let $(a, b) \in \mathbb{A}^2$ satisfies (5.5). Let $F \in L^1_{\text{loc}}(\Omega)$ and let $u \in H^1_{\text{loc}}(\Omega)$ be any solution to*

$$-\Delta u + a|u|^{-(1-m)}u + bu = F, \text{ in } \mathcal{D}'(\Omega). \quad (5.9)$$

Then there exist two positive constants $L = L(|a|, |b|)$ and $M = M(|a|, |b|)$ satisfying the following property. Let $x_0 \in \Omega$ and $\rho_\star > 0$. If $F|_{\Omega \cap B(x_0, \rho_\star)} \in L^2(\Omega \cap B(x_0, \rho_\star))$ then for any $\rho \in [0, \rho_\star)$,

$$\begin{aligned} & \|\nabla u\|_{L^2(\Omega \cap B(x_0, \rho))}^2 + L\|u\|_{L^{m+1}(\Omega \cap B(x_0, \rho))}^{m+1} + L\|u\|_{L^2(\Omega \cap B(x_0, \rho))}^2 \\ & \leq M \left(\left| \int_{\Omega \cap \mathbb{S}(x_0, \rho)} u \overline{\nabla u} \cdot \frac{x - x_0}{|x - x_0|} d\sigma \right| + \int_{\Omega \cap B(x_0, \rho)} |F(x)u(x)| dx \right), \end{aligned} \quad (5.10)$$

where it is additionally assumed that $u \in H^1_0(\Omega)$ if $\rho_\star > \text{dist}(x_0, \Gamma)$.

Proof. Let $x_0 \in \Omega$ and let $\rho_\star > 0$. We set for every $\rho \in [0, \rho_\star)$,

$$I(\rho) = \left| \int_{\mathbb{S}(x_0, \rho)} u \overline{\nabla u} \cdot \frac{x - x_0}{|x - x_0|} d\sigma \right| \text{ and } J(\rho) = \int_{\Omega \cap B(x_0, \rho)} |F(x)u(x)| dx.$$

It follows from Theorem 3.1 that $I, J \in C([0, \rho_\star); \mathbb{R})$ and

$$\left| \|\nabla u\|_{L^2(\Omega \cap B(x_0, \rho))}^2 + \text{Re}(a)\|u\|_{L^{m+1}(\Omega \cap B(x_0, \rho))}^{m+1} + \text{Re}(b)\|u\|_{L^2(\Omega \cap B(x_0, \rho))}^2 \right| \leq I(\rho) + J(\rho), \quad (5.11)$$

$$\left| \text{Im}(a)\|u\|_{L^{m+1}(\Omega \cap B(x_0, \rho))}^{m+1} + \text{Im}(b)\|u\|_{L^2(\Omega \cap B(x_0, \rho))}^2 \right| \leq I(\rho) + J(\rho), \quad (5.12)$$

for any $\rho \in [0, \rho_\star)$. Estimate (5.10) then follows from (5.11), (5.12) and Lemma 4.5 from Bégout and Díaz [2] with $\delta = 0$. \square

²which is the case, for instance, if $(a, b) \in \mathbb{A}^2$ satisfies Assumption 5.2.

Proof of Theorem 5.5. Let $F \in L^2(\Omega)$ with $\text{supp } F \subset \Omega$ and let $u \in H^1(\Omega)$ a solution to (5.7) be given by Theorem 5.3. Set $K = \text{supp } F$ and

$$\mathcal{O}(\varepsilon) = \left\{ x \in \mathbb{R}^N; \exists y \in K \text{ such that } |x - y| < \varepsilon \right\}.$$

Then $K(\varepsilon) = \overline{\mathcal{O}(\varepsilon)}$. Let $\varepsilon_* > 0$ be small enough to have $K(5\varepsilon_*) \subset \Omega$ and let $\varepsilon \in (0, \varepsilon_*]$. Let L and M be given by Lemma 5.6 applied with $\rho_* = 2\varepsilon$. By Theorem 2.1 and estimate (5.8) in Theorem 5.3 above, there exists $\delta_0 = \delta_0(\varepsilon, |a|, |b|, N, m) > 0$ such that if $\|F\|_{L^2(\Omega)} \leq \delta_0$ then $u|_{B(x_0, \varepsilon)} \equiv 0$, for any $x_0 \in \Omega$ such that $B(x_0, 2\varepsilon) \cap K = \emptyset$ and $B(x_0, 2\varepsilon) \subset \Omega$. Let $x_0 \in \overline{K(2\varepsilon)^c} \cap K(3\varepsilon)$. Let $y \in B(x_0, 2\varepsilon)$ and let $z \in K$. By definition of $K(2\varepsilon)$, $\text{dist}(\overline{K(2\varepsilon)^c}, K) = 2\varepsilon$. We then have

$$2\varepsilon = \text{dist}(\overline{K(2\varepsilon)^c}, K) \leq |x_0 - z| \leq |x_0 - y| + |y - z| < 2\varepsilon + |y - z|.$$

It follows that for any $z \in K$, $y \neq z$ so that $y \notin K$. This means that $B(x_0, 2\varepsilon) \cap K = \emptyset$, for any $x_0 \in \overline{K(2\varepsilon)^c} \cap K(3\varepsilon)$. We deduce that for any $x_0 \in \overline{K(2\varepsilon)^c} \cap K(3\varepsilon)$, $u|_{B(x_0, \varepsilon)} \equiv 0$. By compactness, there exist $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \overline{K(2\varepsilon)^c} \cap K(3\varepsilon)$ such that,

$$\overline{K(\varepsilon)^c} \cap \mathcal{O}(4\varepsilon) \subset \bigcup_{j=1}^n B(x_j, \varepsilon) \subset \bigcup_{j=1}^n B(x_j, 2\varepsilon) \subset K(5\varepsilon) \subset \Omega.$$

It follows that $u|_{K(\varepsilon)^c \cap \mathcal{O}(4\varepsilon)} \equiv 0$. Let us define \tilde{u} in Ω by,

$$\tilde{u} = \begin{cases} u, & \text{in } \mathcal{O}(2\varepsilon), \\ 0, & \text{in } \Omega \setminus \mathcal{O}(2\varepsilon). \end{cases}$$

It follows that $\text{supp } \tilde{u} \subset K(\varepsilon)$ and $\tilde{u} \in H_0^1(\Omega)$ is a solution to (5.7). By uniqueness assumption, $\tilde{u} = u$ so that $\text{supp } u \subset K(\varepsilon) \subset \Omega$, which is the desired result. \square

6 Further results on the cases of Dirichlet boundary conditions and the whole space

In Bégout and Díaz [5], the authors study existence, uniqueness, smoothness and localization property for the equations (6.1) below with an external source F belonging to $L^{\frac{m+1}{m}}(\Omega)$ with $0 < m < 1$ (see, for instance, Theorem 3.5 in Bégout and Díaz [5]). In theorems below, we show that the same results hold true with the weaker assumption $F \in L^2(\Omega)$. Indeed, when $|\Omega| < \infty$ and $0 < m < 1$, $L^{\frac{m+1}{m}}(\Omega) \hookrightarrow L^2(\Omega)$ and $L^{\frac{m+1}{m}}(\Omega) \neq L^2(\Omega)$. Hypotheses on $(a, b) \in \mathbb{C}^2$ are the same as in Bégout and Díaz [5], except we have to require $b \neq 0$. Note that from Bégout and Díaz [5] to this paper, there was a change of notation. See Comments 5.1 for precision.

In this section, we will repeatedly refer to Bégout and Díaz [5], so for brevity, we denote by Theorem $\dots\star$ (respectively, Corollary $\dots\star$) the theorems (respectively, the corollaries) in Bégout and Díaz [5].

Results which are stated for $F \in L^{\frac{m+1}{m}}(\Omega)$ are Theorems 1.1 \star , 1.2 \star , 3.1 \star , 3.5 \star , 4.1 \star , 4.4 \star and Corollary 5.3 \star .

Below, we first begin by stating a result analogous to Theorems 4.1 \star , 4.4 \star and Corollary 5.3 \star (existence, uniqueness, smoothness and *a priori* bound).

Theorem 6.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $0 < m < 1$ and let $(a, b) \in \mathbb{A}^2$ satisfies (5.5).*

1. *For any $F \in L^2(\Omega)$, there exists at least one solution $u \in H_0^1(\Omega) \cap L^{m+1}(\Omega)$ to*

$$\begin{cases} -\Delta u + a|u|^{-(1-m)}u + bu = F, & \text{in } L^2(\Omega) + L^{\frac{m+1}{m}}(\Omega), \\ u|_{\Gamma} = 0. \end{cases} \quad (6.1)$$

Symmetry property. *If furthermore, for any $\mathcal{R} \in SO_N(\mathbb{R})$, $\mathcal{R}\Omega = \Omega$ and if F is spherically symmetric then there exists a spherically symmetric solution $u \in H_0^1(\Omega) \cap H_{\text{loc}}^2(\Omega)$ of (6.1). For $N = 1$, this means that if F is an even (respectively, odd) function then u is also an even (respectively, odd) function.*

2. *If furthermore (a, b) satisfies Assumption 5.2 then the solution of (6.1) is unique.*

3. *Let $u \in H_0^1(\Omega) \cap L^{m+1}(\Omega)$ be any solution to (6.1). Then $u \in H_{\text{loc}}^2(\Omega)$. In addition,*

$$\|u\|_{H^1(\Omega)}^2 + \|u\|_{L^{m+1}(\Omega)}^{m+1} \leq M \|F\|_{L^2(\Omega)}^2, \quad (6.2)$$

where $M = M(|a|, |b|)$. Finally, if for some $\alpha \in (0, m]$, $F \in C_{\text{loc}}^{0,\alpha}(\Omega)$ then $u \in C_{\text{loc}}^{2,\alpha}(\Omega)$.

Remark 6.2. One easily checks that if $(a, b) \in \mathbb{A}^2$ satisfies $\text{Re}(a) \geq 0$ and $\text{Re}(a\bar{b}) \geq 0$ then $(a, b) \in \mathbb{A}^2$ verifies (5.5). In this case, uniqueness assumptions imply existence assumptions.

Proof of Theorem 6.1. See Bégout and Díaz [2] (Theorems 2.9, 2.11, 2.12 and 2.15). \square

Now, we give results analogous to Theorems 3.1 \star and 3.5 \star (localization property).

Theorem 6.3. *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $0 < m < 1$ and let $(a, b) \in \mathbb{A}^2$ satisfying (5.5). Let $F \in L_{\text{loc}}^1(\Omega)$, let $u \in H_{\text{loc}}^1(\Omega)$ be any solution in $\mathcal{D}'(\Omega)$ of (5.9), let $x_0 \in \Omega$ and let $\rho_1 > 0$.*

If $\rho_1 > \text{dist}(x_0, \Gamma)$ then assume further that $u \in H_0^1(\Omega)$. Then there exist $E_\star > 0$ and $\varepsilon_\star > 0$ satisfying the following property. Let $\rho_0 \in (0, \rho_1)$. If $\|\nabla u\|_{L^2(\Omega \cap B(x_0, \rho_1))}^2 < E_\star$ and

$$\forall \rho \in (0, \rho_1), \|F\|_{L^2(\Omega \cap B(x_0, \rho))}^2 \leq \varepsilon_\star ((\rho - \rho_0)_+)^p, \quad (6.3)$$

where $p = \frac{2(1+m)+N(1-m)}{1-m}$, then $u|_{\Omega \cap B(x_0, \rho_0)} \equiv 0$. In other words, with the notation of Theorem 2.1 \star (or Theorem 2.1), $\rho_{\max} = \rho_0$.

Theorem 6.4. Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $0 < m < 1$ and let $(a, b) \in \mathbb{A}^2$ satisfying (5.5). Then for any $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon, |a|, |b|, N, m) > 0$ satisfying the following property. Let $F \in L^2(\Omega)$ and let $u \in H_0^1(\Omega) \cap L^{m+1}(\Omega)$ be any solution to (6.1). If $\text{supp } F$ is a compact set and if $\|F\|_{L^2(\Omega)} \leq \delta_0$ then $\text{supp } u \subset \bar{\Omega} \cap K(\varepsilon)$, where $K(\varepsilon)$ is the compact set

$$K(\varepsilon) = \{x \in \mathbb{R}^N; \exists y \in \text{supp } F \text{ such that } |x - y| \leq \varepsilon\}.$$

In particular, if $\varepsilon > 0$ is small enough then $\text{supp } u \subset K(\varepsilon) \subset \Omega$.

Proof of Theorem 6.3. If $\rho_1 > \text{dist}(x_0, \Gamma)$ then we extend u by 0 on $\Omega^c \cap B(x_0, \rho_0)$. The result then comes from Lemma 5.6 and Theorem 2.2. \square

Proof of Theorem 6.4. Apply the proof of Theorem 3.5 \star , p.50-51, with $\|F\|_{L^2(\Omega)} \leq \delta_0$ in Property 3 of Theorem 6.1 above in place of $\|F\|_{L^{\frac{m+1}{m}}(\Omega)} \leq \delta_0$ in Theorem 4.4 \star . \square

We end this section by giving results analogous to Theorems 1.1 \star and 1.2 \star (particular cases).

Theorem 6.5. Let $0 < m < 1$, let $a \in \mathbb{R} \setminus \{0\}$ and let $b \in \mathbb{R}$, $b > 0$. Let $F \in L^2(\mathbb{R}^N)$ with compact support. Then there exists a unique solution $u \in H^1(\mathbb{R}^N) \cap L^{m+1}(\mathbb{R}^N)$ of the problem

$$-\Delta u + ia|u|^{-(1-m)}u + bu = F, \text{ in } L^2(\mathbb{R}^N) + L^{\frac{m+1}{m}}(\mathbb{R}^N).$$

In addition, $u \in H^2(\mathbb{R}^N)$ and u is compactly supported.

Theorem 6.6. Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $0 < m < 1$, let $a \in \mathbb{R} \setminus \{0\}$ and let $b \in \mathbb{R}$, $b > 0$. Let $F \in L^2(\Omega)$ with compact support. Assume that F is small enough in $L^2(\Omega)$. Then there exists a unique solution $u \in H_0^1(\Omega) \cap L^{m+1}(\Omega)$ to the problem

$$\begin{cases} -\Delta u + ia|u|^{-(1-m)}u + bu = F, \text{ in } L^2(\Omega) + L^{\frac{m+1}{m}}(\Omega), \\ u|_\Gamma = 0, \text{ on } \Gamma. \end{cases}$$

In addition, $u \in H^2(\Omega)$ and u is compactly supported in Ω .

Proof of Theorem 6.5. Apply Theorems 6.1 and 3.6 \star . \square

Proof of Theorem 6.6. Apply Theorems 6.1 and 6.4. \square

7 Some planar representations on the complex coefficients assumptions

In this section, we give some geometric interpretation of the values of a and b . For convenience, we repeat the hypotheses (5.5) of existence and 1) of Assumption 5.2 of uniqueness. We recall that,

$$\begin{aligned}\mathbb{A} &= \mathbb{C} \setminus \mathbb{D}, \\ \mathbb{D} &= \{z \in \mathbb{C}; \operatorname{Re}(z) \leq 0 \text{ and } \operatorname{Im}(z) = 0\}.\end{aligned}$$

For existence of solutions to problem (5.6) in Theorems 5.3 and 6.1, we suppose $(a, b) \in \mathbb{C}^2$ satisfies

$$(a, b) \in \mathbb{A} \times \mathbb{A} \quad \text{and} \quad \begin{cases} \operatorname{Im}(a)\operatorname{Im}(b) \geq 0, \\ \text{or} \\ \operatorname{Im}(a)\operatorname{Im}(b) < 0 \text{ and } \operatorname{Re}(b) > \frac{\operatorname{Im}(b)}{\operatorname{Im}(a)}\operatorname{Re}(a), \end{cases} \quad (7.1)$$

while for uniqueness, we assume

$$a \neq 0, \operatorname{Re}(a) \geq 0 \text{ and } \operatorname{Re}(a\bar{b}) \geq 0. \quad (7.2)$$

Existence. Condition (7.1) may easily be interpreted in this way: $[a, b] \cap \mathcal{D} = \emptyset$, where \mathcal{D} is the geometric representation of \mathbb{D} , which is the half-axis of the complex plane where $\operatorname{Re}(z) \leq 0$. See Figures 1 and 2 below.

Uniqueness. Condition (7.2) is trivial. Indeed, we first choose $a \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Re}(a) \geq 0$, and we choose b with respect to a . We see a and b as vectors of \mathbb{R}^2 . Then we write, $\vec{a} = \begin{pmatrix} \operatorname{Re}(a) \\ \operatorname{Im}(a) \end{pmatrix}$, $\vec{b} = \begin{pmatrix} \operatorname{Re}(b) \\ \operatorname{Im}(b) \end{pmatrix}$ and we have

$$\operatorname{Re}(a\bar{b}) = \operatorname{Re}(a)\operatorname{Re}(b) + \operatorname{Im}(a)\operatorname{Im}(b) = \vec{a} \cdot \vec{b}, \quad (7.3)$$

where \cdot denotes the scalar product between two vectors of \mathbb{R}^2 . Then the condition $\operatorname{Re}(a\bar{b}) \geq 0$ is equivalent to $|\angle(\vec{a}, \vec{b})| \leq \frac{\pi}{2}$ rad (see Figure 3 below).

Remark 7.1. Let $(a, b) \in \mathbb{C}^2$. Thanks to (7.3), the following assertions are equivalent.

- 1) $(a, b) \in \mathbb{C}^2$ satisfies (7.1)–(7.2).
- 2) $(a, b) \in \mathbb{A} \times \mathbb{A}$ satisfies (7.2).
- 3) $\left((a, b) \text{ satisfies (7.2)} \right)$ and $\left(\operatorname{Re}(a) = \operatorname{Im}(b) = 0 \implies \operatorname{Re}(b) > 0 \right)$.

In other words, when $\operatorname{Re}(a) \neq 0$, uniqueness hypothesis (7.2) implies existence hypothesis (7.1) (see Figure 4 below).

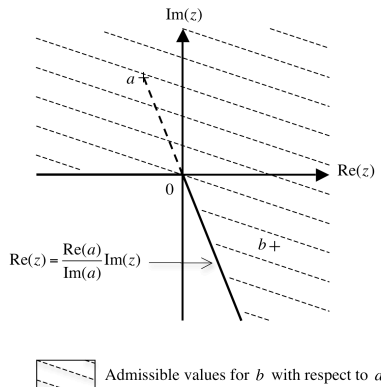


Figure 1: Existence, choice of b

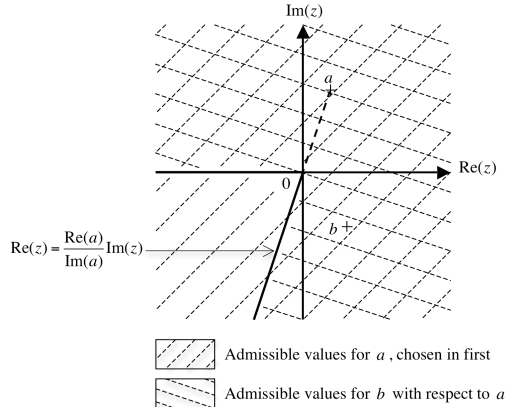


Figure 2: Existence, choice of a and b

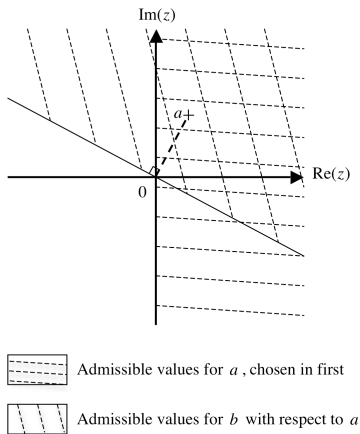


Figure 3: Uniqueness

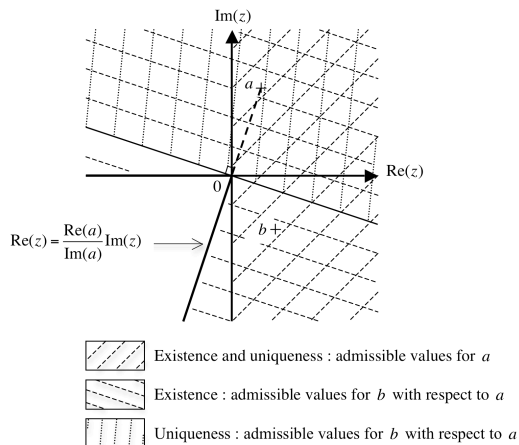


Figure 4: Uniqueness implies existence

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