PRIMITIVE MULTIPLE CURVES: CLASSIFICATION, DEFORMATIONS AND MODULI SPACES OF SHEAVES

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1. Introduction

A *primitive multiple curve* is an algebraic variety Y over $\mathbb{C}$ wich is Cohen-Macaulay, such that the induced reduced variety $C = Y_{red}$ is a smooth irreducible curve, and that every closed point of Y has a neighbourhood that can be embedded in a smooth surface. These curves have been defined and studied by C. Bănică and O. Forster in [1]. The simplest examples are infinitesimal neighbourhoods of smooth curves embedded in a smooth surface (but most primitive multiple curves cannot be globally embedded in smooth surfaces, cf. [2], theorem 7.1).

Let Y be a primitive multiple curve with associated reduced curve C, and suppose that $Y \neq C$. Let $\mathcal{I}_C$ be the ideal sheaf of C in Y. The *multiplicity* of Y is the smallest integer n such that $\mathcal{I}_C^n = 0$. We have then a filtration

$$C = C_1 \subset C_2 \subset \cdots \subset C_n = Y \tag{1}$$

where C_i is the subscheme corresponding to the ideal sheaf $\mathcal{I}_C^i$ and is a primitive multiple curve of multiplicity i. The sheaf $L = \mathcal{I}_C/\mathcal{I}_C^2$ is a line bundle on C, called the *line bundle on C associated to Y*.

Let P be a closed point of C and S a smooth surface containing a neighbourhood of P in Y as a locally closed subvariety. There exists elements y, t of $m_{S,P}$ (the maximal ideal of $\mathcal{O}_{S,P}$) whose images in $m_{S,P}/m_{S,P}^2$ form a basis, and such that for $1 \leq i < n$ we have $\mathcal{I}_{C_i,P} = (y^i)$ ($\mathcal{I}_{C_i,P}$ beeing the ideal sheaf of C_i in Y).

This paper is a survey of the theory of projective primitive multiple curves.

In chapter 1 we describe a parametrization of primitive multiple curves. The case of double curves has been treated in [2], and the case of multiplicities > 2 in [6].

In chapter 2, we give a description of the local structure of coherent sheaves on primitive multiple curves, and a few results on the moduli spaces of semi-stable sheaves on them. Most results come from [5], [7] and [8].

In chapter 3, we study the deformations of primitive multiple curves. In the case of multiplicity 2, the deformations in smooth curves has been treated in [15]. Some results on deformations in reduced reducible curves with the maximal number of components have been obtained in [10].

2. Classification of primitive multiple curves

(See [6])

2.1. Parametrization

Let C be a smooth irreducible curve over $\mathbb{C}$. Let $L \in \mathrm{Pic}(C)$. We can view C as embedded in the dual L^*, seen as a smooth surface, using the zero section. Then the n-th infinitesimal neighbourhood of C in L^* is a primitive multiple curve of multiplicity n with associated line bundle L. We call it the *trivial primitive curve* with associated line bundle L.

The primitive double (i.e. of multiplicity 2) curves are usually called *ribbons*. D. Bayer and D. Eisenbud have obtained in [2] the following classification: if Y is of multiplicity 2, then we have an exact sequence of vector bundles on C

$$0 \longrightarrow L \longrightarrow \Omega_{Y|C} \longrightarrow \omega_C \longrightarrow 0 \tag{2}$$

which splits if and only if Y is the trivial curve. In particular, if C is not projective, then Y is trivial. If C is projective and Y non trivial, then Y is completely determined by the line of $\mathrm{Ext}^1_{\mathcal{O}_C}(\omega_C, L)$ induced by the exact sequence (2). The non trivial primitive double curves with associated line bundle L are thus parametrized by the projective space $\mathbb{P}(\mathrm{Ext}^1_{\mathcal{O}_C}(\omega_C, L))$.

To parametrize the primitive multiple curves of multiplicity $n \geq 2$, we first need to study their local structure. Let Y be a primitive multiple curve of multiplicity n, C its underlying smooth curve, $Z_n = \mathrm{spec}(\mathbb{C}[t]/(t^n))$ and $*$ its unique closed point. For every open subset $U \subset C$, let $Y(U)$ be the corresponding open subset of Y. The local structure of Y is given by

2.1.1. Proposition: *Let $P \in C$ be a closed point. Then there exists an open subset $U \subset C$ containing P and an isomorphism*

$$Y(U) \ \simeq \ U \times Z_n$$

leaving $U \times \{*\}$ invariant.

The primitive multiple curves are then obtained by taking an open cover $(U_i)_{i \in I}$ of C, and automorphisms τ_{ij} of $U_{ij} \times Z_n$ (leaving $U_{ij} \times \{*\}$ invariant) satisfying obvious cocycle conditions. So it is natural to consider the following sheaf $\mathcal{G}_n$ of non abelian groups on C: for every open subset $U \subset C$, $\mathcal{G}_n(U)$ is the group of automorphisms of $U \times Z_n$ leaving $U \times \{*\}$ invariant. We can also view $\mathcal{G}_n(U)$ as the group of automorphisms ϕ of the $\mathbb{C}$-algebra $\mathcal{O}_C(U)[t]/(t^n)$ such that for every $\alpha \in \mathcal{O}_C(U)[t]/(t^n)$, the terms of degree 0 of α and $\phi(\alpha)$ are the same.

We say that two primitive multiple curves Y, Y' of multiplicity n, with the same underlying smooth curve C, are *isomorphic* if there exists an isomorphism $Y \to Y'$ inducing the identity on C. The set of isomorphism classes of such curves can be identified with the cohomology set $H^1(C, \mathcal{G}_n)$ (cf. [13]).

Let U be an open subset of C, such that $\omega_{C|U}$ is trivial, generated by dx, for some $x \in \mathcal{O}_C(U)$. Let $\mu, \nu \in \mathcal{O}_C(U)[t]/(t^{n-1})$, with ν invertible. Then we can define an automorphism $\phi_{\mu,\nu}$ of $\mathcal{O}_C(U)[t]/(t^n)$ by

$$\phi_{\mu,\nu}(\alpha) \ = \ \sum_{k=0}^{n-1} \frac{1}{k!}(\mu t)^k \frac{d^k\alpha}{dx^k} \qquad \text{for every} \quad \alpha \in \mathcal{O}_C(U)$$

(formally we could write $\phi_{\mu,\nu}(\alpha)=\alpha(x+\mu t)$), and

$$\phi_{\mu,\nu}(t) \ = \ \nu t \ .$$

It can be proved that

2.1.2. Proposition: *For every automorphism σ of $\mathcal{O}_C(U)[t]/(t^n)$ there exists unique $\mu,\nu\in\mathcal{O}_C(U)[t]/(t^{n-1})$, with ν invertible, such that $\sigma=\phi_{\mu,\nu}$.*

The product in $\mathcal{G}_n$ is given by

$$\phi_{\mu'\nu'}\circ\phi_{\mu\nu} \ = \ \phi_{\mu"\nu"},$$

with

$$\mu" = \mu'+\nu'\phi_{\mu',\nu'}(\mu), \qquad \nu" = \nu'\phi_{\mu',\nu'}(\nu),$$

and we have $\phi_{\mu\nu}^{-1}=\phi_{\mu',\nu'}$, with

$$\mu' = -\phi_{\mu\nu}^{-1}(\frac{\mu}{\nu}), \qquad \nu' = \phi_{\mu\nu}^{-1}(\frac{1}{\nu}).$$

By associating $\nu_{|C}$ to $\phi_{\mu,\nu}$ we define a surjective morphism $\xi_n:\mathcal{G}_n\to\mathcal{O}_C^*$. The induced map

$$H^1(\xi_n):H^1(C,\mathcal{G}_n)\longrightarrow H^1(C,\mathcal{O}_C^*)=\mathrm{Pic}(C)$$

associates to a primitive multiple curve Y its associated line bundle L on C.

Suppose that $n>2$. Then we have an obvious surjective morphism $\rho_n:\mathcal{G}_n\to\mathcal{G}_{n-1}$, which on U sends $\phi_{\mu,\nu}$ to $\phi_{\overline{\mu},\overline{\nu}}$, where $\overline{\mu},\overline{\nu}$ are the images of μ,ν respectively in $\mathcal{O}_C(U)[t]/(t^{n-2})$. Let T_C denote the tangent vector bundle of C. Using proposition 2.1.2 it is easy to see that

2.1.3. Proposition: *We have* $\ker(\rho_n)\simeq T_C\oplus\mathcal{O}_C$.

Hence we have an exact sequence of sheaves of groups on C

$$0\longrightarrow T_C\oplus\mathcal{O}_C\longrightarrow\mathcal{G}_n\xrightarrow{\rho_n}\mathcal{G}_{n-1}\longrightarrow 0\ .$$

Now the map

$$H^1(\rho_n):H^1(C,\mathcal{G}_n)\longrightarrow H^1(C,\mathcal{G}_{n-1})$$

sends a point representing the primitive multiple curve C_n to the point representing C_{n-1}.

If $g=g_n\in H^1(C,\mathcal{G}_n)$, we will denote by $g_{n-1},\ldots,g_2$ its images in $H^1(C,\mathcal{G}_{n-1}),\ldots,H^1(C,\mathcal{G}_2)$. If g_n corresponds to the primitive curve $Y=C_n$, $g_{n-1},\ldots,g_2$ correspond to the primitive curves $C_{n-1},\ldots,C_2$ of the filtration (1).

Let $g=g_n\in H^1(C,\mathcal{G}_n)$ and (g_{ij}) a cocycle representing g (with respect to an open cover (U_i) of C). Then we can define new sheaves of groups $(\mathcal{G}_n)^g$, $(\mathcal{G}_{n-1})^g$ and $(T_C\oplus\mathcal{O}_C)^g$ on C, obtained by glueing the restrictions of the sheaves on the U_{ij} using the automorphisms g_{ij} (acting by conjugation). The sheaf $(\mathcal{G}_n)^g$ is naturally isomorphic to the sheaf $\mathcal{A}ut_C(C_n)$ of automorphisms of C_n leaving C invariant.

Let C_n be the primitive multiple curve corresponding to g. Let $E(g_2)=(\Omega_{C_2|C})^*$ (i.e. the dual of the vector bundle $\Omega_{C_2|C}$ on C). Then from equation (2) we deduce that

2.1.4. Proposition: *We have* $(T_C\oplus\mathcal{O}_C)^g\simeq E(g_2)\otimes L^{n-1}$.

Now we can examine the fibers of $H^1(\rho_n)$. The theory of cohomology of sheaves of non abelian groups implies that if $g=g_n\in H^1(C,\mathcal{G}_n)$ and $g_{n-1}=H^1(\rho_n)(g_n)$, there exists a canonical surjective map

$$\lambda_g : H^1(C,E(g_2)\otimes L^{n-1})\longrightarrow H^1(\rho_n)^{-1}(g_{n-1})$$

which sends 0 to g, whose fibers are the orbits of an action of $\mathrm{Aut}_C(C_{n-1})$ on $H^1(C,E(g_2)\otimes L^{n-1})$. The proof of the surjectivity of λ_g uses the fact that $(T_C\oplus\mathcal{O}_C)^g$ is a sheaf of abelian groups, whose second cohomology group vanishes. It follows that

2.1.5. Proposition: *Every primitive multiple curve of multiplicity* $n-1$ *can be extended to a primitive multiple curve of multiplicity* n.

Let Y be a primitive multiple curve of multiplicity $n-1$ with associated smooth curve C. Two extensions Z,Z' of Y in primitive multiple curves of multiplicity n are called $(n-1)$-*isomorphic* if there exists an isomorphism $Z\simeq Z'$ inducing the identity on C_{n-1}. Let $\mathcal{H}_Y$ be the set of such extensions, and $Z\in\mathcal{H}_Y$. Then there exists a canonical bjection

$$\Lambda_Z : H^1(C,E(g_2)\otimes L^{n-1})\longrightarrow\mathcal{H}_Y$$

sending 0 to Z. If Z' is another extension of Y, then the composition

$$H^1(C,E(g_2)\otimes L^{n-1})\xrightarrow{\Lambda_Z}\mathcal{H}_Y\xrightarrow{\Lambda_{Z'}^{-1}}H^1(C,E(g_2)\otimes L^{n-1})$$

is a translation. It follows that $\mathcal{H}_Y$ *has a canonical structure of affine space, with associated vector space* $H^1(C,E(g_2)\otimes L^{n-1})$. This means that to a pair (C_n,C'_n) of extensions of C_{n-1} to a primitive multiple curve of multiplicity n, we can associate a well defined $c(C_n,C'_n)\in H^1(E(g_2)\otimes L^{n-1})$ (the vector from C_n to C'_n).

2.2. Canonical sheaves

(cf. [9], [6], 6-)

Let $Y=C_n$ be a primitive multiple curve of multiplicity $n\geq 2$ with underlying smooth curve C and associated line bundle L on C. The canonical sheaf Ω_{C_n} is locally isomorphic to $\mathcal{O}_{C_n}\oplus\mathcal{O}_{C_{n-1}}$. Let $P\in C$ be a closed point, $t\in\mathcal{O}_{C_n,P}$ over a generator of the maximal ideal of $\mathcal{O}_{C,P}$ and $z\in\mathcal{O}_{C_n,P}$ an equation of C. Then dt,dz generate $\Omega_{C_n,P}$: dt generates the factor $\mathcal{O}_{C_n,P}$ and dz the factor $\mathcal{O}_{C_{n-1},P}$ (because $z^{n-1}dz=0$). It follows that $\Omega_{C_n|C_{n-1}}$ is a rank 2 vector bundle on C_{n-1}. It is then easy to see that

2.2.1. Lemma: *The kernel of the canonical morphism* $\Omega_{C_n|C_{n-1}}\to\Omega_{C_{n-1}}$ *is isomorphic to* L^{n-1}.

It follows that we have an exact sequence of coherent sheaves on C_{n-1}

$$0\longrightarrow L^{n-1}\longrightarrow\Omega_{C_n|C_{n-1}}\longrightarrow\Omega_{C_{n-1}}\longrightarrow 0. \qquad (3)$$

The sheaves on the left and on the right are fixed, only the middle depends on C_n. So it is interesting to find which element of $\operatorname{Ext}^1_{\mathcal{O}_{C_{n-1}}}(\Omega_{C_{n-1}}, L^{n-1})$ corresponds to (3). We use the exact sequence

$$H^1(\mathcal{H}om(\Omega_{C_{n-1}}, L^{n-1})) \hookrightarrow \operatorname{Ext}^1_{\mathcal{O}_{C_{n-1}}}(\Omega_{C_{n-1}}, L^{n-1}) \xrightarrow{\pi} H^0(\mathcal{E}xt^1_{\mathcal{O}_{C_{n-1}}}(\Omega_{C_{n-1}}, L^{n-1}))$$

(cf. [14], 7.3). There exists a line bundle $\mathbb{L}$ on C_{n-1} such that $\mathbb{L}_{|C} = L$. We have then a locally free resolution of L^{n-1} on C_{n-1}: $\cdots \mathbb{L}^n \longrightarrow \mathbb{L}^{n-1} \longrightarrow L^{n-1}$, which gives with (3) a locally free resolution of $\Omega_{C_{n-1}}$ on C_{n-1}

$$\cdots \mathbb{L}^n \longrightarrow \mathbb{L}^{n-1} \longrightarrow \Omega_{C_n|C_{n-1}} \longrightarrow \Omega_{C_{n-1}}.$$

Using this resolution it is easy to see that $\mathcal{E}xt^1_{\mathcal{O}_{C_{n-1}}}(\Omega_{C_{n-1}}, L^{n-1}) \simeq \mathcal{O}_C$. We have

$$\mathcal{H}om(\Omega_{C_{n-1}}, L^{n-1}) \simeq \mathcal{H}om(\Omega_{C_{n-1}|C}, L^{n-1}) \simeq \mathcal{H}om(\Omega_{C_2|C}, L^{n-1}) = E(g_2) \otimes L^{n-1} \ .$$

Hence we have an exact sequence

$$0 \longrightarrow H^1(E(g_2) \otimes L^{n-1}) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_{C_{n-1}}}(\Omega_{C_{n-1}}, L^{n-1}) \xrightarrow{\pi} \mathbb{C} \longrightarrow 0 \ .$$

Now let $0 \longrightarrow L^{n-1} \longrightarrow \mathcal{E} \longrightarrow \Omega_{C_{n-1}} \longrightarrow 0$ be an exact sequence, associated to $\theta \in \operatorname{Ext}^1_{\mathcal{O}_{C_{n-1}}}(\Omega_{C_{n-1}}, L^{n-1})$. A local study gives

2.2.2. Lemma: *The sheaf $\mathcal{E}$ is locally free on C_{n-1} if and only if $\pi(\theta) \neq 0$.*

Let $\sigma(C_n) \in \operatorname{Ext}^1_{\mathcal{O}_{C_{n-1}}}(\Omega_{C_{n-1}}, L^{n-1})$ the element corresponding to the exact sequence (3). By correctly choosing the isomorphism of lemma 2.2.1 we can assume that $\pi(\sigma(C_n)) = 1$. Hence, if C'_n is another extension of C_{n-1} to a primitive multiple curve of multiplicity $n-1$, we have $\pi(C_n) - \pi(C'_n) \in H^1(E(g_2) \otimes L^{n-1})$.

Using the representation of primitive multiple curves with cocycles, it is then possible to prove (cf. [9])

2.2.3. Theorem: *Let C_n, C'_n two extensions of C_{n-1} in a primitive multiple curve of multiplicity n. Then we have $\pi(C'_n) - \pi(C_n) = (n-1)c(C_n, C'_n)$.*

3. Coherent sheaves on primitive multiple curves, and moduli spaces of sheaves

Let $Y = C_n$ a primitive multiple curve of multiplicity $n \geq 2$, underlying smooth curve C projective, irreducible, of genus g and associated line bundle L on C. Let $P \in C$ be a closed point, $z \in \mathcal{O}_{Y,P}$ an equation of C and M a $\mathcal{O}_{C_n,P}$-module of finite type. Let $\mathcal{E}$ be a coherent sheaf on C_n.

3.1. Canonical filtrations, generalized rank and degree and the Riemann-Roch theorem

The two *canonical filtrations* are useful tools to study the coherent sheaves on primitive multiple curves.

3.1.1. *First canonical filtration* – The fitst canonical filtration of M is

$$M_n = \{0\} \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = M$$

where for $0 \leq i < n$, M_{i+1} is the kernel of the surjective canonical morphism
$M_i \to M_i \otimes_{\mathcal{O}_{n,P}} \mathcal{O}_{C,P}$. So we have

$$M_i/M_{i+1} \ = \ M_i \otimes_{\mathcal{O}_{n,P}} \mathcal{O}_{C,P}, \qquad M/M_i \ \simeq \ M \otimes_{\mathcal{O}_{n,P}} \mathcal{O}_{C_i,P}, \qquad M_i \ = \ z^i M.$$

If $i > 0$, let $G_i(M) = M_i/M_{i+1}$. The graduate

$$\mathrm{Gr}(M) \ = \ \bigoplus_{i=0}^{n-1} G_i(M) \ = \ \bigoplus_{i=0}^{n-1} z^i M/z^{i+1}M$$

is an $\mathcal{O}_{C,P}$-module. If $1 < i \leq n$, then

- $M_i = \{0\}$ if and only if M is an $\mathcal{O}_{C_i,P}$-module.
- M_i is a $\mathcal{O}_{C_{n-i},P}$-module, and its first canonical filtration is $\{0\} \subset M_n \subset \cdots \subset M_{i+1} \subset M_i$.
- Every morphism of $\mathcal{O}_{C_n,P}$-modules is compatible with the first canonical filtrations of the modules.

One defines similarly the *first canonical filtration of* $\mathcal{E}$: it is the filtration

$$\mathcal{E}_n = 0 \subset \mathcal{E}_{n-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E}$$

such that for $0 \leq i < n$, $\mathcal{E}_{i+1}$ is the kernel of the canonical surjective morphism $\mathcal{E}_i \to \mathcal{E}_{i|C}$. So we have $\mathcal{E}_i/\mathcal{E}_{i+1} = \mathcal{E}_{i|C}$, $\mathcal{E}/\mathcal{E}_i = \mathcal{E}_{|C_i}$. If $i \geq 0$, let $G_i(\mathcal{E}) = \mathcal{E}_i/\mathcal{E}_{i+1}$. the graduate

$$\mathrm{Gr}(\mathcal{E}) \ = \ \bigoplus_{i=0}^{n-1} G_i(\mathcal{E})$$

is a $\mathcal{O}_C$-module. If $1 < i \leq n$ we have

- $\mathcal{E}_i = \mathcal{I}_C^i \mathcal{E}$.
- $\mathcal{E}_i = 0$ if and only if $\mathcal{E}$ is a sheaf on C_i.
- $\mathcal{E}_i$ is a sheaf on C_{n-i}, and its first canonical filtration is $0 \subset \mathcal{E}_n \subset \cdots \subset \mathcal{E}_{i+1} \subset \mathcal{E}_i$.
- Every morphism of coherent sheaves on C_n sends the first canonical filtration of the first sheaf to that of the second.

3.1.2. *Complete type of a coherent sheaf* – The pair

$$\Big(\big(\operatorname{rg}(G_0(\mathcal{E})),\dots,\operatorname{rg}(G_{n-1}(\mathcal{E}))\big),\big(\deg(G_0(\mathcal{E})),\dots,\deg(G_{n-1}(\mathcal{E}))\big)\Big)$$

is called the *complete type* of $\mathcal{E}$.

3.1.3. *Second canonical filtration* – One defines similarly the *second canonical filtration of M*: it is the filtration

$$M^{(0)}=\{0\}\subset M^{(1)}\subset\cdots\subset M^{(n-1)}\subset M^{(n)}=M$$

with $M^{(i)}=\big\{u\in M; z^iu=0\big\}$. If $M_n=\{0\}\subset M_{n-1}\subset\cdots\subset M_1\subset M_0=M$ is the first canonical filtration of M, we have $M_i\subset M^{(n-i)}$ for $0\leq i\leq n$. If $i>0$, let $G^{(i)}(M)=M^{(i)}/M^{(i-1)}$. The graduate

$$\operatorname{Gr}_2(M)\;=\;\bigoplus_{i=1}^{n}G^{(i)}(M)$$

is a $\mathcal{O}_{C,P}$-module. If $1<i\leq n$, then

- $M^{(i)}$ is a $\mathcal{O}_{C_i,P}$-module, and its second canonical filtration is $\{0\}\subset M^{(1)}\subset\cdots\subset M^{(i-1)}\subset M^{(i)}$.
- Every morphism of $\mathcal{O}_{n,P}$-modules sends the second canonical filtration of ther first sheaf to that of the second.

One defines in the same way the *second canonical filtration of $\mathcal{E}$*:

$$\mathcal{E}^{(0)}=\{0\}\subset\mathcal{E}^{(1)}\subset\cdots\subset\mathcal{E}^{(n-1)}\subset\mathcal{E}^{(n)}=\mathcal{E}.$$

If $i>0$, let $G^{(i)}(\mathcal{E})=\mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}$. The graduate

$$\operatorname{Gr}_2(\mathcal{E})\;=\;\bigoplus_{i=1}^{n}G^{(i)}(\mathcal{E})$$

is a $\mathcal{O}_C$-module. If $0<i\leq n$, then

- $\mathcal{E}^{(i)}$ is a sheaf on C_i, and its second canonical filtration is $0\subset\mathcal{E}^{(1)}\subset\cdots\subset\mathcal{E}^{(i-1)}\subset\mathcal{E}^{(i)}$.
- Every morphism of coherent sheaves on C_n sends the second canonical filtration of the first sheaf to that of the second.

3.1.4. *Invariants and the Riemann-Roch theorem* – The integer $R(M)=\operatorname{rk}(Gr(M))$ is called the *generalized rank* of M.

The integer $R(\mathcal{E})=\operatorname{rk}(Gr(\mathcal{E}))$ is called the *generalized rank* of $\mathcal{E}$.

The integer $\operatorname{Deg}(\mathcal{E})=\deg(Gr(\mathcal{E}))$ is called the *generalized degree* of $\mathcal{E}$.

Let $\mathcal{O}(1)$ be a very ample line bundle on C_n and $\mathcal{O}_C(1)=\mathcal{O}(1)_{|C}$. From Riemann-Roch theorem on C we deduce easily

3.1.5. Proposition: *We have $\chi(\mathcal{E})=\operatorname{Deg}(\mathcal{E})+R(\mathcal{E})(1-g)$. The Hilbert polynomial of $\mathcal{E}$ is*

$$P_{\mathcal{E}}(m)\;=\;\operatorname{Deg}(\mathcal{E})+R(\mathcal{E})(1-g)+R(\mathcal{E})\deg(\mathcal{O}_C(1)).m\;.$$

which implies that the canonical rank and degree of $\mathcal{E}$ can be computed by using any filtration of $\mathcal{E}$ whose graduates are sheaves of $\mathcal{O}_C$-modules, and that

3.1.6. Proposition: *The generalized rank and degree are invariant by deformation of the sheaves, and additive, i.e. for every exact sequence*

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

of coherent sheaves on C_n we have

$$R(\mathcal{E}) = R(\mathcal{E}') + R(\mathcal{E}") \quad \textit{and} \quad \mathrm{Deg}(\mathcal{E}) = \mathrm{Deg}(\mathcal{E}') + \mathrm{Deg}(\mathcal{E}").$$

3.1.7. Examples: **1** – Let $\mathcal{E}$ be a locally free sheaf on C_n and $E = \mathcal{E}_{|C}$. Then the two canonical filtrations of $\mathcal{E}$ are the same (i.e we have $\mathcal{E}^{(i)} = \mathcal{E}_{n-i}$ for $1 \leq i \leq n$), and $G_i(\mathcal{E}) = E \otimes L^i$ for $0 \leq i < n$.

2 – Let $\mathcal{F} = \mathcal{I}_P$ be the ideal sheaf of P on C_n. Then we have $\mathcal{F}_i/\mathcal{F}_{i+1} = (\mathcal{O}_C(-P) \otimes L^i) \oplus \mathbb{C}_P$ for $0 \leq i < n-1$, $\mathcal{F}_{n-1} = \mathcal{O}_C(-P) \otimes L^{n-1}$, $\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)} = L^{n-i}$ for $1 \leq i \leq n-1$ and $\mathcal{F}^{(n)}/\mathcal{F}^{(n-1)} = \mathcal{O}_C(-P)$.

3.2. Torsion free sheaves

(cf. [7])

Let $\mathcal{E}$ be a coherent sheaf on C_n. We say that $\mathcal{E}$ *is torsion free* if it is pure of dimension 1, i.e if it is non zero and has no proper subsheaf with finite support.

We denote by $\mathcal{E}^\vee$ the *dual sheaf* of $\mathcal{E}$, i.e $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_{C_n})$. This definition depends on n, i.e if $\mathcal{E}$ is a sheaf on C_i, $1 \leq i < n$, its dual on C_i is not the same as its dual on C_n. For example, if E is a vector bundle on C, then $E^\vee = E^* \otimes L^{n-1}$ (E^* beeing the ordinary dual of E on C). We say that $\mathcal{E}$ is *reflexive* if the canonical morphism $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ is an isomorphism.

3.2.1. Theorem: *Let $\mathcal{E}$ be a coherent sheaf on C_n. Then the following assertions are equivalent:*

(i) *$\mathcal{E}$ is torsion free.*
(ii) *$\mathcal{E}$ is reflexive.*
(iii) *$\mathcal{E}^{(1)}$ is locally free on C.*
(iv) *$\mathcal{E}xt^1_{\mathcal{O}_{C_n}}(\mathcal{E}, \mathcal{O}_{C_n}) = 0$.*

If $\mathcal{E}$ is torsion free, then all the sheaves $G^{(i)}(\mathcal{E})$ are locally free. From theorem 3.2.1 it is easy to deduce that if $\mathcal{E}$ is any coherent sheaf on C_n (even not torsion free) then we have $\mathcal{E}xt^i_{\mathcal{O}_{C_n}}(\mathcal{E}, \mathcal{O}_{C_n}) = 0$ for $i \geq 2$.

3.2.2. *Serre duality for reflexive sheaves* – Since C_n is locally a complete intersection, it has a *dualizing sheaf* ω_{C_n}, which is a line bundle on C_n. We have $\omega_{C_n|C} = \omega_C \otimes L^{1-n}$, and

3.2.3. Theorem: *Let $\mathcal{E}$ be a reflexive coherent sheaf on C_n. Then there exists functorial isomorphisms*

$$H^i(C_n,\mathcal{E}) \ \simeq \ H^{1-i}(C_n,\mathcal{E}^\vee\otimes\omega_{C_n})^*$$

for $i=0,1$.

3.3. Quasi locally free sheaves

(cf. [5], [7])

Let M be a $\mathcal{O}_{C_n,P}$-module of finite type. Then M is called *quasi free* if there exist non negative integers $m_1,\ldots,m_n$ and an isomorphism $M\simeq\oplus_{i=1}^n m_i\mathcal{O}_{C_i,P}$. The integers $m_1,\ldots,m_n$ are uniquely determined: it is easy to recover them from the first canonical filtration of M. We say that $(m_1,\ldots,m_n)$ is the *type* of M.

Let $\mathcal{E}$ be a coherent sheaf on C_n. We say that $\mathcal{E}$ is *quasi free at P* if $\mathcal{E}_P$ is quasi free, and that $\mathcal{E}$ is *quasi locally free* if it is quasi free at every point of C.

3.3.1. Theorem: *The following two assertions are equivalent:*

(i) *The $\mathcal{O}_{n,P}$-module M is quasi free.*
(ii) *$Gr(M)$ is a free $\mathcal{O}_{C,P}$-module, i.e all the M_i/M_{i+1} are free $\mathcal{O}_{C,P}$-modules.*

We have of course a corresponding theorem for sheaves on C_n, i.e. $\mathcal{E}$ is quasi locally free if and only if $Gr(\mathcal{E})$ is a vector bundle on C, if and only if all the $\mathcal{E}_i/\mathcal{E}_{i+1}$ are vector bundles on C.

It follows from theorem 3.3.1 that for any sheaf $\mathcal{E}$, the set of points $Q\in C$ such that $\mathcal{E}$ is quasi free at Q is open and nonempty. Moreover the sequence of integers $m_1,\ldots,m_n$ does not depend on the point of C where $\mathcal{E}$ is quasi free. It is called the *type* of $\mathcal{E}$.

The *complete type* of a coherent sheaf on C_n has been defined in 3.1.2. The type of $\mathcal{E}$ can be deduced from the complete type: if $((r_i),(d_i))$ is the complete type of $\mathcal{E}$ and (m_i) its type, then we have $r_i=m_{i+1}+\cdots+m_n$ for $0\leq i<n$.

For families of quasi locally free sheaves of fixed type, the complete type is invariant by deformation. More precisely

3.3.2. Proposition: *Let Z be an irreducible algebraic variety and $\mathcal{E}$ a coherent sheaf on $Z\times C_n$, flat on Z, such that for every closed point $z\in Z$, $\mathcal{E}_z$ is quasi locally free and of fixed type $(m_1,\ldots,m_n)$. Then the complete type of $\mathcal{E}_z$ is independent of z.*

This means that not only the ranks of the $G_i(\mathcal{E}_z)$ are fixed, but also their degrees.

3.3.3. *Irreducible families* – Let $\mathcal{Y}$ be a nonempty set of isomorphism classes of coherent sheaves on C_n. We say that $\mathcal{Y}$ is *irreducible* if for any $E_0,E_1\in\mathcal{Y}$ there exists an irreducible algebraic variety Z and a coherent sheaf $\mathcal{E}$ on $Z\times C_n$, flat on Z, such that for every closed point z of Z we have $\mathcal{E}_z\in\mathcal{Y}$, and such that there exists two closed points $z_0,z_1\in Z$ such that $\mathcal{E}_{z_0}=E_0$ and $\mathcal{E}_{z_1}=E_1$.

It is well known that the vector bundles of fixed rank and degree on C form an irreducible family. We have an analogous result on primitive multiple curves:

3.3.4. Theorem: *The family of isomorphism classes of quasi locally free sheaves on C_n of fixed complete type is irreducible.*

(The proof is by induction on n.)

3.3.5. *Open families* – Let $\mathcal{Y}$ be a nonempty set of isomorphism classes of coherent sheaves on C_n. We say that $\mathcal{Y}$ is *open* if for any algebraic variety Z and any coherent sheaf $\mathcal{E}$ on $Z \times C_n$, flat on Z, if $z_0 \in Z$ is a closed point such that $\mathcal{E}_{z_0} \in \mathcal{Y}$, there exists an open neighbourhood U of z_0 in Z such that $\mathcal{E}_z \in \mathcal{Y}$ for every closed point $z \in U$.

3.3.6. *Quasi locally free sheaves of rigid type* – In general, the family of all quasi locally free sheaves on C_n with fixed given complete type is not open. For example, all quasi locally free sheaves degenerate to vector bundles on C, hence in general the family of quasi locally free sheaves of complete type $((r, 0, \ldots, 0), (d, 0, \ldots, 0)$ (i.e vector bundles on C of rank r and degree d) is not open.

A quasi locally free sheaf $\mathcal{E}$ on C_n is called *of rigid type* if it is locally free, or locally isomorphic to $a\mathcal{O}_{C_n} \oplus \mathcal{O}_{C_k}$, with $a \geq 0$ and $1 \leq k < n$. So $\mathcal{E}$, of type (m_i), is of rigid type if and only there is at most one integer i such that $1 \leq i < n$ and $m_i > 0$, and in this case $m_i = 1$.

It is easy to see, using proposition 3.3.2 and theorem 3.3.4, that

3.3.7. Proposition: *The family of isomorphism classes of quasi locally free sheaves of rigid type and fixed complete type is irreducible and open.*

3.3.8. *Deformations of quasi locally free sheaves of rigid type* – Let $\mathcal{E}$ be a coherent sheaf on C_n. Let $(\widetilde{\mathcal{E}}, S, s_0, \epsilon)$ be a *semi-universal deformation* of $\mathcal{E}$: $\widetilde{\mathcal{E}}$ is a flat family of sheaves on C_n parametrized by S, $s_0 \in S$ is a closed point and $\epsilon : \widetilde{\mathcal{E}}_{s_0} \simeq \mathcal{E}$ (cf. [22]). The Kodaïra-Spencer map of $\widetilde{\mathcal{E}}$ at s_0

$$\omega_{\widetilde{\mathcal{E}},s_0} : T_{s_0}S \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{C_n}}(\mathcal{E}, \mathcal{E})$$

is an isomorphism.

We say that $\mathcal{E}$ is *smooth* if S is smooth at s_0. This is true if $\mathrm{Ext}^2_{\mathcal{O}_{C_n}}(\mathcal{E}, \mathcal{E}) = \{0\}$, for example if $\mathcal{E}$ is locally free.

We say that $\mathcal{E}$ is *smooth for reduced deformations* if S_{red} is smooth at s_0.

Let $D_{red}(\mathcal{E}) \subset \mathrm{Ext}^1_{\mathcal{O}_{C_n}}(\mathcal{E}, \mathcal{E})$ be the linear subspace corresponding to deformations of $\mathcal{E}$ parametrized by reduced algebraic varieties: $D_{red}(\mathcal{E}) = \omega_{\widetilde{\mathcal{E}},s_0}(T_{s_0}(S_{red}))$. It is the smallest linear subspace $H \subset \mathrm{Ext}^1_{\mathcal{O}_{C_n}}(\mathcal{E}, \mathcal{E})$ having the following property: let Z be a reduced variety, $\mathcal{F}$ a coherent sheaf on $Z \times C_n$, flat on Z, $z \in Z$ a closed point such that $\mathcal{F}_z \simeq \mathcal{E}$, and

$$\omega_{\mathcal{F},z} : T_z Z \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{C_n}}(\mathcal{E}, \mathcal{E})$$

the Kodaïra-Spencer map of $\mathcal{F}$ at z. Then $\mathrm{im}(\omega_{\mathcal{F},z}) \subset H$.

We have a canonical exact sequence
$$0 \longrightarrow H^1(\mathcal{E}nd(\mathcal{E})) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{C_n}}(\mathcal{E},\mathcal{E}) \longrightarrow H^0(\mathcal{E}xt^1_{\mathcal{O}_{C_n}}(\mathcal{E},\mathcal{E})) \longrightarrow 0.$$
Now we have

3.3.9. Theorem: *If $\mathcal{E}$ is a generic quasi locally free sheaf of rigid type, then $D_{red}(\mathcal{E}) = H^1(\mathcal{E}nd(\mathcal{E}))$.*

from which we deduce

3.3.10. Corollary: *If $\mathcal{E}$ is a quasi locally free sheaf of rigid type such that $\dim_{\mathbb{C}}(H^1(\mathcal{E}nd(\mathcal{E})))$ is minimal (i.e is such that for every quasi locally free sheaf $\mathcal{F}$ having the same complete type as $\mathcal{E}$, we have $\dim_{\mathbb{C}}(H^1(\mathcal{E}nd(\mathcal{F}))) \geq \dim_{\mathbb{C}}(H^1(\mathcal{E}nd(\mathcal{E})))$), then we have $D_{red}(\mathcal{E}) = H^1(\mathcal{E}nd(\mathcal{E}))$ and $\mathcal{E}$ is smooth for reduced deformations.*

3.4. Moduli spaces of semi-stable sheaves

(See [7], [8])

It follows from proposition 3.1.5 that the definition of a (semi-)stable sheaf on C_n does not depend on the choice of a very ample line bundle on C_n: a pure coherent sheaf of dimension 1 on C_n is *semi-stable* (resp. *stable*) if and only if for every proper subsheaf $\mathcal{F} \subset \mathcal{E}$ we have
$$\frac{\mathrm{Deg}(\mathcal{F})}{R(\mathcal{F})} \leq \frac{\mathrm{Deg}(\mathcal{E})}{R(\mathcal{E})} \qquad (\text{resp.} \quad <).$$

From now on, we suppose that $\deg(L) < 0$ (otherwise the only stable sheaves on C_n are the stable vector bundles on C).

Let R, D be integers, with $R \geq 1$. Let $\mathcal{M}(R, D)$ denote the moduli space of stable sheaves on C_n of generalized rank R and generalized degree D.

Let $\mathcal{E}$ be a quasi locally free coherent sheaf of rigid type on C_n, and supose that $\mathcal{E}$ is not locally free. Then $\mathcal{E}$ is locally isomorphic to $a\mathcal{O}_{C_n} \oplus \mathcal{O}_{C_k}$, for some integers a, k such that $a \geq 0$ and $1 \leq k < n$. Let
$$E = \mathcal{E}_{|C}, \qquad F = G_k(\mathcal{E}) \otimes L^{-k}.$$
Then we have $\mathrm{rg}(E) = a + 1$, $\mathrm{rg}(F) = a$, and
$$(G_0(\mathcal{E}), G_1(\mathcal{E}), \ldots, G_{n-1}(\mathcal{E})) \;=\; (E, E \otimes L, \ldots, E \otimes L^{k-1}, F \otimes L^k, \ldots, F \otimes L^{n-1}) \;.$$
Hence
$$\mathrm{Deg}(\mathcal{E}) \;=\; k \deg(E) + (n-k) \deg(F) + \big(n(n-1)a + k(k-1)\big) \deg(L)/2 \;.$$
Let $\delta = \delta(\mathcal{E}) = \deg(F)$, $\epsilon = \epsilon(\mathcal{E}) = \deg(E)$. According to proposition 3.3.7 the deformations of $\mathcal{E}$ are quasi locally free sheaves of rigid type, and $a(\mathcal{E})$, $k(\mathcal{E})$, $\delta(\mathcal{E})$, $\epsilon(\mathcal{E})$ are also invariant by deformation. Let
$$R = an + k \;, \qquad D = k\epsilon + (n-k)\delta + \big(n(n-1)a + k(k-1)\big) \deg(L)/2 \;.$$
The stable quasi locally free sheaves of rigid type $\mathcal{F}$ such that $a(\mathcal{F}) = a$, $k(\mathcal{F}) = k$, $\delta(\mathcal{F}) = \delta$, $\epsilon(\mathcal{F}) = \epsilon$ correspond to an open subset of $\mathcal{M}(R, D)$, denoted by $\mathcal{N}(a, k, \delta, \epsilon)$.

Using corollary 3.3.10 one can then prove

3.4.1. Proposition: *The variety $\mathcal{N}(a,k,\delta,\epsilon)$ is irreductible, and the underlying reduced subvariety $\mathcal{N}(a,k,\delta,\epsilon)_{red}$ is smooth. If it is nonempty, then we have*

$$\dim(\mathcal{N}(a,k,\delta,\epsilon)) \;=\; 1-\big(\frac{n(n-1)}{2}a^2+k(n-1)a+\frac{k(k-1)}{2}\big)\deg(L)+(g-1)(na^2+k(2a+1))$$

For every sheaf $\mathcal{F}$ of $\mathcal{N}(a,k,\delta,\epsilon)_{red}$, the tangent space of $\mathcal{N}(a,k,\delta,\epsilon)_{red}$ at $\mathcal{F}$ is canonically isomorphic to $H^1(\mathcal{E}nd(\mathcal{F}))$.

It should be noted that if R and D are fixed, then a and k are uniquely determined, but not δ and ϵ. So $\mathcal{M}(R,D)$ can have several irreducible components.

We now give some results on the *existence* of (semi-)stable sheaves on C_n. For locally free sheaves we have

3.4.2. Theorem: *Let $\mathbb{E}$ be a vector bundle on C_n. If $\mathbb{E}_{|C}$ is semi-stable (resp. stable), then so is $\mathbb{E}$.*

If there exists a semi-stable vector bundle $\mathbb{E}$ of rank R and degree D, then we can write

$$R=rn, \quad D=nd+\frac{n(n-1)}{2}r\deg(L)$$

(with $r=\mathrm{rk}(\mathbb{E}_{|C})$, $d=\deg(\mathbb{E}_{|C})$). So this is only possible if R and $D-\frac{n(n-1)}{2}r\deg(L)$ are multiples of n. In this case the open subset of $\mathcal{M}(R,D)$ corresponding to stable vector bundles is nonempty, irreducible and smooth, of dimension $1+nr^2(g-1)-\frac{n(n-1)}{2}r^2\deg(L)$.

Now we consider the moduli spaces of proposition 3.4.1. We have

3.4.3. Theorem: *If*

$$\frac{\epsilon}{a+1} \;<\; \frac{\delta}{a} \;<\; \frac{\epsilon-(n-k)\deg(L)}{a+1}$$

then $\mathcal{N}(a,k,\delta,\epsilon)$ is nonempty.

4. Deformations of primitive multiple curves

4.1. Deformations in smooth curves

Only deformations of multiplicity 2 primitive multiple curves in smooth curves have been studied, by M. González in [15]. He obtained the

4.1.1. Theorem: *Let Y be a smooth irreducible projective curve and let $\mathcal{E}$ be a line bundle on Y. Assume that there is a smooth irreducible double cover $\pi: X\to Y$ with $\pi_*\mathcal{O}_X/\mathcal{O}_Y=\mathcal{E}$. Then every ribbon $\widetilde{Y}$ with underlying smooth curve Y, with conormal bundle $\mathcal{E}$ and arithmetic genus $p_a(\widetilde{Y})$ is smoothable.*

Note that the conormal bundle $\mathcal{E}$ of Y in $\widetilde{Y}$ is our line bundle L (cf. (2)).

4.1.2. *Inadequacy of deformations in smooth curves* – Let C_n a primitive multiple curve of multiplicity $n \geq 2$, underlying smooth curve C of genus g and associated line bundle L on C. If one wants to study the deformations of moduli spaces of semi-stable sheaves together with the deformations of C_n, deformations of C_n in smooth curves are inapropriate, because one would need to consider only sheaves $\mathcal{E}$ such that $R(\mathcal{E})$ is a multiple of n. To see this, consider a flat family of projective curves $\pi : \mathcal{C} \to S$, parametrized by a neighbourhood of 0 in $\mathbb{C}$, such that $\pi^{-1}(0) = C_n$ and that $\pi^{-1}(z)$ is a smooth irreducible curve if $z \in S\backslash\{0\}$. Let $\mathcal{O}(1)$ be a very ample line bundle on $\mathcal{C}$. Let $z \in S\backslash\{0\}$, and γ the genus of $\mathcal{C}_z$. From the equality $\chi(\mathcal{O}_{\mathcal{C}_z}) = \chi(\mathcal{O}_{C_n})$ we deduce that $1-\gamma = \frac{n(n-1)}{2}\deg(L) + n(1-g)$, and using this and the equality $\chi(\mathcal{O}_{C_n}(1)) = \chi(\mathcal{O}_{\mathcal{C}_z}(1))$, we obtain $\deg(\mathcal{O}_{\mathcal{C}_z}(1)) = n.\deg(\mathcal{O}_C(1))$. Now let $\mathcal{E}$ be a coherent sheaf on $\mathcal{C}$, flat on S. The Hilbert polynomials of $\mathcal{E}_0$ and $\mathcal{E}_e$ are the same, and so are their leading coefficients, and we obtain $\mathrm{rk}(\mathcal{E}_z)\deg(\mathcal{O}_{\mathcal{C}_z}(1)) = R(\mathcal{E}_0)\deg(\mathcal{O}_C(1))$, whence $R(\mathcal{E}_0) = n.\mathrm{rk}(\mathcal{E}_z)$.

4.2. Primitive multiple curves coming from deformations of smooth curves

(See [9])

Let C be a smooth projective irreducible curve. Let T be a smooth curve and $t_0 \in T$ a closed point. Let $\mathcal{D} \to T$ be a flat family of projective smooth irreducible curves such that $C = \mathcal{D}_{t_0}$. Then the n-th infinitesimal neighbourhood of C in $\mathcal{D}$ is a primitive multiple curve C_n of multiplicity n, embedded in the smooth surface $\mathcal{D}$. We say that such a primitive multiple curve *comes from a family of smooth curves*. In this case $\mathcal{I}_C$, the ideal sheaf of C in C_n, is the trivial line bundle on C_{n-1} (so the associated line bundle on C is $\mathcal{O}_C$). In fact we have

4.2.1. Theorem: *Let C_n be a primitive multiple curve of multiplicity n, with underlying smooth curve C. Then C_n comes from a family of smooth curves if and only if the ideal sheaf of C in C_n is trivial on C_{n-1}.*

The proof uses the parametrization of multiple curves given in 2.1 and theorem 2.2.3.

4.3. Deformations in reduced reducible curves

(See [10])

Let $Y = C_n$ be a projective primitive multiple curve of multiplicity $n \geq 2$, underlying smooth curve C and associated line bundle L on C.

Let (S, P) be a germ of smooth curve. Let k a positive integer. Let $\pi : \mathcal{C} \to S$ be a flat morphism, where $\mathcal{C}$ is a reduced algebraic variety, such that

– For every closed point $s \in S$ such that $s \neq P$, the fiber $\mathcal{C}_s$ has k irreducible components, which are smooth and transverse, and any three of these components have no common point.
– The fiber $\mathcal{C}_P$ is isomorphic to Y.

4.3.1. Proposition: **1** – *There exists a germ of smooth curve (S', P') and a non constant morphism $\tau : S' \to S$ such that, if $\pi' : \mathcal{C}' = \pi^*\mathcal{C} \to S'$, $\mathcal{C}'$ has exactly k irreducible components, inducing on every fiber $\mathcal{C}'_{s'}$, $s' \neq P'$ the k irreducible components of $\mathcal{C}'_{s'}$.*

2 – *We have $k \leq n$.*

We call π' a *reducible deformation of Y of length k*. We say that π (or $\mathcal{C}$) is a *maximal reducible deformation of Y* if $k = n$.

We have then

4.3.2. Theorem: *Suppose that π is a maximal reducible deformation of C_n. Then*

1 – *If $\mathcal{C}''$ is the union of $i > 0$ irreducible components of $\mathcal{C}$, and $\pi'' : \mathcal{C}'' \to S$ is the restriction of π, then $\pi''^{-1}(P) \simeq C_i$, and π'' is a maximal reducible deformation of C_i.*

2 – *Let $s \in S \backslash \{P\}$. Then the irreducible components of $\mathcal{C}_s$ have the same genus as C. Moreover, if D_1, D_2 are distinct irreducible components of $\mathcal{C}_s$, then $D_1 \cap D_2$ consists of $-\deg(L)$ points.*

In particular, the n components $\mathcal{C}_1, \ldots, \mathcal{C}_n$ of $\mathcal{C}$ are smooth surfaces, and the restrictions of π, $\mathcal{C}_i \to S$, are flat families of smooth curves with the same fiber C over P.

4.4. FRAGMENTED DEFORMATIONS

(See [10])

We keep the notations of 4.3. Let $\pi : \mathcal{C} \to S$ a maximal reducible deformation of Y. We call it a *fragmented deformation of Y* if $\deg(L) = 0$, i.e. if for every $s \in S \backslash \{P\}$, $\mathcal{C}_s$ is the disjoint union of n smooth curves (cf. theorem 4.3.2). The variety $\mathcal{C}$ appears as a particular case of a *glueing* of $\mathcal{C}_1, \ldots, \mathcal{C}_n$ along C:

4.4.1. Definition: *For $1 \leq i \leq n$, let $\pi_i : \mathcal{C}_i \to S$ be a flat family of smooth projective irreducible curves, with a fixed isomorphism $\pi_i^{-1}(P) \simeq C$. A* glueing *of $\mathcal{C}_1, \cdots, \mathcal{C}_n$* along *$C$ is an algebraic variety $\mathcal{D}$ such that*

- *for $1 \leq i \leq n$, $\mathcal{C}_i$ is isomorphic to a closed subvariety of $\mathcal{D}$, also denoted by $\mathcal{C}_i$, and $\mathcal{D}$ is the union of these subvarieties.*
- *$\coprod_{1 \leq i \leq n} (\mathcal{C}_i \backslash C)$ is an open subset of $\mathcal{D}$.*
- *There exists a morphism $\pi : \mathcal{D} \to S$ inducing π_i on $\mathcal{C}_i$, for $1 \leq i \leq n$.*
- *The subvarieties $C = \pi_i^{-1}(P)$ of $\mathcal{C}_i$ coincide in $\mathcal{D}$.*

All the glueings of $\mathcal{C}_1,\cdots,\mathcal{C}_n$ along C have the same underlying Zariski topological space.

Let $\boldsymbol{\mathcal{A}}$ the *initial glueing* of the $\mathcal{C}_i$ along C. It is an algebraic variety whose points are the same as those of $\mathcal{C}$, i.e.

$$(\coprod_{i=1}^{n}\mathcal{C}_i)/\sim \quad ,$$

where $\sim$ is the equivalence relation: if $x\in\mathcal{C}_i$ and $y\in\mathcal{C}_j$, $x\sim y$ if and only if $x=y$, or if $x\in\mathcal{C}_{iP}\simeq C$, $y\in\mathcal{C}_{jP}\simeq C$ and $x=y$ in C. The structural sheaf is defined by: for every open subset U of $\boldsymbol{\mathcal{A}}$

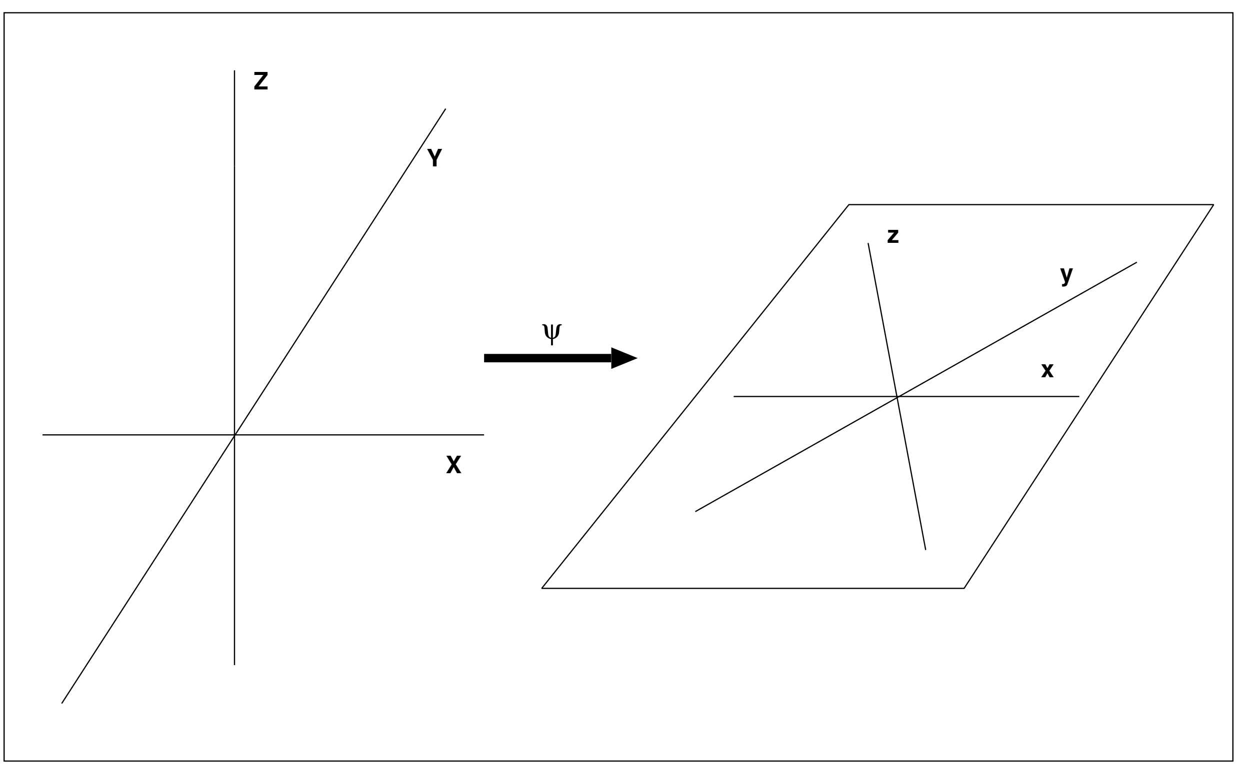
$$\mathcal{O}_{\boldsymbol{\mathcal{A}}}(U) \;=\; \{(\alpha_1,\ldots,\alpha_n)\in\mathcal{O}_{\mathcal{C}_1}(U\cap\mathcal{C}_1)\times\cdots\mathcal{O}_{\mathcal{C}_n}(U\cap\mathcal{C}_n);\alpha_{1|C}=\cdots=\alpha_{n|C}\}.$$

For every glueing $\mathcal{D}$ of $\mathcal{C}_1,\cdots,\mathcal{C}_n$, we have an obvious dominant morphism $\boldsymbol{\mathcal{A}}\to\mathcal{D}$. If follows that the sheaf of rings $\mathcal{O}_{\mathcal{D}}$ can be seen as a subsheaf of $\mathcal{O}_{\boldsymbol{\mathcal{A}}}$.

The fiber $D=\boldsymbol{\mathcal{A}}_0$ is not a primitive multiple curve (if $n>2$): if $\mathcal{I}_{C,D}$ denotes the ideal sheaf of C in D we have $\mathcal{I}_{C,D}^2=0$, and $\mathcal{I}_{C,D}\simeq\mathcal{O}_C\otimes\mathbb{C}^{n-1}$. In fact we have

4.4.2. Proposition: *Let $\mathcal{D}$ be a glueing of $\mathcal{C}_1,\cdots,\mathcal{C}_n$. Then $\pi^{-1}(P)$ is a primitive multiple curve if and only if for every closed point x of C, there exists a neighbourhood of x in $\mathcal{D}$ that can be embedded in a smooth variety of dimension 3.*

The situation is analogous to the following simpler situation: Consider n copies of $\mathbb{C}$ glued at 0. Two extreme examples appear: the trivial glueing $\boldsymbol{\mathcal{A}}_0$ (the set of coordinate lines in $\mathbb{C}^n$), and a set $\mathcal{C}_0$ of n lines in $\mathbb{C}^2$. We can easily construct a bijective morphism $\boldsymbol{\Psi}:\boldsymbol{\mathcal{A}}_0\to\mathcal{C}_0$ sending each coordinate line to a line in the plane



But the two schemes are of course not isomorphic: the maximal ideal of 0 in $\boldsymbol{\mathcal{A}}_0$ needs n generators, but 2 are enough for the maximal ideal of 0 in $\mathcal{C}_0$.

Let $\pi_{\mathcal{C}_0} : \mathcal{C}_0 \to \mathbb{C}$ be a morphism sending each component linearly onto $\mathbb{C}$, and
$\pi_{\boldsymbol{\mathcal{A}}_0} = \pi_{\mathcal{C}_0} \circ \boldsymbol{\Psi} : \boldsymbol{\mathcal{A}}_0 \to \mathbb{C}$. The difference of $\boldsymbol{\mathcal{A}_0}$ and $\mathcal{C}_0$ can be also seen by using the fibers of 0: we have

$$\pi_{\mathcal{C}_0}^{-1}(0) \simeq \mathrm{spec}(\mathbb{C}[t]/(t^n)) \quad \text{and} \quad \pi_{\boldsymbol{\mathcal{A}}_0}^{-1}(0) \simeq \mathrm{spec}(\mathbb{C}[t_1, \ldots, t_n]/(t_1, \ldots, t_n)^2) \ .$$

Let $\mathcal{D}$ a general glueing of n copies of $\mathbb{C}$ at 0, such that there exists a morphism $\pi : \mathcal{D} \to \mathbb{C}$ inducing the identity on each copy of $\mathbb{C}$. It is easy to see that we have $\pi^{-1}(0) \simeq \mathrm{spec}(\mathbb{C}[t]/(t^n))$ if and only if some neighbourhood of 0 in $\mathcal{D}$ can be embedded in a smooth surface.

4.4.3. *Properties of fragmented deformations* – Let $\pi : \mathcal{C} \to S$ be a fragmented deformation of $Y = C_n$. Let $I \subset \{1, \ldots, n\}$ be a proper subset, I^c its complement, and $\mathcal{C}_I \subset \mathcal{C}$ the subscheme union of the $\mathcal{C}_i, i \in I$. then we have

4.4.4. Theorem: *The ideal sheaf $\mathcal{I}_{\mathcal{C}_I}$ of $\mathcal{C}_I$ is isomorphic to $\mathcal{O}_{\mathcal{C}_{I^c}}$.*

In particular, the ideal sheaf $\mathcal{I}_{\mathcal{C}_i}$ of $\mathcal{C}_i$ is generated by a single regular function on $\mathcal{C}$. It is possible to find such a generator such that for $1 \leq j \leq n$, $j \neq i$, its j-th coordinate can be written as $\alpha\pi_j^p$, with $p > 0$ and $\alpha \in H^0(\mathcal{O}_S)$ such that $\alpha(P) \neq 0$. We can then suppose that $\alpha = 1$, and the generator can be written as

$$\mathbf{u}_{ij} \ = \ (u_1, \ldots, u_m),$$

with

$$u_i = 0, \quad u_m \ = \ \alpha_{ij}^{(m)} \pi_m^{p_{im}} \ \text{ for } m \neq i, \quad \alpha_{ij}^{(j)} = 1.$$

Let $p_{ii} = 0$ for $1 \leq i \leq n$. The symmetric matrix $(p_{ij})_{1 \leq i,j \leq n}$ is called the *spectrum* of π (or $\mathcal{C}$).

It follows also from the fact that $\mathcal{I}_{\mathcal{C}_i} = (\mathbf{u}_{ij})$ that the ideal sheaf of C in $Y = C_n$ is isomorphic to $\mathcal{O}_{C_{n-1}}$. Conversely we prove using theorem 4.2.1

4.4.5. Theorem: *If $Y = C_n$ is a primitive multiple curve with underlying smooth curve C such that the ideal sheaf $\mathcal{I}_C$ of C in Y is trivial on C_{n-1}, then there exists a fragmented deformation of Y.*

4.4.6. *n-stars and structure of fragmented deformations* – A *n-star* of (S, P) is a glueing $\boldsymbol{\mathcal{S}}$ of n copies of S at P, together with a morphism $\pi : \boldsymbol{\mathcal{S}} \to S$ which is an identity on each copy of S. All the n-stars have the same underlying Zariski topological space $S(n)$.

A n-star is called *oblate* if some neighbourhood of P can be embedded in a smooth surface. This is the case if and only $\pi^{-1}(0) \simeq \mathrm{spec}(\mathbb{C}[t])/(t^n)$.

Oblate n-stars are analogous to fragmented deformations and simpler.

Let $\pi : \mathcal{C} \to S$ be a fragmented deformation of $Y = C_n$. We associate to it an oblate n-star $\boldsymbol{\mathcal{S}}$ of S: for every open subset U of $S(n)$, $\mathcal{O}_{\boldsymbol{\mathcal{S}}}(U)$ is the set of $(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C}}(U)$ such that $\alpha_i \in \mathcal{O}_S(\pi_i(U \cap \mathcal{C}_i))$ for $1 \leq i \leq n$. We obtain also a canonical morphism

$$\boldsymbol{\Pi} : \mathcal{C} \longrightarrow \boldsymbol{\mathcal{S}}.$$

We can then prove, by using processes of contruction of fragmented deformations and oblate stars by induction on n the

4.4.7. Theorem: *The morphism* $\mathbf{\Pi}$ *is flat.*

Hence $\mathbf{\Pi}$ is a flat family of smooth curves, with $\mathbf{\Pi}^{-1}(P) = C$. The converse is also true, i.e. starting from an oblate n-star of S and a flat family of smooth curves parametrized by it, we obtain a fragmented deformation of a multiple primitive curve of multiplicity n.

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