EXCISION IN EQUIVARIANT COHOMOLOGY WITH EMPHASIS ON SL_2 OVER IMAGINARY QUADRATIC INTEGERS

ETHAN BERKOVE AND ALEXANDER D. RAHM
(WITH AN APPENDIX BY AUREL PAGE)

Abstract. We extend the Torsion Sub-complex Reduction technique that has recently been introduced in [11] to the case where the kernel of a group action is non-trivial. For this purpose, we excise the non-central torsion sub-complex from the equivariant spectral sequence. This new method of excision in equivariant cohomology is elaborated in a general setting in section 4. We demonstrate the power of the method by establishing general dimension formulae for the second page of the equivariant spectral sequence of the action of the SL_2 groups over imaginary quadratic integers on their associated symmetric space in section 7.

1. Introduction

The objects on which we demonstrate the power of this article's refinement of Torsion Sub-complex Reduction, are the Bianchi groups: SL_2 over the ring of integers in an imaginary quadratic number field. Also their central quotients PSL_2, on which the technique under refinement has originally been developed, are called Bianchi groups. Philosophically, the cohomology of the Bianchi groups can be thought of as coming from two different places. Since the Bianchi groups act on a 2-dimensional retract of hyperbolic 3-space $\mathbb{H}^3$, the Bianchi groups have virtual (co)homological dimension 2. Consequently, in dimensions 2 and below, many of the cohomology classes in Bianchi groups come from the topology of the quotient space and are detected with rational coefficients [14]. Above the virtual cohomological dimension, all cohomology classes are torsion and originate from finite subgroups. In particular, one can use the equivariant spectral sequence to determine the (co)homology of Bianchi groups.

Results of the second author made this precise. The article [9] introduced the ℓ-torsion subcomplex and proved that the homology above dimension 2 of a Bianchi group is determined by this sub-complex; in particular, that this homology is a direct sum over generic modules associated to the homeomorphism types of connected components of the sub-complex.

In this work, we are going to show how to extend the results from the projective special linear group to the linear group. We give general dimension formulae for the second page of the equivariant spectral sequence with $\mathbb{Z}/2$–coefficients of the action of the SL_2 groups over imaginary quadratic integers on their associated symmetric space in section 7. The last piece of information that we need in order to know the mod–2 cohomology of the SL_2 Bianchi groups, is the rank of the second page differential; and we explain in section 8 how to read it off from the numerical data collected in the appendix.

Acknowledgments. The first author thanks the De Brún Centre for its hospitality and for funding a stay at NUI Galway devoted to the present work. We are grateful for helpful comments by Matthias Wendt on the core of our excision technique.

2. The non-central torsion sub-complex

In this section we recall the ℓ-torsion sub-complexes theory of [11] and compare it with the *non-central* ℓ-torsion sub-complexes we are going to study.

We require any discrete group Γ under our study to be provided with a cell complex on which it acts cellularly. Let X be a finite-dimensional cell complex with a cellular action of a discrete group Γ, such that each cell stabilizer fixes its cell point-wise. Let ℓ be a prime number.

Date: December 30, 2012.
2000 *Mathematics Subject Classification.* 11F75, Cohomology of arithmetic groups.
The second author is funded by the Irish Research Council.

***Definition* 1.** The *ℓ-torsion sub-complex* of a Γ-cell complex X consists of all the cells of X the stabilizers in Γ of which contain elements of order ℓ.

We further require that the fixed point set X^G is acyclic for every non-trivial finite ℓ-subgroup G of Γ. Then Brown's proposition X.(7.2) [3] specializes as follows.

Proposition 2. *There is an isomorphism between the ℓ-primary parts of the Farrell cohomology of Γ and the Γ-equivariant Farrell cohomology of the ℓ-torsion sub-complex.*

***Definition* 3.** The *non-central ℓ-torsion sub-complex* of a Γ-cell complex X consists of all the cells of X the stabilizers in Γ of which contain elements of order ℓ that are not in the center of Γ.

This yields a correspondence between the non-central ℓ-torsion sub-complex for a group and the ℓ-torsion sub-complex for its central quotient group.

We can reduce the ℓ-torsion sub-complex in a way that proposition 2 still holds.

Condition A. In the ℓ-torsion sub-complex, let σ be a cell of dimension $n-1$, lying in the boundary of precisely two n-cells τ_1 and τ_2, the latter cells representing two different orbits. Assume further that no higher-dimensional cells of the ℓ-torsion sub-complex touch σ; and that the n-cell stabilizers admit an isomorphism $\Gamma_{\tau_1} \cong \Gamma_{\tau_2}$.

Where this condition is fulfilled in the ℓ-torsion sub-complex, we merge the cells τ_1 and τ_2 along σ and do so for their entire orbits, if and only if they meet the following additional condition. Here, we use Zassenhaus's notion for a finite group to be *ℓ-normal*, if the center of one of its Sylow ℓ-subgroups is the center of every Sylow ℓ-subgroup in which it is contained.

Condition B. The group Γ_σ admits a (possibly trivial) normal subgroup T_σ with trivial mod ℓ homology and with quotient group G_σ; and the group Γ_{τ_1} admits a (possibly trivial) normal subgroup T_τ with trivial mod ℓ homology and with quotient group G_τ making the sequences

$$1 \to T_\sigma \to \Gamma_\sigma \to G_\sigma \to 1 \text{ and } 1 \to T_\tau \to \Gamma_{\tau_1} \to G_\tau \to 1$$

exact and satisfying one of the following.

- Either $G_\tau \cong G_\sigma$,
- or G_σ is ℓ-normal and $G_\tau \cong \mathrm{N}_{G_\sigma}(\mathrm{center}(\mathrm{Sylow}_\ell(G_\sigma)))$,
- or both G_σ and G_τ are ℓ-normal and there is a (possibly trivial) group T with trivial mod ℓ homology making the sequence

  $$1 \to T \to \mathrm{N}_{G_\sigma}(\mathrm{center}(\mathrm{Sylow}_\ell(G_\sigma))) \to \mathrm{N}_{G_\tau}(\mathrm{center}(\mathrm{Sylow}_\ell(G_\tau))) \to 1$$

  exact.

Here, we write N_G for taking the normalizer in G; and Sylow_ℓ for picking an arbitrary Sylow ℓ-subgroup. This is well defined because all Sylow ℓ-subgroups are conjugate.

By a "terminal vertex", we will denote a vertex with no adjacent higher-dimensional cells and precisely one adjacent edge in the quotient space, and by "cutting off" the latter edge, we will mean that we remove the edge together with the terminal vertex from our cell complex.

***Definition* 4.** The *reduced ℓ-torsion sub-complex* associated to a Γ-cell complex X is the cell complex obtained by recursively merging orbit-wise all the pairs of cells satisfying conditions A and B; and cutting off edges that admit a terminal vertex together with which they satisfy condition B.

Using Swan's extension [15, final corollary] to Farrell cohomology of the Second Theorem of Grün [5, Satz 5], the following theorem has been proven in [11].

Theorem 5. *There is an isomorphism between the ℓ-primary parts of the Farrell cohomology of Γ and the Γ-equivariant Farrell cohomology of the reduced ℓ-torsion sub-complex.*

3. Spectral sequences and Central Extensions

Our calculations involve two spectral sequences. We use the Lyndon-Hochschild-Serre spectral sequence to determine cohomology of groups via group extensions. We also use the equivariant spectral sequence since many of the groups we consider act cellularly on low-dimensional contractible complexes. We introduce both of them briefly in this section; more details can be found in [1], [3], and [7].

For the development of the Lyndon-Hochschild-Serre spectral sequence we follow the approach in [1]. Given a short exact sequence of groups

$$1 \to H \to \Gamma \to \Gamma/H \to 1 \tag{1}$$

there is an induced surjection of classifying spaces $B_\rho : B_\Gamma \longrightarrow B_{\Gamma/H}$. The mapping cylinder of B_ρ is homotopy equivalent to B_H. One can then apply the Leray-Serre spectral sequence ([7], Chapters 5 and 6) to the resulting fibration. The spectral sequence that emerges has $E_2^{i,j} \cong \mathrm{H}^i(\Gamma/H; \mathrm{H}^j(H; M))$ for untwisted coefficients M and converges to $\mathrm{H}^{i+j}(\Gamma; M)$. We note that this spectral sequence can also be developed more algebraically, like in [3], VII.5.

In the short exact sequence of groups in equation 1, when the normal subgroup H is central in Γ, it is possible to say more. These are central extensions, and in such cases Γ/H acts trivially on the homotopy fiber B_H. Then with field coefficients M, the E_2-term of the resulting spectral sequence has the form

$$E_2^{i,j} \cong \mathrm{H}^i(\Gamma/H; M) \otimes \mathrm{H}^j(H; M).$$

Furthermore, Lemma IV.1.12 in [1], identifies a unique cohomology class k, called the *k-invariant*, which generates the kernel of $B_\pi^* : \mathrm{H}^2(\Gamma/H) \to \mathrm{H}^2(\Gamma)$. The k-invariant is also the cohomology class associated to the extension.

The Leray-Serre spectral sequence, of which the Lyndon-Hochschild-Serre spectral sequence is just one example, has many useful properties. It is compatible with cup products, for example. We will also use the fact that the action of the Steenrod algebra commutes with the differentials in the spectral sequence. That is, for a cohomology class u on the E_r page, $d_r(Sq^I u) = Sq^I d_r(u)$.

For the equivariant spectral sequence, we follow the development in [3, Chapter VII]. Let the group Γ act cellularly on a space X in such a way that the stabilizer of a cell σ fixes the cell pointwise. Consider the *equivariant cohomology groups* $\mathrm{H}^*(\Gamma, C^\bullet(X))$ with coefficients in the cochain complex $C^\bullet(X)$. We define these groups by taking a projective resolution, F, of $\mathbb{F}_2\Gamma$ over $\mathbb{F}_2$, then setting $\mathrm{H}^*(\Gamma, C^\bullet(X)) = \mathrm{H}^*(Hom_{\mathbb{F}_2}(F, C^\bullet(X)))$. When X is a contractible space, the equivariant cohomology groups can be identified with the cohomology of Γ, as

$$\mathrm{H}^*(\Gamma, C^\bullet(X; \mathbb{F}_2)) \cong \mathrm{H}^*(\Gamma, C^\bullet(pt; \mathbb{F}_2)) \cong \mathrm{H}^*(\Gamma; \mathbb{F}_2).$$

Using the horizontal and vertical filtrations of the double complex $Hom_{\mathbb{F}_2}(F, C^\bullet(X))$ we get a spectral sequence with

$$E_1^{i,j} \cong \mathrm{H}^j(\Gamma, C^i(X; \mathbb{F}_2))$$

converging to $H^{i+j}(\Gamma; \mathbb{F}_2)$. Let X_i be a set of representatives of i-cells in X and let Γ_σ be the stabilizer in Γ of a cell σ. Then

$$C^i(X) = \bigoplus_{\sigma \in X_i} \mathbb{F}_2 \cong \bigoplus_{\sigma \in \Gamma \backslash X_i} \mathrm{Coind}_{\Gamma_\sigma}^{\Gamma} \mathbb{F}_2$$

Once we apply Shapiro's Lemma and sum over the representatives of i-cells in $\Gamma\backslash X_i$, the spectral sequence takes the form

$$E_1^{i,j} \cong \bigoplus_{\sigma \in \Gamma \backslash X_i} \mathrm{H}^j(\Gamma_\sigma; \mathbb{F}_2).$$

The equivariant spectral sequence has a number of additional desirable properties that we will use in the sequel.

1. There is a product on the spectral sequence, $E_r^{pq} \otimes E_r^{st} \to E_r^{p+s,q+t}$, which is compatible with the standard cup product on $H^*(\Gamma)$ ([3, VII.5])
2. On the E_2–page, the products in $E_2^{0,*}$, the vertical edge, are compatible with the products in $\prod_{\sigma \in \Sigma_0} H^q(\Gamma_\sigma)$. ([3, X.4.5.vi]).

(3) The differential d_1 is a difference of restriction maps (cohomology analogon of [3, VII.8])

$$\bigoplus_{\sigma \in \Gamma\backslash X_i} \mathrm{H}^j(\Gamma_\sigma; \mathbb{F}_2) \xrightarrow{d_1^{i,j}} \bigoplus_{\sigma \in \Gamma\backslash X_{i+1}} \mathrm{H}^j(\Gamma_\sigma; \mathbb{F}_2).$$

We remark that when $\Gamma\backslash X$ is a line segment or a single edge and vertex loop, then the spectral sequence consists of only two columns and collapses at the E_2 term to give a Wang sequence where the appropriate map is the d_1 differential. This observation allows for the straightforward calculations for cohomology of amalgamated products and HNN extensions.

4. Excision in equivariant cohomology

Let Γ be a discrete group, ℓ be a prime number, n and q be integers. Let X be a cellular Γ-complex on which the cell stabilizers fix their cells pointwise and the center $\mathcal{Z}$ of Γ acts trivially.

***Definition* 6.** The *non-central ℓ-torsion sub-complex* of (X, Γ), which we denote by T, consists of the cells admitting elements of order ℓ in their stabilizer which are not contained in $\mathcal{Z}$.

Throughout this section, we exclusively consider the polynomial part of cohomology with trivial $\mathbb{F}_\ell$-coefficients; furthermore let us write Γ_σ for the stabilizer in Γ of a cell σ.

***Hypothesis* 7.** We require that for any cell $\tau \in T$, the inclusion $\mathcal{Z} \hookrightarrow \Gamma_\tau$ into the cell stabilizer of τ induces a monomorphism on the polynomial parts of the cohomology rings, $\mathrm{H}^*_{\mathrm{poly}}(\Gamma_\tau) \hookrightarrow \mathrm{H}^*_{\mathrm{poly}}(\mathcal{Z})$. Furthermore, for any cell $\sigma \in X$ not in T, we require that the inclusion $\mathcal{Z} \hookrightarrow \Gamma_\sigma$ induces an isomorphism $\mathrm{H}^*_{\mathrm{poly}}(\Gamma_\sigma) \to \mathrm{H}^*_{\mathrm{poly}}(\mathcal{Z})$.

Denote by $\Gamma\backslash X^n$ a system of representatives of the n-cells of X under the action of Γ, and analogously $\Gamma\backslash T^n$. Consider the cokernel of the map of Γ-equivariant co-chain complexes

$$0 \to \bigoplus_{\tau \in \Gamma\backslash T^n} \mathrm{H}^q_{\mathrm{poly}}(\Gamma_\tau) \hookrightarrow \bigoplus_{\tau \in \Gamma\backslash T^n} \mathrm{H}^q_{\mathrm{poly}}(\mathcal{Z}) \to \mathrm{Cokern}^q_n \to 0. \tag{2}$$

Consider the below short exact sequence of co-chain complexes, where the involved polynomial parts of cohomology groups have trivial mod ℓ coefficients:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle d_1^{2,q}} & & \uparrow{\scriptstyle \delta^{2,q}} & & \uparrow{\scriptstyle \partial^{2,q}} & & \\
0 & \longrightarrow & \bigoplus_{\sigma \in \Gamma\backslash X^2} \mathrm{H}^q_{\mathrm{poly}}(\Gamma_\sigma) & \longrightarrow & \bigoplus_{\sigma \in \Gamma\backslash X^2} \mathrm{H}^q_{\mathrm{poly}}(\mathcal{Z}) & \longrightarrow & \mathrm{Cokern}^q_2 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_1^{1,q}} & & \uparrow{\scriptstyle \delta^{1,q}} & & \uparrow{\scriptstyle \partial^{1,q}} & & \\
0 & \longrightarrow & \bigoplus_{\sigma \in \Gamma\backslash X^1} \mathrm{H}^q_{\mathrm{poly}}(\Gamma_\sigma) & \longrightarrow & \bigoplus_{\sigma \in \Gamma\backslash X^1} \mathrm{H}^q_{\mathrm{poly}}(\mathcal{Z}) & \longrightarrow & \mathrm{Cokern}^q_1 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_1^{0,q}} & & \uparrow{\scriptstyle \delta^{0,q}} & & \uparrow{\scriptstyle \partial^{0,q}} & & \\
0 & \longrightarrow & \bigoplus_{\sigma \in \Gamma\backslash X^0} \mathrm{H}^q_{\mathrm{poly}}(\Gamma_\sigma) & \longrightarrow & \bigoplus_{\sigma \in \Gamma\backslash X^0} \mathrm{H}^q_{\mathrm{poly}}(\mathcal{Z}) & \longrightarrow & \mathrm{Cokern}^q_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0.
\end{array} \tag{3}$$

Here, the differentials $\delta^{n,q}$ are given by the topological co-boundary maps of the cell complex X, and the maps $\partial^{n,q}$ are induced by the short exact sequence (2).

Applying the snake lemma, this yields the long exact sequence

$$\begin{array}{ccccccc}
 & \cdots & & & & & \\
 & \longrightarrow & \mathrm{poly}(E_2^{2,q}(X,\Gamma)) & \longrightarrow & \mathrm{poly}(E_2^{2,q}({}_\Gamma\backslash X,\mathcal{Z})) & \longrightarrow & \ker\partial^{2,q}/\mathrm{image}\,\partial^{1,q} \\
 & \longrightarrow & \mathrm{poly}(E_2^{1,q}(X,\Gamma)) & \longrightarrow & \mathrm{poly}(E_2^{1,q}({}_\Gamma\backslash X,\mathcal{Z})) & \longrightarrow & \ker\partial^{1,q}/\mathrm{image}\,\partial^{0,q} \\
0 & \longrightarrow & \mathrm{poly}(E_2^{0,q}(X,\Gamma)) & \longrightarrow & \mathrm{poly}(E_2^{0,q}({}_\Gamma\backslash X,\mathcal{Z})) & \longrightarrow & \ker\partial^{0,q}
\end{array}$$

Here, $\mathrm{poly}(E_2^{n,q}(X,\Gamma))$ is the polynomial part of the (n,q)-th term of the second page of the spectral sequence converging to the Γ-equivariant cohomology of X and $\mathrm{poly}(E_2^{n,q}({}_\Gamma\backslash X,\mathcal{Z}))$ the one for the trivial action of $\mathcal{Z}$ on ${}_\Gamma\backslash X$.

5. The input in the case of the Bianchi groups

The enumeration of the finite subgroups of $SL_2(\mathbb{C})$ is a classical result, and the list of finite subgroups which appear in the Bianchi groups $PSL_2(\mathcal{O}_d)$ is also well known [14]: They are the cyclic groups 1, $\mathbb{Z}/2$ and $\mathbb{Z}/3$, the Klein four group $\mathbf{D}_2$, the symmetric group on three letters $\mathbf{S}_3$, and the alternating group on four letters $\mathbf{A}_4$. The preimages of these groups in $SL_2(\mathcal{O}_d)$ are the cyclic groups $\mathbb{Z}/2$, $\mathbb{Z}/4$, $\mathbb{Z}/6$, the quaternion group $\mathbf{Q}_8$ of order 8 the dicyclic group $\mathbf{Di}$ of order 12, and the binary tetrahedral group $\mathbf{Te}$ of order 24.

We start by recalling some mod-2 cohomology rings. In what follows, $\mathbb{F}_2$ represents the field of two elements.

Theorem 8. [1] *The cohomology rings for finite subgroups of Bianchi groups are given below, where a subscript denotes the degree of the generator.*

$$\begin{aligned}
H^*(\mathbb{Z}/2) &\cong H^*(\mathbf{S}_3) \cong \mathbb{F}_2[x_1] \\
H^*(\mathbb{Z}/3) &\cong H^*(1) \cong \mathbb{F}_2 \\
H^*(\mathbf{D}_2) &\cong \mathbb{F}_2[x_1,y_1] \\
H^*(\mathbf{A}_4) &\cong \mathbb{F}_2[u_2,v_3,w_3]/(u_2^3+v_3^2+w_3^2+v_3w_3)
\end{aligned}$$

We use the Lyndon-Hochschild-Serre spectral sequence to determine the cohomology rings of the finite subgroups of $SL_2(\mathcal{O}_d)$. In particular, for each finite subgroup G of $SL_2(\mathcal{O}_d)$, there is a corresponding finite subgroup Q of $PSL_2(\mathcal{O}_d)$ which fits into the central extension

$$1 \to \mathbb{Z}/2 \longrightarrow G \longrightarrow Q \to 1$$

where $\mathbb{Z}/2$ is the subgroup of $SL_2(\mathcal{O}_d)$ containing $\{\pm I\}$. We note that all finite subgroups of $SL_2(\mathbb{C})$ also sit inside $SU(2)$ and act freely on it. Since $SU(2)$ can be identified with the 3 sphere, the cohomology rings for the finite subgroups of $SL_2(\mathcal{O}_d)$ are periodic of period dividing 4 [3]. In particular, the cohomology rings can all be expressed as a tensor product of a polynomial ring on one generator with an exterior algebra.

Theorem 9. *The following are the mod-2 cohomology rings of the finite subgroups of $SL_2(\mathcal{O}_d)$.*

$$\begin{aligned}
H^*(\mathbb{Z}/4) &\cong H^*(\mathbf{Di}) \cong \mathbb{F}_2[e_2](b_1) \\
H^*(\mathbb{Z}/2) &\cong H^*(\mathbb{Z}/6) \cong \mathbb{F}_2[e_1] \\
H^*(\mathbf{Q}_8) &\cong \mathbb{F}_2[e_4](x_1,y_1,x_2,y_2,x_3) \\
H^*(\mathbf{Te}) &\cong \mathbb{F}_2[e_4](b_3)
\end{aligned}$$

Proof. The cohomology results for cyclic group results are straightforward, and the calculation of the other cohomology rings (and a classification for all periodic groups) are contained in [1]. However, we briefly review the derivations here as the descriptions will be useful when we determine restriction maps.

The dicyclic group **Di** is a semidirect product and fits into the short exact sequence

$$1 \to \mathbb{Z}/3 \longrightarrow \mathbb{Z}/3 \rtimes \mathbb{Z}/4 \longrightarrow \mathbb{Z}/4 \to 1.$$

Since $H^*(\mathbb{Z}/3)$ has trivial mod-2 homology, in the Lyndon-Hochschild-Serre spectral sequence the only non-trivial cohomology is occurs on the x-axis and is isomorphic to $H^*(\mathbb{Z}/4)$.

For $\mathbb{Z}/4$, $\mathbf{Q}_8$ and **Te**, we again use the Lyndon-Hochschild-Serre spectral sequence for central extensions. In the spectral sequence of the extension, let $\mathbb{F}_2[e_1]$ be the cohomology ring corresponding the central subgroup $\mathbb{Z}/2$. On the E_1 page of the spectral sequence, the image of $d_1(e_1)$ can be identified with the k-invariant. In addition, the Lyndon-Hochschild-Serre spectral sequence is compatible with the Steenrod squaring operation, and so the Kudo transgression theorem describes the images of $d_{2n}((e_1)^{2n})$.

We start with the calculation of $H^*(\mathbb{Z}/4)$. Let $H^*(\mathbb{Z}/2) \cong \mathbb{F}_2[x_1]$, and consider the spectral sequence associated to the central extension. In this case, the k-invariant $d_1(e_1) = x_1^2$, and in particular $d_1(e_1 x_i^k) = x_i^{k+2}$ and $d_1(e_1^2) = 0$. As the result, the spectral sequence collapses at the E_2 page with classes in even rows and in columns 0 and 1 only. The class e_1^2 represents the polynomial class in $H^2(\mathbb{Z}/4)$, and the class in $E_2^{0,1}$ is the one-dimensional exterior class.

We next calculate $H^*(\mathbf{Q}_8)$. Since the extension is central, $E_1^{p,q} \cong H^p(\mathbf{D}_2) \otimes H^q(\mathbb{Z}/2)$, and by ([1] Proposition 2.8), the k-invariant $d_1(e_1) = x_1^2 + x_1y_1 + y_1^2 \in H^*(\mathbf{D}_2)$. On the E_2 page, this implies that all classes in odd dimensional rows vanish. Next, by the Kudo transgression theorem,

$$d_2\left(Sq^1(e_1)\right) = d_2(e_1^2) = x_1^2y_1 + x_1y_1^2 = Sq^1(x_1^2 + x_1y_1 + y_1^2).$$

By careful accounting one can show that the classes that remain in E_3 are in columns 0 through 3, and that the spectral sequence has non-zero terms in rows congruent to 0 mod 4. Consequently, all higher differentials vanish and $E_3 \cong E_\infty$. The only classes left correspond to 1, x_1, y_1, x_1^2, y_1^2, and $x_1^2y_1 = x_1y_1^2$ and the product of these classes with powers of e_1^4. The result follows.

Finally, we consider the calculation of $H^*(\mathbf{Te})$, which proceeds in a similar manner as the case for $H^*(\mathbf{Q}_8)$, except now the quotient group is $\mathbf{A}_4$ and $H^*(\mathbf{A}_4) \cong \mathbb{F}_2[u_2, v_3, w_3] \,/(u_2^3 + v_3^2 + w_3^2 + v_3w_3)$. Briefly, $d_1(e_1) = u_2$, and $d_2(e_1^2) = v_3 = Sq^1(u_2)$ [2]. Once again, after a careful check, the spectral sequence collapses at the E_3 page and the only classes left correspond to 1, $w_3 \in E_3^{3,0}$, and the products of these classes with powers of e_1^4. □

Given the following commuting diagram of groups,

$$\begin{array}{ccccc} \mathbb{Z}/2 & \longrightarrow & G_1 & \longrightarrow & Q_1 \\ \| & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_1} \\ \mathbb{Z}/2 & \longrightarrow & G_2 & \longrightarrow & Q_2 \end{array}$$

if one knows the induced map $i_1^* : H^*(Q_2) \longrightarrow H^*(Q_1)$, it is possible to calculate the effect of $i_2^* : H^*(G_2) \longrightarrow H^*(G_1)$. The commutative diagram gives rise to two Lyndon-Hochschild-Serre spectral sequences, one for each extension, and the two are compatible via the isomorphism on the fiber. Specifically, the maps between the spectral sequences are given by i_1^* along the x-axis and the identity along the y-axis. Once the spectral sequences converge, the result gives the effect of the restriction map $res_{G_1}^{G_2} = i_2^*$.

Theorem 10. *The following are the non-trivial restriction maps on cohomology generators for finite subgroups of $SL_2(\mathcal{O}_d)$:*

$$\begin{aligned}
res^{\mathbb{Z}/4}_{\mathbb{Z}/2}(e_2) &= e_1^2 \\
res^{\mathbf{Di}}_{\mathbb{Z}/2}(e_2) &= res^{\mathbf{Di}}_{\mathbb{Z}/6}(e_2) = e_1^2 \\
res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(e_4) &= e_2^2, res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(x_1) = (b_1) \\
res^{\mathbf{Q}_8}_{\mathbb{Z}/2}(e_4) &= e_1^4 \\
res^{\mathbf{Te}}_{\mathbb{Z}/6}(e_4) &= res^{\mathbf{Te}}_{\mathbb{Z}/2}(e_4) = e_1^4 \\
res^{\mathbf{Te}}_{\mathbb{Z}/4}(e_4) &= e_2^2
\end{aligned}$$

In addition, $res^{\mathbb{Z}/6}_{\mathbb{Z}/2}$ and $res^{\mathbf{Di}}_{\mathbb{Z}/4}$ are isomorphisms.

Proof. The isomorphism cases follow as $\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$ and $H^*(S_3) \cong H^*(\mathbb{Z}/2)$. For the rest of the restriction maps, we can prove the results concerning the polynomial classes using the Lyndon-Hochschild-Serre spectral sequence calculation in cohomology associated to the central extension

$$1 \to \mathbb{Z}/2 \longrightarrow G \longrightarrow Q \to 1.$$

In every calculation, the polynomial class of $H^*(G)$ arises from the 0 column of the spectral sequence, so it corresponds to some power of e_1^k. The choice of k invariant of the extension determines when d_{2n} vanishes; at this point, e_1^{2n} survives to E_∞ and represents the polynomial class in the abutment. The result follows since the map between spectral sequences on the fiber (the y-axis) is the identity.

To calculate $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(x_1)$, recall that $H^*(\mathbf{D}_2) = \mathbb{F}_2[s_1, t_1]$, and that if s_1 and $x_1 \in H^*(\mathbb{Z}/2)$ represent the same group element in $\mathbf{D}_2$, then $res^{\mathbf{D}_2}_{\mathbb{Z}/2}(s_1^j) = x_1^j$ and $res^{\mathbf{D}_2}_{\mathbb{Z}/2}(s_1^j t_1^k) = 0$ when $k > 0$. After the Lyndon-Hochschild-Serre spectral sequence is applied, $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(s_1)$ surjects onto the stated class in $H^1(\mathbb{Z}/4)$. We note that the other exterior classes of $H^*(\mathbf{Q}_8)$ in dimensions 2 and 3 must restrict to 0, since from Theorem 9, the E_2 page for the calculation of $H^*(\mathbb{Z}/4)$ is zero from the second column and to the right.

Finally, we need to show that $res^{\mathbf{Te}}_{\mathbb{Z}/4}(b_3) = 0$. We recall from Theorem 9 that in the calculation of $H^*(\mathbf{Te})$, the exterior class $b_3 \in E_\infty^{3,0}$. However, in the calculation of $H^*(\mathbb{Z}/4)$, $E_\infty^{i,j} = 0$ and $i > 1$. Therefore, the restriction map must send b_3 to 0. □

Remark 11. We note that the restriction map is natural with respect to cup products, so the results of Theorem 10 can be used to determine the effect of the restriction map on any cohomology class. In particular, the only non-trivial restriction map on classes with exterior components is $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(e_4^i x_1) = e_2^{2i} b_1$. In addition, in the cohomology of the groups we consider, only pure polynomial classes restrict non-trivially to pure polynomial classes.

We will use the cohomology and restriction information from finite groups to determine the cohomology of the reduced non-central 2-torsion sub-complexes. The key observation is that these sub-complexes correspond to groups which can be described as amalgamated products and HNN extensions. We recall the definition of an HNN extension.

Definition 12. Let $\theta : G_2 \to G_1$ be a injection of G_2 into G_1. An *HNN extension of* G_1 is a group with presentation

$$G_1 *_{G_2} = < t, G_1 \mid t^{-1} g t = \theta(g), g \in G_2 > .$$

The element t is often referred to as the *free letter*.

We note that the notation $G_1 *_{G_2}$ is not completely descriptive, since there may be many choices in G_1 for $(G_2)^t$.

Via Bass-Serre theory, it is known that amalgamated products and HNN extensions both act on trees. In the amalgamated product $G = G_1 *_H G_2$, there is an action where the fundamental domain is given by two vertices with stabilizers G_1 and G_2, connected by an edge with stabilizer H. In the HNN extension $G = G_1 *_{G_2}$, the fundamental domain is a single vertex with a loop

where the vertex stabilizer is G_1 and the edge stabilizer is G_2. We can calculate the cohomology of HNN extensions and amalgamated products using the next result.

Theorem 13. *Let σ_0 and σ_1 be 0-cells and 1-cells in a fundamental domain, and let G_v and G_e be the vertex and edge stabilizers respectively. Then there is a long exact sequence in cohomology*

$$\cdots \xrightarrow{\alpha} \bigoplus_{e\in\sigma_1} H^{i-1}(G_e) \xrightarrow{\delta} H^i(G) \xrightarrow{\beta} \bigoplus_{v\in\sigma_0} H^i(G_v) \xrightarrow{\alpha} \bigoplus_{e\in\sigma_1} H^i(G_e) \xrightarrow{\delta} \cdots \tag{4}$$

The direct sum is over one edge and two vertices if G is an amalgamated product, and over one edge and one vertex if G is an HNN extension.

The following is another result from Bass-Serre theory.

Proposition 14. *In the long exact sequence of Theorem 13, the map β is the restriction map. In an amalgamated product, $\alpha = res^{G_1}_H - res^{G_2}_H$. In an HNN extension, $\alpha = res^{G_1}_{G_2} - \theta^*$, where θ^* is the map induced by conjugating G_2 by the free letter.*

In fact, when the equivariant spectral sequence only has two non-zero columns, the E_1 page degenerates into a Wang sequence which is precisely the long exact sequence (4) in Theorem 13 . Furthermore, the map α is the d_1 differential. We note that although we are primarily interested in an additive calculation of the cohomology, the long exact sequence can also be used to determine ring information, as $\ker(\alpha) = \mathrm{im}(\beta)$, and this is a ring isomorphism.

We will use the results above to determine the cohomology for the four observed non-central reduced 2-torsion sub-complex types. These sub-complexes correspond to the four groups in the following list, determined by the edge stabilizers, the vertex stabilizers, and the way the former inject into the latter.

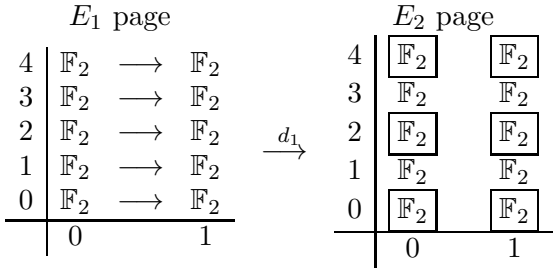
	Type of group	Quotient of non-central reduced 2-torsion sub-complex
(1)	HNN extension $\mathbb{Z}/4*_{\mathbb{Z}/4}$	
(2)	Amalgamated product $\mathbf{Te} *_{\mathbb{Z}/4} \mathbf{Te}$	
(3)	Double HNN extension $\left(\left(\mathbf{Q}_8 *_{\mathbb{Z}/4} \mathbf{Q}_8\right) *_{\mathbb{Z}/4}\right) *_{\mathbb{Z}/4}$	
(4)	Iterated construction $(\mathbf{Q}_8 *_{\mathbb{Z}/4}) *_{\mathbb{Z}/4} \mathbf{Te}$	

We will use the equivariant spectral sequence to carry out our cohomology calculations, where we consider G acting on a tree with quotient spaces and stabilizers given by the sub-complexes in the list. However, it is also possible to determine the cohomology using the long exact sequence 4. We recall Remark 11 which states that in terms of restriction maps on cohomology, polynomial classes and exterior classes only restrict to like classes. As a result, we are able in our calculations to distinguish cohomology in the abutment of the spectral sequence which originates solely from polynomial classes, which we denote by $\mathrm{H}_{\mathrm{poly}}$, from cohomology classes which contains exterior components, $\mathrm{H}_{\mathrm{ext}}$.

(1) Lemma 15. Let $G = \mathbb{Z}/4*_{\mathbb{Z}/4}$. Then the polynomial and exterior components of $H^*(G)$ are:

$$\mathrm{H}^q_{\mathrm{ext}}(G) \;\cong\; \mathbb{F}_2, \quad q > 0; \qquad \mathrm{H}^q_{\mathrm{poly}}(G) \;\cong\; \mathbb{F}_2, \quad q \geq 0$$

Proof. The non-central reduced 2-torsion sub-complex has shape  , with vertex and edge stabilizers both isomorphic to $\mathbb{Z}/4$. Using the restriction maps from Theorem 10 we set up the E_1 and E_2 pages of the equivariant spectral sequence for this HNN extension below. Classes on the E_2 page which arise from polynomial classes are boxed on the right side diagram.

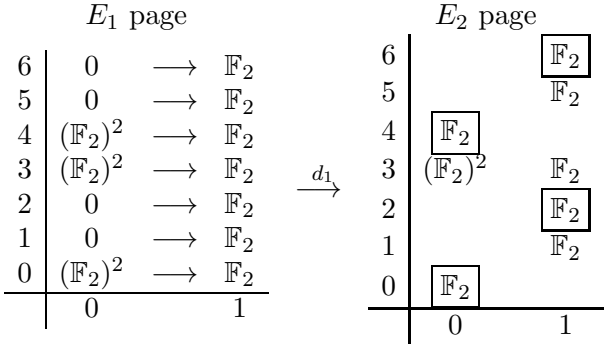


The arrows in the spectral sequence are the d_1 differentials given by $d_1 = res^{\mathbb{Z}/4}_{\mathbb{Z}/4} - \theta^*$. Since both maps have identical effects on $H^*(\mathbb{Z}/4)$, $d_1 = 0$. The polynomial classes occur in the even rows, and the exterior classes are in the odd rows. Passing to the abutment, there are polynomial classes originating in all dimensions starting in dimension 0; exterior classes start in dimension 1. □

(2) Lemma 16. Let $G = \mathbf{Te} *_{\mathbb{Z}/4} \mathbf{Te}$. Then the polynomial and exterior components of $H^*(G)$ are:

$$\mathrm{H}^q_{\mathrm{ext}}(G) \cong \begin{cases} 0, & q = 0 \\ \mathbb{F}_2, & q = 4k, k > 1; \\ 0, & q = 4k+1; \\ \mathbb{F}_2, & q = 4k+2; \\ (\mathbb{F}_2)^2, & q = 4k+3 \end{cases} \qquad \mathrm{H}^q_{\mathrm{poly}}(G) \cong \begin{cases} \mathbb{F}_2, & q = 4k; \\ 0, & q = 4k+1; \\ 0, & q = 4k+2; \\ \mathbb{F}_2, & q = 4k+3 \end{cases}$$

Proof. The non-central reduced 2-torsion sub-complex has shape •—•. Both vertex stabilizers are isomorphic to **Te**, and the edge stabilizer is isomorphic to $\mathbb{Z}/4$. Theorems 9 and 10 allow us to build the E_1 and E_2 pages of the equivariant spectral sequence for this amalgamated product. Both pages are below.



On the E_1 page, the polynomial classes appear in rows $4k$ in the zeroth column and $4k+2$ in the first column, and the only non-zero restriction maps are the ones between these classes. □

(3) Lemma 17. Let $G = \big(\big(\mathbf{Q}_8 *_{\mathbb{Z}/4} \mathbf{Q}_8\big) *_{\mathbb{Z}/4}\big) *_{\mathbb{Z}/4}$. Then the polynomial and exterior components of $H^*(G)$ are:

$$\mathrm{H}^q_{\mathrm{ext}}(G) \cong \begin{cases} 0, & q = 0 \\ (\mathbb{F}_2)^3, & q = 4k, k > 1; \\ (\mathbb{F}_2)^2, & q = 4k+1; \\ (\mathbb{F}_2)^5, & q = 4k+2; \\ (\mathbb{F}_2)^2, & q = 4k+3 \end{cases} \qquad \mathrm{H}^q_{\mathrm{poly}}(G) \cong \begin{cases} \mathbb{F}_2, & q = 4k; \\ (\mathbb{F}_2)^2, & q = 4k+1; \\ 0, & q = 4k+2; \\ (\mathbb{F}_2)^3, & q = 4k+3 \end{cases}$$

Proof. The non-central reduced 2-torsion sub-complex has shape  . The two vertex stabilizers are isomorphic to $\mathbf{Q}_8$, and the three edge stabilizers are all isomorphic to $\mathbb{Z}/4$. Therefore, the E_1 page of the spectral sequence has two copies of $H^*(\mathbf{Q}_8)$ in column 0 and three copies of $H^*(\mathbb{Z}/4)$

in column 1. Our main concern is to determine the action of the d_1 differential from $E_1^{0,1}$ to $E_1^{1,1}$.

By Theorem 10, we know that in dimension 1, a non-zero restriction $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}$ can be traced back to a non-zero restriction $res^{\mathbf{D}_2}_{\mathbb{Z}/2}$. In cohomology, let b_{11}, b_{12}, and b_{13} denote the exterior classes in $\oplus_3 H^1(\mathbb{Z}/4)$ corresponding to the three edge stabilizers, and let $x_{11}, y_{11}, x_{12}, y_{12}$ be the exterior classes in $H^1(\mathbf{Q}_8) \oplus H^1(\mathbf{Q}_8)$ corresponding to the two vertex stabilizers. Further, assume that x_{11} and x_{12} both restrict isomorphically to b_{11} via the correspondence from Theorem 10, and that y_{11} and y_{12} both restrict isomorphically to b_{12}. In $\mathbf{D}_2$, the product of any two non-zero elements is the final non-zero element. Therefore, in cohomology, x_{11}, y_{11}, x_{12} and y_{12} all restrict to b_{13}. That is,

$$\begin{aligned} d_1^{0,1}(x_{11}) &= b_{11} + b_{13} \\ d_1^{0,1}(y_{11}) &= b_{12} + b_{13} \\ d_1^{0,1}(x_{12}) &= b_{11} + b_{13} \\ d_1^{0,1}(y_{12}) &= b_{12} + b_{13}. \end{aligned}$$

So $d_1^{0,1}$ has rank 2. The only other non-zero differential is between polynomial classes in rows $4k$. Let e_{41} and e_{42} be the polynomial generators of $H^4(\mathbf{Q}_8) \oplus H^4(\mathbf{Q}_8)$, and let e_{21}^2, e_{22}^2, e_{23}^2 be the squares of the two-dimensional polynomial generators in $\oplus_3 H^4(\mathbb{Z}/4)$. Based on the geometry of the sub-complex, we have

$$d_1(e_{41}) = d_1(e_{42}) = e_{21}^2 + e_{22}^2 + e_{23}^2.$$

This implies that both the kernel and image of d_1 are 1-dimensional. We can now completely determine the E_1 and E_2 pages of the equivariant spectral sequence.

E_1 page

6	$(\mathbb{F}_2)^4$	$\longrightarrow$	$(\mathbb{F}_2)^3$
5	$(\mathbb{F}_2)^4$	$\longrightarrow$	$(\mathbb{F}_2)^3$
4	$(\mathbb{F}_2)^2$	$\longrightarrow$	$(\mathbb{F}_2)^3$
3	$(\mathbb{F}_2)^2$	$\longrightarrow$	$(\mathbb{F}_2)^3$
2	$(\mathbb{F}_2)^4$	$\longrightarrow$	$(\mathbb{F}_2)^3$
1	$(\mathbb{F}_2)^4$	$\longrightarrow$	$(\mathbb{F}_2)^3$
0	$(\mathbb{F}_2)^2$	$\longrightarrow$	$(\mathbb{F}_2)^3$
	0		1

$\xrightarrow{d_1}$

E_2 page

6	$(\mathbb{F}_2)^4$	$\boxed{(\mathbb{F}_2)^3}$
5	$(\mathbb{F}_2)^2$	$\mathbb{F}_2$
4	$\boxed{\mathbb{F}_2}$	$\boxed{(\mathbb{F}_2)^2}$
3	$(\mathbb{F}_2)^2$	$(\mathbb{F}_2)^3$
2	$(\mathbb{F}_2)^4$	$\boxed{(\mathbb{F}_2)^3}$
1	$(\mathbb{F}_2)^2$	$\mathbb{F}_2$
0	$\boxed{\mathbb{F}_2}$	$\boxed{(\mathbb{F}_2)^2}$
	0	1

□

(4) Lemma 18. Let $G = (\mathbf{Q}_8 *_{\mathbb{Z}/4}) * \mathbf{Te}$. Then the polynomial and exterior components of $H^*(G)$ are:

$$\mathrm{H}^q_{\mathrm{ext}}(G) \cong \begin{cases} 0 & q = 0 \\ (\mathbb{F}_2)^2, & q = 4k, k > 1; \\ \mathbb{F}_2, & q = 4k+1; \\ (\mathbb{F}_2)^3, & q = 4k+2; \\ (\mathbb{F}_2)^2, & q = 4k+3; \end{cases} \qquad \mathrm{H}^q_{\mathrm{poly}}(G) \cong \begin{cases} \mathbb{F}_2, & q = 4k; \\ \mathbb{F}_2, & q = 4k+1; \\ 0, & q = 4k+2; \\ (\mathbb{F}_2)^2, & q = 4k+3 \end{cases}$$

Proof. The non-central reduced 2-torsion sub-complex has shape  . The vertex stabilizer incident to the loop is isomorphic to $\mathbf{Q}_8$. The other vertex stabilizer is isomorphic to $\mathbf{Te}$, and the two edge stabilizers are isomorphic to $\mathbb{Z}/4$. This is the most complicated case, since the fundamental domain for G consists of two vertices and two edges, and the restriction maps arising from the edges have to be taken into account.

We follow the approach used for a similar mod-2 calculation in Lemma 3.1 of [2]. We start with an analysis of the component of $d_1 : H^*(\mathbf{Q}_8) \to H^*(\mathbb{Z}/4)$ where the $\mathbb{Z}/4$ is the stabilizer of

the loop. Referring back to the long exact sequence given in Equation 4, this component of d_1 is the difference $res^{\mathbf{Q}_8}_{\mathbb{Z}/4} - \theta^*$, where the restriction map is induced by subgroup injection on one side of the loop and by the twisting from the HNN extension on the other. We tweak the notation for classes from Theorem 9 by denoting by $b_{11} \in H^1(\mathbb{Z}/4)$ the class in the loop stabilizer and $b_{12} \in H^1(\mathbb{Z}/4)$ the class from the edge stabilizer. (We use the same convention for polynomial classes.) Recalling the mod-2 calculation referenced above we have

$$res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(x_1) = b_{11}, \text{ but } res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(y_1) = 0.$$

That is, the restriction only detects the subgroup on one side of the HNN extension. The map on cohomology induced by θ, on the other hand, detects the subgroup on both sides.

$$\theta^*(x_1) = \theta^*(y_1) = b_{11}.$$

Consequently, the component of d_1 mapping to the loop sends x_1 to 0 and y_1 to b_{11}. In addition, as $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(e_4) = \theta^*(e_4) = e_{21}^2$, the same component of d_1 sends e_4 to 0. The map $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}$ to the unlooped edge with $\mathbb{Z}/4$ stabilizer is given by Theorem 10, although we need to determine whether the exterior class with a non-trivial restriction is x_1, y_1, or $x_1 + y_1$. In fact, a basis for $H^*(\mathbf{Q}_8)$ can be chosen so that $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}$ sends x_1 to 0 and y_1 to b_{12}. Summarizing,

$$\begin{aligned} res^{\mathbf{Q}_8}_{\mathbb{Z}/4\oplus\mathbb{Z}/4}(y_1) &= b_{11} + b_{12} \\ res^{\mathbf{Q}_8}_{\mathbb{Z}/4\oplus\mathbb{Z}/4}(e_4) &= e_{22}^2 \end{aligned}$$

and all other classes are sent to 0. The rest of d_1 is given by the restriction map $res^{\mathbf{Te}}_{\mathbb{Z}/4}$, and this map is non-trivial only on the polynomial class. We have now completely determined the d_1 differential, so we can write down the E_1 and E_2 pages.

E_1 page

6	$(\mathbb{F}_2)^2$	$\longrightarrow$	$(\mathbb{F}_2)^2$
5	$(\mathbb{F}_2)^2$	$\longrightarrow$	$(\mathbb{F}_2)^2$
4	$(\mathbb{F}_2)^2$	$\longrightarrow$	$(\mathbb{F}_2)^2$
3	$(\mathbb{F}_2)^2$	$\longrightarrow$	$(\mathbb{F}_2)^2$
2	$(\mathbb{F}_2)^2$	$\longrightarrow$	$(\mathbb{F}_2)^2$
1	$(\mathbb{F}_2)^2$	$\longrightarrow$	$(\mathbb{F}_2)^2$
0	$(\mathbb{F}_2)^2$	$\longrightarrow$	$(\mathbb{F}_2)^2$
	0		1

$\xrightarrow{d_1}$

E_2 page

6	$(\mathbb{F}_2)^2$	$\boxed{(\mathbb{F}_2)^2}$
5	$\mathbb{F}_2$	$\mathbb{F}_2$
4	$\boxed{\mathbb{F}_2}$	$\boxed{\mathbb{F}_2}$
3	$(\mathbb{F}_2)^2$	$(\mathbb{F}_2)^2$
2	$(\mathbb{F}_2)^2$	$\boxed{(\mathbb{F}_2)^2}$
1	$\mathbb{F}_2$	$\mathbb{F}_2$
0	$\boxed{\mathbb{F}_2}$	$\boxed{\mathbb{F}_2}$
	0	1

□

6. Splitting the exterior components of the cohomology of Bianchi groups

Now, we decompose the exterior part of the cohomology of $SL_2(\mathcal{O}_d)$.

Theorem 19. *The exterior components of the four cohomology rings from Section 5 appear as summands in $H^*(SL_2(\mathcal{O}_d))$ with the same multiplicity as their corresponding components in the quotient of the non-central 2-torsion sub-complex.*

Proof. An argument in Section 6 of [11] explains why the mod-2 cohomology of $PSL_2(\mathcal{O}_d)$ splits into a direct sum above the virtual cohomological dimension. The argument references the calculations in [2], and notes that classes that arise in one component of the 2-torsion subcomplex do not restrict to subgroups in others. This implies that products between classes that come from distinct components multiply trivially in cohomology.

We need to extend this result from the projective special linear group to the linear group. In other words, we need to make sure that the result is compatible with the central extension of $PSL_2(\mathcal{O}_d)$ by $\mathbb{Z}/2$. This follows by an analysis of the 2-torsion sub-complex. We note first that all edge fusions which happen during the construction of the subcomplex remove a vertex

with stabilizer $\mathbb{Z}/2$ or S_3. The adjacent edges which are fused both have stabilizer $\mathbb{Z}/2$. Since $H^*(S_3) \cong H^*(\mathbb{Z}/2)$ and this is an isomorphism of rings, the mod-2 cohomology of the component corresponding to the 2-torsion subcomplex is ring- isomorphic to the mod-2 cohomology coming from the original component.

Since the central $\mathbb{Z}/2$ acts trivially on the retracted cell complex, the non-central 2-torsion subcomplex for $SL_2(\mathcal{O}_d)$ is identical to the 2-torsion subcomplex for $PSL_2(\mathcal{O}_d)$. The stabilizers for $SL_2(\mathcal{O}_d)$ are the $\mathbb{Z}/2$ extensions of the stabilizers of $PSL_2(\mathcal{O}_d)$. In particular, at vertices where there was an edge fusion in the 2-torsion subcomplex in the $PSL_2(\mathcal{O}_d)$ case, there will also be an edge fusion in the non-central 2-torsion subcomplex in the $SL_2(\mathcal{O}_d)$ case, since $H^*(\mathbf{Di}) \cong H^*(\mathbb{Z}/4)$. In addition, these are isomorphic as rings. Therefore, the non-central 2-torsion subcomplex can be reduced as in the $PSL_2(\mathcal{O}_d)$ case presented in [9]. □

7. Application of the technique to the Bianchi groups at the prime 2

We make use of a tessellation $X := \mathcal{H}$ of hyperbolic three-space by a Bianchi group $\Gamma := \mathrm{SL}_2(\mathcal{O}_{-m})$ in which each cell stabilizer fixes its cell pointwise. We are going to apply the method of section 4 in order to obtain the following dimension formula at the prime number $\ell = 2$. Our main result is the following.

Theorem 20. *The polynomial part of the $E_2^{n,q}$–page of the equivariant spectral sequence with $\mathbb{F}_2$–coefficients associated to the action of Γ on $\mathcal{H}$ is concentrated in the columns $n \in \{0,1,2\}$ with the following dimensions over $\mathbb{F}_2$.*

$q = 4k$	1	$\beta_1(\Gamma\backslash\mathcal{H})$	$\beta_2(\Gamma\backslash\mathcal{H})$
$q = 4k+3$	0	$\beta_0(\Gamma\backslash T) - 1 + \beta_1(\Gamma\backslash\mathcal{H}) - \beta_1(\Gamma\backslash T)$	$\beta_2(\Gamma\backslash\mathcal{H})$
$q = 4k+2$	$1 - \mathrm{sign}(v)$	$\beta_1(\Gamma\backslash\mathcal{H}) + v - \mathrm{sign}(v)$	$\beta_2(\Gamma\backslash\mathcal{H})$
$q = 4k+1$	0	$\beta_0(\Gamma\backslash T) - 1 + \beta_1(\Gamma\backslash\mathcal{H}) - \beta_1(\Gamma\backslash T)$	$\beta_2(\Gamma\backslash\mathcal{H})$
	$n = 0$	$n = 1$	$n = 2$

Here, β_i stands for the i–th Betti number of the orbit space in question; and v denotes the number of conjugacy classes of subgroups of quaternionic type $\mathbf{Q}_8$ in $\mathrm{SL}_2(\mathcal{O}_{-m})$, never mind if they are contained in a binary tetrahedral group.

Note that there is a formula for v in terms of the prime divisors of the discriminant of the ring $\mathcal{O}_{-m}$ of integers [6]. Also, recall that for $\mathcal{O}_{-m}$ not the Gaussian or Eisensteinian integers, the naive Euler characteristic $\beta_0(\Gamma\backslash\mathcal{H}) - \beta_1(\Gamma\backslash\mathcal{H}) + \beta_2(\Gamma\backslash\mathcal{H})$ vanishes, so $\beta_2(\Gamma\backslash\mathcal{H}) = \beta_1(\Gamma\backslash\mathcal{H}) - 1$.

Proof in odd degrees q. We note that since T is non-central, it admits no cell stabilizers of type $\mathbb{Z}/2$, so by Theorem 9, no cell stabilizer of T admits polynomial classes in odd degrees q of its cohomology ring. Then for q odd, the short exact sequence (2) becomes an isomorphism of cochain complexes

$$0 \to 0 \to \bigoplus_{\tau \in \Gamma\backslash T^n} \mathrm{H}^q_{\mathrm{poly}}(\mathbb{Z}/2) \to \mathrm{Cokern}^q_n \to 0,$$

which makes the short exact sequence (3) become

$$0 \to \bigoplus_{\sigma \in \Gamma\backslash \mathcal{H}^n} \mathrm{H}^q_{\mathrm{poly}}(\Gamma_\sigma) \to \bigoplus_{\sigma \in \Gamma\backslash \mathcal{H}^n} \mathrm{H}^q_{\mathrm{poly}}(\mathbb{Z}/2) \to \bigoplus_{\sigma \in \Gamma\backslash T^n} \mathrm{H}^q_{\mathrm{poly}}(\mathbb{Z}/2) \to 0.$$

From this, the snake lemma provides us the following long exact sequence, which is concentrated as specified because of the cohomological dimensions of the orbit spaces.

$$\begin{array}{ccccccc}
 & \to & \mathrm{poly}(E_2^{2,q}(\mathcal{H},\Gamma)) & \longrightarrow & \mathrm{H}^2(\Gamma\backslash\mathcal{H};\,\mathbb{F}_2) & \longrightarrow & 0 \longrightarrow \dots \\
 & \to & \mathrm{poly}(E_2^{1,q}(\mathcal{H},\Gamma)) & \longrightarrow & \mathrm{H}^1(\Gamma\backslash\mathcal{H};\,\mathbb{F}_2) & \longrightarrow & \mathrm{H}^1(\Gamma\backslash T;\,\mathbb{F}_2) \to \\
0 & \longrightarrow & \mathrm{poly}(E_2^{0,q}(\mathcal{H},\Gamma)) & \longrightarrow & \mathrm{H}^0(\Gamma\backslash\mathcal{H};\,\mathbb{F}_2) & \longrightarrow & \mathrm{H}^0(\Gamma\backslash T;\,\mathbb{F}_2) \to
\end{array}$$

Since $T \subset \mathcal{H}$, we immediately see that there is a natural inclusion $\mathrm{H}^0({}_\Gamma\backslash\mathcal{H};\, \mathbb{F}_2) \hookrightarrow \mathrm{H}^0({}_\Gamma\backslash T;\, \mathbb{F}_2)$ yielding $\mathrm{poly}(E_2^{0,q}(\mathcal{H},\Gamma)) = 0$.

Furthermore, since rotation axes can be quotiented into loops only by infinite-order-elements of the Bianchi groups, there can be no facets of the Bianchi fundamental polyhedron filling in such a loop. Thus, every loop in ${}_\Gamma\backslash T$ comes from a loop in ${}_\Gamma\backslash\mathcal{H}$; and we have a surjection $\mathrm{H}^1({}_\Gamma\backslash\mathcal{H};\, \mathbb{F}_2) \twoheadrightarrow \mathrm{H}^1({}_\Gamma\backslash T;\, \mathbb{F}_2)$. For this reason, we can conclude that

$$\mathrm{poly}(E_2^{2,q}(\mathcal{H},\Gamma)) \cong \mathrm{H}^2({}_\Gamma\backslash\mathcal{H};\, \mathbb{F}_2);$$

and we read off the claimed formula for the dimension of $\mathrm{poly}(E_2^{1,q}(\mathcal{H},\Gamma))$ from the remaining terms of the long exact sequence. □

Proof in degrees $q = 4k$. By Theorem 10, there is an isomorphism $\mathrm{H}^{4k}_{\mathrm{poly}}(\Gamma_\tau) \cong \mathrm{H}^{4k}_{\mathrm{poly}}(\mathbb{Z}/2)$ for all $\tau \in T$, so $\mathrm{Cokern}_n^{4k} = 0$ for all n; and the short exact sequence (3) reduces to an isomorphism of cochain complexes

$$0 \to \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^n} \mathrm{H}^{4k}_{\mathrm{poly}}(\Gamma_\sigma) \to \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^n} \mathrm{H}^{4k}_{\mathrm{poly}}(\mathbb{Z}/2) \to 0 \to 0.$$

Taking the homology of the cochain complexes yields an isomorphism

$$\mathrm{poly}(E_2^{n,4k}(\mathcal{H},\Gamma)) \cong \mathrm{H}^n({}_\Gamma\backslash\mathcal{H}).$$

□

Proof in degrees $q = 4k+2$. Consider the reduced non-central 2–torsion sub-complex T of a Bianchi group $\mathrm{SL}_2(\mathcal{O}_{-m})$. We split T into the union U of connected components of type and the remaining graph W without connected components of type  . Denote by e, respectively v, the number of edges, respectively vertices, of W. Then we observe

$$\bigoplus_{\tau\in\Gamma\backslash W^0} \mathrm{H}^{4k+2}_{\mathrm{poly}}(\Gamma_\tau) = 0 \text{ and } \bigoplus_{\tau\in\Gamma\backslash W^1} \mathrm{H}^{4k+2}_{\mathrm{poly}}(\Gamma_\tau) \cong (\mathbb{F}_2)^e;$$

whilst we find an isomorphism $\mathrm{H}^{4k+2}_{\mathrm{poly}}(\Gamma_\tau) \cong \mathrm{H}^{4k+2}_{\mathrm{poly}}(\mathbb{Z}/4) \cong \mathrm{H}^{4k+2}_{\mathrm{poly}}(\mathbb{Z}/2)$ for all τ in U. Then in degrees $q = 4k+2$, the short exact sequence (2) of Γ-equivariant co-chain complexes becomes concentrated in the two rows $n = 0$ and $n = 1$; reading as follows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \displaystyle\bigoplus_{\tau\in\Gamma\backslash U^1} \mathrm{H}^{4k+2}_{\mathrm{poly}}(\mathbb{Z}/2) \oplus (\mathbb{F}_2)^e & \longrightarrow & \displaystyle\bigoplus_{\tau\in\Gamma\backslash U^1} \mathrm{H}^{4k+2}_{\mathrm{poly}}(\mathbb{Z}/2) \oplus (\mathbb{F}_2)^e & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle \delta^{0,4k+2}|_{\Gamma\backslash T}} & & \uparrow{\scriptstyle \partial^{0,4k+2}} & & \\
0 & \longrightarrow & \displaystyle\bigoplus_{\tau\in\Gamma\backslash U^0} \mathrm{H}^{4k+2}_{\mathrm{poly}}(\mathbb{Z}/2) \oplus 0 & \xrightarrow{\mathrm{id}\oplus 0} & \displaystyle\bigoplus_{\tau\in\Gamma\backslash U^0} \mathrm{H}^{4k+2}_{\mathrm{poly}}(\mathbb{Z}/2) \oplus (\mathbb{F}_2)^v & \xrightarrow{0\oplus\mathrm{id}} & \mathrm{Cokern}_0^{4k+2} & \longrightarrow & 0,
\end{array}$$

so we have $\mathrm{Cokern}_0^{4k+2} \cong (\mathbb{F}_2)^v$ and $\mathrm{Cokern}_1^{4k+2} = 0$.

Therefore, the short exact sequence (3) becomes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow{\scriptstyle d_1^{2,4k+2}} & & \uparrow{\scriptstyle \delta^{2,4k+2}} & & \uparrow{\scriptstyle \partial^{2,4k+2}} & & \\
0 & \longrightarrow & \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^2} \mathbb{F}_2 & \longrightarrow & \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^2} \mathbb{F}_2 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_1^{1,4k+2}} & & \uparrow{\scriptstyle \delta^{1,4k+2}} & & \uparrow{\scriptstyle \partial^{1,4k+2}} & & \\
0 & \longrightarrow & \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^1} \mathrm{H}^{4k+2}_{\mathrm{poly}}(\Gamma_\sigma) & \longrightarrow & \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^1} \mathbb{F}_2 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_1^{0,4k+2}} & & \uparrow{\scriptstyle \delta^{0,4k+2}} & & \uparrow{\scriptstyle \partial^{0,4k+2}} & & \\
0 & \longrightarrow & \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^0} \mathrm{H}^{4k+2}_{\mathrm{poly}}(\Gamma_\sigma) & \longrightarrow & \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^0} \mathbb{F}_2 & \longrightarrow & (\mathbb{F}_2)^v & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

Applying the snake lemma, this yields the long exact sequence with non-trivial first connecting map

$$\begin{array}{ccccccccc}
 & & \mathrm{poly}(E_2^{2,4k+2}(\mathcal{H},\Gamma)) & \longrightarrow & \mathrm{H}^2({}_\Gamma\backslash\mathcal{H}) & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \mathrm{poly}(E_2^{1,4k+2}(\mathcal{H},\Gamma)) & \longrightarrow & \mathrm{H}^1({}_\Gamma\backslash\mathcal{H}) & \longrightarrow & 0 & & \\
0 & \longrightarrow & \mathrm{poly}(E_2^{0,4k+2}(\mathcal{H},\Gamma)) & \longrightarrow & \mathbb{F}_2 & \longrightarrow & (\mathbb{F}_2)^v & &
\end{array}$$

So,

$$\mathrm{poly}(E_2^{2,4k+2}(\mathcal{H},\Gamma)) \cong \mathrm{H}^2({}_\Gamma\backslash\mathcal{H}).$$

As $\partial^{0,4k+2}$ is the zero map, the isomorphism

$$\bigoplus_{\tau\in\Gamma\backslash W^0} \mathrm{H}^{4k+2}_{\mathrm{poly}}(\mathbb{Z}/2) \to \mathrm{Cokern}_0^{4k+2}$$

induces for $W \neq \emptyset$ an inclusion of the kernel of $\delta^{0,4k+2}$ into $\ker \partial^{0,4k+2} \cong (\mathbb{F}_2)^v$ by composition with the restriction map given by the connected component inclusion $W \subset \mathcal{H}$. Hence, $\mathrm{poly}(E_2^{0,4k+2}(\mathcal{H},\Gamma)) = (\mathbb{F}_2)^{1-\mathrm{sign}(v)}$, where we use the fact that $\mathrm{sign}(v)$ takes the same value as the indicator function of $\{W \neq \emptyset\}$. Therefore, the first connecting map is of rank $v - \mathrm{sign}(v)$. So to yield

$$\dim_{\mathbb{F}_2} \mathrm{poly}(E_2^{1,4k+2}(\mathcal{H},\Gamma)) = \beta^1({}_\Gamma\backslash\mathcal{H}) + v - \mathrm{sign}(v).$$

□

Note that a matrix for the map $\delta^{0,4k+2}|_{{}_\Gamma\backslash T}$ is given by the matrix block $\left(\begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix}\right)$ on the connected components of type  ; by the matrix block $\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)$ on the connected components of type  and by the zero block on ${}_\Gamma\backslash U^0$. However, note that we do not need this explicit form of the map.

7.1. **Example.** $m = 2$, ${}_\Gamma\backslash T =$  . For $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with $m = 2$, the orbit space ${}_\Gamma\backslash\mathcal{H}$ is homotopy equivalent to a cylinder, so we obtain the following polynomial part of the E_2–page for the equivariant spectral sequence of $(\mathcal{H}, \Gamma)$.

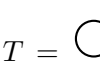
4	$\mathrm{H}^0({}_\Gamma\backslash\mathcal{H})$	$\mathrm{H}^1({}_\Gamma\backslash\mathcal{H})$
3	0	0
2	0	$\mathbb{F}_2 \oplus \mathbb{F}_2$
1	0	0
0	$\mathrm{H}^0({}_\Gamma\backslash\mathcal{H})$	$\mathrm{H}^1({}_\Gamma\backslash\mathcal{H})$
	0	1

Adding the exterior classes

$$\mathrm{exterior}(\mathrm{Tot}_q(E_2^{n,*}(\mathcal{H},\Gamma))) \cong \mathrm{exterior}(\mathrm{Tot}_q(E_2^{n,*}(T,\Gamma))) \quad\cong\quad \begin{cases} 0 & q = 0 \\ (\mathbb{F}_2)^2, & q = 4k, k > 1; \\ \mathbb{F}_2, & q = 4k+1; \\ (\mathbb{F}_2)^3, & q = 4k+2; \\ (\mathbb{F}_2)^2, & q = 4k+3; \end{cases}$$

in the columns $n = 0$ and $n = 1$, we obtain the E_2-page of $(\mathcal{H}, \Gamma)$ concentrated in these two columns. This allows us to conclude that

$$\mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-2}); \mathbb{F}_2) \cong \begin{cases} (\mathbb{F}_2)^2, & q = 5, \\ (\mathbb{F}_2)^3, & q = 4, \\ (\mathbb{F}_2)^4, & q = 3, \\ (\mathbb{F}_2)^3, & q = 2, \\ (\mathbb{F}_2)^2, & q = 1, \\ \mathbb{F}_2, & q = 0. \end{cases}$$

7.2. **Example.** Let $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with $m = 7$. Then ${}_\Gamma\backslash T =$  . Making use of the fact that in our case the quotient space ${}_\Gamma\backslash\mathcal{H}$ is homotopy equivalent to a Möbius strip, the above theorems and considerations yield

$$\mathrm{poly}(E_2^{n,q}(\mathcal{H},\Gamma)) \cong \begin{cases} \mathbb{F}_2, & q \text{ even and } n = 0 \text{ or } 1, \\ 0, & \text{otherwise.} \end{cases}$$

Adding the exterior classes, that are concentrated in the columns $n = 0$ and $n = 1$ and subject to the module types

$$\mathrm{exterior}(E_2^{n,q}(\mathcal{H},\Gamma)) \cong \mathrm{exterior}(E_2^{n,q}(T,\Gamma)) \cong \begin{cases} \mathbb{F}_2, & q \text{ odd}, \\ 0, & q \text{ even}, \end{cases}$$

we obtain the E_2-page of $(\mathcal{H}, \Gamma)$ concentrated in these two columns with $E_2^{n,q}(\mathcal{H},\Gamma) \cong \mathbb{F}_2$ for all q. This allows us to conclude that

$$\mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-7}); \mathbb{F}_2) \cong \begin{cases} (\mathbb{F}_2)^2, & q > 0, \\ \mathbb{F}_2, & q = 0. \end{cases}$$

7.3. **Example.** Let $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with $m = 11$. Then ${}_\Gamma\backslash T =$  . Making use of the fact that in our case the quotient space ${}_\Gamma\backslash\mathcal{H}$ is homotopy equivalent to a Möbius strip, the latter $\mathrm{poly}(E_2^{n,q}(\mathcal{H},\Gamma))$–page becomes

$q = 4k$	$\mathbb{F}_2$	$\mathbb{F}_2$	0
$q = 4k+3$	0	$\mathbb{F}_2$	0
$q = 4k+2$	0	$\mathbb{F}_2 \oplus \mathbb{F}_2$	0
$q = 4k+1$	0	$\mathbb{F}_2$	0
	$n = 0$	$n = 1$	$n = 2$

Adding the exterior classes $\mathrm{H}^q_{\mathrm{ext}}(\bullet\!\!-\!\!\!-\!\!\bullet) \cong \begin{cases} 0, & q=0 \\ \mathbb{F}_2, & q=4k, k>1; \\ 0, & q=4k+1; \\ \mathbb{F}_2, & q=4k+2; \\ (\mathbb{F}_2)^2, & q=4k+3, \end{cases}$ we obtain

$$\mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-11}); \mathbb{F}_2) \cong \begin{cases} (\mathbb{F}_2)^3, & q=4k+4, \\ (\mathbb{F}_2)^4, & q=4k+3, \\ (\mathbb{F}_2)^2, & q=4k+2, \\ \mathbb{F}_2, & q=4k+1, \\ \mathbb{F}_2, & q=0. \end{cases}$$

The last three examples have been checked in HAP [4] with the cellular complex imported from Bianchi.gp [12]; and the example $m=2$ additionally by a paper-and-pencil calculation by the first author using classical methods. We also observe that our last three examples are compatible with the homology with Steinberg coefficients calculated in [14] up to a minor typo that has infiltrated the case $m=2$ of the publisher's version probably during the manual re-typesetting process of the era.

7.4. **Example.** Consider a Bianchi group $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with $\mathrm{sign}(v)=1$. Then the polynomial part of the $E_\infty^{n,q}$–page of the equivariant spectral sequence with $\mathbb{F}_2$–coefficients associated to the action of Γ on $\mathcal{H}$ is concentrated in the columns $n \in \{0,1,2\}$ with the following dimensions over $\mathbb{F}_2$.

$q=4k+4$	$1-\mathrm{rank}(d_2)$	$\beta_1(\Gamma\backslash\mathcal{H})$	$\beta_2(\Gamma\backslash\mathcal{H})$
$q=4k+3$	0	$\beta_0(\Gamma\backslash T)-1+\beta_1(\Gamma\backslash\mathcal{H})-\beta_1(\Gamma\backslash T)$	$\beta_2(\Gamma\backslash\mathcal{H})-\mathrm{rank}(d_2)$
$q=4k+2$	0	$\beta_1(\Gamma\backslash\mathcal{H})+v-1$	$\beta_2(\Gamma\backslash\mathcal{H})$
$q=4k+1$	0	$\beta_0(\Gamma\backslash T)-1+\beta_1(\Gamma\backslash\mathcal{H})-\beta_1(\Gamma\backslash T)$	$\beta_2(\Gamma\backslash\mathcal{H})$
$q=0$	1	$\beta_1(\Gamma\backslash\mathcal{H})$	$\beta_2(\Gamma\backslash\mathcal{H})$
	$n=0$	$n=1$	$n=2$

This yields

$$\dim_{\mathbb{F}_2}\mathrm{H}^q_{\mathrm{poly}}(\Gamma;\mathbb{F}_2) \cong \begin{cases} -\mathrm{rank}(d_2)+\beta_0(\Gamma\backslash T)+\beta_1(\Gamma\backslash\mathcal{H})-\beta_1(\Gamma\backslash T)+\beta_2(\Gamma\backslash\mathcal{H}), & q=4k+4, \\ \beta_1(\Gamma\backslash\mathcal{H})+v-1+\beta_2(\Gamma\backslash\mathcal{H}), & q=4k+3, \\ \beta_0(\Gamma\backslash T)-1+\beta_1(\Gamma\backslash\mathcal{H})-\beta_1(\Gamma\backslash T)+\beta_2(\Gamma\backslash\mathcal{H}), & q=4k+2, \\ \beta_1(\Gamma\backslash\mathcal{H})+\beta_2(\Gamma\backslash\mathcal{H})-\mathrm{rank}(d_2), & q=4k+1, \\ \beta_1(\Gamma\backslash\mathcal{H}), & q=1. \end{cases}$$

More specifically, let $\Gamma\backslash T = \bullet\!\!-\!\!\!-\!\!\bullet$. Then adding the exterior classes we obtain

$$\dim_{\mathbb{F}_2}\mathrm{H}^q(\Gamma;\mathbb{F}_2) \cong \begin{cases} 2-\mathrm{rank}(d_2)+\beta_1(\Gamma\backslash\mathcal{H})+\beta_2(\Gamma\backslash\mathcal{H}), & q=4k+4, \\ 3+\beta_1(\Gamma\backslash\mathcal{H})+\beta_2(\Gamma\backslash\mathcal{H}), & q=4k+3, \\ 1+\beta_1(\Gamma\backslash\mathcal{H})+\beta_2(\Gamma\backslash\mathcal{H}), & q=4k+2, \\ \beta_1(\Gamma\backslash\mathcal{H})+\beta_2(\Gamma\backslash\mathcal{H})-\mathrm{rank}(d_2), & q=4k+1, \\ \beta_1(\Gamma\backslash\mathcal{H}), & q=1. \end{cases}$$

The type $\Gamma\backslash T = \bullet\!\!-\!\!\!-\!\!\bullet$ is taken for $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with $m \in \{19, 43, 67, 163\}$. In these cases, we can let HAP run, producing the results

$$\dim_{\mathbb{F}_2}\mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-m});\mathbb{F}_2) \cong \begin{cases} 2+\beta_1(\Gamma\backslash\mathcal{H})+\beta_2(\Gamma\backslash\mathcal{H}), & q=4k+4, \\ 3+\beta_1(\Gamma\backslash\mathcal{H})+\beta_2(\Gamma\backslash\mathcal{H}), & q=4k+3, \\ 1+\beta_1(\Gamma\backslash\mathcal{H})+\beta_2(\Gamma\backslash\mathcal{H}), & q=4k+2, \\ \beta_1(\Gamma\backslash\mathcal{H})+\beta_2(\Gamma\backslash\mathcal{H}), & q=4k+1, \\ \beta_1(\Gamma\backslash\mathcal{H}), & q=1, \end{cases}$$

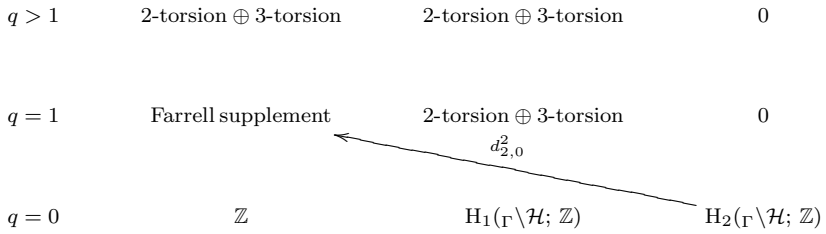
where the Betti numbers are

m	19	43	67	163
$\beta_1(\Gamma\backslash\mathcal{H})$	1	2	3	7
$\beta_2(\Gamma\backslash\mathcal{H})$	0	1	2	6.

Compatibility with our theoretical result is reached for $\mathrm{rank}(d_2^{0,4k}) = 0$.

8. The second page differential

The last piece of information that we need is the second page differential of the spectral sequence. For this purpose, we compute the d^2–differential of the equivariant homology spectral sequence associated to the action of $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$ on hyperbolic 3–space $\mathcal{H}$ with integral coefficients. We use the cell structure which is subdivided until each cells σ is fixed pointwise by its stabilizer Γ_σ. The $E^2_{p,q}$–page of this spectral sequence is concentrated as follows in the three columns $p \in \{0, 1, 2\}$.



where the "Farrell supplement" is the cokernel of the map

$$\bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^0} \mathrm{H}_1(\Gamma_\sigma;\mathbb{Z}) \xleftarrow{d^1_{1,1}} \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^1} \mathrm{H}_1(\Gamma_\sigma;\mathbb{Z}).$$

induced by inclusion of cell stabilizers. The Farrell supplement and $\mathrm{H}_1(\Gamma\backslash\mathcal{H};\mathbb{Z})$ have been computed on a database of Bianchi groups [13].

For the cases in this database for which the origin of $d^2_{2,0}$ is non-trivial and its target contains 2–torsion, Aurel Page has computed the Abelianization $\Gamma^{ab} \cong \mathrm{H}_1(\Gamma;\mathbb{Z})$, i.e. the commutator factor subgroup, of $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$. As the above spectral sequence converges to the group homology of Γ with integer coefficients, we obtain a short exact sequence

$$0 \to \text{Farrell supplement}/_{\text{image } d^2_{2,0}} \to \mathrm{PSL}_2(\mathcal{O}_{-m})^{ab} \to \mathrm{H}_1(\Gamma\backslash\mathcal{H};\mathbb{Z}) \to 1;$$

and we deduce the image of $d^2_{2,0}$ from it in the appendix.

***Observation* 21.** Within the scope of the database, the dimension over $\mathbb{F}_2$ of the 2–primary part of the image of the $d^2_{2,0}$–differential is at most the number of connected components of type  in the 2–torsion subcomplex quotient. It is clear from $\mathrm{H}_1(\mathcal{A}_4;\mathbb{Z}) \cong \mathbb{Z}/3$ that the target of $d^2_{2,0}$ has no 2–torsion on the connected components of type  ; and the same property follows for connected components of type  by a lemma in [9] specifying the matrix block induced in the $d^1_{1,1}$–differential by the inclusions into $\mathcal{D}_2$. The connected components on which our observation is backed only by the numerical results, are the ones of type  .

Appendix A. Numerical results

The program [8] computes a presentation of the Bianchi groups. Aurel Page has carried this out and calculated the commutator factor groups (Abelianizations) in the cases that we needed for the deduction of the image of $d^2_{2,0}$ described in section 8 and that we have achieved in the below tables. We insert the quotient of the reduced non-central 2–torsion sub-complex from [11] and [12]: Let A denote the number of connected components of homeomorphism type , let

B the one for , C for  and D for  .

$-$discriminant	m	$P\mathrm{SL}_2(\mathcal{O}_{-m})^{ab}$	$\mathrm{SL}_2(\mathcal{O}_{-m})^{ab}$	image $d^2_{2,0}$	A	B	C	D
35	35	$3*\mathbb{Z}\oplus\mathbb{Z}/3$	$3*\mathbb{Z}\oplus\mathbb{Z}/3$	$\mathbb{Z}/2$	1	0	0	0
40	10	$3*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	$3*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	$\mathbb{Z}/3$	0	0	1	0
47	47	$5*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$5*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	0	1	0	0	0
52	13	$3*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	$3*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	0	0	0	1	0
55	55	$5*\mathbb{Z}\oplus\mathbb{Z}/2$	$5*\mathbb{Z}\oplus\mathbb{Z}/4$	$\mathbb{Z}/3$	1	0	0	0
56	14	$5*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$5*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$\mathbb{Z}/2$	1	1	0	0
68	17	$5*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$5*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	0	0	1	1	0
79	79	$6*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	$6*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/4$	$\mathbb{Z}/2$	3	0	0	0
84	21	$7*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$7*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$\mathbb{Z}/2\oplus\mathbb{Z}/3$	3	0	0	0
87	87	$8*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$8*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	0	1	0	0	0
88	22	$5*\mathbb{Z}\oplus\mathbb{Z}/2$	$5*\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/3$	1	1	0	0
91	91	$5*\mathbb{Z}$	$5*\mathbb{Z}$	$\mathbb{Z}/2$	1	0	0	0
95	95	$9*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$9*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	0	1	0	0	0
103	103	$7*\mathbb{Z}\oplus\mathbb{Z}/2$	$7*\mathbb{Z}\oplus\mathbb{Z}/4$	0	1	0	0	0
104	26	$8*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$8*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$\mathbb{Z}/3$	0	0	1	0
111	111	$10*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$10*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	0	1	0	0	0
115	115	$7*\mathbb{Z}$	$7*\mathbb{Z}$	$\mathbb{Z}/2\oplus\mathbb{Z}/3$	1	0	0	0
116	29	$9*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$9*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	0	0	0	1	0
119	119	$11*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$11*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	$\mathbb{Z}/2$	2	0	0	0
120	30	$10*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$10*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$(\mathbb{Z}/2)^2\oplus(\mathbb{Z}/3)^2$	3	0	0	0
127	127	$8*\mathbb{Z}\oplus\mathbb{Z}/2$	$8*\mathbb{Z}\oplus\mathbb{Z}/4$	0	1	0	0	0
132	33	$10*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$10*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$\mathbb{Z}/2\oplus(\mathbb{Z}/3)^2$	3	2	0	0
136	34	$8*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	$8*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	$(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$	2	0	0	2
143	143	$12*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$12*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	$\mathbb{Z}/3$	1	0	0	0
148	37	$8*\mathbb{Z}\oplus 3*\mathbb{Z}/2$	$8*\mathbb{Z}\oplus 3*\mathbb{Z}/2$	$\mathbb{Z}/2$	2	0	1	0
151	151	$10*\mathbb{Z}\oplus\mathbb{Z}/2$	$10*\mathbb{Z}\oplus\mathbb{Z}/4$	0	1	0	0	0
152	38	$10*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$10*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	0	1	1	0	0
155	155	$10*\mathbb{Z}\oplus\mathbb{Z}/3$	$10*\mathbb{Z}\oplus\mathbb{Z}/3$	$\mathbb{Z}/2$	1	0	0	0
159	159	$14*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$14*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	0	1	0	0	0
164	41	$12*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$12*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	0	0	1	1	0
167	167	$13*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$13*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	0	1	0	0	0
168	42	$13*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$13*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$	3	0	0	0
183	183	$14*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$14*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	0	1	0	0	0
184	46	$11*\mathbb{Z}\oplus\mathbb{Z}/2$	$11*\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/3$	2	0	0	0
191	191	$15*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$15*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	0	1	0	0	0
195	195	$15*\mathbb{Z}\oplus\mathbb{Z}/3$	$15*\mathbb{Z}\oplus\mathbb{Z}/3$	$(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$	2	0	0	0
199	199	$13*\mathbb{Z}\oplus\mathbb{Z}/2$	$13*\mathbb{Z}\oplus\mathbb{Z}/4$	0	1	0	0	0
203	203	$12*\mathbb{Z}\oplus\mathbb{Z}/3$	$12*\mathbb{Z}\oplus\mathbb{Z}/3$	$\mathbb{Z}/2$	1	0	0	0
212	53	$14*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$14*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	0	0	0	1	0
215	215	$18*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$18*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	0	1	0	0	0
219	219	$13*\mathbb{Z}\oplus\mathbb{Z}/3$	$13*\mathbb{Z}\oplus\mathbb{Z}/3$	$\mathbb{Z}/2$	1	2	0	0
223	223	$15*\mathbb{Z}\oplus\mathbb{Z}/2$	$15*\mathbb{Z}\oplus\mathbb{Z}/4$	$(\mathbb{Z}/2)^2$	3	0	0	0
228	57	$16*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$16*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$	3	2	0	0
231	231	$21*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$21*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	$\mathbb{Z}/2\oplus\mathbb{Z}/3$	2	0	0	0
232	58	$12*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	$12*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	$\mathbb{Z}/3$	0	0	1	0
235	235	$13*\mathbb{Z}\oplus\mathbb{Z}/2$	$13*\mathbb{Z}\oplus\mathbb{Z}/2$	$(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$	3	0	0	0
239	239	$18*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$18*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	0	1	0	0	0
244	61	$15*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	$15*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	0	0	0	1	0
247	247	$14*\mathbb{Z}\oplus\mathbb{Z}/2$	$14*\mathbb{Z}\oplus\mathbb{Z}/4$	0	1	0	0	0
248	62	$16*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$16*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$\mathbb{Z}/2$	2	0	0	0
255	255	$23*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$23*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	$\mathbb{Z}/2\oplus(\mathbb{Z}/3)^2$	2	0	0	0
259	259	$14*\mathbb{Z}$	$14*\mathbb{Z}$	$\mathbb{Z}/2\oplus\mathbb{Z}/3$	1	0	0	0
260	65	$20*\mathbb{Z}\oplus 4*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$20*\mathbb{Z}\oplus 4*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$\mathbb{Z}/2\oplus\mathbb{Z}/3$	1	0	2	0
263	263	$18*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$18*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	$\mathbb{Z}/2\oplus\mathbb{Z}/3$	1	0	0	0
264	66	$20*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$20*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$\mathbb{Z}/2\oplus(\mathbb{Z}/3)^2$	2	2	0	0
271	271	$17*\mathbb{Z}\oplus\mathbb{Z}/2$	$17*\mathbb{Z}\oplus\mathbb{Z}/4$	0	1	0	0	0
276	69	$23*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$23*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$\mathbb{Z}/2\oplus\mathbb{Z}/3$	3	0	0	0
280	70	$19*\mathbb{Z}\oplus\mathbb{Z}/2$	$19*\mathbb{Z}\oplus\mathbb{Z}/2$	$(\mathbb{Z}/2)^2\oplus(\mathbb{Z}/3)^2$	3	0	0	0
287	287	$21*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$21*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	$\mathbb{Z}/2$	2	0	0	0
291	291	$17*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$17*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	0	1	2	0	0
292	73	$16*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	$16*\mathbb{Z}\oplus 2*\mathbb{Z}/2$	$\mathbb{Z}/3$	0	1	1	0
295	295	$19*\mathbb{Z}\oplus\mathbb{Z}/2$	$19*\mathbb{Z}\oplus\mathbb{Z}/4$	$\mathbb{Z}/3$	1	0	0	0
296	74	$19*\mathbb{Z}\oplus 4*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$19*\mathbb{Z}\oplus 4*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$\mathbb{Z}/3$	0	0	1	0
299	299	$18*\mathbb{Z}\oplus 2*\mathbb{Z}/3$	$18*\mathbb{Z}\oplus 2*\mathbb{Z}/3$	$\mathbb{Z}/2\oplus(\mathbb{Z}/3)^2$	1	0	0	0
303	303	$22*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$22*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	0	1	0	0	0
308	77	$23*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$23*\mathbb{Z}\oplus 2*\mathbb{Z}/2\oplus\mathbb{Z}/3$	$\mathbb{Z}/2\oplus\mathbb{Z}/3$	3	0	0	0
311	311	$23*\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$	$23*\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3$	0	1	0	0	0

$-\text{disc.}$	m	$P\mathrm{SL}_2\left(\mathcal{O}_{-m}\right)^{ab}$	$\mathrm{SL}_2\left(\mathcal{O}_{-m}\right)^{ab}$	$\operatorname{image} d_{2,0}^2$	A	B	C	D
312	78	$22 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$22 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	3	0	0	0
319	319	$20 * \mathbb{Z} \oplus \mathbb{Z}/2$	$20 * \mathbb{Z} \oplus \mathbb{Z}/4$	$\mathbb{Z}/3$	1	0	0	0
323	323	$16 * \mathbb{Z} \oplus \mathbb{Z}/3$	$16 * \mathbb{Z} \oplus \mathbb{Z}/3$	$\mathbb{Z}/2$	1	2	0	0
327	327	$24 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$24 * \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	0	1	0	0	0
328	82	$17 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2$	$17 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2$	$\mathbb{Z}/3$	1	1	1	0
335	335	$26 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$26 * \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	0	1	0	0	0
340	85	$23 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2$	$23 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2$	$(\mathbb{Z}/3)^2$	0	0	2	0
344	86	$21 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$21 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	0	1	1	0	0
355	355	$20 * \mathbb{Z}$	$20 * \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	1	0	0	0
356	89	$24 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$24 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	0	0	1	1	0
359	359	$25 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$25 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	$\mathbb{Z}/2$	3	0	0	0
367	367	$20 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/3$	$20 * \mathbb{Z} \oplus \mathbb{Z}/4 \oplus 2 * \mathbb{Z}/3$	$\mathbb{Z}/3$	1	0	0	0
371	371	$22 * \mathbb{Z} \oplus \mathbb{Z}/3$	$22 * \mathbb{Z} \oplus \mathbb{Z}/3$	$\mathbb{Z}/2$	1	0	0	0
372	93	$27 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$27 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	3	0	0	0
376	94	$22 * \mathbb{Z} \oplus \mathbb{Z}/2$	$22 * \mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	2	0	0	0
383	383	$25 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$25 * \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	0	1	0	0	0
388	97	$21 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2$	$21 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2$	$\mathbb{Z}/3$	0	1	1	0
391	391	$25 * \mathbb{Z} \oplus \mathbb{Z}/2$	$25 * \mathbb{Z} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	2	0	0	0
395	395	$24 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$24 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$\mathbb{Z}/2$	1	0	0	0
399	399	$33 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$33 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	4	0	0	0
403	403	$19 * \mathbb{Z}$	$19 * \mathbb{Z}$	$\mathbb{Z}/2$	1	0	0	0
404	101	$28 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$28 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$\mathbb{Z}/2$	2	0	1	0
407	407	$29 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$29 * \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	$\mathbb{Z}/3$	1	0	0	0
408	102	$27 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/3$	$27 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/3$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^4$	1	2	0	0
415	415	$28 * \mathbb{Z} \oplus \mathbb{Z}/2$	$28 * \mathbb{Z} \oplus \mathbb{Z}/4$	$\mathbb{Z}/3$	1	0	0	0
420	105	$41 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$41 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$(\mathbb{Z}/2)^6 \oplus (\mathbb{Z}/3)^3$	8	0	0	0
424	106	$23 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2$	$23 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2$	$\mathbb{Z}/3$	0	0	1	0
427	427	$21 * \mathbb{Z} \oplus \mathbb{Z}/2$	$21 * \mathbb{Z} \oplus \mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	3	0	0	0
431	431	$29 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$29 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	0	1	0	0	0
435	435	$31 * \mathbb{Z} \oplus \mathbb{Z}/3$	$31 * \mathbb{Z} \oplus \mathbb{Z}/3$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^5$	2	0	0	0
436	109	$25 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2$	$25 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2$	0	0	0	1	0
439	439	$26 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2$	$26 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/4$	$(\mathbb{Z}/2)^2$	5	0	0	0
440	110	$32 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$32 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	3	0	0	0
443	443	$21 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$21 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	0	1	1	0	0
447	447	$32 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$32 * \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	0	1	0	0	0
452	113	$27 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$27 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	0	0	1	1	0
455	455	$39 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$39 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	$\mathbb{Z}/3$	2	0	0	0
456	114	$32 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$32 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^3$	2	2	0	0
463	463	$23 * \mathbb{Z} \oplus \mathbb{Z}/2$	$23 * \mathbb{Z} \oplus \mathbb{Z}/4$	0	1	0	0	0
471	471	$34 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$34 * \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	0	1	0	0	0
472	118	$25 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2$	$25 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2$	$\mathbb{Z}/3$	1	1	0	0
479	479	$33 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus 3 * \mathbb{Z}/3$	$33 * \mathbb{Z} \oplus \mathbb{Z}/4 \oplus 3 * \mathbb{Z}/3$	0	1	0	0	0
483	483	$33 * \mathbb{Z} \oplus \mathbb{Z}/3$	$33 * \mathbb{Z} \oplus \mathbb{Z}/3$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	2	0	0	0
487	487	$24 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/13$	$24 * \mathbb{Z} \oplus \mathbb{Z}/4 \oplus 2 * \mathbb{Z}/13$	0	1	0	0	0
488	122	$28 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$28 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$\mathbb{Z}/3$	0	0	1	0
499	499	$22 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/3$	$22 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/3$	0	2	1	0	0
520	130	$32 * \mathbb{Z} \oplus 6 * \mathbb{Z}/2$	$32 * \mathbb{Z} \oplus 6 * \mathbb{Z}/2$	$(\mathbb{Z}/3)^2$	0	0	2	0
532	133	$33 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2$	$33 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$	3	0	0	0
555	555	$39 * \mathbb{Z} \oplus \mathbb{Z}/3$	$39 * \mathbb{Z} \oplus \mathbb{Z}/3$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	2	0	0	0
568	142	$29 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2 \oplus \mathbb{Z}/4$	$29 * \mathbb{Z} \oplus 3 * \mathbb{Z}/2 \oplus \mathbb{Z}/4$	$(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3$	6	0	0	0
595	595	$37 * \mathbb{Z} \oplus \mathbb{Z}/3$	$37 * \mathbb{Z} \oplus \mathbb{Z}/3$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^3$	2	0	0	0
667	667	$32 * \mathbb{Z}$	$32 * \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	1	0	0	0
696	174	$50 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$50 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$	3	0	0	0
715	715	$43 * \mathbb{Z}$	$43 * \mathbb{Z}$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$	2	0	0	0
723	723	$41 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$41 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus \mathbb{Z}/3$	$\mathbb{Z}/2$	1	2	0	0
760	190	$46 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2$	$46 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$	3	0	0	0
763	763	$38 * \mathbb{Z}$	$38 * \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	1	0	0	0
795	795	$55 * \mathbb{Z} \oplus \mathbb{Z}/3$	$55 * \mathbb{Z} \oplus \mathbb{Z}/3$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$	2	0	0	0
955	955	$50 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/3$	$50 * \mathbb{Z} \oplus 4 * \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/3$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	1	0	0	0
1003	1003	$48 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/3$	$48 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/3$	$\mathbb{Z}/3$	1	2	0	0
1027	1027	$48 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus 3 * \mathbb{Z}/3$	$48 * \mathbb{Z} \oplus 2 * \mathbb{Z}/2 \oplus 3 * \mathbb{Z}/3$	$\mathbb{Z}/2$	1	0	0	0
1227	1227	$69 * \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/3$	$69 * \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/3$	0	1	2	0	0
1243	1243	$58 * \mathbb{Z} \oplus 4 * \mathbb{Z}/3$	$58 * \mathbb{Z} \oplus 4 * \mathbb{Z}/3$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	1	2	0	0
1387	1387	$62 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/3 \oplus 2 * \mathbb{Z}/167$	$62 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/3 \oplus 2 * \mathbb{Z}/167$	$\mathbb{Z}/3$	1	2	0	0
1411	1411	$64 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/16 \oplus 2 * \mathbb{Z}/43$	$64 * \mathbb{Z} \oplus \mathbb{Z}/2 \oplus 2 * \mathbb{Z}/16 \oplus 2 * \mathbb{Z}/43$	$\mathbb{Z}/3$	1	2	0	0
1507	1507	$70 * \mathbb{Z} \oplus 2 * \mathbb{Z}/3 \oplus 4 * \mathbb{Z}/5$	$70 * \mathbb{Z} \oplus 2 * \mathbb{Z}/3 \oplus 4 * \mathbb{Z}/5$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	1	2	0	0
1555	1555	$80 * \mathbb{Z} \oplus 8 * \mathbb{Z}/4 \oplus 2 * \mathbb{Z}/11$	$80 * \mathbb{Z} \oplus 8 * \mathbb{Z}/4 \oplus 2 * \mathbb{Z}/11$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	1	0	0	0

References

[1] Alejandro Adem and R. James Milgram, *Cohomology of finite groups*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 309, Springer-Verlag, Berlin, 2004. MR2035696 (2004k:20109)

[2] Ethan Berkove, *The mod-2 cohomology of the Bianchi groups*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4585–4602. MR1675241 (2001b:11043)

[3] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. MR1324339 (96a:20072)

[4] Graham Ellis, *Homological algebra programming*, Computational group theory and the theory of groups, Contemp. Math., vol. 470, Amer. Math. Soc., Providence, RI, 2008, pp. 63–74, DOI 10.1090/conm/470/09186, (to appear in print). MR2478414 (2009k:20001)

[5] Otto Grün, *Beiträge zur Gruppentheorie. I.*, J. Reine Angew. Math. **174** (1935), 1-14 (German).JFM 61.0096.03

[6] Norbert Krämer, *Imaginärquadratische Einbettung von Maximalordnungen rationaler Quaternionenalgebren, und die nichtzyklischen endlichen Untergruppen der Bianchi-Gruppen*, preprint, 2012 `http://hal.archives-ouvertes.fr/hal-00720823/en/` (German).

[7] John McCleary, *User's Guide to Spectral Sequences*, Mathematics Lecture Series, vol. 12, Publish or Perish, Inc., Washington, Delaware, 1985.

[8] Aurel Page, *Computing arithmetic Kleinian groups*, preprint, `http://hal.archives-ouvertes.fr/hal-00703043`, 2012.

[9] Alexander D. Rahm, *The homological torsion of* PSL_2 *of the imaginary quadratic integers*, Trans. Amer. Math. Soc. **365** (2013), no. 3, 1603–1635. MR3003276

[10] ———, *Homology and K-theory of the Bianchi groups*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 11-12, 615–619. MR2817377 (2012e:20116)

[11] ———, *Accessing the Farrell-Tate cohomology of discrete groups*, preprint, `http://hal.archives-ouvertes.fr/hal-00618167`, 2012.

[12] Alexander D. Rahm, *Bianchi.gp*, Open source program (GNU general public license), validated by the CNRS: `http://www.projet-plume.org/fiche/bianchigp` Part of the Pari/GP Development Center scripts library, 2010.

[13] Alexander D. Rahm, *Higher torsion in the Abelianization of the full Bianchi groups*, preprint, `http://arxiv.org/abs/1207.7133v1s`, 2012.

[14] Joachim Schwermer and Karen Vogtmann, *The integral homology of* SL_2 *and* PSL_2 *of Euclidean imaginary quadratic integers*, Comment. Math. Helv. **58** (1983), no. 4, 573–598, DOI 10.1007/BF02564653. MR728453 (86d:11046)

[15] Richard G. Swan, *The p-period of a finite group*, Illinois J. Math. **4** (1960), 341–346. MR0122856 (23 #A188)

Mathematics Department, Lafayette College, Quad Drive, Easton, PA 18042, berkovee@lafayette.edu,
National University of Ireland at Galway, Department of Mathematics, `http://www.maths.nuigalway.ie/~rahm/`