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REFINEMENT OF THE BENOIST THEOREM ON THE SIZE OF DINI SUBDIFFERENTIALS

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ABSTRACT. Given a lower semicontinuous function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, we prove that the set of points of \mathbb{R}^n where the lower Dini subdifferential has convex dimension k is countably $(n - k)$ -rectifiable. In this way, we extend a theorem of Benoist (see [1, Theorem 3.3]), and as a corollary we obtain a classical result concerning the singular set of locally semiconcave functions.

1. Introduction. Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be any lower semicontinuous function, the lower Dini subdifferential of f at x in the domain of f (denoted by $\text{dom}(f)$) is defined by

$$\partial^- f(x) = \left\{ \zeta \in \mathbb{R}^n \mid \liminf_{y \rightarrow x} \frac{f(y) - f(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \geq 0 \right\}.$$

As it is well-known, for every $x \in \text{dom}(f)$, the set $\partial^- f(x)$ is a possibly empty convex subset of \mathbb{R}^n . Now let $k \in \{1, \dots, n\}$ be fixed; we call k -dimensional Dini singular set of f , denoted by $\mathcal{D}^k(f)$, the set of $x \in \text{dom}(f)$ such that $\partial^- f(x)$ is a nonempty convex set of dimension k . Moreover, we call Dini singular set of f , the set defined by

$$\mathcal{D}(f) := \bigcup_{k \in \{1, \dots, n\}} \mathcal{D}^k(f).$$

Before stating our result, we recall that, given $r \in \{0, 1, \dots, n\}$, the set $C \subset \mathbb{R}^n$ is called a r -rectifiable set if there exists a Lipschitz continuous function $\phi : \mathbb{R}^r \rightarrow \mathbb{R}^n$ such that $C \subset \phi(\mathbb{R}^r)$. In addition, C is called countably r -rectifiable if it is the union of a countable family of r -rectifiable sets. The aim of the present short note is to extend a result by Benoist, who proved that $\mathcal{D}(f)$ is countably $(n - 1)$ -rectifiable (see [1, Theorem 3.3]), and to obtain as a corollary a classical result on locally semiconcave functions. We prove the following result.

Theorem 1.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Then for every $k \in \{1, \dots, n\}$, the set $\mathcal{D}^k(f)$ is countably $(n - k)$ -rectifiable.*

Let us now recall briefly the notions of semiconcave and locally semiconcave functions; we refer the reader to the book [2] for further details on semiconcavity (see also [4]). Let Ω be an open and convex subset of \mathbb{R}^n , $u : \Omega \rightarrow \mathbb{R}$ be a continuous

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function, and C be a nonnegative constant. We say that u is C -semiconcave or semiconcave on Ω if

$$\mu u(y) + (1 - \mu)u(x) - u(\mu x + (1 - \mu)y) \leq \frac{\mu(1 - \mu)C}{2}|x - y|^2, \quad (1)$$

for any $\mu \in [0, 1]$, and any $x, y \in \mathbb{R}^n$. Consider now an open subset Ω of \mathbb{R}^n ; the function $u : \Omega \rightarrow \mathbb{R}$ is called locally semiconcave on Ω , if for every $x \in \Omega$, there is an open and convex neighborhood of x where u is semiconcave. For every $k \in \{1, \dots, n\}$, we call k -dimensional singular set of u , denoted by $\Sigma^k(u)$, the set of $x \in \Omega$ such that the Clarke generalized gradient of u at x , denoted by $\partial u(x)$, is a convex set of dimension k (see [2, 3]). In fact, it is easy to deduce from (1), that for any locally semiconcave function $u : \Omega \rightarrow \mathbb{R}$ on an open subset Ω of \mathbb{R}^n , the sets $\partial u(x)$ and $(-\partial^- u(x))$ coincide at any $x \in \Omega$ (see [2, Theorem 3.3.6 p. 59]). This implies that $\Sigma^k(u) = \mathcal{D}^k(-u)$ for every $k \in \{1, \dots, n\}$ and yields the following result.

Corollary 1. *Let Ω be an open subset of \mathbb{R}^n and $u : \Omega \rightarrow \mathbb{R}$ be a locally semiconcave function. Then for every $k \in \{1, \dots, n\}$, the set $\Sigma^k(u)$ is countably $(n - k)$ -rectifiable.*

Our proofs combine techniques developed by Benoist in [1] and Cannarsa, Sinestrari in [2].

Notations: Throughout this paper, we denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively, the Euclidean scalar product and norm in \mathbb{R}^n . For any $x \in \mathbb{R}^n$ and any $r > 0$, we set $B(x, r) := \{y \in \mathbb{R}^n \mid |y - x| < r\}$ and $\bar{B}(x, r) := \{y \in \mathbb{R}^n \mid |y - x| \leq r\}$. Finally, we use the abbreviations $B_r := B(0, r)$, $\bar{B}_r := \bar{B}(0, r)$, $B := B_1$, and $\bar{B} := \bar{B}_1$.

2. Preliminary results. Let $k \in \{1, \dots, n-1\}$, we call k -planes the k -dimensional subspaces of \mathbb{R}^n . Given a k -plane Π , we denote by Π^\perp its orthogonal complement in \mathbb{R}^n . Given $x \in \mathbb{R}^n$, we denote by $p_\Pi(x)$ and $p_{\Pi^\perp}(x)$ the orthogonal projections of x onto Π and Π^\perp respectively. If Π, Π' are two given k -planes, we set

$$d(\Pi, \Pi') := \|p_\Pi - p_{\Pi'}\|,$$

where $\|\cdot\|$ is the operator norm of a linear operator in \mathbb{R}^n . We notice that the set of k -planes, denoted by \mathcal{P}^k , equipped with the distance d , is a compact metric space. Hence it admits a dense countable family $\{\Pi_i^k\}_{i \geq 1}$. In the sequel, we denote by $B_d^k(\Pi, \epsilon)$ the set of $\Pi' \in \mathcal{P}^k$ such that $d(\Pi, \Pi') \leq \epsilon$.

Given a compact set $K \subset \mathbb{R}^n$, we recall that the support function σ_K of K is defined by

$$\forall h \in \mathbb{R}^n, \quad \sigma_K(h) := \max \{\langle w, h \rangle \mid w \in K\}.$$

We notice that if $\text{conv}(K)$ denotes the convex hull of K , then we have

$$\sigma_{\text{conv}(K)} = \sigma_K.$$

Moreover if K, K' are two compact sets such that $K \subset K'$, then $\sigma_K \leq \sigma_{K'}$.

Given a k -plane Π , we define the function $\bar{\sigma}_\Pi : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\forall h \in \mathbb{R}^n, \quad \bar{\sigma}_\Pi(h) := \max \{\langle w, h \rangle \mid w \in \Pi \cap \bar{B}\}.$$

The following result is useful for the proof of our theorem.

Lemma 2.1. *Let Π, Π' be two k -planes and $h \in \mathbb{R}^n$, then we have*

$$|\bar{\sigma}_\Pi(h) - \bar{\sigma}_{\Pi'}(h)| \leq d(\Pi, \Pi')|h|. \quad (2)$$

Proof. There is $w \in \Pi \cap \bar{B}$ such that $\bar{\sigma}_\Pi(h) = \langle w, h \rangle$. Set

$$d := |p_{\Pi'}(w)|.$$

Notice, that since $w \in \bar{B}$, we have necessarily $d \leq 1$, which means that $p_{\Pi'}(w)$ belongs to $\Pi' \cap \bar{B}$. Hence we have

$$\begin{aligned} \bar{\sigma}_{\Pi'}(h) &\geq \langle p_{\Pi'}(w), h \rangle \\ &= \langle p_{\Pi'}(w) - p_\Pi(w), h \rangle + \langle w, h \rangle \\ &\geq -\|p_{\Pi'}(w) - p_\Pi(w)\| |h| + \bar{\sigma}_\Pi(h) \\ &\geq -\|p_{\Pi'} - p_\Pi\| |w| |h| + \bar{\sigma}_\Pi(h) \\ &\geq -d(\Pi, \Pi') |h| + \bar{\sigma}_\Pi(h). \end{aligned}$$

We deduce that $\bar{\sigma}_{\Pi'}(h) - \bar{\sigma}_\Pi(h) \geq -d(\Pi, \Pi') |h|$. By symmetry, we obtain the inequality (2). \square

3. Proof of the theorem. Let $k \in \{1, \dots, n\}$ be fixed. Let us choose a sequence $(v_j)_{j \geq 1}$ which is dense in \mathbb{R}^n and let us define, for $\omega = (r, i, j, l) \in I := (\mathbb{N}^*)^4$, the set D_ω constituted of elements x belonging to the closed ball \bar{B}_r such that $f(x) \leq r$, and such that there exist $\Pi \in B_d^k(\Pi_i, \frac{1}{4r})$, $\rho \geq \frac{9}{r}$ and $\zeta \in \bar{B}(v_j, \frac{1}{2r})$ satisfying:

$$\forall y \in B\left(x, \frac{1}{l}\right), \quad f(y) \geq f(x) + \langle \zeta, y - x \rangle + \rho \bar{\sigma}_\Pi(y - x) - \frac{1}{2r} |y - x|. \quad (3)$$

Lemma 3.1. *We have the following inclusion:*

$$\mathcal{D}^k(f) \subset \bigcup_{\omega \in I} D_\omega.$$

Proof. Denote by e_1^k, \dots, e_k^k the standard basis in \mathbb{R}^k and choose a constant $\nu^k > 0$ such that

$$\bar{B}_{\nu^k}^k \subset \text{conv}(\pm e_1^k, \dots, \pm e_k^k), \quad (4)$$

where $\bar{B}_{\nu^k}^k$ denotes the closed ball centered at the origin with radius ν^k in \mathbb{R}^k . Let $x \in \mathcal{D}^k(f)$; there are $\zeta \in \mathbb{R}^n$ and $\mu > 0$ such that the convex set $\partial^- f(x)$ contains the k -ball \mathcal{B} defined as,

$$\mathcal{B} := \bar{B}(\zeta, \mu) \cap H,$$

where H denotes the affine subspace of dimension k which is spanned by $\partial^- f(x)$ in \mathbb{R}^n . Choose $r \geq 1$ such that $|x| \leq r$, $f(x) \leq r$, and $\mu \geq \frac{9}{\nu^k r}$. By (4), there are k vectors $e_1, \dots, e_k \in \mathbb{R}^n$ of norm 1 such that

$$\bar{B}_{\nu^k \mu} \cap \Pi \subset \mu E \subset \bar{B}_\mu, \quad (5)$$

where Π and E are defined by

$$\Pi := \text{SPAN}\{e_1, \dots, e_k\} \quad \text{and} \quad E := \text{conv}(\pm e_1, \dots, \pm e_k).$$

For every $m \in \{1, \dots, k\}$ and every $\epsilon = \pm 1$, the vector $\zeta + \mu \epsilon e_m$ belongs to \mathcal{B} , then there exists a neighborhood $\mathcal{V}_{m, \epsilon}$ of x such that

$$\forall y \in \mathcal{V}_{m, \epsilon}, \quad f(y) \geq f(x) + \langle \zeta + \mu \epsilon e_m, y - x \rangle - \frac{1}{2r} |y - x|.$$

Hence we deduce that for every $y \in \bigcap_{m \in \{1, \dots, k\}, \epsilon = \pm 1} \mathcal{V}_{m, \epsilon}$, we have

$$\begin{aligned} f(y) &\geq f(x) + \langle \zeta, y - x \rangle \\ &\quad + \max\{\mu \langle \epsilon e_m, y - x \rangle \mid m = 1, \dots, k, \epsilon = \pm 1\} - \frac{1}{2r} |y - x|. \end{aligned}$$

But by (5), we have for every $h \in \mathbb{R}^n$,

$$\max \{ \mu \langle \epsilon \epsilon_m, h \rangle \mid m = 1, \dots, k, \epsilon = \pm 1 \} = \sigma_{\mu E}(h) \geq \sigma_{(\bar{B}_{\nu^k \mu} \cap \Pi)}(h) = \nu^k \mu \bar{\sigma}_\Pi(h).$$

We conclude easily by density of the families $\{\Pi_i^k\}_{i \geq 1}, \{v_j\}_{j \geq 1}$. \square

Set for every $i \geq 1$, the cone

$$K_i := \left\{ h \in \mathbb{R}^n \mid \bar{\sigma}_{\Pi_i}(h) \leq \frac{1}{2} \|h\| \right\}.$$

We have the following lemma.

Lemma 3.2. *For every $\omega = (r, i, j, l) \in I$ and every $x \in D_\omega$, we have*

$$D_\omega \cap \bar{B}\left(x, \frac{1}{l}\right) \subset \{x\} + K_i.$$

Proof. Let $y \in D_\omega \cap \bar{B}\left(x, \frac{1}{l}\right)$ be fixed. There are $\Pi_y \in B_d^k\left(\Pi_i, \frac{1}{4r}\right)$, $\rho_y \geq \frac{9}{r}$ and $\zeta_y \in \bar{B}\left(v_j, \frac{1}{2r}\right)$ such that

$$\forall z \in \bar{B}\left(y, \frac{1}{l}\right), \quad f(z) \geq f(y) + \langle \zeta_y, z - y \rangle + \rho_y \bar{\sigma}_{\Pi_y}(z - y) - \frac{1}{2r} |z - y|.$$

In particular, for $z = x$, this implies

$$\begin{aligned} f(x) &\geq f(y) + \langle \zeta_y, x - y \rangle + \rho_y \bar{\sigma}_{\Pi_y}(x - y) - \frac{1}{2r} |y - x| \\ &\geq f(y) + \langle \zeta_y, x - y \rangle - \frac{1}{2r} |y - x|. \end{aligned} \quad (6)$$

But since $x \in D_\omega$, there are $\Pi_x \in B_d^k\left(\Pi_i, \frac{1}{4r}\right)$, $\rho_x \geq \frac{9}{r}$ and $\zeta_x \in \bar{B}\left(v_j, \frac{1}{2r}\right)$ such that

$$f(y) \geq f(x) + \langle \zeta_x, y - x \rangle + \rho_x \bar{\sigma}_{\Pi_x}(y - x) - \frac{1}{2r} |y - x|. \quad (7)$$

Summing the inequalities (6) and (7), we obtain

$$0 \geq \langle \zeta_x - \zeta_y, y - x \rangle + \rho_x \bar{\sigma}_{\Pi_x}(y - x) - \frac{1}{r} |y - x|.$$

But $|\zeta_x - \zeta_y| \leq \frac{1}{r}$, hence

$$\rho_x \bar{\sigma}_{\Pi_x}(y - x) \leq \frac{2}{r} |y - x|.$$

Which gives by (2)

$$\begin{aligned} \bar{\sigma}_{\Pi_i}(y - x) &= (\bar{\sigma}_{\Pi_i}(y - x) - \bar{\sigma}_{\Pi_x}(y - x)) + \bar{\sigma}_{\Pi_x}(y - x) \\ &\leq d(\Pi_i, \Pi_x) |y - x| + \frac{2}{\rho_x r} |y - x| \\ &\leq \frac{1}{4r} |y - x| + \frac{1}{4} |y - x| \\ &\leq \frac{1}{2} |y - x|. \end{aligned}$$

\square

Lemma 3.3. *Let $\omega = (r, i, j, l) \in I$ and $\bar{x} \in D_\omega$ be fixed; set*

$$A := p_{\Pi_i^\perp} \left(D_\omega \cap \bar{B} \left(\bar{x}, \frac{1}{2l} \right) \right).$$

For every $y \in A$, there exists a unique $z = z_y \in \Pi_i$ such that

$$y + z \in D_\omega \cap \bar{B} \left(\bar{x}, \frac{1}{2l} \right).$$

Moreover, the mapping $\psi_\omega : y \in A \mapsto z_y$ is Lipschitz continuous.

Proof. First of all, for every $y \in A$, there is, by definition of A , some $x \in D_\omega \cap \bar{B} \left(\bar{x}, \frac{1}{2l} \right)$ such that $y = p_{\Pi_i^\perp}(x)$. Since $x - y \in \Pi_i$, this proves the existence of z_y . To prove the uniqueness, we argue by contradiction. Let $y \in A$, assume that there are $z \neq z' \in \Pi_i$ such that $y + z$ and $y + z'$ belong to $D_\omega \cap \bar{B} \left(\bar{x}, \frac{1}{2l} \right)$. Since $y + z \in D_\omega$, by the previous lemma, we know that

$$D_\omega \cap \bar{B} \left(y + z, \frac{1}{l} \right) \subset \{y + z\} + K_i.$$

But since both $y + z$ and $y + z'$ belong to $\bar{B} \left(\bar{x}, \frac{1}{2l} \right)$, $y + z'$ belongs clearly to $D_\omega \cap \bar{B} \left(y + z, \frac{1}{l} \right)$. Hence $y + z' \in \{y + z\} + K_i$. Which means that $(y + z') - (y + z) = z' - z$ belongs to K_i . But since $z' - z \in \Pi_i$, we have that $\bar{\sigma}_{\Pi_i}(z' - z) = |z' - z| > \frac{1}{2}|z' - z|$. We find a contradiction. Let us now prove that the map ψ_ω is Lipschitz continuous. Let $y, y' \in A$ be fixed. By the proof above we know that $\psi_\omega(y) = x - y$ (resp. $\psi_\omega(y') = x' - y'$) where x is such that $y = p_{\Pi_i^\perp}(x)$ (resp. $y' = p_{\Pi_i^\perp}(x')$). Set $z := \psi_\omega(y)$, $z' := \psi_\omega(y')$ and $h := x' - x$. Since $x = y + z$ and $x' = y' + z'$ where $y, y' \in \Pi_i^\perp$ and $z, z' \in \Pi_i$, we have $|h|^2 = |z' - z|^2 + |y' - y|^2$. But $\bar{\sigma}_{\Pi_i}(h) = |z' - z| \leq \frac{1}{2}|h|$. Hence we obtain that

$$|z' - z| \leq |x' - x| = |h| \leq \frac{2}{\sqrt{3}}|y' - y|.$$

The proof of the lemma is completed. \square

From the lemma above, for every $\omega = (r, i, j, l) \in I$ and every $\bar{x} \in D_\omega$, the map $\phi : A \rightarrow \mathbb{R}^n$ defined as,

$$\forall y \in A, \quad \phi(y) = y + \psi_\omega(y),$$

is Lipschitz continuous and satisfies

$$D_\omega \cap \bar{B} \left(\bar{x}, \frac{1}{2l} \right) \subset \phi(A).$$

Since $A \subset \Pi_i^\perp$, such a map can be extended into a Lipschitz continuous map from Π_i^\perp into \mathbb{R}^n . Since Π_i^\perp has dimension $(n - k)$, we deduce that the set $D_\omega \cap \bar{B} \left(\bar{x}, \frac{1}{2l} \right)$ is $(n - k)$ -rectifiable. The fact that any set D_ω can be covered by a finite number of balls of radius $\frac{1}{2l}$ completes the proof of the theorem.

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