



**HAL**  
open science

# Refinement of the Benoist theorem on the size of Dini subdifferentials

Ludovic Rifford

► **To cite this version:**

Ludovic Rifford. Refinement of the Benoist theorem on the size of Dini subdifferentials. Communications on Pure and Applied Mathematics, 2008, 7 (1), pp.119-124. hal-00769097

**HAL Id: hal-00769097**

**<https://hal.science/hal-00769097>**

Submitted on 28 Dec 2012

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## REFINEMENT OF THE BENOIST THEOREM ON THE SIZE OF DINI SUBDIFFERENTIALS

LUDOVIC RIFFORD

Université de Nice-Sophia Antipolis, Laboratoire J.A. Dieudonné,  
Parc Valrose, 06108 Nice Cedex 02, France

(Communicated by Yacine Chitour)

ABSTRACT. Given a lower semicontinuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ , we prove that the set of points of  $\mathbb{R}^n$  where the lower Dini subdifferential has convex dimension  $k$  is countably  $(n - k)$ -rectifiable. In this way, we extend a theorem of Benoist (see [1, Theorem 3.3]), and as a corollary we obtain a classical result concerning the singular set of locally semiconcave functions.

**1. Introduction.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be any lower semicontinuous function, the lower Dini subdifferential of  $f$  at  $x$  in the domain of  $f$  (denoted by  $\text{dom}(f)$ ) is defined by

$$\partial^- f(x) = \left\{ \zeta \in \mathbb{R}^n \mid \liminf_{y \rightarrow x} \frac{f(y) - f(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \geq 0 \right\}.$$

As it is well-known, for every  $x \in \text{dom}(f)$ , the set  $\partial^- f(x)$  is a possibly empty convex subset of  $\mathbb{R}^n$ . Now let  $k \in \{1, \dots, n\}$  be fixed; we call  $k$ -dimensional Dini singular set of  $f$ , denoted by  $\mathcal{D}^k(f)$ , the set of  $x \in \text{dom}(f)$  such that  $\partial^- f(x)$  is a nonempty convex set of dimension  $k$ . Moreover, we call Dini singular set of  $f$ , the set defined by

$$\mathcal{D}(f) := \bigcup_{k \in \{1, \dots, n\}} \mathcal{D}^k(f).$$

Before stating our result, we recall that, given  $r \in \{0, 1, \dots, n\}$ , the set  $C \subset \mathbb{R}^n$  is called a  $r$ -rectifiable set if there exists a Lipschitz continuous function  $\phi : \mathbb{R}^r \rightarrow \mathbb{R}^n$  such that  $C \subset \phi(\mathbb{R}^r)$ . In addition,  $C$  is called countably  $r$ -rectifiable if it is the union of a countable family of  $r$ -rectifiable sets. The aim of the present short note is to extend a result by Benoist, who proved that  $\mathcal{D}(f)$  is countably  $(n - 1)$ -rectifiable (see [1, Theorem 3.3]), and to obtain as a corollary a classical result on locally semiconcave functions. We prove the following result.

**Theorem 1.1.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a lower semicontinuous function. Then for every  $k \in \{1, \dots, n\}$ , the set  $\mathcal{D}^k(f)$  is countably  $(n - k)$ -rectifiable.*

Let us now recall briefly the notions of semiconcave and locally semiconcave functions; we refer the reader to the book [2] for further details on semiconcavity (see also [4]). Let  $\Omega$  be an open and convex subset of  $\mathbb{R}^n$ ,  $u : \Omega \rightarrow \mathbb{R}$  be a continuous

---

2000 *Mathematics Subject Classification.* Primary: 49J52, 26B35; Secondary: 26BXX.  
*Key words and phrases.* Nonsmooth analysis, generalized differentials.

function, and  $C$  be a nonnegative constant. We say that  $u$  is  $C$ -semiconcave or semiconcave on  $\Omega$  if

$$\mu u(y) + (1 - \mu)u(x) - u(\mu x + (1 - \mu)y) \leq \frac{\mu(1 - \mu)C}{2}|x - y|^2, \quad (1)$$

for any  $\mu \in [0, 1]$ , and any  $x, y \in \mathbb{R}^n$ . Consider now an open subset  $\Omega$  of  $\mathbb{R}^n$ ; the function  $u : \Omega \rightarrow \mathbb{R}$  is called locally semiconcave on  $\Omega$ , if for every  $x \in \Omega$ , there is an open and convex neighborhood of  $x$  where  $u$  is semiconcave. For every  $k \in \{1, \dots, n\}$ , we call  $k$ -dimensional singular set of  $u$ , denoted by  $\Sigma^k(u)$ , the set of  $x \in \Omega$  such that the Clarke generalized gradient of  $u$  at  $x$ , denoted by  $\partial u(x)$ , is a convex set of dimension  $k$  (see [2, 3]). In fact, it is easy to deduce from (1), that for any locally semiconcave function  $u : \Omega \rightarrow \mathbb{R}$  on an open subset  $\Omega$  of  $\mathbb{R}^n$ , the sets  $\partial u(x)$  and  $(-\partial^- u(x))$  coincide at any  $x \in \Omega$  (see [2, Theorem 3.3.6 p. 59]). This implies that  $\Sigma^k(u) = \mathcal{D}^k(-u)$  for every  $k \in \{1, \dots, n\}$  and yields the following result.

**Corollary 1.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and  $u : \Omega \rightarrow \mathbb{R}$  be a locally semiconcave function. Then for every  $k \in \{1, \dots, n\}$ , the set  $\Sigma^k(u)$  is countably  $(n - k)$ -rectifiable.*

Our proofs combine techniques developed by Benoist in [1] and Cannarsa, Sinestrari in [2].

Notations: Throughout this paper, we denote by  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$ , respectively, the Euclidean scalar product and norm in  $\mathbb{R}^n$ . For any  $x \in \mathbb{R}^n$  and any  $r > 0$ , we set  $B(x, r) := \{y \in \mathbb{R}^n \mid |y - x| < r\}$  and  $\bar{B}(x, r) := \{y \in \mathbb{R}^n \mid |y - x| \leq r\}$ . Finally, we use the abbreviations  $B_r := B(0, r)$ ,  $\bar{B}_r := \bar{B}(0, r)$ ,  $B := B_1$ , and  $\bar{B} := \bar{B}_1$ .

**2. Preliminary results.** Let  $k \in \{1, \dots, n-1\}$ , we call  $k$ -planes the  $k$ -dimensional subspaces of  $\mathbb{R}^n$ . Given a  $k$ -plane  $\Pi$ , we denote by  $\Pi^\perp$  its orthogonal complement in  $\mathbb{R}^n$ . Given  $x \in \mathbb{R}^n$ , we denote by  $p_\Pi(x)$  and  $p_{\Pi^\perp}(x)$  the orthogonal projections of  $x$  onto  $\Pi$  and  $\Pi^\perp$  respectively. If  $\Pi, \Pi'$  are two given  $k$ -planes, we set

$$d(\Pi, \Pi') := \|p_\Pi - p_{\Pi'}\|,$$

where  $\|\cdot\|$  is the operator norm of a linear operator in  $\mathbb{R}^n$ . We notice that the set of  $k$ -planes, denoted by  $\mathcal{P}^k$ , equipped with the distance  $d$ , is a compact metric space. Hence it admits a dense countable family  $\{\Pi_i^k\}_{i \geq 1}$ . In the sequel, we denote by  $B_d^k(\Pi, \epsilon)$  the set of  $\Pi' \in \mathcal{P}^k$  such that  $d(\Pi, \Pi') \leq \epsilon$ .

Given a compact set  $K \subset \mathbb{R}^n$ , we recall that the support function  $\sigma_K$  of  $K$  is defined by

$$\forall h \in \mathbb{R}^n, \quad \sigma_K(h) := \max \{\langle w, h \rangle \mid w \in K\}.$$

We notice that if  $\text{conv}(K)$  denotes the convex hull of  $K$ , then we have

$$\sigma_{\text{conv}(K)} = \sigma_K.$$

Moreover if  $K, K'$  are two compact sets such that  $K \subset K'$ , then  $\sigma_K \leq \sigma_{K'}$ .

Given a  $k$ -plane  $\Pi$ , we define the function  $\bar{\sigma}_\Pi : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$\forall h \in \mathbb{R}^n, \quad \bar{\sigma}_\Pi(h) := \max \{\langle w, h \rangle \mid w \in \Pi \cap \bar{B}\}.$$

The following result is useful for the proof of our theorem.

**Lemma 2.1.** *Let  $\Pi, \Pi'$  be two  $k$ -planes and  $h \in \mathbb{R}^n$ , then we have*

$$|\bar{\sigma}_\Pi(h) - \bar{\sigma}_{\Pi'}(h)| \leq d(\Pi, \Pi')|h|. \quad (2)$$

*Proof.* There is  $w \in \Pi \cap \bar{B}$  such that  $\bar{\sigma}_\Pi(h) = \langle w, h \rangle$ . Set

$$d := |p_{\Pi'}(w)|.$$

Notice, that since  $w \in \bar{B}$ , we have necessarily  $d \leq 1$ , which means that  $p_{\Pi'}(w)$  belongs to  $\Pi' \cap \bar{B}$ . Hence we have

$$\begin{aligned} \bar{\sigma}_{\Pi'}(h) &\geq \langle p_{\Pi'}(w), h \rangle \\ &= \langle p_{\Pi'}(w) - p_\Pi(w), h \rangle + \langle w, h \rangle \\ &\geq -\|p_{\Pi'}(w) - p_\Pi(w)\| |h| + \bar{\sigma}_\Pi(h) \\ &\geq -\|p_{\Pi'} - p_\Pi\| |w| |h| + \bar{\sigma}_\Pi(h) \\ &\geq -d(\Pi, \Pi') |h| + \bar{\sigma}_\Pi(h). \end{aligned}$$

We deduce that  $\bar{\sigma}_{\Pi'}(h) - \bar{\sigma}_\Pi(h) \geq -d(\Pi, \Pi') |h|$ . By symmetry, we obtain the inequality (2).  $\square$

**3. Proof of the theorem.** Let  $k \in \{1, \dots, n\}$  be fixed. Let us choose a sequence  $(v_j)_{j \geq 1}$  which is dense in  $\mathbb{R}^n$  and let us define, for  $\omega = (r, i, j, l) \in I := (\mathbb{N}^*)^4$ , the set  $D_\omega$  constituted of elements  $x$  belonging to the closed ball  $\bar{B}_r$  such that  $f(x) \leq r$ , and such that there exist  $\Pi \in B_d^k(\Pi_i, \frac{1}{4r})$ ,  $\rho \geq \frac{9}{r}$  and  $\zeta \in \bar{B}(v_j, \frac{1}{2r})$  satisfying:

$$\forall y \in B\left(x, \frac{1}{l}\right), \quad f(y) \geq f(x) + \langle \zeta, y - x \rangle + \rho \bar{\sigma}_\Pi(y - x) - \frac{1}{2r} |y - x|. \quad (3)$$

**Lemma 3.1.** *We have the following inclusion:*

$$\mathcal{D}^k(f) \subset \bigcup_{\omega \in I} D_\omega.$$

*Proof.* Denote by  $e_1^k, \dots, e_k^k$  the standard basis in  $\mathbb{R}^k$  and choose a constant  $\nu^k > 0$  such that

$$\bar{B}_{\nu^k}^k \subset \text{conv}(\pm e_1^k, \dots, \pm e_k^k), \quad (4)$$

where  $\bar{B}_{\nu^k}^k$  denotes the closed ball centered at the origin with radius  $\nu^k$  in  $\mathbb{R}^k$ . Let  $x \in \mathcal{D}^k(f)$ ; there are  $\zeta \in \mathbb{R}^n$  and  $\mu > 0$  such that the convex set  $\partial^- f(x)$  contains the  $k$ -ball  $\mathcal{B}$  defined as,

$$\mathcal{B} := \bar{B}(\zeta, \mu) \cap H,$$

where  $H$  denotes the affine subspace of dimension  $k$  which is spanned by  $\partial^- f(x)$  in  $\mathbb{R}^n$ . Choose  $r \geq 1$  such that  $|x| \leq r$ ,  $f(x) \leq r$ , and  $\mu \geq \frac{9}{\nu^k r}$ . By (4), there are  $k$  vectors  $e_1, \dots, e_k \in \mathbb{R}^n$  of norm 1 such that

$$\bar{B}_{\nu^k \mu} \cap \Pi \subset \mu E \subset \bar{B}_\mu, \quad (5)$$

where  $\Pi$  and  $E$  are defined by

$$\Pi := \text{SPAN}\{e_1, \dots, e_k\} \quad \text{and} \quad E := \text{conv}(\pm e_1, \dots, \pm e_k).$$

For every  $m \in \{1, \dots, k\}$  and every  $\epsilon = \pm 1$ , the vector  $\zeta + \mu \epsilon e_m$  belongs to  $\mathcal{B}$ , then there exists a neighborhood  $\mathcal{V}_{m, \epsilon}$  of  $x$  such that

$$\forall y \in \mathcal{V}_{m, \epsilon}, \quad f(y) \geq f(x) + \langle \zeta + \mu \epsilon e_m, y - x \rangle - \frac{1}{2r} |y - x|.$$

Hence we deduce that for every  $y \in \bigcap_{m \in \{1, \dots, k\}, \epsilon = \pm 1} \mathcal{V}_{m, \epsilon}$ , we have

$$\begin{aligned} f(y) &\geq f(x) + \langle \zeta, y - x \rangle \\ &\quad + \max\{\mu \langle \epsilon e_m, y - x \rangle \mid m = 1, \dots, k, \epsilon = \pm 1\} - \frac{1}{2r} |y - x|. \end{aligned}$$

But by (5), we have for every  $h \in \mathbb{R}^n$ ,

$$\max \{ \mu \langle \epsilon \epsilon_m, h \rangle \mid m = 1, \dots, k, \epsilon = \pm 1 \} = \sigma_{\mu E}(h) \geq \sigma_{(\bar{B}_{\nu^k \mu} \cap \Pi)}(h) = \nu^k \mu \bar{\sigma}_\Pi(h).$$

We conclude easily by density of the families  $\{\Pi_i^k\}_{i \geq 1}, \{v_j\}_{j \geq 1}$ .  $\square$

Set for every  $i \geq 1$ , the cone

$$K_i := \left\{ h \in \mathbb{R}^n \mid \bar{\sigma}_{\Pi_i}(h) \leq \frac{1}{2} \|h\| \right\}.$$

We have the following lemma.

**Lemma 3.2.** *For every  $\omega = (r, i, j, l) \in I$  and every  $x \in D_\omega$ , we have*

$$D_\omega \cap \bar{B}\left(x, \frac{1}{l}\right) \subset \{x\} + K_i.$$

*Proof.* Let  $y \in D_\omega \cap \bar{B}\left(x, \frac{1}{l}\right)$  be fixed. There are  $\Pi_y \in B_d^k\left(\Pi_i, \frac{1}{4r}\right)$ ,  $\rho_y \geq \frac{9}{r}$  and  $\zeta_y \in \bar{B}\left(v_j, \frac{1}{2r}\right)$  such that

$$\forall z \in \bar{B}\left(y, \frac{1}{l}\right), \quad f(z) \geq f(y) + \langle \zeta_y, z - y \rangle + \rho_y \bar{\sigma}_{\Pi_y}(z - y) - \frac{1}{2r} |z - y|.$$

In particular, for  $z = x$ , this implies

$$\begin{aligned} f(x) &\geq f(y) + \langle \zeta_y, x - y \rangle + \rho_y \bar{\sigma}_{\Pi_y}(x - y) - \frac{1}{2r} |y - x| \\ &\geq f(y) + \langle \zeta_y, x - y \rangle - \frac{1}{2r} |y - x|. \end{aligned} \quad (6)$$

But since  $x \in D_\omega$ , there are  $\Pi_x \in B_d^k\left(\Pi_i, \frac{1}{4r}\right)$ ,  $\rho_x \geq \frac{9}{r}$  and  $\zeta_x \in \bar{B}\left(v_j, \frac{1}{2r}\right)$  such that

$$f(y) \geq f(x) + \langle \zeta_x, y - x \rangle + \rho_x \bar{\sigma}_{\Pi_x}(y - x) - \frac{1}{2r} |y - x|. \quad (7)$$

Summing the inequalities (6) and (7), we obtain

$$0 \geq \langle \zeta_x - \zeta_y, y - x \rangle + \rho_x \bar{\sigma}_{\Pi_x}(y - x) - \frac{1}{r} |y - x|.$$

But  $|\zeta_x - \zeta_y| \leq \frac{1}{r}$ , hence

$$\rho_x \bar{\sigma}_{\Pi_x}(y - x) \leq \frac{2}{r} |y - x|.$$

Which gives by (2)

$$\begin{aligned} \bar{\sigma}_{\Pi_i}(y - x) &= (\bar{\sigma}_{\Pi_i}(y - x) - \bar{\sigma}_{\Pi_x}(y - x)) + \bar{\sigma}_{\Pi_x}(y - x) \\ &\leq d(\Pi_i, \Pi_x) |y - x| + \frac{2}{\rho_x r} |y - x| \\ &\leq \frac{1}{4r} |y - x| + \frac{1}{4} |y - x| \\ &\leq \frac{1}{2} |y - x|. \end{aligned}$$

$\square$

**Lemma 3.3.** *Let  $\omega = (r, i, j, l) \in I$  and  $\bar{x} \in D_\omega$  be fixed; set*

$$A := p_{\Pi_i^\perp} \left( D_\omega \cap \bar{B} \left( \bar{x}, \frac{1}{2l} \right) \right).$$

*For every  $y \in A$ , there exists a unique  $z = z_y \in \Pi_i$  such that*

$$y + z \in D_\omega \cap \bar{B} \left( \bar{x}, \frac{1}{2l} \right).$$

*Moreover, the mapping  $\psi_\omega : y \in A \mapsto z_y$  is Lipschitz continuous.*

*Proof.* First of all, for every  $y \in A$ , there is, by definition of  $A$ , some  $x \in D_\omega \cap \bar{B} \left( \bar{x}, \frac{1}{2l} \right)$  such that  $y = p_{\Pi_i^\perp}(x)$ . Since  $x - y \in \Pi_i$ , this proves the existence of  $z_y$ . To prove the uniqueness, we argue by contradiction. Let  $y \in A$ , assume that there are  $z \neq z' \in \Pi_i$  such that  $y + z$  and  $y + z'$  belong to  $D_\omega \cap \bar{B} \left( \bar{x}, \frac{1}{2l} \right)$ . Since  $y + z \in D_\omega$ , by the previous lemma, we know that

$$D_\omega \cap \bar{B} \left( y + z, \frac{1}{l} \right) \subset \{y + z\} + K_i.$$

But since both  $y + z$  and  $y + z'$  belong to  $\bar{B} \left( \bar{x}, \frac{1}{2l} \right)$ ,  $y + z'$  belongs clearly to  $D_\omega \cap \bar{B} \left( y + z, \frac{1}{l} \right)$ . Hence  $y + z' \in \{y + z\} + K_i$ . Which means that  $(y + z') - (y + z) = z' - z$  belongs to  $K_i$ . But since  $z' - z \in \Pi_i$ , we have that  $\bar{\sigma}_{\Pi_i}(z' - z) = |z' - z| > \frac{1}{2}|z' - z|$ . We find a contradiction. Let us now prove that the map  $\psi_\omega$  is Lipschitz continuous. Let  $y, y' \in A$  be fixed. By the proof above we know that  $\psi_\omega(y) = x - y$  (resp.  $\psi_\omega(y') = x' - y'$ ) where  $x$  is such that  $y = p_{\Pi_i^\perp}(x)$  (resp.  $y' = p_{\Pi_i^\perp}(x')$ ). Set  $z := \psi_\omega(y)$ ,  $z' := \psi_\omega(y')$  and  $h := x' - x$ . Since  $x = y + z$  and  $x' = y' + z'$  where  $y, y' \in \Pi_i^\perp$  and  $z, z' \in \Pi_i$ , we have  $|h|^2 = |z' - z|^2 + |y' - y|^2$ . But  $\bar{\sigma}_{\Pi_i}(h) = |z' - z| \leq \frac{1}{2}|h|$ . Hence we obtain that

$$|z' - z| \leq |x' - x| = |h| \leq \frac{2}{\sqrt{3}}|y' - y|.$$

The proof of the lemma is completed.  $\square$

From the lemma above, for every  $\omega = (r, i, j, l) \in I$  and every  $\bar{x} \in D_\omega$ , the map  $\phi : A \rightarrow \mathbb{R}^n$  defined as,

$$\forall y \in A, \quad \phi(y) = y + \psi_\omega(y),$$

is Lipschitz continuous and satisfies

$$D_\omega \cap \bar{B} \left( \bar{x}, \frac{1}{2l} \right) \subset \phi(A).$$

Since  $A \subset \Pi_i^\perp$ , such a map can be extended into a Lipschitz continuous map from  $\Pi_i^\perp$  into  $\mathbb{R}^n$ . Since  $\Pi_i^\perp$  has dimension  $(n - k)$ , we deduce that the set  $D_\omega \cap \bar{B} \left( \bar{x}, \frac{1}{2l} \right)$  is  $(n - k)$ -rectifiable. The fact that any set  $D_\omega$  can be covered by a finite number of balls of radius  $\frac{1}{2l}$  completes the proof of the theorem.

**Acknowledgements.** The author is very indebted to Joël Benoist for fruitful discussions.

**REFERENCES**

- [1] J. Benoist, *The size of the Dini subdifferential*, Proc. Amer. Math. Soc, **129** (2000), 525–530.
- [2] P. Cannarsa and C. Sinestrari, “Semiconcave functions, Hamilton-Jacobi equations, and optimal control,” Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston Inc., Boston, MA, 2004.
- [3] F.H. Clarke, Yu.S. Ledyev, R.J. Stern, and P.R. Wolenski, “Nonsmooth Analysis and Control Theory,” Graduate Texts in Mathematics, vol. 178. Springer-Verlag, New York, 1998.
- [4] L. Rifford, “Nonholonomic Variations: An Introduction to Sub-Riemannian Geometry,” Monograph. In progress.

Received September 2006; revised May 2007.

*E-mail address:* rifford@math.unice.fr