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RANGE OF THE GRADIENT OF A SMOOTH BUMP FUNCTION IN FINITE DIMENSIONS

LUDOVIC RIFFORD

(Communicated by Jonathan M. Borwein)

ABSTRACT. This paper proves the semi-closedness of the range of the gradient for sufficiently smooth bumps in the Euclidean space.

Let \mathbb{R}^N be the Euclidean space of dimension N . A *bump* on \mathbb{R}^N is a function from \mathbb{R}^N into \mathbb{R} with a bounded nonempty support. The aim of this short paper is to partially answer an open question suggested by Borwein, Fabian, Kortezov and Loewen in [1]. Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be a C^1 -smooth bump function; does $f'(\mathbb{R}^N)$ equal the closure of its interior? We are not able to provide an answer, but we can prove the following result.

Theorem 0.1. *Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be a C^{N+1} -smooth bump. Then $f'(\mathbb{R}^N)$ is the closure of its interior.*

We do not know if the hypothesis on the regularity of the bump f is optimal in our theorem when $N \geq 3$. However, the result can be improved for $N = 2$; Gaspari [3] proved by specific two-dimensional arguments that the conclusion holds if the bump is only assumed to be C^2 -smooth on the plane. Again we cannot say if we need the bump function to be C^2 for $N = 2$. We proceed now to prove our theorem.

1. PROOF OF THEOREM 0.1

For the sequel, we set $F := f' = \nabla f$. Moreover, since the theorem is obvious for $N = 1$ we will assume that $N \geq 2$. The proof is based on a refinement of Sard's Theorem that can be found in Federer [2]. Let us denote by B_k, C_k ($k \in \{0, \dots, N\}$) the sets defined as follows:

$$B_k := \{x \in \mathbb{R}^N : \text{rank } DF(x) \leq k\},$$

$$C_k := \{x \in \mathbb{R}^N : \text{rank } DF(x) = k\}.$$

Of course $C_k \subseteq B_k$ and $B_N = \mathbb{R}^N$. Theorem 3.4.3 in [2] says that if the function F is C^N -smooth, then for all $k = 0, \dots, N - 1$,

$$(1.1) \quad \mathcal{H}^{k+1}(F(B_k)) = \mathcal{H}^{k+1}(F(C_k)) = 0,$$

where \mathcal{H}^{k+1} denotes the $(k + 1)$ -dimensional Hausdorff measure.

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Fix \bar{x} in \mathbb{R}^N and let us prove that $F(\bar{x})$ belongs to the closure of $\text{int}(F(\mathbb{R}^N))$. Since it is well known that $0 \in \text{int}(F(\mathbb{R}^N))$ (see Wang [6]), we can assume that $F(\bar{x}) \neq 0$. Our proof begins with the following lemma.

Lemma 1.1. *There exists a neighbourhood \mathcal{V} of $F(\bar{x})$ relative to $F(\mathbb{R}^N)$ and an integer $\bar{k} \in \{1, \dots, N\}$ such that for any $x \in F^{-1}(\mathcal{V})$, $\text{rank } DF(x) \leq \bar{k}$ and there exists a sequence $(v_n)_{n \in \mathbb{N}}$ in \mathcal{V} which converges to $F(\bar{x})$ such that*

$$(1.2) \quad F^{-1}(v_n) \subseteq \text{int}(C_{\bar{k}}).$$

Proof. Let us fix V an open neighbourhood of $F(\bar{x})$ relative to $F(\mathbb{R}^N)$ and denote by k_0 the max of the k 's in $\{0, 1, \dots, n\}$ which satisfy $V \cap F(C_k) \neq \emptyset$.

First of all we remark that $k_0 > 0$. As a matter of fact, suppose that for any $k \geq 1$, $V \cap F(C_k) = \emptyset$, that is, for any y in $F^{-1}(V)$, $\text{rank } DF(y) = 0$. Since $F^{-1}(V)$ is open this implies that F is constant on $F^{-1}(V)$ and hence that $F(\bar{x})$ is isolated in $F(\mathbb{R}^N)$. So, we get a contradiction by arc-connectedness of $F(\mathbb{R}^N)$ (and since $F(\bar{x}) \neq 0$ and $0 \in F(\mathbb{R}^N)$). Consequently, we deduce that there exists $y \in \mathbb{R}^N$ such that $F(y) \in V$ and $\text{rank } DF(y) = k_0 > 0$. Furthermore for all $z \in F^{-1}(V)$, $\text{rank } DF(z) \leq k_0$. Hence by lower semicontinuity of $z \mapsto \text{rank } DF(z)$, this implies that $\text{rank } DF$ is constant in a neighbourhood of y (because $\{z : \text{rank } DF(z) \geq k_0\}$ is open). Therefore, by the rank theorem (see Rudin [4, Theorem 9.20]), V has the structure of a k_0 -dimensional manifold near $F(y)$, and hence $\mathcal{H}^{k_0}(V) > 0$. Thus by (1.1), $V \setminus F(B_{k_0-1})$ is nonempty. We conclude that for any v in the latter set,

$$F(z) = v \implies \text{rank } DF(z) = k_0;$$

in addition z has a neighbourhood on which $\text{rank } DF \leq \bar{k}$ by choice of \bar{k} , and the set where $\text{rank } DF \geq \bar{k}$ is open. Consequently such a v satisfies $F^{-1}(v) \subseteq \text{int}(C_{\bar{k}})$. Repeating this argument with a decreasing sequence on neighbourhoods, we get a decreasing sequence of k_0 -values in $\{1, \dots, n\}$ which has to be stationary. Hence the proof is easy to complete. \square

We now claim the following lemma.

Lemma 1.2. *The constant of Lemma 1.1 satisfies $\bar{k} = N$.*

Proof. Let us remark that since $F = f' = \nabla f$, the Jacobian of F at any point y in \mathbb{R}^N is actually the Hessian of the function f . We argue by contradiction and so we assume that $\bar{k} < N$.

By the previous remark, for any $y \in \mathbb{R}^N$, $DF(y)$ is a symmetric matrix, the nontrivial vector subspaces $\text{Ker } DF(y)$ and $\text{Im } DF(y)$ are orthogonal, and $DF(y)$ induces an automorphism on $\text{Im } DF(y)$. Let us fix $n \in \mathbb{N}$. By Lemma 1.1 and by the constant rank theorem (see for instance Spivak [5] page 65) we deduce that $M_n := \{y : F(y) = v_n\}$ is a submanifold of \mathbb{R}^N of dimension $N - k$ and at least C^2 -smooth (since F is C^N -smooth and $N \geq 2$). Furthermore since f is a bump, M_n is a compact submanifold.

Now since M_n is a C^2 submanifold of \mathbb{R}^N there exists an open tubular neighbourhood $\mathcal{U} \subset \mathcal{V}$ of M_n and a C^2 -smooth function $r : \mathcal{U} \rightarrow M_n$ which is the projection on the set M_n such that for any $x \in \mathcal{U}$, $x - r(x) \in N_{r(x)}M_n$, where for any $p \in M_n$, N_pM_n denotes the normal space of M_n at p . In addition, from the properties of the constant \bar{k} , by reducing \mathcal{U} if necessary, we can assume that for

any $x \in \mathcal{U}$, $\text{rank } DF(x) = \bar{k}$. We set the following function on the neighbourhood \mathcal{U} :

$$\begin{aligned} \Phi : \mathcal{U} &\rightarrow \mathbb{R}^N, \\ x &\mapsto DF(r(x))(x - r(x)). \end{aligned}$$

We now need the following result.

Lemma 1.3. *If M_n is a compact C^2 submanifold of \mathbb{R}^N , then for all ξ in the unit sphere \mathbb{S}^{N-1} , there exists $p \in M_n$ such that $\xi \in N_p M_n$.*

Proof. Consider for any $l \in \mathbb{N}$, $p_l := \text{proj}_{M_n}(l\xi)$, where $\text{proj}_{M_n}(\cdot)$ denotes the projection map on the closed set M_n . Since the submanifold M_n is C^2 , the vector $\frac{l\xi - p_l}{\|l\xi - p_l\|}$ belongs to $N_{p_l} M_n$. Moreover by compactness of M_n we can assume that $p_l \rightarrow \bar{p}$ when l tends to infinity. Now since the sequence $(p_l)_{l \in \mathbb{N}}$ is bounded, we have that $\lim_{l \rightarrow \infty} \frac{l\xi - p_l}{\|l\xi - p_l\|} = \xi$. By continuity of the normal bundle NM_n , we easily conclude that $\xi \in N_{\bar{p}} M_n$. \square

Returning to the proof of Lemma 1.2, Lemma 1.3 immediately implies that for all $\xi \in \mathbb{S}^{N-1}$, there exists $p \in M_n$ and $v \in N_p M_n$ such that $v = \xi$. Furthermore the map $DF(p)$ is an automorphism on $N_p M_n$, hence there exists $w \in N_p M_n$ such that $DF(p)(w) = v$. We conclude that for any t small enough (s.t. $p + tw \in \mathcal{U}$), $DF(p)(tw) = t\xi$ and hence that $\Phi(p + tw) = t\xi$. Furthermore since M_n is compact and since the map $p \mapsto [DF(p)|_{N_p M_n}]^{-1}$ is continuous on M_n , we deduce that $\|w\|$ is bounded above. Hence by compactness on M_n , we get that for some $t_0 > 0$ the ball $B(0, t_0)$ is included in $\Phi(\mathcal{U})$; hence $\Phi(\mathcal{U})$ has a nonempty interior. Therefore (since the function Φ is smooth enough) Sard's Theorem gives us the existence of regular values of Φ in \mathbb{R}^N . So there exists $\bar{y} \in \mathcal{U}$ such that $\text{rank } D\Phi(\bar{y}) = N$. Consequently there exists $\rho > 0$ such that the map Φ is one-to-one on $\mathcal{W} = B(\bar{y}, \rho)$ (the ball centered at \bar{y} with radius ρ).

For any $l \in \mathbb{N}^*$, we set $y_l := r(\bar{y}) + \frac{1}{l}(\bar{y} - r(\bar{y}))$. The constant rank theorem implies that for any l the set $V_l := \{y \in \mathcal{U} : F(y) = F(y_l)\}$ is a submanifold of \mathcal{U} of dimension $N - \bar{k}$. (Of course V_l might be noncompact in \mathcal{U} , i.e. \bar{V}_l not included in \mathcal{U} .) On the other hand, by Lipschitz continuity of $DF(\cdot)$ and since $N - \bar{k} > 0$, there exists a neighbourhood \mathcal{Y} of the segment $[\bar{y}, r(\bar{y})]$ in $\text{co}\{\mathcal{W} \cup r(\mathcal{W})\}$ and a Lipschitz continuous map $X : \mathcal{Y} \rightarrow \mathbb{R}^N$ such that for any $x \in \mathcal{Y}$,

$$X(x) \in \ker DF(x) \text{ and } \|X(x)\| = 1.$$

If we denote by $\theta_X(y, \tau)$ the local flow of the vector field X on \mathcal{Y} , we get that for any τ small enough, $\theta_X(y_l, \tau) \in V_l$. On the other hand, Gronwall's Lemma easily yields the following (we omit the proof):

Lemma 1.4. *There exist two positive constants K, μ such that for any $l \in \mathbb{N}^*$ and for any $\tau \leq \mu$, we have*

$$(1.3) \quad \theta_X(y_l, \tau) \in \text{co} \left\{ B\left(\bar{y}, \frac{\rho}{2}\right) \cup r\left(B\left(\bar{y}, \frac{\rho}{2}\right)\right) \right\},$$

$$(1.4) \quad \frac{\|\theta_X(y_l, \tau) - r(\theta_X(y_l, \tau))\|}{\|y_l - r(y_l)\|} \in [e^{-K\tau}, e^{K\tau}].$$

We now conclude the proof of Lemma 1.2. We set for any $l \in \mathbb{N} \setminus \{0\}$, $z_l := \theta_X(y_l, \mu)$. First remark that if μ is small enough, then we have (recall that $\|X\| = 1$)

$$\begin{aligned} & \langle X(\theta_X(y_l, s)), X(y_l) \rangle \geq \frac{1}{2} \\ \implies & \left\langle \int_0^\mu X(\theta_X(y_l, s)) ds, X(y_l) \right\rangle \geq \frac{\mu}{2} \\ (1.5) \quad \implies & \|z_l - y_l\| \geq \frac{\mu}{2}. \end{aligned}$$

By considering a converging subsequence of $(z_l)_{l \in \mathbb{N}^*}$ if necessary we can compute

$$\begin{aligned} \lim_{l \rightarrow \infty} \frac{F(y_l) - F(r(y_l))}{\|z_l - r(z_l)\|} &= \lim_{l \rightarrow \infty} \frac{F(z_l) - F(r(z_l))}{\|z_l - r(z_l)\|} \\ &= \lim_{l \rightarrow \infty} DF(r(z_l)) \left(\frac{z_l - r(z_l)}{\|z_l - r(z_l)\|} \right) \\ &= DF(\bar{z})(\bar{\zeta}), \end{aligned}$$

where $\lim_{l \rightarrow \infty} z_l = \bar{z} = r(\bar{z}) \in M_n$ and $\lim_{l \rightarrow \infty} \frac{z_l - r(z_l)}{\|z_l - r(z_l)\|} = \bar{\zeta} \in N_{\bar{z}}M_n$. We deduce that

$$\begin{aligned} DF(r(\bar{y}))(\bar{y} - r(\bar{y})) &= \lim_{l \rightarrow \infty} l(F(y_l) - F(r(y_l))) \\ &= \lim_{l \rightarrow \infty} l\|z_l - r(z_l)\| \frac{F(y_l) - F(r(y_l))}{\|z_l - r(z_l)\|} \\ &= c\|\bar{y} - r(\bar{y})\|DF(\bar{z})(\bar{\zeta}) \\ &= DF(\bar{z})(c\|\bar{y} - r(\bar{y})\|\bar{\zeta}), \end{aligned}$$

with $c = \lim_{l \rightarrow \infty} \frac{\|z_l - r(z_l)\|}{\|y_l - r(y_l)\|}$.

The computations prove that $\Phi(\bar{y}) = \Phi(\bar{z} + c\|\bar{y} - r(\bar{y})\|\bar{\zeta})$. Furthermore by (1.3) and (1.5), \bar{z} belongs to $r(\mathcal{W})$ and $\|\bar{z} - r(\bar{y})\| > 0$. Consequently since Φ is injective on \mathcal{W} , it remains to prove that $\bar{z} + c\|\bar{y} - r(\bar{y})\|\bar{\zeta}$ is in \mathcal{W} to get a contradiction. By (1.4) taking μ smaller if necessary, we get the result of Lemma 1.2. \square

The proof of Theorem 0.1 is now easy. Since $\bar{k} = N$, for any $n \in \mathbb{N}$ the different values v_n of Lemma 1.1 belong to the interior of $f'(\mathbb{R}^N)$ and moreover the sequence $(v_n)_{n \in \mathbb{N}}$ converges to $F(\bar{x})$. This proves the theorem.

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