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# Quasilinear Lane-Emden equations with absorption and measure data

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**Abstract** We study the existence of solutions to the equation  $-\Delta_p u + g(x, u) = \mu$  when  $g(x, \cdot)$  is a nondecreasing function and  $\mu$  a measure. We characterize the good measures, i.e. the ones for which the problem has a renormalized solution. We study particularly the cases where  $g(x, u) = |x|^{-\beta} |u|^{q-1} u$  and  $g(x, u) = \text{sgn}(u)(e^{\tau|u|^\lambda} - 1)$ . The results state that a measure is good if it is absolutely continuous with respect to an appropriate Lorentz-Bessel capacities.

## 1 Introduction

Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain containing 0 and  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Caratheodory function. We assume that for almost all  $x \in \Omega$ ,  $r \mapsto g(x, r)$  is nondecreasing and odd. In this article we consider the following problem

$$\begin{aligned} -\Delta_p u + g(x, u) &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega \end{aligned} \quad (1.1)$$

where  $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ , ( $1 < p < N$ ), is the p-Laplacian and  $\mu$  a bounded measure. A measure for which the problem admits a solution, in an appropriate class, is called a *good measure*. When  $p = 2$  and  $g(x, u) = g(u)$  the problem has been considered by Benilan and Brezis [4] in the subcritical case that is when any bounded measure is good. They prove that such is the case if  $N \geq 3$  and  $g$  satisfies

$$\int_1^\infty g(s) s^{-\frac{N-1}{N-2}} ds < \infty. \quad (1.2)$$

The supercritical case, always with  $p = 2$ , has been considered by Baras and Pierre [3] when  $g(u) = |u|^{q-1} u$  and  $q > 1$ . They prove that the corresponding problem to (1.1) admits a solution (always unique in that case) if and only if the measure  $\mu$  is absolutely continuous with respect to the Bessel capacity  $C_{2,q'}$  ( $q' = q/(q-1)$ ). In the case  $p \neq 2$  it is shown by Bidaut-Véron [6] that if problem (1.1) with  $\beta = 0$  and  $g(s) = |s|^{q-1} s$  ( $q > p-1 > 0$ ) admits a solution, then  $\mu$  is absolutely continuous with respect to any capacity  $C_{p, \frac{q}{q+1-p} + \epsilon}$  for any  $\epsilon > 0$ .

In this article we introduce a new class of Bessel capacities which are modelled on Lorentz spaces  $L^{s,q}$  instead of  $L^q$  spaces. If  $G_\alpha$  is the Bessel kernel of order  $\alpha > 0$ , we denote by  $L^{\alpha,s,q}(\mathbb{R}^N)$  the Besov space which is the space of functions  $\phi = G_\alpha * f$  for some  $f \in L^{s,q}(\mathbb{R}^N)$  and we set  $\|\phi\|_{\alpha,s,q} = \|f\|_{s,q}$  (a norm which is defined by using rearrangements). Then we set

$$C_{\alpha,s,q}(E) = \inf\{\|f\|_{s,q} : f \geq 0, G_\alpha * f \geq 1 \text{ on } E\} \quad (1.3)$$

for any Borel set  $E$ . We say that a measure  $\mu$  in  $\Omega$  is absolutely continuous with respect to the capacity  $C_{\alpha,s,q}$  if ,

$$\forall E \subset \Omega, E \text{ Borel}, C_{\alpha,s,q}(E) = 0 \implies |\mu|(E) = 0. \quad (1.4)$$

We also introduce the Wolff potential of a positive measure  $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$  by

$$\mathbf{W}_{\alpha,s}[\mu](x) = \int_0^\infty \left( \frac{\mu(B_t(x))}{t^{N-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dt}{t} \quad (1.5)$$

if  $\alpha > 0$ ,  $1 < s < \alpha^{-1}N$ . When we are dealing with bounded domains  $\Omega \subset B_R$  and  $\mu \in \mathfrak{M}_+(\Omega)$ , it is useful to introduce truncated Wolff potentials.

$$\mathbf{W}_{\alpha,s}^R[\mu](x) = \int_0^R \left( \frac{\mu(B_t(x))}{t^{N-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dt}{t} \quad (1.6)$$

We prove the following existence results concerning

$$\begin{aligned} -\Delta_p u + |x|^{-\beta} g(u) &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega \end{aligned} \quad (1.7)$$

**Theorem 1.1** *Assume  $1 < p < N$ ,  $q > p - 1$  and  $0 \leq \beta < N$  and  $\mu$  is a bounded Radon measure in  $\Omega$ .*

1- *If  $g(s) = |s|^{q-1} s$ , then (1.7) admits a renormalized solution if  $\mu$  is absolutely continuous with respect to the capacity  $C_{p, \frac{Nq}{Nq-(p-1)(N-\beta)}, \frac{q}{q+1-p}}$ .*

2- *If  $g$  satisfies*

$$\int_1^\infty g(s) s^{-q-1} ds < \infty \quad (1.8)$$

*then (1.7) admits a renormalized solution if  $\mu$  is absolutely continuous with respect to the capacity  $C_{p, \frac{Nq}{Nq-(p-1)(N-\beta)}, 1}$ .*

*Furthermore, in both case there holds*

$$-cW_{1,p}^{2\text{diam}(\Omega)}[\mu^-](x) \leq u(x) \leq cW_{1,p}^{2\text{diam}(\Omega)}[\mu^+](x) \quad \text{for almost all } x \in \Omega. \quad (1.9)$$

*where  $c$  is a positive constant depending on  $p$  and  $N$ .*

In order to deal with exponential nonlinearities we introduce for  $0 < \alpha < N$  the fractional maximal operator (resp. the truncated fractional maximal operator), defined for a positive measure  $\mu$  by

$$\mathbf{M}_\alpha[\mu](x) = \sup_{t>0} \frac{\mu(B_t(x))}{t^{N-\alpha}}, \quad \left( \text{resp } \mathbf{M}_{\alpha,R}[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-\alpha}} \right), \quad (1.10)$$

and the  $\eta$ -fractional maximal operator (resp. the truncated  $\eta$ -fractional maximal operator)

$$\mathbf{M}_\alpha^\eta[\mu](x) = \sup_{t>0} \frac{\mu(B_t(x))}{t^{N-\alpha} h_\eta(t)}, \quad \left( \text{resp } \mathbf{M}_{\alpha,R}^\eta[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-\alpha} h_\eta(t)} \right), \quad (1.11)$$

where  $\eta \geq 0$  and

$$h_\eta(t) = \begin{cases} (-\ln t)^{-\eta} & \text{if } 0 < t < \frac{1}{2} \\ (\ln 2)^{-\eta} & \text{if } t \geq \frac{1}{2} \end{cases} \quad (1.12)$$

**Theorem 1.2** *Assume  $1 < p < N$ ,  $\tau > 0$  and  $\lambda \geq 1$ . Then there exists  $M > 0$  depending on  $N, p, \tau$  and  $\lambda$  such that if a measure in  $\Omega$ ,  $\mu = \mu^+ - \mu^-$  can be decomposed as follows*

$$\mu^+ = f_1 + \nu_1 \quad \text{and} \quad \mu^- = f_2 + \nu_2, \quad (1.13)$$

where  $f_j \in L^1_+(\Omega)$  and  $\nu_j \in \mathfrak{M}^b_+(\Omega)$  ( $j = 1, 2$ ), and if

$$\left\| \mathbf{M}_{p, 2\text{diam}(\Omega)}^{\frac{(p-1)(\lambda-1)}{\lambda}}[\nu_j] \right\|_{L^\infty(\Omega)} < M, \quad (1.14)$$

there exists a renormalized solution to

$$\begin{aligned} -\Delta_p u + \text{sign}(u) \left( e^{\tau|u|^\lambda} - 1 \right) &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega. \end{aligned} \quad (1.15)$$

and satisfies (1.9).

Our study is based upon delicate estimates on Wolff potentials and  $\eta$ -fractional maximal operators which are developed in the first part of this paper.

## 2 Lorentz spaces and capacities

### 2.1 Lorentz spaces

Let  $(X, \Sigma, \alpha)$  be a measured space. If  $f : X \rightarrow \mathbb{R}$  is a measurable function, we set  $S_f(t) := \{x \in X : |f|(x) > t\}$  and  $\lambda_f(t) = \alpha(S_f(t))$ . The decreasing rearrangement  $f^*$  of  $f$  is defined by

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leq t\}.$$

It is well known that  $(\Phi(f))^* = \Phi(f^*)$  for any continuous and nondecreasing function  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . We set

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(\tau) d\tau \quad \forall t > 0.$$

and, for  $1 \leq s < \infty$  and  $1 < q \leq \infty$ ,

$$\|f\|_{L^{s,q}} = \begin{cases} \left( \int_0^\infty t^{\frac{q}{s}} (f^{**}(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{t>0} t^{\frac{1}{s}} f^{**}(t) & \text{if } q = \infty \end{cases} \quad (2.1)$$

It is known that  $L^{s,q}(X, \alpha)$  is a Banach space when endowed with the norm  $\|\cdot\|_{L^{s,q}}$ . Furthermore there holds (see e.g. [12])

$$\left\| t^{\frac{1}{s}} f^* \right\|_{L^q(\mathbb{R}^+, \frac{dt}{t})} \leq \|f\|_{L^{s,q}} \leq \frac{s}{s-1} \left\| t^{\frac{1}{s}} f^* \right\|_{L^q(\mathbb{R}^+, \frac{dt}{t})}, \quad (2.2)$$

the left-hand side inequality being valid only if  $s > 1$ . Finally, if  $f \in L^{s,q}(\mathbb{R}^N)$  (with  $1 \leq q, s < \infty$  and  $\alpha$  being the Lebesgue measure) and if  $\{\rho_n\} \subset C_c^\infty(\mathbb{R}^N)$  is a sequence of mollifiers,  $f * \rho_n \rightarrow f$  and  $(f \chi_{B_n}) * \rho_n \rightarrow f$  in  $L^{s,q}(\mathbb{R}^N)$ , where  $\chi_{B_n}$  is the indicator function of the ball  $B_n$  centered at the origin of radius  $n$ . In particular  $C_c^\infty(\mathbb{R}^N)$  is dense in  $L^{s,q}(\mathbb{R}^N)$ .

## 2.2 Wolff potentials, fractional and $\eta$ -fractional maximal operators

If  $D$  is either a bounded domain or whole  $\mathbb{R}^N$ , we denote by  $\mathfrak{M}(D)$  (resp  $\mathfrak{M}^b(D)$ ) the set of Radon measure (resp. bounded Radon measures) in  $D$ . Their positive cones are  $\mathfrak{M}_+(D)$  and  $\mathfrak{M}_+^b(D)$  respectively. If  $0 < R \leq \infty$  and  $\mu \in \mathfrak{M}_+(D)$  and  $R \geq \text{diam}(D)$ , we define, for  $\alpha > 0$  and  $1 < s < \alpha^{-1}N$ , the  $R$ -truncated Wolff-potential by

$$\mathbf{W}_{\alpha,s}^R[\mu](x) = \int_0^R \left( \frac{\mu(B_t(x))}{t^{N-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dt}{t} \quad \text{for a.e. } x \in \mathbb{R}^N. \quad (2.3)$$

If  $h_\eta(t) = \min\{(-\ln t)^{-\eta}, (\ln 2)^{-\eta}\}$  and  $0 < \alpha < N$ , the truncated  $\eta$ -fractional maximal operator is

$$\mathbf{M}_{\alpha,R}^\eta[\mu](x) = \sup_{0 < t < R} \frac{\mu(B_t(x))}{t^{N-\alpha} h_\eta(t)} \quad \text{for a.e. } x \in \mathbb{R}^N. \quad (2.4)$$

If  $R = \infty$ , we drop it in expressions (2.3) and (2.4). In particular

$$\mu(B_t(x)) \leq t^{N-\alpha} h_\eta(t) \mathbf{M}_{\alpha,R}^\eta[\mu](x). \quad (2.5)$$

We also define  $\mathbf{G}_\alpha$  the Bessel potential of a measure  $\mu$  by

$$\mathbf{G}_\alpha[\mu](x) = \int_{\mathbb{R}^N} G_\alpha(x-y) d\mu(y) \quad \forall x \in \mathbb{R}^N, \quad (2.6)$$

where  $G_\alpha$  is the Bessel kernel of order  $\alpha$  in  $\mathbb{R}^N$ .

**Definition 2.1** We denote by  $L^{\alpha,s,q}(\mathbb{R}^N)$  the Besov space the space of functions  $\phi = G_\alpha * f$  for some  $f \in L^{s,q}(\mathbb{R}^N)$  and we set  $\|\phi\|_{\alpha,s,q} = \|f\|_{s,q}$ . If we set

$$C_{\alpha,s,q}(E) = \inf\{\|f\|_{s,q} : f \geq 0, G_\alpha * f \geq 1 \text{ on } E\}, \quad (2.7)$$

then  $C_{\alpha,s,q}$  is a capacity, see [1].

## 2.3 Estimates on potentials

In the sequel, we denote by  $|A|$  the  $N$ -dimensional Lebesgue measure of a measurable set  $A$  and, if  $F, G$  are functions defined in  $\mathbb{R}^N$ , we set  $\{F > a\} := \{x \in \mathbb{R}^N : F(x) > a\}$ ,  $\{G \leq b\} := \{x \in \mathbb{R}^N : G(x) \leq b\}$  and  $\{F > a, G \leq b\} := \{F > a\} \cap \{G \leq b\}$ . The following result is an extension of [14, Th 1.1]

**Proposition 2.2** Let  $0 \leq \eta < p-1$ ,  $0 < \alpha p < N$  and  $r > 0$ . There exist  $c_0 > 0$  depending on  $N, \alpha, p, \eta$  and  $\epsilon_0 > 0$  depending on  $N, \alpha, p, \eta, r$  such that, for all  $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$  with  $\text{diam}(\text{supp}(\mu)) \leq r$  and  $R \in (0, \infty]$ ,  $\epsilon \in (0, \epsilon_0]$ ,  $\lambda > (\mu(\mathbb{R}^N))^{\frac{1}{p-1}} l(r, R)$  there holds,

$$\begin{aligned} & \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu] > 3\lambda, (\mathbf{M}_{\alpha p, R}^\eta[\mu])^{\frac{1}{p-1}} \leq \epsilon\lambda \right\} \right| \\ & \leq c_0 \exp\left(-\left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 \epsilon^{-\frac{p-1}{p-1-\eta}}\right) |\{\mathbf{W}_{\alpha,p}^R[\mu] > \lambda\}|. \end{aligned} \quad (2.8)$$

where  $l(r, R) = \frac{N-\alpha p}{p-1} \left( \min\{r, R\}^{-\frac{N-\alpha p}{p-1}} - R^{-\frac{N-\alpha p}{p-1}} \right)$  if  $R < \infty$ ,  $l(r, R) = \frac{N-\alpha p}{p-1} r^{-\frac{N-\alpha p}{p-1}}$  if  $R = \infty$ . Furthermore, if  $\eta = 0$ ,  $\epsilon_0$  is independent of  $r$  and (2.8) holds for all  $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$  with compact support in  $\mathbb{R}^N$  and  $R \in (0, \infty]$ ,  $\epsilon \in (0, \epsilon_0]$ ,  $\lambda > 0$ .

*Proof.* Case  $R = \infty$ . Let  $\lambda > 0$ ; since  $\mathbf{W}_{\alpha,p}[\mu]$  is lower semicontinuous, the set

$$D_\lambda := \{\mathbf{W}_{\alpha,p}[\mu] > \lambda\}$$

is open. By Whitney covering lemma, there exists a countable set of closed cubes  $\{Q_i\}_i$  such that  $D_\lambda = \cup_i Q_i$ ,  $\overset{\circ}{Q}_i \cap \overset{\circ}{Q}_j = \emptyset$  for  $i \neq j$  and

$$\text{diam}(Q_i) \leq \text{dist}(Q_i, D_\lambda^c) \leq 4 \text{diam}(Q_i).$$

Let  $\epsilon > 0$  and  $F_{\epsilon,\lambda} = \{\mathbf{W}_{\alpha,p}[\mu] > 3\lambda, (M_{\alpha p}^\eta[\mu])^{\frac{1}{p-1}} \leq \epsilon\lambda\}$ . We claim that there exist  $c_0 = c_0(N, \alpha, p, \eta) > 0$  and  $\epsilon_0 = \epsilon_0(N, \alpha, p, \eta, r) > 0$  such that for any  $Q \in \{Q_i\}_i$ ,  $\epsilon \in (0, \epsilon_0)$  and  $\lambda > (\mu(\mathbb{R}^N))^{\frac{1}{p-1}} l(r, \infty)$  there holds

$$|F_{\epsilon,\lambda} \cap Q| \leq c_0 \exp\left(-\left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \epsilon^{-\frac{p-1}{p-1-\eta}} \alpha p \ln 2\right) |Q|. \quad (2.9)$$

The first we show that there exists  $c_1 > 0$  depending on  $N, \alpha, p$  and  $\eta$  such that for any  $Q \in \{Q_i\}_i$  there holds

$$F_{\epsilon,\lambda} \cap Q \subset E_{\epsilon,\lambda} \quad \forall \epsilon \in (0, c_1], \lambda > 0 \quad (2.10)$$

where

$$E_{\epsilon,\lambda} = \left\{x \in Q : \mathbf{W}_{\alpha,p}^{5 \text{diam}(Q)}[\mu](x) > \lambda, (M_{\alpha p}^\eta[\mu](x))^{\frac{1}{p-1}} \leq \epsilon\lambda\right\}. \quad (2.11)$$

Infact, take  $Q \in \{Q_i\}_i$  such that  $Q \cap F_{\epsilon,\lambda} \neq \emptyset$  and let  $x_Q \in D_\lambda^c$  such that  $\text{dist}(x_Q, Q) \leq 4 \text{diam}(Q)$  and  $\mathbf{W}_{\alpha,p}[\mu](x_Q) \leq \lambda$ . For  $k \in \mathbb{N}$ ,  $r_0 = 5 \text{diam}(Q)$  and  $x \in F_{\epsilon,\lambda} \cap Q$ , we have

$$\int_{2^k r_0}^{2^{k+1} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} = A + B$$

where

$$A = \int_{2^k r_0}^{2^k \frac{1+2^{k+1}}{1+2^k} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \quad \text{and} \quad B = \int_{2^k \frac{1+2^{k+1}}{1+2^k} r_0}^{2^{k+1} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Since

$$\mu(B_t(x)) \leq t^{N-\alpha p} h_\eta(t) M_{\alpha p}^\eta[\mu](x) \leq t^{N-\alpha p} h_\eta(t) (\epsilon\lambda)^{p-1}. \quad (2.12)$$

Then

$$B \leq \int_{2^k \frac{1+2^{k+1}}{1+2^k} r_0}^{2^{k+1} r_0} \left(\frac{t^{N-\alpha p} h_\eta(t) (\epsilon\lambda)^{p-1}}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} = \epsilon\lambda \int_{2^k \frac{1+2^{k+1}}{1+2^k} r_0}^{2^{k+1} r_0} (h_\eta(t))^{\frac{1}{p-1}} \frac{dt}{t}$$

Replacing  $h_\eta(t)$  by its value we obtain  $B \leq c_2 \epsilon\lambda 2^{-k}$  after a lengthy computation where  $c_2$  depends only on  $p$  and  $\eta$ . Since  $\delta := \left(\frac{2^k}{2^{k+1}}\right)^{\frac{N-\alpha p}{p-1}}$ , then  $1 - \delta \leq c_3 2^{-k}$  where  $c_3$  depends only on  $\frac{N-\alpha p}{p-1}$ , thus

$$\begin{aligned} (1 - \delta)A &\leq c_3 2^{-k} \int_{2^k r_0}^{2^{k+1} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq c_3 2^{-k} \epsilon\lambda \int_{2^k r_0}^{2^{k+1} r_0} (h_\eta(t))^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq c_4 2^{-k} \epsilon\lambda, \end{aligned}$$

where  $c_4 = c_4(N, \alpha, p, \eta) > 0$ .

By a change of variables and using that for any  $x \in F_{\epsilon, \lambda} \cap Q$  and  $t \in [r_0(1+2^k), r_0(1+2^{k+1})]$ ,  $B_{\frac{2^k t}{1+2^k}}(x) \subset B_t(x_Q)$ , we get

$$\delta A = \int_{r_0(1+2^k)}^{r_0(1+2^{k+1})} \left( \frac{\mu(B_{\frac{2^k t}{1+2^k}}(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \int_{r_0(1+2^k)}^{r_0(1+2^{k+1})} \left( \frac{\mu(B_t(x_Q))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Therefore

$$\int_{2^k r_0}^{2^{k+1} r_0} \left( \frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_5 2^{-k} \epsilon \lambda + \int_{r_0(1+2^k)}^{r_0(1+2^{k+1})} \left( \frac{\mu(B_t(x_Q))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t},$$

with  $c_5 = c_5(N, \alpha, p, \eta) > 0$ . This implies

$$\int_{r_0}^{\infty} \left( \frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq 2c_5 \epsilon \lambda + \int_{2r_0}^{\infty} \left( \frac{\mu(B_t(x_Q))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq (1 + 2c_5 \epsilon) \lambda, \quad (2.13)$$

since  $\mathbf{W}_{\alpha, p}[\mu](x_Q) \leq \lambda$ . If  $\epsilon \in (0, c_1]$  with  $c_1 = (2c_5)^{-1}$  then

$$\int_{r_0}^{\infty} \left( \frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq 2\lambda$$

which implies (2.10).

Now, we let  $\lambda > (\mu(\mathbb{R}^N))^{\frac{1}{p-1}} l(r, \infty)$ . Let  $B_1$  be a ball with radius  $r$  such that  $\text{supp}(\mu) \subset B_1$ . We denote  $B_2$  by the ball concentric to  $B_1$  with radius  $2r$ . Since  $x \notin B_2$ ,

$$\mathbf{W}_{\alpha, p}[\mu](x) = \int_r^{\infty} \left( \frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq (\mu(\mathbb{R}^N))^{\frac{1}{p-1}} l(r, \infty).$$

Thus, we obtain  $D_\lambda \subset B_2$ . In particular,  $r_0 = 5 \text{diam}(Q) \leq 20r$ .

Next we set  $m_0 = \frac{\max(1, \ln(40r))}{\ln 2}$ , so that  $2^{-m} r_0 \leq 2^{-1}$  if  $m \geq m_0$ . Then for any  $x \in E_{\epsilon, \lambda}$

$$\begin{aligned} \int_{2^{-m} r_0}^{r_0} \left( \frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} &\leq \epsilon \lambda \int_{2^{-m} r_0}^{r_0} (h_\eta(t))^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \epsilon \lambda \int_{2^{-m} r_0}^{2^{-m_0} r_0} (-\ln t)^{\frac{-\eta}{p-1}} \frac{dt}{t} + \epsilon \lambda \int_{2^{-m_0} r_0}^{r_0} (\ln 2)^{\frac{-\eta}{p-1}} \frac{dt}{t} \\ &\leq m_0 \epsilon \lambda + \frac{(p-1)((m-m_0) \ln 2)^{1-\frac{\eta}{p-1}}}{p-1-\eta} \epsilon \lambda. \end{aligned}$$

For the last inequality we have used  $a^{1-\frac{\eta}{p-1}} - b^{1-\frac{\eta}{p-1}} \leq (a-b)^{1-\frac{\eta}{p-1}}$  valid for any  $a \geq b \geq 0$ . Therefore,

$$\int_{2^{-m} r_0}^{r_0} \left( \frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon \lambda \quad \forall m \in \mathbb{N}, m > m_0^{\frac{p-1}{p-1-\eta}}. \quad (2.14)$$



Set

$$g_i(x) = \int_{2^{-i}r_0}^{2^{-i+1}r_0} \left( \frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t},$$

then

$$\begin{aligned} \mathbf{W}_{\alpha,p}^{r_0}[\mu](x) &\leq \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon \lambda + \mathbf{W}_{\alpha,p}^{2^{-m}r_0}[\mu](x) \\ &\leq \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon \lambda + \sum_{i=m+1}^{\infty} g_i(x) \end{aligned}$$

for all  $m > m_0^{\frac{p-1}{p-1-\eta}}$ . We deduce that, for  $\beta > 0$ ,

$$\begin{aligned} |E_{\epsilon,\lambda}| &\leq \left| \left\{ x \in Q : \sum_{i=m+1}^{\infty} g_i(x) > \left( 1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon \right) \lambda \right\} \right| \\ &\leq \left| \left\{ x \in Q : \sum_{i=m+1}^{\infty} g_i(x) > 2^{-\beta(i-m-1)} (1-2^{-\beta}) \left( 1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon \right) \lambda \right\} \right| \\ &\leq \sum_{i=m+1}^{\infty} \left| \left\{ x \in Q : g_i(x) > 2^{-\beta(i-m-1)} (1-2^{-\beta}) \left( 1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon \right) \lambda \right\} \right|. \end{aligned} \tag{2.15}$$

Next we claim that

$$|\{x \in Q : g_i(x) > s\}| \leq \frac{c_6(N,\eta)}{s^{p-1}} 2^{-i\alpha p} |Q| (\epsilon\lambda)^{p-1}. \tag{2.16}$$

To see that, we pick  $x_0 \in E_{\epsilon,\lambda}$  and we use the Chebyshev's inequality

$$\begin{aligned} |\{x \in Q : g_i(x) > s\}| &\leq \frac{1}{s^{p-1}} \int_Q |g_i|^{p-1} dx \\ &= \frac{1}{s^{p-1}} \int_Q \left( \int_{r_0 2^{-i}}^{r_0 2^{-i+1}} \left( \frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \right)^{p-1} dx \\ &\leq \frac{1}{s^{p-1}} \int_Q \frac{\mu(B_{r_0 2^{-i+1}}(x))}{(r_0 2^{-i})^{N-\alpha p}} := A. \end{aligned}$$

Thanks to Fubini's theorem, the last term  $A$  of the above inequality can be rewritten as

$$\begin{aligned} A &= \frac{1}{s^{p-1}} \frac{1}{(r_0 2^{-i})^{N-\alpha p}} \int_Q \int_{\mathbb{R}^N} \chi_{B_{r_0 2^{-i+1}}(x)}(y) d\mu(y) dx \\ &= \frac{1}{s^{p-1}} \frac{1}{(r_0 2^{-i})^{N-\alpha p}} \int_{Q+B_{r_0 2^{-i+1}}(0)} \int_Q \chi_{B_{r_0 2^{-i+1}}(y)}(x) dx d\mu(y) \\ &\leq \frac{1}{s^{p-1}} \frac{1}{(r_0 2^{-i})^{N-\alpha p}} \int_{Q+B_{r_0 2^{-i+1}}(0)} |B_{r_0 2^{-i+1}}(y)| d\mu(y) \\ &\leq c_7(N) \frac{1}{s^{p-1}} 2^{-i\alpha p} r_0^{\alpha p} \mu(Q + B_{r_0 2^{-i+1}}(0)) \\ &\leq c_7(N) \frac{1}{s^{p-1}} 2^{-i\alpha p} r_0^{\alpha p} \mu(B_{r_0(1+2^{-i+1})}(x_0)), \end{aligned}$$

since  $Q+B_{r_0}2^{-i+1}(0) \subset B_{r_0(1+2^{-i+1})}(x_0)$ . Using the fact that  $\mu(B_t(x_0)) \leq (\ln 2)^{-\eta} t^{N-\alpha p} (\epsilon \lambda)^{p-1}$  for all  $t > 0$  and  $r_0 = 5 \operatorname{diam}(Q)$ , we obtain

$$A \leq c_8(N, \eta) \frac{1}{s^{p-1}} 2^{-i\alpha p} r_0^{\alpha p} (r_0(1+2^{-i+1}))^{N-\alpha p} (\epsilon \lambda)^{p-1} \leq c_9(N, \eta) \frac{1}{s^{p-1}} 2^{-i\alpha p} |Q| (\epsilon \lambda)^{p-1},$$

which is (2.16). Consequently, (2.15) can be rewritten as

$$\begin{aligned} |E_{\epsilon, \lambda}| &\leq \sum_{i=m+1}^{\infty} \frac{c_6(N, \eta)}{\left(2^{-\beta(i-m-1)}(1-2^{-\beta}) \left(1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon\right) \lambda\right)^{p-1}} 2^{-i\alpha p} (\epsilon \lambda)^{p-1} |Q| \\ &\leq c_6(N, \eta) 2^{-(m+1)\alpha p} \left(\frac{\epsilon}{1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon}\right)^{p-1} |Q| (1-2^{-\beta})^{-p+1} \sum_{i=m+1}^{\infty} 2^{(\beta(p-1)-\alpha p)(i-m-1)}. \end{aligned} \quad (2.17)$$

If we choose  $\beta = \beta(\alpha, p)$  so that  $\beta(p-1) - \alpha p < 0$ , we obtain

$$|E_{\epsilon, \lambda}| \leq c_{10} 2^{-m\alpha p} \left(\frac{\epsilon}{1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon}\right)^{p-1} |Q| \quad \forall m > m_0^{\frac{p-1}{p-1-\eta}} \quad (2.18)$$

where  $c_{10} = c_{10}(N, \alpha, p, \eta) > 0$ . Put  $\epsilon_0 = \min\left\{\frac{1}{\frac{4(p-1)}{p-1-\eta} m_0+1}, c_1\right\}$ . For any  $\epsilon \in (0, \epsilon_0]$  we choose  $m \in \mathbb{N}$  such that

$$\left(\frac{p-1-\eta}{2(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \left(\frac{1}{\epsilon} - 1\right)^{\frac{p-1}{p-1-\eta}} - 1 < m \leq \left(\frac{p-1-\eta}{2(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \left(\frac{1}{\epsilon} - 1\right)^{\frac{p-1}{p-1-\eta}}.$$

Then

$$\left(\frac{\epsilon}{1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon}\right)^{p-1} \leq 1$$

and

$$2^{-m\alpha p} \leq 2^{\alpha p - \alpha p \left(\frac{p-1-\eta}{2(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \left(\frac{1}{\epsilon} - 1\right)^{\frac{p-1}{p-1-\eta}}} \leq 2^{\alpha p} \exp\left(-\alpha p \ln 2 \left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \epsilon^{-\frac{p-1}{p-1-\eta}}\right).$$

Combining these inequalities with (2.18) and (2.10), we get (2.9).

In the case  $\eta = 0$  we still have for any  $m \in \mathbb{N}$ ,  $\lambda, \epsilon > 0$  and  $x \in E_{\epsilon, \lambda}$

$$\mathbf{W}_{\alpha, p}^{r_0}[\mu](x) \leq m\epsilon\lambda + \sum_{i=m+1}^{\infty} g_i(x)$$

Accordingly (2.18) reads as

$$|E_{\epsilon, \lambda}| \leq c_{10} 2^{-m\alpha p} \left(\frac{\epsilon}{1 - m\epsilon}\right)^{p-1} |Q| \quad \forall m \in \mathbb{N}, \lambda, \epsilon > 0 \text{ with } m\epsilon < 1.$$

Put  $\epsilon_0 = \min\{\frac{1}{2}, c_1\}$ . For any  $\epsilon \in (0, \epsilon_0]$  and  $m \in \mathbb{N}$  satisfies  $\epsilon^{-1} - 2 < m \leq \epsilon^{-1} - 1$ , we finally get from (2.10 )

$$|F_{\epsilon, \lambda} \cap Q| \leq |E_{\epsilon, \lambda}| \leq c_{10} 2^{2\alpha p} \exp(-\alpha p \epsilon^{-1} \ln 2) |Q|, \quad (2.19)$$

which ends the proof in the case  $R = \infty$ .

*Case  $R < \infty$ .* For  $\lambda > 0$ ,  $D_\lambda = \{\mathbf{W}_{\alpha, p}^R > \lambda\}$  is open. Using again Whitney covering lemma, there exists a countable set of closed cubes  $\mathcal{Q} := \{Q_i\}$  such that  $\cup_i Q_i = D_\lambda$ ,  $\overset{\circ}{Q}_i \cap \overset{\circ}{Q}_j = \emptyset$  for  $i \neq j$  and  $\text{dist}(Q_i, D_\lambda^c) \leq 4 \text{diam}(Q_i)$ . If  $Q \in \mathcal{Q}$  : is such that  $\text{diam}(Q) > \frac{R}{8}$ , there exists a finite number  $n_Q$  of closed dyadic cubes  $\{P_{j, Q}\}_{j=1}^{n_Q}$  such that  $\cup_{j=1}^{n_Q} P_{j, Q} = Q$ ,  $P_{i, Q} \cap P_{j, Q} = \emptyset$  if  $i \neq j$  and  $\frac{R}{16} < \text{diam}(P_{j, Q}) \leq \frac{R}{8}$ . We set  $\mathcal{Q}' = \{Q \in \mathcal{Q} : \text{diam}(Q) \leq \frac{R}{8}\}$ ,  $\mathcal{Q}'' = \{P_{i, Q} : 1 \leq i \leq n_Q, Q \in \mathcal{Q}, \text{diam}(Q) > \frac{R}{8}\}$  and  $\mathcal{F} = \mathcal{Q}' \cup \mathcal{Q}''$ .

For  $\epsilon > 0$  we denote again  $F_{\epsilon, \lambda} = \left\{ \mathbf{W}_{\alpha, p}^R[\mu] > 3\lambda, (\mathbf{M}_{\alpha p, R}^\eta[\mu])^{\frac{1}{p-1}} \leq \epsilon\lambda \right\}$ . Let  $Q \in \mathcal{F}$  such that  $F_{\epsilon, \lambda} \cap Q \neq \emptyset$  and  $r_0 = 5 \text{diam}(Q)$ .

If  $\text{dist}(D_\lambda^c, Q) \leq 4 \text{diam}(Q)$ , that is if there exists  $x_Q \in D_\lambda^c$  such that  $\text{dist}(x_Q, Q) \leq 4 \text{diam}(Q)$  and  $\mathbf{W}_{\alpha, p}^R[\mu](x_Q) \leq \lambda$ , we find, by the same argument as in the case  $R = \infty$ , (2.13 ), that for any  $x \in F_{\epsilon, \lambda} \cap Q$  there holds

$$\int_{r_0}^R \left( \frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq (1 + c_{11}\epsilon)\lambda. \quad (2.20)$$

where  $c_{11} = c_{11}(N, \alpha, p, \eta) > 0$ .

If  $\text{dist}(D_\lambda^c, Q) > 4 \text{diam}(Q)$ , we have  $\frac{R}{16} < \text{diam}(Q) \leq \frac{R}{8}$  since  $Q \in \mathcal{Q}''$ . Then, for all  $x \in F_{\epsilon, \lambda} \cap Q$ , there holds

$$\begin{aligned} \int_{r_0}^R \left( \frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} &\leq \int_{\frac{5R}{16}}^R \left( \frac{t^{N-\alpha p} (\ln 2)^{-\eta} (\epsilon\lambda)^{p-1}}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &= (\ln 2)^{-\frac{\eta}{p-1}} \ln \frac{16}{5} \epsilon\lambda \\ &\leq 2\epsilon\lambda. \end{aligned} \quad (2.21)$$

Thus, if we take  $\epsilon \in (0, c_{12}]$  with  $c_{12} = \min\{1, c_{11}^{-1}\}$ , we derive

$$F_{\epsilon, \lambda} \cap Q \subset E_{\epsilon, \lambda}, \quad (2.22)$$

where

$$E_{\epsilon, \lambda} = \left\{ \mathbf{W}_{\alpha, p}^{r_0}[\mu] > \lambda, (\mathbf{M}_{\alpha p, R}^\eta[\mu])^{\frac{1}{p-1}} \leq \epsilon\lambda \right\}.$$

Furthermore, since  $x \notin B_2$ ,

$$\mathbf{W}_{\alpha, p}^R[\mu](x) = \int_{\min\{r, R\}}^R \left( \frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq (\mu(\mathbb{R}^N))^{\frac{1}{p-1}} l(r, R).$$

Thus, if  $\lambda > (\mu(\mathbb{R}^N))^{\frac{1}{p-1}} l(r, R)$  then  $D_\lambda \subset B_2$  which implies  $r_0 = 5 \text{diam}(\mathbb{Q}) \leq 20r$ . The end of the proof is as in the case  $R = \infty$ .  $\square$

In the next result we list a series of equivalent norms concerning Radon measures.

**Theorem 2.3** *Assume  $\alpha > 0$ ,  $0 < p-1 < q < \infty$ ,  $0 < \alpha p < N$  and  $0 < s \leq \infty$ . Then there exists a constant  $c_{13} = c_{13}(N, \alpha, p, q, s) > 0$  such that for any  $R \in (0, \infty]$  and  $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$ , there holds*

$$c_{13}^{-1} \|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)} \leq \|\mathbf{M}_{\alpha p,R}[\mu]\|_{L^{\frac{p-1}{q}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{13} \|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)}. \quad (2.23)$$

For any  $R > 0$ , there exists  $c_{14} = c_{14}(N, \alpha, p, q, s, R) > 0$  such that for any  $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$ ,

$$c_{14}^{-1} \|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)} \leq \|\mathbf{G}_{\alpha p}[\mu]\|_{L^{\frac{p-1}{q}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{14} \|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)}. \quad (2.24)$$

In (2.24),  $\|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)}$  can be replaced by  $\|\mathbf{M}_{\alpha p,R}[\mu]\|_{L^{\frac{p-1}{q}, \frac{s}{p-1}}(\mathbb{R}^N)}$ .

*Proof.* We denote  $\mu_n$  by  $\chi_{B_n} \mu$  for  $n \in \mathbb{N}^*$ .

*Step 1* We claim that

$$\|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)} \leq c'_{13} \|\mathbf{M}_{\alpha p,R}[\mu]\|_{L^{\frac{p-1}{q}, \frac{s}{p-1}}(\mathbb{R}^N)}. \quad (2.25)$$

From Proposition 2.2 there exist positive constants  $c_0 = c_0(N, \alpha, p)$ ,  $a = a(\alpha, p)$  and  $\epsilon_0 = \epsilon_0(N, \alpha, p)$  such that for all  $n \in \mathbb{N}^*$ ,  $t > 0$ ,  $0 < R \leq \infty$  and  $0 < \epsilon \leq \epsilon_0$ , there holds

$$\left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > 3t, (\mathbf{M}_{\alpha p,R}^\eta[\mu_n])^{\frac{1}{p-1}} \leq \epsilon t \right\} \right| \leq c_0 \exp(-a\epsilon^{-1}) \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > t \right\} \right|. \quad (2.26)$$

In the case  $0 < s < \infty$  and  $0 < q < \infty$ , we have

$$\left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > 3t \right\} \right|^{\frac{s}{q}} \leq c_{15} \exp\left(-\frac{s}{q} a\epsilon^{-1}\right) \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > t \right\} \right|^{\frac{s}{q}} + c_{15} \left| \left\{ (\mathbf{M}_{\alpha p,R}^\eta[\mu_n])^{\frac{1}{p-1}} > \epsilon t \right\} \right|^{\frac{s}{q}}.$$

with  $c_{15} = c_{15}(N, \alpha, p, q, s) > 0$ .

Multiplying by  $t^{s-1}$  and integrating over  $(0, \infty)$ , we obtain

$$\begin{aligned} \int_0^\infty t^s \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > 3t \right\} \right|^{\frac{s}{q}} \frac{dt}{t} &\leq c_{15} \exp\left(-\frac{s}{q} a\epsilon^{-1}\right) \int_0^\infty t^s \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > t \right\} \right|^{\frac{s}{q}} \frac{dt}{t} \\ &\quad + c_{15} \int_0^\infty t^s \left| \left\{ \mathbf{M}_{\alpha p,R}^\eta[\mu_n] > (\epsilon t)^{p-1} \right\} \right|^{\frac{s}{q}} \frac{dt}{t}. \end{aligned}$$

By a change of variable, we derive

$$\begin{aligned} \left( 3^{-s} - c_{15} \exp\left(-\frac{s}{q} a\epsilon^{-1}\right) \right) \int_0^\infty t^s \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > t \right\} \right|^{\frac{s}{q}} \frac{dt}{t} \\ \leq \frac{c_{15} \epsilon^{-s}}{p-1} \int_0^\infty t^{\frac{s}{p-1}} \left| \left\{ \mathbf{M}_{\alpha p,R}^\eta[\mu_n] > t \right\} \right|^{\frac{s}{q}} \frac{dt}{t}. \end{aligned}$$

We choose  $\epsilon$  small enough so that  $3^{-s} - c_{15} \exp\left(-\frac{s}{q} a \epsilon^{-1}\right) > 0$ , we derive from (2.2) and  $\|t^{1/s_1} f^*\|_{L^{s_2}(\mathbb{R}, \frac{dt}{t})} = s_1^{1/s_2} \|\lambda_f^{1/s_1} t\|_{L^{s_2}(\mathbb{R}, \frac{dt}{t})}$  for any  $f \in L^{s_1, s_2}(\mathbb{R}^N)$  with  $0 < s_1 < \infty, 0 < s_2 \leq \infty$

$$\|\mathbf{W}_{\alpha, p}^R[\mu_n]\|_{L^{q, s}(\mathbb{R}^N)} \leq c'_{13} \|\mathbf{M}_{\alpha p, R}[\mu_n]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)},$$

and (2.25) follows by Fatou's lemma. Similarly, we can prove (2.25) in the case  $s = \infty$ .

*Step 2* We claim that

$$\|\mathbf{W}_{\alpha, p}^R[\mu]\|_{L^{q, s}(\mathbb{R}^N)} \geq c''_{13} \|\mathbf{M}_{\alpha p, R}[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)}. \quad (2.27)$$

For  $R > 0$  we have

$$\begin{aligned} \mathbf{W}_{\alpha, p}^{2R}[\mu_n](x) &= \mathbf{W}_{\alpha, p}^R[\mu_n](x) + \int_R^{2R} \left( \frac{\mu_n(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \mathbf{W}_{\alpha, p}^R[\mu_n](x) + \left( \frac{\mu_n(B_{2R}(x))}{R^{N-\alpha p}} \right)^{\frac{1}{p-1}}. \end{aligned} \quad (2.28)$$

Thus

$$\left| \left\{ x : \mathbf{W}_{\alpha, p}^{2R}[\mu_n](x) > 2t \right\} \right| \leq \left| \left\{ x : \mathbf{W}_{\alpha, p}^R[\mu_n](x) > t \right\} \right| + \left| \left\{ x : \frac{\mu_n(B_{2R}(x))}{R^{N-\alpha p}} > t^{p-1} \right\} \right|,$$

Consider  $\{z_j\}_{j=1}^m \subset B_2$  such that  $B_2 \subset \bigcup_{i=1}^m B_{\frac{1}{2}}(z_i)$ . Thus  $B_{2R}(x) \subset \bigcup_{i=1}^m B_{\frac{R}{2}}(x + Rz_i)$  for any  $x \in \mathbb{R}^N$  and  $R > 0$ . Then

$$\begin{aligned} \left| \left\{ x : \frac{\mu_n(B_{2R}(x))}{R^{N-\alpha p}} > t^{p-1} \right\} \right| &\leq \left| \left\{ x : \sum_{i=1}^m \frac{\mu_n(B_{\frac{R}{2}}(x + Rz_i))}{R^{N-\alpha p}} > t^{p-1} \right\} \right| \\ &\leq \sum_{i=1}^m \left| \left\{ x : \frac{\mu_n(B_{\frac{R}{2}}(x + Rz_i))}{R^{N-\alpha p}} > \frac{1}{m} t^{p-1} \right\} \right| \\ &\leq \sum_{i=1}^m \left| \left\{ x - Rz_i : \frac{\mu_n(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}} > \frac{1}{m} t^{p-1} \right\} \right| \\ &= m \left| \left\{ x : \frac{\mu_n(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}} > \frac{1}{m} t^{p-1} \right\} \right|. \end{aligned}$$

Moreover from (2.28)

$$\left( \frac{\mu_n(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}} \right)^{\frac{1}{p-1}} \leq 2 \mathbf{W}_{\alpha, p}^R[\mu_n](x),$$

thus

$$\left| \left\{ x : \frac{\mu_n(B_{2R}(x))}{R^{N-\alpha p}} > t^{p-1} \right\} \right| \leq m \left| \left\{ x : \mathbf{W}_{\alpha, p}^R[\mu_n](x) > \frac{1}{2m^{\frac{1}{p-1}}} t \right\} \right|.$$

This leads to

$$|\{x : \mathbf{W}_{\alpha,p}^{2R}[\mu_n](x) > 2t\}| \leq (m+1) \left| \left\{ x : \mathbf{W}_{\alpha,p}^R[\mu_n](x) > \frac{1}{2m^{\frac{1}{p-1}}}t \right\} \right| \quad \forall t > 0$$

This implies

$$\|\mathbf{W}_{\alpha,p}^{2R}[\mu_n]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{16} \|\mathbf{W}_{\alpha,p}^R[\mu_n]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)},$$

with  $c_{16} = c_{16}(N, \alpha, p, q, s) > 0$ . By Fatou's lemma, we get

$$\|\mathbf{W}_{\alpha,p}^{2R}[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{16} \|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)}. \quad (2.29)$$

On the other hand, from the identity in (2.28) we derive that for any  $\rho \in (0, R)$ ,

$$\mathbf{W}_{\alpha,p}^{2R}[\mu](x) \geq \mathbf{W}_{\alpha,p}^{2\rho}[\mu](x) \geq c_{17} \sup_{0 < \rho \leq R} \left( \frac{\mu(B_\rho(x))}{\rho^{N-\alpha p}} \right)^{\frac{1}{p-1}},$$

with  $c_{17} = c_{17}(N, \alpha, p) > 0$ , from which follows

$$\mathbf{W}_{\alpha,p}^{2R}[\mu](x) \geq c_{17} (\mathbf{M}_{\alpha p, R}[\mu](x))^{\frac{1}{p-1}}. \quad (2.30)$$

Combining (2.29) and (2.30) we obtain (2.27) and then (2.23). Notice that the estimates are independent of  $R$  and thus valid if  $R = \infty$ .

*Step 3* We claim that (2.24) holds. By the previous result we have also

$$c_{18}^{-1} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq \|\mathbf{M}_{\alpha p, R}[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{18} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)}. \quad (2.31)$$

where  $c_{18} = c_{18}(N, \alpha, p, q, s) > 0$ . For  $R > 0$ , the Bessel kernel satisfies [18, V-3-1]

$$c_{19}^{-1} \left( \frac{\chi_{B_R}(x)}{|x|^{N-\alpha p}} \right) \leq G_{\alpha p}(x) \leq c_{19} \left( \frac{\chi_{B_{\frac{R}{2}}}(x)}{|x|^{N-\alpha p}} \right) + c_{19} e^{-\frac{|x|}{2}} \quad \forall x \in \mathbb{R}^N,$$

where  $c_{19} = c_{19}(N, \alpha, p, R) > 0$ . Therefore

$$c_{19}^{-1} \left( \frac{\chi_{B_R}}{|\cdot|^{N-\alpha p}} \right) * \mu \leq \mathbf{G}_{\alpha p}[\mu] \leq c_{19} \left( \frac{\chi_{B_{\frac{R}{2}}}}{|\cdot|^{N-\alpha p}} \right) * \mu + c_{19} e^{-\frac{|\cdot|}{2}} * \mu. \quad (2.32)$$

By integration by parts, we get

$$\left( \frac{\chi_{B_R}}{|\cdot|^{N-\alpha p}} \right) * \mu(x) = (N - \alpha p) \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu](x) + \frac{\mu(B_R(x))}{R^{N-\alpha p}} \geq (N - \alpha p) \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu](x),$$

which implies

$$c_{20} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq \|\mathbf{G}_{\alpha p}[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)}. \quad (2.33)$$

where  $c_{20} = c_{20}(N, \alpha, p, q, s) > 0$ . Furthermore  $e^{-\frac{|x|}{2}} \leq c_{21} \chi_{B_{\frac{R}{2}}} * e^{-\frac{|\cdot|}{2}}(x)$  where  $c_{21} = c_{21}(N, R) > 0$ , thus

$$e^{-\frac{|\cdot|}{2}} * \mu \leq c_{21} \left( \chi_{B_{\frac{R}{2}}} * e^{-\frac{|\cdot|}{2}} \right) * \mu = c_{21} e^{-\frac{|\cdot|}{2}} * \left( \chi_{B_{\frac{R}{2}}} * \mu \right).$$

Since

$$\chi_{B_{\frac{R}{2}}} * \mu(x) = \mu(B_{\frac{R}{2}}(x)) \leq c_{22} \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu](x)$$

where  $c_{22} = c_{22}(N, \alpha, p, R) > 0$ , we derive with  $c_{23} = c_{21}c_{22}$

$$e^{-\frac{|\cdot|}{2}} * \mu \leq c_{23} e^{-\frac{|\cdot|}{2}} * \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu].$$

Using Young inequality, we obtain

$$\begin{aligned} \left\| e^{-\frac{|\cdot|}{2}} * \mu \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} &\leq c_{23} \left\| e^{-\frac{|\cdot|}{2}} * \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \\ &\leq c_{24} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \left\| e^{-\frac{|\cdot|}{2}} \right\|_{L^{1, \infty}(\mathbb{R}^N)} \\ &\leq c_{25} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)}. \end{aligned} \quad (2.34)$$

where  $c_{25} = c_{25}(N, \alpha, p, R) > 0$ .

Since by integration by parts there holds as above

$$\left( \frac{\chi_{B_{\frac{R}{2}}}}{|\cdot|^{N-\alpha p}} \right) * \mu(x) = (N - \alpha p) \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu](x) + 2^{N-\alpha p} \frac{\mu(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}} \leq c_{26} \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu](x),$$

where  $c_{26} = c_{26}(N, \alpha, p) > 0$  we obtain

$$\left\| \left( \frac{\chi_{B_R}}{|\cdot|^{N-\alpha p}} \right) * \mu \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{27} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}}. \quad (2.35)$$

where  $c_{27} = c_{27}(N, \alpha, p, q, s) > 0$ . Thus

$$\left\| \mathbf{G}_{\alpha p}[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{28} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}}. \quad (2.36)$$

where  $c_{28} = c_{28}(N, \alpha, p, q, s, R) > 0$ .

follows by combining (2.32), (2.34) and (2.35). Then, combining (2.33), (2.36) and using (2.31), (2.23) we obtain (2.24).  $\square$

*Remark.* Proposition 5.1 in [17] is a particular case of the previous result.

**Theorem 2.4** *Let  $\alpha > 0$ ,  $p > 1$ ,  $0 \leq \eta < p - 1$ ,  $0 < \alpha p < N$  and  $r > 0$ . Set  $\delta_0 = \left( \frac{p-1-\eta}{12(p-1)} \right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2$ . Then there exists  $c_{29} > 0$ , depending on  $N, \alpha, p, \eta$  and  $r$  such that*

for any  $R \in (0, \infty]$ ,  $\delta \in (0, \delta_0)$ ,  $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$ , any ball  $B_1 \subset \mathbb{R}^N$  with radius  $\leq r$  and ball  $B_2$  concentric to  $B_1$  with radius double  $B_1$ 's radius, there holds

$$\frac{1}{|B_2|} \int_{B_2} \exp \left( \delta \frac{(\mathbf{W}_{\alpha,p}^R[\mu_{B_1}](x))^{\frac{p-1}{p-1-\eta}}}{\|\mathbf{M}_{\alpha,p,R}^\eta[\mu_{B_1}]\|_{L^\infty(B_1)}^{\frac{p-1}{p-1-\eta}}} \right) dx \leq \frac{c_{29}}{\delta_0 - \delta} \quad (2.37)$$

where  $\mu_{B_1} = \chi_{B_1} \mu$ . Furthermore, if  $\eta = 0$ ,  $c_{29}$  is independent of  $r$ .

*Proof.* Let  $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$  such that  $M := \|\mathbf{M}_{\alpha,p,R}^\eta[\mu_{B_1}]\|_{L^\infty(B_1)} < \infty$ . By Proposition 2.2- (2.8) with  $\mu = \mu_{B_1}$ , there exist  $c_0 > 0$  depending on  $N, \alpha, p, \eta$  and  $\epsilon_0 > 0$  depending on  $N, \alpha, p, \eta$  and  $r$  such that, for all  $R \in (0, \infty]$ ,  $\epsilon \in (0, \epsilon_0]$ ,  $t > (\mu_{B_1}(\mathbb{R}^N))^{\frac{1}{p-1}} l(r', R)$  where  $r'$  is radius of  $B_1$  there holds,

$$\begin{aligned} & \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_{B_1}] > 3t, (\mathbf{M}_{\alpha,p,R}^\eta[\mu_{B_1}])^{\frac{1}{p-1}} \leq \epsilon t \right\} \right| \\ & \leq c_0 \exp \left( - \left( \frac{p-1-\eta}{4(p-1)} \right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 \epsilon^{-\frac{p-1}{p-1-\eta}} \right) \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_{B_1}] > t \right\} \right|. \end{aligned} \quad (2.38)$$

Since  $(\mu_{B_1}(\mathbb{R}^N))^{\frac{1}{p-1}} l(r', R) \leq \frac{N-\alpha p}{p-1} (\ln 2)^{-\frac{\eta}{p-1}} M^{\frac{1}{p-1}}$ , thus in (2.8) we can choose

$$\epsilon = t^{-1} \left\| \mathbf{M}_{\alpha,p,R}^\eta[\mu_{B_1}] \right\|_{L^\infty(\mathbb{R}^N)}^{\frac{1}{p-1}} = t^{-1} M^{\frac{1}{p-1}} \quad \forall t > \max \left\{ \epsilon_0^{-1}, \frac{N-\alpha p}{p-1} (\ln 2)^{-\frac{\eta}{p-1}} \right\} M^{\frac{1}{p-1}}$$

and as in the proof of Proposition 2.2,  $\left\{ \mathbf{W}_{\alpha,p}^R[\mu_{B_1}] > t \right\} \subset B_2$ .

Then

$$\left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_{B_1}] > 3t \right\} \cap B_2 \right| \leq c_0 \exp \left( - \left( \frac{p-1-\eta}{4(p-1)} \right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 M^{-\frac{1}{p-1-\eta}} t^{\frac{p-1}{p-1-\eta}} \right) |B_2|. \quad (2.39)$$

This can be written under the form

$$\left| \left\{ F > t \right\} \cap B_2 \right| \leq |B_2| \chi_{(0,t_0]} + c_0 \exp(-\delta_0 t) |B_2| \chi_{(t_0,\infty)}(t). \quad (2.40)$$

where  $F = M^{-\frac{1}{p-1-\eta}} (\mathbf{W}_{\alpha,p}^R[\mu_{B_1}])^{\frac{p-1}{p-1-\eta}}$  and  $t_0 = \left( 3 \max \left\{ \epsilon_0^{-1}, \frac{N-\alpha p}{p-1} (\ln 2)^{-\frac{\eta}{p-1}} \right\} \right)^{\frac{p-1}{p-1-\eta}}$ .

Take  $\delta \in (0, \delta_0)$ , by Fubini's theorem

$$\int_{B_2} \exp(\delta F(x)) dx = \delta \int_0^\infty \exp(\delta t) \left| \left\{ F > t \right\} \cap B_2 \right| dt$$

Thus,

$$\begin{aligned} \int_{B_2} \exp(\delta F(x)) dx & \leq \delta \int_0^{t_0} \exp(\delta t) dt |B_2| + c_0 \delta \int_{t_0}^\infty \exp(-(\delta_0 - \delta)t) dt |B_2| \\ & \leq (\exp(\delta t_0) - 1) |B_2| + \frac{c_0 \delta}{\delta_0 - \delta} |B_2| \end{aligned}$$

which is the desired inequality.  $\square$

*Remark.* By the proof of Proposition 2.2, we see that  $\epsilon_0 \geq \frac{c_{30}}{\max(1, \ln 40r)}$  where  $c_{30} = c_{30}(N, \alpha, p, \eta) > 0$ . Thus,  $t_0 \leq c_{31} (\max(1, \ln 40r))^{\frac{p-1}{p-1-\eta}}$ . Therefore  $c_{29} \leq c_{32} \exp \left( c_{33} (\max(1, \ln 40r))^{\frac{p-1}{p-1-\eta}} \right)$  where  $c_{32}$  and  $c_{33}$  depend on  $N, \alpha, p$  and  $\eta$ .



## 2.4 Approximation of measures

The next result is an extension of a classical result of Feyel and de la Pradelle [11]. This type of result has been intensively used in the framework of Sobolev spaces since the pioneering work of Baras and Pierre [3], but apparently it is new in the case of Bessel-Lorentz spaces. We recall that a sequence of bounded measures  $\{\mu_n\}$  in  $\Omega$  converges to some bounded measure  $\mu$  in  $\Omega$  in the *narrow topology* of  $\mathfrak{M}^b(\Omega)$  if

$$\lim_{n \rightarrow \infty} \int_{\Omega} \phi d\mu_n = \int_{\Omega} \phi d\mu \quad \forall \phi \in C_b(\Omega) := C(\Omega) \cap L^\infty(\Omega). \quad (2.41)$$

**Theorem 2.5** *Assume  $\Omega$  is an open subset of  $\mathbb{R}^N$ . Let  $\alpha > 0$ ,  $1 < s < \infty$ ,  $1 \leq q < \infty$  and  $\mu \in \mathfrak{M}_+(\Omega)$ . If  $\mu$  is absolutely continuous with respect to  $C_{\alpha,s,q}$  in  $\Omega$ , there exists a nondecreasing sequence  $\{\mu_n\} \subset \mathfrak{M}_+(\Omega) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$ , with compact support in  $\Omega$  which converges to  $\mu$  weakly in the sense of measures. Furthermore, if  $\mu \in \mathfrak{M}_+^b(\Omega)$ , then  $\mu_n \rightarrow \mu$  in the narrow topology.*

*Proof. Step 1.* Assume that  $\mu$  has compact support. Let  $\phi \in L^{\alpha,s,q}(\mathbb{R}^N)$  and  $\tilde{\phi}$  its  $C_{\alpha,s,q}$ -quasicontinuous representative. Since  $\mu$  is absolutely continuous with respect to  $C_{\alpha,s,q}$ , we can define the mapping

$$\phi \mapsto P(\phi) = \int_{\mathbb{R}^N} \tilde{\phi}^+ d\mu|_{\Omega}$$

where  $\mu|_{\Omega}$  is the extension of  $\mu$  by 0 in  $\Omega^c$ . By Fatou's lemma,  $P$  is lower semicontinuous on  $L^{\alpha,s,q}(\mathbb{R}^N)$ . Furthermore it is convex and positively homogeneous of degree 1. If  $Epi(P)$  denotes the epigraph of  $P$ , i.e.

$$Epi(P) = \{(\phi, t) \in L^{\alpha,s,q}(\mathbb{R}^N) \times \mathbb{R} : t \geq P(\phi)\},$$

it is a closed convex cone. Let  $\epsilon > 0$  and  $\phi_0 \in C_c^\infty$ ,  $\phi_0 \geq 0$ . Since  $(\phi_0, P(\phi_0) - \epsilon) \notin Epi(P)$ , there exist  $\ell \in (L^{\alpha,s,q}(\mathbb{R}^N))'$ ,  $a$  and  $b$  in  $\mathbb{R}$  such that

$$a + bt + \ell(\phi) \leq 0 \quad \forall (\phi, t) \in Epi(P), \quad (2.42)$$

$$a + b(P(\phi_0) - \epsilon) + \ell(\phi_0) > 0. \quad (2.43)$$

Since  $(0, 0) \in Epi(P)$ ,  $a \leq 0$ . Since  $(s\phi, st) \in Epi(P)$  for all  $s > 0$ ,  $s^{-1}a + bt + \ell(\phi) \leq 0$ , which implies

$$bt + \ell(\phi) \leq 0 \quad \forall (\phi, t) \in Epi(P).$$

Finally, since  $(0, 1) \in Epi(P)$ ,  $b \leq 0$ . But if  $b = 0$  we would have  $\ell(\phi) \leq -a$  for all  $\phi \in L^{\alpha,s,q}(\mathbb{R}^N)$ . which would lead to  $\ell = 0$  and  $a > 0$  from (2.43), a contradiction. Therefore  $b < 0$ . Then, we put  $\theta(\phi) = -\frac{\ell(\phi)}{b}$  and derive that, for any  $(\phi, t) \in Epi(P)$ , there holds  $\theta(\phi) \leq t$ , and in particular

$$\theta(\phi) \leq P(\phi) \quad \forall \phi \in L^{\alpha,s,q}(\mathbb{R}^N). \quad (2.44)$$

Since  $\phi \leq 0 \implies P(\phi) = 0$ ,  $\theta$  is a positive linear functional on  $L^{\alpha,s,q}(\mathbb{R}^N)$ . Furthermore

$$\sup_{\substack{\phi \in C_c^\infty(\mathbb{R}^N) \\ \|\phi\|_{L^\infty} \leq 1}} |\theta(\phi)| = \sup_{\substack{\phi \in C_c^\infty(\mathbb{R}^N) \\ \|\phi\|_{L^\infty} \leq 1}} \theta(\phi) \leq \sup_{\substack{\phi \in C_c^\infty(\mathbb{R}^N) \\ \|\phi\|_{L^\infty} \leq 1}} P(\phi) = P(1) = \mu(\Omega).$$

By the Riesz representation theorem, there exists  $\sigma \in \mathfrak{M}_+(\mathbb{R}^N)$  such that

$$\theta(\phi) = \int_{\mathbb{R}^N} \phi d\sigma \quad \forall \phi \in C_c^\infty(\mathbb{R}^N). \quad (2.45)$$

Inequality (2.44) implies  $0 \leq \sigma \leq \mu|_\Omega$ . Thus  $\text{supp}(\sigma) \subset \text{supp}(\mu|_\Omega) = \text{supp}(\mu)$  and  $\sigma$  vanishes on Borel subsets of  $C_{\alpha,s,q}$  capacity zero, as  $\mu$  does it, besides (2.45) also values for all  $\phi \in C^\infty(\mathbb{R}^N)$ . From (2.43), we have

$$\int_{\mathbb{R}^N} \tilde{\phi}_0 d\sigma = \theta(\phi_0) > P(\phi_0) - \epsilon + \frac{a}{b} \geq \int_{\mathbb{R}^N} \tilde{\phi}_0 d\mu|_\Omega - \epsilon.$$

This implies

$$0 \leq \int_{\mathbb{R}^N} \tilde{\phi}_0 d(\mu|_\Omega - \sigma) \leq \epsilon. \quad (2.46)$$

It remains to prove that  $\sigma \in (L^{\alpha,s,q}(\mathbb{R}^N))'$ . For all  $f \in C_c^\infty(\mathbb{R}^N)$ ,  $f \geq 0$ , there holds

$$\int_{\mathbb{R}^N} \mathbf{G}_\alpha[f] d\sigma = \theta(\mathbf{G}_\alpha[f]) \leq \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \|\mathbf{G}_\alpha[f]\|_{L^{\alpha,s,q}(\mathbb{R}^N)}, \quad (2.47)$$

since  $\theta = -b^{-1}\ell$  and  $\ell \in (L^{\alpha,s,q}(\mathbb{R}^N))'$ . Now, given  $f \in L^{s,q}(\mathbb{R}^N)$ ,  $f \geq 0$  and a sequence of molifiers  $\{\rho_n\}$ ,  $(\chi_{B_n} f) * \rho_n \in C_c^\infty(\mathbb{R}^N)$  and  $(\chi_{B_n} f) * \rho_n \rightarrow f$  in  $L^{s,q}(\mathbb{R}^N)$ , where  $\chi_{B_n}$  is the indicator function of the ball  $B_n$  centered at the origin of radius  $n$ . Furthermore, there is a subsequence  $\{n_k\}$  such that  $\lim_{n_k \rightarrow \infty} \mathbf{G}_\alpha[(\chi_{B_{n_k}} f) * \rho_{n_k}](x) \rightarrow \mathbf{G}_\alpha[f](x)$ ,  $C_{\alpha,s,q}$ -quasi everywhere. Using Fatou's lemma and lower semicontinuity of the norm

$$\begin{aligned} \int_{\mathbb{R}^N} \mathbf{G}_\alpha[f] d\sigma &\leq \liminf_{n_k \rightarrow \infty} \int_{\mathbb{R}^N} \mathbf{G}_\alpha[(\chi_{B_{n_k}} f) * \rho_{n_k}] d\sigma \\ &\leq \liminf_{n_k \rightarrow \infty} \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \left\| \mathbf{G}_\alpha[(\chi_{B_{n_k}} f) * \rho_{n_k}] \right\|_{L^{\alpha,s,q}(\mathbb{R}^N)} \\ &\leq \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \|\mathbf{G}_\alpha[f]\|_{L^{\alpha,s,q}(\mathbb{R}^N)}. \end{aligned}$$

Therefore (2.47) also holds for all  $f \in L^{s,q}(\mathbb{R}^N)$ ,  $f \geq 0$ . Consequently  $\sigma \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$  satisfies

$$\left| \int_{\mathbb{R}^N} \mathbf{G}_\alpha[f] d\sigma \right| \leq \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \|\mathbf{G}_\alpha[f]\|_{L^{\alpha,s,q}(\mathbb{R}^N)} \quad \forall f \in L^{s,q}(\mathbb{R}^N). \quad (2.48)$$

*Step 2.* We assume that  $\mu$  has no longer compact support. Set  $\Omega_n = \{x \in \Omega : \text{dist}(x, \Omega^c) \geq n^{-1}, |x| \leq n\}$ , then  $\Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1} \subset \Omega$  for  $n \geq n_0$  such that  $\Omega_{n_0} \neq \emptyset$ . Let  $\{\phi_n\} \subset C_c^\infty(\mathbb{R}^N)$  be an increasing sequence such that  $0 \leq \phi_n \leq 1$ ,  $\phi_n = 1$  in a neighborhood of  $\overline{\Omega}_n$  and  $\text{supp}(\phi_n) \subset \Omega_{n+1}$ . and let  $\nu_n = \phi_n \mu$ . For  $n \geq n_0$  there is  $\sigma_n \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$  with  $0 \leq \sigma_n \leq \nu_n$  and

$$\frac{1}{n} > \int_{\Omega} \phi_n d(\nu_n - \sigma_n) \geq \int_{\Omega_n} d(\nu_n - \sigma_n) = \int_{\Omega_n} d(\mu - \sigma_n).$$

We set  $\mu_n = \sup\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ , then  $\{\mu_n\}$  is nondecreasing and  $\text{supp}(\mu_n) \subset \Omega_{n+1}$ , and  $\mu_n \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap (L^{\alpha, s, q}(\mathbb{R}^N))'$ . Finally, let  $\phi \in C_c(\Omega)$  and  $m \in \mathbb{N}^*$  such that  $\text{supp}(\phi) \subset \Omega_m$ . For all  $n \geq m$ , we have

$$\left| \int_{\Omega} \phi d\mu_n - \int_{\Omega} \phi d\mu \right| \leq \left| \int_{\Omega_n} d(\mu - \mu_n) \right| \|\phi\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{n} \|\phi\|_{L^\infty(\mathbb{R}^N)}.$$

Thus  $\mu_n \rightharpoonup \mu$  weakly in the sense of measures.

*Step 3.* Assume that  $\mu \in \mathfrak{M}_+^b(\Omega)$ . Then  $\mu_n(\Omega) \leq \mu(\Omega)$ . Thus

$$\mu_n(\Omega) = \mu_n(\Omega_{n_0}) + \sum_{k=n_0}^{\infty} \mu_n(\overline{\Omega}_{k+1} \setminus \Omega_k)$$

Since the sequence  $\{\mu_n\}$  is nondecreasing and  $\lim_{k \rightarrow \infty} \mu_n(\overline{\Omega}_{k+1} \setminus \Omega_k) = \mu(\overline{\Omega}_{k+1} \setminus \Omega_k)$  by the previous construction, we obtain by monotone convergence

$$\lim_{n \rightarrow \infty} \mu_n(\Omega) = \mu(\Omega_{n_0}) + \sum_{k=n_0}^{\infty} \mu(\overline{\Omega}_{k+1} \setminus \Omega_k) = \mu(\Omega)$$

Next we consider  $\phi \in C_b(\Omega) := C(\Omega) \cap L^\infty(\Omega)$ , then

$$\left| \int_{\Omega} \phi d\mu_n - \int_{\Omega} \phi d\mu \right| \leq \left| \int_{\Omega} d(\mu - \mu_n) \right| \|\phi\|_{L^\infty(\Omega)} \leq (\mu(\Omega) - \mu_n(\Omega)) \|\phi\|_{L^\infty(\Omega)} \rightarrow 0.$$

Thus  $\mu_n \rightharpoonup \mu$  in the narrow topology of measures.  $\square$

As a consequence of Theorem 2.5 and Theorem 2.3 we obtain the following.

**Theorem 2.6** *Let  $p-1 < s_1 < \infty$ ,  $p-1 < s_2 \leq \infty$ ,  $0 < \alpha p < N$ ,  $R > 0$  and  $\mu \in \mathfrak{M}_+(\Omega)$ . If  $\mu$  is absolutely continuous with respect to the capacity  $C_{\alpha p, \frac{s_1}{s_1-p+1}, \frac{s_2}{s_2-p+1}}$ , there exists a nondecreasing sequence  $\{\mu_n\} \subset \mathfrak{M}_+(\Omega)$  with compact support in  $\Omega$  which converges to  $\mu$  in the weak sense of measures and such that  $\mathbf{W}_{\alpha, p}^R[\mu_n] \in L^{s_1, s_2}(\mathbb{R}^N)$ , for all  $n$ . Furthermore, if  $\mu \in \mathfrak{M}_+^b(\Omega)$ ,  $\mu_n$  converges to  $\mu$  in the narrow topology.*

*Proof.* By Theorem 2.5 there exists a nondecreasing sequence  $\{\mu_n\}$  of nonnegative measures with compact support in  $\Omega$ , all elements of  $(L^{\alpha p, \frac{s_1}{s_1-p+1}, \frac{s_2}{s_2-p+1}}(\mathbb{R}^N))'$ , which converges weakly to  $\mu$ . If  $\mu \in \mathfrak{M}_+^b(\Omega)$ , the convergence holds in the narrow topology. Noting that for a positive measure  $\sigma$  in  $\mathbb{R}^N$ ,

$$\mathbf{G}_{\alpha p}[\sigma] \in L^{\frac{s_1}{p-1}, \frac{s_2}{p-1}}(\mathbb{R}^N) \iff \sigma \in (L^{\alpha p, \frac{s_1}{s_1-p+1}, \frac{s_2}{s_2-p+1}}(\mathbb{R}^N))',$$

it implies  $\mathbf{G}_{\alpha p}[\mu_n] \in L^{\frac{s_1}{p-1}, \frac{s_2}{p-1}}(\mathbb{R}^N)$ . Then, by Theorem 2.3,  $\mathbf{W}_{\alpha, p}^R[\mu_n] \in L^{s_1, s_2}(\mathbb{R}^N)$ .  $\square$

### 3 Renormalized solutions

#### 3.1 Classical results

Although the notion of renormalized solutions is becoming more and more present in the theory of quasilinear equations with measure data, it has not yet acquainted a popularity which could avoid us to present some of its main aspects. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ . If  $\mu \in \mathfrak{M}^b(\Omega)$ , we denote by  $\mu^+$  and  $\mu^-$  respectively its positive and negative part. We denote by  $\mathfrak{M}_0(\Omega)$  the space of measures in  $\Omega$  which are absolutely continuous with respect to the  $c_{1,p}^\Omega$ -capacity defined on a compact set  $K \subset \Omega$  by

$$c_{1,p}^\Omega(K) = \inf \left\{ \int_\Omega |\nabla \phi|^p dx : \phi \geq \chi_K, \phi \in C_c^\infty(\Omega) \right\}. \quad (3.1)$$

We also denote  $\mathfrak{M}_s(\Omega)$  the space of measures in  $\Omega$  with support on a set of zero  $c_{1,p}^\Omega$ -capacity. Classically, any  $\mu \in \mathfrak{M}^b(\Omega)$  can be written in a unique way under the form  $\mu = \mu_0 + \mu_s$  where  $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}^b(\Omega)$  and  $\mu_s \in \mathfrak{M}_s(\Omega)$ . We recall that any  $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}^b(\Omega)$  can be written under the form  $\mu_0 = f - \operatorname{div} g$  where  $f \in L^1(\Omega)$  and  $g \in L^{p'}(\Omega)$ .

For  $k > 0$  and  $s \in \mathbb{R}$  we set  $T_k(s) = \max\{\min\{s, k\}, -k\}$ . We recall that if  $u$  is a measurable function defined and finite a.e. in  $\Omega$ , such that  $T_k(u) \in W_0^{1,p}(\Omega)$  for any  $k > 0$ , there exists a measurable function  $v : \Omega \rightarrow \mathbb{R}^N$  such that  $\nabla T_k(u) = \chi_{|u| \leq k} v$  a.e. in  $\Omega$  and for all  $k > 0$ . We define the gradient  $\nabla u$  of  $u$  by  $v = \nabla u$ . We recall the definition of a renormalized solution given in [10].

**Definition 3.1** *Let  $\mu = \mu_0 + \mu_s \in \mathfrak{M}^b(\Omega)$ . A measurable function  $u$  defined in  $\Omega$  and finite a.e. is called a renormalized solution of*

$$\begin{aligned} -\Delta_p u &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (3.2)$$

if  $T_k(u) \in W_0^{1,p}(\Omega)$  for any  $k > 0$ ,  $|\nabla u|^{p-1} \in L^r(\Omega)$  for any  $0 < r < \frac{N}{N-1}$ , and  $u$  has the property that for any  $k > 0$  there exist  $\lambda_k^+, \lambda_k^- \in \mathfrak{M}_+^b(\Omega) \cap \mathfrak{M}_0(\Omega)$ , respectively concentrated on the sets  $u = k$  and  $u = -k$ , with the property that  $\lambda_k^+ \rightarrow \mu_s^+$ ,  $\lambda_k^- \rightarrow \mu_s^-$  in the narrow topology of measures, such that

$$\int_{\{|u| < k\}} |\nabla u|^{p-2} \nabla u \nabla \phi dx = \int_{\{|u| < k\}} \phi d\mu_0 + \int_\Omega \phi d\lambda_k^+ - \int_\Omega \phi d\lambda_k^-, \quad (3.3)$$

for every  $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ .

*Remark.* If  $u$  is a renormalized solution of problem (3.2) and  $\mu \in \mathfrak{M}_+^b(\Omega)$ , then  $u \geq 0$  in  $\Omega$ . Indeed, taking  $k > m > 0$  and  $\phi = T_m(\max\{-u, 0\})$ , then  $0 \leq \phi \leq m$  and we have

$$\begin{aligned} \int_{\{|u| < k\}} |\nabla u|^{p-2} \nabla u \nabla \phi dx &= \int_{\{|u| < k\}} T_m(\max\{-u, 0\}) d\mu_0 + \int_\Omega T_m(\max\{-u, 0\}) d\lambda_k^+ \\ &\quad - \int_\Omega T_m(\max\{-u, 0\}) d\lambda_k^- \\ &\geq -m\lambda_k^-(\Omega). \end{aligned}$$

Thus

$$\int_{\Omega} |\nabla T_m(\max\{-u, 0\})|^p \leq m\lambda_k^-(\Omega)$$

Letting  $k \rightarrow \infty$ , we obtain  $\nabla T_m(\max\{-u, 0\}) = 0$  a.e., thus  $u \geq 0$  a.e. in  $\Omega$ .

We recall the following important results, see [10, Th 4.1, Sec 5.1].

**Theorem 3.2** *Let  $\{\mu_n\} \subset \mathfrak{M}^b(\Omega)$  be a sequence such that  $\sup_n |\mu_n|(\Omega) < \infty$  and let  $\{u_n\}$  be renormalized solutions of*

$$\begin{aligned} -\Delta_p u_n &= \mu_n && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.4)$$

*Then, up to a subsequence,  $\{u_n\}$  converges a.e. to a solution  $u$  of  $-\Delta_p u = \mu$  in the sense of distributions in  $\Omega$ , for some measure  $\mu \in \mathfrak{M}^b(\Omega)$ , and for every  $k > 0$ ,  $k^{-1} \int_{\Omega} |\nabla T_k(u)|^p \leq M$  for some  $M > 0$ .*

Finally we recall the following fundamental stability result of [10] which extends Theorem 3.2.

**Theorem 3.3** *Let  $\mu = \mu_0 + \mu_s^+ - \mu_s^- \in \mathfrak{M}^b(\Omega)$ , with  $\mu_0 = f - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$ ,  $\mu_s^+, \mu_s^- \in \mathfrak{M}_s^+(\Omega)$ . Assume there are sequences  $\{f_n\} \subset L^1(\Omega)$ ,  $\{g_n\} \subset (L^{p'}(\Omega))^N$ ,  $\{\eta_n^1\}, \{\eta_n^2\} \subset \mathfrak{M}_+^b(\Omega)$  such that  $f_n \rightharpoonup f$  weakly in  $L^1(\Omega)$ ,  $g_n \rightarrow g$  in  $L^{p'}(\Omega)$  and  $\operatorname{div} g_n$  is bounded in  $\mathfrak{M}^b(\Omega)$ ,  $\eta_n^1 \rightharpoonup \mu_s^+$  and  $\eta_n^2 \rightharpoonup \mu_s^-$  in the narrow topology. If  $\mu_n = f_n - \operatorname{div} g_n + \eta_n^1 - \eta_n^2$  and  $u_n$  is a renormalized solution of (3.4), then, up to a subsequence,  $u_n$  converges a.e. to a renormalized solution  $u$  of (3.2). Furthermore  $T_k(u_n) \rightarrow T_k(u)$  in  $W_0^{1,p}(\Omega)$ .*

## 3.2 Applications

We present below some interesting consequences of the above theorem.

**Corollary 3.4** *Let  $\mu \in \mathfrak{M}^b(\Omega)$  with compact support in  $\Omega$  and  $\omega \in \mathfrak{M}^b(\Omega)$ . Let  $\{f_n\} \subset L^1(\Omega)$  which converges weakly to  $f \in L^1(\Omega)$  and  $\mu_n = \rho_n * \mu$  where  $\{\rho_n\}$  is a sequence of mollifiers. If  $u_n$  is a renormalized solution of*

$$\begin{aligned} -\Delta_p u_n &= f_n + \mu_n + \omega && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (3.5)$$

*then, up to a subsequence,  $u_n$  converges to a renormalized solution of*

$$\begin{aligned} -\Delta_p u &= f + \mu + \omega && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.6)$$

*Proof.* We write  $\omega = \tilde{h} - \operatorname{div} \tilde{g} + \omega_s^+ - \omega_s^-$  and  $\mu = h - \operatorname{div} g + \mu_s^+ - \mu_s^-$ , with  $h, \tilde{h} \in L^1(\Omega)$ ,  $g, \tilde{g} \in (L^{p'}(\Omega))^N$ ,  $h, g, \mu_s^+$  and  $\mu_s^-$  with support in a compact set  $K \subset \Omega$ . For  $n_0$  large enough,  $\rho_n * h, \rho_n * g, \rho_n * \mu_s^+$  and  $\rho_n * \mu_s^-$  have also their support in a fixed compact subset

of  $\Omega$  for all  $n \geq n_0$ . Moreover  $\rho_n * h \rightarrow h$  and  $\rho_n * g \rightarrow g$  in  $L^1(\Omega)$  and  $(L^{p'}(\Omega))^N$  respectively and  $\operatorname{div} \rho_n * g \rightarrow \operatorname{div} g$  in  $W^{-1,p'}(\Omega)$ . Therefore

$$f_n + \mu_n + \omega = f_n + \tilde{h} + \rho_n * h - \operatorname{div}(\tilde{g} + \rho_n * g) + \omega_s^+ + \rho_n * \mu_s^+ - \omega_s^- - \rho_n * \mu_s^-$$

is an approximation of the measure  $f + \mu + \omega$  in the sense of Theorem 3.3. This implies the claim.  $\square$

**Corollary 3.5** *Let  $\mu_i \in \mathfrak{M}_+^b(\Omega)$ ,  $i = 1, 2$ , and  $\{\mu_{i,n}\} \subset \mathfrak{M}_+^b(\Omega)$  be a nondecreasing and converging to  $\mu_i$  in  $\mathfrak{M}_+^b(\Omega)$ . Let  $\{f_n\} \subset L^1(\Omega)$  which converges to some  $f$  weakly in  $L^1(\Omega)$ . Let  $\{\vartheta_n\} \subset \mathfrak{M}^b(\Omega)$  which converges to some  $\vartheta \in \mathfrak{M}_s(\Omega)$  in the narrow topology. For any  $n \in \mathbb{N}$  let  $u_n$  be a renormalized solution of*

$$\begin{aligned} -\Delta_p u_n &= f_n + \mu_{1,n} - \mu_{2,n} + \vartheta_n && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.7)$$

Then, up to a subsequence,  $u_n$  converges a.e. to a renormalized solution of problem

$$\begin{aligned} -\Delta_p u &= f + \mu_1 - \mu_2 + \vartheta && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.8)$$

The proof of this results is based upon two lemmas

**Lemma 3.6** *For any  $\mu \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_+^b(\Omega)$  there exists  $f \in L^1(\Omega)$  and  $h \in W^{-1,p'}(\Omega)$  such that  $\mu = f + h$  and*

$$\|f\|_{L^1(\Omega)} + \|h\|_{W^{-1,p'}(\Omega)} + \|h\|_{\mathfrak{M}^b(\Omega)} \leq 5\mu(\Omega). \quad (3.9)$$

*Proof.* Following [9] and the proof of [7, Th 2.1], one can write  $\mu = \phi\gamma$  where  $\gamma \in W^{-1,p'}(\Omega) \cap \mathfrak{M}_+^b(\Omega)$  and  $0 \leq \phi \in L^1(\Omega, \gamma)$ . Let  $\{\Omega_n\}_{n \in \mathbb{N}_*}$  be an increasing sequence of compact subsets of  $\Omega$  such that  $\cup_n \Omega_n = \Omega$ . We define the sequence of measures  $\{\nu_n\}_{n \in \mathbb{N}_*}$  by

$$\begin{aligned} \nu_n &= T_n(\chi_{\Omega_n} \phi) \gamma - T_{n-1}(\chi_{\Omega_{n-1}} \phi) \gamma \quad \text{for } n \geq 2 \\ \nu_1 &= T_1(\chi_{\Omega_1} \phi) \gamma. \end{aligned}$$

Since  $\nu_k \geq 0$ , then  $\sum_{k=1}^{\infty} \nu_k = \mu$  with strong convergence in  $\mathfrak{M}^b(\Omega)$ ,  $\|\nu_k\|_{\mathfrak{M}^b(\Omega)} = \nu_k(\Omega)$

and  $\sum_{k=1}^{\infty} \|\nu_k\|_{\mathfrak{M}^b(\Omega)} = \mu(\Omega)$ . Let  $\{\rho_n\}$  be a sequence of mollifiers. We may assume that  $\eta_n = \rho_n * \nu_n \in C_c^\infty(\Omega)$ ,

$$\|\eta_n - \nu_n\|_{W^{-1,p'}(\Omega)} \leq 2^{-n} \mu(\Omega)$$

Set  $f_n = \sum_{k=1}^n \eta_k$ , then  $\|f_n\|_{L^1(\Omega)} \leq \sum_{k=1}^n \|\eta_k\|_{L^1(\Omega)} \leq \sum_{k=1}^n \|\nu_k\|_{\mathfrak{M}^b(\Omega)} \leq \mu(\Omega)$ . If we define

$f = \lim_{n \rightarrow \infty} f_n$ , then  $f \in L^1(\Omega)$  with  $\|f\|_{L^1(\Omega)} \leq \mu(\Omega)$ . Set  $h_n = \sum_{k=1}^n (\nu_k - \eta_k)$ , then

$h_n \in W^{-1,p'}(\Omega) \cap \mathfrak{M}^b(\Omega)$ ,  $\|h_n\|_{W^{-1,p'}(\Omega)} \leq 2\mu(\Omega)$  and  $h_n$  converges strongly in  $W^{-1,p'}(\Omega)$  to some  $h$  which satisfies  $\|h\|_{W^{-1,p'}(\Omega)} \leq 2\mu(\Omega)$ . Since  $\mu = f + h$  and  $\|h\|_{\mathfrak{M}^b(\Omega)} \leq 2\mu(\Omega)$ , the result follows.  $\square$

**Lemma 3.7** *Let  $\mu \in \mathfrak{M}_+^b(\Omega)$ . If  $\{\mu_n\} \subset \mathfrak{M}_+^b(\Omega)$  is a nondecreasing sequence which converges to  $\mu$  in  $\mathfrak{M}^b(\Omega)$ , there exist  $F_n, F \in L^1(\Omega)$ ,  $G_n, G \in W^{-1,p'}(\Omega)$  and  $\mu_{n_s}, \mu_s \in \mathfrak{M}_s(\Omega)$  such that*

$$\mu_n = \mu_{n_0} + \mu_{n_s} = F_n + G_n + \mu_{n_s} \quad \text{and} \quad \mu = \mu_0 + \mu_s = F + G + \mu_s,$$

*such that  $F_n \rightarrow F$  in  $L^1(\Omega)$ ,  $G_n \rightarrow G$  in  $W^{-1,p'}(\Omega)$  and in  $\mathfrak{M}^b(\Omega)$  and  $\mu_{n_s} \rightarrow \mu_s$  in  $\mathfrak{M}^b(\Omega)$ , and*

$$\|F_n\|_{L^1(\Omega)} + \|G_n\|_{W^{-1,p'}(\Omega)} + \|G_n\|_{\mathfrak{M}^b(\Omega)} + \|\mu_{n_s}\|_{\mathfrak{M}^b(\Omega)} \leq 6\mu(\Omega). \quad (3.10)$$

*Proof.* Since  $\{\mu_n\}$  is nondecreasing  $\{\mu_{n_0}\}$  and  $\{\mu_{n_s}\}$  share this property. Clearly

$$\|\mu - \mu_n\|_{\mathfrak{M}^b(\Omega)} = \|\mu_0 - \mu_{n_0}\|_{\mathfrak{M}^b(\Omega)} + \|\mu_s - \mu_{n_s}\|_{\mathfrak{M}^b(\Omega)},$$

thus  $\mu_{n_0} \rightarrow \mu_0$  and  $\mu_{n_s} \rightarrow \mu_s$  in  $\mathfrak{M}^b(\Omega)$ . Furthermore  $\|\mu_{n_s}\|_{\mathfrak{M}^b(\Omega)} \leq \mu_s(\Omega) \leq \mu(\Omega)$ . Set  $\tilde{\mu}_{0_0} = 0$  and  $\tilde{\mu}_{n_0} = \mu_{n_0} - \mu_{n-1_0}$  for  $n \in \mathbb{N}_*$ . From Lemma 3.6, for any  $n \in \mathbb{N}$ , one can find  $f_n \in L^1(\Omega)$ ,  $h_n \in W^{-1,p'}(\Omega) \cap \mathfrak{M}^b(\Omega)$  such that  $\tilde{\mu}_{n_0} = f_n + h_n$  and

$$\|f_n\|_{L^1(\Omega)} + \|h_n\|_{W^{-1,p'}(\Omega)} + \|h_n\|_{\mathfrak{M}^b(\Omega)} \leq 5\tilde{\mu}_{n_0}(\Omega).$$

If we define  $F_n = \sum_{k=1}^n f_k$  and  $G_n = \sum_{k=1}^n h_k$ , then  $\mu_{n_0} = F_n + G_n$  and

$$\|F_n\|_{L^1(\Omega)} + \|G_n\|_{W^{-1,p'}(\Omega)} + \|G_n\|_{\mathfrak{M}^b(\Omega)} \leq 5\tilde{\mu}_{0_0}(\Omega).$$

Therefore the convergence statements and (3.10) hold.  $\square$

*Proof of Corollary 3.5.* We set  $\nu_n = f_n + \mu_{n,1} - \mu_{n,2} + \vartheta_n$  and  $\nu = f + \mu_1 - \mu_2 + \vartheta$ . From Lemma 3.7 we can write

$$\nu_n = f_n + F_{1n} - F_{2n} + G_{1n} - G_{2n} + \mu_{1n_s} - \mu_{2n_s} + \vartheta_n$$

and

$$\nu = f + F_1 - F_2 + G_1 - G_2 + \mu_{1s} - \mu_{2s} + \vartheta,$$

and the convergence properties listed in the lemma hold. Therefore we can apply Theorem 3.3 and the conclusion follows.  $\square$

In the next result we prove the main pointwise estimates on renormalized solutions.

**Theorem 3.8** *Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$ . Then there exists a constant  $c > 0$ , dependent on  $p$  and  $N$  such that if  $\mu \in \mathfrak{M}^b(\Omega)$  and  $u$  is a renormalized solution of problem (3.2) there holds*

$$-c\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\mu^-] \leq u(x) \leq c\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\mu^+] \quad \text{a.e. in } \Omega. \quad (3.11)$$

*Proof.* We claim there exist renormalized solutions  $u_1$  and  $u_2$  of problem (3.2) with respective data  $\mu^+$  and  $\mu^-$  such that

$$-u_2 \leq u \leq u_1 \quad \text{a.e. in } \Omega. \quad (3.12)$$

We use the decomposition  $\mu = \mu^+ - \mu^- = (\mu_0^+ - \mu_s^+) - (\mu_0^- - \mu_s^-)$ . We put  $u_k = T_k(u)$ ,  $\mu_k = \mathbf{1}_{\{|u| < k\}} \mu_0 + \lambda_k^+ - \lambda_k^-$ ,  $v_k = \mathbf{1}_{\{|u| < k\}} \mu_0^+ + \lambda_k^+$ . Since  $\mu_k \in \mathfrak{M}_0(\Omega)$ , problem (3.2) with data  $\mu_k$  admits a unique renormalized solution (see [7]), and clearly  $u_k$  is such a solution. Since  $v_k \in \mathfrak{M}_0(\Omega)$ , problem (3.2) with data  $v_k$  admits a unique solution  $u_{k,1}$  which is furthermore nonnegative and dominates  $u_k$  a.e. in  $\Omega$ . From Corollary 3.5,  $\{u_{k,1}\}$  converges a.e. in  $\Omega$  to a renormalized solution  $u_1$  of (3.2) with data  $\mu^+$  and  $u \leq u_1$ . Similarly  $-u \leq u_2$  where  $u_2$  is a renormalized solution of (3.2) with  $\mu^-$ . Finally, from [17, Th 6.9] there is a positive constant  $c$  dependent only on  $p$  and  $N$  such that

$$u_1(x) \leq c \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\mu^+] \quad \text{and} \quad u_2(x) \leq c \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\mu^-] \quad \text{a.e. in } \Omega. \quad (3.13)$$

This implies the claim.  $\square$

## 4 Equations with absorption terms

### 4.1 The general case

Let  $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$  be a Caratheodory function such that the map  $s \mapsto g(x, s)$  is nondecreasing and odd for almost all  $x \in \Omega$ . If  $U$  is a function defined in  $\Omega$  we define the function  $g \circ U$  in  $\Omega$  by

$$g \circ U(x) = g(x, U(x)) \quad \text{for almost all } x \in \Omega.$$

We consider the problem

$$\begin{aligned} -\Delta_p u + g \circ u &= \mu & \text{in } \Omega \\ u &= 0 & \text{in } \partial\Omega. \end{aligned} \quad (4.14)$$

where  $\mu \in \mathfrak{M}^b(\Omega)$ . We say that  $u$  is a *renormalized solution* of problem (4.14) if  $g \circ u \in L^1(\Omega)$  and  $u$  is a renormalized solution of

$$\begin{aligned} -\Delta_p u &= \mu - g \circ u & \text{in } \Omega \\ u &= 0 & \text{in } \partial\Omega. \end{aligned} \quad (4.15)$$

**Theorem 4.1** *Let  $\mu_i \in \mathfrak{M}_+^b(\Omega)$ ,  $i = 1, 2$ , such that there exists a nondecreasing sequences  $\{\mu_{i,n}\} \subset \mathfrak{M}_+^b(\Omega)$ , with compact support in  $\Omega$ , converging to  $\mu_i$  and  $g \circ (c \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\mu_{i,n}]) \in L^1(\Omega)$  with the same constant  $c$  as in Theorem 3.8. Then there exists a renormalized solution of*

$$\begin{aligned} -\Delta_p u + g \circ u &= \mu_1 - \mu_2 & \text{in } \Omega \\ u &= 0 & \text{in } \partial\Omega, \end{aligned} \quad (4.16)$$

such that

$$-c \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\mu_2](x) \leq u(x) \leq c \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\mu_1](x) \quad \text{a.e. in } \Omega. \quad (4.17)$$



**Lemma 4.2** Assume  $g$  belongs to  $L^\infty(\Omega \times \mathbb{R})$ , besides the assumptions of Theorem 4.1. Let  $\lambda_i \in \mathfrak{M}_+^b(\Omega)$  ( $i = 1, 2$ ), with compact support in  $\Omega$ . Then there exist renormalized solutions  $u, u_i, v_i$  ( $i = 1, 2$ ) to problems

$$\begin{aligned} -\Delta_p u + g \circ u &= \lambda_1 - \lambda_2 && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega, \end{aligned} \quad (4.18)$$

$$\begin{aligned} -\Delta_p u_i + g \circ u_i &= \lambda_i && \text{in } \Omega \\ u_i &= 0 && \text{in } \partial\Omega, \end{aligned} \quad (4.19)$$

$$\begin{aligned} -\Delta_p v_i &= \lambda_i && \text{in } \Omega \\ v_i &= 0 && \text{in } \partial\Omega, \end{aligned} \quad (4.20)$$

such that

$$\begin{aligned} -c\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\lambda_2](x) &\leq -v_2(x) \leq -u_2(x) \leq u(x) \\ &\leq u_1(x) \leq v_1(x) \leq c\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\lambda_1](x) \end{aligned} \quad (4.21)$$

for almost all  $x \in \Omega$ .

*Proof.* Let  $\{\rho_n\}$  be a sequence of mollifiers,  $\lambda_{i,n} = \rho_n * \lambda_i$ , ( $i = 1, 2$ ) and  $\lambda_n = \lambda_{1,n} - \lambda_{2,n}$ . Then, for  $n_0$  large enough,  $\lambda_{1,n}$ ,  $\lambda_{2,n}$  and  $\lambda_n$  are bounded with compact support in  $\Omega$  for all  $n \geq n_0$  and by minimization there exist unique solutions in  $W_0^{1,p}(\Omega)$  to problems

$$\begin{aligned} -\Delta_p u_n + g \circ u_n &= \lambda_n && \text{in } \Omega \\ u_n &= 0 && \text{in } \partial\Omega, \end{aligned}$$

$$\begin{aligned} -\Delta_p u_{i,n} + g \circ u_{i,n} &= \lambda_{i,n} && \text{in } \Omega \\ u_{i,n} &= 0 && \text{in } \partial\Omega, \end{aligned}$$

$$\begin{aligned} -\Delta_p v_{i,n} &= \lambda_{i,n} && \text{in } \Omega \\ v_{i,n} &= 0 && \text{in } \partial\Omega, \end{aligned}$$

and by the maximum principle, they satisfy

$$-v_{2,n}(x) \leq -u_{2,n}(x) \leq u_n(x) \leq u_{1,n}(x) \leq v_{1,n}(x), \quad \forall x \in \Omega, \quad \forall n \geq n_0. \quad (4.22)$$

Since the  $\lambda_i$  are bounded measure and  $g \in L^\infty(\Omega \times \mathbb{R})$  the the sequences of measures  $\{\lambda_{1,n} - \lambda_{2,n} - g \circ u_n\}$ ,  $\{\lambda_{i,n} - g \circ u_{i,n}\}$  and  $\{\lambda_{i,n}\}$  are uniformly bounded in  $\mathfrak{M}^b(\Omega)$ . Thus, by Theorem 3.2 there exists a subsequence, still denoted by the index  $n$  such that  $\{u_n\}$ ,  $\{u_{i,n}\}$ ,  $\{v_{i,n}\}$  converge a.e. in  $\Omega$  to functions  $\{u\}$ ,  $\{u_i\}$ ,  $\{v_i\}$  ( $i = 1, 2$ ) when  $n \rightarrow \infty$ . Furthermore  $g \circ u_n$  and  $g \circ u_{i,n}$  converge in  $L^1(\Omega)$  to  $g \circ u$  and  $g \circ u_i$  respectively. By Corollary 3.4, we can assume that  $\{u\}$ ,  $\{u_i\}$ ,  $\{v_i\}$  are renormalized solutions of (4.18)-(4.20), and by Theorem 3.8,  $v_i(x) \leq c\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\lambda_i](x)$ , a.e. in  $\Omega$ . Thus we get (4.21).  $\square$

**Lemma 4.3** Let  $g$  satisfy the assumptions of Theorem 4.1 and let  $\lambda_i \in \mathfrak{M}_+^b(\Omega)$  ( $i = 1, 2$ ), with compact support in  $\Omega$  such that  $g \circ (c\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\lambda_i]) \in L^1(\Omega)$ , where  $c$  is the constant

of Theorem 4.1. Then there exist renormalized solutions  $u, u_i$  of the problems (4.18)-(4.19) such that

$$-c\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\lambda_2](x) \leq -u_2(x) \leq u(x) \leq u_1(x) \leq c\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\lambda_1](x) \quad (4.23)$$

for almost all  $x \in \Omega$ . Furthermore, if  $\omega_i, \theta_i$  have the same properties as the  $\lambda_i$  and satisfy  $\omega_i \leq \lambda_i \leq \theta_i$ , one can find solutions  $u_{\omega_i}$  and  $u_{\theta_i}$  of problems (4.19) with right-hand respective side  $\omega_i$  and  $\theta_i$ , such that  $u_{\omega_i} \leq u_i \leq u_{\theta_i}$ .

*Proof.* From Lemma 4.2 there exist renormalized solutions  $u_n, u_{i,n}$  to problems

$$\begin{aligned} -\Delta_p u_n + T_n(g \circ u_n) &= \lambda_1 - \lambda_2 && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta_p u_{i,n} + T_n(g \circ u_{i,n}) &= \lambda_i && \text{in } \Omega \\ u_{i,n} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

$i = 1, 2$ , and they satisfy

$$-c\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\lambda_2](x) \leq -u_{2,n}(x) \leq u_n(x) \leq u_{1,n}(x) \leq c\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\lambda_1](x). \quad (4.24)$$

Since  $\int_{\Omega} |g \circ u_n| dx \leq \lambda_1(\Omega) + \lambda_2(\Omega)$  and  $\int_{\Omega} g \circ u_{i,n} dx \leq \lambda_i(\Omega)$  thus as in Lemma 4.2 one can choose a subsequence, still denoted by the index  $n$  such that  $\{u_n, u_{1,n}, u_{2,n}\}$  converges a.e. in  $\Omega$  to  $\{u, u_1, u_2\}$  for which (4.24) is satisfied a.e. in  $\Omega$ . Since  $g \circ (c\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\lambda_i]) \in L^1(\Omega)$  we derive from (4.24) and the dominated convergence theorem that  $T_n(g \circ u_n) \rightarrow g \circ u$  and  $T_n(g \circ u_{i,n}) \rightarrow g \circ u_i$  in  $L^1(\Omega)$ . It follows from Theorem 3.3 that  $u$  and  $u_i$  are respective solutions of (4.18), (4.19). The last statement follows from the same assertion in Lemma 4.2.  $\square$

*Proof of Theorem 4.1.* From Lemma 4.3, there exist renormalized solutions  $u_n, u_{i,n}$  to problems

$$\begin{aligned} -\Delta_p u_n + g \circ u_n &= \mu_{1,n} - \mu_{2,n} && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta_p u_{i,n} + g \circ u_{i,n} &= \mu_{i,n} && \text{in } \Omega \\ u_{i,n} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

$i = 1, 2$  such that  $\{u_{i,n}\}$  is nonnegative and nondecreasing and they satisfy

$$-c\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\mu_2](x) \leq -u_{2,n}(x) \leq u_n(x) \leq u_{1,n}(x) \leq c\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\mu_1](x) \quad (4.25)$$

a.e. in  $\Omega$ . As in the proof of Lemma 4.3, up to the same subsequence,  $\{u_{1,n}\}, \{u_{2,n}\}$  and  $\{u_n\}$  converge to  $u_1, u_2$  and  $u$  a.e. in  $\Omega$ . Since  $g \circ u_{i,n}$  are nondecreasing, positive and  $\int_{\Omega} g \circ u_{i,n} dx \leq \mu_{i,n}(\Omega) \leq \mu_i(\Omega)$ , it follows from the monotone convergence theorem that  $\{g \circ u_{i,n}\}$  converges to  $g \circ u_i$  in  $L^1(\Omega)$ . Finally, since  $|g \circ u_n| \leq g \circ u_1 + g \circ u_2$ ,  $\{g \circ u_n\}$  converges to  $g \circ u$  in  $L^1(\Omega)$  by dominated convergence. Applying Corollary 3.5 we conclude that  $u$  is a renormalized solution of (4.16) and that (4.17) holds.  $\square$

## 4.2 Proofs of Theorem 1.1 and Theorem 1.2

We are now in situation of proving the two theorems stated in the introduction.

*Proof of Theorem 1.1.* 1- Since  $\mu$  is absolutely continuous with respect to the capacity  $C_{p, \frac{Nq}{Nq-(p-1)(N-\beta)}, \frac{q}{q+1-p}}$ ,  $\mu^+$  and  $\mu^-$  share this property. By Theorem 2.6 there exist two nondecreasing sequences  $\{\mu_{1,n}\}$  and  $\{\mu_{2,n}\}$  of positive bounded measures with compact support in  $\Omega$  which converge to  $\mu^+$  and  $\mu^-$  respectively and which have the property that  $\mathbf{W}_{1,p}^R[\mu_{i,n}] \in L^{\frac{Nq}{N-\beta}, q}(\mathbb{R}^N)$ , for  $i = 1, 2$  and all  $n \in \mathbb{N}$ . Furthermore, with  $R = \text{diam}(\Omega)$ ,

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{1}{|x|^\beta} (\mathbf{W}_{1,p}^{2R}[\mu_{i,n}](x))^q dx &\leq \int_0^\infty \left(\frac{1}{|\cdot|^\beta}\right)^*(t) \left((\mathbf{W}_{1,p}^{2R}[\mu_{i,n}])^*(t)\right)^q dt \\ &\leq c_{34} \int_0^\infty \frac{1}{t^{\frac{\beta}{N}}} \left((\mathbf{W}_{1,p}^{2R}[\mu_{i,n}])^*(t)\right)^q dt \\ &\leq c_{34} \left\| \mathbf{W}_{1,p}^{2R}[\mu_{i,n}] \right\|_{L^{\frac{Nq}{N-\beta}, q}(\mathbb{R}^N)}^q \\ &< \infty. \end{aligned} \tag{4.26}$$

Then the result follows from Theorem 4.1.

2- Because  $\mu$  is absolutely continuous with respect to the capacity  $C_{p, \frac{Nq}{Nq-(p-1)(N-\beta)}, 1}$ , so are  $\mu^+$  and  $\mu^-$ . Applying again Theorem 2.6 there exist two nondecreasing sequences  $\{\mu_{1,n}\}$  and  $\{\mu_{2,n}\}$  of positive bounded measures with compact support in  $\Omega$  which converge to  $\mu^+$  and  $\mu^-$  respectively and such that  $\mathbf{W}_{1,p}^R[\mu_{i,n}] \in L^{\frac{Nq}{N-\beta}, 1}(\mathbb{R}^N)$ . This implies in particular

$$(\mathbf{W}_{1,p}^{2R}[\mu_{i,n}](\cdot))^*(t) \leq c_{35} t^{-\frac{N-\beta}{Nq}}, \quad \forall t > 0, \tag{4.27}$$

for some  $c_{34} > 0$ . Therefore, by Theorem 2.3

$$\begin{aligned} \int_{\Omega} \frac{1}{|x|^\beta} g(c \mathbf{W}_{1,p}^{2R}[\mu_{i,n}](x)) dx &\leq \int_0^{|\Omega|} \left(\frac{1}{|\cdot|^\beta}\right)^*(t) g\left(c (\mathbf{W}_{1,p}^{2R}[\mu_{i,n}])^*(t)\right) dt \\ &\leq c_{36} \int_0^{|\Omega|} \frac{1}{t^{\frac{\beta}{N}}} g\left(c (\mathbf{W}_{1,p}^{2R}[\mu_{i,n}])^*(t)\right) dt \\ &\leq c_{36} \int_0^{|\Omega|} \frac{1}{t^{\frac{\beta}{N}}} g\left(c_{35} c t^{-\frac{N-\beta}{Nq}}\right) dt \\ &\leq c_{37} \int_a^\infty g(t) t^{-q-1} dt \\ &< \infty, \end{aligned} \tag{4.28}$$

where  $a > 0$  depends on  $|\Omega|$ ,  $c_{35}c$ ,  $N$ ,  $\beta$ ,  $q$ . Thus the result follows by Theorem 4.1.  $\square$

*Proof of Theorem 1.2.* Again we take  $R = \text{diam}(\Omega)$ . Let  $\{\Omega_n\}_{n \in \mathbb{N}_*}$  be an increasing sequence of compact subsets of  $\Omega$  such that  $\cup_n \Omega_n = \Omega$ . We define  $\mu_{i,n} = T_n(\chi_{\Omega_n} f_i) + \chi_{\Omega_n} \nu_i$  ( $i = 1, 2$ ). Then  $\{\mu_{1,n}\}$  and  $\{\mu_{2,n}\}$  are nondecreasing sequences of elements of  $\mathfrak{M}_+^b(\Omega)$  with

compact support, and they converge to  $\mu^+$  and  $\mu^-$  respectively. Since for any  $\epsilon > 0$  there exists  $c_\epsilon > 0$  such that

$$(\mathbf{W}_{1,p}^{2R}[\mu_{i,n}])^\lambda \leq c_\epsilon n^{\frac{\lambda}{p-1}} + (1 + \epsilon) (\mathbf{W}_{1,p}^{2R}[\nu_i])^\lambda, \quad (4.29)$$

a.e. in  $\Omega$ , it follows

$$\exp\left(\tau (c\mathbf{W}_{1,p}^{2R}[\mu_{i,n}])^\lambda\right) \leq c_{\epsilon,n,c} \exp\left(\tau(1 + \epsilon) (c\mathbf{W}_{1,p}^{2R}[\nu_i])^\lambda\right). \quad (4.30)$$

If there holds

$$\left\| \mathbf{M}_{p,2R}^{\frac{(p-1)(\lambda-1)}{\lambda}}[\nu_i] \right\|_{L^\infty(\Omega)} < \left( \frac{p \ln 2}{\tau(12\lambda c)^\lambda} \right)^{\frac{p-1}{\lambda}}, \quad (4.31)$$

we can choose  $\epsilon > 0$  small enough so that

$$\tau(1 + \epsilon)c^\lambda < \frac{p \ln 2}{(12\lambda)^\lambda \left\| \mathbf{M}_{p,2R}^{\frac{(p-1)(\lambda-1)}{\lambda}}[\nu_i] \right\|_{L^\infty(\Omega)}^{\frac{\lambda}{p-1}}}.$$

Hence, by Theorem 2.4 with  $\eta = \frac{(p-1)(\lambda-1)}{\lambda}$ ,  $\exp\left(\tau(1 + \epsilon) (c\mathbf{W}_{1,p}^{2R}[\nu_i])^\lambda\right) \in L^1(\Omega)$ , which implies  $\exp\left(\tau (c\mathbf{W}_{1,p}^{2diam(\Omega)}[\mu_{i,n}])^\lambda\right) \in L^1(\Omega)$ . We conclude by Theorem 4.1.  $\square$

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