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Quasilinear Lane-Emden equations with absorption and measure data

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Abstract We study

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Key words: quasilinear elliptic equations, Wolff potential, maximal functions, Borel measures, Lorentz spaces, Lorentz-Bessel capacities.

Abstract We study the existence of solutions to the equation $-\Delta_p u + g(x, u) = \mu$ when $g(x, \cdot)$ is a nondecreasing function and μ a measure. We characterize the good measures, i.e. the ones for which the problem as a renormalized solution. We study particularly the cases where $g(x, u) = |x|^\beta |u|^{q-1} u$ and $g(x, u) = |x|^\beta \operatorname{sgn}(u)(e^{\tau|u|^\lambda} - 1)$. The results state that a

measure is good if it is absolutely continuous with respect to an appropriate Lorentz-Bessel capacities.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain containing 0 and $g : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a Caratheodory function. We assume that for almost all $x \in \Omega$, $r \mapsto g(x, r)$ is nondecreasing and vanishes at 0. In this article we consider the following problem

$$\begin{aligned} -\Delta_p u + g(x, u) &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega \end{aligned} \quad (1.1)$$

where $\Delta_p = \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right)$, ($1 < p < N$), is the p-Laplacian and μ a bounded measure. A measure for which the problem admits a solution, in an appropriate class, is called a *good measure*. When $p = 2$ and $g(x, u) = g(u)$ the problem has been considered by Benilan and Brezis [4] in the subcritical case case that is when any bounded measure is good. They prove that such is the case if $N \geq 3$ and g satisfies

$$\int_1^\infty (g(s) - g(-s)) s^{-\frac{N-1}{N-2}} ds < \infty. \quad (1.2)$$

The supercritical case, always with $p = 2$, has been considered by Baras and Pierre when $g(u) = |u|^{q-1} u$ and $q > 1$. They prove that the corresponding problem to (1.1) admits a solution (always unique in that case) if and only if the measure μ is absolutely continuous with respect to the Bessel capacity $C_{2,q'}$ ($q' = q/(q-1)$). In the case $p \neq 2$ it is shown by Bidaut-Véron [6] that if problem (1.1) with $\beta = 0$ and $g(s) = |s|^{q-1} s$ ($q > p-1 > 0$) admits a solution, then μ is absolutely continuous with respect to any capacity $C_{p, \frac{q}{q+1-p} + \epsilon}$ for any $\epsilon > 0$.

In this article we introduce a new class of Bessel capacities which are modelled on Lorentz spaces $L^{s,q}$ instead of L^q spaces. If G_α is the Bessel kernel of order $\alpha > 0$, we denote by $L^{\alpha,s,q}(\mathbb{R}^N)$ the Besov space which is the space of functions $\phi = G_\alpha * f$ for some $f \in L^{s,q}(\mathbb{R}^N)$ and we set $\|\phi\|_{\alpha,s,q} = \|f\|_{s,q}$ (a norm which is defined by using rearrangements). Then we set

$$C_{\alpha,s,q}(E) = \inf\{\|f\|_{s,q} : f \geq 0, G_\alpha * f \geq 1 \text{ on } E\} \quad (1.3)$$

for any Borel set E . We say that a measure μ in Ω is absolutely continuous with respect to the capacity $C_{\alpha,s,q}$ if ,

$$\forall E \subset \Omega, E \text{ Borel}, C_{\alpha,s,q}(E) = 0 \implies |\mu|(E) = 0. \quad (1.4)$$

We also introduce the Wolff potential of a positive measure $\mu \in \mathfrak{M}(\mathbb{R}^N)$ by

$$\mathbf{W}_{\alpha,s}[\mu](x) = \int_0^\infty \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dt}{t} \quad (1.5)$$

if $\alpha > 0$, $1 < s < \alpha^{-1}N$. When we are dealing with bounded domains $\Omega \subset B_R$ and $\mu \in \mathfrak{M}(\Omega)$, it is useful to introduce truncated Wolff potentials.

$$\mathbf{W}_{\alpha,s}^R[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dt}{t} \quad (1.6)$$

We prove the following existence results concerning

$$\begin{aligned} -\Delta_p u + |x|^{-\beta} g(u) &= \mu & \text{in } \Omega \\ u &= 0 & \text{in } \partial\Omega \end{aligned} \quad (1.7)$$

Theorem 1.1 *Assume $1 < p < N$, $q > p - 1$ and $0 \leq \beta < N$ and μ is a bounded Radon measure in Ω .*

1- *If $g(s) = |s|^{q-1} s$, then (1.7) admits a renormalized solution if μ is absolutely continuous with respect to the capacity $C_{p, \frac{Nq}{Nq-(p-1)(N-\beta)}, \frac{q}{q+1-p}}$.*

2- *If g satisfies*

$$\int_1^\infty g(s) s^{-q-1} ds < \infty \quad (1.8)$$

then (1.7) admits a renormalized solution if μ is absolutely continuous with respect to the capacity $C_{p, \frac{Nq}{Nq-(p-1)(N-\beta)}, 1}$.

Furthermore, in both case there holds

$$-KW_{1,p}^{2\text{diam}(\Omega)}[\mu^-](x) \leq u(x) \leq KW_{1,p}^{2\text{diam}(\Omega)}[\mu^+](x) \quad \text{for almost all } x \in \Omega. \quad (1.9)$$

In order to deal with exponential nonlinearities we introduce for $0 < \alpha < N$ the fractional maximal operator (resp. the truncated fractional maximal operator), defined for a positive measure μ by

$$\mathbf{M}_\alpha[\mu](x) = \sup_{t>0} \frac{\mu(B_t(x))}{t^{N-\alpha}}, \quad \left(\text{resp } \mathbf{M}_{\alpha,R}[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-\alpha}} \right), \quad (1.10)$$

and the η -fractional maximal operator (resp. the truncated η -fractional maximal operator)

$$\mathbf{M}_\alpha^\eta[\mu](x) = \sup_{t>0} \frac{\mu(B_t(x))}{t^{N-\alpha} h_\eta(t)}, \quad \left(\text{resp } \mathbf{M}_{\alpha,R}^\eta[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-\alpha} h_\eta(t)} \right), \quad (1.11)$$

where $\eta \geq 0$ and

$$h_\eta(t) = \begin{cases} (-\ln t)^{-\eta} & \text{if } 0 < t < \frac{1}{2} \\ (\ln 2)^{-\eta} & \text{if } t \geq \frac{1}{2} \end{cases} \quad (1.12)$$

Theorem 1.2 *Assume $1 < p < N$, $q > p - 1$, $\tau > 0$ and $\lambda \geq 1$. Then there exists $C > 0$ such that if a measure in Ω , $\mu = \mu^+ - \mu^-$ can be decomposed as follows*

$$\mu^+ = f_1 + \nu_1 \quad \text{and} \quad \mu^- = f_2 + \nu_2, \quad (1.13)$$

where $f_j \in L^1_+(\Omega)$ and $\nu_j \in \mathfrak{M}^b_+(\Omega)$ ($j = 1, 2$), and if

$$\left\| \mathbf{M}_{p, 2\text{diam}(\Omega)}^{\frac{(p-1)(\lambda-1)}{\lambda}}[\nu_j] \right\|_{L^\infty(\Omega)} \leq C, \quad (1.14)$$

there exists a renormalized solution to

$$\begin{aligned} -\Delta_p u + \text{sgn}(s) \left(e^{\tau|s|^\lambda} - 1 \right) &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega. \end{aligned} \quad (1.15)$$

Our study is based upon delicate estimates on Wolff potentials and η -fractional maximal operators which are developed in the first part of this paper.

2 Lorentz spaces and capacities

2.1 Lorentz spaces

Let (X, Σ, α) be a measured space. If $f : X \mapsto \mathbb{R}$ is a measurable function, we set $S_f(t) := \{x \in X : f(x) > t\}$ and $\lambda_f(t) = \alpha(S_f(t))$. The decreasing rearrangement f^* of f is defined by

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leq t\}.$$

It is well known that $(\Phi(f))^* = \Phi(f^*)$ for any continuous and nondecreasing even function $\Phi : \mathbb{R} \mapsto \mathbb{R}$. We set

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(\tau) d\tau \quad \forall t > 0.$$

and, for $1 \leq s < \infty$ and $1 < q \leq \infty$,

$$\|f\|_{L^{s,q}} = \begin{cases} \left(\int_0^\infty t^{sq} (f^{**}(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{t>0} t^s f^{**}(t) & \text{if } q = \infty \end{cases} \quad (2.1)$$

It is known that $L^{s,q}(X, \alpha)$ is a Banach space when endowed with the norm $\|\cdot\|_{L^{s,q}}$. Furthermore there holds (see e.g. [12])

$$\left\| t^{\frac{1}{s}} f^* \right\|_{L^q(\mathbb{R}^+, \frac{dt}{t})} \leq \|f\|_{L^{s,q}} \leq \frac{s}{s-1} \left\| t^{\frac{1}{s}} f^* \right\|_{L^q(\mathbb{R}^+, \frac{dt}{t})}, \quad (2.2)$$

the left-hand side inequality being valid only if $s > 1$. Finally, if $f \in L^{s,q}(\mathbb{R}^N)$ (with $1 \leq q < \infty$ and α being the Lebesgue measure) and if $\{\rho_n\} \subset C_0^\infty(\mathbb{R}^N)$ is a sequence of mollifiers $f * \rho_n \rightarrow f$ and $(f \chi_{B_n} * \rho_n \rightarrow f$ in $L^{s,q}(\mathbb{R}^N)$). In particular $C_0^\infty(\mathbb{R}^N)$ is dense in $L^{s,q}(\mathbb{R}^N)$.

2.2 Wolff potentials, fractional and η -fractional maximal operators

If D is either a bounded domain or whole \mathbb{R}^N , we denote by $\mathfrak{M}(D)$ (resp $\mathfrak{M}^b(D)$) the set of Radon measure (resp. bounded Radon measures) in D . Their positive cones are $\mathfrak{M}_+(D)$ and $\mathfrak{M}_+^b(D)$ respectively. If $0 < R \leq \infty$ and $\mu \in \mathfrak{M}_+(D)$ and $R \geq \text{diam}(D)$, we define, for $\alpha > 0$ and $0 < s < \alpha^{-1}N$, the R -truncated Wolff-potential by

$$\mathbf{W}_{\alpha,s}^R[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dt}{t} \quad \text{for a.e. } x \in \mathbb{R}^N. \quad (2.3)$$

If $h_\eta(t) = \min\{(-\ln t)^{-\eta}, (\ln 2)^{-\eta}\}$, the truncated η -fractional maximal operator is

$$\mathbf{M}_{\alpha,R}^\eta[\mu](x) = \sup_{0 < t < R} \frac{\mu(B_t(x))}{t^{N-\alpha} h_\eta(t)} \quad \text{for a.e. } x \in \mathbb{R}^N. \quad (2.4)$$

If $R = \infty$, we drop it in expressions (2.3) and (2.4). In particular

$$\mu(B_t(x)) \leq t^{N-\alpha} h_\eta(t) \mathbf{M}_{\alpha,R}^\eta[\mu](x). \quad (2.5)$$

We also define the \mathbf{G}_α the Bessel potential of a measure μ by

$$\mathbf{G}_\alpha[\mu](x) = \int_{\mathbb{R}^N} G_\alpha(x-y) d\mu(y) \quad \forall x \in \mathbb{R}^N, \quad (2.6)$$

where G_α is the Bessel kernel of order α in \mathbb{R}^N .

Definition 2.1 We denote by $L^{\alpha,s,q}(\mathbb{R}^N)$ the Besov space the space of functions $\phi = G_\alpha * f$ for some $f \in L^{s,q}(\mathbb{R}^N)$ and we set $\|\phi\|_{\alpha,s,q} = \|f\|_{s,q}$. If we set

$$C_{\alpha,s,q}(E) = \inf\{\|f\|_{s,q} : f \geq 0, G_\alpha * f \geq 1 \text{ on } E\}, \quad (2.7)$$

then $C_{\alpha,s,q}$ is a capacity.

2.3 Estimates on potentials

In the sequel, we denote by $|A|$ the N -dimensional Lebesgue measure of a measurable set A and, if F, G are functions defined in \mathbb{R}^N , we set $\{F > a\} := \{x \in \mathbb{R}^N : F(x) > a\}$, $\{G \leq b\} := \{x \in \mathbb{R}^N : G(x) \leq b\}$ and $\{F > a, G \leq b\} := \{F > a\} \cap \{G \leq b\}$. The following result is an extension of [14, Th 1.1]

Proposition 2.2 Let $0 \leq \eta < p-1$, $0 < \alpha p < N$ and $r > 0$. There exist $c_0 > 0$ and $\epsilon_0 > 0$ depending on N, r, p, α, η such that, for all $R \in (0, \infty]$, $\epsilon \in (0, \epsilon_0]$, $\lambda > 0$ and $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$, there holds.

$$\begin{aligned} & \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu] > 3\lambda, (\mathbf{M}_{\alpha p, R}^\eta[\mu])^{\frac{1}{p-1}} \leq \epsilon\lambda \right\} \cap B_r \right| \\ & \leq c_0 \exp\left(-\left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 \epsilon^{-\frac{p-1}{p-1-\eta}}\right) |\{\mathbf{W}_{\alpha,p}^R[\mu] > \lambda\} \cap B_r|. \end{aligned} \quad (2.8)$$

Furthermore, if $\eta = 0$, c_0 is independent of r .

Proof. Case $R = \infty$. Let $\lambda > 0$; since $\mathbf{W}_{\alpha,p}[\mu]$ is lower semicontinuous, the set

$$D = D_{\lambda,r} := \{x \in B_r : \mathbf{W}_{\alpha,p}[\mu](x) > \lambda\}$$

is open. By Whitney covering lemma, there exists a countable set of closed cubes $\{Q_i\}_i$ such that $D = \cup_i Q_i$, $\overset{\circ}{Q}_i \cap \overset{\circ}{Q}_j = \emptyset$ for $i \neq j$ and

$$\text{diam}(Q_i) \leq \text{dist}(Q_i, D^c) \leq 4 \text{diam}(Q_i).$$

Let $\epsilon > 0$ and $F_\epsilon = \left\{ \mathbf{W}_{\alpha,p}[\mu] > 3\lambda, (\mathbf{M}_{\alpha p}^\eta[\mu])^{\frac{1}{p-1}} \leq \epsilon\lambda \right\} \cap B_r$. We claim that there exist $c = c(N, \alpha, \beta, p, \eta) > 0$ and $\epsilon_0 \in (0, 1)$ depending also on N, α, β, p and η such that for any $Q \in \{Q_i\}_i$ there holds

$$|F_\epsilon \cap Q| \leq c \exp\left(-\left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \epsilon^{-\frac{p-1}{p-1-\eta}} \alpha p \ln 2\right) |Q| \quad \forall \epsilon \in (0, \epsilon_0). \quad (2.9)$$

Take $Q \in \{Q_i\}_i$ such that $Q \cap F_\epsilon \neq \emptyset$ and let $x_Q \in D^c$ such that $\text{dist}(x_Q, Q) \leq 4 \text{diam}(Q)$ and $\mathbf{W}_{\alpha,p}[\mu](x_Q) \leq \lambda$. For $k \in \mathbb{N}$, $r_0 = 5 \text{diam}(Q)$ and $x \in F_\epsilon \cap Q$, we have

$$\int_{2^k r_0}^{2^{k+1} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = A + B$$

where

$$A = \int_{2^k r_0}^{2^k \frac{1+2^{k+1}}{1+2^k} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}$$

and

$$B = \int_{2^k \frac{1+2^{k+1}}{1+2^k} r_0}^{2^{k+1} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Since

$$\mu(B_t(x)) \leq t^{N-\alpha p} h_\eta(t) M_{\alpha p}^\eta(x) \leq t^{N-\alpha p} h_\eta(t) (\epsilon\lambda)^{p-1}. \quad (2.10)$$

Then

$$B \leq \int_{2^k \frac{1+2^{k+1}}{1+2^k} r_0}^{2^{k+1} r_0} \left(\frac{t^{N-\alpha p} h_\eta(t) (\epsilon\lambda)^{p-1}}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \epsilon\lambda \int_{2^k \frac{1+2^{k+1}}{1+2^k} r_0}^{2^{k+1} r_0} (h_\eta(t))^{\frac{1}{p-1}} \frac{dt}{t}$$

Replacing $h_\eta(t)$ by its value we obtain $B \leq c\epsilon\lambda 2^{-k}$ after a lengthy computation, where c depends only on $N, \alpha p$ and η . Since $\delta := \left(\frac{2^k}{2^{k+1}}\right)^{\frac{N-\alpha p}{p-1}}$, then $1 - \delta \leq c_1 2^{-k}$ where c_1 depends only on $\frac{N-\alpha p}{p-1}$, thus

$$\begin{aligned} (1 - \delta)A &\leq c_1 2^{-k} \int_{2^k r_0}^{2^{k+1} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq c_1 2^{-k} \epsilon\lambda \int_{2^k r_0}^{2^{k+1} r_0} (h_\eta(t))^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq c_2 2^{-k} \epsilon\lambda. \end{aligned}$$

By a change of variables and using that for any $x \in F_\epsilon \cap Q$ and $t \in [r_0(1+2^k), r_0(1+2^{k+1})]$, $B_{\frac{2^k}{1+2^k}}(x) \subset B_t(x_Q)$, we get

$$\delta A = \int_{r_0(1+2^k)}^{r_0(1+2^{k+1})} \left(\frac{\mu(B_{\frac{2^k}{1+2^k}}(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \int_{r_0(1+2^k)}^{r_0(1+2^{k+1})} \left(\frac{\mu(B_t(x_Q))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Therefore

$$\int_{2^k r_0}^{2^{k+1} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_3 2^{-k} \epsilon \lambda + \int_{r_0(1+2^k)}^{r_0(1+2^{k+1})} \left(\frac{\mu(B_t(x_Q))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t},$$

with $c_3 = c_3(N, \alpha, p, \eta) > 0$. This implies

$$\int_{r_0}^{\infty} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq 2c_3 \epsilon \lambda + \int_{2r_0}^{\infty} \left(\frac{\mu(B_t(x_Q))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq (1 + c_4 \epsilon) \lambda, \quad (2.11)$$

since $\mathbf{W}_{\alpha, p}[\mu](x_Q) \leq \lambda$. If $\epsilon \in (0, c_4^{-1}]$, then

$$\int_{r_0}^{\infty} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq 2\lambda$$

which implies, since $x \in F_\epsilon$,

$$\mathbf{W}_{\alpha, p}^{r_0}[\mu](x) = \int_0^{r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} > \lambda$$

and finally

$$F_\epsilon \cap Q \subset E_\epsilon, \quad (2.12)$$

where

$$E_\epsilon = \left\{ x \in Q : \mathbf{W}_{\alpha, p}^{r_0}[\mu](x) > \lambda, (M_{\alpha p}^\eta[\mu](x))^{\frac{1}{p-1}} \leq \epsilon \lambda \right\}. \quad (2.13)$$

Next we set $m_0 = \frac{\max(1, \ln(20r))}{\ln 2}$, so that $2^{-m} r_0 \leq 2^{-1}$ if $m \geq m_0$. Then

$$\begin{aligned} \int_{2^{-m} r_0}^{r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} &\leq \epsilon \lambda \int_{2^{-m} r_0}^{r_0} (h_\eta(t))^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \epsilon \lambda \int_{2^{-m} r_0}^{2^{-m_0} r_0} (-\ln t)^{\frac{-\eta}{p-1}} \frac{dt}{t} + \epsilon \lambda \int_{2^{-m_0} r_0}^{r_0} (\ln 2)^{\frac{-\eta}{p-1}} \frac{dt}{t} \\ &\leq m_0 \epsilon \lambda + \frac{(p-1)((m-m_0) \ln 2)^{1-\frac{\eta}{p-1}}}{p-1-\eta} \epsilon \lambda. \end{aligned}$$

For the last inequality we have used $a^{1-\frac{\eta}{p-1}} - b^{1-\frac{\eta}{p-1}} \leq (a-b)^{1-\frac{\eta}{p-1}}$ valid for any $a \geq b \geq 0$. Therefore,

$$\int_{2^{-m} r_0}^{r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon \lambda \quad \forall m \in \mathbb{N}, m > m_0^*, \quad (2.14)$$

where $m_0^* = \left(\frac{m_0}{2 - (\ln 2) \frac{p-1-\eta}{p-1}} \right)^{\frac{p-1}{p-1-\eta}}$. Set

$$g_i(x) = \int_{2^{-i}r_0}^{2^{-i+1}r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t},$$

then

$$\begin{aligned} \mathbf{W}_{\alpha,p}^{r_0}[\mu](x) &\leq \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon \lambda + \mathbf{W}_{\alpha,p}^{2^{-m}r_0}[\mu](x) \\ &\leq \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon \lambda + \sum_{i=m+1}^{\infty} g_i(x) \end{aligned}$$

for all $m \geq m_0^*$. We deduce that, for $\beta > 0$,

$$\begin{aligned} |E_\epsilon| &\leq \left| \left\{ x \in Q : \sum_{i=m+1}^{\infty} g_i(x) > \left(1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon \right) \lambda \right\} \right| \\ &\leq \left| \left\{ x \in Q : \sum_{i=m+1}^{\infty} g_i(x) > 2^{-\beta(i-m-1)} (1 - 2^{-\beta}) \left(1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon \right) \lambda \right\} \right| \\ &\leq \sum_{i=m+1}^{\infty} \left| \left\{ x \in Q : g_i(x) > 2^{-\beta(i-m-1)} (1 - 2^{-\beta}) \left(1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon \right) \lambda \right\} \right|. \end{aligned} \tag{2.15}$$

Next we claim that

$$|\{x \in Q : g_i(x) > s\}| \leq \frac{c(N)}{s^{p-1}} 2^{-i\alpha p} |Q| (\epsilon \lambda)^{p-1}. \tag{2.16}$$

To see that, we pick $x_0 \in Q$ and we use the Chebyshev's inequality

$$\begin{aligned} |\{x \in Q : g_i(x) > s\}| &\leq \frac{1}{s^{p-1}} \int_Q |g_i|^{p-1} dx \\ &= \frac{1}{s^{p-1}} \int_Q \left(\int_{r_0 2^{-i}}^{r_0 2^{-i+1}} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \right)^{p-1} dx \\ &\leq \frac{1}{s^{p-1}} \int_Q \frac{\mu(B_{r_0 2^{-i+1}}(x))}{(r_0 2^{-i})^{N-\alpha p}} := A. \end{aligned}$$

Thanks to Fubini's theorem, the last term A of the above inequality can be rewritten as

$$\begin{aligned}
A &= \frac{1}{s^{p-1}} \frac{1}{(r_0 2^{-i})^{N-\alpha p}} \int_Q \int_{\mathbb{R}^N} \mathbf{1}_{B_{r_0 2^{-i+1}}(x)} d\mu(y) dx \\
&= \frac{1}{s^{p-1}} \frac{1}{(r_0 2^{-i})^{N-\alpha p}} \int_{Q+B_{r_0 2^{-i+1}}(0)} \int_Q \mathbf{1}_{B_{r_0 2^{-i+1}}(x)} dx d\mu(y) \\
&\leq \frac{1}{s^{p-1}} \frac{1}{(r_0 2^{-i})^{N-\alpha p}} \int_{Q+B_{r_0 2^{-i+1}}(0)} |B_{r_0 2^{-i+1}}(x)| d\mu(y) \\
&\leq c_1(N) \frac{1}{s^{p-1}} 2^{-i\alpha p} r_0^{\alpha p} \mu(Q + B_{r_0 2^{-i+1}}(0)) \\
&\leq c_1(N) \frac{1}{s^{p-1}} 2^{-i\alpha p} r_0^{\alpha p} \mu(B_{r_0(1+2^{-i+1})}(x_0)),
\end{aligned}$$

since $Q + B_{r_0 2^{-i+1}}(0) \subset B_{r_0(1+2^{-i+1})}(x_0)$. Using the fact that $\mu(B_t(x_0)) \leq 4t^{N-\alpha p}(\epsilon\lambda)^{p-1}$ for all $t > 0$ and $r_0 = 5 \text{diam}(Q)$, we obtain

$$A \leq c_2(N) \frac{1}{s^{p-1}} 2^{-i\alpha p} r_0^{\alpha p} (r_0(1+2^{-i+1}))^{N-\alpha p} (\epsilon\lambda)^{p-1} \leq c_3(N) \frac{1}{s^{p-1}} 2^{-i\alpha p} |Q| (\epsilon\lambda)^{p-1},$$

which is (2.16). Consequently, (2.15) can be rewritten as

$$\begin{aligned}
|E_\epsilon| &\leq \sum_{i=m+1}^{\infty} \frac{c(N)}{\left(2^{-\beta(i-m-1)(1-2^{-\beta})} \left(1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon\right) \lambda\right)^{p-1}} 2^{-i\alpha p} (\epsilon\lambda)^{p-1} |Q| \\
&\leq c_4(N) 2^{-m\alpha p} \left(\frac{\epsilon}{1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon}\right)^{p-1} |Q| \sum_{i=m+1}^{\infty} 2^{(\beta(p-1)-\alpha p)(i-m-1)}.
\end{aligned} \tag{2.17}$$

If we choose β so that $(\beta(p-1) - \alpha p) < 0$, we obtain from (2.12)

$$|F_\epsilon \cap Q| \leq |E_\epsilon| \leq c_5 2^{-m\alpha p} \left(\frac{\epsilon}{1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon}\right)^{p-1} |Q| \quad \forall m \geq m_0^* \tag{2.18}$$

where $c_5 = c_5(N, \alpha, \beta, p, \eta) > 0$. Put $\epsilon_0 = \frac{1}{2} \min \left\{ \frac{1}{1+2^{\frac{p-1}{p-1-\eta}} m_0}, c_5^{-1} \right\}$. For any $\epsilon \in (0, \epsilon_0)$ we choose $m \in \mathbb{N}$ such that

$$\left(\frac{p-1-\eta}{2(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \left(\frac{1}{\epsilon} - 1\right)^{\frac{p-1}{p-1-\eta}} - 1 < m \leq \left(\frac{p-1-\eta}{2(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \left(\frac{1}{\epsilon} - 1\right)^{\frac{p-1}{p-1-\eta}}. \tag{2.19}$$

Then

$$\left(\frac{\epsilon}{1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \epsilon}\right)^{p-1} \leq 1$$

and

$$2^{-m\alpha p} \leq 2^{\alpha p - \left(\frac{p-1-\eta}{2(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \left(\frac{1}{\epsilon} - 1\right)^{\frac{p-1}{p-1-\eta}}} \leq 2^{\alpha p} \exp \left(-\alpha p \ln 2 \left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \epsilon^{-\frac{p-1}{p-1-\eta}} \right). \tag{2.20}$$

Combinbing this inequality with (2.18), we get (2.9).

In the case $\eta = 0$ we still have

$$\mathbf{W}_{\alpha,p}^{r_0}[\mu](x) \leq 2m\epsilon\lambda + \sum_{i=m+1}^{\infty} g_i(x).$$

Accordingly (2.18) reads as

$$|F_\epsilon \cap Q| \leq c_5 2^{-m\alpha p} \left(\frac{\epsilon}{1-2m\epsilon} \right)^{p-1} |Q| \quad \forall m \geq 2^{\frac{1}{p-1}} m_0.$$

If $\epsilon_0 = \frac{1}{2} \min\{1, c_5^{-1}\}$ and $m \in \mathbb{N}$ satisfies $\epsilon^{-1} - 2 < m \leq \epsilon^{-1} - 1$, we finally get

$$|F_\epsilon \cap Q| \leq c_5 \exp(-\alpha p \epsilon^{-1} \ln 2), \quad (2.21)$$

which ends the proof in the case $R = \infty$.

Case $R < \infty$. For $\lambda > 0$, $D = \{x \in B_r : \mathbf{W}_{\alpha,p}^R(x) > \lambda\}$ is open. Using again Whitney covering lemma, there exist a countable set of closed cubes $\mathcal{Q} := \{Q_i\}$ such that $\cup_i Q_i = D$, $\overset{\circ}{Q}_i \cap \overset{\circ}{Q}_j = \emptyset$ for $i \neq j$ and $\text{dist}(Q_i, D^c) \leq 4 \text{diam}(Q_i)$. If $Q \in \mathcal{Q}$: is such that $\text{diam}(Q) > \frac{R}{8}$, there exists a finite number n_Q of closed dyadic cubes $\{P_{j,Q}\}_{j=1}^{n_Q}$ such that $\cup_{j=1}^{n_Q} Q_j = Q$, $P_{i,Q} \cap P_{j,Q} = \emptyset$ if $i \neq j$ and $\frac{R}{8} < \text{diam}(P_{j,Q}) \leq \frac{R}{8}$. We set $\mathcal{Q}' = \{Q \in \mathcal{Q} : \text{diam}(Q) \leq \frac{R}{8}\}$, $\mathcal{Q}'' = \{Q \in \mathcal{Q} : \text{diam}(Q) > \frac{R}{8}\}$ and

$$\mathcal{F} = \mathcal{Q}' \cup \{P_{i,Q} : 1 \leq i \leq n_Q, Q \in \mathcal{Q}''\}.$$

For $\epsilon > 0$ we denote again $F_\epsilon = \left\{ \mathbf{W}_{\alpha,p}^R[\mu] > 3\lambda, (\mathbf{M}_{\alpha p}^\eta[\mu])^{\frac{1}{p-1}} \leq \epsilon\lambda \right\} \cap B_r$. Let $Q \in \mathcal{F}$ such that $F_\epsilon \cap Q \neq \emptyset$ and $r_0 = 5 \text{diam}(Q)$.

If $\text{dist}(D^c, Q) \leq 4 \text{diam}(Q)$, that is if there exists $x_Q \in D^c$ such that $\text{dist}(x_Q, Q) \leq 4 \text{diam}(Q)$ and $\mathbf{W}_{\alpha,p}^R(x_Q) \leq \lambda$, we find, by the same argument as in the case $R = \infty$, (2.11), that for any $x \in F_\epsilon \cap Q$ there holds

$$\int_{r_0}^R \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq (1 + c_4\epsilon)\lambda. \quad (2.22)$$

If $\text{dist}(D^c, Q) > 4 \text{diam}(Q)$, we have $\frac{R}{8} < \text{diam}(Q) \leq \frac{R}{8}$ since $Q \in \mathcal{F}$. Then, for all $x \in F_\epsilon \cap Q$, there holds

$$\begin{aligned} \mathbf{W}_{\alpha,p}^R[\mu](x) &= \int_{r_0}^R \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \int_{\frac{5r_0}{16}}^R \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \frac{11}{5} \mathbf{M}_{\alpha p, R}^\eta[\mu](x) \\ &\leq (1 + c_4\epsilon)\lambda. \end{aligned} \quad (2.23)$$

Finally, if we take $\epsilon \in (0, c_4^{-1}]$, we derive

$$|F_\epsilon \cap Q| \leq \left| \left\{ x \in Q : \mathbf{W}_{\alpha,p}^{r_0}[\mu](x) > \lambda, \mathbf{M}_{\alpha p, R}^\eta[\mu](x) \leq \epsilon \lambda \right\} \right|. \quad (2.24)$$

The end of the proof is as in the case $R = \infty$. \square

In the next result we list a series of equivalent norms concerning Radon measures.

Theorem 2.3 *Assume $\alpha > 0$, $0 < p - 1 < q < \infty$, $0 < \alpha p < N$ and $0 < s \leq \infty$. Then there exists a constant $c_6 > 0$ such that for any $R \in (0, \infty]$ and $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$, there holds*

$$c_6^{-1} \|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)} \leq \|\mathbf{M}_{\alpha p, R}[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_6 \|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)}. \quad (2.25)$$

For any $R > 0$, there exists $c_7 > 0$ such that for any $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$,

$$c_7^{-1} \|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)} \leq \|\mathbf{G}_{\alpha p}[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_7 \|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)}. \quad (2.26)$$

In (2.26), $\|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)}$ can be replaced by $\|\mathbf{M}_{\alpha p, R}[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)}$.

Proof. Step 1

$$\|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)} \leq c'_6 \|\mathbf{M}_{\alpha p, R}[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)}. \quad (2.27)$$

For $n \in \mathbb{N}^*$, set $\mu_n = \mathbf{1}_{B_n} \mu$. From Proposition 2.2 there exist constants $C, a > 0$ and $\epsilon_0 \in (0, 1)$ such that for all $n \in \mathbb{N}^*$, $\lambda > 0$, $r > 0$, $0 < R \leq \infty$ and $0 < \epsilon \leq \epsilon_0$, there holds

$$\left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > 3\lambda, (\mathbf{M}_{\alpha p, R}^\eta[\mu_n])^{\frac{1}{p-1}} \leq \epsilon \lambda \right\} \cap B_r \right| \leq C \exp(-a\epsilon^{-1}) \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > \lambda \right\} \cap B_r \right|. \quad (2.28)$$

Letting $r \rightarrow \infty$, it follows

$$\left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > 3\lambda, (\mathbf{M}_{\alpha p, R}^\eta[\mu_n])^{\frac{1}{p-1}} \leq \epsilon \lambda \right\} \right| \leq C \exp(-a\epsilon^{-1}) \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > \lambda \right\} \right|. \quad (2.29)$$

In the case $0 < s < \infty$ and $0 < q < \infty$, we have

$$\left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > 3t \right\} \right|^{\frac{s}{q}} \leq c_7 \exp(-a\epsilon^{-1}) \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > t \right\} \right|^{\frac{s}{q}} + M \left| \left\{ (\mathbf{M}_{\alpha p, R}^\eta[\mu_n])^{\frac{1}{p-1}} > \epsilon t \right\} \right|^{\frac{s}{q}}.$$

Multiplying by t^{s-1} and integrating over $(0, \infty)$, we obtain

$$\begin{aligned} \int_0^\infty t^s \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > 3t \right\} \right|^{\frac{s}{q}} \frac{dt}{t} &\leq c_7 \exp(-a\epsilon^{-1}) \int_0^\infty t^s \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > t \right\} \right|^{\frac{s}{q}} \frac{dt}{t} \\ &\quad + M \int_0^\infty t^s \left| \left\{ (\mathbf{M}_{\alpha p, R}^\eta[\mu_n])^{\frac{1}{p-1}} > \epsilon t \right\} \right|^{\frac{s}{q}} \frac{dt}{t}. \end{aligned}$$

By a change of variable, we derive

$$\begin{aligned} (3^{-s} - c_7 \exp(-a\epsilon^{-1})) \int_0^\infty t^s \left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > t \right\} \right|^{\frac{s}{q}} \frac{dt}{t} \\ \leq \frac{M}{(p-1)\epsilon^s} \int_0^\infty t^s \left| \left\{ (\mathbf{M}_{\alpha p, R}^\eta[\mu_n])^{\frac{1}{p-1}} > t \right\} \right|^{\frac{s}{q}} \frac{dt}{t}. \end{aligned}$$

We choose ϵ small enough so that $3^{-s} - c_7 \exp(-a\epsilon^{-1}) > 0$ and we derive

$$\|\mathbf{W}_{\alpha,p}^R[\mu_n]\|_{L^{q,s}(\mathbb{R}^N)} \leq c'_6 \|\mathbf{M}_{\alpha p,R}[\mu_n]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)},$$

and (2.27) follows by Fatou's lemma. Similarly, we can prove (2.27) in the case $s = \infty$.

Step 2 We claim that

$$\|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{q,s}(\mathbb{R}^N)} \geq c''_6 \|\mathbf{M}_{\alpha p,R}[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)}. \quad (2.30)$$

For $R > 0$ we have for some $c_8 = c_8(N, \alpha, p) > 0$

$$\begin{aligned} \mathbf{W}_{\alpha,p}^{2R}[\mu](x) &= \mathbf{W}_{\alpha,p}^R[\mu](x) + \int_R^{2R} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \mathbf{W}_{\alpha,p}^R[\mu](x) + c_8 \left(\frac{\mu(B_{2R}(x))}{R^{N-\alpha p}} \right)^{\frac{1}{p-1}}. \end{aligned} \quad (2.31)$$

Thus

$$\left| \left\{ x : \mathbf{W}_{\alpha,p}^{2R}[\mu](x) > 2t \right\} \right| \leq \left| \left\{ x : \mathbf{W}_{\alpha,p}^R[\mu](x) > t \right\} \right| + \left| \left\{ x : \frac{\mu(B_{2R}(x))}{R^{N-\alpha p}} > c_9 t^{p-1} \right\} \right|,$$

with $c_9 = c_8^{1-p}$. Consider $\{z_j\}_{j=1}^m \subset B_2$ such that $B_2 \subset \bigcup_{i=1}^m B_{\frac{1}{2}}(z_i)$. Thus $B_{2R}(x) \subset \bigcup_{i=1}^m B_{\frac{R}{2}}(x + Rz_i)$ for any $x \in \mathbb{R}^N$ and $R > 0$. Then

$$\begin{aligned} \left| \left\{ x : \frac{\mu(B_{2R}(x))}{R^{N-\alpha p}} > c_9 t^{p-1} \right\} \right| &\leq \left| \left\{ x : \sum_{i=1}^m \frac{\mu(B_{\frac{R}{2}}(x + Rz_i))}{R^{N-\alpha p}} > c_9 t^{p-1} \right\} \right| \\ &\leq \sum_{i=1}^m \left| \left\{ x : \frac{\mu(B_{\frac{R}{2}}(x + Rz_i))}{R^{N-\alpha p}} > \frac{c_9 t^{p-1}}{m} \right\} \right| \\ &\leq \sum_{i=1}^m \left| \left\{ x - Rz_i : \frac{\mu(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}} > \frac{c_9 t^{p-1}}{m} \right\} \right| \\ &= m \left| \left\{ x : \frac{\mu(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}} > \frac{c_9 t^{p-1}}{m} \right\} \right|. \end{aligned}$$

Moreover from (2.31)

$$\left(\frac{\mu(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}} \right)^{\frac{1}{p-1}} \leq c_{10} \mathbf{W}_{\alpha,p}^R[\mu](x),$$

thus

$$\left| \left\{ x : \frac{\mu(B_{2R}(x))}{R^{N-\alpha p}} > c_9 t^{p-1} \right\} \right| \leq m \left| \left\{ x : \mathbf{W}_{\alpha,p}^R[\mu](x) > c_{11} t \right\} \right|.$$

This leads to

$$\left| \left\{ x : \mathbf{W}_{\alpha,p}^{2R}[\mu](x) > 2t \right\} \right| \leq (m+1) \left| \left\{ x : \mathbf{W}_{\alpha,p}^R[\mu](x) > c_{12} t \right\} \right|, \quad (2.32)$$

where $c_{12} = \min\{1, c_{11}\}$ depends only on N, α and p . This implies

$$\|\mathbf{W}_{\alpha,p}^{2R}[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{13} \|\mathbf{W}_{\alpha,p}^R[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)}. \quad (2.33)$$

On the other hand, from the identity in (2.31) we derive that for any $\rho \in (0, R)$,

$$\mathbf{W}_{\alpha,p}^{2R}[\mu](x) \geq \mathbf{W}_{\alpha,p}^{2\rho}[\mu](x) \geq c_{14} \sup_{0 < \rho \leq R} \left(\frac{\mu(B_\rho(x))}{\rho^{N-\alpha p}} \right)^{\frac{1}{p-1}}, \quad (2.34)$$

from which follows

$$\mathbf{W}_{\alpha,p}^{2R}[\mu](x) \geq c_{14} (\mathbf{M}_{\alpha p}^R[\mu](x))^{\frac{1}{p-1}}. \quad (2.35)$$

Combining (2.33) and (2.35) we obtain (2.30) and then (2.25). Notice that the estimates are independent of R and thus valid if $R = \infty$.

Step 3 We claim that (2.26) holds. By the previous result we have also

$$c_{15}^{-1} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq \|\mathbf{M}_{\alpha p, R}[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{15} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)}. \quad (2.36)$$

For $R > 0$, the Bessel kernel satisfies [18, V-3-1]

$$c_{15} \left(\frac{\mathbf{1}_{B_R}(x)}{|x|^{N-\alpha p}} \right) \leq G_{\alpha p}(x) \leq c_{16} \left(\frac{\mathbf{1}_{B_{\frac{R}{2}}}(x)}{|x|^{N-\alpha p}} \right) + c_{16} e^{-\frac{|x|}{2}} \quad \forall x \in \mathbb{R}^N,$$

where c_{15} depends also N, α, p and R . Therefore

$$c_{15} \left(\frac{\mathbf{1}_{B_R}}{|\cdot|^{N-\alpha p}} \right) * \mu \leq \mathbf{G}_{\alpha p}[\mu] \leq c_{16} \left(\frac{\mathbf{1}_{B_{\frac{R}{2}}}}{|\cdot|^{N-\alpha p}} \right) + c_{16} e^{-\frac{|\cdot|}{2}} * \mu \quad \forall x \in \mathbb{R}^N. \quad (2.37)$$

By integration by parts, we get

$$\left(\frac{\mathbf{1}_{B_R}}{|\cdot|^{N-\alpha p}} \right) * \mu(x) = (N - \alpha p) \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu](x) + \frac{\mu(B_R(x))}{R^{N-\alpha p}} \geq (N - \alpha p) \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu](x),$$

which implies

$$c'_{15} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq \|\mathbf{G}_{\alpha p}[\mu]\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)}. \quad (2.38)$$

Furthermore $e^{-\frac{|x|}{2}} \leq c_{17} \mathbf{1}_{B_{\frac{R}{2}}} * e^{-\frac{|\cdot|}{2}}(x)$ where c_{17} depends on N and R , thus

$$e^{-\frac{|\cdot|}{2}} * \mu \leq c_{17} \left(\mathbf{1}_{B_{\frac{R}{2}}} * e^{-\frac{|\cdot|}{2}} \right) * \mu = c_{17} e^{-\frac{|\cdot|}{2}} * \left(\mathbf{1}_{B_{\frac{R}{2}}} * \mu \right).$$

Since

$$\mathbf{1}_{B_{\frac{R}{2}}} * \mu(x) = \mu(B_{\frac{R}{2}}(x)) \leq c_{18} \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu](x)$$

where c_{18} depends N, α, p and R , we derive with $c_{19} = c_{18} c_{17}$

$$e^{-\frac{|\cdot|}{2}} * \mu \leq c_{19} e^{-\frac{|\cdot|}{2}} * \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu].$$

Using Young inequality, we obtain

$$\begin{aligned}
\left\| e^{-\frac{|\cdot|}{2}} * \mu \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} &\leq c_{19} \left\| e^{-\frac{|\cdot|}{2}} * \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \\
&\leq c_{19} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \left\| e^{-\frac{|\cdot|}{2}} \right\|_{L^{1, \infty}(\mathbb{R}^N)} \\
&\leq c_{20} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)}.
\end{aligned} \tag{2.39}$$

Since by integration by parts there holds as above

$$\left(\frac{\mathbf{1}_{B_{\frac{R}{2}}}}{|\cdot|^{N-\alpha p}} \right) * \mu(x) = (N - \alpha p) \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu](x) + 2^{N-\alpha p} \frac{\mu(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}} \leq c_{21} \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu](x),$$

we obtain

$$\left\| \left(\frac{\mathbf{1}_{B_R}}{|\cdot|^{N-\alpha p}} \right) * \mu \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{21} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}}. \tag{2.40}$$

Thus

$$\left\| \mathbf{G}_{\alpha p}[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{22} \left\| \mathbf{W}_{\frac{\alpha p}{2}, 2}^R[\mu] \right\|_{L^{\frac{q}{p-1}, \frac{s}{p-1}}}. \tag{2.41}$$

follows by combining (2.37) and (2.40). Using (2.37) we obtain the claim. \square

Remark. Proposition 5.1 in [17] is a particular case of the previous result.

Theorem 2.4 *Let $\alpha > 0$, $p > 1$, $0 \leq \eta < p - 1$, $0 < \alpha p < N$ and $r > 0$. Then there exists $c_{23} > 0$, depending on α , p , η , N and r such that for any $R \in (0, \infty]$, $\delta \in (0, \left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2)$, any $x_0 \in B_r$ and r' such that $B_{2r'}(x_0) \subset B_r$ there holds*

$$\frac{1}{|B_{2,0}|} \int_{B_{2,0}} \exp \left(\delta \frac{(\mathbf{W}_{\alpha, p}^R[\mu_{B_0}](x))^{\frac{p-1}{p-1-\eta}}}{\|\mathbf{M}_{\alpha p, R}^\eta[\mu_{B_0}]\|_{L^\infty(B_0)}^{\frac{1}{p-1-\eta}}} \right) dx \leq \frac{c_{23}}{\left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 - \delta} \tag{2.42}$$

where we have set $B_0 = B_{r'}(x_0)$, $B_{2,0} = B_{2r'}(x_0)$ and $\mu_{B_0} = \mathbf{1}_{B_0} \mu$.

Proof. Let $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$ such that $\|\mathbf{M}_{\alpha p, R}^\eta[\mu_{B_0}]\|_{L^\infty(B_0)} < \infty$. We apply Proposition 2.2-(2.8) with $\mu = \mu_{B_0}$, $t_0 = \epsilon_0^{-1} M$ where

$$M := \epsilon_0^{-1} \left\| \mathbf{M}_{\alpha p, R}^\eta[\mu_{B_0}] \right\|_{L^\infty(\mathbb{R}^N)}^{\frac{1}{p-1}} = \left\| \mathbf{M}_{\alpha p, R}^\eta[\mu_{B_0}] \right\|_{L^\infty(B_0)}^{\frac{1}{p-1}},$$

$t > t_0$ and $\epsilon = t^{-1} M < \epsilon_0$. Then

$$\begin{aligned}
&|\{\mathbf{W}_{\alpha, p}^R[\mu_{B_0}] > 3t\} \cap B_r| \\
&\leq c_0 \exp \left(- \left(\frac{p-1-\eta}{4(p-1)} \right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 M^{-\frac{p-1}{p-1-\eta}} t^{-\frac{p-1}{p-1-\eta}} \right) |\{\mathbf{W}_{\alpha, p}^R[\mu_{B_0}] > t\} \cap B_r|.
\end{aligned} \tag{2.43}$$

If $x \notin B_{2,0}$ and $r' \geq R$ then $\mathbf{W}_{\alpha,p}^R[\mu_{B_0}](x) = 0$ while if $0 < r' < R$, then

$$\mathbf{W}_{\alpha,p}^R[\mu_{B_0}](x) = \int_r^R \left(\frac{\mu_{B_0}(B_t(x))}{t^{N-\alpha p}} \right) \frac{dt}{t} \leq \frac{N-\alpha p}{p-1} (\mu(B_0))^{\frac{1}{p-1}} r^{-\frac{N-\alpha p}{p-1}} \leq \frac{N-\alpha p}{p-1} M.$$

Then, for any $t > \left(\epsilon_0^{-1} + \frac{N-\alpha p}{p-1} \right) M$, $\{\mathbf{W}_{\alpha,p}^R[\mu_{B_0}] > t\} \subset B_{2,0}$, consequently, for any such t ,

$$|\{\mathbf{W}_{\alpha,p}^R[\mu_{B_0}] > 3t\} \cap B_{2,0}| \leq c_0 \exp\left(-\left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 M^{-\frac{p-1}{p-1-\eta}} t^{-\frac{p-1}{p-1-\eta}}\right) |B_{2,0}|. \quad (2.44)$$

This can be written under the form

$$\begin{aligned} & |\{\mathbf{W}_{\alpha,p}^R[\mu_{B_0}] > t\} \cap B_{2,0}| \\ & \leq c_0 \exp\left(-\left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 M^{-\frac{p-1}{p-1-\eta}} t^{-\frac{p-1}{p-1-\eta}}\right) |B_{2,0}| \quad \forall t > 3 \left(\epsilon_0^{-1} + \frac{N-\alpha p}{p-1} \right) M. \end{aligned} \quad (2.45)$$

If $\delta \in (0, \left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2)$, we get by Fubini's theorem

$$\begin{aligned} & \int_{B_{2,0}} \exp\left(\delta M^{-\frac{p-1}{p-1-\eta}} (\mathbf{W}_{\alpha,p}^R[\mu_{B_0}])(x)^{\frac{p-1}{p-1-\eta}}\right) dx \\ & = \frac{p-1}{p-1-\eta} \delta M^{-\frac{1}{p-1-\eta}} \int_0^\infty t^{-\frac{p-1}{p-1-\eta}-1} \exp\left(\delta M^{-\frac{p-1}{p-1-\eta}} t^{\frac{p-1}{p-1-\eta}}\right) |\{\mathbf{W}_{\alpha,p}^R[\mu_{B_0}] > t\} \cap B_{2,0}| dt \\ & = \frac{p-1}{p-1-\eta} \delta M^{-\frac{1}{p-1-\eta}} \left(\int_0^{3\left(\epsilon_0^{-1} + \frac{N-\alpha p}{p-1}\right)M} \dots + \int_{3\left(\epsilon_0^{-1} + \frac{N-\alpha p}{p-1}\right)M}^\infty \right) \\ & = I + II. \end{aligned}$$

Then

$$\begin{aligned} I & \leq \frac{p-1}{p-1-\eta} \delta M^{-\frac{1}{p-1-\eta}} |B_{2,0}| \int_0^{3\left(\epsilon_0^{-1} + \frac{N-\alpha p}{p-1}\right)M} t^{-\frac{p-1}{p-1-\eta}-1} \exp\left(\delta M^{-\frac{p-1}{p-1-\eta}} t^{\frac{p-1}{p-1-\eta}}\right) dt \\ & \leq |B_{2,0}| \int_0^{\delta\left(3\left(\epsilon_0^{-1} + \frac{N-\alpha p}{p-1}\right)M\right)^{\frac{p-1}{p-1-\eta}}} e^s ds \\ & \leq c_{24} |B_{2,0}|. \end{aligned}$$

It follows from (2.45) that

$$\begin{aligned}
II &\leq c_0 \frac{p-1}{p-1-\eta} \delta M^{-\frac{1}{p-1-\eta}} |B_{2,0}| \\
&\quad \times \int_{3(\epsilon_0^{-1} + \frac{N-\alpha p}{p-1})M}^{\infty} t^{-\frac{p-1}{p-1-\eta}-1} \exp\left(\left(\delta - \left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2\right) M^{-\frac{p-1}{p-1-\eta}} t^{\frac{p-1}{p-1-\eta}}\right) dt \\
&= c_0 |B_{2,0}| \int_{3(\epsilon_0^{-1} + \frac{N-\alpha p}{p-1})M}^{\infty} \frac{p-1}{p-1-\eta} \left(-\left(\left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 - \delta\right) s\right) ds \\
&\leq \frac{c_0 |B_{2,0}|}{\left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 - \delta}.
\end{aligned}$$

Combining the estimates on I and II , we derive

$$\int_{B_{2,0}} \exp\left(\delta M^{-\frac{p-1}{p-1-\eta}} (\mathbf{W}_{\alpha,p}^R[\mu_{B_0}](x))^{\frac{p-1}{p-1-\eta}}\right) dx \leq \frac{c_{25} |B_{2,0}|}{\left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 - \delta}, \quad (2.46)$$

which is the desired inequality. \square

2.4 Approximation of measures

The next result is an extension of a classical result of Feyel and de la Pradelle [11]. This type of result has been intensively used in the framework of Sobolev spaces since the pioneering work of Baras and Pierre [3], but apparently it is new in the case of Bessel-Lorentz spaces. We recall that a sequence of bounded measures $\{\mu_n\}$ in Ω converges to some bounded measure μ in Ω in the *narrow topology* of $\mathfrak{M}(\Omega)$ if

$$\lim_{n \rightarrow \infty} \int_{\Omega} \phi d\mu_n = \int_{\Omega} \phi d\mu \quad \forall \phi \in C_b(\Omega) := C(\Omega) \cap L^\infty(\Omega). \quad (2.47)$$

Theorem 2.5 *Assume Ω is an open subset of \mathbb{R}^N . Let $\alpha > 0$, $1 < s < \infty$, $1 \leq q < \infty$ and $\mu \in \mathfrak{M}_+(\Omega)$. If μ is absolutely continuous with respect to $C_{\alpha,s,q}$ in Ω , there exists an increasing sequence $\{\mu_n\} \subset \mathfrak{M}_+^b(\Omega) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$, with compact support in Ω which converges to μ weakly in the sense of measures. Furthermore, if $\mu \in \mathfrak{M}_+^b(\Omega)$, then $\mu_n \rightharpoonup \mu$ in the narrow topology.*

Proof. Step 1. Assume that μ has compact support. Let $\phi \in L^{\alpha,s,q}(\mathbb{R}^N)$ and $\tilde{\phi}$ its $C_{\alpha,s,q}$ -quasicontinuous representative. Since μ is absolutely continuous with respect to $C_{\alpha,s,q}$, we can define the mapping

$$\phi \mapsto P(\phi) = \int_{\mathbb{R}^N} \tilde{\phi}^+ d\mu|_{\Omega}$$

where $\mu|_{\Omega}$ is the extension of μ by 0 in Ω^c . By Fatou's lemma, P is lower semicontinuous on $L^{\alpha,s,q}(\mathbb{R}^N)$. Furthermore it is convex and positively homogeneous of degree 1. If $\text{Epi}(P)$ denotes the epigraph of P , i.e.

$$\text{Epi}(P) = \{(\phi, t) \in L^{\alpha,s,q}(\mathbb{R}^N) \times \mathbb{R} : t \geq P(\phi)\},$$

it is a closed convex cone. Let $\epsilon > 0$ and $\phi_0 \in C_c^\infty$, $\phi_0 \geq 0$. Since $(\phi_0, P(\phi_0) - \epsilon) \notin Epi(P)$, there exist $\ell \in (L^{\alpha,s,q}(\mathbb{R}^N))'$, a and b in \mathbb{R} such that

$$a + bt + P(\phi) \leq 0 \quad \forall (\phi, t) \in Epi(P), \quad (2.48)$$

$$a + b(P(\phi_0) - \epsilon) + \ell(\phi_0) > 0. \quad (2.49)$$

Since $(0, 0) \in Epi(P)$, $a \leq 0$. Since $(s\phi, st) \in Epi(P)$ for all $s > 0$, $s^{-1}a + bt + P(\phi) \leq 0$, which implies

$$bt + P(\phi) \leq 0 \quad \forall (\phi, t) \in Epi(P).$$

Finally, since $(0, 1) \in Epi(P)$, $b \leq 0$. But if $b = 0$ we would have $P(\phi) \leq -a$ for all $\phi \in L^{\alpha,s,q}(\mathbb{R}^N)$. which would lead to $\ell = 0$ and $a > 0$ from (2.49), a contradiction. Therefore $b < 0$. Then, we put $\theta(\phi) = -\frac{\ell(\phi)}{b}$ and derive that, for any $(\phi, t) \in Epi(P)$, there holds $\theta(\phi) \leq t$, and in particular

$$\theta(\phi) \leq P(\phi) \quad \forall \phi \in L^{\alpha,s,q}(\mathbb{R}^N). \quad (2.50)$$

Since $\phi \leq 0 \implies P(\phi) = 0$, θ is a positive linear functional on $L^{\alpha,s,q}(\mathbb{R}^N)$. Furthermore

$$\sup_{\substack{\phi \in C_c^\infty(\mathbb{R}^N) \\ \|\phi\|_{L^\infty} \leq 1}} |\theta(\phi)| = \sup_{\substack{\phi \in C_c^\infty(\mathbb{R}^N) \\ \|\phi\|_{L^\infty} \leq 1}} \theta(\phi) \leq \sup_{\substack{\phi \in C_c^\infty(\mathbb{R}^N) \\ \|\phi\|_{L^\infty} \leq 1}} P\theta(\phi) = P(1) = \mu(\Omega).$$

By the Riesz representation theorem, there exists $\sigma \in \mathfrak{M}_+(\mathbb{R}^N)$ such that

$$\theta(\phi) = \int_{\mathbb{R}^N} \phi d\sigma \quad \forall \phi \in C_c^\infty(\mathbb{R}^N). \quad (2.51)$$

Inequality (2.50) implies $0 \leq \sigma \leq \mu|_\Omega$. Therefore $supp(\sigma) \subset supp(\mu|_\Omega) = supp(\mu)$ and σ vanishes on Borel subsets of $C_{\alpha,s,q}$ capacity zero, as μ does it. From (2.49), we have

$$\int_{\mathbb{R}^N} \tilde{\phi}_0 d\sigma = \theta(\phi_0) > P(\phi_0) - \epsilon + \frac{a}{b} \geq \int_{\mathbb{R}^N} \tilde{\phi}_0 d\mu|_\Omega - \epsilon.$$

This implies

$$0 \leq \int_{\mathbb{R}^N} \tilde{\phi}_0 d(\mu|_\Omega - \sigma) \leq \epsilon. \quad (2.52)$$

It remains to prove that $\sigma \in (L^{\alpha,s,q}(\mathbb{R}^N))'$. Let $\chi \in C_0^\infty(\mathbb{R}^N)$, $\chi \geq 0$, with value 1 in a neighborhood of $supp(\mu)$ and support in $\{x : \text{dist}(x, supp(\mu)) \leq 1\}$; then for any $f \in C_0^\infty(\mathbb{R}^N)$, $f \geq 0$, $\chi \mathbf{G}_\alpha[f] \in C_0^\infty(\mathbb{R}^N)$ and is nonnegative.

$$\int_{\mathbb{R}^N} \mathbf{G}_\alpha[f] d\sigma = \int_{\mathbb{R}^N} \chi \mathbf{G}_\alpha[f] d\sigma = \tau(\chi \mathbf{G}_\alpha[f]) = \theta(\mathbf{G}_\alpha[f]) - \theta((1 - \chi)\mathbf{G}_\alpha[f]) = \theta(\mathbf{G}_\alpha[f]),$$

since $1 - \chi$ vanishes on $supp(\mu)$. This implies that for all $f \in C_0^\infty(\mathbb{R}^N)$, $f \geq 0$, there holds

$$\int_{\mathbb{R}^N} \mathbf{G}_\alpha[f] d\sigma \leq \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \|\mathbf{G}_\alpha[f]\|_{L^{\alpha,s,q}(\mathbb{R}^N)}, \quad (2.53)$$

since $\theta = -b^{-1}\ell$ and $\ell \in (L^{\alpha,s,q}(\mathbb{R}^N))'$. Now, given $f \in L^{\alpha,s,q}(\mathbb{R}^N)$, $f \geq 0$ and a sequence of mollifiers $\{\rho_n\}$, $(\mathbf{1}_{B_n}f) * \rho_n \in C_c^\infty(\mathbb{R}^N)$ and $(\mathbf{1}_{B_n}f) * \rho_n \rightarrow f$ in $L^{\alpha,s,q}(\mathbb{R}^N)$. Furthermore, there is a subsequence $\{n_k\}$ such that $\lim_{n_k \rightarrow \infty} \mathbf{G}_\alpha[(\mathbf{1}_{B_{n_k}}f) * \rho_{n_k}](x) \rightarrow \mathbf{G}_\alpha[f](x)$, $C_{\alpha,s,q}$ -quasi everywhere. Using Fatou's lemma and lower semicontinuity of the norm

$$\begin{aligned} \int_{\mathbb{R}^N} \mathbf{G}_\alpha[f] d\sigma &\leq \liminf_{n_k \rightarrow \infty} \int_{\mathbb{R}^N} \mathbf{G}_\alpha[(\mathbf{1}_{B_{n_k}}f) * \rho_{n_k}] d\sigma \\ &\leq \liminf_{n_k \rightarrow \infty} \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \left\| \mathbf{G}_\alpha[(\mathbf{1}_{B_{n_k}}f) * \rho_{n_k}] \right\|_{L^{\alpha,s,q}(\mathbb{R}^N)} \\ &\leq \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \|\mathbf{G}_\alpha[f]\|_{L^{\alpha,s,q}(\mathbb{R}^N)}. \end{aligned}$$

Therefore (2.53) can be rewritten under the form

$$\int_{\mathbb{R}^N} \mathbf{G}_\alpha[f] d\sigma \leq \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \|\mathbf{G}_\alpha[f]\|_{L^{\alpha,s,q}(\mathbb{R}^N)} \quad \forall f \in L^{s,q}(\mathbb{R}^N), f \geq 0. \quad (2.54)$$

Consequently $\sigma \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$ satisfies

$$\left| \int_{\mathbb{R}^N} \mathbf{G}_\alpha[f] d\sigma \right| \leq \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \|\mathbf{G}_\alpha[f]\|_{L^{\alpha,s,q}(\mathbb{R}^N)} \quad \forall f \in L^{s,q}(\mathbb{R}^N). \quad (2.55)$$

Step 2. We assume that μ has no longer compact support. Set $\Omega_n = \{x : \text{dist}(x, \Omega^c) \geq n^{-1}\}$, then $\Omega_n \subset \overline{\Omega_n} \subset \Omega_{n+1} \subset \Omega$ for $n \geq n_0$ such that $\Omega_{n_0} \neq \emptyset$. Let $\{\phi_n\} \subset C_c^\infty(\mathbb{R}^N)$ be an increasing sequence such that $\phi_n \geq 0$, $\phi_n = 1$ in a neighborhood of $\overline{\Omega_n}$ and $\text{supp}(\mu_n) \subset \Omega_{n+1}$. and let $\nu_n = \phi_n \mu$. For $n \geq n_0$ there is $\sigma_n \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$ with $0 \leq \sigma_n \leq \nu_n$ and

$$\frac{1}{n} > \int_{\Omega} \phi_n d(\nu_n - \sigma_n) \geq \int_{\Omega_n} d(\nu_n - \sigma_n) = \int_{\Omega_n} d(\mu - \sigma_n).$$

If we set $\mu_n = \sup\{\sigma_1, \sigma_2, \dots, \sigma_n\}$, then $\{\mu_n\}$ is increasing and $\text{supp}(\mu_n) \subset \Omega_{n+1}$, and for any $f \in L^{s,q}(\mathbb{R}^N)$, there holds

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \mathbf{G}_\alpha[f] d\mu_n \right| &\leq \int_{\mathbb{R}^N} \mathbf{G}_\alpha[|f|] d\mu_n \\ &\leq \sum_{k=1}^n \int_{\mathbb{R}^N} \mathbf{G}_\alpha[|f|] d\sigma_k \\ &\leq \left(\sum_{k=1}^n \|\sigma_k\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \right) \|f\|_{L^{s,q}(\mathbb{R}^N)}. \end{aligned} \quad (2.56)$$

Finally, let $\phi \in C_c(\Omega)$ and $m \in \mathbb{N}^*$ such that $\text{supp}(\phi) \subset \Omega_m$. For all $n \geq m$, we have

$$\left| \int_{\Omega} \phi d\mu_n - \int_{\Omega} \phi d\mu \right| \leq \left| \int_{\Omega} \phi d(\mu - \mu_n) \right| \|\phi\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{n} \|\phi\|_{L^\infty(\mathbb{R}^N)}.$$

Thus $\mu_n \rightharpoonup \mu$ weakly in the sense of measures.

Step 3. Assume that $\mu \in \mathfrak{M}_+^b(\Omega)$. Then $\mu_n(\Omega) \leq \mu(\Omega)$. Thus

$$\mu_n(\Omega) = \mu_n(\Omega_{n_0}) + \sum_{k=n_0}^{\infty} \mu_n(\overline{\Omega}_{k+1} \setminus \Omega_k)$$

Since the sequence $\{\mu_n\}$ is increasing and $\lim_{k \rightarrow \infty} \mu_n(\overline{\Omega}_{k+1} \setminus \Omega_k) = \mu(\overline{\Omega}_{k+1} \setminus \Omega_k)$ by the previous construction, we obtain by monotone convergence

$$\lim_{n \rightarrow \infty} \mu_n(\Omega) = \mu(\Omega_{n_0}) + \sum_{k=n_0}^{\infty} \mu(\overline{\Omega}_{k+1} \setminus \Omega_k) = \mu(\Omega)$$

Next we consider $\phi \in C_b(\Omega) := C(\Omega) \cap L^\infty(\Omega)$, then

$$\left| \int_{\Omega} \phi d\mu_n - \int_{\Omega} \phi d\mu \right| \leq \left| \int_{\Omega} d(\mu - \mu_n) \right| \|\phi\|_{L^\infty(\Omega)} \leq (\mu(\Omega) - \mu_n(\Omega)) \|\phi\|_{L^\infty(\Omega)} \rightarrow 0.$$

Thus $\mu_n \rightarrow \mu$ in the narrow topology of measures. \square

As a consequence of Theorem 2.5 and Theorem 2.3 we obtain the following.

Theorem 2.6 *Let $p-1 < s_1 < \infty$, $p-1 < s_2 \leq \infty$, $0 < \alpha p < N$, $R > 0$ and $\mu \in \mathfrak{M}_+(\Omega)$. If μ is absolutely continuous with respect to the capacity $C_{\alpha p, \frac{s_1}{s_1-p+1}, \frac{s_2}{s_2-p+1}}$, there exists an increasing sequence $\{\mu_n\} \subset \mathfrak{M}_+(\Omega)$ with compact support in Ω which converges to μ in the weak sense of measures and such that $\mathbf{W}_{\alpha, p}^R[\mu_n] \in L^{s_1, s_2}(\mathbb{R}^N)$, for all n . Furthermore, if $\mu \in \mathfrak{M}_+^b(\Omega)$, μ_n converges to μ in the narrow topology.*

Proof. By Theorem 2.5 there exists an increasing sequence $\{\mu_n\}$ of nonnegative measures with compact support in Ω , all elements of $(L^{\alpha p, \frac{s_1}{s_1-p+1}, \frac{s_2}{s_2-p+1}}(\mathbb{R}^N))'$, which converges weakly to μ . If $\mu \in \mathfrak{M}_+^b(\Omega)$, the convergence holds in the narrow topology. Noting that for a positive measure σ in \mathbb{R}^N ,

$$\mathbf{G}_{\alpha p}[\sigma] \in L^{\frac{s_1}{p-1}, \frac{s_2}{p-1}}(\mathbb{R}^N) \iff \sigma \in (L^{\alpha p, \frac{s_1}{s_1-p+1}, \frac{s_2}{s_2-p+1}}(\mathbb{R}^N))',$$

it implies $\mathbf{G}_{\alpha p}[\mu_n] \in L^{\frac{s_1}{p-1}, \frac{s_2}{p-1}}(\mathbb{R}^N)$. Then, by Theorem 2.3, $\mathbf{W}_{\alpha, p}^R[\mu_n] \in L^{s_1, s_2}(\mathbb{R}^N)$. \square

3 Renormalized solutions

3.1 Classical results

Although the notion of renormalized solutions is becoming more and more present in the theory of quasilinear equations with measure data, it has not yet acquainted a popularity which could avoid us to present some of its main aspects. If $\mu \in \mathfrak{M}^b(\Omega)$, we denote by μ^+ and μ^- respectively its positive and negative part. We denote by $\mathfrak{M}_0(\Omega)$ the space of measures in Ω which are absolutely continuous with respect to the $c_{1, p}^\Omega$ -capacity defined on a compact set $K \subset \Omega$ by

$$c_{1, p}^\Omega(K) = \inf \left\{ \int_{\Omega} |\nabla \phi|^p dx : \phi \geq \mathbf{1}_K, \phi \in C_c^\infty(\Omega) \right\}. \quad (3.1)$$

We also denote $\mathfrak{M}_s(\Omega)$ the space of measures in Ω with support on a set of zero $c_{1,p}^\Omega$ -capacity. Classically, any $\mu \in \mathfrak{M}^b(\Omega)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}^b(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. We recall that any $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}^b(\Omega)$ can be written under the form $\mu_0 = f - \operatorname{div} g$ where $f \in L^1(\Omega)$ and $g \in L^{p'}(\Omega)$. Furthermore, if μ is positive, one can find such a decomposition with $f \geq 0$ (see [7]).

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s, k\}, -k\}$. We recall that if u is a measurable function defined and finite a.e. in Ω , such that $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, there exists a measurable function $v : \Omega \rightarrow \mathbb{R}^N$ such that $\nabla T_k(u) = \mathbf{1}_{|u| \leq k} v$ a.e. in Ω and for all $k > 0$. We define the gradient ∇u of u by $v = \nabla u$. We recall the definition of a renormalized solution given in [10]

Definition 3.1 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}^b(\Omega)$. A measurable function u defined in Ω and finite a.e. is called a renormalized solution of*

$$\begin{aligned} -\Delta_p u &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (3.2)$$

if $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{N}{N-1}$, and u has the property that for any $k > 0$ there exist $\lambda_k^+, \lambda_k^- \in \mathfrak{M}_+^b(\Omega) \cap \mathfrak{M}_0(\Omega)$, respectively concentrated on the sets $u = k$ and $u = -k$, with the property that $\lambda_k^+ \rightharpoonup \mu_s^+$, $\lambda_k^- \rightharpoonup \mu_s^-$ in the narrow topology of measures, such that

$$\int_{\{|u| < k\}} |\nabla u|^{p-2} \nabla \phi \, dx = \int_{\{|u| < k\}} \phi \, d\mu_0 + \int_{\Omega} \phi \, d\lambda_k^+ - \int_{\Omega} \phi \, d\lambda_k^-, \quad (3.3)$$

for every $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark. If u is a renormalized solution of problem (3.2) and $\mu \in \mathfrak{M}_+^b(\Omega)$, then $u \geq 0$ in Ω . Indeed, taking $k > m > 0$ and $\phi = T_m(\max\{-u, 0\})$, then $0 \leq \phi \leq m$ and we have

$$\begin{aligned} \int_{\{|u| < k\}} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx &\geq \int_{\{|u| < k\}} T_m(\max\{-u, 0\}) \, d\mu_0 + \int_{\Omega} T_m(\max\{-u, 0\}) \, d\lambda_k^+ \\ &\quad - \int_{\Omega} T_m(\max\{-u, 0\}) \, d\lambda_k^- \\ &\geq -m \lambda_k^-(\Omega). \end{aligned}$$

Thus

$$\int_{\Omega} |\nabla T_m(\max\{-u, 0\})|^p \leq m \lambda_k^-(\Omega)$$

Letting $k \rightarrow \infty$, we obtain $\nabla T_m(\max\{-u, 0\}) = 0$ a.e., thus $u \geq 0$ a.e. in Ω .

We recall the following important results, see [10, Th 4.1, Sec 5.1].

Theorem 3.2 *1- Let $\{u_n\}$ be a sequence of measurable functions defined in Ω and finite a.e. such that, for every $k > 0$ and $n \in \mathbb{N}$, $T_k(u_n) \in W_0^{1,p}(\Omega)$, $|\nabla u_n|^{p-1} \in L^r(\Omega)$ for*

any $0 < r < \frac{N}{N-1}$ and , $k^{-1} \int_{\Omega} |\nabla T_k(u_n)|^p \leq M$, for some constant M . Then, up to a subsequence, $\{u_n\}$ and $\{\nabla u_n\}$ are Cauchy sequences in measure, $\{u_n\}$ converges a.e. to a function u , finite a.e. in Ω , such that $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$. Furthermore $T_k(u_n)$ converges to $T_k(u)$ weakly in $W_0^{1,p}(\Omega)$, ∇u_n converges to ∇u a.e., $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{N}{N-1}$ and $|\nabla u_n|^{p-2} \nabla u_n$ converges to $|\nabla u|^{p-2} \nabla u$ in $L^r(\Omega)$.

2- Let $\{\mu_n\} \subset \mathfrak{M}^b(\Omega)$ be a sequence such that $\sup_n |\mu_n|(\Omega) < \infty$ and let $\{u_n\}$ be renormalized solutions of

$$\begin{aligned} -\Delta_p u_n &= \mu_n && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.4)$$

Then, up to a subsequence, $\{u_n\}$ converges a.e. to a solution u of $-\Delta_p u = \mu$ in the sense of distributions in Ω , and for every $k > 0$, $k^{-1} \int_{\Omega} |\nabla T_k(u)|^p \leq M$ for some M .

Finally we recall the following fundamental stability result of [10] which extends Theorem 3.2-2.

Theorem 3.3 Let $\mu = \mu_0 + \mu_s^+ - \mu_s^- \in \mathfrak{M}^b(\Omega)$, with $\mu_0 = f - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$, $\mu_s^+, \mu_s^- \in \mathfrak{M}_s^+(\Omega)$. Assume there are sequences $\{f_n\} \subset L^1(\Omega)$, $\{g_n\} \subset (L^{p'}(\Omega))^N$, $\{\rho_n\}, \{\eta_n\} \subset \mathfrak{M}_+^b(\Omega)$ such that $f_n \rightharpoonup f$ weakly in $L^1(\Omega)$, $g_n \rightarrow g$ in $L^{p'}(\Omega)$ and $\operatorname{div} g_n$ is bounded in $\mathfrak{M}^b(\Omega)$, $\rho_n \rightharpoonup \mu_s^+$ and $\eta_n \rightharpoonup \mu_s^-$ in the narrow topology. If $\mu_n = f_n - \operatorname{div} g_n + \rho_n - \eta_n$ and u_n is a renormalized solution of (3.4), then, up to a subsequence, u_n converges a.e. to a renormalized solution u of (3.2). Furthermore $T_k(u_n) \rightarrow T_k(u)$ in $W_0^{1,p}(\Omega)$.

3.2 Applications

We present below some interesting consequences of the above theorem.

Corollary 3.4 Let $\mu \in \mathfrak{M}^b(\Omega)$ with compact support in Ω and $\omega \in \mathfrak{M}^b(\Omega)$. Let $\{f_n\} \subset L^1(\Omega)$ which converges weakly to $f \in L^1(\Omega)$ and $\mu_n = \eta_n * \mu$ where $\{\eta_n\}$ is a sequence of mollifiers. If u_n is a renormalized solution of

$$\begin{aligned} -\Delta_p u_n &= f_n + \mu_n + \omega && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (3.5)$$

then, up to a subsequence, u_n converges to a renormalized solution of

$$\begin{aligned} -\Delta_p u &= f + \mu + \omega && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.6)$$

Proof. We write $\omega = \tilde{h} - \operatorname{div} \tilde{g} + \omega_s^+ - \omega_s^-$ and $\mu = h - \operatorname{div} g + \mu_s^+ - \mu_s^-$, with $h, \tilde{h} \in L^1(\Omega)$, $g, \tilde{g} \in (L^{p'}(\Omega))^N$, h, g, μ_s^+ and μ_s^- with support in a compact set $K \subset \Omega$. If the support of η_n is included in the ball $B_{n^{-1}}$ with \mathbb{N}_* , for $n \geq n_0$, $h * \eta_n, g * \eta_n$ and $\mu_s * \eta_n$ have also their support in a fixed compact subset of Ω . Moreover $h * \eta_n \rightarrow h$ and $g * \eta_n \rightarrow g$ in $L^1(\Omega)$ and $(L^{p'}(\Omega))^N$ respectively and $\operatorname{div} g * \eta_n \rightarrow \operatorname{div} g$ in $W^{-1,p'}(\Omega)$. Therefore

$$f_n + \mu_n + \omega = f_n + \tilde{h} - \operatorname{div}(\tilde{g} + g * \eta_n) + \omega_s^+ + \mu_s^+ * \eta_n - \omega_s^- - \mu_s^- * \eta_n$$

is an approximation of the measure $f + \mu + \omega$ in the sense of Theorem 3.3. This implies the claim. \square

Corollary 3.5 Let $\mu_i \in \mathfrak{M}_+^b(\Omega)$, $i = 1, 2$, and $\{\mu_{i,n}\} \subset \mathfrak{M}_+^b(\Omega)$ increasing and converging to μ_i in $\mathfrak{M}_+^b(\Omega)$. Let $\{f_n\} \subset L^1(\Omega)$ which converges to some f weakly in $L^1(\Omega)$. Let $\{\rho_n\} \subset \mathfrak{M}^b(\Omega)$ which converges to some $\rho \in \mathfrak{M}_s(\Omega)$ in the narrow topology. For any $n \in \mathbb{N}$ let u_n be a renormalized solution of

$$\begin{aligned} -\Delta_p u_n &= f_n + \mu_{n,1} - \mu_{n,2} + \rho_n && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.7)$$

Then, up to a subsequence, u_n converges a.e. to a renormalized solution of problem

$$\begin{aligned} -\Delta_p u &= f + \mu_1 - \mu_2 + \rho && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.8)$$

The proof of this results is based upon two lemmas

Lemma 3.6 For any $\mu \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_+^b(\Omega)$ there exists $f \in L^1(\Omega)$ and $h \in W^{-1,p'}(\Omega)$ such that $\mu = f + h$ and

$$\|f\|_{L^1(\Omega)} + \|g\|_{W^{-1,p'}(\Omega)} + \|g\|_{\mathfrak{M}^b(\Omega)} \leq 5\mu(\Omega). \quad (3.9)$$

Proof. Following [9] and the proof of [7, Th 2.1], one can write $\mu = \phi\gamma$ where $\gamma \in W^{-1,p'}(\Omega) \cap \mathfrak{M}_+^b(\Omega)$ and $\phi \in L^1(\Omega, \gamma)$. Let $\{K_n\}_{n \in \mathbb{N}_*}$ be an increasing sequence of compact subsets of Ω such that $\cup_n K_n = \Omega$. We define the sequence of measures $\{\nu_n\}_{n \in \mathbb{N}_*}$ by

$$\begin{aligned} \nu_n &= T_n(\mathbf{1}_{K_n}\phi)\gamma - T_{n-1}(\mathbf{1}_{K_{n-1}}\phi)\gamma \quad \text{for } n \geq 2 \\ \nu_1 &= T_1(\mathbf{1}_{K_1}\phi)\gamma. \end{aligned}$$

Since $\nu_k \geq 0$, then $\sum_{k=1}^{\infty} \nu_k = \mu$ with strong convergence in $\mathfrak{M}^b(\Omega)$, $\|\nu_k\|_{\mathfrak{M}^b(\Omega)} = \nu_k(\Omega)$ and

$\sum_{k=1}^{\infty} \|\nu_k\|_{\mathfrak{M}^b(\Omega)} = \mu(\Omega)$. If $\{\eta_n\}$ is a sequence of mollifiers, $\phi_n = \mu_n * \eta_n \in C_c^\infty(\Omega)$, $\phi_n \geq 0$, we may assume that

$$\|\phi_n - \nu_n\|_{W^{-1,p'}(\Omega)} \leq 2^{-n}\mu(\Omega).$$

Set $f_n = \sum_{k=1}^n \phi_k$, then $\|f_n\|_{L^1(\Omega)} \leq \sum_{k=1}^n \|\phi_k\|_{L^1(\Omega)} \leq \sum_{k=1}^n \|\nu_k\|_{\mathfrak{M}^b(\Omega)} \leq \mu(\Omega)$. If we define

$f = \lim_{n \rightarrow \infty} f_n$, then $f \in L^1(\Omega)$ with $\|f\|_{L^1(\Omega)} \leq \mu(\Omega)$. Set $h_n = \sum_{k=1}^n (\nu_k - \phi_k)$, then

$h_n \in W^{-1,p'}(\Omega) \cap \mathfrak{M}^b(\Omega)$, $\|h_n\|_{W^{-1,p'}(\Omega)} \leq 2\mu(\Omega)$ and h_n converges strongly in $W^{-1,p'}(\Omega)$ to some h which satisfies $\|h\|_{W^{-1,p'}(\Omega)} \leq 2\mu(\Omega)$. Since $\mu = f + h$ and $\|h\|_{\mathfrak{M}^b(\Omega)} \leq 2\mu(\Omega)$, the result follows. \square

Lemma 3.7 Let $\mu \in \mathfrak{M}_+^b(\Omega)$. If $\{\mu_n\} \subset \mathfrak{M}_+^b(\Omega)$ is an increasing sequence which converges to μ in $\mathfrak{M}^b(\Omega)$, there exist $F_n, F \in L^1(\Omega)$ and $G_n, G \in W^{-1,p'}(\Omega)$ such that

$$\mu_n = \mu_{n0} + \mu_{ns} = F_n + G_n + \mu_{ns} \quad \text{and} \quad \mu = \mu_0 + \mu_s = F + G + \mu_s,$$

such that $F_n \rightarrow F$ in $L^1(\Omega)$, $G_n \rightarrow G$ in $W^{-1,p'}(\Omega)$ and in $\mathfrak{M}^b(\Omega)$ and $\mu_{ns} \rightarrow \mu_s$ in $\mathfrak{M}^b(\Omega)$, and

$$\|F_n\|_{L^1(\Omega)} + \|G_n\|_{W^{-1,p'}(\Omega)} + \|G_n\|_{\mathfrak{M}^b(\Omega)} + \|\mu_{n,s}\|_{\mathfrak{M}^b(\Omega)} \leq 6\mu(\Omega). \quad (3.10)$$

Proof. Since $\{\mu_n\}$ is increasing $\{\mu_{n,0}\}$ and $\{\mu_{n,s}\}$ share this property. Clearly

$$\|\mu - \mu_n\|_{\mathfrak{M}^b(\Omega)} = \|\mu_0 - g_{m_{n,0}}\|_{\mathfrak{M}^b(\Omega)} + \|\mu_s - g_{m_{n,s}}\|_{\mathfrak{M}^b(\Omega)},$$

thus $\mu_{n,0} \rightarrow \mu_0$ and $\mu_{n,s} \rightarrow \mu_s$ in $\mathfrak{M}^b(\Omega)$. Furthermore $\|\mu_{n,s}\|_{\mathfrak{M}^b(\Omega)} \leq \mu_s(\Omega) \leq \mu(\Omega)$. Set $\tilde{\mu}_{0,0} = 0$ and $\tilde{\mu}_{n,0} = \mu_{n,0} - \mu_{n-1,0}$ for $n \in \mathbb{N}_*$. From Lemma 3.6, for any $n \in \mathbb{N}$, one can find $f_n \in L^1(\Omega)$, $h_n \in W^{-1,p'}(\Omega) \cap \mathfrak{M}^b(\Omega)$ such that $\tilde{\mu}_{n,0} = f_n + h_n$ and

$$\|f_n\|_{L^1(\Omega)} + \|h_n\|_{W^{-1,p'}(\Omega)} + \|h_n\|_{\mathfrak{M}^b(\Omega)} \leq 5\tilde{\mu}_{n,0}(\Omega).$$

If we define $F_n = \sum_{k=1}^n f_k$ and $G_n = \sum_{k=1}^n h_k$, then $\mu_{n,0} = F_n + G_n$ and the convergence statements hold. Furthermore

$$\|F_n\|_{L^1(\Omega)} + \|G_n\|_{W^{-1,p'}(\Omega)} + \|G_n\|_{\mathfrak{M}^b(\Omega)} \leq 5\tilde{\mu}_0(\Omega).$$

Therefore (3.10) holds. \square

Proof of Corollary 3.5. We set $\nu_n = f_n + \mu_{n,1} - \mu_{n,2} + \rho_n$ and $\nu = f + \mu_1 - \mu_2 + \rho$. From Lemma 3.7 we can write

$$\nu_n = f_n + F_{1,n} - F_{2,n} + G_{1,n} - G_{2,n} + \mu_{1,n,s} - \mu_{2,n,s} + \rho_n$$

and

$$\nu = f + F_1 - F_2 + G_1 - G_2 + \mu_{1,s} - \mu_{2,s} + \rho,$$

and the convergence properties listed in the lemma hold. Therefore we can apply Theorem 3.3 and the conclusion follows. \square

In the next result we prove the main pointwise estimates on renormalized solutions.

Theorem 3.8 *Let Ω be a bounded domain of \mathbb{R}^N . Then there exists a constant c_{22} , dependent on p and Ω such that if $\mu \in \mathfrak{M}^b(\Omega)$ and u is a renormalized solution of problem (3.2) there holds*

$$-c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\mu^-] \leq u(x) \leq c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\mu^+] \quad \text{a.e. in } \Omega. \quad (3.11)$$

Proof. We claim there exists renormalized solutions u_1 and u_2 of problem (3.2) with respective data μ^+ and μ^- such that

$$-u_2 \leq u \leq u_1 \quad \text{a.e. in } \Omega. \quad (3.12)$$

We use the decomposition $\mu = \mu^+ - \mu^- = (\mu_0^+ - \mu_s^+) - (\mu_0^- - \mu_s^-)$. We put $u_k = T_k(u)$, $\mu_k = \mathbf{1}_{\{|u| < k\}} \mu_0 + \lambda_k^+ - \lambda_k^-$, $v_k = \mathbf{1}_{\{|u| < k\}} \mu_0^+ + \lambda_k^+$. Since $\mu_k \in \mathfrak{M}_0(\Omega)$, problem (3.2) with data μ_k admits a unique renormalized solution (see [7]), and clearly u_k is such a solution. Since $v_k \in \mathfrak{M}_0(\Omega)$, problem (3.2) with data v_k admits a unique solution $u_{k,1}$ which is furthermore nonnegative and dominates u_k a.e. in Ω . From Corollary 3.5, $\{u_{k,1}\}$ converges a.e. in Ω to a renormalized solution u_1 of (3.2) with data μ^+ and $u \leq u_1$. Similarly $-u \leq u_2$ where u_2 is a renormalized solution of (3.2) with μ^- . Finally, from [17, Th 6.9] there is a positive constant c_{22} dependent only on p and Ω such that

$$u_1(x) \leq c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\mu^+] \quad \text{and} \quad u_2(x) \leq c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\mu^-] \quad \text{a.e. in } \Omega. \quad (3.13)$$

This implies the claim. \square

4 Equations with absorption terms

4.1 The general case

Let $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be a Caratheodory function such that the map $s \mapsto g(x, s)$ is nondecreasing and odd for almost all $x \in \Omega$. If U is a function defined in Ω we define the function $g \circ U$ in Ω by

$$g \circ U(x) = g(x, U(x)) \quad \text{for almost all } x \in \Omega.$$

We consider the problem

$$\begin{aligned} -\Delta_p u + g \circ u &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega. \end{aligned} \quad (4.14)$$

where $\mu \in \mathfrak{M}^b(\Omega)$. We say that u is a *renormalized solution* of problem (4.14) if $g \circ u \in L^1(\Omega)$ and u is a renormalized solution of

$$\begin{aligned} -\Delta_p u &= \mu - g \circ u && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega. \end{aligned} \quad (4.15)$$

Theorem 4.1 *Let $\mu_i \in \mathfrak{M}_+^b(\Omega)$, $i = 1, 2$, such that there exist nondecreasing sequences $\{\mu_{i,n}\} \subset \mathfrak{M}_+^b(\Omega)$, with compact support in Ω , converging to μ_i and $g \circ (c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\mu_{i,n}]) \in L^1(\Omega)$ with the same constant c_{22} as in Theorem 3.8. Then there exists a renormalized solution of*

$$\begin{aligned} -\Delta_p u + g \circ u &= \mu_1 - \mu_2 && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega, \end{aligned} \quad (4.16)$$

such that

$$-c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\mu_2] \leq u(x) \leq c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\mu_1] \quad \text{a.e. in } \Omega. \quad (4.17)$$

Lemma 4.2 *Assume g belongs to $L^\infty(\Omega \times \mathbb{R})$, besides the assumptions of Theorem 4.1. Let $\lambda_i \in \mathfrak{M}_+^b(\Omega)$ ($i = 1, 2$), with compact support in Ω . Then there exists renormalized solutions u, u_i, v_i ($i = 1, 2$) to problems*

$$\begin{aligned} -\Delta_p u + g \circ u &= \lambda_1 - \lambda_2 && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega, \end{aligned} \quad (4.18)$$

$$\begin{aligned} -\Delta_p u_i + g \circ u_i &= \lambda_i && \text{in } \Omega \\ u_i &= 0 && \text{in } \partial\Omega, \end{aligned} \quad (4.19)$$

$$\begin{aligned} -\Delta_p v_i &= \lambda_i && \text{in } \Omega \\ v_i &= 0 && \text{in } \partial\Omega, \end{aligned} \quad (4.20)$$

such that

$$\begin{aligned} -c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } (\Omega)}[\lambda_2](x) &\leq -v_2(x) \leq -u_2(x) \leq u(x) \\ &\leq u_1(x) \leq v_1(x) \leq c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } (\Omega)}[\lambda_1](x) \end{aligned} \quad (4.21)$$

for almost all $x \in \Omega$.

Proof. Let $\{\rho_n\}$ be a sequence of mollifiers, $\lambda_{i,n} = \rho_n * \lambda_i$, ($i = 1, 2$) and $\lambda_n = \lambda_{1,n} - \lambda_{2,n}$. Then $\lambda_{1,n}$, $\lambda_{2,n}$ and λ_n are bounded with compact support in Ω (for $n \geq n_0$) and by minimization there exist unique solutions in $W_0^{1,p}(\Omega)$ to problems

$$\begin{aligned} -\Delta_p u_n + g \circ u_n &= \lambda_n && \text{in } \Omega \\ u_n &= 0 && \text{in } \partial\Omega, \end{aligned}$$

$$\begin{aligned} -\Delta_p u_{i,n} + g \circ u_{i,n} &= \lambda_{i,n} && \text{in } \Omega \\ u_{i,n} &= 0 && \text{in } \partial\Omega, \end{aligned}$$

$$\begin{aligned} -\Delta_p v_{i,n} &= \lambda_{i,n} && \text{in } \Omega \\ v_{i,n} &= 0 && \text{in } \partial\Omega, \end{aligned}$$

and by the maximum principle, they satisfy

$$-v_{2,n}(x) \leq -u_{2,n}(x) \leq u_n(x) \leq u_{1,n}(x) \leq v_{1,n}(x), \quad \forall x \in \Omega, \quad \forall n \geq n_0. \quad (4.22)$$

Since the λ_i are bounded measure and $g \in L^\infty(\Omega) \times \mathbb{R}$ the the sequences of measures $\{\lambda_{1,n} - \lambda_{2,n} - g \circ u_n\}$, $\{\lambda_{i,n} - g \circ u_{i,n}\}$ and $\{\lambda_{i,n}\}$ are uniformly bounded in $\mathfrak{M}^b(\Omega)$. Thus, by Theorem 3.2 there exists a subsequence, still denoted by the index n such that $\{u_n\}$, $\{u_{i,n}\}$, $\{v_{i,n}\}$ converge a.e. in Ω to functions $\{u\}$, $\{u_i\}$, $\{v_i\}$ ($i = 1, 2$) when $n \rightarrow \infty$. Furthermore $g \circ u_n$ and $g \circ u_{i,n}$ converge in $L^1(\Omega)$ to $g \circ u$ and $g \circ u_i$ respectively. By Corollary 3.4, we can assume that $\{u\}$, $\{u_i\}$, $\{v_i\}$ are renormalized solutions of (4.18)-(4.20), and by Theorem 3.8, $v_i(x) \leq c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\lambda_i]$, a.e. in Ω . Thus we get (4.21). \square

Lemma 4.3 *Let g satisfy the assumptions of Theorem 4.1 and let $\lambda_i \in \mathfrak{M}_+^b(\Omega)$ ($i = 1, 2$), with compact support in Ω such that $g \circ (c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\lambda_i]) \in L^1(\Omega)$, where c_{22} is the constant of Theorem 4.1. Then there exist renormalized solutions u , u_i of the problems (4.18)-(4.19) such that*

$$-c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\lambda_2](x) \leq -u_2(x) \leq u(x) \leq u_1(x) \leq c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\lambda_1](x) \quad (4.23)$$

for almost all $x \in \Omega$. Furthermore, if ω_i , θ_i have the same properties as the λ_i and satisfy $\omega_i \leq \lambda_i \leq \theta_i$, one can find solutions u_{ω_i} and u_{θ_i} of problems (4.19) with right-hand respective side ω_i and θ_i , such that $u_{\omega_i} \leq u_i \leq u_{\theta_i}$.

Proof. From Lemma 4.2 There exist renormalized solutions u_n , $u_{i,n}$ to problems

$$\begin{aligned} -\Delta_p u_n + T_n(g \circ u_n) &= \lambda_1 - \lambda_2 && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta_p u_{i,n} + T_n(g \circ u_{i,n}) &= \lambda_i && \text{in } \Omega \\ u_{i,n} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

$i = 1, 2$, and they satisfy

$$-c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\lambda_2](x) \leq -u_{2,n}(x) \leq u_n(x) \leq u_{1,n}(x) \leq c_{22} \mathbf{W}_{1,p}^{2 \text{ diam } \Omega}[\lambda_1](x). \quad (4.24)$$

As in Lemma 4.2 one can choose a subsequence, still denoted by the index n such that $\{u_n, u_{1,n}, u_{2,n}\}$ converges a.e. in Ω to $\{u, u_1, u_2\}$ for which (4.24) is satisfied a.e. in Ω . Since $g \circ (c_{22} \mathbf{W}_{1,p}^{2 \text{ diam}(\Omega)}[\lambda_i]) \in L^1(\Omega)$ we derive from (4.24) and the dominated convergence theorem that $T_n(g \circ u_n) \rightarrow g \circ u$ and $T_n(g \circ u_{i,n}) \rightarrow g \circ u_i$ in $L^1(\Omega)$. It follows from Theorem 3.3 that u and u_i are respective solutions of (4.18), (4.19). The last statement follows from the same assertion in Lemma 4.2. \square

Proof of Theorem 4.1. From Lemma 4.3, there exist renormalized solutions $u_n, u_{i,n}$ to problems

$$\begin{aligned} -\Delta_p u_n + g \circ u_n &= \mu_{1,n} - \mu_{2,n} && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta_p u_{i,n} + g \circ u_{i,n} &= \mu_{i,n} && \text{in } \Omega \\ u_{i,n} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

$i = 1, 2$ such that $\{u_{i,n}\}$ is nonnegative and increasing and they satisfy

$$-c_{22} \mathbf{W}_{1,p}^{2 \text{ diam}(\Omega)}[\mu_2](x) \leq -u_{2,n}(x) \leq u_n(x) \leq u_{1,n}(x) \leq c_{22} \mathbf{W}_{1,p}^{2 \text{ diam}(\Omega)}[\mu_1](x) \quad (4.25)$$

a.e. in Ω . As in the proof of Lemma 4.3, up to the same subsequence, $\{u_{1,n}\}, \{u_{2,n}\}$ and $\{u_n\}$ converge to u_1, u_2 and u a.e. in Ω . Since $g \circ u_{i,n}$ are increasing, positive and $\int_{\Omega} g \circ u_{i,n} dx \leq \mu_{i,n}(\Omega) \leq \mu_i(\Omega)$, it follows from the monotone convergence theorem that $\{g \circ u_{i,n}\}$ converges to $g \circ u_i$ in $L^1(\Omega)$. Finally, since $|g \circ u_n| \leq g \circ u_1 + g \circ u_2$, $\{g \circ u_n\}$ converges to $g \circ u$ in $L^1(\Omega)$ by dominated convergence. Applying Corollary 3.5 we conclude that u is a renormalized solution of (4.16) and that (4.17) holds. \square

4.2 Proofs of Theorem 1.1 and Theorem 1.2

We are now in situation of proving the two theorems stated in the introduction.

Proof of Theorem 1.1. **1-** Since μ is absolutely continuous with respect to the capacity $C_{p, \frac{Nq}{Nq-(p-1)(N-\beta)}, \frac{q}{q+1-p}}$, μ^+ and μ^- share this property. By Theorem 2.6 there exist two increasing sequences $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ of positive bounded measures with compact support in Ω which converge to μ^+ and μ^- respectively and which have the property that $\mathbf{W}_{1,p}^R[\mu_{i,n}] \in L^{\frac{Nq}{N-\beta}, q}(\mathbb{R}^N)$, for $i = 1, 2$ and all $n \in \mathbb{N}$. Furthermore, with $R = \text{diam}(\Omega)$,

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{1}{|x|^\beta} (\mathbf{W}_{1,p}^{2R}[\mu_{i,n}](x))^q dx &\leq \int_0^\infty \left(\frac{1}{|\cdot|^\beta}\right)^*(t) \left((\mathbf{W}_{1,p}^{2R}[\mu_{i,n}])^*(t)\right)^q dt \\ &\leq c \int_0^\infty \frac{1}{t^{\frac{\beta}{N}}} \left((\mathbf{W}_{1,p}^{2R}[\mu_{i,n}])^*(t)\right)^q dt \\ &\leq c \|\mathbf{W}_{1,p}^{2R}[\mu_{i,n}]\|_{L^{\frac{Nq}{N-\beta}, q}(\mathbb{R}^N)} \\ &< \infty. \end{aligned} \quad (4.26)$$

Then the result follows from Theorem 4.1.

2- Because μ is absolutely continuous with respect to the capacity $C_{p, \frac{Nq}{Nq-(p-1)(N-\beta)}, 1}$, so are μ^+ and μ^- . Applying again Theorem 2.6 there exist two increasing sequences $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ of positive bounded measures with compact support in Ω which converge to μ^+ and μ^- respectively and such that $\mathbf{W}_{1,p}^{2R}[\mu_{i,n}] \in L^{\frac{Nq}{N-\beta}, 1}(\mathbb{R}^N)$. This implies in particular

$$(\mathbf{W}_{1,p}^{2R}[\mu_{i,n}] (\cdot))^* (t) \leq ct^{-\frac{N-\beta}{Nq}}, \quad \forall t > 0, \quad (4.27)$$

for some $c > 0$. Therefore, by Theorem 2.3,

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{1}{|x|^\beta} g(c_{22} \mathbf{W}_{1,p}^{2R}[\mu_{i,n}](x)) dx &\leq \int_0^{|\Omega|} \left(\frac{1}{|\cdot|^\beta}\right)^* (t) g\left(c_{22} (\mathbf{W}_{1,p}^{2R}[\mu_{i,n}])^* (t)\right) dt \\ &\leq \int_0^{|\Omega|} \frac{1}{t^{\frac{\beta}{N}}} g\left(c_{22} (\mathbf{W}_{1,p}^{2R}[\mu_{i,n}])^* (t)\right) dt \\ &\leq \int_0^{|\Omega|} \frac{1}{t^{\frac{\beta}{N}}} g\left(cc_{22} t^{-\frac{N-\beta}{Nq}}\right) dt \\ &\leq \int_a^\infty g(t) t^{-q-1} dt \\ &< \infty, \end{aligned} \quad (4.28)$$

where $a > 0$ depends on $|\Omega|, cc_{22}, N, \beta, q$. Thus the result follows by Theorem 4.1. \square

Proof of Theorem 1.2. Again we take $R = \text{diam}(\Omega)$. We define $\mu_{i,n} = T_n(\mathbf{1}_{\Omega_n} f_i) + \mathbf{1}_{\Omega_n} \nu_i$ ($i = 1, 2$). Then $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ are increasing sequences of elements of $\mathfrak{M}_+^b(\Omega)$ with compact support, and they converge to μ^+ and μ^- respectively. Since for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$(\mathbf{W}_{1,p}^{2R}[T_n(\mathbf{1}_{\Omega_n} f_i) + \mathbf{1}_{\Omega_n} \nu_i])^\lambda \leq C_\epsilon n^{\frac{\lambda}{p-1}} + (1 + \epsilon) (\mathbf{W}_{1,p}^{2R}[\nu_i])^\lambda, \quad (4.29)$$

a.e. in Ω , it follows

$$\exp\left(\tau (\mathbf{W}_{1,p}^{2R}[T_n(\mathbf{1}_{\Omega_n} f_i) + \mathbf{1}_{\Omega_n} \nu_i])^\lambda\right) \leq C_{\epsilon,n} \exp\left(\tau(1 + \epsilon) (\mathbf{W}_{1,p}^{2R}[\nu_i])^\lambda\right). \quad (4.30)$$

If there holds

$$\left\| \mathbf{M}_{p, 2R}^{\frac{(p-1)(\lambda-1)}{\lambda}}[\nu_i] \right\|_{L^\infty(\Omega)} < \left(\frac{p \ln 2}{\tau(18\lambda K)^\lambda} \right)^{\frac{p-1}{\lambda}}, \quad (4.31)$$

we can choose $\epsilon > 0$ small enough so that

$$\tau(1 + \epsilon) K^\lambda \leq \frac{\alpha p \ln 2}{(12\lambda)^\lambda \left\| \mathbf{M}_{p, 2R}^{\frac{(p-1)(\lambda-1)}{\lambda}}[\nu_i] \right\|_{L^\infty(\Omega)}}.$$

Hence, by Theorem 2.4 with $\eta = \frac{(p-1)(\lambda-1)}{\lambda}$, $\exp\left(\tau(1 + \epsilon) (K \mathbf{W}_{1,p}^{2R}[\nu_i])^\lambda\right) \in L^1(\Omega)$, which implies

$$\exp\left(\tau (K \mathbf{W}_{1,p}^{2R}[T_n(\mathbf{1}_{\Omega_n} f_i) + \mathbf{1}_{\Omega_n} \nu_i])^\lambda\right) = \exp\left(\tau (K \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\mu_{i,n}])^\lambda\right) \in L^1(\Omega).$$

We conclude by Theorem 4.1. □

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