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THE GAUSS-BONNET OPERATOR OF AN INFINITE GRAPH

COLETTE ANNÉ AND NABILA TORKI-HAMZA

ABSTRACT. — We propose a general condition for self-adjointness of the Gauß-Bonnet operator $D = d + \delta$ based on the notion of negligible boundary introduced by Gaffney. This gives self-adjointness of the Laplace operator both for functions or 1-forms on infinite graphs. This is used to extend Flanders result concerning solutions of Kirchhoff laws.

RÉSUMÉ. Nous proposons une condition générale qui assure le caractère auto-adjoint de l'opérateur de Gauss-Bonnet $D = d + \delta$, basée sur la notion de bord négligeable introduite par Gaffney. Comme conséquence, l'opérateur de Laplace agissant sur les fonctions ou les 1-formes de graphes infinis. Nous utilisons ce cadre pour étendre le résultat de Flanders à propos des solutions des lois de Kirchhoff.

1. INTRODUCTION

Operators on infinite graphs are of large interest and a lot of recent works deals with this subject. One approach can be to study how technics of spectral geometry can be extended on graphs regarded as simplicial complex of dimension one. We refer to Dodziuk [D84, DK87] for general presentation of this approach and to [CdV98, CTT11] for the geometric point of view.

We consider here only connected locally finite graphs and study Kirchhoff laws. Flanders has first studied this question on infinite graphs, see [F71]. This question can be reduced to the study of a Dirac type operator : the Gauß-Bonnet operator $D = d + \delta$. We give a general condition on the graph for this operator to be self-adjoint: the graph has to be complete homogeneous (definitions are given in section 3.3). This condition covers the situation of [M09] and [CTT11].

Then, in this case, we can formulate a theorem in the same way as Anghel for non-compact Riemannian manifolds in [A93]: when the operator D is *positive at infinity* (see section 4.2 for the definition) then its image is closed. This gives new examples of infinite graphs on which Flanders problem admits a unique solution.

2. PRELIMINARIES

2.1. Definitions on Graphs. A graph K is a simplicial complex of dimension one. We denote by \mathcal{V} the set of vertices and \mathcal{E} the set of *oriented edges*, considered as a subset of $\mathcal{V} \times \mathcal{V}$. We assume that \mathcal{E} is symmetric without loops :

$$v \in \mathcal{V} \Rightarrow (v, v) \notin \mathcal{E}, \quad (v_1, v_2) \in \mathcal{E} \Rightarrow (v_2, v_1) \in \mathcal{E}.$$

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Choosing an orientation consists of defining a partition of \mathcal{E} :

$$\begin{aligned}\mathcal{E}^+ \sqcup \mathcal{E}^- &= \mathcal{E} \\ (v_1, v_2) \in \mathcal{E}^+ &\iff (v_2, v_1) \in \mathcal{E}^-.\end{aligned}$$

For $e = (v_1, v_2) \in \mathcal{E}$, we define

$$e^+ = v_2, e^- = v_1, -e = (v_2, v_1).$$

2.1.1. A *path* between two vertices x, y in \mathcal{V} is a finite set of edges $e_1, \dots, e_n, n \geq 1$ such that

$$e_1^- = x, e_n^+ = y \text{ and, if } n \geq 2, \forall j, 1 \leq j \leq (n-1) \Rightarrow e_j^+ = e_{j+1}^-.$$

Notice that each path has a beginning and an end, and that an edge is a path.

Let's denote Γ_{xy} the set of the paths from the vertex x to the vertex y .

2.1.2. The graph is *connected* if two vertices are always related by a path, *ie.* if Γ_{xy} is non empty for all x, y in \mathcal{V} .

2.1.3. The graph is *locally finite* if each vertex belongs to a finite number of edges. The *degree* or *valence* of a vertex $x \in \mathcal{V}$ is the cardinal of the set $\{e \in \mathcal{E}; e^+ = x\}$.

Remark 1. *All the graphs we shall consider on the sequel will be connected, locally finite, so with countably many vertices.*

2.2. **Functions and forms.** The 0-cochains are just real functions on \mathcal{V} , we denote their set $C^0(K)$.

The 1-cochains or forms are odd real functions on \mathcal{E} we denote their set $C^1(K)$.

Thus we have

$$\begin{aligned}C^0(K) &= \mathbb{R}^{\mathcal{V}}, \\ C^1(K) &= \{\varphi : \mathcal{E} \rightarrow \mathbb{R}, \varphi(-e) = -\varphi(e)\}.\end{aligned}$$

The sets of cochains with finite support are denoted by $C_0^0(K)$, $C_0^1(K)$.

To obtain Hilbert spaces we need weights, let's give

$$c : \mathcal{V} \rightarrow \mathbb{R}_+^*,$$

and

$$r : \mathcal{E} \rightarrow \mathbb{R}_+^* \text{ even}$$

so $r(-e) = r(e)$.

They define scalar products :

$$\begin{aligned}\forall f, g \in C_0^0(K); \langle f, g \rangle &= \sum_{v \in \mathcal{V}} c(v) f(v) g(v) \\ \forall \varphi, \psi \in C_0^1(K); \langle \varphi, \psi \rangle &= \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) \varphi(e) \psi(e)\end{aligned} \tag{1}$$

Remark 2. *As the product $r(e)\varphi(e)\psi(e)$ is even in (1), the term $\frac{1}{2}$ allows to recover the usual definition.*

Let's finally define the Hilbert spaces

$$\begin{aligned} L_2(\mathcal{V}) &= \overline{C_0^0(K)}, \\ L_2(\mathcal{E}) &= \overline{C_0^1(K)}. \end{aligned}$$

2.3. Operators.

2.3.1. *The difference operator.* It is the operator

$$d : C_0^0(K) \rightarrow C_0^1(K),$$

given by

$$d(f)(e) = f(e^+) - f(e^-) \quad (2)$$

2.3.2. *The coboundary operator.* It is δ the formal adjoint of d .

Thus it satisfies

$$\langle df, \varphi \rangle = \langle f, \delta\varphi \rangle \quad (3)$$

for all $f \in C_0^0(K)$ and $\varphi \in C_0^1(K)$.

Lemma 3. *The coboundary operator $\delta : C_0^1(K) \rightarrow C_0^0(K)$, is defined by the formula*

$$\delta(\varphi)(x) = \frac{1}{c(x)} \sum_{e, e^+=x} r(e)\varphi(e). \quad (4)$$

Proof. — Using the equation (3), we have

$$\frac{1}{2} \sum_{e \in \mathcal{E}} r(e) (f(e^+) - f(e^-)) \varphi(e) = \frac{1}{2} \sum_{x \in \mathcal{V}} f(x) \left(\sum_{e^+=x} r(e)\varphi(e) - \sum_{e^-=x} r(e)\varphi(e) \right)$$

But $r\varphi$ is odd and \mathcal{E} symmetric, so

$$\sum_{e^-=x} r(e)\varphi(e) = - \sum_{e^+=x} r(e)\varphi(e).$$

We remark that the sum entering in the formula (4) of δ is finite due to the hypothesis that the graph is locally finite. \square

Remark 4. *The operator d is defined by (2) in all $C^0(K)$, but to define δ in all $C^1(K)$, we need an hypothesis on K : the graph should be locally finite.*

With these two operators we can form the following two operators.

2.3.3. *The Gauß-Bonnet operator.* It is the endomorphism

$$D = d + \delta : C_0^0(K) \oplus C_0^1(K) \curvearrowright.$$

This operator is symmetric and of Dirac type.

2.3.4. *Laplacian.* By definition, it is

$$\Delta = D^2 : C_0^0(K) \oplus C_0^1(K) \curvearrowright.$$

This operator preserves the direct sum $C_0^0(K) \oplus C_0^1(K)$, so we can write

$$\Delta = \Delta_0 + \Delta_1.$$

2.4. **Metrics.** A *metric* is an even function

$$a : \mathcal{E} \rightarrow \mathbb{R}_+^*,$$

it defines a distance on the graph K in the following way.

One first defines the *length of a path* : for $\gamma = (e_1, \dots, e_n)$

$$l_a(\gamma) = \sum_{j=1}^n \sqrt{a(e_j)}.$$

Then the *metric distance* between two vertices x, y is given by

$$d_a(x, y) = \inf_{\gamma \in \Gamma_{xy}} l_a(\gamma).$$

3. CLOSABILITY AND SELF-ADJOINTNESS

3.1. Closability.

Lemma 5. *If the graph K is connected and locally finite the operators d and δ are closable.*

Proof. — Let's suppose that there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $C_0^0(K)$ such that $\|f_n\| \rightarrow 0$ and $(d(f_n))_n$ converges. Let's denote by φ this limit.

We have to show that $\varphi = 0$. If

$$\|f_n\| + \|d(f_n) - \varphi\| \rightarrow 0,$$

then for each vertex v , $f_n(v)$ converges to 0 and for each edge e , $d(f_n)(e)$ converges to $\varphi(e)$ but by the first statement and the expression of d , for each edge e , $d(f_n)(e)$ converges to 0.

The same can be done for δ : convergence in norm to 0 of a sequence $(\varphi_n)_n$ implies ponctual convergence to 0 which implies pontual convergence of $\delta(\varphi_n)$ to 0 because of locally finiteness of the graph ; if $\delta(\varphi_n)$ converges in norm, it must be to 0. \square

Thus, we can consider different extensions of these operators in the framework of Hilbert spaces (see [RS80]).

The smallest extension is the closure, denoted $\bar{d} = d_{min}$ (resp. $\bar{\delta} = \delta_{min}$ and $\bar{D} = D_{min}$) has the domain

$$\text{Dom}(d_{min}) = \{f \in L_2(\mathcal{V}), \exists (f_n)_{n \in \mathbb{N}}, f_n \in C_0^0(K), \lim_{n \rightarrow \infty} (L_2)f_n = f, \lim_{n \rightarrow \infty} (L_2)d(f_n) \text{ exists} \} \quad (5)$$

for such an f , one puts

$$d_{min}(f) = \lim_{n \rightarrow \infty} d(f_n).$$

The largest is $d_{max} = \delta^*$, the adjoint operator of δ_{min} , (resp. $\delta_{max} = d^*$, the adjoint operator of d_{min} .)

3.2. Essential Self-adjointness.

3.2.1. If $d_{min} = d_{max}$ and $\delta_{min} = \delta_{max}$ then D is essentially self-adjoint. Indeed, D is a direct sum and if $F = (f, \varphi) \in \text{Dom}(D^*)$ then $\varphi \in \text{Dom}(d^*)$ and $f \in \text{Dom}(\delta^*)$ and then, by hypothesis, $\varphi \in \text{Dom}(\delta_{min})$ and $f \in \text{Dom}(d_{min})$, thus $F \in \text{Dom}(\bar{D})$.

3.2.2. If D is essentially self-adjoint, then $\text{Im}(D \pm i)$ is dense and $(\bar{D} \pm i)$ are invertible. This is a result for essentially self-adjoint operators (Corollary of Theorem VIII.3 in [RS80]). By the second property we know that and its

$$\exists C_2 > 0, \forall F \in \text{Dom}(\bar{D}), \|F\|_{L_2} \leq C_2 \|(D \pm i)(F)\|_{L_2}.$$

3.2.3. As a consequence, if D is essentially self-adjoint, Δ is also essentially self-adjoint. Indeed, by the theorem of von Neumann, $(\bar{D})^2 = D^* \bar{D}$ is self-adjoint when $D^* = \bar{D}$ and it is an extension of Δ ; then its domain contains the domain of $\bar{\Delta}$, the closure of Δ . But

$$\text{Dom}(\bar{\Delta}) \subset \text{Dom}((\bar{D})^2) \Rightarrow \text{Dom}((\bar{D})^2) \subset \text{Dom}(\Delta^*).$$

In fact, the converse is also true: let $\Psi \in \text{Dom}(\Delta^*)$, then

$$\exists C_1 > 0, \forall F \in C_0^0(K) \oplus C_0^1(K), |\langle (\Delta + 1)(F), \Psi \rangle| \leq C_1 \|F\|_{L_2}.$$

We shall now use that $\text{Im}(D + i)$ is dense. For all $G = (D + i)(F)$, $F \in C_0^0(K) \oplus C_0^1(K)$,

$$\begin{aligned} |\langle (D - i)G, \Psi \rangle| &= |\langle (\Delta + 1)F, \Psi \rangle| \leq C_1 \|F\|_{L_2} \leq C_1 C_2 \|G\|_{L_2} \\ &\Rightarrow \exists C > 0, \forall G \in \text{Im}(D + i), |\langle D(G), \Psi \rangle| \leq C \|G\|_{L_2}. \end{aligned} \quad (6)$$

Thus, by the Hahn-Banach theorem (and the density of $\text{Im}(D + i)$), this inequality extends

$$\forall F \in \text{Dom}(\bar{D}), |\langle D(F), \Psi \rangle| \leq C \|F\|_{L_2}.$$

This means that $\Psi \in \text{Dom}(\bar{D})$ because \bar{D} is self-adjoint. It is then clear that $D(\Psi) \in \text{Dom}(\bar{D})$:

$$\forall F \in C_0^0(K) \oplus C_0^1(K), |\langle D(F), D(\Psi) \rangle| = |\langle \Delta(F), \Psi \rangle| \leq (C_1 + \|\Psi\|_{L_2}) \|F\|_{L_2}.$$

Finally we have proved

$$\text{Dom}(\Delta^*) \subset \text{Dom}((\bar{D})^2) \Rightarrow \text{Dom}((\bar{D})^2) \subset \text{Dom}(\bar{\Delta})$$

because $\Delta^{**} = \bar{\Delta}$, and then $\bar{\Delta} = (\bar{D})^2$ is self-adjoint, see also [G55].

3.3. Sufficient condition for self-adjointness.

3.3.1. *Geometric hypothesis for the graph K .* We introduce two notions:

Completeness of the graph K means that there exists a growing sequence of finite sets $(B_n)_{n \in \mathbb{N}}$ such that $\mathcal{V} = \dot{\cup} B_n$ and there exist related functions χ_n satisfying the following three conditions

- (i) $\chi_n \in C_0^0(K)$, $0 \leq \chi_n \leq 1$
- (ii) $v \in B_n \Rightarrow \chi_n(v) = 1$
- (iii) $\forall p \in \mathbb{N}, \exists n_o, n \geq n_o \Rightarrow \forall e \in \mathcal{E}$, such that e^+ or $e^- \in B_p$, $d\chi_n(e) = 0$.

Homogeneity means that the following condition is satisfied

$$\exists C > 0, \forall n \in \mathbb{N}, x \in \mathcal{V}, \frac{1}{c(x)} \sum_{e, e^\pm = x} r(e) d\chi_n(e)^2 \leq C. \quad (7)$$

For this type of graphs one has

$$\begin{aligned} \forall f \in L_2(\mathcal{V}), \lim_{n \rightarrow \infty} \langle \chi_n f, f \rangle &= \|f\|^2 \\ \forall \varphi \in L_2(\mathcal{E}), \|\varphi\|^2 &= \lim_{n \rightarrow \infty} \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) \chi_n(e^+) \varphi(e)^2 \\ \text{and } \lim_{n \rightarrow \infty} \sum_{e \in \text{supp}(d\chi_n)} r(e) \varphi(e)^2 &= 0. \end{aligned}$$

This definition seems strange but it covers as well the situation of [CTT11] where it is supposed that the graph is complete for the metric

$$a(e) = \frac{\min(c(e^+), c(e^-))}{r(e)} \quad (8)$$

and with bounded valence :

$$\exists A > 0, \forall v \in \mathcal{V}, \#\{e \in \mathcal{E}, e^+ = v\} \leq A.$$

In this case we take, for $x_0 \in \mathcal{V}$ fixed,

$$B_n = \{y \in \mathcal{V}; d_a(x_0, y) \leq n\}$$

and

$$\chi_n(x) = \min(1, d_a(x, B_{n+1}^C)).$$

Thus

$$|d\chi_n(e)|^2 \leq \min(1, a(e)).$$

And it covers also the situation of [M09] where the hypotheses taken give that

$$\sup_{B_n} \frac{1}{c(x)} \sum_{e, e^\pm = x} r(e) d\chi_n(e)^2 = o(1)$$

for some χ_n satisfying $d\chi_n(e)^2 = O(n^{-2})$.

Proposition 6. *Let K be a connected, locally finite graph. If K is complete and homogeneous, then the operator D is essentially self-adjoint.*

To prove this result, we follow the method of Gaffney [G55] and show that the graph has a *negligible boundary*, ie. satisfies the property

$$\forall f \in \text{Dom}(d_{max}), \varphi \in \text{Dom}(\delta_{max}), \quad \langle df, \varphi \rangle = \langle f, \delta(\varphi) \rangle. \quad (9)$$

3.3.2. *Condition (9) gives self-adjointness of D .* We show that condition (9) implies that the closed operators d and δ have no extension. Indeed suppose, for instance, $\text{Dom}(d_{\min}) \subsetneq \text{Dom}(d_{\max})$, but these two spaces are complete for the norm operator

$$\|f\|_d^2 = \|f\|^2 + \|d(f)\|^2 \quad (\text{issued of a scalar product}).$$

Let $\alpha \in \text{Dom}(d_{\min})^\perp$ for this scalar product. It means that $\alpha \in \text{Dom}(d_{\max})$ and

$$\forall f \in C_0^0(K), \quad \langle \alpha, f \rangle + \langle d(\alpha), d(f) \rangle = 0 \quad (10)$$

$$\Rightarrow d(\alpha) \in \text{Dom}(d^*) \text{ and } d^* \circ d(\alpha) = -\alpha \quad (11)$$

But $d^* = \delta_{\max}$, applying (9) with $\varphi = d(\alpha)$ gives :

$$\|\alpha\|^2 = - \langle \delta \circ d(\alpha), \alpha \rangle = - \langle d(\alpha), d(\alpha) \rangle \leq 0 \Rightarrow \alpha = 0.$$

We have proved that $\text{Dom}(d_{\min}) = \text{Dom}(d_{\max})$.

The proof for δ works on the same way (because $\delta^* = d_{\max}$).

3.3.3. *Proof of Proposition 6.* We need the following formula :

$$\forall f, g \in C^0(K), \forall e \in \mathcal{E}, \quad d(fg)(e) = f(e^+)dg(e) + df(e)g(e^-). \quad (12)$$

We can then calculate

$$\forall f \in \text{Dom}(d_{\max}), \varphi \in \text{Dom}(\delta_{\max})$$

$$\langle f, \delta(\varphi) \rangle = \lim_{n \rightarrow \infty} \langle \chi_n f, \delta(\varphi) \rangle = \lim_{n \rightarrow \infty} \langle d(\chi_n f), \varphi \rangle$$

$$\text{but, by (12)} \quad \langle d(\chi_n f), \varphi \rangle = \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) \chi_n(e^+) df(e) \varphi(e) + \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) f(e^-) d\chi_n(e) \varphi(e).$$

The hypothesis of completeness gives that

$$\lim_{n \rightarrow \infty} \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) \chi_n(e^+) df(e) \varphi(e) = \langle df, \varphi \rangle .$$

For the second term (which must tend to 0), we write

$$\sum_{e \in \mathcal{E}} r(e) f(e^-) d\chi_n(e) \varphi(e) \leq \sqrt{\sum_{e \in \text{supp}(d\chi_n)} r(e) \varphi(e)^2} \sqrt{\sum_{e \in \text{supp}(d\chi_n)} r(e) f(e^-)^2 d\chi_n(e)^2}$$

$$\text{and } \sum_{e \in \text{supp}(d\chi_n)} r(e) f(e^-)^2 d\chi_n(e)^2 = \sum_{x \in \mathcal{V}} f(x)^2 \left(\sum_{e, e^- = x} r(e) d\chi_n(e)^2 \right) \leq C \sum_{x \in \mathcal{V}} c(x) f(x)^2$$

by hypothesis of homogeneity. On the other hand, we know, by completeness, that

$$\lim_{n \rightarrow \infty} \sum_{e \in \text{supp}(d\chi_n)} r(e) \varphi(e)^2 = 0.$$

Thus

$$\lim_{n \rightarrow \infty} \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) f(e^-) d\chi_n(e) \varphi(e) = 0.$$

Remark 7. *The case studied in [T11] namely a complete graph for the metric*

$$a(e) = \frac{\sqrt{c(e^+)c(e^-)}}{r(e)}$$

and with a valence bounded by A can be proved directly with the same kind of calculus. Indeed the condition satisfied now is

$$\exists C > 0, \forall e \in \mathcal{E}, n \in \mathbb{N}, \quad r(e)d\chi_n(e)^2 \leq C\sqrt{c(e^+)c(e^-)}$$

We write

$$\begin{aligned} \sum_{e \in \mathcal{E}} r(e)f(e^-)d\chi_n(e)\varphi(e) &= \frac{1}{2} \sum_{e \in \mathcal{E}} r(e)(f(e^+) + f(e^-))d\chi_n(e)\varphi(e) \\ &\leq \sqrt{\sum_{e \in \text{supp}(d\chi_n)} r(e)\varphi(e)^2} \sqrt{\sum_{e \in \text{supp}(d\chi_n)} r(e)(f(e^+) + f(e^-))^2 d\chi_n(e)^2} \\ &\quad \text{and} \quad \sum_{e \in \text{supp}(d\chi_n)} r(e)(f(e^+) + f(e^-))^2 d\chi_n(e)^2 \\ &= \sum_{e \in \text{supp}(d\chi_n)} r(e) \left((f(e^+) - f(e^-))^2 + 4f(e^+)f(e^-) \right) d\chi_n(e)^2 \\ &= \sum_{e \in \text{supp}(d\chi_n)} r(e)|d(f)(e)|^2 + 4 \sum_{v \in \mathcal{V}} f(x) \sum_{e^+=x} r(e)f(e^-)d\chi_n(e)^2 \end{aligned}$$

the first term tends to 0 by completeness and the second is bounded as follows

$$\begin{aligned} \sum_{x \in \mathcal{V}} f(x) \sum_{e^+=x} r(e)f(e^-)d\chi_n(e)^2 &\leq C \sum_{x \in \mathcal{V}} |f(x)| \sum_{e \in \text{supp } d\chi_n, e^+=x} |f(e^-)|\sqrt{c(e^+)c(e^-)} \\ &\leq AC \sum_{x \in \mathcal{V}, \exists e \in \text{supp } d\chi_n, e^+=x} c(x)f(x)^2 \end{aligned}$$

because, as \mathcal{E} is symmetric, one has

$$\sum_{x \in \mathcal{V}, \exists e \in \text{supp } d\chi_n, e^+=x} c(x)f(x)^2 = \sum_{x \in \mathcal{V}, \exists e \in \text{supp } d\chi_n, e^-=x} c(x)f(x)^2$$

So the second term also tends to 0, because of completeness and bounded valence.

4. FLANDERS THEOREM

In 1971, Flanders published a very nice result [F71] concerning resistive networks. The problem is the following : Let i be a finite current source, *i.e.* an element of $C_0^0(K)$, and E' a finite voltage source, *i.e.* an element of $C_0^1(K)$, is there a resulting current flow, and is it unique? *i.e.* find solutions I of the problem (Kirchhoff's laws)

$$\delta(I) + i = 0, \quad \text{and} \quad \forall Z, \partial Z = 0 \quad \int_Z E' = \int_Z I \quad (13)$$

Here Z is a cycle, *i.e.* a 1-chain (a formal finite sum of oriented edges) with no boundary ($\partial(e) = e^+ - e^-$). Formally $Z = \sum_{e \in \mathcal{E}^+} z_e e$, $z_e \in \mathbb{Z}$ or

$$Z = \frac{1}{2} \sum_{e \in \mathcal{E}} z_e e, \quad z_e \in \mathbb{Z} \quad \text{with} \quad z_e = -z_{-e} \quad \text{and} \quad \partial Z = \sum_{x \in \mathcal{V}} \left(\sum_{e^+=x} z_e \right) x. \quad (14)$$

Flanders studies this problem for an infinite graph with weight $c = 1$ and r is called resistance. He shows that this problem has a unique L_2 -solution if i has zero mean value $\sum_{v \in \mathcal{V}} i(v) = 0$.

In the framework we have introduced, this question is related to the question of the Hodge decomposition. Indeed we have to look for $I = E_0 + I_0$ such that E_0 is the harmonic component of E' and I_0 satisfies $-i = \delta(I_0)$ and $\int_Z I_0 = 0$ on cycles.

4.1. Sketch of proof. The easiest is to fix the periods : in a graph that satisfies the previous conditions which give self-adjointness, the space $\text{Ker } \delta$ is closed in $L_2(\mathcal{E})$, let E_0 be the orthogonal projection of E' on $\text{Ker } \delta$.

Lemma 8. *For any $E \in L_2(\mathcal{E})$ orthogonal to $\text{Ker } \delta$ and any L_2 -cycle Z*

$$\int_Z E = 0.$$

Indeed, a cycle Z defines an element E_Z in $\text{Ker } \delta$ by the formula

$$Z = \sum_{e \in \mathcal{E}^+} z_e e, \quad z_e \in \mathbb{Z} \quad \Rightarrow \quad E_Z = \sum_{e \in \mathcal{E}^+} \frac{z_e}{r_e} e^*.$$

where the cochain e^* is defined by $e^*(e) = 1$ and $e^*(e') = 0$ if $e' \neq \pm e$. Then,

$$\int_Z E = \langle E_Z, E \rangle$$

and we say that *the cycle is L_2 if $E_Z \in L_2(\mathcal{E})$* . Now, if Z is a cycle, then $E_Z \in \text{Ker } \delta$ as a consequence of (14).

As a consequence

$$\forall Z \text{ } L_2\text{-cycle}, \quad \int_Z E' = \int_Z E_0.$$

Now, the existence of I_0 is related of the property of $-i$ to be in the image of Δ . In the case where i has finite support, we can do as follows : let K_0 be a finite connected subgraph of K (vertex of K_0 are vertex of K and edges of K_0 are edges of K). We suppose that the support of i is included in K_0 . Denote by d_0 the difference operator of K_0 . The Laplacian Δ_0 of K_0 is self-adjoint and $\text{Im } \Delta_0 = \text{Ker } \Delta_0^\perp$. Thus, as $\text{Ker } \Delta_0 = \mathbb{R}$ consists of constant functions

$$\langle i, 1 \rangle = 0 \Rightarrow \exists f \in C^0(K_0), \quad -i = \Delta_0(f).$$

Let $\varphi \in C_0^1(K)$ be the prolongation of $d_0 f$ by 0 on the edges that don't belong to K_0 . This form is certainly different from $d f$ but $\delta \varphi = -i$.

We define now I_0 as the orthogonal projection of φ on the orthogonal of $\text{Ker } \delta$, it means that I_0 differs from φ by an element of $\text{Ker } \delta$ and that $I_0 \in \text{Ker } \delta^\perp$. Using the Lemma above, we conclude that :

$$\delta I_0 = -i \text{ and } \forall Z \text{ } L_2\text{-cycle}, \quad \int_Z I_0 = 0.$$

Remark 9. *In the case of Flanders, where E' has finite support, we only take care of finite cycles, but the proof extends to $E' \in L_2(\mathcal{E})$ if we consider only L_2 -cycles. The question is how extend on more general i . It is related to the property of closeness of $\text{Im}(\Delta)$, what we explore below.*

4.2. Anghel's hypothesis. In [A93], N. Anghel shows that a Dirac type operator D defined on a complete manifold is Fredholm if and only if D^2 is *positive at infinity*.

Let's define a subgraph of a graph and the complementary of a subgraph.

Definition 10. A subgraph of a graph K is a graph $K_0 = (\mathcal{V}_0, \mathcal{E}_0)$ such that $\mathcal{V}_0 \subset \mathcal{V}$ and $\mathcal{E}_0 \subset \mathcal{E}$. For such a subgraph we define the complementary graph $K_0^c = (\mathcal{V}^c, \mathcal{E}^c)$ as follows

$$\mathcal{V}^c = \mathcal{V} \setminus \mathcal{V}_0, \quad \mathcal{E}^c = \{e \in \mathcal{E} \setminus \mathcal{E}_0, \partial(e) \subset \mathcal{V}^c\}.$$

Remark 11. (1) In particular boundary points of edges in \mathcal{E}_0 belong to \mathcal{V}_0 .
(2) As a consequence of the definition, \mathcal{E}^c avoids the edges with one end in \mathcal{V}^c and one in \mathcal{V}_0 .

Following [KL10], we define the *boundary* of a subgraph K_0 by

$$\partial(K_0) = \mathcal{E} \setminus (\mathcal{E}_0 \cup \mathcal{E}^c).$$

Definition 12. We say that a Dirac type operator is *positive at infinity* if there exists a finite connected subgraph $K_0 = (\mathcal{V}_0, \mathcal{E}_0)$ of K such that

$$\exists C > 0, \quad \forall (f, \varphi) \in L_2(\mathcal{V}^c) \times L_2(\mathcal{E}^c) \cap \text{Dom}(D), \quad \|(f, \varphi)\| \leq C \|D(f, \varphi)\|. \quad (15)$$

(Remark that this definition give rather positivity of Δ .)

Proposition 13. If the graph (connected and locally finite) is complete and homogeneous and if its Gauß-Bonnet operator

$$D = d + \delta$$

(which is essentially self-adjoint) satisfies the condition (15), then $\text{Im}(D)$ is closed and, as a consequence, holds the Hodge property :

$$L_2(\mathcal{E}) = \text{Ker } \delta \oplus \text{Im}(d), \quad L_2(\mathcal{V}) = \text{Ker } d \oplus \text{Im}(\delta). \quad (16)$$

Proof. — The condition (15) implies that the closed restriction operator D^c of D on K_0^c :

$$D^c : \text{Dom}(D^c) \subset L_2(\mathcal{V}^c) \times L_2(\mathcal{E}^c) \rightarrow L_2(\mathcal{V}^c) \times L_2(\mathcal{E}^c)$$

is continuous (for the operator norm), injective and with closed image. By the inversion theorem, there exists

$$P : L_2(\mathcal{V}^c) \times L_2(\mathcal{E}^c) \rightarrow \text{Dom}(D^c)$$

such that

$$P \circ D = \mathbb{I} \quad \text{and} \quad \mathbb{I} - D \circ P,$$

is the orthogonal projector on the subspace $\text{Im}(D^c)^\perp$.

Let now $\psi \in \overline{\text{Im}(D)}$. It means :

$$\exists \text{ a sequence } (\sigma_n)_{n \in \mathbb{N}} \text{ in } \text{Dom}(D), \quad \sigma_n \in \text{Ker}(D)^\perp, \quad \text{and} \quad \lim_{n \rightarrow \infty} D(\sigma_n) = \psi.$$

The sequence (σ_n) is bounded. If not, by extraction we can construct

$$\varphi_n = \frac{\sigma_n}{\|\sigma_n\|}$$

a subsequence tending to $+\infty$ which satisfies

$$\|\varphi_n\| = 1, \quad \lim_{n \rightarrow \infty} D(\varphi_n) = 0.$$

Then the restriction of $D(\varphi_n)$ to K_0^c also converge to 0 in $L_2(\mathcal{V}^c) \times L_2(\mathcal{E}^c)$. But the set of vertices not in \mathcal{V}^c and the set of edges not in \mathcal{E}^c are finite. As φ_n is bounded, we can by extraction suppose that all their values in these finite sets converge, and by the same argument we can suppose that the value of φ_n on the vertices which are boundary points of edges in $\partial(K_0)$ converge. By local finiteness we conclude that $D(\varphi_n|_{K_0^c})$ converges.

By (15), then also $\varphi_n|_{K_0^c}$ converges, thus finally φ_n converges, let φ be the limit, it satisfies

$$\|\varphi\| = 1, \quad \varphi \in \text{Ker}(D)^\perp, \quad D(\varphi) = 0 \quad \text{absurd.}$$

So we can suppose that (σ_n) is bounded, then by the same kind of reasoning, we show that $(\sigma_n)_n$ admits a subsequence which converges, let σ be this limit. As D is closed and $D(\sigma_n)$ converges, then $\sigma \in \text{Dom}(D)$ and $D(\sigma) = \psi$. \square

We see that the reasoning is separated for 0-forms and 1-forms. This gives :

Corollary 14. *Let K be a graph (connected and locally finite) complete and homogeneous so its Gauß-Bonnet operator $D = d + \delta$ is essentially self-adjoint. If d satisfies the condition*

$$\exists C > 0, \quad \forall f \in L_2(\mathcal{V}^c) \cap \text{Dom}(d), \quad \|f\| \leq C \|df\| \quad (17)$$

for the complementary of some finite graph, then $\text{Im } d$ is closed and

$$L_2(\mathcal{E}) = \text{Ker } \delta \oplus \text{Im}(d).$$

And there exists a similar statement for δ .

5. EXAMPLES

It is clear that if K possesses infinitely many cycles (as infinite ladders, or infinite grids) it does not work. A family of examples could be a graph with finite geometry: there exists a finite subgraph K_0 such that K_0^c is a (finite) union of disconnected branches.

Proposition 15. *If the connected graph K admits a finite subgraph such that its complementary is a finite union of trees with constant valence larger than 3, then, considered with the weights constant equal to 1 on vertex and edges, it is complete homogeneous and $\text{Im } d$ is closed.*

Proof. — We will prove that d is positive at infinity, *ie.* on each tree. Let U be a tree with a base point and valence $p + 1$, $p \geq 2$. We apply Corollary 17 of [KL10], taking the notations of this paper: in our case $D_U = p + 1$ is finite, so it suffices to show that the isoperimetric constant α_U is positive.

Recall that

$$\alpha_U = \inf_{W \subset U, \text{finite}} \frac{\#(\partial W)}{\#W}.$$

For a tree, one has a notion of *height*: the base point is of height 0, and for another point its height is the necessary number of edges to join it to the base point.

Let W be a finite set of vertices of U , we shall show by recurrence on $\#W$ that

$$\#(\partial W) \geq \#W.$$

If $\#W = 1$, then $\#(\partial W) = p + 1$. If $\#W = n \geq 1$, let $x \in W$ be a point of highest height in W and y is the point just below. Then define $W' = W - \{x\}$ so $\#W' = \#W - 1$ and

$$y \in W \Rightarrow \#(\partial W) = p - 1 + \#(\partial W')$$

$$y \notin W \Rightarrow \#(\partial W) = p + 1 + \#(\partial W')$$

In all cases, applying the recurrence hypothesis, we get:

$$\#(\partial W) \geq p - 1 + \#(\partial W') \geq p - 1 + \#W - 1 \geq \#W.$$

□

Corollary 16. *Such a graph (as in the proposition 15) satisfies also that $\text{Im } \delta$ is closed and $\text{Ker } d = \{0\}$ (because constants are not in L_2) so δ is surjective.*

As a consequence, for such a graph Flanders problem (13) has always a unique solution.

Proof. — Indeed, if (17) is satisfied, then

$$\forall f \in \text{Dom}(\Delta^c) \subset L_2(\mathcal{V}^c), \quad \|f\| \leq C^2 \|\Delta(f)\|. \quad (18)$$

Thus, by the same reasoning as before the range of Δ acting on functions is closed. Now if $(\varphi_n)_n$ is a sequence of 1-forms such that $\delta(\varphi_n)$ converges, we can apply the Hodge decomposition 16 at φ_n , because of the Proposition 15:

$$\exists f_n \in \text{Dom}(d) \text{ such that } \delta \circ d(f_n) \in L_2(\mathcal{V}) \text{ and converges.}$$

But we can extract a subsequence of $(f_n)_n$ which converges, because of (18). □

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