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Abstract

In this paper we prove the discrete compactness property for a discontinuous Galerkin approximation of Maxwell's system on quite general tetrahedral meshes. As a consequence, a discrete Friedrichs inequality is obtained and the convergence of the discrete eigenvalues to the continuous ones is deduced using the theory of collectively compact operators. Some numerical experiments confirm the theoretical predictions.

Keywords: DG method, Maxwell's system, discrete compactness, eigenvalue approximation.

AMS Subject Classification: 65N25, 65N30.

1 Introduction

In this paper we deal with the discrete compactness property for a discontinuous Galerkin approximation of Maxwell's system on quite general tetrahedral meshes. This property has been studied for standard Galerkin approximation for a quite large family of edge elements on two and three dimensional

domains [21, 7, 9, 26, 15]. But to our knowledge this property is not yet proved for the discontinuous Galerkin method. We here concentrate on the interior penalty method introduced in [19] (see also [24] for two-dimensional domains).

The success of DG methods is today well recognized and is mainly due to its flexibility in the choice of the approximation space, since it allows the use of meshes with hanging nodes and local spaces of different orders. This renders this method well suited for $h-p$ adaptivity. Furthermore the implementation of DG methods is quite easy by using elementwise shape functions, a great advantage for elements of high order, such a property is not satisfied for standard edge elements.

Our proof of the discrete compactness property is based on the same property for the standard Galerkin approximation proved in [26] and the use of a decomposition of the discontinuous approximation space into a continuous one and its orthogonal for an appropriate inner product similar to [19, 20] (but different from the one used in these references). The discrete Friedrichs inequality follows from this discrete compactness property and a contradiction argument. The convergence of the discrete eigenvalues to the continuous ones is deduced using the theory of collectively compact operators [1], which requires pointwise convergence of the sequence of the discrete operators (see [26] for the use of this approach for the standard Galerkin approximation of Maxwell's system). In our case, the collectively compact property is deduced from the discrete compactness property and the pointwise convergence is obtained by introducing mixed formulations and using a variant of the second Strang lemma.

Let us notice that we restrict ourselves to the h-version of the method, even if we consider a discontinuous Galerkin method for polynomials of order less than k, for any positive integer k, we do not estimate the dependence of the constant with respect to k. The extension of our analysis to the so-called $h-p$ version needs more investigations (see [8] for rectangular meshes). The convergence analysis of the DG method for the source problem by using the results of this paper also requires some further analysis.

For the numerical experiments, since the null space of the operator is relatively large, we have used a discrete regularization method that allows us to work in the setting of positive definite matrices (see [3, 18] for the standard edge elements).

The paper is organized as follows. Section 2 introduces the problem and

some notation. The discretization and the discontinuous Galerkin method are given in Section 3. Section 4 contains the main results of the paper, namely the proof of the discrete compactness property and of the discrete Friedrichs inequality. The discrete eigenvalue problem is presented in section 5. In section 6, we first recall some results about collectively compact operators and then used them to deduce the convergence of the eigenvalues. Finally some numerical experiments are presented in section 7, that confirm the theoretical predictions. The obtained numerical eigenvalues are furthermore comparable with the ones given in the existing literature.

2 Preliminaries and notation

Let us fix a bounded domain Ω of $\mathbb{R}^3$ with a Lipschitz polyhedral boundary Γ. We denote by $\boldsymbol{n}$ the unit outward normal vector along Γ. For the sake of simplicity we assume that Γ is simply connected.

On this domain we consider the following eigenvalue problem: Find an electric field $\boldsymbol{E} \neq 0$ and an electric eigenvalue λ such that

$$\operatorname{curl}\operatorname{curl} \boldsymbol{E} = \lambda \boldsymbol{E} \text{ in } \Omega, \tag{1}$$

$$\operatorname{div} \boldsymbol{E} = 0 \text{ in } \Omega, \tag{2}$$

$$\boldsymbol{E} \times \boldsymbol{n} = 0 \text{ on } \Gamma. \tag{3}$$

This eigenvalue problem has been studied extensively in the literature, see e. g. [23, 25]. It is well known that under the above assumptions, the electric eigenvalues are real and positive and that the set of eigenvalues is discrete. Moreover if λ is an electric eigenvalue, then the eigenspace is finite dimensional.

For further purposes, we recall the variational formulation of this problem. It requires the use of the following (real) function spaces:

$$\begin{aligned} \mathcal{H}_0(\operatorname{curl};\Omega) &= \{\boldsymbol{u} \in L^2(\Omega)^3 : \operatorname{curl} \boldsymbol{u} \in L^2(\Omega)^3 \text{ and } \boldsymbol{u} \times \boldsymbol{n} = 0 \text{ on } \Gamma\}, \\ X &= \{\boldsymbol{u} \in \mathcal{H}_0(\operatorname{curl};\Omega) : \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}, \end{aligned}$$

both spaces being equipped with the norm

$$\|\boldsymbol{u}\|_{\mathcal{H}_0(\operatorname{curl};\Omega)} = (\|\boldsymbol{u}\|^2 + \|\operatorname{curl} \boldsymbol{u}\|^2)^{1/2},$$

where here and below $\|\cdot\|$ means the $L^2(\Omega)^3$ (or $L^2(\Omega)$ according to the context) norm. Furthermore for any $\boldsymbol{u}, \boldsymbol{v} \in L^2(\Omega)^3$, we will denote

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v},$$

their standard inner product.

In view of (2), the weak formulation of the electric eigenvalue problem is to find $\boldsymbol{E} \in X$, $\boldsymbol{E} \neq 0$ and a real number λ such that

$$(\operatorname{curl} \boldsymbol{E}, \operatorname{curl} \boldsymbol{v}) = \lambda(\boldsymbol{E}, \boldsymbol{v}), \forall \boldsymbol{v} \in X. \tag{4}$$

Let us reformulate this problem in a more operator form. More precisely, introduce the operator

$$A : L^2(\Omega)^3 \to X : \boldsymbol{u} \to A\boldsymbol{u},$$

where $A\boldsymbol{u}$ is the unique element in X that satisfies

$$(\operatorname{curl} A\boldsymbol{u}, \operatorname{curl} \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}), \forall \boldsymbol{v} \in X. \tag{5}$$

This problem is meaningful since the bilinear form $a(\boldsymbol{u}, \boldsymbol{v}) = (\operatorname{curl} \boldsymbol{u}, \operatorname{curl} \boldsymbol{v})$ is coercive on X [22, 25] and therefore problem (5) has a unique solution by the Lax-Milgram Lemma.

Using the operator A, the eigenvalue problem (4) is equivalent to the problem of finding $\boldsymbol{E} \in X$, $\boldsymbol{E} \neq 0$ such that

$$A\boldsymbol{E} = \mu \boldsymbol{E}, \tag{6}$$

with $\mu = \frac{1}{\lambda}$.

Let us finish this introduction with some notation used in the whole paper: For shortness the $L^2(D)^3$-norm will be denoted by $\|\cdot\|_D$. In the case $D = \Omega$, we will drop the index Ω. The usual norm and seminorm of $H^t(D)$ are denoted by $\|\cdot\|_{t,D}$ and $|\cdot|_{t,D}$.

3 Discontinuous Galerkin discretization

In this section, we introduce the interior penalty DG discretization of problem (4). To this end, let us introduce the next notation.

The domain Ω is discretized by a discrete family of conforming meshes $\mathcal{T}_h$, $h \in \Lambda$, made of tetrahedra. The discrete set Λ is equal to $\{h_n : n \in \mathbb{N}\}$, where we suppose that the sequence $(h_n)_{n\in\mathbb{N}}$ is strictly decreasing and satisfies $h_n > 0$, for all n and $h_n \to 0$ as $n \to \infty$. In other words the meshes are built progressively and become finer and finer.

The family is supposed to be regular in Ciarlet's sense [12], i.e. there exists $\sigma > 0$ such that the ratio

$$\frac{h_T}{\rho_T} \leq \sigma, \forall T \in \mathcal{T}_h, h \in \Lambda,$$

where we recall that h_T is the diameter of T and ρ_T is the diameter of the largest ball inscribed into T. As usual $h = \max_{T\in\mathcal{T}_h} h_T$.

As in [26], we further assume that the meshes are weakly quasi-uniform, i.e., if

$$h_{min} = \min_{T\in\mathcal{T}_h} h_T,$$

then there exists $\mu^\star \in (0, 1)$ such that

$$hh_{min}^{-\mu^\star} \to 0 \text{ as } h \to 0.$$

As pointed out in [26], quasi-uniform meshes are weakly quasi-uniform. Moreover graded meshes of Raugel's type used in the presence of corner and edge singularities (see for instance [2]) are also weakly quasi-uniform.

Elements will be denoted by T, its faces are denoted by E. The set of all (interior and boundary) faces of the mesh will be denoted by $\mathcal{E}$. The measure of an element or face is denoted by $|T| := \text{meas}_3(T)$ and $|E| := \text{meas}_2(E)$, respectively. For any face E we will denote by h_E its diameter (which is equivalent to the diameter h_T of any element T containing E by the regularity of the mesh). For each element $T \in \mathcal{T}_h$, denote by $\boldsymbol{n}_T$ the unit outward normal vector along ∂T.

For our further analysis we need to define some jumps and means through any $E \in \mathcal{E}$ of the mesh. For $E \in \mathcal{E}$ such that $E \subset \Omega$, denote by T^+ and T^- the two elements of $\mathcal{T}_h$ containing E. Let $\boldsymbol{v}$ be a vector-valued function defined on $T^+ \cup T^-$, and which is in H^1 inside each element $T^\pm$. We denote by $\boldsymbol{v}^\pm$ the traces of $\boldsymbol{v}$ on E taken from $T^\pm$, respectively. Then we define the mean of $\boldsymbol{v}$ on E by

$$\{\{\boldsymbol{v}\}\} = \frac{\boldsymbol{v}^+ + \boldsymbol{v}^-}{2},$$

while its tangential jump on E is defined as follows:

$$\underline{[\![v]\!]}_T = \boldsymbol{n}_{T^+} \times \boldsymbol{v}^+ + \boldsymbol{n}_{T^-} \times \boldsymbol{v}^-.$$

For a boundary face E, i. e., $E \subset \partial\Omega$, there exists a unique element $T^+ \in \mathcal{T}_h$ such that $E \subset \partial T^+$. Therefore the mean and the tangential jumps are defined as before by taking $\boldsymbol{v}^- = 0$.

Following [19], we consider the following discontinuous Galerkin approximation of the continuous eigenvalue problem: Given a mesh $\mathcal{T}_h$ and a polynomial degree $k \geq 1$, we consider the approximation space

$$V_h = \{v \in L^2(\Omega)^3 : v_{|T} \in \mathbb{P}^k(T)^3, \forall T \in \mathcal{T}_h\},$$

where $\mathbb{P}^k(T)$ denotes the space of polynomials of total degree at most k on T.

On this space, we define the discontinuous curl operator curl_h:

$$(\mathrm{curl}_h\, \boldsymbol{u})_{|T} = \mathrm{curl}(\boldsymbol{u}_{|T}), \forall T \in \mathcal{T}_h,$$

and the bilinear form $a_h(.,.)$:

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) &:= (\mathrm{curl}_h\, \boldsymbol{u}, \mathrm{curl}_h\, \boldsymbol{v}) - \int_{\mathcal{E}} \underline{[\![\boldsymbol{u}]\!]}_T \cdot \{\{\mathrm{curl}_h\, \boldsymbol{v}\}\} \\ &- \int_{\mathcal{E}} \underline{[\![\boldsymbol{v}]\!]}_T \cdot \{\{\mathrm{curl}_h\, \boldsymbol{u}\}\} + \alpha \int_{\mathcal{E}} \mathrm{h}^{-1} \underline{[\![\boldsymbol{u}]\!]}_T \cdot \underline{[\![\boldsymbol{v}]\!]}_T, \end{aligned}$$

where the positive parameter α is the interior penalty stabilization parameter and that will be chosen large enough to ensure coerciveness of the bilinear form a_h (see Lemma 5.1 below). Here we use the notation

$$\int_{\mathcal{E}} \varphi := \sum_{E \in \mathcal{E}} \int_E \varphi(x) ds(x),$$

and h is the function defined on each face $E \in \mathcal{E}$ by

$$\mathrm{h}(x) = h_E, \forall x \in E.$$

The discontinous Galerkin approximation of the eigenvalue problem (4) reads now: Find $\boldsymbol{E}_h \in V_h$, $\boldsymbol{E}_h \neq 0$ and $\lambda_h \in \mathbb{R}$ such that

$$a_h(\boldsymbol{E}_h, \boldsymbol{v}_h) = \lambda_h(\boldsymbol{E}_h, \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in V_h. \tag{7}$$

4 The discrete Friedrichs inequality

Following [19, 20] we consider the space

$$V(h) = \mathcal{H}_0(\mathrm{curl};\Omega) + V_h,$$

that we equipped with the norm

$$\|\boldsymbol{u}\|_{DG,h}^2 = \|\boldsymbol{u}\|^2 + |\boldsymbol{u}|_{DG,h}^2,$$

where the semi-norm is defined by

$$|\boldsymbol{u}|_{DG,h}^2 = \|\,\mathrm{curl}_h\,\boldsymbol{u}\|^2 + \int_{\mathcal{E}} \mathrm{h}^{-1} |\underline{[\![\boldsymbol{u}]\!]_T}|^2.$$

Now we consider $V_h^c = V_h \cap \mathcal{H}_0(\mathrm{curl};\Omega)$ and decompose V_h into

$$V_h = V_h^c \oplus V_h^\perp, \tag{8}$$

where the decomposition is orthogonal with respect to the following inner product

$$(\boldsymbol{u},\boldsymbol{v})_{DG,h\,new} = h^{-2}(\boldsymbol{u},\boldsymbol{v}) + (\mathrm{curl}_h\,\boldsymbol{u},\mathrm{curl}_h\,\boldsymbol{v}) + \int_{\mathcal{E}} \mathrm{h}^{-1} \underline{[\![\boldsymbol{u}]\!]_T} \cdot \underline{[\![\boldsymbol{v}]\!]_T}.$$

In other words, $V_h^\perp$ is the orthogonal complement of V_h^c into V_h for this inner product. Note that our orthogonal decomposition is different from the one from [19, 20] but is motivated by the following result proved in Proposition 4.5 of [19] (or in the Appendix of [20]):

Theorem 4.1 *There exists a positive constant C (independent of h) such that for any $\boldsymbol{v} \in V_h$, there exists $\boldsymbol{v}^c \in V_h^c$ satisfying*

$$\|\boldsymbol{v} - \boldsymbol{v}^c\|_{DG,h} \leq C \left(\int_{\mathcal{E}} \mathrm{h}^{-1} |\underline{[\![\boldsymbol{v}]\!]_T}|^2 \right)^{1/2},$$
$$\|\boldsymbol{v} - \boldsymbol{v}^c\| \leq Ch \left(\int_{\mathcal{E}} \mathrm{h}^{-1} |\underline{[\![\boldsymbol{v}]\!]_T}|^2 \right)^{1/2}.$$

The following approximation property and norm equivalence (compare with Proposition 4.6 from [19] and Theorem 2 of [20]) will be the basis of our analysis below.

Corollary 4.2 *There exists a positive constant C (independent of h) such that for all $\boldsymbol{v} \in V_h$, the orthogonal projection $\pi_h \boldsymbol{v}$ of $\boldsymbol{v}$ on V_h^c wrt to the inner product $(\cdot,\cdot)_{DG,h\,new}$ satisfies*

$$\|\boldsymbol{v} - \pi_h \boldsymbol{v}\|_{DG,h\,new} \leq C \left(\int_{\mathcal{E}} \mathrm{h}^{-1} |\underline{[\![\boldsymbol{v}]\!]_T}|^2) \right)^{1/2},$$

where $\|\cdot\|_{DG,h\,new} = (\cdot,\cdot)^{1/2}_{DG,h\,new}$.

Proof: The requested estimate directly follows from Theorem 4.1 because it shows that there exists $\boldsymbol{v}^c \in V_h^c$ such that

$$\|\boldsymbol{v} - \boldsymbol{v}^c\|_{DG,h\,new} \leq C \left(\int_{\mathcal{E}} \mathrm{h}^{-1} |\underline{[\![\boldsymbol{v}]\!]_T}|^2) \right)^{1/2},$$

and clearly $\|\boldsymbol{v} - \pi_h \boldsymbol{v}\|_{DG,h\,new} \leq \|\boldsymbol{v} - \boldsymbol{v}^c\|_{DG,h\,new}$. ∎

Note that this corollary shows in particular that

$$\|\boldsymbol{v} - \pi_h \boldsymbol{v}\| \leq Ch \left(\int_{\mathcal{E}} \mathrm{h}^{-1} |\underline{[\![\boldsymbol{v}]\!]_T}|^2) \right)^{1/2},$$

which is not the case if we take the orthogonal projection wrt the natural inner product associated with $\|\cdot\|_{DG,h}$.

Corollary 4.3 *There exists a positive constant C (independent of h) such that*

$$C\|\boldsymbol{v}_h\|_{DG,h} \leq \|\boldsymbol{v}_h\|_{V_h^\perp} \leq \|\boldsymbol{v}_h\|_{DG,h}, \forall \boldsymbol{v}_h \in V_h^\perp,$$

where the norm $\|\cdot\|_{V_h^\perp}$ is defined

$$\|\boldsymbol{v}_h\|^2_{V_h^\perp} := \int_{\mathcal{E}} \mathrm{h}^{-1} |\underline{[\![\boldsymbol{v}_h]\!]_T}|^2, \forall \boldsymbol{v}_h \in V_h^\perp.$$

Proof: For $\boldsymbol{v}_h \in V_h^\perp$, its projection $\pi_h \boldsymbol{v}_h = 0$ and by Corollary 4.2, we obtain

$$\|\boldsymbol{v}_h\|_{DG,h\,new} \leq C\|\boldsymbol{v}_h\|_{V_h^\perp},$$

for some $C > 0$. The proof is completed by the simple estimate $\|\boldsymbol{v}_h\|_{DG,h} \leq \max\{1, h_0\}\|\boldsymbol{v}_h\|_{DG,h\,new}$. ∎

Let us finally introduce the spaces:

$$\begin{aligned} S_h &= \{p_h \in H_0^1(\Omega) : p_{h|T} \in \mathbb{P}^{k+1}(T), \forall T \in \mathcal{T}_h\}, \\ X_h &= \{\boldsymbol{v}_h \in V_h : (\boldsymbol{v}_h, \nabla p_h) = 0, \forall p_h \in S_h\}. \end{aligned}$$

The space X_h corresponds to the subspace of V_h of functions which are "discrete divergence free". Note further that

$$\nabla S_h \subset V_h^c. \tag{9}$$

Now we are ready to prove a property which is similar to the standard discrete compactness property.

Theorem 4.4 *Let $(h_n)_{n\in\mathbb{N}} \subset \Lambda^{\mathbb{N}}$ satisfy $h_n \to 0$ as n goes to infinity. Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be a sequence such that $\boldsymbol{u}_n$ belongs to X_{h_n} and satisfying*

$$\|\boldsymbol{u}_n\|_{DG,h_n} \leq C, \forall n \in \mathbb{N}, \tag{10}$$

for some $C > 0$. Then using the orthogonal decomposition

$$\boldsymbol{u}_n = \boldsymbol{u}_n^c + \boldsymbol{u}_n^\perp \tag{11}$$

from (8), with $\boldsymbol{u}_n^c \in V_{h_n}^c$ and $\boldsymbol{u}_n^\perp \in V_{h_n}^\perp$. Then

$$\boldsymbol{u}_n^c \in X_{h_n}^c := \{\boldsymbol{v}_h \in V_{h_n}^c : (\boldsymbol{v}_h, \nabla p_{h_n}) = 0, \forall p_{h_n} \in S_{h_n}\}, \tag{12}$$

and there exists a subsequence, still denoted by $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, and $\boldsymbol{u} \in X$ such that

$$\boldsymbol{u}_n \to \boldsymbol{u} \text{ strongly in } L^2(\Omega)^3 \text{ as } n \to \infty, \tag{13}$$

$$\boldsymbol{u}_n^c \to \boldsymbol{u} \text{ weakly in } \mathcal{H}_0(\mathrm{curl};\Omega) \text{ as } n \to \infty. \tag{14}$$

Proof: As $\boldsymbol{u}_n \in X_{h_n}$, it satisfies

$$(\boldsymbol{u}_n, \nabla q) = 0, \forall q \in S_{h_n},$$

and by (11), we get

$$(\boldsymbol{u}_n^c, \nabla q) + (\boldsymbol{u}_n^\perp, \nabla q) = 0, \forall q \in S_{h_n}.$$

For $q \in S_{h_n}$, by (9) we have $\mathrm{curl}_{h_n} \nabla q = \mathrm{curl}\, \nabla q = 0$ and $\underline{[\![\nabla q]\!]_T} = 0$ on each edge E, therefore

$$h_n^{-2}(\boldsymbol{u}_n^\perp, \nabla q) = (\boldsymbol{u}_n^\perp, \nabla q)_{DG,h_n new} = 0, \forall q \in S_{h_n},$$

again thanks to (9). The two above identities imply that

$$(\boldsymbol{u}_n^c, \nabla q) = 0, \forall q \in S_{h_n},$$

which shows the inclusion (12).

Now the decomposition (11) yields

$$\|\boldsymbol{u}_n^c\|_{DG,h_n} \leq \|\boldsymbol{u}_n\|_{DG,h_n} + \|\boldsymbol{u}_n^\perp\|_{DG,h_n}, \forall n \in \mathbb{N},$$

and by Corollary 4.3, there exists a positive constant C (independent of h_n) such that

$$\|\boldsymbol{u}_n^c\|_{DG,h_n} \leq \|\boldsymbol{u}_n\|_{DG,h_n} + C\|\boldsymbol{u}_n^\perp\|_{V_{h_n}^\perp}, \forall n \in \mathbb{N}.$$

As $\underline{[\![\boldsymbol{u}_n^\perp]\!]_T} = \underline{[\![\boldsymbol{u}_n]\!]_T}$, we deduce that

$$\|\boldsymbol{u}_n^c\|_{DG,h_n} \leq (1+C)\|\boldsymbol{u}_n\|_{DG,h_n}, \forall n \in \mathbb{N}.$$

By (10) we then have

$$\|\boldsymbol{u}_n^c\|_{\mathcal{H}_0(\mathrm{curl};\Omega)} = \|\boldsymbol{u}_n^c\|_{DG,h_n} \leq C, \forall n \in \mathbb{N}.$$

Consequently the sequence $(\boldsymbol{u}_n^c)_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_0(\mathrm{curl};\Omega)$. Hence Theorem 4.1 of [26] (also valid for the second family of edge elements on tetrahedra, see the beginning of section 3 of [26]) yields a subsequence, still denoted by $(\boldsymbol{u}_n^c)_{n\in\mathbb{N}}$, and $\boldsymbol{u} \in X$ satisfying (14) and

$$\boldsymbol{u}_n^c \to \boldsymbol{u} \text{ strongly in } L^2(\Omega)^3 \text{ as } n \to \infty. \tag{15}$$

But according to Corollary 4.2,

$$\|\boldsymbol{u}_n^\perp\| \leq Ch_n\|\boldsymbol{u}_n\|_{DG,h_n} \leq Ch_n,$$

which shows that

$$\boldsymbol{u}_n^\perp \to 0 \text{ strongly in } L^2(\Omega)^3 \text{ as } n \to \infty. \tag{16}$$

The property (13) is proved. ∎

This "discontinuous discrete compactness" result allows us to prove the discrete Friedrichs inequality:

Theorem 4.5 *There exists a positive constant $C > 0$ such that for all $h \in \Lambda$, one has*

$$\|\boldsymbol{u}_h\| \leq C|\boldsymbol{u}_h|_{DG,h}, \forall \boldsymbol{u}_h \in X_h. \tag{17}$$

Proof: Assume that (17) does not hold, then there exists a sequence $\boldsymbol{u}_n \in X_{h_n}, n \in \mathbb{N}$ such that

$$\|\boldsymbol{u}_n\| = 1, \forall n \in \mathbb{N}, \tag{18}$$

$$|\boldsymbol{u}_n|_{DG,h_n} = \frac{1}{n}, \forall n \in \mathbb{N}. \tag{19}$$

By the definition of the semi-norm $|\cdot|_{DG,h_n}$, we then have

$$\int_{\mathcal{E}} \mathrm{h}^{-1} |\underline{[\![\boldsymbol{u}_n]\!]_T}|^2 \leq \frac{1}{n^2}.$$

Using the (orthogonal) decomposition (11) of $\boldsymbol{u}_n$, we obtain

$$\int_{\mathcal{E}} \mathrm{h}^{-1} |\underline{[\![\boldsymbol{u}_n^{\perp}]\!]_T}|^2 \leq \frac{1}{n^2},$$

because $\underline{[\![\boldsymbol{u}_n^c]\!]_T} = 0$. Now with the help of Corollary 4.3, we deduce that

$$\|\boldsymbol{u}_n^{\perp}\|_{DG,h_n} \leq \frac{C_1}{n}, \tag{20}$$

for some positive constant C_1.

From (18) and (19), the sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ satisfies

$$\|\boldsymbol{u}_n\|_{DG,h_n} \leq C_2, \forall n \in \mathbb{N},$$

for some positive constant C_2. Therefore by Theorem 4.4, there exists a subsequence, still denoted by $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, and $\boldsymbol{u} \in X$ satisfying (13) and (14).

By (13) and (18), we deduce that

$$\|\boldsymbol{u}\| = 1. \tag{21}$$

Now from (14), we may write

$$\|\operatorname{curl} \boldsymbol{u}\| \leq \lim_{n\to\infty} \|\operatorname{curl} \boldsymbol{u}_n^c\|.$$

But the triangular inequality leads to

$$\begin{aligned} \|\operatorname{curl} \boldsymbol{u}_n^c\| &\leq \|\operatorname{curl}_{h_n} \boldsymbol{u}_n\| + \|\operatorname{curl}_{h_n} \boldsymbol{u}_n^{\perp}\| \\ &\leq |\boldsymbol{u}_n|_{DG,h_n} + \|\boldsymbol{u}_n^{\perp}\|_{DG,h_n} \end{aligned}$$

and by (19) and (20), we conclude that

$$\| \operatorname{curl} \boldsymbol{u}_n^c \| \to 0 \text{ as } n \to \infty.$$

Hence

$$\operatorname{curl} \boldsymbol{u} = 0 \text{ in } \Omega,$$

and since $\boldsymbol{u}$ belongs to X, $\boldsymbol{u} = 0$ by the standard Friedrichs inequality. This is a contradiction with (21). ∎

5 The discrete eigenvalue problem

Now we can come back to the discrete eigenvalue problem (7) and show its well posedness.

Lemma 5.1 *There exists a parameter $\alpha_{min} > 0$, independent of the mesh size h such that for $\alpha \geq \alpha_{min} > 0$, and $h \in \Lambda$, the bilinear form a_h is strongly coercive on X_h, namely there exists $\beta > 0$ independent of the mesh size h such that*

$$a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \beta \|\boldsymbol{u}_h\|^2_{DG,h}, \forall \boldsymbol{u}_h \in X_h.$$

Proof: Using standard inverse estimates (see Lemma 3.1 of [19]), for $\alpha \geq \alpha_{min} > 0$, there exists $\beta_0 > 0$ independent of the mesh size h such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \beta_0 |\boldsymbol{u}_h|^2_{DG,h}, \forall \boldsymbol{u}_h \in V_h.$$

The conclusion follows from Theorem 4.5. ∎

In view of this lemma, we reduce problem (7) to the space X_h.

Lemma 5.2 *If $\lambda_h \neq 0$, then (7) is equivalent to find $\boldsymbol{E}_h \in X_h$ solution of*

$$a_h(\boldsymbol{E}_h, \boldsymbol{v}_h) = \lambda_h(\boldsymbol{E}_h, \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in X_h. \tag{22}$$

Proof: $\boxed{\Rightarrow}$ Let $\boldsymbol{E}_h \in V_h$ be a solution of (7). We only need to show that it belongs to X_h. To this end, we simply take $\boldsymbol{v}_h = \nabla q_h$ with $q_h \in S_h$ as test function in (7) to obtain

$$a_h(\boldsymbol{E}_h, \nabla q_h) = \lambda_h(\boldsymbol{E}_h, \nabla q_h).$$

Since $a_h(\boldsymbol{E}_h, \nabla q_h) = 0$ and $\lambda_h \neq 0$, we conclude that

$$(\boldsymbol{E}_h, \nabla q_h) = 0, \forall q_h \in S_h.$$

$\boxed{\Leftarrow}$ Let $\boldsymbol{E}_h \in X_h$ be a solution of (22). As X_h is a closed subspace of V_h, any $\boldsymbol{v}_h \in V_h$ may be split up

$$\boldsymbol{v}_h = \boldsymbol{v}_h^0 + \nabla q_h,$$

where $\boldsymbol{v}_h^0$ belongs to X_h and q_h in S_h. Consequently, we obtain

$$\begin{aligned} a_h(\boldsymbol{E}_h, \boldsymbol{v}_h) &= a_h(\boldsymbol{E}_h, \boldsymbol{v}_h^0) = \lambda_h(\boldsymbol{E}_h, \boldsymbol{v}_h^0) \\ &= \lambda_h\{(\boldsymbol{E}_h, \boldsymbol{v}_h) - (\boldsymbol{E}_h, \nabla q_h)\} = \lambda_h(\boldsymbol{E}_h, \boldsymbol{v}_h), \end{aligned}$$

using successively the property $a_h(\boldsymbol{E}_h, \nabla q_h) = 0$, the identity (22) and the fact that $\boldsymbol{E}_h \in X_h$. ∎

From Lemma 5.1, the eigenvalues problem (22) has only positive eigenvalues. Hence from Lemma 5.2, problem (7) has only nonegative eigenvalues. Thus from the computational point of view, we can solve problem (7) and keep the positive eigenvalues, since only these ones have a physical meaning. Owing to Lemma 5.2, our theoretical analysis can be based on (22).

As in the continuous case, we rewrite problem (22) as a discrete operator equation. Namely we introduce the discrete operator

$$A_h : L^2(\Omega)^3 \to X_h : \boldsymbol{u} \to A_h\boldsymbol{u},$$

where $A_h\boldsymbol{u} \in X_h$ is the unique solution of

$$a_h(A_h\boldsymbol{u}, \boldsymbol{v}_h) = (\boldsymbol{u}, \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in X_h. \tag{23}$$

Thanks to Lemma 5.1 this problem is well-posed and hence A_h is well-defined.

Lemma 5.3 *$\boldsymbol{E}_h \in X_h$ is solution of (22) for $\lambda_h > 0$ if and only if $E_h \in L^2(\Omega)^3$ satisfies*

$$A_h\boldsymbol{E}_h = \mu_h\boldsymbol{E}_h, \tag{24}$$

with $\mu_h = \frac{1}{\lambda_h}$.

Proof: $\boxed{\Rightarrow}$ Let $\boldsymbol{E}_h \in X_h$ be a solution of (22). By the definition of A_h, we may write for any $\boldsymbol{v}_h \in X_h$

$$a_h(A_h\boldsymbol{E}_h, \boldsymbol{v}_h) = (\boldsymbol{E}_h, \boldsymbol{v}_h) = \frac{1}{\lambda_h} a_h(\boldsymbol{E}_h, \boldsymbol{v}_h).$$

This leads to the conclusion since $a_h(\cdot, \cdot)$ is an inner product on X_h.

$\boxed{\Leftarrow}$ Let $\boldsymbol{E}_h \in L^2(\Omega)^3$ be a solution of (24). Then $\boldsymbol{E}_h = \frac{1}{\mu_h} A_h\boldsymbol{E}_h$, which belongs to X_h. The conclusion then follows from this identity and (23). ∎

6 Collective compactness and pointwise convergence

Following [26], we prove the convergence of the positive eigenvalues of the discrete problem (7) to the positive eigenvalues of the continuous problem (4) by using the theory of collectively compact and pointwise convergent operators [1, 27, 11]. For completeness, we first recall a basis result that will be used later on.

Definition 6.1 *Let X, Y be two normed spaces and Λ a subset of $\mathbb{R}$. A set $\mathcal{A} = \{A_h : X \to Y, h \in \Lambda\}$ of linear operators from X to Y is called collectively compact if for each bounded set U of X, the range*

$$\mathcal{A}(U) := \{A_h u : u \in U, h \in \Lambda\}$$

is relatively compact in Y.

With this definition, we can recall the next convergence result proved in [27].

Theorem 6.2 *Let X be a Hilbert space of inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$, and let $\Lambda = \{h_n : n \in \mathbb{N}\}$ be a discrete subset of $\mathbb{R}$ such that $h_n \to 0$ as $n \to \infty$. Assume that the set $\mathcal{A} = \{A_h : X \to X, h \in \Lambda\}$ of linear selfadjoint operators in X is collectively compact. Assume furthermore that there exists a selfadjoint and compact operator A in X such that A_h converges pointwise to A (i.e., for all $x \in X$, $A_h x \to Ax$ in X as $h \to 0$). Let μ be an eigenvalue of A of multiplicity m and denote by $\{\phi_j\}_{j=1}^m$ the set of associated orthogonal eigenvectors. Then for any $\varepsilon > 0$ such that the disk $B(\mu,\varepsilon)$ of radius ε and center μ contains no other eigenvalues of A, there exists h_ε such that for all $h < h_\varepsilon$, A_h has exactly m eigenvalues (repeated according to their multiplicity) in $B(\mu,\varepsilon)$. Moreover for $h < h_\varepsilon$, if we denote by $\mu_{h,j}, j = 1,\cdots,m$, the set of the eigenvalues of A_h in $B(\mu,\varepsilon)$, then for all $j = 1,\cdots,m$, there exists a positive constant C such that*

$$(25)\ |\mu - \mu_{h,j}| \le C\left(\sum_{j,l=1}^{m} |((A - A_h)\phi_j, \phi_l))_X| + (\sum_{j=1}^{m} \|(A - A_h)\phi_j\|_X^2)^{1/2}\right).$$

The remainder of the section consists in checking that the set of operators A_h defined by (23) satisfies the hypotheses of the above Theorem.

Theorem 6.3 *The set* $\mathcal{A} := \{A_h : h \in \Lambda\}$ *is collectively compact.*

Proof: Let U be a bounded set of $L^2(\Omega)^3$. By the definition of A_h, for any $\boldsymbol{u} \in U$, $A_h\boldsymbol{u} \in X_h$ satisfies

$$a_h(A_h\boldsymbol{u}, \boldsymbol{v}_h) = (\boldsymbol{u}, \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in X_h.$$

By Lemma 5.1, we then have

$$\|A_h\boldsymbol{u}\|^2_{DG,h} \leq \beta^{-1} a_h(A_h\boldsymbol{u}, A_h\boldsymbol{u}) = \beta^{-1}(\boldsymbol{u}, A_h\boldsymbol{u}).$$

By Cauchy-Schwarz's inequality and the boundedness assumption of U, we obtain

$$\|A_h\boldsymbol{u}\|_{DG,h} \leq C, \forall h \in \Lambda, \tag{26}$$

for some $C > 0$ (independent of h).

This estimate and Theorem 4.4 imply that the

$$W = \{A_h\boldsymbol{u} : \boldsymbol{u} \in U, h \in \Lambda\},$$

is precompact in $L^2(\Omega)^3$. Indeed let us fix a sequence $(A_{h_n}\boldsymbol{u}_n)_{n\in\mathbb{N}}$ in W. Then either $h_n \to 0$ as $n \to \infty$ and since (26) guarantees that (10) holds, we conclude by Theorem 4.4; or h_n does not tend to zero as $n \to \infty$ and in that case the set $\{h_n : n \in \mathbb{N}\}$ is finite, in this last case we conclude by finite dimensionality. ∎

Let us go on with the pointwise convergence:

Theorem 6.4 *Let* $\boldsymbol{u} \in L^2(\Omega)^3$*, then*

$$A_h\boldsymbol{u} \to A\boldsymbol{u} \text{ strongly in } L^2(\Omega)^3, \text{ as } h \to 0,$$

where we recall that A was defined by (5).

Proof: We follow the proof of Theorem 4.5 of [26] with the necessary adaptations due to the discontinuous scheme. First following some recent ideas from [4, 19, 20], we introduce the lifting operator

$$\mathcal{L} : V(h) \to V_h : \boldsymbol{v} \to \mathcal{L}(\boldsymbol{v}),$$

where $\mathcal{L}(\boldsymbol{v})$ is the unique element in V_h such that

$$(\mathcal{L}(\boldsymbol{v}), \boldsymbol{w}_h) = \int_{\mathcal{E}} \underline{[\![\boldsymbol{v}]\!]_T} \cdot \{\{\boldsymbol{w}_h\}\}, \forall \boldsymbol{w}_h \in V_h.$$

Introduce further the modified bilinear form

$$\begin{aligned}\tilde{a}_h(\boldsymbol{u},\boldsymbol{v}) &:= (\mathrm{curl}_h\,\boldsymbol{u},\mathrm{curl}_h\,\boldsymbol{v}) - (\mathcal{L}(\boldsymbol{u}),\mathrm{curl}_h\,\boldsymbol{v})\\ &- (\mathcal{L}(\boldsymbol{v}),\mathrm{curl}_h\,\boldsymbol{u}) + \alpha\int_{\mathcal{E}} \mathrm{h}^{-1}\underline{[\![\boldsymbol{u}]\!]_T}\cdot\underline{[\![\boldsymbol{v}]\!]_T}.\end{aligned}$$

We may remark that

$$\begin{aligned}\tilde{a}_h(\boldsymbol{u}_h,\boldsymbol{v}_h) &= a_h(\boldsymbol{u}_h,\boldsymbol{v}_h), \forall \boldsymbol{u}_h,\boldsymbol{v}_h\in V_h,\\ \tilde{a}_h(\boldsymbol{u},\boldsymbol{v}) &= (\mathrm{curl}\,\boldsymbol{u},\mathrm{curl}\,\boldsymbol{v}), \forall \boldsymbol{u},\boldsymbol{v}\in \mathcal{H}_0(\mathrm{curl};\Omega).\end{aligned}$$

Now we transform the discrete problem (23) and the continuous problem (5) in mixed problems: If $A_h\boldsymbol{u}\in X_h$ is solution of (23), then the pair $(A_h\boldsymbol{u},p_h)\in V_h\times S_h$ is the unique solution of

$$\left\{\begin{array}{l} a_h(A_h\boldsymbol{u},\boldsymbol{v}_h)+b(\boldsymbol{v}_h,p_h)=(\boldsymbol{u},\boldsymbol{v}_h),\forall \boldsymbol{v}_h\in V_h,\\ b(A_h\boldsymbol{u},q_h)=0,\forall q_h\in S_h,\end{array}\right. \tag{27}$$

where $b(\boldsymbol{v},p)=(\nabla p,\boldsymbol{v})$. Indeed the second identity means that $A_h\boldsymbol{u}\in X_h$ and the first one reduces to (23), when the test functions are taken in X_h because $b(\boldsymbol{v}_h,p_h)=0$ for $p_h\in S_h$ and $\boldsymbol{v}_h\in X_h$. This mixed problem has a unique solution because a_h is coercive on X_h, which is precisely the kernel of b in V_h, and b satisfies the discrete inf-sup condition

$$\sup_{\boldsymbol{v}_h\in V_h}\frac{b(\boldsymbol{v}_h,p_h)}{\|\boldsymbol{v}_h\|_{DG,h}}\geq |p_h|_{1,\Omega},$$

for all $p_h\in S_h$, since $\boldsymbol{v}_h=\nabla p_h$ yields

$$\frac{b(\boldsymbol{v}_h,p_h)}{\|\boldsymbol{v}_h\|_{DG,h}}=|p_h|_{1,\Omega}.$$

Note that in the above problem we can replace a_h by $\tilde{a}_h$.

In a similar manner if $A\boldsymbol{u}\in X$ is solution of (5), then the pair $(A\boldsymbol{u},p)\in \mathcal{H}_0(\mathrm{curl};\Omega)\times H_0^1(\Omega)$ is the unique solution of

$$\left\{\begin{array}{l}(\mathrm{curl}\,A\boldsymbol{u},\mathrm{curl}\,\boldsymbol{v})+b(\boldsymbol{v},p)=(\boldsymbol{u},\boldsymbol{v}),\forall \boldsymbol{v}\in\mathcal{H}_0(\mathrm{curl};\Omega),\\ b(A\boldsymbol{u},q)=0,\forall q\in H_0^1(\Omega).\end{array}\right. \tag{28}$$

Since V_h is not included into $\mathcal{H}_0(\mathrm{curl};\Omega)$, we may look at (27) as a non-conforming approximation of (28). Therefore by Proposition II.2.16 of [10],

we may write

$$\begin{aligned}\|A\boldsymbol{u}-A\boldsymbol{u}_h\|_{DG,h} &\leq C\Big(\inf_{\boldsymbol{v}_h\in V_h}\|A\boldsymbol{u}-\boldsymbol{v}_h\|_{DG,h}+\inf_{p_h\in S_h}|p-p_h|_{1,\Omega}\\ &+ \sup_{\boldsymbol{w}_h\in V_h}\frac{|\tilde{a}_h(A\boldsymbol{u},\boldsymbol{w}_h)+b(\boldsymbol{w}_h,p)-(\boldsymbol{u},\boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|_{DG,h}}\Big).\end{aligned}$$

The two first terms of this right-hand side are so-called approximation error terms and the third one is the consistency error term.

Since V_h^c is included into V_h we can estimate the first approximation error term by

$$\inf_{\boldsymbol{v}_h\in V_h}\|A\boldsymbol{u}-\boldsymbol{v}_h\|_{DG,h}\leq\inf_{\boldsymbol{v}_h\in V_h^c}\|A\boldsymbol{u}-\boldsymbol{v}_h\|_{DG,h}=\inf_{\boldsymbol{v}_h\in V_h^c}\|A\boldsymbol{u}-\boldsymbol{v}_h\|_{\mathcal{H}_0(\mathrm{curl};\Omega)}.$$

Since V_h^c is dense in $\mathcal{H}_0(\mathrm{curl};\Omega)$ (see Theorem 4.5 of [26]) the above right-hand side tends to zero as $h\to 0$, hence

$$\inf_{\boldsymbol{v}_h\in V_h}\|A\boldsymbol{u}-\boldsymbol{v}_h\|_{DG,h}\to 0 \text{ as } h\to 0.$$

Similarly the density of S_h in $H_0^1(\Omega)$ leads to

$$\inf_{p_h\in S_h}|p-p_h|_{1,\Omega}\to 0 \text{ as } h\to 0.$$

It remains to estimate the consistency term. Let us denote by

$$r_h(\boldsymbol{w}_h)=\tilde{a}_h(A\boldsymbol{u},\boldsymbol{w}_h)+b(\boldsymbol{w}_h,p)-(\boldsymbol{u},\boldsymbol{w}_h).$$

By the definition of $\tilde{a}_h$, it can be written

$$r_h(\boldsymbol{w}_h)=(\mathrm{curl}\,A\boldsymbol{u},\mathrm{curl}_h\,\boldsymbol{w}_h)-(\mathcal{L}(\boldsymbol{w}_h),\mathrm{curl}\,A\boldsymbol{u})+b(\boldsymbol{w}_h,p)-(\boldsymbol{u},\boldsymbol{w}_h).$$

Introducing Π_h the L^2-orthogonal projection on V_h, we may write

$$r_h(\boldsymbol{w}_h)=(\mathrm{curl}\,A\boldsymbol{u},\mathrm{curl}_h\,\boldsymbol{w}_h)-(\mathcal{L}(\boldsymbol{w}_h),\Pi_h\,\mathrm{curl}\,A\boldsymbol{u})+b(\boldsymbol{w}_h,p)-(\boldsymbol{u},\boldsymbol{w}_h).$$

Elementwise integration by parts leads to

$$\begin{aligned}r_h(\boldsymbol{w}_h) &= \sum_{T\in\mathcal{T}_h}\int_T\mathrm{curl}\,\mathrm{curl}\,A\boldsymbol{u}\cdot\boldsymbol{w}_h+\int_{\mathcal{E}}\underline{[\![\boldsymbol{w}_h]\!]_T}\cdot\{\{\mathrm{curl}\,A\boldsymbol{u}\}\}\\ &- (\mathcal{L}(\boldsymbol{w}_h),\Pi_h\,\mathrm{curl}\,A\boldsymbol{u})+b(\boldsymbol{w}_h,p)-(\boldsymbol{u},\boldsymbol{w}_h).\end{aligned}$$

As (28) implies that $\operatorname{curl}\operatorname{curl} A\boldsymbol{u} + \nabla p = \boldsymbol{u}$, we finally arrive at

$$r_h(\boldsymbol{w}_h) = \int_{\mathcal{E}} \underline{[\![\boldsymbol{w}_h]\!]_T} \cdot \{\{ \operatorname{curl} A\boldsymbol{u} - \Pi_h \operatorname{curl} A\boldsymbol{u}\}\}.$$

Cauchy-Schwarz's inequality and Lemma 4.3 of [19] yield

$$\begin{aligned} |r_h(\boldsymbol{w}_h)| &\leq \sum_{E\in\mathcal{E}} \|\underline{[\![\boldsymbol{w}_h]\!]_T}\|_E \|\{\{ \operatorname{curl} A\boldsymbol{u} - \Pi_h \operatorname{curl} A\boldsymbol{u}\}\}\|_E \\ &\leq C \sum_{T\in\mathcal{T}_h} h_T^{t-1/2} \|\operatorname{curl} A\boldsymbol{u}\|_{t,T} \sum_{E\in\mathcal{E}, E\subset T} \|\underline{[\![\boldsymbol{w}_h]\!]_T}\|_E, \end{aligned}$$

for any $t \in]1/2, 1]$. Using discrete Cauchy-Schwarz's inequality and the definition of the norm $\|\cdot\|_{DG,h}$, we obtain

$$|r_h(\boldsymbol{w}_h)| \leq Ch^t \|\operatorname{curl} A\boldsymbol{u}\|_{t,\Omega} \|\boldsymbol{w}_h\|_{DG,h}.$$

The Theorem will be proved if one can show that $\operatorname{curl} A\boldsymbol{u}$ belongs to $H^t(\Omega)$ for some $t > 1/2$. As $A\boldsymbol{u}$ belongs to X and $\operatorname{curl}\operatorname{curl} A\boldsymbol{u} = \boldsymbol{u} - \nabla p$ belongs to $L^2(\Omega)^3$, we see that

$$\operatorname{curl} A\boldsymbol{u} \in X_T(\Omega),$$

where $X_T(\Omega)$ is defined by (see [13])

$$X_T(\Omega) := \{\boldsymbol{v} \in L^2(\Omega)^3 : \operatorname{curl} \boldsymbol{v} \in L^2(\Omega)^3, \operatorname{div} \boldsymbol{v} \in L^2(\Omega) \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

The conclusion follows from the well known embedding $X_T(\Omega) \hookrightarrow H^t(\Omega)$ for some $t > 1/2$, consequence of Theorem 1.1 from [13] for instance (see also [6, 5]). ∎

Corollary 6.5 *Let λ be a positive eigenvalue of the continuous problem (4) of multiplicity m. Then for h small enough the discrete problem (7) has exactly m positive eigenvalues $\lambda_{h,j}$, $j = 1, \cdots, m$ (repeated according to their multiplicity) such that*

$$\lambda_{h,j} \to \lambda \text{ as } h \to 0, \forall j = 1, \cdots, m.$$

Proof: We have verified in Theorems 6.3 and 6.4 the assumptions of Theorem 6.2 for the set $\mathcal{A}$ introduced in Theorems 6.3. Denote by $\mu = \frac{1}{\lambda}$ the corresponding eigenvalue of A (see section 2). By Theorem 6.2, for h small

enough, A_h has exactly m positive eigenvalues $\mu_{h,j}$, $j = 1, \cdots, m$, moreover the estimate (25) implies

$$|\mu - \mu_{h,j}| \leq C \left(\sum_{j=1}^{m} \|(A - A_h)\phi_j\|_X^2 \right)^{1/2}.$$

By the pointwise convergence of A_h to A, we deduce that

$$\mu_{h,j} \to \mu \text{ as } h \to 0, \forall j = 1, \cdots, m,$$

and we conclude by Lemma 5.3. ∎

7 Some numerical tests

For the sake of simplicity, our theoretical results are tested on two-dimensional domains. In fact using the results from [15, 24], our theoretical results also hold for two-dimensional domains, meshed by a regular family (in Ciarlet's sense) of triangles. For our tests we take $k = 1$ and compute the first fifth eigenvalues for the square domain $(0,1)^2$ and the L-shape domain $(-1,1)^2 \setminus (0,1)^2$, as well as the first four eigenvalues for the sector domain given in polar coordinates by $\{(r,\theta) \in \mathbb{R}^2; 0 \leq r \leq 1, 0 \leq \theta \leq \frac{3\pi}{2}\}$. For the first domain, analytical solutions are known (see for instance [8]) and the non zero eigenvalues of problem (1) are given by

$$\lambda_{j,l} = (j^2 + l^2)\pi^2, j + l \geq 1,$$

in other words, for $j \geq 1$, the number $j^2\pi^2$ is a double eigenvalue, while for $j, l \geq 1$, $j \neq l$, the number $(j^2 + l^2)\pi^2$ is a simple eigenvalue. For the second domain, no analytical solution exists but we compare our results with the computational eigenvalues presented by the benchmark of M. Dauge [14] obtained by using a Galerkin approximation of the Neumann eigenvalues with a geometrical refined mesh near the corner and polynomials of high degree. For the third domain, analytical solutions are known (see for instance [15]).

Since the discrete problem (7) has a quite large kernel and therefore does not correspond to an eigenvalue problem for positive definite matrices we use the discrete regularization technique (compare with [3, 18]): First we remark that the kernel K_h of problem (7) is explicitly known, namely

$$K_h = \nabla S_h.$$

Taking into account Lemma 5.2 we further remark that λ_h is a nonzero eigenvalue of (7) if and only if there exists $\boldsymbol{E}_h \in X_h$ solution of

$$a_h(\boldsymbol{E}_h, \boldsymbol{v}_h) = \lambda_h(\boldsymbol{E}_h, \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in V_h. \tag{29}$$

Now we fix a basis $\{\lambda_j\}_{j=1}^J$ of S_h and a basis $\{\boldsymbol{v}_i\}_{i=1}^I$ of V_h (clearly $J < I$, but J is quite large). Writing

$$\boldsymbol{E}_h = \sum_{i=1}^{I} x_i \boldsymbol{v}_i,$$

we see that the eigenvalue problem (29) is equivalent to

$$AX = \lambda_h BX, \tag{30}$$

where $X = (x_i)_{i=1}^I$ is the unknown vector, $A = (a(\boldsymbol{v}_i, \boldsymbol{v}_{i'}))_{1 \leq i,i' \leq I}$ is the "rigidity" matrix and $B = ((\boldsymbol{v}_i, \boldsymbol{v}_{i'}))_{1 \leq i,i' \leq I}$ is the mass matrix. Using the basis of S_h we can say that $\boldsymbol{E}_h$ belongs to X_h if and only if

$$C^\top X = 0, \tag{31}$$

where $C = (c_{ij})_{1 \leq i \leq I, 1 \leq j \leq J}$, with $c_{ij} = (\boldsymbol{v}_i, \nabla \lambda_j)$.

In summary the eigenvalue problem is reduced to find X satisfying (31) and solution of (30).

At this stage taking an arbitrary weight $s > 0$, we see that X satisfying (31) and solution of (30) also satisfies

$$(A + sCC^\top)X = \lambda_h BX. \tag{32}$$

The main point is that $A + sCC^\top$ is a positive definite matrix. Indeed we clearly have

$$X^\top (A + sCC^\top)X \geq 0, \forall X \in \mathbb{R}^I.$$

On the other hand if

$$X^\top (A + sCC^\top)X = 0,$$

then

$$X^\top AX = 0 \text{ and } \|C^\top X\| = 0,$$

and consequently X belongs to the kernel of A and to its orthogonal, so that $X = 0$.

The second advantage of the eigenvalue problem (32) is that the zero eigenvalues of problem (30) become positive but depend linearly on s (while the non zero eigenvalues of problem (30) are unchanged). Consequently the nonzero eigenvalues of interest can be captured by choosing the parameter s large enough (or by taking different values of s). In that respect, our method is similar to the discrete regularization method for standard edge elements [3, 18] and can be compared with the so-called grad-div regularizated method for the continuous Maxwell's problem where the term $s \int_\Omega \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}$ is added to the standard bilinear form of the curl [16, 13].

The approximated eigenvalues are then produced by solving the problem (32) using Matlab; the matrices being generated by our finite element code Simula+. For both problems we have fixed the penalty parameter α equal to 100 and the weight $s = 3200$ in order that the first eigenvalues are the expected ones.

In Figure 1 we have plotted the first fifth eigenvalues for the square in comparison with the exact ones for different meshes, while in Figure 2 the error is given as a function of the DoF. This last figure demonstrates the convergence of the discrete eigenvalues to the exact ones with a convergence rate of 1 in DoF^{-1} (i.e. of order 2 in h). Figures 3 and 4 present the same results for the L-shape domain, by taking as exact values the one given by M. Dauge [14]. Here again, a convergence rate of order 1 in DoF^{-1} (or of order 2 in h) is obtained. We should only notice the particular behaviour of the first eigenvalue convergence rate : The graph of the error for this smallest eigenvalue exhibits a sudden rise after a dramatic fall, and the convergence rate is no more equal to 1 with respect to DoF^{-1}. This phenomenom occurs for another domain (the slit domain) in [15], for which the smallest eigenvalue also corresponds to a singular eigenfield. Moreover, also for the L-shape, numerical results obtained in [17] confirm that there exists a strong correlation between the convergence rate and the regularity of the corresponding eigenfunction. In Figure 5 we have plotted the first four eigenvalues for the sector domain in comparison with the exact ones for different meshes, while in Figure 6 the error is given as a function of the DoF. For this last example, the convergence properties are the same as the ones obtained for the L-shape domain. Note further that these results are comparable with those from [15].

The tests presented in this section have been performed with the help of the SIMULA+ finite element code (MACS, University of Valenciennes and LPMM, University and ENSAM of Metz, both in France)

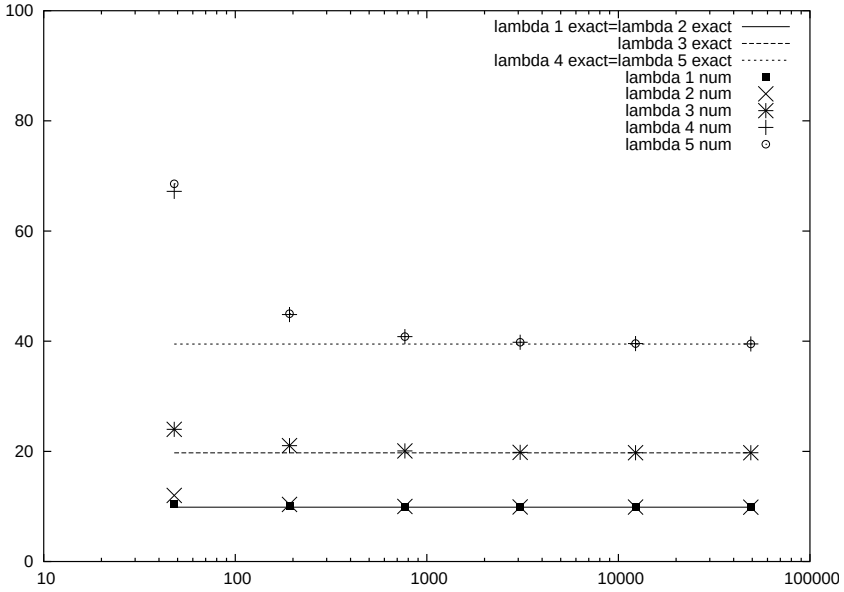

Figure 1: Eigenvalues wrt DoF for the unit square domain

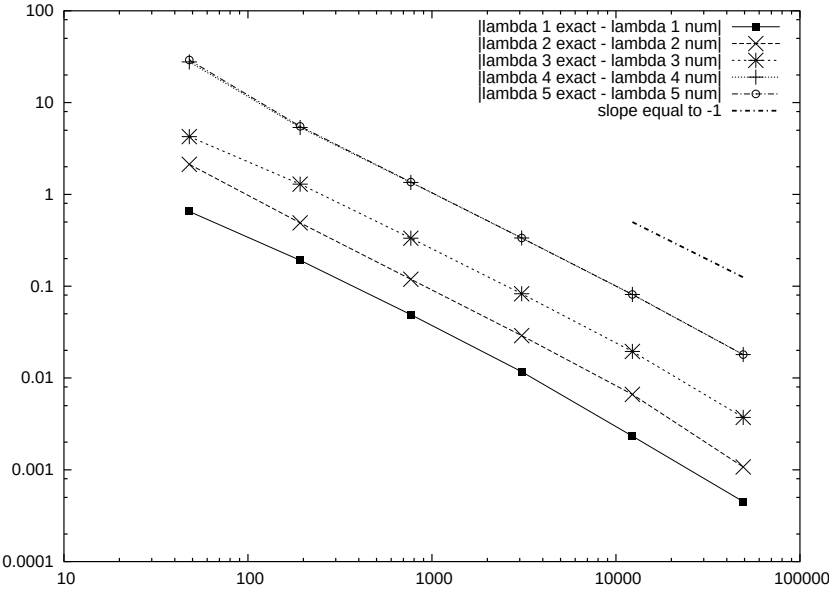

Figure 2: Convergence rates in DoF for the unit square domain

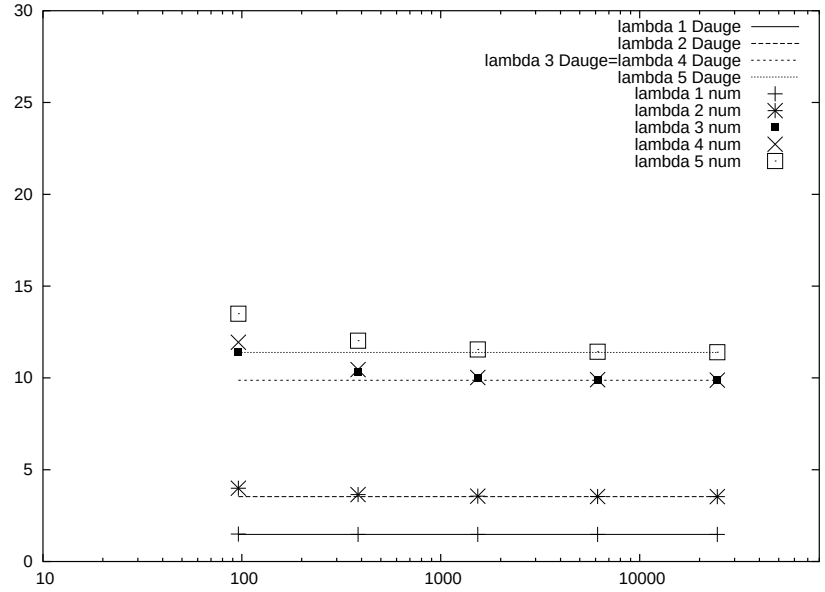

Figure 3: Eigenvalues wrt DoF for the L-shape domain

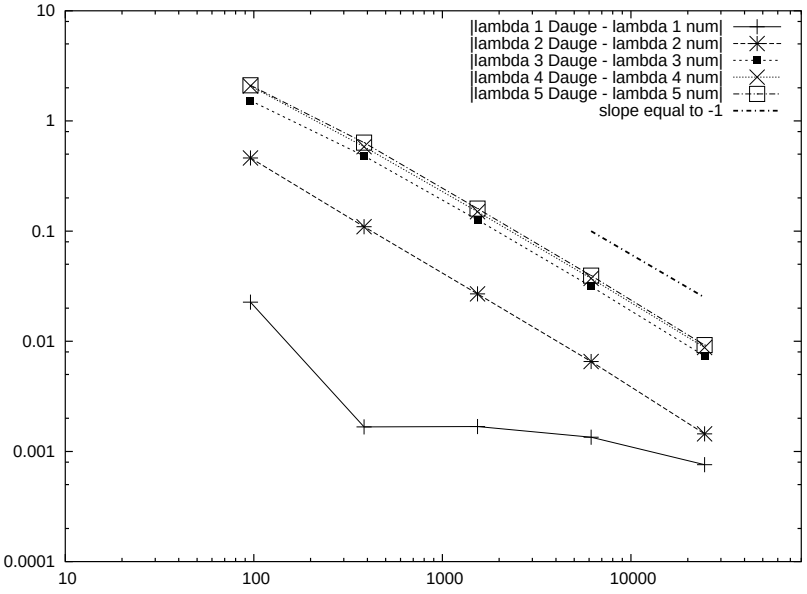

Figure 4: Convergence rates in DoF for the L-shape domain

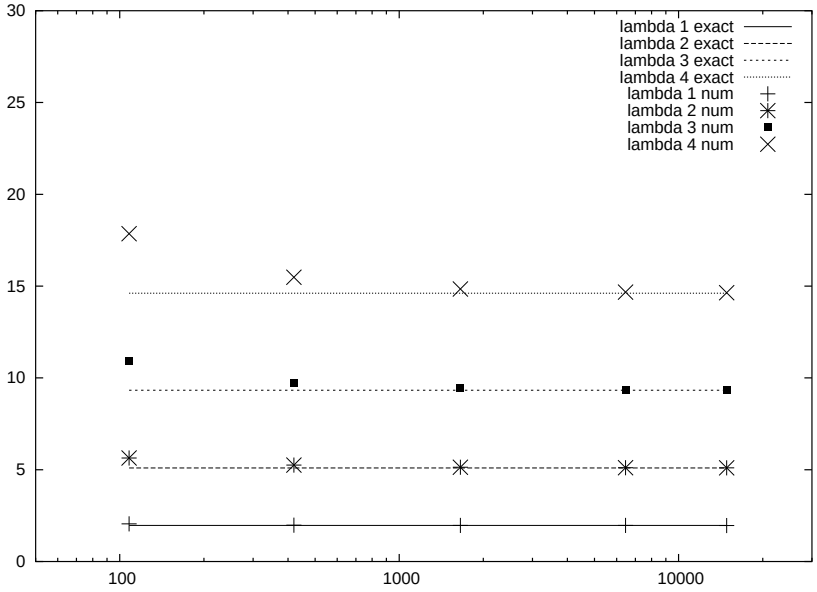

Figure 5: Eigenvalues wrt DoF for the sector domain

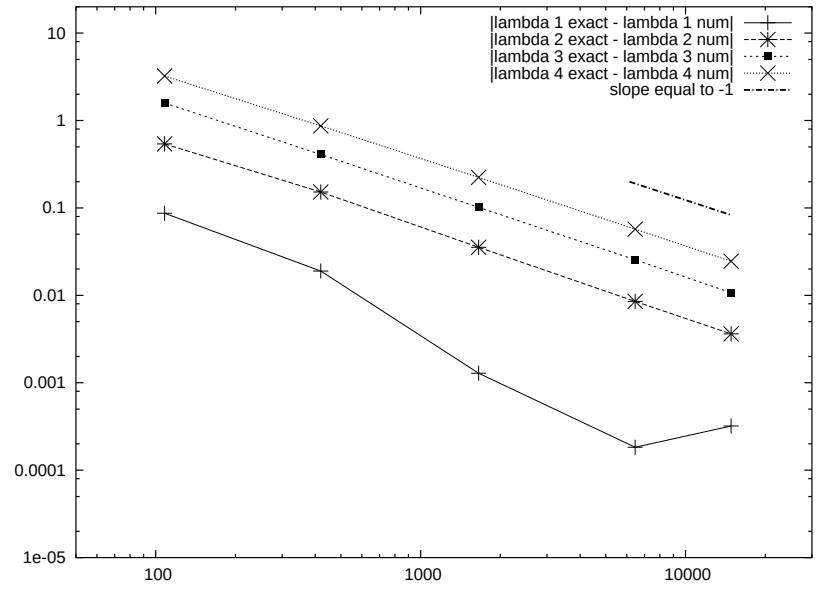

Figure 6: Convergence rates in DoF for the sector domain

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