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A posteriori error estimations of a coupled mixed and standard Galerkin method for second order operators

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Abstract

In this paper, we consider a discretization method proposed by Wieners and Wohlmuth [26] (see also [16]) for second order operators, which is a coupling between a mixed method in a sub-domain and a standard Galerkin method in the remaining part of the domain. We perform an a posteriori error analysis of residual type of this method, by combining some arguments from a posteriori error analysis of Galerkin methods and mixed methods. The reliability and efficiency of the estimator are proved. Some numerical tests are presented and confirm the theoretical error bounds.

Key words: A posteriori estimates, coupled method
PACS: 65M60, 65M12, 65M15

1 Introduction

Let us fix a bounded domain Ω of $\mathbb{R}^2$, with a polygonal boundary. For the sake of simplicity we assume that Ω is simply connected. The case of a multiply connected domain can be treated as in [12].

In this paper we consider the following second order problem: For $f \in L^2(\Omega)$,

let $\theta \in H_0^1(\Omega)$ be the unique solution of

$$\operatorname{div}(A\nabla\theta) = -f \text{ in } \Omega, \tag{1}$$

where the matrix $A \in L^\infty(\Omega, \mathbb{R}^{2\times 2})$ is supposed to be symmetric and uniformly positive definite.

The domain Ω is decomposed into two nonoverlapping polygonal subdomains Ω_1 and Ω_2 such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$. We also assume that $\partial\Omega_2 \cap \partial\Omega$ is of positive measure. We further denote by Γ the (relative) interior of $\partial\Omega_1 \cap \partial\Omega_2$, the interface between Ω_1 and Ω_2 (for further purposes, it is easier to assume that Γ is open). On Ω_1 a mixed formulation of problem (1) is introduced, while on Ω_2 a standard Galerkin method is used, see [26,16]. This means that we introduce as new unknowns $\underline{\xi} = (A\nabla\theta)_{|\Omega_1}$, $\theta_1 = \theta_{|\Omega_1}$ and $\theta_2 = \theta_{|\Omega_2}$. These unknowns will be coupled through the interface by the conditions

$$\theta_1 = \theta_2, \quad \underline{\xi} \cdot \underline{n} = A\nabla\theta_2 \cdot \underline{n} \text{ on } \Gamma,$$

where $\underline{n}$ is the unit normal vector along Γ that is directed from Ω_1 to Ω_2.

For shortness, we denote by $(\cdot,\cdot)_i, i = 1, 2$, the L^2-inner product in Ω_i, namely

$$(\theta, \chi)_i = \int_{\Omega_i} \theta(x)\chi(x)\, dx, \forall \theta, \chi \in L^2(\Omega_i).$$

Obviously we use the same notation for vector fields. Similarly we denote by $(\cdot,\cdot)_\Gamma$, the L^2-inner product in Γ.

The coupling between the mixed and standard formulations leads to the following saddle point problem [26,16]: Find $u = (\underline{\xi}, \theta_2)$ in X and $p = \theta_1$ in M solutions of

$$\begin{cases} a(u,v) + b(v,p) = (f,\chi)_2, \forall v = (\eta,\chi) \in X, \\ b(u,q) = -(f,q)_1, \quad \forall v \in M, \end{cases} \tag{2}$$

where

$$\begin{aligned} X &:= H(\operatorname{div}, \Omega_1) \times H_D^1(\Omega_2), \\ M &:= L^2(\Omega_1), \\ H(\operatorname{div}, \Omega_1) &:= \{\eta \in [L^2(\Omega_1)]^2 : \operatorname{div}\eta \in L^2(\Omega_1)\}, \\ H_D^1(\Omega_2) &:= \{\chi \in H^1(\Omega_2) : \chi = 0 \text{ on } \partial\Omega \cap \partial\Omega_2\}. \end{aligned}$$

The space X is endowed with the natural norm

$$\|(\eta,\chi)\|_X^2 := \|\eta\|_{L^2(\Omega_1)}^2 + \|\operatorname{div}\eta\|_{L^2(\Omega_1)}^2 + \|\nabla\chi\|_{L^2(\Omega_2)}^2.$$

Moreover the bilinear forms a and b are defined by

$$\begin{aligned} a(u,v) &:= (A^{-1}\underline{\xi},\eta)_1 + (A\nabla\theta_2,\nabla\chi)_2 \\ &\quad - (\eta\cdot\underline{n},\theta_2)_\Gamma + (\underline{\xi}\cdot\underline{n},\chi)_\Gamma, \forall u=(\underline{\xi},\theta_2), v=(\eta,\chi)\in X, \\ b(u,q) &:= (\operatorname{div}\underline{\xi},q)_1, \forall u=(\underline{\xi},\theta_2)\in X, q\in M. \end{aligned}$$

Since the bilinear form a is coercive on X and the so-called inf-sup condition is satisfied (see section 2.1 of [16] or [26]), problem (2) has a unique solution [24, p.16], which is clearly given by $\underline{\xi} = (A\nabla\theta)_{|\Omega_1}$, $\theta_1 = \theta_{|\Omega_1}$ and $\theta_2 = \theta_{|\Omega_2}$, when θ is the unique solution of (1).

Problem (2) is approximated in a (not necessarily conforming) finite element space $X_h \times M_h$ of $X \times M$ based on triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of the domains Ω_1 and Ω_2 made of isotropic elements. Under appropriate properties described below, the discrete problem has a unique discrete solution $(u_h, p_h) \in X_h \times M_h$. We then consider an efficient and reliable residual a posteriori error estimator for the errors $\underline{\epsilon}_1 = \underline{\xi} - \underline{\xi}_h$ in the $L^2(\Omega_2)$-norm, $e = \theta_1 - \theta_{1h}$ in the $L^2(\Omega_1)$-norm and $e_2 = \nabla\theta_2 - \nabla_h\theta_{2h}$ in the $L^2(\Omega_2)$-norm.

A posteriori error estimations are highly recommended for problem (1) since the solution presents corner singularities [10,11,13,17,21] or boundary layers [18,19], that can be even difficult to describe explicitly (if A has large oscillations for instance). A priori error estimations can then be compromised since they require the explicit knowledge of the singularities or boundary layers.

On one hand, a posteriori error estimators of standard Galerkin methods for elliptic boundary value problems is in our days well understood (see for instance [25] and the references cited there). The analysis of isotropic a posteriori error estimators for the mixed finite element method were initiated in [2,4,1] and definitively fixed in [22]. On the other hand, the coupling between mixed and standard Galerkin methods can have some interests. Namely it might be interesting for the coupling of different models and materials, for meshes construction reasons (simpler domains can be easily meshed independently), and finally the quantity $A\nabla\theta$ of physical interest could be required only on a part of the domain (recall that by a mixed method this quantity is directly approximated and is then obtained without any postprocessing). Such a coupling were initiated in [26] and its a priori error analysis were performed in [16]. But to our knowledge, the a posteriori error analysis of this coupled method was not done yet. Therefore our goal is to derive this a posteriori error analysis in a quite large setting allowing to include standard mixed methods in Ω_1 and (not necessarily conforming) Galerkin methods in Ω_2.

Since the meshes do not necessarily align at the interface, the resulting spaces are necessarily nonconforming. Therefore our a posteriori error analysis com-

bines some ideas from [9] developed for the a posteriori error analysis of nonconforming Galerkin methods with the techniques from [4,22] for the a posteriori error analysis of mixed methods.

For the sake of simplicity we have restricted ourselves to the case of 2D problems and to the use of isotropic meshes. Combining the results from [7,22,8] with our approach below, all presented results hold for 3D domains and for anisotropic meshes (fulfilling standard assumptions from [15,22,8]).

The schedule of the paper is the following one: Section 2 recalls the discretization of our problem and introduces some natural conditions on the finite element spaces. In section 3 we recall some interpolation error estimates for Clément type interpolants. Since the meshes do not fit at the interface we pay some attentions on the bubble functions along this interface. Moreover some specific surjectivity results of the divergence operator are proved. The efficiency and reliability of the error are established in section 4. Finally section 5 is devoted to numerical tests which confirm our theoretical analysis.

Let us finish this introduction with notation used in the whole paper: For shortness the $L^2(D)$-norm will be denoted by $\|\cdot\|_D$. In the case $D = \Omega$, the index Ω will be dropped. The usual norm and seminorm of $H^1(D)$ are denoted by $\|\cdot\|_{1,D}$ and $|\cdot|_{1,D}$, respectively. The notation $\underline{u}$ means that the quantity u is a vector and $\nabla\underline{u}$ means the matrix $(\partial_j u_i)_{1\le i,j\le d}$ (i being the index of row and j the index of column). For a vector function $\underline{u}$ we denote by $\operatorname{curl}\underline{u} = \partial_1 u_2 - \partial_2 u_1$. On the other hand for a scalar function ϕ we write $\underline{\operatorname{curl}}\ \phi = (\partial_2\phi, -\partial_1\phi)$. Finally, the notation $a \lesssim b$ and $a \sim b$ means the existence of positive constants C_1 and C_2 (which are independent of the mesh $\mathcal{T}$ and of the function under consideration) such that $a \le C_2 b$ and $C_1 b \le a \le C_2 b$, respectively.

2 Discretization of the problem

The domain $\Omega_i, i = 1, 2$ is discretized by a conforming mesh $\mathcal{T}_i$, cf. [5]. All elements are either triangles or rectangles. We assume that both triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ are regular (or isotropic) but they do not need to fit at the interface Γ. For convenience we denote by $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, the mesh in the whole Ω.

An element will be denoted by T, T_i or T', its edges are denoted by E (supposed to be open). The set of all edges included in $\Omega_i, i = 1, 2$ or belonging to the boundary $\partial\Omega \setminus \Gamma$ of the triangulation $\mathcal{T}_i$ will be denoted by $\mathcal{E}_i$. Clearly if an edge E of an element $T \in \mathcal{T}_1$ is included into Γ, then E is not necessarily an edge of an element T' of $\mathcal{T}_2$. Therefore the set $\mathcal{E}_\Gamma$ is the set of intersection

of such edges, namely

$$\mathcal{E}_\Gamma = \{E_1 \cap E_2 : E_i \text{ is an edge of } T_i \in \mathcal{T}_i, i = 1, 2\}.$$

The measure of an element or edge is denoted by $|T| := \text{meas}_2(T)$ and $|E| := \text{meas}_1(E)$, respectively. As usual h_T is the diameter of T and $h_E = |E|$ is the diameter of E.

Let $\underline{x}$ denote a nodal point of $\mathcal{T}_i, i = 1, 2$ (i.e. a vertex of an element of $\mathcal{T}_i$), and let $\mathcal{N}_{\bar{\Omega}_i}$ be the set of nodes of the mesh $\mathcal{T}_i$.

For an edge E of an element T, let $\underline{n} = (n_x, n_y)^\top$ be the *outer normal vector*. Furthermore, for each edge E we fix one of the two normal vectors and denote it by $\underline{n}_E$. We introduce additionally the *tangent vector* $\underline{t} = \underline{n}^\perp := (-n_y, n_x)^\top$ such that it is oriented positively (with respect to T). Similarly we set $\underline{t}_E := \underline{n}_E^\perp$.

The *jump* of some (scalar or vector valued) function v across an edge E at a point $\underline{y} \in E$ is then defined as

$$[\![v(\underline{y})]\!]_E := \begin{cases} \lim\limits_{\alpha \to 0+} v(\underline{y} + \alpha \underline{n}_E) - v(\underline{y} - \alpha \underline{n}_E) & \text{for an interior edge } E, \\ v(\underline{y}) & \text{for an edge } E \subset \partial\Omega_i \cap \partial\Omega. \end{cases}$$

Note that the sign of $[\![v]\!]_E$ depends on the orientation of $\underline{n}_E$. However, terms such as a gradient jump $[\![\nabla v\, \underline{n}_E]\!]_E$ are independent of this orientation.

Furthermore one requires local subdomains (also known as patches). If $T \in \mathcal{T}$, let ω_T be the union of all elements T' of $\mathcal{T}$ such that $T' \cap T$ is an edge of T or of T'. Similarly if $E \in \mathcal{E}_i$ (resp. $\underline{x} \in \mathcal{N}_{\bar{\Omega}_i}$) let ω_E (resp. $\omega_{\underline{x}}$) be the union of elements of $\mathcal{T}$ having E as edge (resp. as node). Finally if $E \in \mathcal{E}_\Gamma$, then E is included into an edge E_i of an element T_i of $\mathcal{T}_i$, $i = 1$ and 2, and therefore we set $\omega_E = T_1 \cup T_2$.

2.1 Finite element spaces assumptions

The space $H^1_D(\Omega_2)$ is approximated by a (not necessarily conforming) finite element space V_h but it is large enough to contain the space of piecewise $\mathbb{P}_1$ (or $\mathbb{Q}^1$) continuous functions on the triangulation $\mathcal{T}_2$. Namely if we set

$\mathcal{P}_T = \mathbb{P}_1(T)$ if T is a triangle,
$\mathcal{P}_T = \mathbb{Q}_1(T)$ if T is a rectangle,

then we set

$$S(\Omega_2, \mathcal{T}_2) := \{v_h \in C(\bar{\Omega}_2) : \ v_h|_T \in \mathcal{P}_T, \forall T \in \mathcal{T}_2\} \ \subset H^1(\Omega_2). \tag{3}$$

Our assumption on V_h is then

$$S(\Omega_2, \mathcal{T}_2) \cap H^1_D(\Omega_2) \subset V_h. \tag{4}$$

For a nonconforming space V_h we further assume that the following Crouzeix-Raviart property holds (see [7]):

$$\int_E [\![v_h]\!]_E = 0, \forall v_h \in V_h. \tag{5}$$

These assumptions are quite weak and allows the use of standard conforming element (like conforming piecewise $\mathbb{P}_k$ (or $\mathbb{Q}_k$) finite element spaces, $k \geq 1$) or standard nonconforming elements (like Crouzeix-Raviart elements [9,7,23]).

Note that the Crouzeix-Raviart property (5) directly implies that the discontinuous H^1-seminorm is a norm on V_h, namely

$$\forall v_h \in V_h : \|\nabla_h v_h\|^2_{\Omega_2} = 0 \Rightarrow v_h = 0, \tag{6}$$

where ∇_h is the broken gradient of v_h:

$$(\nabla_h v_h)_{|T} := \nabla(v_{h|T}), \forall T \in \mathcal{T}_2.$$

Now the pair $(H(\mathrm{div}\,, \Omega_1), M)$ involved in the mixed method in Ω_1 is approximated by a pair (X_{1h}, M_h) that satisfies the next properties:

$$\{\underline{q} \in H(\mathrm{div}\,, \Omega_1) : \underline{q}_{\,|T} \in [\mathbb{P}_0(T)]^d, \forall T \in \mathcal{T}_1\} \subset X_{1h}, \tag{7}$$
$$X_{1h} \subset \{\underline{q} \in H(\mathrm{div}\,, \Omega_1) : \underline{q}_{\,|T} \in [H^1(T)]^d, \forall T \in \mathcal{T}_1\}, \tag{8}$$
$$\{v \in L^2(\Omega_1) : v_{|K} \in \mathbb{P}_0(K), \forall K \in \mathcal{T}_1\} \subset M_h, \tag{9}$$
$$\mathrm{div}\, X_{1h} = M_h. \tag{10}$$

We suppose that the commuting diagram property holds [3,4]: There exists an interpolation operator $\Pi_h : W \to X_{1h}$, where $W = H(\mathrm{div}\,, \Omega_1) \cap L^s(\Omega_1)$,

with $s > 2$, such that the next diagram commutes

$$\begin{array}{ccc} W & \stackrel{\text{div}}{\rightarrow} & M \\ \Pi_h \downarrow & & \downarrow \rho_h \\ X_{1h} & \stackrel{\text{div}}{\rightarrow} & M_h, \end{array} \tag{11}$$

where ρ_h is the $L^2(\Omega_1)$-orthogonal projection on M_h. This property implies in particular that

$$\text{div}\,(Id - \Pi_h)W \perp M_h. \tag{12}$$

This orthogonality holds for the $L^2(\Omega_1)$ inner product, and Id means the identity operator.

We further assume that the interpolant satifies the global stability estimate

$$\|\Pi_h \underline{q}\,\|_{\Omega_1} \lesssim \|\underline{q}\,\|_{1,\Omega_1}, \forall \underline{q} \in [H^1(\Omega_1)]^2. \tag{13}$$

It is well known (see e.g. Lemma 3.6 of [22]) that this assumption added to (10) and (11) lead to the uniform discrete inf-sup condition.

Finally we assume that Π_h satisfies the approximation property

$$\int_E v_h(\underline{q} - \Pi_h \underline{q}) \cdot \underline{n}_E = 0, \forall \underline{q} \in W, v_h \in M_h, E \in \mathcal{E}_1 \cup \mathcal{E}_\Gamma. \tag{14}$$

Such properties are satisfied by standard elements, like the Raviart-Thomas elements (in short RT), the Brezzi-Douglas-Marini elements (BDM), and the Brezzi-Douglas-Fortin-Marini elements (BDFM).

For any element $T \in \mathcal{T}_1$, we recall in the next table the finite dimensional spaces $D_k(T)$ and $M_k(T)$, where $k \in \mathbb{N}$, for the RT, BDM and BDFM elements.

Name	Element	$M_k(T)$	$D_k(T)$
RT	Triangle	$RT_k := [\mathbb{P}_k]^d + \underline{x}\,\widetilde{\mathbb{P}}_k$	$\mathbb{P}_k$
RT	Rectangle	$\mathbb{P}_{k+1,k} \times \mathbb{P}_{k,k+1}$	$\mathbb{Q}_k$
BDM	Triangle	$[\mathbb{P}_{k+1}]^d$	$\mathbb{P}_k$
BDFM	Triangle	$\{\underline{q} \in [\mathbb{P}_{k+1}]^d : \underline{q} \cdot \underline{n} \in \mathcal{R}_k(\partial T)\}$	$\mathbb{P}_k$

Here $\mathbb{P}_{k+1,k}$ means the space of polynomials of degree $k+1$ in x_1 and of degree k in x_2, $\widetilde{\mathbb{P}}_k$ means the space of homogeneous polynomials of degree k, while $\mathcal{R}_k(\partial T)$ denotes the space of functions defined in ∂T which are polynomials of degree at most k on each edge of T. With these sets we may define

$$M_h := \{v_h \in M : v_{h|T} \in D_k(T), \forall T \in \mathcal{T}_1\}, \tag{15}$$
$$X_{1h} := \{\underline{p}_h \in H(\mathrm{div}\,, \Omega_1) : \underline{p}_{h|T} \in M_k(T), \forall T \in \mathcal{T}_1\}. \tag{16}$$

For these element pairs (X_{1h}, M_h), the assumptions (10), (11), (13) and (14) are checked in section III.3 of [3]; while (7), (8) and (9) clearly hold.

Our next upper error bound uses the following orthogonality property:

$$\int_E \theta_h(\underline{v} - \Pi_h \underline{v}) \cdot \underline{n}_E = 0, \forall \underline{v} \in [H^1(\Omega_1)]^2, \theta_h \in V_h, E \in \mathcal{E}_\Gamma. \tag{17}$$

This general assumption is made if Γ is not "smooth", and can be avoided in some particular cases (see subsection 3.3 below). Clearly the assumption (17) holds if V_h is made of piecewise $\mathbb{P}_1$ (or $\mathbb{Q}_1$) elements and X_{1h} is made of RT_1 (or BDM_1 or $BDFM_1$) elements.

Finally the approximation space of X is defined by

$$X_h = X_{1h} \times V_h,$$

which is a (not necessarily conforming) approximation of X.

2.2 Discrete formulation

The discrete problem associated with (2) is to find $(u_h, p_h) \in X_h \times M_h$ such that

$$\begin{cases} a_h(u_h, v_h) + b(v_h, p_h) = (f, \chi_h)_2, \forall v_h \in X_h, \\ b(u_h, q_h) = -(f, q_h)_1, \quad \forall q_h \in M_h, \end{cases} \tag{18}$$

where

$$\begin{aligned} a_h(u_h, v_h) := & (A^{-1}\underline{\xi}_h, \eta_h)_1 + (A\nabla_h \theta_{2h}, \nabla_h \chi_h)_2 \\ & - (\eta_h \cdot \underline{n}, \theta_{2h})_\Gamma + (\underline{\xi}_h \cdot \underline{n}, \chi_h)_\Gamma, \forall u_h = (\underline{\xi}_h, \theta_{2h}), v_h = (\eta_h, \chi_h) \in X_h. \end{aligned}$$

Since a_h is coercive on X_h (due to (6)) and since the discrete inf-sup condition holds, this problem has a unique solution.

Let us recall that the errors are defined by

$$E := u - u_h = (\underline{\xi} - \underline{\xi}_h, \theta_2 - \theta_{2h}),\ e := \theta_1 - \theta_{1h} = p - p_h.$$

Therefore by (2) we directly get the defect equations:

$$\begin{cases} a_h(E, v) + b(v, e) = (f, \chi)_2 - a_h(u_h, v) - b(v, p_h), \forall v = (\eta, \chi) \in X, \\ b(E, q) = -(f, q)_1 - b(u_h, q), \ \ \forall q \in M. \end{cases} \tag{19}$$

In particular taking $v = v_h \in X \cap X_h$ and $q = q_h \in M_h$, owing to (18) we obtain the Galerkin orthogonality relations

$$a_h(E, v_h) + b(v_h, e) = 0, \forall v_h \in X_h \cap X, \tag{20}$$
$$b(E, q_h) = 0, \ \ \forall q_h \in M_h. \tag{21}$$

For further purposes, we introduce the error $\underline{\epsilon}$ defined by

$$\underline{\epsilon} = \begin{cases} \underline{\xi} - \underline{\xi}_h & \text{in } \Omega_1, \\ A(\nabla\theta_2 - \nabla_h\theta_{2h}) & \text{in } \Omega_2. \end{cases}$$

This expression may be understood as the error on the gradient of θ. Its introduction is the key point of our analysis.

3 Analytical tools

3.1 Bubble functions, extension operators, and inverse inequalities

For the analysis of the lower bound we need to use some bubble functions and extension operators that satisfy certain properties. A special attention will be paid for the edges along the interface Γ since there the meshes do not fit together (or the full mesh is nonconforming in a neighbourhood of Γ). For that reason, we require the meshes to have the same size along Γ, namely

$$\forall T_1 \in \mathcal{T}_1, T_2 \in \mathcal{T}_2 : T_1 \cap T_2 = E \in \mathcal{E}_\Gamma \text{ then } |E| \sim h_{T_1} \sim h_{T_2}. \tag{22}$$

We need two types of bubble functions, namely b_T and b_E associated with an element T and an edge E, respectively. For a triangle T, denoting by $\lambda_{a_i^T}$, $i = 1, \cdots, 3$, the barycentric coordinates of T and by $a_i^{E,T}, i = 1, 2$ the vertices of the edge $E \subset \partial T$ we recall that

$$b_T = 9 \prod_{i=1}^{3} \lambda_{a_i^T} \text{ and } b_{E,T} = 4 \prod_{i=1}^{2} \lambda_{a_i^{E,T}}.$$

Similarly for a rectangle T and an edge E of T, b_T is the unique element in $\mathbb{Q}^2(T)$ such that

$$b_T = 0 \text{ on } \partial T,$$

and equal to 1 at the center of gravity of T; while the function $b_{E,T}$ is the unique element in $\mathbb{Q}^2(T)$ such that

$$b_{E,T} = 0 \text{ on } \partial T \setminus E,$$

and equal to 1 at the center of gravity of E.

For an edge $E \in \mathcal{E}_i$, $i = 1$ or 2, the bubble function b_E is defined on ω_E by

$$b_{E|T} = b_{E,T} \text{ on } T \subset \omega_E.$$

One recalls that

$$b_T = 0 \text{ on } \partial T, \qquad b_E = 0 \text{ on } \partial \omega_E, \qquad \|b_T\|_{\infty,T} = \|b_E\|_{\infty,\omega_E} = 1.$$

If an edge $E' \in \mathcal{E}_\Gamma$ with $E' \subset T \in \mathcal{T}_i$, $i = 1$ or 2, is not a full edge of T, then its associated bubble function $b_{E',T}$ has to be modified (compare with section 4.3 of [8]). Indeed in that case, we introduce an artificial element T' such that $T' \subset T$, E' is a full edge of T' and that satisfies

$$|T'| \sim |T|, \quad h_T \lesssim h_{T'}. \tag{23}$$

If T is a triangle, then T' is the triangle obtained by joining E' to the vertex of T opposite to the edge E of T containing E' (see Figure 1). If T is a rectangle, then T' is the rectangle defined by $T' = E' \times I$, when $T = E \times I$, E being the edge E of T containing E' (see Figure 2). Recalling that our mesh assumption (22) means that $|E'| \sim |E|$, we directly get the properties (23).

With the help of this artificial element, we define $b_{E',T}$ as follows:

$$b_{E',T} = \begin{cases} b_{E',T'} \text{ on } T', \\ 0 \text{ on } T \setminus T'. \end{cases}$$

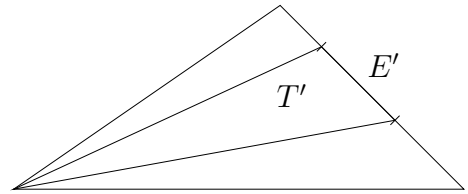

Fig. 1. Definition of T' for a triangle T.

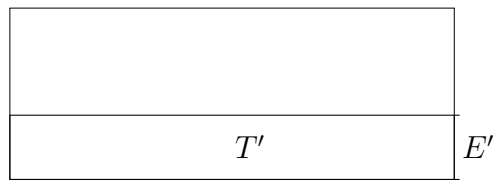

Fig. 2. Definition of T' for a rectangle T.

We finally define b_E on ω_E as before. Remark that the builded function b_E belongs to $H_0^1(\omega_E)$.

For an edge $\hat{E}$ of the reference element $\hat{T}$ included into the $\hat{x}$-axis, the extension $\mathrm{F}_{\mathrm{ext}}(\hat{v}_{\hat{E}})$ of $\hat{v}_{\hat{E}} \in C(\hat{E})$ to $\hat{T}$ is defined by $\mathrm{F}_{\mathrm{ext}}(\hat{v}_{\hat{E}})(\hat{x}, \hat{y}) = \hat{v}_{\hat{E}}(\hat{x})$. For an edge $E \in \mathcal{E}_i$, $i = 1$ or 2, which is an edge of an element $T \in \mathcal{T}_i$ and $v_E \in C(\bar{E})$, $\mathrm{F}_{\mathrm{ext}}(v_E)$ is obtained using the affine mapping that sends $\hat{T}$ to T and $\hat{E}$ to E. For $E \in \mathcal{E}_\Gamma$, we proceed similarly by using the artificial element T' and extension by zero outside T'.

Now we may recall the so-called inverse inequalities that are proved using classical scaling techniques (cf. [25] for the standard case and Lemma 4.9 of [8] for the edges of Γ).

Lemma 3.1 (Inverse inequalities) *Assume that (22) holds. Let $T \in \mathcal{T}_i$ and $E \in \mathcal{E}_i \cup \mathcal{E}_\Gamma$, $i = 1$ or 2. Let $v_T \in \mathbb{P}_{k_0}(T)$ and $v_E \in \mathbb{P}_{k_1}(E)$, for some nonnegative integers k_0 and k_1. Then the following inequalities hold, the inequality constants depending on the polynomial degree k_0 or k_1 but not on T, E or v_T, v_E.*

$$\|v_T b_T^{1/2}\|_T \sim \|v_T\|_T, \tag{24}$$

$$\|\nabla(v_T b_T)\|_T \lesssim h_T^{-1}\|v_T\|_T, \tag{25}$$

$$\|v_E b_E^{1/2}\|_E \sim \|v_E\|_E, \tag{26}$$

$$\|\mathrm{F}_{\mathrm{ext}}(v_E) b_E\|_{\omega_E} \lesssim h_E^{1/2}\|v_E\|_E, \tag{27}$$

$$\|\nabla(\mathrm{F}_{\mathrm{ext}}(v_E) b_E)\|_{\omega_E} \lesssim h_E^{-1/2}\|v_E\|_E. \tag{28}$$

For our analysis we need some interpolation operators that map a function from $H^1(\Omega_2)$ to the usual space $S(\Omega_2, \mathcal{T}_2)$. Hence Lagrange interpolation is unsuitable, but Clément like interpolant is more appropriate.

Recall that the nodal basis function $\varphi_{\underline{x}} \in S(\Omega_2, \mathcal{T}_2)$ associated with a node $\underline{x}$ is uniquely determined by the condition

$$\varphi_{\underline{x}}(\underline{y}) = \delta_{\underline{x},\underline{y}} \qquad \forall \underline{y} \in \mathcal{N}_{\bar{\Omega}_2}.$$

Next, the Clément interpolation operator will be defined via the basis functions $\varphi_{\underline{x}} \in S(\Omega_2, \mathcal{T}_2)$.

Definition 3.2 (Clément interpolation operator) *We define the* Clément interpolation operator $\mathrm{I}_{\mathrm{Cl}} : H^1(\Omega_2) \to S(\Omega_2, \mathcal{T}_2)$ *by*

$$\mathrm{I}_{\mathrm{Cl}} v := \sum_{\underline{x} \in \mathcal{N}_{\bar{\Omega}_2}} \frac{1}{|\omega_{\underline{x}}|} \left(\int_{\omega_{\underline{x}}} v \right) \varphi_{\underline{x}}.$$

Finally we may state the interpolation estimates.

Lemma 3.3 (Clément interpolation estimates) *For any $v \in H^1(\Omega_2)$ and any $T \in \mathcal{T}_2$ we have*

$$\|v - \mathrm{I}_{\mathrm{Cl}} v\|_T \lesssim h_T \|\nabla v\|_{\omega_T}, \tag{29}$$

$$\|v - \mathrm{I}_{\mathrm{Cl}} v\|_E \lesssim h_T^{1/2} \|\nabla v\|_T, \forall E \text{ edge of } T. \tag{30}$$

Proof: The proof of the estimates (29) and (30) is given in [6] and simply use some scaling arguments. ■

Note that if $v \in H^1_D(\Omega_2)$, then $\mathrm{I}_{\mathrm{Cl}} v$ does no more satisfy the Dirichlet boundary condition on $\partial\Omega_2 \cap \partial\Omega$. Therefore we define another Clément interpolant $\mathrm{I}^0_{\mathrm{Cl}} z$ in order to satisfy this boundary condition. Namely $\mathrm{I}^0_{\mathrm{Cl}} : H^1_D(\Omega_2) \to S(\Omega_2, \mathcal{T}_2) \cap H^1_D(\Omega_2)$ is defined by

$$\mathrm{I}^0_{\mathrm{Cl}} v := \sum_{\underline{x} \in \mathcal{N}_{\Omega_2 \cup \Gamma}} \frac{1}{|\omega_{\underline{x}}|} \left(\int_{\omega_{\underline{x}}} v \right) \varphi_{\underline{x}},$$

where $\mathcal{N}_{\Omega_2 \cup \Gamma}$ is the set of nodes in Ω_2 and on Γ. Clearly the estimates from Lemma 3.3 remain valid for this second Clément interpolation operator.

Obviously for a function in $H^1(\Omega_1)$ we can define its Clément interpolant based on the triangulation $\mathcal{T}_1$ and similar results hold.

Thanks to the assumption (9) the same proof shows that the projection operator ρ_h on M_h satisfies an estimate like (29), namely we have the

Lemma 3.4 *For all $v \in H^1(\Omega_1)$ we have*

$$\|v - \rho_h v\|_T \lesssim h_T \|\nabla v\|_T, \forall T \in \mathcal{T}_1. \tag{31}$$

3.3 Surjectivity of the divergence operator

Here we focus on the surjectivity of the divergence operator from $[H^1(\Omega)]^2$ to $L^2(\Omega_1)$. First we consider the general case (proved in Lemma 3.5 of [22]) and then consider a particular case.

Lemma 3.5 *Let g be an arbitrary function in $L^2(\Omega_1)$, then there exists $\underline{v} \in [H^1(\Omega)]^2$ such that*

$$\operatorname{div} \underline{v} = g \text{ in } \Omega_1, \tag{32}$$
$$\operatorname{div} \underline{v} = 0 \text{ in } \Omega_2, \tag{33}$$
$$\|\underline{v}\|_{1,\Omega} \lesssim \|g\|_{\Omega_1}. \tag{34}$$

Proof: We follow the proof of Lemma 3.5 of [22] but slightly adapted in order to guarantee (33). Consider a domain D with a smooth boundary such that $\bar{\Omega} \subset D$. We extend g by zero outside Ω_1 to get $\tilde{g}$ in $L^2(D)$. Let $\psi \in H_0^1(D)$ be the unique solution of

$$\Delta\psi = \tilde{g} \text{ in } D.$$

As $\tilde{g} \in L^2(D)$ and D has a smooth boundary, ψ belongs to $H^2(D)$ with the estimate

$$\|\psi\|_{2,D} \lesssim \|\tilde{g}\|_D = \|g\|. \tag{35}$$

Therefore $\underline{v}$ defined in Ω by

$$\underline{v} = \nabla\psi \text{ in } \Omega$$

belongs to $[H^1(\Omega)]^2$ and satisfies (32), (33) as well as (34) as a consequence of (35). ■

This lemma does not take into account the smoothness of Ω_1 and therefore no boundary conditions on $\underline{v}$ are imposed on the boundary of Ω_1. For further

purposes, let us look at the case when Ω_1 is convex near Γ (convex in the following sense) :

Lemma 3.6 *Assume that Ω_1 is convex near Γ in the sense that there exists a convex domain D of $\mathbb{R}^2$ such that $\bar{\Omega}_1 \subset D$ and $\Gamma \subset \partial D$. Let g be an arbitrary function in $L^2(\Omega_1)$, then there exists $\underline{v} \in [H^1(\Omega_1)]^2$ satisfying (32), (34) and the boundary condition*

$$\underline{v} \cdot \underline{n} = 0 \text{ on } \Gamma. \tag{36}$$

Proof: Fix a domain D as in the statement of the Lemma. We extend g by a constant outside Ω to get $\tilde{g}$ in $L^2(D)$ with a zero mean, i.e.,

$$\int_D \tilde{g} = 0.$$

Let $\psi \in H^1(D)$ satisfy $\int_D \psi = 0$ and be the unique solution of the Neumann problem

$$\begin{aligned} &\Delta\psi = \tilde{g} \text{ in } D, \\ &\frac{\partial \psi}{\partial n} = 0 \text{ on } \partial D. \end{aligned}$$

As $\tilde{g} \in L^2(D)$ and D is convex, ψ belongs to $H^2(D)$ with the estimate (see [20] or Theorems 3.2.4.1 and 3.2.4.2 of [14])

$$\|\psi\|_{2,D} \lesssim \|\tilde{g}\|_D \lesssim \|g\|_{\Omega_1}. \tag{37}$$

Therefore $\underline{v}$ defined in Ω_1 by

$$\underline{v} = \nabla\psi \text{ in } \Omega_1$$

belongs to $[H^1(\Omega_1)]^2$ and satisfies all requested properties. ■

Corollary 3.7 *Let the assumption of Lemma 3.6 be satisfied, and assume that X_{1h} is made of $RT_k, k \geq 0$ (or BDM_k or $BDFM_k$) elements. Then the interpolant $\Pi_h \underline{v}$ of the function $\underline{v}$ from Lemma 3.6 satisfies*

$$\Pi_h \underline{v} \cdot \underline{n} = 0 \text{ on } \Gamma.$$

Consequently the identity (17) holds for this function $\underline{v}$.

4 Error estimators

4.1 Residual error estimators

The exact element residual R_T is defined as follows:

$$R_T = (f + \operatorname{div} \underline{\xi}_h)_{|T} = (f - \rho_h f)_{|T} \text{ if } T \in \mathcal{T}_1,$$
$$R_T = (f + \operatorname{div} A\nabla_h \theta_{2h})_{|T} \text{ if } T \in \mathcal{T}_2.$$

From the first expression we see that for $T \in \mathcal{T}_1$, the exact element residual R_T is already an approximation term. For the sake of simplicity if $T \in \mathcal{T}_2$, we do not replace the exact element residual R_T by an approximate element residual r_T. Nevertheless our analysis below could be made in that case as well.

For $\underline{\xi}_h \in X_{1,h}$ and $\theta_{2h} \in V_h$ we define the tangential and normal jumps across an edge E by

$$J_{E,t} := \begin{cases} [\![A^{-1}\underline{\xi}_h \cdot \underline{t}_E]\!]_E & \text{if } E \in \mathcal{E}_1, \\ [\![\nabla_h \theta_{2h} \cdot \underline{t}_E]\!]_E & \text{if } E \in \mathcal{E}_2, \\ (\nabla_h \theta_{2h} - A^{-1}\underline{\xi}_h) \cdot \underline{t}_E & \text{if } E \in \mathcal{E}_\Gamma, \end{cases}$$

$$J_{E,n} := \begin{cases} 0 & \text{if } E \in \mathcal{E}_2 \cap \partial\Omega, \\ [\![A\nabla_h \theta_{2h} \cdot \underline{n}_E]\!]_E & \text{if } E \in \mathcal{E}_2 \setminus \partial\Omega, \\ (A\nabla_h \theta_{2h} - \underline{\xi}_h) \cdot \underline{n}_E & \text{if } E \in \mathcal{E}_\Gamma. \end{cases}$$

Definition 4.1 (Residual error estimator) *For any $T \in \mathcal{T}_1$, the local residual error estimator is defined by*

$$\eta_T^2 := h_T^2 \|\operatorname{curl}(A^{-1}\underline{\xi}_h)\|_T^2 + h_T^2 \min_{q_h \in M_h} \|A^{-1}\underline{\xi}_h - \nabla q_h\|_T^2 + \sum_{E \subset \partial T} h_E \|J_{E,t}\|_E^2.$$

On the other hand for any $T \in \mathcal{T}_2$, the local residual error estimator is defined by

$$\eta_T^2 := h_T^2 \|R_T\|_T^2 + \sum_{E \subset \partial T} h_E (\|J_{E,t}\|_E^2 + \|J_{E,n}\|_E^2).$$

The global residual error estimator is simply

$$\eta^2 := \sum_{T \in \mathcal{T}} \eta_T^2.$$

Furthermore the local and global approximation terms are denoted by

$$\zeta_T = h_T \|f - \rho_h f\|_T, \forall T \in \mathcal{T}_1, \quad \zeta^2 = \sum_{T \in \mathcal{T}_1} \zeta_T^2.$$

If for all $T \in \mathcal{T}_2$ an approximate element residual r_T is used, then

$$\zeta_T = h_T \|r_T - R_T\|_T, \forall T \in \mathcal{T}_2,$$

and the global approximation term should be defined by

$$\zeta^2 = \sum_{T \in \mathcal{T}} \zeta_T^2.$$

4.2 Proof of the upper error bound

The use of Lemma 3.5 allows to prove the following error bound on $e = \theta_1 - \theta_{1h}$.

Lemma 4.2 *The next estimate holds:*

$$\|\theta_1 - \theta_{1h}\|_{\Omega_1} \lesssim \|A^{-1}(\underline{\xi} - \underline{\xi}_h)\|_{\Omega_1} + \|\nabla \theta_2 - \nabla_h \theta_{2h}\|_{\Omega_2} + \eta. \tag{38}$$

Proof: Owing to Lemma 3.5 there exists a solution $\underline{v} \in [H^1(\Omega)]^2$ of (32) with $g = e$ and that satisfies (33) and (34). By (32) we may write

$$\|e\|_{\Omega_1}^2 = \int_{\Omega_1} (\theta_1 - \theta_{1h}) \operatorname{div} \underline{v}.$$

By Green's formula and the fact that $\nabla \theta_1 = A^{-1} \underline{\xi}$ (recall that $\theta_1 = 0$ on $\partial \Omega_1 \setminus \Gamma$) we get

$$\|e\|_{\Omega_1}^2 = -\int_{\Omega_1} (A^{-1} \underline{\xi}) \cdot \underline{v} + \int_{\Gamma} \theta_1 \underline{v} \cdot \underline{n} - \int_{\Omega_1} \theta_{1h} \operatorname{div} \underline{v}.$$

Now using the commuting property (12) we obtain

$$\|e\|_{\Omega_1}^2 = -\int_{\Omega_1} (A^{-1} \underline{\xi}) \cdot \underline{v} + \int_{\Gamma} \theta_1 \underline{v} \cdot \underline{n} - \int_{\Omega_1} \theta_{1h} \operatorname{div} \Pi_h \underline{v}.$$

The discrete mixed formulation (18) with $v_h = (\Pi_h \underline{v}, 0)$ then leads to

$$\begin{aligned}\|e\|^2_{\Omega_1} =& -\int_{\Omega_1} (A^{-1}(\underline{\xi} - \underline{\xi}_h)) \cdot \underline{v} - \int_{\Omega_1} (A^{-1}\underline{\xi}_h) \cdot (\underline{v} - \Pi_h \underline{v}) \\ &+ \int_{\Gamma} (\theta_1 \underline{v} \cdot \underline{n} - \theta_{2h} \Pi_h \underline{v} \cdot \underline{n}).\end{aligned}$$

Green's formula on each element and properties (12) and (14) imply that

$$\sum_{T \in \mathcal{T}_1} \int_T \nabla q_h \cdot (\underline{v} - \Pi_h \underline{v}) = 0, \forall q_h \in M_h.$$

Therefore, we have

$$\begin{aligned}\|e\|^2_{\Omega_1} =& -\int_{\Omega_1} (A^{-1}(\underline{\xi} - \underline{\xi}_h)) \cdot \underline{v} \\ &- \sum_{T \in \mathcal{T}_1} \int_T (A^{-1}\underline{\xi}_h - \nabla q_h) \cdot (\underline{v} - \Pi_h \underline{v}) \\ &+ \int_{\Gamma} (\theta_1 \underline{v} \cdot \underline{n} - \theta_{2h} \Pi_h \underline{v} \cdot \underline{n}), \forall q_h \in M_h.\end{aligned}$$

As $\theta_1 = \theta_2$ on Γ and recalling the property (17), namely

$$\int_{\Gamma} \theta_{2h} (\underline{v} - \Pi_h \underline{v}) \cdot \underline{n} = 0,$$

we arrive at

$$\begin{aligned}\|e\|^2_{\Omega_1} =& -\int_{\Omega_1} (A^{-1}(\underline{\xi} - \underline{\xi}_h)) \cdot \underline{v} \\ &- \sum_{T \in \mathcal{T}_1} \int_T (A^{-1}\underline{\xi}_h - \nabla q_h) \cdot (\underline{v} - \Pi_h \underline{v}) \\ &+ \int_{\Gamma} (\theta_2 - \theta_{2h}) \underline{v} \cdot \underline{n}, \forall q_h \in M_h.\end{aligned} \tag{39}$$

We now transform the last term of this right-hand side. Indeed, elementwise integration by parts and the property (33) yield

$$\int_{\Gamma} (\theta_2 - \theta_{2h}) \underline{v} \cdot \underline{n} = \int_{\Omega_2} \nabla_h (\theta_2 - \theta_{2h}) \cdot \underline{v}$$

$$- \sum_{E \in \mathcal{E}_2} \int_E [\![\theta_2 - \theta_{2h}]\!]_E \underline{v} \cdot \underline{n}_E.$$

Since θ_2 is "continuous" through the interior edges of Ω_2 and using the Crouzeix-Raviart property (5), we obtain

$$\begin{aligned}\int_\Gamma (\theta_2 - \theta_{2h}) \underline{v} \cdot \underline{n} = & \int_{\Omega_2} \nabla_h (\theta_2 - \theta_{2h}) \cdot \underline{v} \\ & + \sum_{E \in \mathcal{E}_2} \int_E [\![\theta_{2h}]\!]_E (\underline{v} - \mathcal{M}_E \underline{v}) \cdot \underline{n}_E,\end{aligned}$$

where $\mathcal{M}_E \underline{v} = |E|^{-1} \int_E \underline{v}$ is the mean of $\underline{v}$. Inserting this identity in (39) we arrive at

$$\begin{aligned}\|e\|_{\Omega_1}^2 = & - \int_{\Omega_1} (A^{-1}(\underline{\xi} - \underline{\xi}_h)) \cdot \underline{v} \\ & - \sum_{T \in \mathcal{T}_1} \int_T (A^{-1} \underline{\xi}_h - \nabla q_h) \cdot (\underline{v} - \Pi_h \underline{v}) \\ & + \int_{\Omega_2} \nabla_h (\theta_2 - \theta_{2h}) \cdot \underline{v} \\ & + \sum_{E \in \mathcal{E}_2} \int_E [\![\theta_{2h}]\!]_E (\underline{v} - \mathcal{M}_E \underline{v}) \cdot \underline{n}_E, \forall q_h \in M_h.\end{aligned}$$

Now Cauchy-Schwarz's inequality leads to

$$\begin{aligned}\|e\|_{\Omega_1}^2 \leq & \|A^{-1}(\underline{\xi} - \underline{\xi}_h)\|_{\Omega_1} \|\underline{v}\|_{\Omega_1} + \|\nabla_h (\theta_2 - \theta_{2h})\|_{\Omega_2} \|\underline{v}\|_{\Omega_2} \\ & + \sum_{T \in \mathcal{T}_1} \|A^{-1} \underline{\xi}_h - \nabla q_h\|_T \|\underline{v} - \Pi_h \underline{v}\|_T \\ & + \sum_{E \in \mathcal{E}_2} \|[\![\theta_{2h}]\!]_E\|_E \|\underline{v} - \mathcal{M}_E \underline{v}\|_E, \forall q_h \in M_h.\end{aligned}$$

Scaling arguments yield

$$\begin{aligned}&\|\underline{v} - \Pi_h \underline{v}\|_T \lesssim h_T \|\nabla \underline{v}\|_{\omega_T}, \\ &\|\underline{v} - \mathcal{M}_E \underline{v}\|_E \lesssim h_E^{1/2} \|\nabla \underline{v}\|_{T_E}, \forall E \in \mathcal{E}_2,\end{aligned}$$

where T_E is one triangle of $\mathcal{T}_2$ having E as edge. In the same manner due to (5), we have

$$\|[\![\theta_{2h}]\!]_E\|_E \lesssim h_E \|[\![\nabla_h \theta_{2h} \cdot \underline{t}_E]\!]_E\|_E, \forall E \in \mathcal{E}_2.$$

These three estimates in the previous one allow to obtain

$$\|e\|_{\Omega_1}^2 \lesssim \Big(\|A^{-1}(\underline{\xi} - \underline{\xi}_h)\|_{\Omega_1} + \|\nabla\theta_2 - \nabla_h\theta_{2h}\|_{\Omega_2}$$
$$+ (\sum_{T\in\mathcal{T}_1} h_T^2\|A^{-1}\underline{\xi}_h - \nabla q_h\|_T^2)^{1/2} + (\sum_{E\in\mathcal{E}_2} h_E^3\|[\![\nabla_h\theta_{2h}\cdot \underline{t}_E]\!]_E\|_E^2)^{1/2}\Big)\|\underline{v}\|_{1,\Omega},$$

for any $q_h \in M_h$. The conclusion follows from the estimate (34). ■

From the above proof and Corollary 3.7 we see that the assumption (17) can be avoided if Ω_1 is convex near Γ and if the space X_{1h} is well chosen.

It remains to estimate the error on $\underline{\epsilon}_1$ and on e_2. These estimates are obtained using a Helmholtz like decomposition of the error $\underline{\epsilon}$.

Lemma 4.3 *There exist $z \in H_0^1(\Omega)$ and $\beta \in H^1(\Omega)$ such that*

$$\underline{\epsilon} = A\nabla z + \underline{\mathrm{curl}}\ \beta, \tag{40}$$

with the estimates

$$|z|_{1,\Omega} \lesssim \|\underline{\epsilon}\|, \tag{41}$$
$$|\beta|_{1,\Omega} \lesssim \|\underline{\epsilon}\|. \tag{42}$$

Proof: First we consider $z \in H_0^1(\Omega)$ as the unique solution of $\mathrm{div}\,(A\nabla z) = \mathrm{div}\,\underline{\epsilon}$, i.e., solution of

$$\int_\Omega (A\nabla z)\cdot\nabla w = \int_\Omega \underline{\epsilon}\cdot\nabla w, \forall w \in H_0^1(\Omega),$$

which clearly satisfies (41). Secondly we remark that $\underline{\epsilon} - A\nabla z$ is divergence free so by Theorem I.3.1 of [13], there exists $\beta \in H^1(\Omega)$ such that

$$\underline{\mathrm{curl}}\ \beta = \underline{\epsilon} - A\nabla z$$

with the estimate

$$|\beta|_{1,\Omega} \lesssim \|\underline{\epsilon} - A\nabla z\|,$$

which leads to (42) thanks to (41). ■

Lemma 4.4 *The next estimate holds*

$$\|\underline{\epsilon}\| \lesssim \eta + \zeta. \tag{43}$$

Proof: By (40) we may write

$$\int_\Omega (A^{-1}\underline{\epsilon})\cdot\underline{\epsilon} = \int_\Omega \underline{\epsilon}\cdot\nabla z + \int_\Omega \underline{\epsilon}\cdot A^{-1}\underline{\mathrm{curl}}\ \beta. \tag{44}$$

We now estimate separately the two terms of this right-hand side. For the first one, applying Green's formula in Ω_1 and on each triangle T of Ω_2, we get

$$\begin{aligned}\int_\Omega \underline{\epsilon}\cdot\nabla z = &-\int_{\Omega_1} z\mathrm{div}\,(\underline{\xi}-\underline{\xi}_h) - \int_\Gamma \underline{\xi}_h\cdot\underline{n}z \\ &-\sum_{T\in\mathcal{T}_2}\int_T z\mathrm{div}\,A\nabla(\theta_2-\theta_{2h}) \\ &-\sum_{E\in\mathcal{E}_2}\int_E [\![A\nabla_h\theta_{2h}\cdot\underline{n}]\!]_E z \\ &+\sum_{E\in\mathcal{E}_\Gamma}\int_E A\nabla_h\theta_{2h}\cdot\underline{n}z.\end{aligned} \tag{45}$$

On the other hand the first identity of (18) with $v_h=(0,\mathrm{I}^0_{\mathrm{Cl}}z)$ leads to

$$(A\nabla_h\theta_{2h},\nabla\mathrm{I}^0_{\mathrm{Cl}}z)_2+\int_\Gamma \underline{\xi}_h\cdot\underline{n}\mathrm{I}^0_{\mathrm{Cl}}z=(f,\mathrm{I}^0_{\mathrm{Cl}}z)_2.$$

Applying elementwise Green's formula we then obtain

$$\begin{aligned}&-\sum_{T\in\mathcal{T}_2}\int_T(f+\mathrm{div}\,A\nabla\theta_{2h})\mathrm{I}^0_{\mathrm{Cl}}z \\ &+\sum_{E\in\mathcal{E}_2}\int_E [\![A\nabla_h\theta_{2h}\cdot\underline{n}]\!]_E\mathrm{I}^0_{\mathrm{Cl}}z \\ &+\sum_{E\in\mathcal{E}_\Gamma}\int_E(\underline{\xi}_h-A\nabla_h\theta_{2h})\cdot\underline{n}\mathrm{I}^0_{\mathrm{Cl}}z=0.\end{aligned}$$

Inserting this identity in (45), we arrive at

$$\begin{aligned}\int_\Omega \underline{\epsilon}\cdot\nabla z = &\int_{\Omega_1} z(f+\mathrm{div}\,\underline{\xi}_h) \\ &+\sum_{T\in\mathcal{T}_2}\int_T(f+\mathrm{div}\,A\nabla\theta_{2h})(z-\mathrm{I}^0_{\mathrm{Cl}}z) \\ &-\sum_{E\in\mathcal{E}_2}\int_E [\![A\nabla_h\theta_{2h}\cdot\underline{n}]\!]_E(z-\mathrm{I}^0_{\mathrm{Cl}}z) \\ &-\sum_{E\in\mathcal{E}_\Gamma}\int_E(\underline{\xi}_h-A\nabla_h\theta_{2h})\cdot\underline{n}(z-\mathrm{I}^0_{\mathrm{Cl}}z).\end{aligned}$$

Finally we remark that the second identity of (18) means that

$$\mathrm{div}\,\underline{\xi}_h=-\rho_h f,$$

and therefore the above identity becomes (recalling that $z - \mathrm{I}^0_{\mathrm{Cl}} z$ is equal to 0 on $\partial\Omega$)

$$\begin{aligned}\int_\Omega \underline{\epsilon} \cdot \nabla z = &\int_{\Omega_1} (z - \rho_h z)(f + \operatorname{div} \underline{\xi}_h) \\ &+ \sum_{T \in \mathcal{T}_2} \int_T (f + \operatorname{div} A\nabla\theta_{2h})(z - \mathrm{I}^0_{\mathrm{Cl}} z) \\ &- \sum_{E \in \mathcal{E}_2 \setminus \partial\Omega} \int_E [\![A\nabla_h \theta_{2h} \cdot \underline{n}]\!]_E (z - \mathrm{I}^0_{\mathrm{Cl}} z) \\ &- \sum_{E \in \mathcal{E}_\Gamma} \int_E (\underline{\xi}_h - A\nabla_h\theta_{2h}) \cdot \underline{n} (z - \mathrm{I}^0_{\mathrm{Cl}} z).\end{aligned}$$

Continuous and discrete Cauchy-Schwarz's inequalities then yield

$$\begin{aligned}\left|\int_\Omega \underline{\epsilon} \cdot \nabla z\right| \lesssim &\left(\sum_{T \in \mathcal{T}_1} h_T^{-2} \|z - \rho_h z\|_T^2\right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_1} h_T^2 \|f + \operatorname{div} \underline{\xi}_h\|_T^2\right)^{\frac{1}{2}} \\ &+ \left(\sum_{T \in \mathcal{T}_2} h_T^{-2} \|z - \mathrm{I}^0_{\mathrm{Cl}} z\|_T^2\right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_2} h_T^2 \|f + \operatorname{div} A\nabla\theta_{2h}\|_T^2\right)^{\frac{1}{2}} \\ &+ \left(\sum_{E \in \mathcal{E}_2} h_E^{-1} \|z - \mathrm{I}^0_{\mathrm{Cl}} z\|_E^2\right)^{\frac{1}{2}} \left(\sum_{E \in \mathcal{E}_2 \setminus \partial\Omega} h_E \|[\![A\nabla_h\theta_{2h} \cdot \underline{n}]\!]_E\|_E^2\right)^{\frac{1}{2}} \\ &+ \left(\sum_{E \in \mathcal{E}_\Gamma} h_E^{-1} \|z - \mathrm{I}^0_{\mathrm{Cl}} z\|_E^2\right)^{\frac{1}{2}} \left(\sum_{E \in \mathcal{E}_\Gamma} h_E \|(\underline{\xi}_h - A\nabla_h\theta_{2h}) \cdot \underline{n}\|_E^2\right)^{\frac{1}{2}}.\end{aligned} \tag{46}$$

Lemmas 3.3 and 3.4 and the estimate (41) finally lead to

$$\left|\int_\Omega \underline{\epsilon} \cdot \nabla z\right| \lesssim (\eta + \zeta)\|\underline{\epsilon}\|. \tag{47}$$

For the second term of the right-hand side of (44) we first apply the Galerkin orthogonality relation (20) with $v_h = (\underline{\mathrm{curl}}\ \mathrm{I}_{\mathrm{Cl}}\beta, 0) \in X_h \cap X$, to see that

$$(A^{-1}(\underline{\xi} - \underline{\xi}_h), \underline{\mathrm{curl}}\ \mathrm{I}_{\mathrm{Cl}}\beta)_1 - \int_\Gamma \underline{\mathrm{curl}}\ \mathrm{I}_{\mathrm{Cl}}\beta \cdot \underline{n}(\theta_2 - \theta_{2h}) = 0. \tag{48}$$

On the other hand applying Green's formula in each element T of Ω_2, we have

$$\int_{\Omega_2} \nabla_h(\theta_2 - \theta_{2h}) \cdot \underline{\mathrm{curl}}\ \mathrm{I}_{\mathrm{Cl}}\beta = -\int_{\Gamma} \underline{\mathrm{curl}}\ \mathrm{I}_{\mathrm{Cl}}\beta \cdot \underline{n}(\theta_2 - \theta_{2h})$$
$$+ \sum_{E\in\mathcal{E}_2} \int_E \underline{\mathrm{curl}}\ \mathrm{I}_{\mathrm{Cl}}\beta \cdot \underline{n}[\![\theta_{2h}]\!]_E.$$

Due to the Crouzeix-Raviart property (5), this second term is zero and therefore

$$\int_{\Omega_2} \nabla_h(\theta_2 - \theta_{2h}) \cdot \underline{\mathrm{curl}}\ \mathrm{I}_{\mathrm{Cl}}\beta = -\int_{\Gamma} \underline{\mathrm{curl}}\ \mathrm{I}_{\mathrm{Cl}}\beta \cdot \underline{n}(\theta_2 - \theta_{2h}). \tag{49}$$

The two identities (48) and (49) imply that

$$\int_{\Omega} \underline{\epsilon} \cdot A^{-1}\underline{\mathrm{curl}}\ \mathrm{I}_{\mathrm{Cl}}\beta = 0,$$

and therefore we may write

$$\int_{\Omega} \underline{\epsilon} \cdot A^{-1}\underline{\mathrm{curl}}\ \beta = \int_{\Omega} \underline{\epsilon} \cdot A^{-1}\underline{\mathrm{curl}}\ (\beta - \mathrm{I}_{\mathrm{Cl}}\beta).$$

Splitting the integral and using the definition of $\underline{\epsilon}$, we see that

$$\int_{\Omega} \underline{\epsilon} \cdot A^{-1}\underline{\mathrm{curl}}\ \beta = \int_{\Omega} \nabla\theta \cdot \underline{\mathrm{curl}}\ (\beta - \mathrm{I}_{\mathrm{Cl}}\beta) \tag{50}$$
$$- \int_{\Omega_1} A^{-1}\underline{\xi}_h \cdot \underline{\mathrm{curl}}\ (\beta - \mathrm{I}_{\mathrm{Cl}}\beta) - \int_{\Omega_2} \nabla_h\theta_{2h} \cdot \underline{\mathrm{curl}}\ (\beta - \mathrm{I}_{\mathrm{Cl}}\beta).$$

For the first term of this right-hand side, Green's formula in Ω directly yields

$$\int_{\Omega} \nabla\theta \cdot \underline{\mathrm{curl}}\ (\beta - \mathrm{I}_{\mathrm{Cl}}\beta) = 0,$$

reminding that $\theta = 0$ on $\partial\Omega$. For the second and third terms applying Green's formula on each element T we get

$$\int_{\Omega_1} A^{-1}\underline{\xi}_h \cdot \underline{\mathrm{curl}}\ (\beta - \mathrm{I}_{\mathrm{Cl}}\beta) + \int_{\Omega_2} \nabla_h\theta_{2h} \cdot \underline{\mathrm{curl}}\ (\beta - \mathrm{I}_{\mathrm{Cl}}\beta) = \sum_{T\in\mathcal{T}_1} \int_T \mathrm{curl}\,(A^{-1}\underline{\xi}_h) \cdot (\beta - \mathrm{I}_{\mathrm{Cl}}\beta)$$
$$- \sum_{E\in\mathcal{E}_1\cup\mathcal{E}_2\cup\mathcal{E}_\Gamma} \int_E J_{E,t} \cdot (\beta - \mathrm{I}_{\mathrm{Cl}}\beta).$$

Continuous and discrete Cauchy-Schwarz's inequalities then yield

$$\left|\int_\Omega \underline{\epsilon}\cdot A^{-1}\underline{\mathrm{curl}}\,\beta\right| \le \left(\sum_{T\in\mathcal{T}_1} h_T^2\|\mathrm{curl}\,(A^{-1}\underline{\xi}_h)\|_T^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_1} h_T^{-2}\|\beta-\mathrm{I}_{\mathrm{Cl}}\beta\|_T^2\right)^{1/2}$$
$$+\left(\sum_{E\in\mathcal{E}_1\cup\mathcal{E}_2\cup\mathcal{E}_\Gamma} h_E\|J_{E,t}\|_E^2\right)^{1/2}\left(\sum_{E\in\mathcal{E}_1\cup\mathcal{E}_2\cup\mathcal{E}_\Gamma} h_E^{-1}\|\beta-\mathrm{I}_{\mathrm{Cl}}\beta\|_E^2\right)^{1/2}.$$

By Lemma 3.3 we obtain

$$\left|\int_\Omega \underline{\epsilon}\cdot A^{-1}\underline{\mathrm{curl}}\,\beta\right| \lesssim \eta\|\nabla\beta\|.$$

According to (42) we arrive at the estimate

$$\left|\int_\Omega \underline{\epsilon}\cdot A^{-1}\underline{\mathrm{curl}}\,\beta\right| \lesssim \eta\|\underline{\epsilon}\,\|. \tag{51}$$

The conclusion directly follows from the identity (44) and the estimates (47) and (51). ■

Using the two above Lemmas we have obtained the

Theorem 4.5 (Upper error bound) *The error is bounded globally from above by*

$$\|\theta_1-\theta_{1h}\|_{\Omega_1}+\|A\nabla\theta_1-\underline{\xi}_h\|_{\Omega_1}+\|\nabla\theta_2-\nabla_h\theta_{2h}\|_{\Omega_2}\ \lesssim \eta+\zeta. \tag{52}$$

4.3 Proof of the lower error bound

The main point is the next error equation (compare with Lemma 3.1 of [9]).

Lemma 4.6 *For all $w\in H_0^1(\Omega)$ and $\varphi\in H^1(\Omega)$, we have*

$$\begin{aligned}\int_\Omega A^{-1}\underline{\epsilon}\cdot(A\nabla w+\underline{\mathrm{curl}}\,\varphi) &= \sum_{T\in\mathcal{T}}\int_T R_T w\\ &-\sum_{T\in\mathcal{T}_1}\int_T \mathrm{curl}\,(A^{-1}\underline{\xi}_h)\varphi\\ &-\sum_{E\in\mathcal{E}_1\cup\mathcal{E}_2\cup\mathcal{E}_\Gamma}\int_E J_{E,t}\varphi-\sum_{E\in\mathcal{E}_2\cup\mathcal{E}_\Gamma}\int_E J_{E,n}w.\end{aligned}$$

Proof: Use Green's formula on each element T of $\mathcal{T}_i$, $i=1,2$. ■

Combining some arguments from [9] (see also [23]) and from [4] (or [15,7,22]), we can state the

Theorem 4.7 (Lower error bound) *Assume that (22) holds. Assume further that there exists $k \in \mathbb{N}$ such that $(A^{-1}\underline{\xi}_h)_{|T}$ belongs to $\mathbb{P}_k$, for all $T \in \mathcal{T}_1$. Then for all elements T, the following local lower error bound holds:*

$$\eta_T \lesssim \|\underline{\xi} - \underline{\xi}_h\|_{\omega_T} + \|\theta_1 - \theta_{1h}\|_T + \|\nabla\theta_2 - \nabla_h\theta_{2h}\|_{\omega_T} + \sum_{T' \in \mathcal{T}_1, T' \subset \omega_T} \zeta_{T'}. \quad (53)$$

Proof: The Curl residual $\|\mathrm{curl}\,(A^{-1}\underline{\xi}_h)\|_T$ was estimated in Theorem 5.2 of [22], where it was proved that

$$h_T \|\mathrm{curl}\,(A^{-1}\underline{\xi}_h)\|_T \lesssim \|\underline{\xi} - \underline{\xi}_h\|_T, \forall T \in \mathcal{T}_1. \quad (54)$$

Similarly Theorem 5.2 of [22] estimates the tangential jump for edges from $\mathcal{E}_1$ and for the element residual $\|A^{-1}\underline{p}_h - \nabla\theta_{1h}\|_T$, namely

$$h_E^{1/2} \|J_{E,t}\|_E \lesssim \|\underline{\xi} - \underline{\xi}_h\|_{\omega_E}, \forall E \in \mathcal{E}_1, \quad (55)$$

$$h_T \|A^{-1}\underline{\xi}_h - \nabla\theta_{1h}\|_T \lesssim \|\underline{\xi} - \underline{\xi}_h\|_T + \|\theta_1 - \theta_{1h}\|_T, \forall T \in \mathcal{T}_1. \quad (56)$$

Similarly using the arguments from [9], namely by taking $v = R_T b_T$ and $\varphi = 0$ in Lemma 4.6 for the element residual $\|R_T\|_T$ for $T \in \mathcal{T}_2$; by taking $v = 0$ and $\varphi = b_E J_{E,t}$ in Lemma 4.6 for the tangential jump $\|J_{E,t}\|_E$ for $E \in \mathcal{E}_2$, and by taking $v = b_E J_{E,n}$ and $\varphi = 0$ in Lemma 4.6 for the normal jump $\|J_{E,n}\|_E$ for $E \in \mathcal{E}_2$ and using inverse estimates, we obtain

$$h_T \|R_T\|_T \lesssim \|\nabla\theta_2 - \nabla\theta_{2h}\|_T, \forall T \in \mathcal{T}_2, \quad (57)$$

$$h_E^{1/2}(\|J_{E,t}\|_E + \|J_{E,n}\|_E) \lesssim \|\nabla\theta_2 - \nabla_h\theta_{2h}\|_{\omega_E}, \forall E \in \mathcal{E}_2. \quad (58)$$

It therefore remains to estimate the normal and tangential jumps for edges on Γ. This is proved as before. We give the details for the sake of completeness.

For $E \in \mathcal{E}_\Gamma$, we set

$$w_E := \mathrm{F}_{\mathrm{ext}}(J_{E,t}) b_E,$$

which belongs to $H_0^1(\omega_E)$. The inverse inequality (26) yields

$$\|J_{E,t}\|_E^2 \lesssim \int_E J_{E,t} w_E.$$

Now we apply Lemma 4.6 with $w = w_E$ and $\varphi = 0$ and obtain

$$\int_E J_{E,t} \cdot w_E = \int_{\omega_E} \underline{\epsilon} \cdot \nabla w_E - \sum_{T \in \mathcal{T}, T \subset \omega_E} \int_T R_T w_E.$$

Using Cauchy-Schwarz's inequality we obtain

$$\|J_{E,t}\|_E^2 \leq \|\underline{\epsilon}\|_{\omega_E} \|\nabla w_E\|_{\omega_E} + \sum_{T \in \mathcal{T}, T \subset \omega_E} \|R_T\|_T \|w_E\|_T.$$

Using the inverse inequality (27) and (28) we get

$$\|J_{E,t}\|_E \lesssim h_E^{-1/2} \|\underline{\epsilon}\|_{\omega_E} + \sum_{T \in \mathcal{T}, T \subset \omega_E} h_T^{1/2} \|R_T\|_T.$$

By the estimate (57) and the definition of ζ_T, we obtain

$$h_E^{1/2} \|J_{E,t}\|_E \lesssim \|\underline{\epsilon}\|_{\omega_E} + \sum_{T \in \mathcal{T}_1, T \subset \omega_E} \zeta_T, \forall E \in \mathcal{E}_\Gamma. \tag{59}$$

Similarly using Lemma 4.6 with $v = 0$ and $\varphi = \mathrm{F}_{\mathrm{ext}}(J_{E,n}) b_E$, inverse estimates and (54) we have

$$h_E^{1/2} \|J_{E,n}\|_E \lesssim \|\underline{\epsilon}\|_{\omega_E}, \forall E \in \mathcal{E}_\Gamma. \tag{60}$$

The estimates (54) to (60) provide the desired bound (53). ■

5 Numerical experiments

The following experiments will underline and confirm our theoretical predictions. In the first example, we consider a regular solution in a non convex domain, while the second example treats the case of a singular solution in a non convex domain.

5.1 The regular solution

The first example consists in solving the two dimensional equation (1) with $A = Id$ on the L-shape domain Ω, defined by $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, with $\Omega_1 = (0,2)^2$ and $\Omega_2 = (2,3) \times (0,1)$. Each of these two squares is discretized using cartesian and uniform meshes composed of triangles. The accuracy of the mesh is characterized by the parameter n, corresponding to the number of segments

of the mesh on the left boundary of Ω_1 (see Fig. 3 for $n = 8$). As a consequence, we have for each edge E of the mesh the property $h_E \sim \dfrac{1}{n}$.

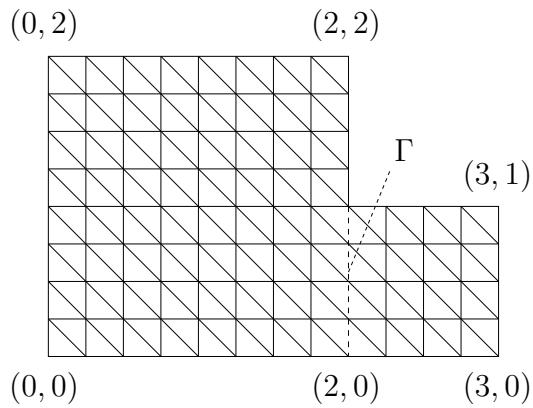

Fig. 3. The computational domain Ω and the associated mesh for $n = 8$, regular case.

We consider the discrete formulation (18), and look for $(u_h, p_h) \in X_h \times M_h$, with V_h made of piecewise $\mathbb{P}_1$ (conforming) elements, X_{1h} made of RT_0 elements, and M_h made of $\mathbb{P}_0$ elements. If we define $ns1$ the number of edges contained in the mesh of $\overline{\Omega}_1$, $nt1$ the number of triangles contained in the mesh of $\overline{\Omega}_1$, and $nn2$ the number of nodes contained in the mesh of $\overline{\Omega}_2 \setminus \partial\Omega$, we can write the unknowns of the discretized problem in the form

$$\underline{\xi}_h = \sum_{i=1}^{ns1} \alpha_i \, (\underline{\xi}_h)_i \,, \quad \theta_{2h} = \sum_{i=1}^{nn2} \beta_i \, \Phi_i \,, \quad p_h = \sum_{i=1}^{nt1} \gamma_i \, \Psi_i \,,$$

when $(\underline{\xi}_h)_i \, (1 \leq i \leq ns1)$, $\Phi_i \, (1 \leq i \leq nn2)$ and $\Psi_i \, (1 \leq i \leq nt1)$ are the global basis functions associated with the spaces X_{1h}, V_h and M_h, respectively. Moreover, if we denote by α, β and γ the vectors made of the coefficients α_i, β_i and γ_i respectively, then the linear system corresponding to the discrete formulation (18) can be written

$$\begin{bmatrix} A_1 & M & N \\ M^T & B_2 & 0 \\ N^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ F_\beta \\ F_\gamma \end{bmatrix} \tag{61}$$

with

$$
\begin{aligned}
(A_1)_{ij} &= ((\underline{\xi}_h)_i, (\underline{\xi}_h)_j)_1, \\
M_{ij} &= -((\underline{\xi}_h)_i.\underline{n}, \Phi_j)_\Gamma, \\
N_{ij} &= (\mathrm{div}\,(\underline{\xi}_h)_i, \Psi_j)_1, \\
(B_2)_{ij} &= -(\nabla\Phi_i, \nabla\Phi_j)_2, \\
(F_\beta)_j &= -(f, \Phi_j)_2, \\
(F_\gamma)_j &= -(f, \Psi_j)_1.
\end{aligned}
$$

The resulting linear system is then solved using the preconditioning solver GMRES.

For this test we choose the exact solution $\theta(x,y) = x(2-x)(3-x)(2-y)(1-y)y$, for which $\theta_{|\partial\Omega} = 0$. Let us denote by DoF the total number of degrees of freedom associated with the finite element triangulation. If we set $h = 2/n$, then we have $DoF = O(h^{-2})$. We plot in Figure 4 the global error as a function of DoF, this error being defined by

$$||(\theta_1-\theta_{1h}, \underline{\xi}-\underline{\xi}_h, \theta_2-\theta_{2h})||_{global} = ||\theta_1-\theta_{1h}||_{\Omega_1}+||\underline{\xi}-\underline{\xi}_h||_{\Omega_1}+||\nabla\theta_2-\nabla_h\theta_{2h}||_{\Omega_2}.$$

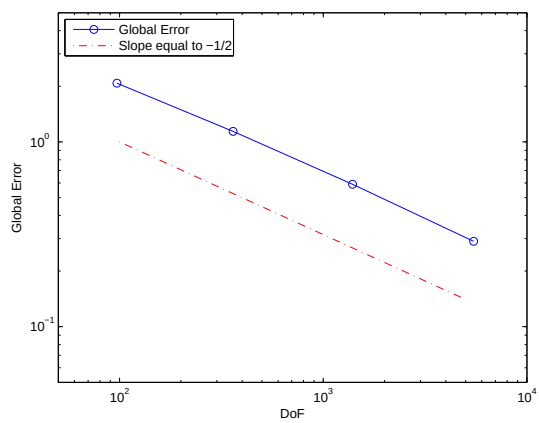

Fig. 4. Global error as a function of DoF, regular case.

We clearly see that we have $||(\theta_1-\theta_{1h}, \underline{\xi}-\underline{\xi}_h, \theta_2-\theta_{2h})||_{global} = O(DoF^{-1/2}) = O(h)$, which corresponds to the expected rate of convergence [16, Th. 3.2] since the exact solution θ belongs to $H^2(\Omega)$. This first result illustrates the optimal convergence of the numerical method that we use to solve the discrete problem (18).

Now, in order to verify the upper error bound, we plot in Figure 5 the ratio

q_{up} defined by

$$q_{up} = \frac{||(\theta_1 - \theta_{1h}, \underline{\xi} - \underline{\xi}_h, \theta_2 - \theta_{2h})||_{global}}{\eta}.$$

From this figure, we see that q_{up} is uniformly bounded with respect to DoF.

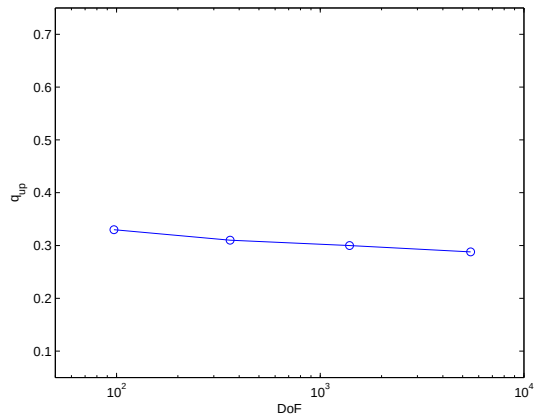

Fig. 5. q_{up} as a function of DoF, regular case.

This confirms the theoretical result of Theorem 4.5 and means that the estimator is reliable.

Finally, in order to verify the lower error bound, we plot in Figure 6 the ratio q_{low} defined by

$$q_{low} = \max_{T \in \mathcal{T}} \frac{\eta_T}{||\theta_1 - \theta_{1h}||_T + ||\underline{\xi} - \underline{\xi}_h||_{\omega_T} + ||\nabla\theta_2 - \nabla_h\theta_{2h}||_{\omega_T}}.$$

Once again, we see that q_{low} is uniformly bounded with respect to DoF and

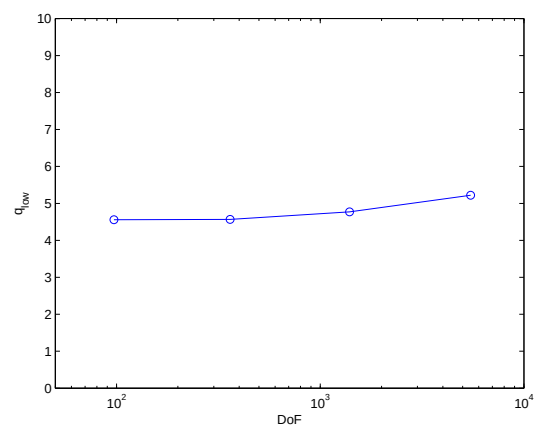

Fig. 6. q_{low} as a function of DoF, regular case.

confirms the theoretical result of Theorem 4.7. The estimator is then efficient.

The second example consists in solving the two dimensional equation (1) with $A = Id$ on the domain Ω displayed in Figure 7 with $\Omega_1 = \{(r\cos\varphi, r\sin\varphi) : 0 < r < 1, \pi/2 < \varphi < 3\pi/2\}$ and $\Omega_2 = \{(r\cos\varphi, r\sin\varphi) : 0 < r < 1, 0 < \varphi < \pi/2\}$. In that case, we use unstructured meshes, and Γ is the segment between the points $(0,0)$ and $(0,1)$.

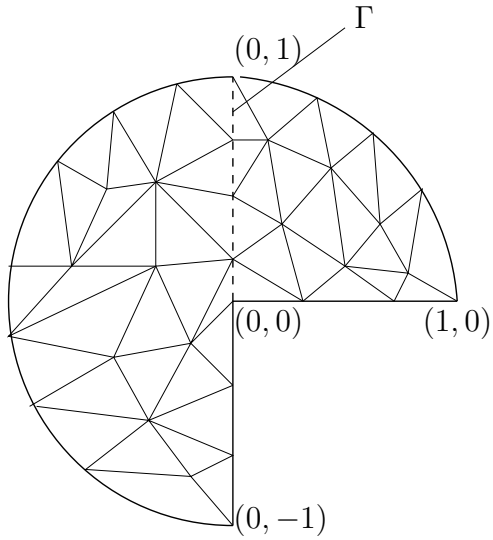

Fig. 7. The computational domain Ω and the associated mesh, singular case.

Defining r as the distance to the origin ($r = \sqrt{x^2+y^2}$) and φ the angle in the usual polar coordinates system, the test is performed with the singular solution $\theta(x,y) = r^{2/3}(1-r)\sin(\frac{2\varphi}{3})$. Once again, we have $\theta_{|\partial\Omega} = 0$, but the solution is singular, namely $\theta \notin H^2(\Omega)$.

Figures 8 to 10 are respectively similar to Figures 4 to 6. We observe that the numerical solution converges towards the exact one with a rate of convergence of $0(DoF^{-1/3}) = O(h^{2/3})$, which corresponds to the a priori error analysis theory [16, Th. 3.2] because $\theta \in H^{3/2}(\Omega)$. Once again the ratios q_{up} and q_{low} remain uniformly bounded with respect to DoF. This confirms that our error estimator is reliable and efficient, even for singular solutions as theoretically expected.

The tests presented in this section have been performed with the help of the finite element code Simula+ developed by the LAMAV laboratory (University of Valenciennes, France) and the LPMM laboratory (ENSAM of Metz,

France).

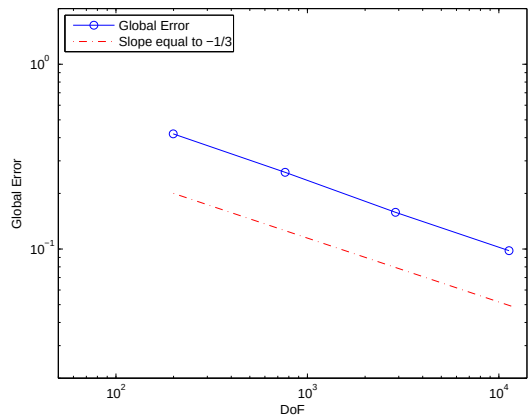

Fig. 8. Global error as a function of DoF, singular case.

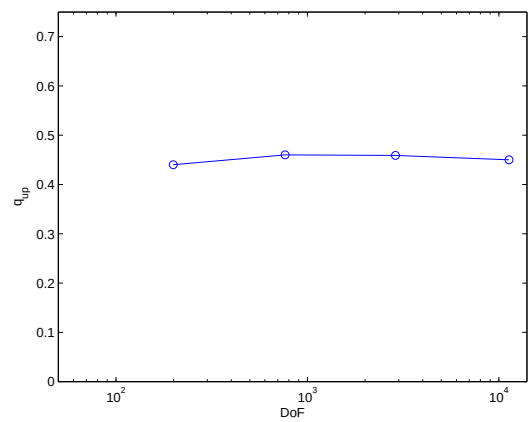

Fig. 9. q_{up} as a function of DoF, singular case.

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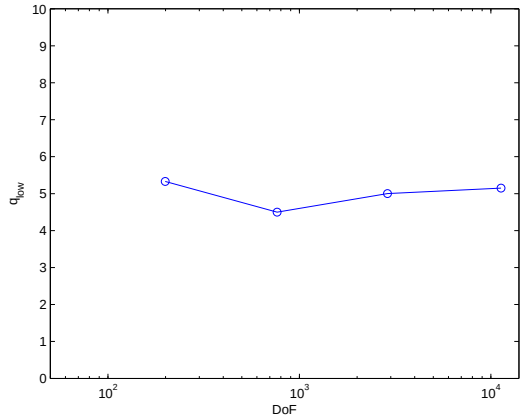

Fig. 10. q_{low} as a function of DoF, singular case.

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