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A posteriori error estimator based on gradient recovery by averaging for discontinuous Galerkin methods

Emmanuel Creusé*, Serge Nicaise†

March 19, 2010

Abstract

We consider some (anisotropic and piecewise constant) diffusion problems in domains of \mathbb{R}^2 , approximated by a discontinuous Galerkin method with polynomials of any fixed degree. We propose an a posteriori error estimator based on gradient recovery by averaging. It is shown that this estimator gives rise to an upper bound where the constant is one up to some additional terms that guarantee reliability. The lower bound is also established. Moreover these additional terms are negligible when the recovered gradient is superconvergent. The reliability and efficiency of the proposed estimator is confirmed by some numerical tests.

Key Words A posteriori estimator, Discontinuous Galerkin finite elements.

AMS (MOS) subject classification 65N30; 65N15, 65N50,

1 Introduction

Among other methods, the finite element method is one of the more popular that is commonly used in the numerical realization of different problems appearing in engineering applications, like the Laplace equation, the Lamé system, the Stokes system, the Maxwell system, etc.... (see [7, 8, 24]). More recently discontinuous Galerkin finite element methods become very attractive since they present some advantages. For example, they allow to do some "p refinement", by locally increasing the polynomial degree of the approximation if needed. They can moreover use non-conform meshes allowing hanging-nodes, making the mesh generation easier for concrete industrial cases. In our days a vast literature exists on

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the subject, we refer to [3, 10] and the references cited there. Adaptive techniques based on a posteriori error estimators have become indispensable tools for such methods. For continuous Galerkin finite element methods, there now exists a vast amount of literature on a posteriori error estimation for problems in mechanics or electromagnetism and obtaining locally defined a posteriori error estimates. We refer to the monographs [2, 4, 25, 33] for a good overview on this topic. On the other hand a similar theory for discontinuous methods is less developed, let us quote [5, 12, 17, 18, 20, 28, 31].

Usually upper and lower bounds are proved in order to guarantee the reliability and the efficiency of the proposed estimator. Most of the existing approaches involve constants depending on the shape regularity of the elements and/or of the jumps in the coefficients; but these dependences are often not given. Only a small number of approaches gives rise to estimates with explicit constants, let us quote [2, 6, 21, 23, 26, 27, 16] for continuous methods. For discontinuous methods, we may cite the recent papers [1, 22, 9, 14, 15].

Our goal is therefore to consider second order elliptic operators with discontinuous diffusion coefficients in two-dimensional domains with mixed boundary conditions and a discontinuous Galerkin method with polynomials of any degree. Inspired from the paper [16], which treats the case of continuous diffusion coefficients approximated by a continuous Galerkin method, we further derive an a posteriori estimator with an explicit constant in the upper bound (more precisely 1) up to some additional terms that are usually superconvergent and some oscillating terms. The approach, called gradient recovery by averaging [16] is based on the construction of a Zienkiewicz/Zhu estimator, namely the difference in an appropriate norm of $a\nabla_h u_h - Gu_h$, where $\nabla_h u_h$ is the broken gradient of u_h and Gu_h is a $H(\text{div})$ -conforming approximation of this variable. Here special attention has to be paid due to the assumption that a may be discontinuous. Moreover the non conforming part of the error is managed using a Helmholtz decomposition of the error and a standard Oswald interpolation operator [20, 1]. Furthermore using standard inverse inequalities, we show that our estimator is locally efficient. Two interests of this approach are first the simplicity of the construction of Gu_h , and secondly its superconvergence property (validated by numerical tests).

The schedule of the paper is as follows: We recall in section 2 the diffusion problem, its numerical approximation and an appropriate Helmholtz decomposition of the error. Section 3 is devoted to the introduction of the estimator based on gradient averaging and the proofs of the upper and lower bounds. The upper bound directly follows from the construction of the estimator and some results from [16], while the lower bound requires the use of some inverse inequalities and a special construction of Gu_h . Finally in section 4 some numerical tests are presented that confirm the reliability and efficiency of our estimator and the superconvergence of Gu_h to $a\nabla u$.

Let us finish this introduction with some notation used in the remainder of the paper: On D , the $L^2(D)$ -norm will be denoted by $\|\cdot\|_D$. In the case $D = \Omega$, we will drop the index Ω . The usual norm and semi-norm of $H^s(D)$ ($s \geq 0$) are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. Finally, the notation $a \lesssim b$ and $a \sim b$ means the existence of positive constants C_1 and C_2 , which are independent of the mesh size and of the quantities a and b under consideration such that $a \lesssim C_2 b$ and $C_1 b \lesssim a \lesssim C_2 b$, respectively. In other words,

the constants may depend on the aspect ratio of the mesh, the diffusion matrix, as well as the polynomial degree l (see below).

2 The boundary value problem and its discretization

Let Ω be a bounded open domain of \mathbb{R}^2 with a Lipschitz boundary Γ that we suppose to be polygonal. We further assume that Ω is simply connected and that Γ is connected. We consider the following elliptic second order boundary value problem with non homogeneous mixed boundary conditions:

$$\begin{cases} -\operatorname{div}(a \nabla u) &= f \text{ in } \Omega, \\ u &= g_D \text{ on } \Gamma_D, \\ a \nabla u \cdot n &= g_N \text{ on } \Gamma_N, \end{cases} \quad (1)$$

where $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For convenience we suppose that $\operatorname{meas} \Gamma_D > 0$.

In the sequel, we suppose that a is a symmetric positive definite matrix which is piecewise constant, namely we assume that there exists a partition \mathcal{P} of Ω into a finite set of Lipschitz polygonal domains $\Omega_1, \dots, \Omega_J$ such that, on each Ω_j , $a = a_j$ where a_j is a symmetric positive definite matrix. The variational formulation of (1) involves the bilinear form

$$B(u, v) = \int_{\Omega} a \nabla u \cdot \nabla v$$

and the Hilbert space

$$H_D^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}.$$

Given $f \in L^2(\Omega)$, $g_D \in H^{\frac{1}{2}}(\Gamma_D)$ and $g_N \in L^2(\Gamma_N)$, the weak formulation consists in finding $u \in w + H_D^1(\Omega)$ such that

$$B(u, v) = \int_{\Omega} f v + \int_{\Gamma_N} g_N v, \quad \forall v \in H_D^1(\Omega), \quad (2)$$

where $w \in H^1(\Omega)$ is a lifting for g_D , i.e., $w = g_D$ on Γ_D . Invoking the positiveness of a , the bilinear form B is coercive on $H_D^1(\Omega)$ with respect to the norm $\left(\int_{\Omega} |a^{1/2} \nabla u|^2\right)^{1/2}$ and this coerciveness guarantees that problem (2) has a unique solution by the Lax-Milgram lemma.

2.1 Discontinuous Galerkin approximated problem

Following [20, 3], we consider the following discontinuous Galerkin approximation of our continuous problem: We consider a triangulation \mathcal{T} of Ω , that is a "partition" of Ω made of triangles T (closed subsets of $\bar{\Omega}$) whose edges are denoted by e . We assume that this

triangulation is regular, i.e., for any element T , the ratio $\frac{h_T}{\rho_T}$ is bounded by a constant $\sigma > 0$ independent of T and of the mesh size $h = \max_{T \in \mathcal{T}} h_T$, where h_T is the diameter of T and ρ_T the diameter of its largest inscribed ball. We further assume that \mathcal{T} is conforming with the partition \mathcal{P} of Ω , i.e., the matrix a being constant on each $T \in \mathcal{T}$, we then denote by a_T the value of a restricted to an element T . With each edge e of the triangulation, we denote by h_e its length, we associate a unit normal vector n_e (whose orientation can be arbitrary chosen) and the so-called patch $\omega_e = \cup_{e \in T} T$, the union of triangles having e as edge. For a triangle T , n_T stands for the outer unit normal vector of T . \mathcal{E} (resp. \mathcal{N}) represents the set of edges (resp. vertices) of the triangulation. In the sequel, we need to distinguish between edges included into Ω , Γ_D or Γ_N , in other words, we set

$$\begin{aligned}\mathcal{E}_{int} &= \{e \in \mathcal{E} : e \subset \Omega\}, \\ \mathcal{E}_D &= \{e \in \mathcal{E} : e \subset \Gamma_D\}, \\ \mathcal{E}_N &= \{e \in \mathcal{E} : e \subset \Gamma_N\}.\end{aligned}$$

For shortness, we also write $\mathcal{E}_{ID} = \mathcal{E}_{int} \cup \mathcal{E}_D$.

Problem (2) is approximated by the (discontinuous) finite element space:

$$X_h = \{v_h \in L^2(\Omega) | v_h|_T \in \mathbb{P}_\ell(T), T \in \mathcal{T}\}, \quad (3)$$

where ℓ is a fixed positive integer.

For our further analysis we need to define some jumps and means of elements of X_h through any $e \in \mathcal{E}$ of the triangulation. For $e \in \mathcal{E}$ such that $e \subset \Omega$, we denote by T^+ and T^- the two elements of \mathcal{T} containing e . Let $q \in X_h$, we denote by q^\pm , the traces of q taken from T^\pm , respectively. Then we define the mean of q on e by

$$\{\{q\}\} = \frac{q^+ + q^-}{2}.$$

For $v \in [X_h]^d$, we denote similarly

$$\{\{v\}\} = \frac{v^+ + v^-}{2}.$$

The jump of q on e is now defined as follows:

$$\llbracket q \rrbracket = q^+ n_{T^+} + q^- n_{T^-}.$$

Remark that the jump $\llbracket q \rrbracket$ of q is vector-valued.

For a boundary edge e , i. e., $e \subset \partial\Omega$, there exists a unique element $T^+ \in \mathcal{T}$ such that $e \subset \partial T^+$. Therefore the mean and jump of q are defined by $\{\{q\}\} = q^+$ and $\llbracket q \rrbracket = q^+ n_{T^+}$.

For $q \in X_h$, we define its broken gradient $\nabla_h q$ in Ω by :

$$(\nabla_h q)|_T = \nabla q|_T, \forall T \in \mathcal{T}.$$

The space X_h is equipped with the norm

$$\|q\|_{DG,h} := \left(\sum_{T \in \mathcal{T}} \|a^{1/2} \nabla q\|_T^2 \right)^{1/2} + \left(\sum_{e \in \mathcal{E}_{ID}} h_e^{-1} \|\llbracket q \rrbracket\|_e^2 \right)^{1/2}.$$

Later on we also need the continuous counterpart of X_h , namely we introduce

$$S_h = \{v_h \in C(\bar{\Omega}) | v_h|_T \in \mathbb{P}_\ell(T), T \in \mathcal{T}\},$$

as well as

$$S_{h,1} = \{v_h \in C(\bar{\Omega}) | v_h|_T \in \mathbb{P}_1(T), T \in \mathcal{T}\}.$$

We further need

$$X_{h,1} = \{v_h \in L^2(\Omega) | v_h|_T \in \mathbb{P}_1(T), T \in \mathcal{T}\}.$$

With these notations, we define the bilinear form $B_h(\cdot, \cdot)$ as follows:

$$\begin{aligned} B_h(u_h, v_h) &:= \sum_{T \in \mathcal{T}} \int_T a \nabla u_h \cdot \nabla v_h - \sum_{e \in \mathcal{E}_{ID}} \int_e (\{a \nabla_h v_h\}) \cdot \llbracket u_h \rrbracket + \{a \nabla_h u_h\} \cdot \llbracket v_h \rrbracket \\ &+ \sum_{e \in \mathcal{E}_{ID}} \gamma_e h_e^{-1} \int_e \llbracket u_h \rrbracket \cdot \llbracket v_h \rrbracket, \quad \forall u_h, v_h \in X_h, \end{aligned}$$

where the positive parameters γ_e are chosen large enough to ensure coerciveness of the bilinear form B_h on X_h (see, e.g., Lemma 2.1 of [20]), namely according to Theorem 3 of [34] (and the arguments from section 3 of [30]), the following choices yield the coerciveness of B_h :

$$\gamma_e > c(\ell + 1)(\ell + 2) \max_{T \subset \omega_e} C_T, \quad (4)$$

where C_T is the largest eigenvalue of the diffusion matrix a_T and c is the positive constant such that

$$h_e^2 \leq c|T|, \forall T \subset \omega_e,$$

that exists due to the regularity assumption on the mesh.

The discontinuous Galerkin approximation of problem (2) reads now: Find $u_h \in X_h$, such that

$$B_h(u_h, v_h) = F(v_h), \quad (5)$$

where

$$F(v_h) = \int_{\Omega} f v_h + \sum_{e \in \mathcal{E}_D} \int_e g_D (\gamma h_e^{-1} v_h - a \nabla v_h \cdot n_T) + \int_{\Gamma_N} g_N v_h, \quad \forall v_h \in X_h.$$

As our approximated scheme is a non conforming one (i.e. the solution does not belong to $H_D^1(\Omega)$), as usual we need to use an appropriate Helmholtz decomposition of the error (see Lemma 3.2 of [13] or Theorem 1 of [1] in 2D).

Lemma 2.1 (Helmholtz decomposition of the error) *We have the following error decomposition*

$$a\nabla_h(u - u_h) = a\nabla\varphi + \operatorname{curl}\chi, \quad (6)$$

with $\chi \in H^1(\Omega)$ such that

$$\operatorname{curl}\chi \cdot n = 0 \text{ on } \Gamma_N, \quad (7)$$

and $\varphi \in H_D^1(\Omega)$. Moreover the next identity holds:

$$\|a^{1/2}\nabla_h(u - u_h)\|^2 = \|a^{1/2}\nabla_h\varphi\|^2 + \|a^{-1/2}\operatorname{curl}\chi\|^2. \quad (8)$$

For the detailed proof of this Lemma we refer to Lemma 2.1 of [9].

3 The a posteriori error analysis based on gradient recovery by averaging

Error estimators can be constructed in many different ways as, for example, using residual type error estimators which measure locally the jump of the discrete flux [20]. A different method, based on equilibrated fluxes, consists in solving local Neumann boundary value problems [2] or in using Raviart-Thomas interpolant [1, 9, 14, 15]. Here, as an alternative we introduce a gradient recovery by averaging and define an error estimator based on a $H(\operatorname{div})$ -conforming approximation of this variable. In comparison with [16], we here allow the case of discontinuous diffusion coefficient and use a discontinuous Galerkin method.

Inspired from [16] the conforming part of the estimator η_{CF} involves the difference between the broken gradient $a\nabla_h u_h$ and its smoothed version Gu_h , where Gu_h is for the moment any element in $X_{h,1}^2$ satisfying

$$Gu_h \in H(\operatorname{div}, \Omega) = \{v \in L^2(\Omega)^2 : \operatorname{div} v \in L^2(\Omega)\}, \quad (9)$$

$$(Gu_h)|_{\Omega_j} \in H^1(\Omega_j), \forall j = 1, \dots, J. \quad (10)$$

Hence conforming part of the estimator η_{CF} is defined by

$$\eta_{CF}^2 = \sum_{T \in \mathcal{T}} \eta_{CF,T}^2, \quad (11)$$

where the indicator $\eta_{CF,T}$ is defined by

$$\eta_{CF,T} = \|a^{-1/2}(a\nabla u_h - Gu_h)\|_T.$$

For the nonconforming part of the error, we associate with u_h , its Oswald interpolation operator, namely the unique element $w_h \in S_h$ defined in the following natural way (see Theorem 2.2 of [20]): to each node n of the mesh corresponding to the Lagrangian-type degrees of freedom of S_h , the value of w_h is the average of the values of u_h at this node n if it belongs to $\Omega \cup \Gamma_N$ (i.e., $w_h(n) = \frac{\sum_{n \in T} |T| u_h|_T(n)}{\sum_{n \in T} |T|}$) and the value of g_D at this node if

it belongs to $\bar{\Gamma}_D$ (here we assume that $g_D \in C(\bar{\Gamma}_D)$). Then the non conforming indicator $\eta_{NC,T}$ is simply

$$\eta_{NC,T} = \|a^{1/2}\nabla(w_h - u_h)\|_T.$$

The non conforming part of the estimator is then

$$\eta_{NC}^2 = \sum_{T \in \mathcal{T}} \eta_{NC,T}^2. \quad (12)$$

Similarly we introduce the estimator corresponding to jumps of u_h :

$$\eta_J^2 = \sum_{e \in \mathcal{E}_{ID}} \eta_{J,e}^2,$$

with

$$\eta_{J,e}^2 = \begin{cases} \frac{1}{h_e} \|[[u_h]]\|_e^2 & \text{if } e \in \mathcal{E}_{int}, \\ \frac{1}{h_e} \|u_h - g_D\|_e^2 & \text{if } e \in \mathcal{E}_D. \end{cases}$$

As in [16], we introduce some additional superconvergent security parts. In order to define them properly we recall that for a node $x \in \mathcal{N}$, we denote by λ_x the standard hat function (defined as the unique element in $S_{h,1}$ such that $\lambda_x(y) = \delta_{x,y}$ for all $y \in \mathcal{N}$), let ω_x be the patch associated with x , which is simply the support of λ_x and let h_x be the diameter of ω_x (which is equivalent to the diameter h_K of any triangle K included into ω_x). We now denote by r the element residual

$$r = f + \operatorname{div}(Gu_h)$$

and for all $x \in \mathcal{N}$, we set

$$\bar{r}_x = \begin{cases} (\int_{\omega_x} \lambda_x)^{-1} \int_{\omega_x} r \lambda_x & \text{if } x \in \mathcal{N} \setminus \mathcal{N}_D, \\ 0 & \text{if } x \in \mathcal{N}_D. \end{cases}$$

We further use a multilevel decomposition of $S_{h,1}$, namely we suppose that we start from a coarse grid \mathcal{T}_0 and that the successive triangulations are obtained by using the bisection method, see [29, 16]. This means that we obtain a finite sequence of nested triangulations \mathcal{T}_ℓ , $\ell = 0, \dots, L$ such that $\mathcal{T}_L = \mathcal{T}$. Denoting by S_ℓ the space

$$S_\ell = \{v \in C(\bar{\Omega}) | v|_T \in \mathbb{P}_1(T), T \in \mathcal{T}_\ell\},$$

then we have

$$S_\ell \subset S_{\ell+1} \text{ and } S_{h,1} = \cup_{\ell=0}^L S_\ell = S_L.$$

Furthermore if we denote by \mathcal{N}_ℓ the nodes of the triangulation \mathcal{T}_ℓ , we have

$$\mathcal{N}_\ell \subset \mathcal{N}_{\ell+1}.$$

As usual for all $z \in \mathcal{N}_\ell$ we denote by $\lambda_{\ell z}$ the hat function associated with z , namely the unique element in S_ℓ such that

$$\lambda_{\ell z}(z') = \delta_{zz'} \forall z' \in \mathcal{N}_\ell.$$

For all $\ell \geq 1$ we finally set

$$\tilde{\mathcal{N}}_\ell = (\mathcal{N}_\ell \setminus \mathcal{N}_{\ell-1}) \cup \{z \in \mathcal{N}_{\ell-1} : \lambda_{\ell z} \neq \lambda_{\ell-1 z}\},$$

and $\tilde{\mathcal{N}}_0 = \mathcal{N}_0$. It should be noticed (see for instance [16]) that to each $z \in \tilde{\mathcal{N}}_\ell$, the corresponding hat function $\lambda_{\ell z}$ does not belong to $S_{\ell-1}$.

Now we define $\bar{\rho}$ and $\bar{\gamma}$ by

$$\begin{aligned} \bar{\rho}^2 &= \sum_{x \in \mathcal{N}} \rho_x^2, \\ \bar{\gamma}^2 &= \sum_{\ell=0}^L \sum_{z \in \tilde{\mathcal{N}}_\ell \setminus \Gamma_D} \gamma_{\ell z}^2, \end{aligned}$$

where

$$\begin{aligned} \rho_x^2 &= h_x^2 \int_{\omega_x} |r - \bar{r}_x|^2 \lambda_x + h_x \int_{\omega_x \cap \Gamma_N} |Gu_h \cdot n - g_N|^2 \lambda_x, \\ \gamma_{\ell z} &= |\langle R, \lambda_{\ell z} \rangle|, \end{aligned}$$

R being the residual defined by

$$\langle R, \varphi \rangle = \int_{\Omega} Gu_h \cdot \nabla \varphi - \int_{\Omega} f \varphi - \int_{\Gamma_N} g_N \varphi, \forall \varphi \in H^1(\Omega).$$

3.1 Upper bound

Theorem 3.1 *Assume that there exists $w_h \in X_h \cap H^1(\Omega)$ such that $g_D = w_h|_{\Gamma_D}$. Let $u \in w + H_D^1(\Omega)$ be a solution of problem (2) and let u_h be its discontinuous Galerkin approximation, i.e. $u_h \in X_h$ solution of (5). Then there exists $C > 0$ such that*

$$\|a^{1/2} \nabla_h(u - u_h)\| \leq (\eta_{CF}^2 + \eta_{NC}^2)^{1/2} + C(\bar{\rho} + \bar{\gamma}), \quad (13)$$

and consequently

$$\|u - u_h\|_{DG,h} \leq (\eta_{CF}^2 + \eta_{NC}^2)^{1/2} + \eta_J + C(\bar{\rho} + \bar{\gamma}). \quad (14)$$

Remark 3.2 *Let us note that under a superconvergence property of $\|a^{-1/2}(Gu_h - a\nabla u)\|$, $\bar{\rho}$ and $\bar{\gamma}$ will be proved to be negligible quantities (see Theorem 3.5 below), so that the error is in this case asymptotically bounded above by the estimator without any multiplicative constant. This superconvergence property is observed in most of practical cases, as for example in our numerical tests (see section 4). Moreover, theoretical results for different continuous finite element methods on structured and unstructured meshes have been established (see for example [19, 32, 35]), but, to our knowledge, not yet for discontinuous methods on unstructured multi-dimensional meshes.*

Proof: From the Helmholtz decomposition of the error we have

$$\|a^{1/2}\nabla_h(u - u_h)\|^2 = \|a^{1/2}\nabla\varphi\|^2 + \|a^{-1/2}\operatorname{curl}\chi\|^2. \quad (15)$$

We are then reduced to estimate each term of this right-hand side.

For the non conforming part, we proceed as in [1], namely by Green's formula we have

$$\begin{aligned} \|a^{-1/2}\operatorname{curl}\chi\|^2 &= \int_{\Omega} \nabla_h(u - u_h) \cdot \operatorname{curl}\chi \\ &= - \int_{\Omega} \nabla_h u_h \cdot \operatorname{curl}\chi + \int_{\Gamma_D} g_D \operatorname{curl}\chi \cdot n \\ &= \int_{\Omega} \nabla_h(w_h - u_h) \cdot \operatorname{curl}\chi, \end{aligned}$$

since $\int_{\Omega} \nabla w_h \cdot \operatorname{curl}\chi = \int_{\Gamma_D} g_D \operatorname{curl}\chi \cdot n$. By Cauchy-Schwarz's inequality we directly obtain

$$\|a^{-1/2}\operatorname{curl}\chi\|^2 \leq \eta_{NC} \|a^{-1/2}\operatorname{curl}\chi\|. \quad (16)$$

For the conforming part, we write

$$\begin{aligned} \|a^{1/2}\nabla\varphi\|^2 &= \int_{\Omega} a\nabla_h(u - u_h) \cdot \nabla\varphi \\ &= \int_{\Omega} (a\nabla u - Gu_h) \cdot \nabla\varphi + \int_{\Omega} (Gu_h - a\nabla_h u_h) \cdot \nabla\varphi. \end{aligned}$$

By Cauchy-Schwarz's inequality we obtain

$$\|a^{1/2}\nabla\varphi\|^2 \leq \|a^{-1/2}(a\nabla_h u_h - Gu_h)\| \|a^{1/2}\nabla\varphi\| + \left| \int_{\Omega} (a\nabla u - Gu_h) \cdot \nabla\varphi \right|. \quad (17)$$

Using problem (2), the second term of this right-hand side is bounded by the residual, indeed

$$- \int_{\Omega} (a\nabla u - Gu_h) \cdot \nabla\varphi = \int_{\Omega} Gu_h \cdot \nabla\varphi - \int_{\Omega} f\varphi - \int_{\Gamma_N} g_N\varphi = \langle R, \varphi \rangle.$$

Using the arguments from Theorem 4.1 of [16], we have

$$|\langle R, \varphi \rangle| \leq C(\bar{\rho} + \bar{\gamma}) \|a^{1/2}\nabla\varphi\|. \quad (18)$$

Coming back to the identity (15), and using the estimates (16), (17) and (18) we conclude by discrete Cauchy-Schwarz's inequality and again using (15):

$$\begin{aligned} \|a^{1/2}\nabla_h(u - u_h)\|^2 &\leq \eta_{NC} \|a^{-1/2}\operatorname{curl}\chi\| + (\eta_{CF} + C(\bar{\rho} + \bar{\gamma})) \|a^{1/2}\nabla\varphi\| \\ &\leq (\eta_{NC}^2 + \eta_{CF}^2)^{1/2} (\|a^{-1/2}\operatorname{curl}\chi\|^2 + \|a^{1/2}\nabla\varphi\|^2)^{1/2} \\ &\quad + C(\bar{\rho} + \bar{\gamma}) \|a^{1/2}\nabla\varphi\| \\ &\leq [(\eta_{NC}^2 + \eta_{CF}^2)^{1/2} + C(\bar{\rho} + \bar{\gamma})] \|a^{1/2}\nabla_h(u - u_h)\|. \end{aligned}$$

■

3.2 Lower bound

Our lower bound is based on the equivalence of the local L^2 -norm of any element in X_h with a local L^2 -norm in the interfaces. First of all for any vertex x of one Ω_j and belonging to more than one sub-domain, we introduce the following local notation: let Ω_i , $i = 1, \dots, n$, $n \geq 2$, the sub-domains that have x as vertex. We further denote by n_i the unit normal vector along the interface I_i between Ω_i and Ω_{i+1} (modulo n if x is inside the domain Ω) and oriented from Ω_i and Ω_{i+1} . Now we are able to prove the following lemma:

Lemma 3.3 *Assume that x is a vertex of one Ω_j and belonging to more than one sub-domain, and use the notations introduced above. Then there exists a positive constant C that depends only on the geometrical situation of the Ω_i 's near x such that for all $v^{(i)} \in \mathbb{R}^2$, $i = 1, \dots, n$, there exist vectors $g(v)^{(i)} \in \mathbb{R}^2$, $i = 1, \dots, n$ satisfying*

$$(g(v)^{(i+1)} - g(v)^{(i)}) \cdot n_i = 0, \forall i = 1, \dots, n, \quad (19)$$

and such that the following estimate holds

$$\sum_{i=1}^n |v^{(i)} - g(v)^{(i)}| \leq C \sum_{i=1}^n |[v \cdot n]_i|, \quad (20)$$

where here $|\cdot|$ means the Euclidean norm and $[v \cdot n]_i$ means the normal jump of v along the interface I_i :

$$[v \cdot n]_i = (v^{(i+1)} - v^{(i)}) \cdot n_i, \forall i = 1, \dots, n.$$

Proof: First introduce the following subspace of \mathbb{R}^{2n} :

$$W = \{v = (v^{(i)})_{i=1}^n : v^{(i)} \in \mathbb{R}^2 \text{ and satisfying } [v \cdot n]_i = 0, \forall i = 1, \dots, n\}.$$

We take $g(v) = \Pi_W v$, the orthogonal projection of $v = (v^{(i)})_{i=1}^n$ into W . By construction $g(v)$ trivially satisfies (19). On the other hand the estimate (20) is equivalent to

$$|v - \Pi_W v| \leq C \sum_{i=1}^n |[v \cdot n]_i|,$$

which is easily proved by a contradiction argument and the fact that we are in a finite dimensional space. \blacksquare

Using the above lemma, we are now able to prove the asymptotic nondeterioration of the smoothed gradient if the following choice for Gu_h is made: We distinguish the following different possibilities for $x \in \mathcal{N}$.

1) First for all vertex x of the mesh (i.e. vertex of at least one triangle) such that x is inside one Ω_j , we set

$$(Gu_h)_{|\Omega_j}(x) = \frac{1}{|\omega_x|} \sum_{x \in T} |T| a_T \nabla u_{h|T}(x). \quad (21)$$

2) Second if x belongs to the boundary of Ω and to the boundary of only one Ω_j (hence it does not belong to the boundary of another Ω_k), we define $(Gu_h)|_{\Omega_j}(x)$ as before.

3) If x belongs to an interface between two different sub-domain Ω_j and Ω_k but is not a vertex of these sub-domains, then we denote by $n_{j,k}$ the unit normal vector pointing from Ω_j to Ω_k and set $t_{j,k}$ the unit orthogonal vector of $n_{j,k}$ so that $(n_{j,k}, t_{j,k})$ is a direct basis of \mathbb{R}^2 ; in that case we set

$$(Gu_h)|_{\Omega_j}(x) \cdot n_{j,k} = (Gu_h)|_{\Omega_k}(x) \cdot n_{j,k} = \frac{1}{|\omega_x|} \sum_{x \in T} |T| a_T \nabla u_h|_T(x) \cdot n_{j,k}, \quad (22)$$

$$(Gu_h)|_{\Omega_j}(x) \cdot t_{j,k} = \frac{1}{|\omega_x \cap \Omega_j|} \sum_{T \subset \Omega_j: x \in T} |T| a_T \nabla u_h|_T(x) \cdot t_{j,k}, \quad (23)$$

$$(Gu_h)|_{\Omega_k}(x) \cdot t_{j,k} = \frac{1}{|\omega_x \cap \Omega_k|} \sum_{T \subset \Omega_k: x \in T} |T| a_T \nabla u_h|_T(x) \cdot t_{j,k}. \quad (24)$$

4) Finally if x is a vertex of at least two sub-domains Ω_j , for the sake of simplicity we suppose that each triangle T having x as vertex is included into one Ω_j , and we take

$$(Gu_h)|_{\Omega_j}(x) = g(v)^{(j)} \forall j \in \mathcal{J}_x, \quad (25)$$

where $\mathcal{J}_x = \{j \in \{1, \dots, J\} : x \in \bar{\Omega}_j\}$, $g(v)^{(j)}$ were defined in the previous Lemma 3.3 with here v given by $v = (a_j \nabla u_h|_T(x))_{j \in \mathcal{J}_x}$.

With these choices, we take

$$(Gu_h)|_{\Omega_j} = \sum_{x \in \mathcal{N} \cap \bar{\Omega}_j} (Gu_h)|_{\Omega_j}(x) \lambda_x, \forall j = 1, \dots, J, \quad (26)$$

where $(Gu_h)|_{\Omega_j}(x)$ was defined before.

The main point is that by construction Gu_h satisfies the requirements (9) and (10) but moreover we have the next asymptotic nondeterioration result:

Theorem 3.4 *If $\ell \leq 2$ (see (3)), then for each element $T \in \mathcal{T}$ the following estimate holds*

$$\|a_T^{-1/2}(Gu_h - a_T \nabla u)\|_T \lesssim \|u - u_h\|_{DG, \omega_T} + \text{osc}(f, \omega_T), \quad (27)$$

where ω_T denotes the patch consisting of all the triangles of \mathcal{T} having a nonempty intersection with T and

$$\|v\|_{DG, \omega_T}^2 = \|a^{1/2} \nabla_h v\|_{\omega_T}^2 + \gamma \sum_{e \in \mathcal{E}_{ID}: e \subset \omega_T} h_e^{-1} \|[[v]]\|_e^2,$$

and

$$\text{osc}(f, \omega_T)^2 = h_T^2 \sum_{T' \subset \omega_T} \|f - \Pi_{T'} f\|_{T'}^2,$$

where $\Pi_{T'} f$ is the $L^2(T')$ -orthogonal projection of f onto $\mathbb{P}_1(T')$.

Proof: By the triangle inequality we may write

$$\begin{aligned} \|a_T^{-1/2}(Gu_h - a_T \nabla u)\|_T &\leq \|a_T^{-1/2}(Gu_h - a_T \nabla u_h)\|_T + \|a_T^{-1/2}(a_T \nabla u_h - a_T \nabla u)\|_T \\ &\leq \|a_T^{-1/2}(Gu_h - a_T \nabla u_h)\|_T + \|u - u_h\|_{DG,T}. \end{aligned}$$

Therefore it remains to estimate the first term of this right-hand side. For that purpose, since $T \subset \Omega_j$ for a unique $j \in \{1, \dots, J\}$, we may write

$$(Gu_h - a_T \nabla u_h)|_T = \sum_{x \in T} \{(Gu_h)|_{\Omega_j}(x) - a_j \nabla u_h|_T(x)\} \lambda_x.$$

As $0 \leq \lambda_x \leq 1$, and since the triangulation is regular, we get

$$\|a_T^{-1/2}(Gu_h - a_T \nabla u_h)\|_T \lesssim \sum_{x \in T} |(Gu_h)|_{\Omega_j}(x) - a_j \nabla u_h|_T(x)| h_T. \quad (28)$$

We are then reduced to estimate the factor $|(Gu_h)|_{\Omega_j}(x) - a_j \nabla u_h|_T(x)|$ for all nodes x of T . For that purpose, we distinguish four different cases:

1) If $x \in \Omega_j$, then we use an argument similar to the one from Proposition 4.2 of [16] adapted to the DG situation. By the definition of Gu_h , we have

$$(Gu_h)|_{\Omega_j}(x) = \frac{1}{|\omega_x|} \sum_{T' \subset \omega_x} |T'| a_j \nabla u_h|_{T'}(x),$$

because in this case all $T' \subset \omega_x$ are included into Ω_j . As a consequence, we obtain

$$(Gu_h)|_{\Omega_j}(x) - a_j \nabla u_h|_T(x) = \frac{1}{|\omega_x|} \sum_{T' \subset \omega_x} |T'| a_j (\nabla u_h|_{T'}(x) - \nabla u_h|_T(x)),$$

and therefore

$$|(Gu_h)|_{\Omega_j}(x) - a_j \nabla u_h|_T(x)| \lesssim \sum_{T' \subset \omega_x} |a_j (\nabla u_h|_{T'}(x) - \nabla u_h|_T(x))|.$$

For each $T' \subset \omega_x$, there exists a path of triangles of ω_x , written $T_i, i = 0, \dots, n$ such that

$$\begin{aligned} T_0 = T, T_n = T', T_i \neq T_j, \forall i \neq j, \\ T_i \cap T_{i+1} \text{ is an common edge } \forall i = 1, \dots, n-1. \end{aligned}$$

Hence by the triangle inequality we can estimate

$$|a_j (\nabla u_h|_{T'}(x) - \nabla u_h|_T(x))| \leq \sum_{i=0}^{n-1} |a_j (\nabla u_h|_{T_{i+1}}(x) - \nabla u_h|_{T_i}(x))|.$$

Now for each term, since a_j is symmetric and positive definite, we have

$$|a_j (\nabla u_h|_{T_{i+1}}(x) - \nabla u_h|_{T_i}(x))| \lesssim | \{a_j (\nabla u_h|_{T_{i+1}}(x) - \nabla u_h|_{T_i}(x))\} \cdot n_i | + | (\nabla u_h|_{T_{i+1}}(x) - \nabla u_h|_{T_i}(x)) \cdot t_i |,$$

where n_i is one fixed unit normal vector along the edge $T_i \cap T_{i+1}$ and t_i is one fixed unit tangent vector along this edge.

All together we have shown that

$$|(Gu_h)|_{\Omega_j}(x) - a_j \nabla u_h|_T(x)| h_T \lesssim h_T \sum_{e \in \mathcal{E}_{int}: e \subset \omega_x} \{ | \llbracket a_j \nabla u_h(x) \cdot n \rrbracket_e | + | \llbracket \nabla - hu_h(x) \cdot t \rrbracket_e | \}.$$

Using a norm equivalence and an inverse inequality we obtain

$$|(Gu_h)|_{\Omega_j}(x) - a_j \nabla u_h|_T(x)| h_T \lesssim \sum_{e \in \mathcal{E}_{int}: e \subset \omega_x} \{ h_e^{1/2} \| \llbracket a_j \nabla_h u_h \cdot n \rrbracket_e \|_e + h_e^{-1/2} \| \llbracket u_h \rrbracket_e \| \}. \quad (29)$$

2) If the node x belongs to the boundary of Ω and to the boundary of a unique Ω_j , since $(Gu_h)|_{\Omega_j}(x)$ is defined as in the first case, the above arguments lead to (29).

3) If x is a vertex of different sub-domains Ω_j , then by Lemma 3.3, we have

$$|(Gu_h)|_{\Omega_j}(x) - a_j \nabla u_h|_T(x)| h_T \lesssim h_T \sum_{e \in \mathcal{E}_{int}: e \subset \omega_x} | \llbracket a \nabla_h u_h(x) \cdot n \rrbracket_e |, \quad (30)$$

and therefore as before we conclude that (29) holds.

4) Finally if x belongs to an interface between two subdomains and is not a vertex of them, then it is not difficult to show that (30) holds (due to the regularity of the mesh), and consequently (29) is still valid.

Summarizing the different cases, by (28) and (29), we have

$$\| a_T^{-1/2} (Gu_h - a_T \nabla u_h) \|_T \lesssim \sum_{e \in \mathcal{E}_{int}: e \subset \omega_x} \{ h_e^{1/2} \| \llbracket a \nabla_h u_h \cdot n \rrbracket_e \|_e + h_e^{-1/2} \| \llbracket u_h \rrbracket_e \| \}. \quad (31)$$

The first term of this right hand side is a part of the standard residual error estimator and it is by now standard that (using appropriate bubble functions and Green's formula)

$$h_e^{1/2} \| \llbracket a \nabla_h u_h \cdot n \rrbracket_e \|_e \lesssim \| a \nabla_h (u - u_h) \|_{\omega_e} + \text{osc}(f, \omega_e), \forall e \in \mathcal{E}_{int}.$$

The second term is part of the DG-norm. Therefore the above estimate in (31) leads to (27). \blacksquare

Now using the same arguments than in Proposition 4.1 of [16], we have

Theorem 3.5 *For all $T \in \mathcal{T}$, $x \in \mathcal{N}$ and $\ell \geq 0, z \in \mathcal{N}_\ell$, we have*

$$\begin{aligned} \eta_{CF,T} &\leq \| a_T^{1/2} \nabla (u_h - u) \|_T + \| a^{-1/2} (Gu_h - a \nabla u) \|_T, \\ \bar{\rho}_x &\lesssim \| a^{-1/2} (Gu_h - a \nabla u) \|_{\omega_x} + \text{osc}(f, \omega_x) + \text{osc}(g_N, \omega_x), \\ \bar{\gamma}_{\ell z} &\lesssim \| a^{-1/2} (Gu_h - a \nabla u) \|_{\omega_{\ell z}}, \end{aligned}$$

where

$$\text{osc}(g_N, \omega_x)^2 = \sum_{e \subset \omega_x \cap \Gamma_N} h_e \| g_N - \Pi_e g_N \|_T^2,$$

$\Pi_e g_N$ being the $L^2(e)$ -orthogonal projection of g_N onto $\mathbb{P}_2(e)$.

For the non conforming part of the estimator, we make use of Theorem 2.2 of [20] to directly obtain the

Theorem 3.6 *Let the assumptions of Theorem 3.1 be satisfied. For each element $T \in \mathcal{T}$ the following estimate holds*

$$\eta_{NC;T} \lesssim a_T^{1/2} \|u - u_h\|_{DG,\omega_T}. \quad (32)$$

A direct consequence of these three Theorems is the next local lower bound:

Theorem 3.7 *Let the assumptions of Theorems 3.1 and 3.4 be satisfied. For each element $T \in \mathcal{T}$ the following estimate holds*

$$\eta_{CF;T} + \eta_{NC;T} + \eta_{J;T} + \sum_{x \in T} (\bar{\rho}_x + \bar{\gamma}_x) \lesssim \|u - u_h\|_{DG,\omega_T} + \text{osc}(f, \omega_T) + \text{osc}(g_N, \omega_T),$$

where $\bar{\gamma}_x = \bar{\gamma}_{Lx}$ recalling that L is such that $\mathcal{N}_L = \mathcal{N}$.

Remark 3.8 Note that the lower bound on the non conforming estimator (see (32)) involves a constant that depends on the aspect ratio of the mesh and of the penalization parameter γ , and is specific to the discontinuous Galerkin method. Consequently it prevents the estimator to be asymptotically exact, as in the continuous Galerkin method [16]. Nevertheless the numerical tests show quite satisfactory effectivity indices (see below).

As in Proposition 4.3 of [16], one has

$$\bar{\gamma} \lesssim \|a^{-1/2}(Gu_h - a\nabla u)\|,$$

and therefore a global lower bound can be obtained:

Theorem 3.9 *Let the assumptions of Theorems 3.1 and 3.4 be satisfied. Then the following global lower bound holds*

$$\eta_{CF} + \eta_{NC} + \eta_J + \bar{\rho} + \bar{\gamma} \lesssim \|u - u_h\|_{DG,h} + \text{osc}(f, \Omega) + \text{osc}(g_N, \Omega).$$

4 Numerical results

4.1 The polynomial solution

In order to illustrate our theoretical predictions, this first numerical test consists in validating our computations on a simple case, using an uniform refinement process. Let Ω be the square $(-1, 1)^2$, $\Gamma_D = \partial\Omega$, $a = Id$ and f defined such that the exact solution u is given by :

$$u(x, y) = (x + 1)(x - 1)(y + 1)(y - 1).$$

Let us recall that u_h is the finite element solution, $e(u_h) = \|u - u_h\|_{DG,h}$ the error, $\eta(u_h) = (\eta_{CF}^2 + \eta_{NC}^2)^{1/2} + \eta_J$ the estimator and Gu_h the approximated value of $a\nabla u$ given by (26). We also define CV_{error} (resp. CV_{recov}) as the convergence rate of the error $e(u_h)$ (resp. of the quantity $\|Gu_h - \nabla u\|$) with respect to $DoF^{-1/2}$ from one line of the table to the following one.

Computations are performed using a global mesh refinement process from an initial cartesian grid, using $\gamma_e \equiv \gamma = 20$ to ensure (4). First, it can be seen from Table 1 that the convergence rate of the numerical method is equal to one, as theoretically expected. Then, the superconvergence property of the term $\|Gu_h - \nabla u\|$ is actually observed with a convergence rate of 1.50. Finally, the reliability of the estimator is ensured since the ratio in the last column (the so-called effectivity index), converges fastly towards the constant 1.70.

k	DoF	$e(u_h)$	CV_{error}	$\ Gu_h - \nabla u\ $	CV_{recov}	$\frac{\eta(u_h)}{e(u_h)}$
1	384	3.80E-01		2.78E-01		1.60
2	1536	1.90E-01	1.00	1.04E-01	1.42	1.67
3	6144	9.51E-02	1.00	3.73E-02	1.47	1.70
4	24576	4.74E-02	1.00	1.32E-02	1.49	1.70
5	98304	2.37E-02	1.00	4.69E-03	1.50	1.70
6	393216	1.19E-02	1.00	1.66E-02	1.50	1.70

Table 1: The polynomial solution (uniform refinement).

4.2 The interior and boundary layer solution

The following numerical test consists in solving the interior and boundary layer example given in [16]. Let Ω the square $(-1, 1)^2$, $\Gamma_D = \partial\Omega$, $a = Id$ and f defined such that :

$$u(x, y) = \arctan(60(x^2 + y^2 - 1))$$

is the exact solution (see Figure 1). Let us note that the boundary layer crosses the boundary $\partial\Omega$ and that the loading term oscillates across it, what constitutes the difficulty of the computation. This time, an adaptive mesh refinement strategy is used based on the estimator $\eta_T = \eta_{CF,T} + \eta_{NC,T} + \eta_{J,T}$ and the marking procedure

$$\eta_T > 0.75 \max_{T'} \eta_{T'}$$

and a standard refinement procedure with a limitation on the minimal angle. Once again, we choose $\gamma_e \equiv \gamma = 20$. Several mesh levels are displayed on Figure 2, to show the capability of the algorithm to track the high gradients regions. Furthermore, quantitative

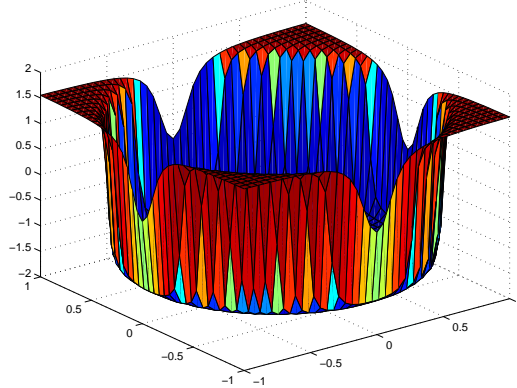


Figure 1: Interior and boundary layer solution.

results are displayed on Table 2. Once again, the superconvergence property of $\|Gu_h - \nabla u\|$ is asymptotically observed, as well as the reliability of the estimator, provided the mesh is fine enough around the boundary layer.

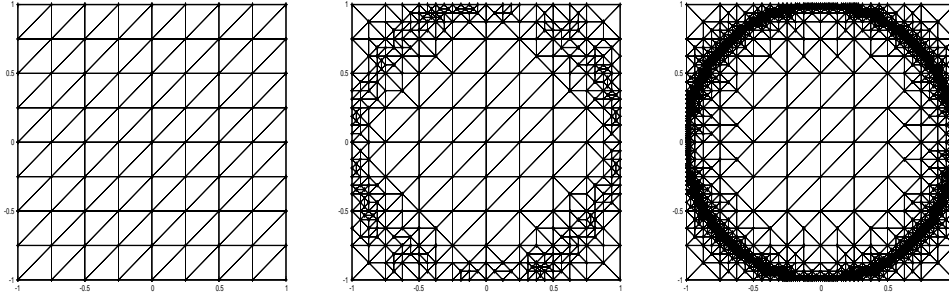


Figure 2: Mesh levels 1, 10 and 20, interior and boundary layer solution.

4.3 The discontinuous case

This section is devoted to the case of the discontinuous coefficient a . Namely, the domain $\Omega = (-1, 1)^2$ with $\Gamma_D = \Gamma$ is decomposed into 4 sub-domains Ω_i , $i = 1, \dots, 4$, with $\Omega_1 = (0, 1) \times (0, 1)$, $\Omega_2 = (-1, 0) \times (0, 1)$, $\Omega_3 = (-1, 0) \times (-1, 0)$ and $\Omega_4 = (0, 1) \times (-1, 0)$. In that case we take discontinuous coefficient a , namely we take $a = a_i$ on Ω_i , with $a_1 = a_3 = Id$ and $a_2 = a_4 = C Id$ with C to be specified. For this second test, and using usual polar coordinates centered at $(0, 0)$, the exact solution is equal to the singular function $u(x, y) = r^\alpha \phi(\theta)$, where $\alpha \in (0, 1)$ and ϕ are chosen such that u is harmonic on each sub-domain Ω_i , $i = 1, \dots, 4$, and satisfies the jump conditions :

$$[u] = 0 \text{ and } [a \nabla u \cdot n] = 0$$

k	DoF	$e(u_h)$	CV_{error}	$\ Gu_h - \nabla u\ $	CV_{recov}	$\frac{\eta(u_h)}{e(u_h)}$
1	384	1.02E+02		4.67E+01		1.10
7	1323	4.59E+01	1.29	3.36E+01	0.53	0.94
15	6069	1.71E+01	1.29	1.67E+01	0.91	1.35
22	22458	7.84E+00	1.19	6.55E+00	1.43	1.67
28	94260	3.56E+00	1.10	1.84E+00	1.77	1.62
34	366180	1.80E+00	1.01	7.14E-01	1.40	1.60

Table 2: The interior and boundary layer solution (adaptive refinement).

on the interfaces. Non-homogeneous Dirichlet boundary conditions on Γ are fixed accordingly. It is easy to see (see for instance [11]) that α is the root of the transcendental equation

$$\tan \frac{\alpha \pi}{4} = \sqrt{1/C}.$$

Since $\alpha < 1$, this solution has a singular behavior around the point $(0, 0)$. For this test, we also compute the standard ZZ smoothed gradient G^*u_h belonging to $S_{h,1}^{(2)} = \{v_h \in C(\bar{\Omega})^2 | v_h|_T \in \mathbb{P}_1^2(T), T \in \mathcal{T}\}$ and characterized by its value at each node of the mesh given by :

$$(G^*u_h)(x) = \frac{1}{|\omega_x|} \sum_{x \in T} |T| a_T \nabla u_h|_T(x). \quad (33)$$

For the case $C = 5$, we choose $\gamma_e \equiv \gamma = 20$ and figure 3 shows some of the meshes obtained during the local refinement process, which is the same as the one used in section 4.2. Moreover, table 3 displays the corresponding quantitative results. The smoothed gradient Gu_h is superconvergent, while the effectivity index converges towards 1.75, which is quite satisfactory and comparable with results from [9, 15] as well as those of the previous tests in sections 4.1 and 4.2.

For the case $C = 100$, we choose $\gamma_e \equiv \gamma = 500$ and figure 4 shows some of the meshes obtained during the local refinement process. These ones are more refined around the singularity than the case $C = 5$. Table 4 displays the corresponding quantitative results. Once again, the smoothed gradient Gu_h is superconvergent, provided the mesh resolution is fine enough. The effectivity index requires more iterations to converge, but is already equal to 2.82 for the finer mesh resolution.

Let us finally note that for both cases, the standard ZZ smoothed gradient G^*u_h is no more superconvergent towards $a\nabla u$, as in the case $a = Id$. This is not surprising since the statement of Theorem 3.4 is not valid for G^*u_h if $a \neq Id$.

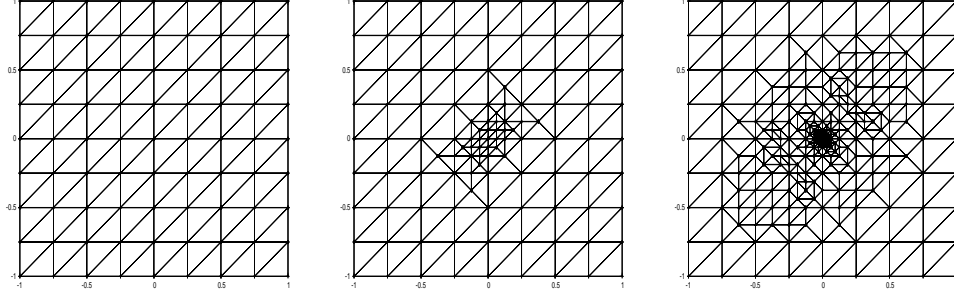


Figure 3: Mesh levels 1, 3 and 10, singular solution for $C = 5$.

k	DoF	$e(u_h)$	CV_{error}	$\ Gu_h - \nabla u\ $	CV_{recov}	$\frac{\ G^*u_h - \nabla u\ }{e(u_h)}$	$\frac{\eta(u_h)}{e(u_h)}$
1	384	3.06E-01		3.00E-01		9.99E-01	1.85
8	1812	9.72E-02	1.47	6.95E-02	1.88	1.00E+00	1.76
12	7254	4.75E-02	1.03	2.41E-02	1.52	9.33E-01	1.75
34	49503	1.78E-02	1.02	6.80E-03	1.31	9.33E-01	1.75
64	201411	8.89E-03	0.99	2.90E-03	1.21	9.36E-01	1.75

Table 3: Discontinuous coefficient a : $C = 5$, $\gamma = 20$, singular solution (local refinement).

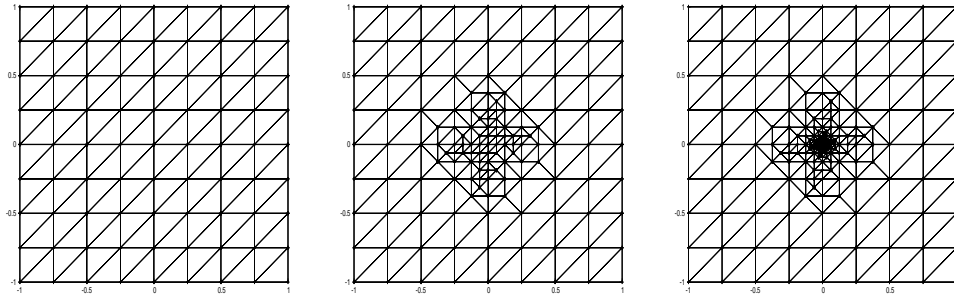


Figure 4: Mesh levels 1, 3 and 10, singular solution for $C = 100$.

k	DoF	$e(u_h)$	CV_{error}	$\ Gu_h - \nabla u\ $	CV_{recov}	$\frac{\ G^*u_h - \nabla u\ }{e(u_h)}$	$\frac{\eta(u_h)}{e(u_h)}$
1	384	3.39E-01		3.38E-01		9.99E-01	4.10
3	1596	3.16E-01	0.10	3.13E-01	0.11	9.99E-01	4.07
12	6996	1.81E-01	0.75	1.74E-01	0.80	9.97E-01	4.01
25	36054	6.88E-02	1.17	6.20E-02	1.26	1.02E+00	3.68
38	123642	3.16E-02	1.26	2.41E-02	1.53	1.29E+00	2.82

Table 4: Discontinuous coefficient a : $C = 100$, $\gamma = 500$, singular solution (local refinement).

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