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# Locally identifying coloring in bounded expansion classes of graphs \*

Daniel Gonçalves<sup>†</sup>, Aline Parreau<sup>‡</sup>, Alexandre Pinlou<sup>†§</sup>

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## Abstract

A proper vertex coloring of a graph is said to be *locally identifying* if the sets of colors in the closed neighborhood of any two adjacent non-twin vertices are distinct. The lid-chromatic number of a graph is the minimum number of colors used by a locally identifying vertex-coloring. In this paper, we prove that for any graph class of bounded expansion, the lid-chromatic number is bounded. Classes of bounded expansion include minor closed classes of graphs. For these latter classes, we give an alternative proof to show that the lid-chromatic number is bounded. This leads to an explicit upper bound for the lid-chromatic number of planar graphs. This answers in a positive way a question of Esperet *et al.* [L. Esperet, S. Gravier, M. Montassier, P. Ochem and A. Parreau. Locally identifying coloring of graphs. *Electronic Journal of Combinatorics*, **19(2)**, 2012.].

## 1 Introduction

A vertex-coloring is said to be *locally identifying* if (i) the vertex-coloring is proper (i.e. no adjacent vertices receive the same color), and (ii) for any adjacent vertices  $u, v$ , the set of colors assigned to the closed neighborhood of  $u$  differs from the set of colors assigned to the closed neighborhood of  $v$  whenever these neighborhoods are distinct. The *locally identifying chromatic number* of the graph  $G$  (or lid-chromatic number, for short), denoted by  $\chi_{lid}(G)$ , is the smallest number of colors required in any locally identifying coloring of  $G$ .

Locally identifying colorings of graphs have been recently introduced by Esperet et al. [6] and later studied by Foucaud et al. [7]. They are related to identifying codes [8, 9], distinguishing colorings [1, 3, 5] and locating-colorings [4]. For

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example, upper bounds on lid-chromatic number have been obtained for bipartite graphs,  $k$ -trees, outerplanar graphs and bounded degree graphs. An open question asked by Esperet et al. [6] was to know whether  $\chi_{lid}$  is bounded for the class of planar graphs. In this paper, we answer positively to this question proving more generally that  $\chi_{lid}$  is bounded for any class of bounded expansion.

In Section 3, we first give a tight bound of  $\chi_{lid}$  in term of the tree-depth. Then we use the fact that any class of bounded expansion admits a low tree-depth coloring (that is a  $k$ -coloring such that each triplet of colors induces a graph of tree-depth 3, for some constant  $k$ ) to prove that it has bounded lid-chromatic number.

In Section 4, we focus on minor closed classes of graphs which have bounded expansion and give an alternative bound on the lid-chromatic number, which gives an explicit bound for planar graphs.

The next section is devoted to introduce notation and preliminary results.

## 2 Notation and preliminary results

Let  $G = (V, E)$  be a graph. For any vertex  $u$ , we denote by  $N_G(u)$  its *neighborhood* in  $G$  and by  $N_G[u]$  its *closed neighborhood* in  $G$  ( $u$  together with its adjacent vertices). The notion of neighborhood can be extended to sets as follows: for  $X \subseteq V$ ,  $N_G[X] = \{w \in V(G) \mid \exists v \in X, w \in N[v]\}$  and  $N_G(X) = N_G[X] \setminus X$ . When the considered graph is clearly identified, the subscript is dropped.

The *degree* of vertex  $u$  is the size of its neighborhood. The *distance* between two vertices  $u$  and  $v$  is the number of edges in a shortest path between  $u$  and  $v$ . For  $X \subseteq V$ , we denote by  $G[X]$  the subgraph of  $G$  *induced by*  $X$ .

We say that two vertices  $u$  and  $v$  are *twins* if  $N[u] = N[v]$  (although they are often called *true twins* in the literature, we call them *twins* for convenience). In particular,  $u$  and  $v$  are adjacent vertices. Note that if  $u$  and  $v$  are adjacent but not twins, there exists a vertex  $w$  which is adjacent to exactly one vertex among  $\{u, v\}$ , i.e.  $w \in N[u] \Delta N[v]$  (where  $\Delta$  is the symmetric difference between sets). We say that  $w$  *distinguishes*  $u$  and  $v$ , or simply  $w$  *distinguishes* the edge  $uv$ . For a subset  $X \subseteq V$ , we say that a subset  $Y \subseteq V$  *distinguishes*  $X$  if for every pair  $u, v$  of non-twin vertices of  $X$ , there exists a vertex  $w \in Y$  that distinguishes the edge  $uv$ .

Let  $c : V \rightarrow \mathbb{N}$  be a vertex-coloring of  $G$ . The coloring  $c$  is *proper* if adjacent vertices have distinct colors. We denote by  $\chi(G)$  the *chromatic number* of  $G$ , i.e. the minimum number of colors in a proper coloring of  $G$ . For any  $X \subseteq V$ , let  $c(X)$  be the set of colors that appear on the vertices of  $X$ . A *locally identifying coloring* (lid-coloring for short) of  $G$  is a proper vertex-coloring  $c$  of  $G$  such that for any two adjacent vertices  $u$  and  $v$  that are not twins (i.e.  $N[u] \neq N[v]$ ), we have  $c(N[u]) \neq c(N[v])$ . A graph  $G$  is  *$k$ -lid-colorable* if it admits a locally identifying coloring using at most  $k$  colors and the minimum number of colors needed for any locally identifying coloring of  $G$  is the *locally identifying chromatic number* (lid-chromatic number for short) denoted by  $\chi_{lid}(G)$ . For a vertex  $u$ , we say that  $u$  *sees* color  $a$  if  $a \in c(N[u])$ . For two adjacent vertices

$u$  and  $v$ , a color that is in the set  $c(N[u]) \Delta c(N[v])$  separates  $u$  and  $v$ , or simply separates the edge  $uv$ . The notion of chromatic number (resp. lid-chromatic number) can be extended to a class of graphs  $\mathcal{C}$  as follows:  $\chi(\mathcal{C}) = \sup\{\chi(G), G \in \mathcal{C}\}$  (resp.  $\chi_{lid}(\mathcal{C}) = \sup\{\chi_{lid}(G), G \in \mathcal{C}\}$ ).

The following theorem is due to Bondy [2]:

**Theorem 1** (Bondy's theorem [2]). *Let  $\mathcal{A} = \{A_1, \dots, A_n\}$  be a collection of  $n$  distinct subsets of a finite set  $X$ . There exists a subset  $X'$  of  $X$  of size at most  $n - 1$  such that the sets  $A_i \cap X'$  are all distinct.*

**Corollary 2.** *Let  $C$  be a  $n$ -clique subgraph of  $G$ . There exists a vertex subset  $S(C) \subseteq V(G)$  of size at most  $n - 1$  that distinguishes all the pair of non-twin vertices of  $C$ .*

**Proof.** Let  $C$  be a  $n$ -clique subgraph of  $G$  induced by the vertex set  $V(C) = \{v_1, v_2, \dots, v_n\}$ . Let  $\mathcal{A} = \{N[v_i] \mid v_i \in V(C)\}$  be a collection of distinct subsets of the finite set  $X = \bigcup_{1 \leq i \leq n} N[v_i]$ . Note that some  $v_i$ 's might be twins in  $G$  (i.e.  $N[v_i] = N[v_j]$  for some  $v_i, v_j \in V(C)$ ) and therefore  $|\mathcal{A}|$  could be smaller than  $n$ . By Bondy Theorem, there exists  $S(C) \subseteq X$  of size at most  $|\mathcal{A}| - 1 \leq n - 1$  such that for any distinct elements  $A_1, A_2$  of  $\mathcal{A}$ , we have  $A_1 \cap S(C) \neq A_2 \cap S(C)$ .

Let us prove that  $S(C)$  is a set of vertices that distinguish all the pairs of non-twin vertices of  $C$ . For a pair of non-twin vertices  $v_i, v_j$  of  $C$ , we have  $N[v_i] \neq N[v_j]$ . By definition of  $S(C)$ , we have  $N[v_i] \cap S(C) \neq N[v_j] \cap S(C)$ , then there exists  $w \in S(C)$  that belongs to  $N[v_i] \Delta N[v_j]$ . Therefore,  $w$  distinguishes the edge  $v_i v_j$ .  $\square$

### 3 Bounded expansion classes of graphs

A rooted tree is a tree with a special vertex, called the *root*. The *height* of a vertex  $x$  in a rooted tree is the number of vertices on a path from the root to  $x$  (hence, the height of the root is 1). The *height* of a rooted tree  $T$  is the maximum height of the vertices of  $T$ . If  $x$  and  $y$  are two vertices of  $T$ ,  $x$  is an *ancestor* of  $y$  in  $T$  if  $x$  belongs to the path between  $y$  and the root. The *closure*  $\text{clos}(T)$  of a rooted tree  $T$  is the graph with vertex set  $V(T)$  and edge set  $\{xy \mid x \text{ is an ancestor of } y \text{ in } T, x \neq y\}$ . The *tree-depth*  $\text{td}(G)$  of a connected graph  $G$  is the minimum height of a rooted tree  $T$  such that  $G$  is a subgraph of  $\text{clos}(T)$ . If  $G$  is not connected, the tree-depth of  $G$  is the maximum tree-depth of its connected components.

Let  $p$  be a fixed integer. A *low tree-depth coloring* of a graph  $G$  (relatively to  $p$ ) is a coloring of the vertices of  $G$  such that the union of any  $i \leq p$  color classes induces a graph of tree-depth at most  $i$ . Let  $\chi_p^{\text{td}}(G)$  be the minimum number of colors required in such a coloring. Note that as tree-depth one graphs and tree-depth two graphs are respectively the stables and star forests,  $\chi_1^{\text{td}}$  and  $\chi_2^{\text{td}}$  respectively correspond to the usual chromatic number and the star chromatic number.

In the following of this section, we first give a tight bound on the lid-chromatic number in terms of tree-depth.

**Proposition 3.** *For any graph  $G$ ,  $\chi_{lid}(G) \leq 2td(G) - 1$  and this is tight.*

Using this bound, we then bound the lid-chromatic number in terms of  $\chi_3^{td}$ .

**Theorem 4.** *For any graph  $G$ ,*

$$\chi_{lid}(G) \leq 6^{\binom{td(G)}{3}}.$$

Classes of graphs of bounded expansion have been introduced by Nešetřil and Ossona de Mendez [10]. These classes contain minor closed classes of graphs and any class of graphs defined by an excluded topological minor. Actually, these classes of graphs are closely related to low tree-depth colorings:

**Theorem 5** (Theorem 7.1 [10]). *A class of graphs  $\mathcal{C}$  has bounded expansion if and only if  $\chi_p^{td}(\mathcal{C})$  is bounded for any  $p$ .*

We therefore deduce the following corollary from Theorems 4 and 5:

**Corollary 6.** *For any class  $\mathcal{C}$  of bounded expansion,  $\chi_{lid}(\mathcal{C})$  is bounded.*

It is in particular true for a class of bounded tree-width. A consequence is that  $\chi_{lid}$  is bounded for chordal graphs by a function of the clique number (which is equals to the tree-width plus 1 for a chordal graph). It is conjectured by Esperet *et al.* [6] that  $\chi_{lid}(G) \leq 2\omega(G)$  if  $G$  is chordal.

We now prove Proposition 3.

**Proof of Proposition 3.** Let us first prove that the bound is tight. Consider the graph  $H_n$  obtained from a complete graph, with vertex set  $\{a_1, \dots, a_n\}$ , by adding a pendant vertex  $b_i$  to every  $a_i$  but one, say for  $1 \leq i < n$ . The tree-depth of this graph is at least  $n$  as it contains a  $n$ -clique. Indeed, given a rooted tree  $T$ , two vertices at the same height are non-adjacent in  $\text{clos}(T)$ , we thus need at least  $n$  levels. Actually the tree-depth of this graph is at most  $n$  since the tree  $T$  rooted at  $a_1$ , and such that  $a_i$  has two sons  $a_{i+1}$  and  $b_i$ , for  $1 \leq i < n$ , has height  $n$  and is such that  $\text{clos}(T)$  contains  $H_n$  as a subgraph.

Let us show that in any lid-coloring of  $H_n$  all the vertices must have distinct colors, and thus use  $2n - 1 = 2td(H_n) - 1$  colors. Indeed, two vertices  $a_i$  must have different colors as the coloring is proper. A vertex  $b_j$  cannot use the same color as a vertex  $a_i$ , as otherwise the vertex  $a_j$  would only see the  $n$  colors used in the clique, just as  $a_n$ . Similarly if two vertices  $b_i$  and  $b_j$  would use the same color, the vertices  $a_i$  and  $a_j$  would see the same set of colors.

Let us now focus on the upper bound. We prove the result for a connected graph and by induction on the tree-depth of  $G$ , denoted by  $k$ . The result is clear for  $k = 1$  (the graph is a single vertex).

Let  $G$  be a graph of tree-depth  $k > 1$  and let  $T$  be a rooted tree of height  $k$  such that  $G$  is a subgraph of  $\text{clos}(T)$ . If  $T$  is a path, the result is clear since there are only  $k$  vertices. So assume that  $T$  is not a path, and let  $r$  be the root

of  $T$ . Let  $s$  be the smallest height such that there are at least two vertices of height  $s + 1$ . We name  $r_i$ , for  $i \in \{1, \dots, s\}$ , the unique vertex of height  $i$ . Let  $R = \{r_1, \dots, r_s\}$ . Note that each of the vertices of  $R$  is adjacent to all the vertices of  $\text{clos}(T)$ . Therefore, we can choose the way we label the  $s$  vertices in  $R$  (i.e. we can choose the height of each of them in  $T$ ) without changing  $\text{clos}(T)$ .

Necessarily,  $G \setminus R$  has at least two connected components. Let  $G_1, \dots, G_\ell$  be its connected components and thus  $\ell \geq 2$ . We choose  $T$  such that  $s$  is minimal. It implies that for each  $i \in \{1, \dots, s\}$ ,  $r_i$  has neighbors in all the components  $G_1, \dots, G_\ell$ . Indeed, if it is not the case, by permuting the elements of  $R$  (this is possible by the above remark), we can assume without loss of generality that  $r_s$  does not have a neighbor in  $G_\ell$ . Therefore, the set of edges  $e(r_s, G_\ell) = \{r_s x : x \in V(G_\ell)\}$  of  $\text{clos}(T)$  are not used by  $G$ . Then let  $T'$  be the tree obtained from  $T$  by moving the whole component  $G_\ell$  one level up in such a way that the root of the subtree corresponding to  $G_\ell$  is now the son of  $r_{s-1}$  (instead of  $r_s$  previously). Note that  $\text{clos}(T')$  is isomorphic to  $\text{clos}(T) \setminus e(r_s, G_\ell)$  and thus  $G$  is a subgraph of  $\text{clos}(T')$ . This new tree  $T'$  has two vertices at height  $s$ , contradicting the minimality of  $s$ .

Any connected component  $G_j$  has tree-depth at most  $k' = k - s < k$ . By induction, for each  $j \in \{1, \dots, \ell\}$ , there exists a lid-coloring  $c_j$  of  $G_j$  using colors in  $\{1, \dots, 2k' - 1\}$ . For each  $c_j$ , there is a minimum value  $s_j$  such that every vertex  $r_i$  sees a color in  $\{1, \dots, s_j\}$  in  $G_j$ . We choose a  $(2k' - 1)$ -lid-coloring  $c_j$  of  $G_j$  such that  $s_j$  is minimized. Note that for each color  $a \leq s_j$ , there exists  $r_i \in R$  such that  $r_i$  sees color  $a$  in  $G_j$  but no other color of  $\{1, \dots, s_j\}$ . Otherwise, after permuting colors  $a$  and  $s_j$ , every vertex  $r_i \in R$  would see a color in  $\{1, \dots, s_j - 1\}$ , contradicting the minimality of  $s_j$ . Assume without loss of generality that  $s_1 \geq s_2 \geq \dots \geq s_\ell$ .

We replace in  $c_1$  the colors  $1, 2, \dots, s_1$  by  $1', 2', \dots, s'_1$ . Note that now each vertex  $r_i$  sees a color in  $\{1', \dots, s'_1\}$  (in  $G_1$ ) and a color in  $\{1, \dots, s_2\}$  (in  $G_2$ ). Furthermore, the other vertices of  $G$  (that is the vertices in  $G_1, \dots, G_\ell$ ) do not have this property since  $s_1 \geq s_2$ . Thus at this step every edge  $xy$  with  $x$  in some  $G_j$  is separated.

Now we color each vertex  $r_i$  with color  $i^*$ . Let  $c : V(G) \rightarrow \{1^*, \dots, s^*\} \cup \{1', \dots, s'_1\} \cup \{1, \dots, 2k' - 1\}$  be the current coloring of  $G$ .

Note that now every distinguishable edge  $xy$  in some  $G_j$  is separated. Indeed, either  $xy$  was distinguished in  $G_j$  and it has been separated by  $c_j$ , or  $xy$  is distinguished by some  $r_i$  and it is separated by the color  $i^*$ . Note also that  $c$  is a proper coloring.

It remains to deal with the edges  $r_i r_j$ . For that purpose we will refine some color classes. In the following lemma we show that such refinements do not damage what we have done so far.

**Claim.** *Consider a graph  $G$  and a coloring  $\varphi : V(G) \rightarrow \{1, \dots, k\}$ . Consider any refinement  $\varphi'$  of  $\varphi$ , obtained from  $\varphi$  by recoloring with color  $k + 1$  some vertices colored  $i$ , for some  $i$ . Any edge  $xy$  of  $G$  properly colored (resp. separated) by  $\varphi$  is properly colored (resp. separated) by  $\varphi'$ .*

Indeed if  $\varphi(x) \neq \varphi(y)$  then  $\varphi'(x) \neq \varphi'(y)$ , and if  $i \in \varphi(N[x]) \Delta \varphi(N[y])$  then

$i$  or  $k + 1 \in \varphi'(N[x]) \Delta \varphi'(N[y])$ .

Let us define a relation  $\mathcal{R}$  among vertices in  $R$  by  $r_i \mathcal{R} r_j$  if and only if  $c(N[r_i]) = c(N[r_j])$ . Let  $R_1, \dots, R_{\bar{s}}$  be the equivalence classes of the relation  $\mathcal{R}$  (note that each  $R_i$  forms a clique since every  $r_i$  has distinct colors). We have  $\bar{s} \geq s_1$ . Indeed, by definition of  $s_1$  and the coloring  $c_1$ , for each color  $a \in \{1', \dots, s'_1\}$ , there exists  $r_i \in R$  that sees  $a$  in  $G_1$  but no other color of  $\{1', \dots, s'_1\}$ . This vertex  $r_i$  belongs to some equivalence class  $R_j$  and thus all the vertices of  $R_j$  sees color  $a$  in  $G_1$  but no other color of  $\{1', \dots, s'_1\}$ .

By Corollary 2, there is a vertex set  $S(R_i)$  of size at most  $|R_i| - 1$  which distinguishes all pairs of non-twin vertices in  $R_i$ . We give to the vertices of  $S(R_i)$  new distinct colors. By the previous claim, this last operation does not damage the coloring, and now all the distinguishable edges are separated.

Since for this last operation we need  $s - \bar{s}$  new colors, since we used  $2k' - 1$  colors  $\{1, \dots, 2k' - 1\}$ ,  $s_1$  colors  $\{1', \dots, s'_1\}$  and  $s$  colors  $\{1^*, \dots, s^*\}$ , the total number of colors is  $(s - \bar{s}) + (2k' - 1) + s_1 + s = 2k - 1 + s_1 - \bar{s} \leq 2k - 1$ . This concludes the proof of the theorem.  $\square$

We are now ready to prove Theorem 4:

**Proof of Theorem 4.** Let  $\alpha$  be a low tree-depth coloring of  $G$  with parameter  $p = 3$  and using  $\chi_3^{\text{td}}(G)$  colors. Let  $A = \{\alpha_1, \alpha_2, \alpha_3\}$  be a triplet of three distinct colors and let  $H_A$  be the subgraph of  $G$  induced by the vertices colored by a color of  $A$ . Since  $H_A$  has tree-depth at most 3, by Proposition 3,  $H_A$  admits a lid-coloring  $c_A$  with five colors (says colors 1 to 5). We extend  $c_A$  to the whole graph by giving color 0 to the vertices in  $V(G) \setminus V(H_A)$ .

Let  $A_1, A_2, \dots, A_k$  be the  $k = \binom{\chi_3^{\text{td}}(G)}{3}$  distinct triplets of colors. We now construct a coloring  $c$  of  $G$  giving to each vertex  $x$  of  $G$  the  $k$ -uplet

$$(c_{A_1}(x), c_{A_2}(x), \dots, c_{A_k}(x)).$$

The coloring  $c$  is using  $6^k$  colors. Clearly it is a proper coloring: each pair of adjacent vertices will be in some common graph  $H_A$  and will receive distinct colors in this graph. Let  $x$  and  $y$  be two adjacent vertices with  $N[x] \neq N[y]$ . Let  $w$  be a vertex adjacent to only one vertex among  $x$  and  $y$ . Let  $A = \{\alpha(x), \alpha(y), \alpha(w)\}$ . Vertices  $x$  and  $y$  are not twins in the graph  $H_A$ . Hence  $c_A(N[x]) \neq c_A(N[y])$  and therefore,  $c(N[x]) \neq c(N[y])$ .  $\square$

## 4 Minor closed classes of graphs

Let  $G$  and  $H$  be two graphs.  $H$  is a *minor* of  $G$  if  $H$  can be obtained from  $G$  with successive edge deletions, vertex deletions and edge contractions. A class  $\mathcal{C}$  is *minor closed* if for any graph  $G$  of  $\mathcal{C}$ , for any minor  $H$  of  $G$ , we have  $H \in \mathcal{C}$ . The class  $\mathcal{C}$  is *proper* if it is not the class of all graphs. Let  $H$  be a graph. A  *$H$ -minor free graph* is a graph that does not have  $H$  as a minor. We denote by  $\mathcal{X}_n$  the  $K_n$ -minor-free class of graphs. It is clear that any proper minor closed class of graphs is included in the class  $\mathcal{X}_n$  for some  $n$ . It is folklore that any proper minor closed class of graphs  $\mathcal{C}$  has a bounded chromatic number  $\chi(\mathcal{C})$ .

The class of graphs of bounded expansion includes all the proper minor closed classes of graphs. Thus, by Corollary 6, proper minor closed classes have bounded lid-chromatic number. In this section, we focus on these latter classes and give an alternative upper bound on the lid-chromatic number. This gives us an explicit upper bound for the lid-chromatic number of planar graphs.

Consider any proper minor closed class of graphs  $\mathcal{C}$ . Since  $\mathcal{C}$  is proper, there exists  $n$  such that  $\mathcal{C}$  does not contain  $K_n$ , that is  $\mathcal{C} \subseteq \mathcal{K}_n$ . Let  $\mathcal{C}^N$  be the class of graphs defined by  $H \in \mathcal{C}^N$  if and only if there exists  $G \in \mathcal{C}$  and  $v \in G$  such that  $H = G[N(v)]$ . Note that  $\mathcal{C}^N$  is a minor-closed class of graphs. Indeed, given any  $H \in \mathcal{C}^N$ , let  $G \in \mathcal{C}$  and  $v \in V(G)$  such that  $H = G[N(v)]$ . Let  $H'$  be any minor of  $H$ . Since  $\mathcal{C}$  is minor-closed and  $H$  is a subgraph of  $G$ , there exists a minor  $G'$  of  $G$  such that  $H' = G'[N(v)]$ . Therefore,  $H'$  belongs to  $\mathcal{C}^N$ .

We prove the following result on minor-closed classes of graphs:

**Theorem 7.** *Let  $\mathcal{C}$  be a proper minor closed class of graphs and let  $n \geq 3$  be such that  $\mathcal{C} \subseteq \mathcal{K}_n$ . Then*

$$\chi_{lid}(\mathcal{C}) \leq 4 \cdot \chi_{lid}(\mathcal{C}^N) \cdot \chi(\mathcal{C})^{n-3}$$

The class of trees is exactly the class  $\mathcal{K}_3$ . Esperet et al. [6] proved the following result.

**Proposition 8** ([6]).  $\chi_{lid}(\mathcal{K}_3) \leq 4$ .

It is clear that  $\mathcal{K}_3^N$  is the class of stable graphs and therefore,  $\chi_{lid}(\mathcal{K}_3^N) = 1$ . Note that Theorem 7 implies Proposition 8.

Assume that  $\chi_{lid}(\mathcal{K}_{n-1})$  is bounded for some  $n \geq 4$ . It is clear that  $\mathcal{K}_n^N = \mathcal{K}_{n-1}$ . Then, by Theorem 7, we have  $\chi_{lid}(\mathcal{K}_n) \leq 4 \cdot \chi_{lid}(\mathcal{K}_{n-1}) \cdot \chi(\mathcal{K}_n)^{n-3}$ . Since  $\chi_{lid}(\mathcal{K}_{n-1})$  and  $\chi(\mathcal{K}_n)$  are bounded,  $\chi_{lid}(\mathcal{K}_n)$  is bounded.

Esperet et al. [6] also proved the following result.

**Proposition 9** ([6]). *If  $G$  is an outerplanar graph,  $\chi_{lid}(G) \leq 20$ .*

We can then deduce from Theorem 7 and Proposition 9 the following corollary:

**Corollary 10.** *Let  $\mathcal{P}$  be the class of planar graphs. Then  $\chi_{lid}(\mathcal{P}) \leq 1280$ .*

**Proof.** Any graph  $G \in \mathcal{P}$  is  $\{K_{3,3}, K_5\}$ -minor free and thus  $\mathcal{P}$  is a proper minor closed class of graphs. Moreover, the neighborhood of any vertex of  $G \in \mathcal{P}$  is an outerplanar graph. By Proposition 9, we have  $\chi_{lid}(\mathcal{P}^N) \leq 20$ . Furthermore, the Four-Color-Theorem gives  $\chi(\mathcal{P}) = 4$ . By Theorem 7,  $\chi_{lid}(\mathcal{P}) \leq 4 \times 20 \times 4^2 = 1280$ .  $\square$

We finally give the proof of Theorem 7.

**Proof of Theorem 7.** Let  $G \in \mathcal{C}$  and let  $u$  be a vertex of minimum degree. For any  $i$ , define  $V_{u,i}$  as the set of vertices of  $G$  at distance exactly  $i$  from  $u$  and let  $G_{u,i} = G[V_{u,i}]$ . Let  $s$  be the largest distance from a vertex of  $V$  to  $u$ . In other words, there are  $s + 1$  nonempty sets  $V_{u,i}$  (note that  $V_{u,0} = \{u\}$ ).



For any  $i$ , contracting in  $G$  the subgraph  $G[V_{u,0} \cup V_{u,1} \cup \dots \cup V_{u,i-1}]$  in a single vertex  $x$  gives a graph  $G' \in \mathcal{C}$  such that  $x$  is exactly adjacent to every vertex of  $G_{u,i}$ . Therefore, for any  $i$ ,  $G_{u,i} \in \mathcal{C}^N$ . Hence,  $\chi_{lid}(G_{u,i}) \leq \chi_{lid}(\mathcal{C}^N)$  for any  $i$ . Moreover,  $\mathcal{C}^N \subseteq \mathcal{K}_{n-1}$ . Indeed, suppose that there exists  $H \in \mathcal{C}^N$  that admits  $K_{n-1}$  as a minor. Therefore there exists  $G \in \mathcal{C}$  such that  $H = G[N(v)]$  for some  $v \in G$ . Taking  $v$  together with its neighborhood would give  $K_n$  as a minor, that contradicts the fact that  $\mathcal{C} \subseteq \mathcal{K}_n$ . Hence, any  $G_{u,i} \in \mathcal{K}_{n-1}$ .

We construct a lid-coloring of  $G$  using  $4 \cdot \chi_{lid}(\mathcal{C}^N) \cdot \chi(\mathcal{C})^{n-3}$  colors. This coloring is constructed with three different colorings of the vertices of  $G$ :  $c_1$  which uses 4 colors,  $c_2$  which uses  $\chi_{lid}(\mathcal{C}^N)$  colors and  $c_3$  which is itself composed of  $n-3$  colorings with  $\chi(\mathcal{C})$  colors. The final color  $c(v)$  of a vertex  $v$  will be the triplet  $(c_1(v), c_2(v), c_3(v))$ . Hence the coloring  $c$  uses at most  $4\chi_{lid}(\mathcal{C}^N)\chi(\mathcal{C})^{n-3}$  colors. The coloring  $c_1$  is used to separate the pairs of vertices that lie in distinct sets  $V_{u,i}$ . The coloring  $c_2$  separates the pairs of vertices that lie in the same set  $V_{u,i}$  and are not twins in  $G_{u,i}$ . Finally, the coloring  $c_3$  separates the pairs of vertices that lie in the same set  $V_{u,i}$ , that are twins in  $G_{u,i}$  but that are not twins in  $G$ .

The coloring  $c_1$  is simply defined by  $c_1(v) \equiv i \pmod{4}$  if  $v \in V_{u,i}$ .

To define  $c_2$ , we define for each  $i$ ,  $0 \leq i \leq s$ , a lid-coloring  $c_2^i$  of  $G_{u,i}$  using colors 1 to  $\chi_{lid}(\mathcal{C}^N)$ . Then  $c_2$  is defined by  $c_2(v) = c_2^i(v)$  if  $v \in V_{u,i}$ .

We now define the coloring  $c_3$ . Let  $V_{u,i}^{id}$  be the set of vertices of  $V_{u,i}$  that have a twin in  $G_{u,i}$ :

$$V_{u,i}^{id} = \{v \in V_{u,i} \mid \exists w \in V_{u,i}, N_{G_{u,i}}[v] = N_{G_{u,i}}[w]\}.$$

Let  $G_{u,i}^{id} = G_{u,i}[V_{u,i}^{id}]$ . Since the relation ‘‘be twin’’ is transitive (i.e. if  $u$  and  $v$  are twins, and  $v$  and  $w$  are twins, then  $u$  and  $w$  are twins), then  $G_{u,i}^{id}$  is clearly a union of cliques. In addition, since  $G_{u,i} \in \mathcal{K}_{n-1}$ , the connected components of  $G_{u,i}^{id}$  are cliques of size at most  $n-2$ .

Let  $C$  be a clique of  $G_{u,i}^{id}$ . By Corollary 2, there exists a subset  $S(C) \subseteq V(G)$  of at most  $n-3$  vertices that distinguishes all the pairs of non-twin vertices of  $C$ . Note that by definition of  $C$ ,  $S(C) \cap V_{u,i} = \emptyset$ , and thus  $S(C) \subseteq V_{u,i-1} \cup V_{u,i+1}$ .

Let  $\mathcal{S} = \{(v, C) \mid v \in S(C) \text{ and } C \text{ is a clique in a graph } G_{u,i}^{id}\}$ . We partition  $\mathcal{S}$  in  $s \times (n-3)$  sets  $S_i^k$ ,  $1 \leq i \leq s$ ,  $1 \leq k \leq n-3$ , such that:

- if  $(v, C) \in S_i^k$  for some  $k$ , then  $v \in V_{u,i}$ ;
- if  $(v, C)$  and  $(w, C')$  are two elements of  $S_i^k$ , then  $C \neq C'$ .

This partition can be done because each set  $S(C)$  has size at most  $n-3$ .

For each  $S_i^k = \{(x_1, C_1), (x_2, C_2), \dots, (x_t, C_t)\}$ , we define a graph  $H_i^k$  as follows. We start from the graph induced by  $V_{u,i} \cup V(C_1) \cup V(C_2) \cup \dots \cup V(C_t)$ . Then, for each  $(x_j, C_j)$  in  $S_i^k$ , we contract  $C_j$  in a single vertex  $y_j$  and finally, we contract the edge  $x_j y_j$  on the vertex  $x_j$ . Note that  $V_{u,i}$  is the vertex set of  $H_i^k$ . Note also that  $H_i^k \in \mathcal{C}$  since it is obtained from a subgraph of  $G$  by successive edge-contractions. Therefore,  $\chi(H_i^k) \leq \chi(\mathcal{C})$ .

We now define a proper coloring  $c_3^{i,k}$  of  $H_i^k$  with colors 1 to  $\chi(\mathcal{C})$ . Let  $c_3^k$  be the coloring of vertices of  $G$  defined by  $c_3^k(v) = c_3^{i,k}(v)$  if  $v \in V_{u,i}$ . Finally,  $c_3$  is defined by  $c_3(v) = (c_3^1(v), \dots, c_3^{n-3}(v))$ , and the final color of  $v$  is  $c(v) = (c_1(v), c_2(v), c_3(v))$ .

We now prove that  $c$  is a lid-coloring of  $G$ . First,  $c$  is a proper coloring. Indeed, two adjacent vertices that are not in the same set  $V_{u,i}$  lie in consecutive sets  $V_{u,i}$  and  $V_{u,i+1}$  and thus have different colors in  $c_1$ , and two adjacent vertices in the same set  $V_{u,i}$  have different colors in  $c_2$  (which induces a proper coloring on  $V_{u,i}$ ).

Let now  $x$  and  $y$  be two adjacent vertices with  $N[x] \neq N[y]$ . We will prove that  $c(N[x]) \neq c(N[y])$ . We distinguish three cases.

Case 1:  $x \in V_{u,i}$  and  $y \in V_{u,i+1}$ .

If  $x = u$ , then  $y$  has a neighbor  $v$  in  $V_{u,i+2} = V_{u,2}$ . Indeed,  $u$  is taken with minimum degree, so  $y$  has at least as many neighbors as  $u$  and does not have the same neighborhood than  $u$ , implying that  $y$  has a neighbor in  $V_{u,2}$ . Then  $c_1(v) = 2 \notin c_1(N[u])$  and so  $c(N[x]) \neq c(N[y])$ .

Otherwise,  $x$  has neighbor  $v$  in  $V_{u,i-1}$  and  $c_1(v) \equiv i-1 \pmod{4} \in c_1(N[x])$ . On the other hand, all the neighbors of  $y$  belong to  $V_{u,i} \cup V_{u,i+1} \cup V_{u,i+2}$  and therefore  $c_1(N[y]) \subseteq \{i, i+1, i+2 \pmod{4}\}$ . Thus,  $c(N[x]) \neq c(N[y])$ .

Case 2:  $x$  and  $y$  belong to  $V_{u,i}$  and they are not twins in  $V_{u,i}$  (i.e.  $N_{V_{u,i}}[x] \neq N_{V_{u,i}}[y]$ ).

By definition of the coloring  $c_2^i$ , there exists a color  $a$  that separates  $x$  and  $y$ , i.e.  $a \in c_2^i(N_{V_{u,i}}[x]) \Delta c_2^i(N_{V_{u,i}}[y])$ . Then we necessarily have  $c(N[x]) \neq c(N[y])$ .

Case 3:  $x$  and  $y$  belong to  $V_{u,i}$  and they are twins in  $V_{u,i}$  (i.e.  $N_{V_{u,i}}[x] = N_{V_{u,i}}[y]$ ).

In this case, vertices  $x$  and  $y$  are in the set  $V_{u,i}^{id}$ . Let  $C$  be the clique of  $G_{u,i}$  containing  $x$  and  $y$ . Let  $v \in S(C)$  that distinguishes  $x$  and  $y$ ; thus,  $v \in V_{u,j}$  for  $j = i-1$  or  $j = i+1$ . Wlog,  $v \in N[x]$  but  $v \notin N[y]$ . Let  $S_j^k$  be the part of  $\mathcal{S}$  that contains  $(v, C)$ . Suppose that there exists a neighbor  $w$  of  $y$  such that  $c(v) = c(w)$ . Then  $w$  lies in  $V_{u,j}$  because of the coloring  $c_1$ . However, in the graph  $H_j^k$ , the vertex  $v$  is adjacent to all the neighbors of  $y$  in  $V_{u,j}$ , and in particular is adjacent to  $w$ ; therefore,  $c_3^{j,k}(v) \neq c_3^{j,k}(w)$ , a contradiction. Therefore, the vertex  $y$  does not have any neighbor that has the same color as  $v$ . Hence,  $c(v) \notin c(N[y])$ , and  $c(N[x]) \neq c(N[y])$ .

□

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