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Locally identifying coloring in bounded expansion classes of graphs *

Daniel Gonçalves[†], Aline Parreau[‡], Alexandre Pinlou^{†§}

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Abstract

A proper vertex coloring of a graph is said to be *locally identifying* if the sets of colors in the closed neighborhood of any two adjacent non-twin vertices are distinct. The lid-chromatic number of a graph is the minimum number of colors used by a locally identifying vertex-coloring. In this paper, we prove that for any graph class of bounded expansion, the lid-chromatic number is bounded. Classes of bounded expansion include minor closed classes of graphs. For these latter classes, we give an alternative proof to show that the lid-chromatic number is bounded. This leads to an explicit upper bound for the lid-chromatic number of planar graphs. This answers in a positive way a question of Esperet *et al.* [L. Esperet, S. Gravier, M. Montassier, P. Ochem and A. Parreau. Locally identifying coloring of graphs. *Electronic Journal of Combinatorics*, **19(2)**, 2012.].

1 Introduction

A vertex-coloring is said to be *locally identifying* if (i) the vertex-coloring is proper (i.e. no adjacent vertices receive the same color), and (ii) for any adjacent vertices u, v , the set of colors assigned to the closed neighborhood of u differs from the set of colors assigned to the closed neighborhood of v whenever these neighborhoods are distinct. The *locally identifying chromatic number* of the graph G (or lid-chromatic number, for short), denoted by $\chi_{lid}(G)$, is the smallest number of colors required in any locally identifying coloring of G .

Locally identifying colorings of graphs have been recently introduced by Esperet et al. [6] and later studied by Foucaud et al. [7]. They are related to identifying codes [8, 9], distinguishing colorings [1, 3, 5] and locating-colorings [4]. For

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example, upper bounds on lid-chromatic number have been obtained for bipartite graphs, k -trees, outerplanar graphs and bounded degree graphs. An open question asked by Esperet et al. [6] was to know whether χ_{lid} is bounded for the class of planar graphs. In this paper, we answer positively to this question proving more generally that χ_{lid} is bounded for any class of bounded expansion.

In Section 3, we first give a tight bound of χ_{lid} in term of the tree-depth. Then we use the fact that any class of bounded expansion admits a low tree-depth coloring (that is a k -coloring such that each triplet of colors induces a graph of tree-depth 3, for some constant k) to prove that it has bounded lid-chromatic number.

In Section 4, we focus on minor closed classes of graphs which have bounded expansion and give an alternative bound on the lid-chromatic number, which gives an explicit bound for planar graphs.

The next section is devoted to introduce notation and preliminary results.

2 Notation and preliminary results

Let $G = (V, E)$ be a graph. For any vertex u , we denote by $N_G(u)$ its *neighborhood* in G and by $N_G[u]$ its *closed neighborhood* in G (u together with its adjacent vertices). The notion of neighborhood can be extended to sets as follows: for $X \subseteq V$, $N_G[X] = \{w \in V(G) \mid \exists v \in X, w \in N[v]\}$ and $N_G(X) = N_G[X] \setminus X$. When the considered graph is clearly identified, the subscript is dropped.

The *degree* of vertex u is the size of its neighborhood. The *distance* between two vertices u and v is the number of edges in a shortest path between u and v . For $X \subseteq V$, we denote by $G[X]$ the subgraph of G *induced by* X .

We say that two vertices u and v are *twins* if $N[u] = N[v]$ (although they are often called *true twins* in the literature, we call them *twins* for convenience). In particular, u and v are adjacent vertices. Note that if u and v are adjacent but not twins, there exists a vertex w which is adjacent to exactly one vertex among $\{u, v\}$, i.e. $w \in N[u] \Delta N[v]$ (where Δ is the symmetric difference between sets). We say that w *distinguishes* u and v , or simply w *distinguishes* the edge uv . For a subset $X \subseteq V$, we say that a subset $Y \subseteq V$ *distinguishes* X if for every pair u, v of non-twin vertices of X , there exists a vertex $w \in Y$ that distinguishes the edge uv .

Let $c : V \rightarrow \mathbb{N}$ be a vertex-coloring of G . The coloring c is *proper* if adjacent vertices have distinct colors. We denote by $\chi(G)$ the *chromatic number* of G , i.e. the minimum number of colors in a proper coloring of G . For any $X \subseteq V$, let $c(X)$ be the set of colors that appear on the vertices of X . A *locally identifying coloring* (lid-coloring for short) of G is a proper vertex-coloring c of G such that for any two adjacent vertices u and v that are not twins (i.e. $N[u] \neq N[v]$), we have $c(N[u]) \neq c(N[v])$. A graph G is *k -lid-colorable* if it admits a locally identifying coloring using at most k colors and the minimum number of colors needed for any locally identifying coloring of G is the *locally identifying chromatic number* (lid-chromatic number for short) denoted by $\chi_{lid}(G)$. For a vertex u , we say that u *sees* color a if $a \in c(N[u])$. For two adjacent vertices

u and v , a color that is in the set $c(N[u]) \Delta c(N[v])$ separates u and v , or simply separates the edge uv . The notion of chromatic number (resp. lid-chromatic number) can be extended to a class of graphs \mathcal{C} as follows: $\chi(\mathcal{C}) = \sup\{\chi(G), G \in \mathcal{C}\}$ (resp. $\chi_{lid}(\mathcal{C}) = \sup\{\chi_{lid}(G), G \in \mathcal{C}\}$).

The following theorem is due to Bondy [2]:

Theorem 1 (Bondy's theorem [2]). *Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be a collection of n distinct subsets of a finite set X . There exists a subset X' of X of size at most $n - 1$ such that the sets $A_i \cap X'$ are all distinct.*

Corollary 2. *Let C be a n -clique subgraph of G . There exists a vertex subset $S(C) \subseteq V(G)$ of size at most $n - 1$ that distinguishes all the pair of non-twin vertices of C .*

Proof. Let C be a n -clique subgraph of G induced by the vertex set $V(C) = \{v_1, v_2, \dots, v_n\}$. Let $\mathcal{A} = \{N[v_i] \mid v_i \in V(C)\}$ be a collection of distinct subsets of the finite set $X = \bigcup_{1 \leq i \leq n} N[v_i]$. Note that some v_i 's might be twins in G (i.e. $N[v_i] = N[v_j]$ for some $v_i, v_j \in V(C)$) and therefore $|\mathcal{A}|$ could be smaller than n . By Bondy Theorem, there exists $S(C) \subseteq X$ of size at most $|\mathcal{A}| - 1 \leq n - 1$ such that for any distinct elements A_1, A_2 of \mathcal{A} , we have $A_1 \cap S(C) \neq A_2 \cap S(C)$.

Let us prove that $S(C)$ is a set of vertices that distinguish all the pairs of non-twin vertices of C . For a pair of non-twin vertices v_i, v_j of C , we have $N[v_i] \neq N[v_j]$. By definition of $S(C)$, we have $N[v_i] \cap S(C) \neq N[v_j] \cap S(C)$, then there exists $w \in S(C)$ that belongs to $N[v_i] \Delta N[v_j]$. Therefore, w distinguishes the edge $v_i v_j$. \square

3 Bounded expansion classes of graphs

A rooted tree is a tree with a special vertex, called the *root*. The *height* of a vertex x in a rooted tree is the number of vertices on a path from the root to x (hence, the height of the root is 1). The *height* of a rooted tree T is the maximum height of the vertices of T . If x and y are two vertices of T , x is an *ancestor* of y in T if x belongs to the path between y and the root. The *closure* $\text{clos}(T)$ of a rooted tree T is the graph with vertex set $V(T)$ and edge set $\{xy \mid x \text{ is an ancestor of } y \text{ in } T, x \neq y\}$. The *tree-depth* $\text{td}(G)$ of a connected graph G is the minimum height of a rooted tree T such that G is a subgraph of $\text{clos}(T)$. If G is not connected, the tree-depth of G is the maximum tree-depth of its connected components.

Let p be a fixed integer. A *low tree-depth coloring* of a graph G (relatively to p) is a coloring of the vertices of G such that the union of any $i \leq p$ color classes induces a graph of tree-depth at most i . Let $\chi_p^{\text{td}}(G)$ be the minimum number of colors required in such a coloring. Note that as tree-depth one graphs and tree-depth two graphs are respectively the stables and star forests, χ_1^{td} and χ_2^{td} respectively correspond to the usual chromatic number and the star chromatic number.

In the following of this section, we first give a tight bound on the lid-chromatic number in terms of tree-depth.

Proposition 3. *For any graph G , $\chi_{lid}(G) \leq 2td(G) - 1$ and this is tight.*

Using this bound, we then bound the lid-chromatic number in terms of χ_3^{td} .

Theorem 4. *For any graph G ,*

$$\chi_{lid}(G) \leq 6^{\binom{td(G)}{3}}.$$

Classes of graphs of bounded expansion have been introduced by Nešetřil and Ossona de Mendez [10]. These classes contain minor closed classes of graphs and any class of graphs defined by an excluded topological minor. Actually, these classes of graphs are closely related to low tree-depth colorings:

Theorem 5 (Theorem 7.1 [10]). *A class of graphs \mathcal{C} has bounded expansion if and only if $\chi_p^{td}(\mathcal{C})$ is bounded for any p .*

We therefore deduce the following corollary from Theorems 4 and 5:

Corollary 6. *For any class \mathcal{C} of bounded expansion, $\chi_{lid}(\mathcal{C})$ is bounded.*

It is in particular true for a class of bounded tree-width. A consequence is that χ_{lid} is bounded for chordal graphs by a function of the clique number (which is equals to the tree-width plus 1 for a chordal graph). It is conjectured by Esperet *et al.* [6] that $\chi_{lid}(G) \leq 2\omega(G)$ if G is chordal.

We now prove Proposition 3.

Proof of Proposition 3. Let us first prove that the bound is tight. Consider the graph H_n obtained from a complete graph, with vertex set $\{a_1, \dots, a_n\}$, by adding a pendant vertex b_i to every a_i but one, say for $1 \leq i < n$. The tree-depth of this graph is at least n as it contains a n -clique. Indeed, given a rooted tree T , two vertices at the same height are non-adjacent in $\text{clos}(T)$, we thus need at least n levels. Actually the tree-depth of this graph is at most n since the tree T rooted at a_1 , and such that a_i has two sons a_{i+1} and b_i , for $1 \leq i < n$, has height n and is such that $\text{clos}(T)$ contains H_n as a subgraph.

Let us show that in any lid-coloring of H_n all the vertices must have distinct colors, and thus use $2n - 1 = 2td(H_n) - 1$ colors. Indeed, two vertices a_i must have different colors as the coloring is proper. A vertex b_j cannot use the same color as a vertex a_i , as otherwise the vertex a_j would only see the n colors used in the clique, just as a_n . Similarly if two vertices b_i and b_j would use the same color, the vertices a_i and a_j would see the same set of colors.

Let us now focus on the upper bound. We prove the result for a connected graph and by induction on the tree-depth of G , denoted by k . The result is clear for $k = 1$ (the graph is a single vertex).

Let G be a graph of tree-depth $k > 1$ and let T be a rooted tree of height k such that G is a subgraph of $\text{clos}(T)$. If T is a path, the result is clear since there are only k vertices. So assume that T is not a path, and let r be the root

of T . Let s be the smallest height such that there are at least two vertices of height $s + 1$. We name r_i , for $i \in \{1, \dots, s\}$, the unique vertex of height i . Let $R = \{r_1, \dots, r_s\}$. Note that each of the vertices of R is adjacent to all the vertices of $\text{clos}(T)$. Therefore, we can choose the way we label the s vertices in R (i.e. we can choose the height of each of them in T) without changing $\text{clos}(T)$.

Necessarily, $G \setminus R$ has at least two connected components. Let G_1, \dots, G_ℓ be its connected components and thus $\ell \geq 2$. We choose T such that s is minimal. It implies that for each $i \in \{1, \dots, s\}$, r_i has neighbors in all the components G_1, \dots, G_ℓ . Indeed, if it is not the case, by permuting the elements of R (this is possible by the above remark), we can assume without loss of generality that r_s does not have a neighbor in G_ℓ . Therefore, the set of edges $e(r_s, G_\ell) = \{r_s x : x \in V(G_\ell)\}$ of $\text{clos}(T)$ are not used by G . Then let T' be the tree obtained from T by moving the whole component G_ℓ one level up in such a way that the root of the subtree corresponding to G_ℓ is now the son of r_{s-1} (instead of r_s previously). Note that $\text{clos}(T')$ is isomorphic to $\text{clos}(T) \setminus e(r_s, G_\ell)$ and thus G is a subgraph of $\text{clos}(T')$. This new tree T' has two vertices at height s , contradicting the minimality of s .

Any connected component G_j has tree-depth at most $k' = k - s < k$. By induction, for each $j \in \{1, \dots, \ell\}$, there exists a lid-coloring c_j of G_j using colors in $\{1, \dots, 2k' - 1\}$. For each c_j , there is a minimum value s_j such that every vertex r_i sees a color in $\{1, \dots, s_j\}$ in G_j . We choose a $(2k' - 1)$ -lid-coloring c_j of G_j such that s_j is minimized. Note that for each color $a \leq s_j$, there exists $r_i \in R$ such that r_i sees color a in G_j but no other color of $\{1, \dots, s_j\}$. Otherwise, after permuting colors a and s_j , every vertex $r_i \in R$ would see a color in $\{1, \dots, s_j - 1\}$, contradicting the minimality of s_j . Assume without loss of generality that $s_1 \geq s_2 \geq \dots \geq s_\ell$.

We replace in c_1 the colors $1, 2, \dots, s_1$ by $1', 2', \dots, s'_1$. Note that now each vertex r_i sees a color in $\{1', \dots, s'_1\}$ (in G_1) and a color in $\{1, \dots, s_2\}$ (in G_2). Furthermore, the other vertices of G (that is the vertices in G_1, \dots, G_ℓ) do not have this property since $s_1 \geq s_2$. Thus at this step every edge xy with x in some G_j is separated.

Now we color each vertex r_i with color i^* . Let $c : V(G) \rightarrow \{1^*, \dots, s^*\} \cup \{1', \dots, s'_1\} \cup \{1, \dots, 2k' - 1\}$ be the current coloring of G .

Note that now every distinguishable edge xy in some G_j is separated. Indeed, either xy was distinguished in G_j and it has been separated by c_j , or xy is distinguished by some r_i and it is separated by the color i^* . Note also that c is a proper coloring.

It remains to deal with the edges $r_i r_j$. For that purpose we will refine some color classes. In the following lemma we show that such refinements do not damage what we have done so far.

Claim. *Consider a graph G and a coloring $\varphi : V(G) \rightarrow \{1, \dots, k\}$. Consider any refinement φ' of φ , obtained from φ by recoloring with color $k + 1$ some vertices colored i , for some i . Any edge xy of G properly colored (resp. separated) by φ is properly colored (resp. separated) by φ' .*

Indeed if $\varphi(x) \neq \varphi(y)$ then $\varphi'(x) \neq \varphi'(y)$, and if $i \in \varphi(N[x]) \Delta \varphi(N[y])$ then

i or $k + 1 \in \varphi'(N[x]) \Delta \varphi'(N[y])$.

Let us define a relation \mathcal{R} among vertices in R by $r_i \mathcal{R} r_j$ if and only if $c(N[r_i]) = c(N[r_j])$. Let $R_1, \dots, R_{\bar{s}}$ be the equivalence classes of the relation \mathcal{R} (note that each R_i forms a clique since every r_i has distinct colors). We have $\bar{s} \geq s_1$. Indeed, by definition of s_1 and the coloring c_1 , for each color $a \in \{1', \dots, s'_1\}$, there exists $r_i \in R$ that sees a in G_1 but no other color of $\{1', \dots, s'_1\}$. This vertex r_i belongs to some equivalence class R_j and thus all the vertices of R_j sees color a in G_1 but no other color of $\{1', \dots, s'_1\}$.

By Corollary 2, there is a vertex set $S(R_i)$ of size at most $|R_i| - 1$ which distinguishes all pairs of non-twin vertices in R_i . We give to the vertices of $S(R_i)$ new distinct colors. By the previous claim, this last operation does not damage the coloring, and now all the distinguishable edges are separated.

Since for this last operation we need $s - \bar{s}$ new colors, since we used $2k' - 1$ colors $\{1, \dots, 2k' - 1\}$, s_1 colors $\{1', \dots, s'_1\}$ and s colors $\{1^*, \dots, s^*\}$, the total number of colors is $(s - \bar{s}) + (2k' - 1) + s_1 + s = 2k - 1 + s_1 - \bar{s} \leq 2k - 1$. This concludes the proof of the theorem. \square

We are now ready to prove Theorem 4:

Proof of Theorem 4. Let α be a low tree-depth coloring of G with parameter $p = 3$ and using $\chi_3^{\text{td}}(G)$ colors. Let $A = \{\alpha_1, \alpha_2, \alpha_3\}$ be a triplet of three distinct colors and let H_A be the subgraph of G induced by the vertices colored by a color of A . Since H_A has tree-depth at most 3, by Proposition 3, H_A admits a lid-coloring c_A with five colors (says colors 1 to 5). We extend c_A to the whole graph by giving color 0 to the vertices in $V(G) \setminus V(H_A)$.

Let A_1, A_2, \dots, A_k be the $k = \binom{\chi_3^{\text{td}}(G)}{3}$ distinct triplets of colors. We now construct a coloring c of G giving to each vertex x of G the k -uplet

$$(c_{A_1}(x), c_{A_2}(x), \dots, c_{A_k}(x)).$$

The coloring c is using 6^k colors. Clearly it is a proper coloring: each pair of adjacent vertices will be in some common graph H_A and will receive distinct colors in this graph. Let x and y be two adjacent vertices with $N[x] \neq N[y]$. Let w be a vertex adjacent to only one vertex among x and y . Let $A = \{\alpha(x), \alpha(y), \alpha(w)\}$. Vertices x and y are not twins in the graph H_A . Hence $c_A(N[x]) \neq c_A(N[y])$ and therefore, $c(N[x]) \neq c(N[y])$. \square

4 Minor closed classes of graphs

Let G and H be two graphs. H is a *minor* of G if H can be obtained from G with successive edge deletions, vertex deletions and edge contractions. A class \mathcal{C} is *minor closed* if for any graph G of \mathcal{C} , for any minor H of G , we have $H \in \mathcal{C}$. The class \mathcal{C} is *proper* if it is not the class of all graphs. Let H be a graph. A *H -minor free graph* is a graph that does not have H as a minor. We denote by \mathcal{X}_n the K_n -minor-free class of graphs. It is clear that any proper minor closed class of graphs is included in the class \mathcal{X}_n for some n . It is folklore that any proper minor closed class of graphs \mathcal{C} has a bounded chromatic number $\chi(\mathcal{C})$.

The class of graphs of bounded expansion includes all the proper minor closed classes of graphs. Thus, by Corollary 6, proper minor closed classes have bounded lid-chromatic number. In this section, we focus on these latter classes and give an alternative upper bound on the lid-chromatic number. This gives us an explicit upper bound for the lid-chromatic number of planar graphs.

Consider any proper minor closed class of graphs \mathcal{C} . Since \mathcal{C} is proper, there exists n such that \mathcal{C} does not contain K_n , that is $\mathcal{C} \subseteq \mathcal{K}_n$. Let \mathcal{C}^N be the class of graphs defined by $H \in \mathcal{C}^N$ if and only if there exists $G \in \mathcal{C}$ and $v \in G$ such that $H = G[N(v)]$. Note that \mathcal{C}^N is a minor-closed class of graphs. Indeed, given any $H \in \mathcal{C}^N$, let $G \in \mathcal{C}$ and $v \in V(G)$ such that $H = G[N(v)]$. Let H' be any minor of H . Since \mathcal{C} is minor-closed and H is a subgraph of G , there exists a minor G' of G such that $H' = G'[N(v)]$. Therefore, H' belongs to \mathcal{C}^N .

We prove the following result on minor-closed classes of graphs:

Theorem 7. *Let \mathcal{C} be a proper minor closed class of graphs and let $n \geq 3$ be such that $\mathcal{C} \subseteq \mathcal{K}_n$. Then*

$$\chi_{lid}(\mathcal{C}) \leq 4 \cdot \chi_{lid}(\mathcal{C}^N) \cdot \chi(\mathcal{C})^{n-3}$$

The class of trees is exactly the class \mathcal{K}_3 . Esperet et al. [6] proved the following result.

Proposition 8 ([6]). $\chi_{lid}(\mathcal{K}_3) \leq 4$.

It is clear that \mathcal{K}_3^N is the class of stable graphs and therefore, $\chi_{lid}(\mathcal{K}_3^N) = 1$. Note that Theorem 7 implies Proposition 8.

Assume that $\chi_{lid}(\mathcal{K}_{n-1})$ is bounded for some $n \geq 4$. It is clear that $\mathcal{K}_n^N = \mathcal{K}_{n-1}$. Then, by Theorem 7, we have $\chi_{lid}(\mathcal{K}_n) \leq 4 \cdot \chi_{lid}(\mathcal{K}_{n-1}) \cdot \chi(\mathcal{K}_n)^{n-3}$. Since $\chi_{lid}(\mathcal{K}_{n-1})$ and $\chi(\mathcal{K}_n)$ are bounded, $\chi_{lid}(\mathcal{K}_n)$ is bounded.

Esperet et al. [6] also proved the following result.

Proposition 9 ([6]). *If G is an outerplanar graph, $\chi_{lid}(G) \leq 20$.*

We can then deduce from Theorem 7 and Proposition 9 the following corollary:

Corollary 10. *Let \mathcal{P} be the class of planar graphs. Then $\chi_{lid}(\mathcal{P}) \leq 1280$.*

Proof. Any graph $G \in \mathcal{P}$ is $\{K_{3,3}, K_5\}$ -minor free and thus \mathcal{P} is a proper minor closed class of graphs. Moreover, the neighborhood of any vertex of $G \in \mathcal{P}$ is an outerplanar graph. By Proposition 9, we have $\chi_{lid}(\mathcal{P}^N) \leq 20$. Furthermore, the Four-Color-Theorem gives $\chi(\mathcal{P}) = 4$. By Theorem 7, $\chi_{lid}(\mathcal{P}) \leq 4 \times 20 \times 4^2 = 1280$. \square

We finally give the proof of Theorem 7.

Proof of Theorem 7. Let $G \in \mathcal{C}$ and let u be a vertex of minimum degree. For any i , define $V_{u,i}$ as the set of vertices of G at distance exactly i from u and let $G_{u,i} = G[V_{u,i}]$. Let s be the largest distance from a vertex of V to u . In other words, there are $s + 1$ nonempty sets $V_{u,i}$ (note that $V_{u,0} = \{u\}$).

For any i , contracting in G the subgraph $G[V_{u,0} \cup V_{u,1} \cup \dots \cup V_{u,i-1}]$ in a single vertex x gives a graph $G' \in \mathcal{C}$ such that x is exactly adjacent to every vertex of $G_{u,i}$. Therefore, for any i , $G_{u,i} \in \mathcal{C}^N$. Hence, $\chi_{lid}(G_{u,i}) \leq \chi_{lid}(\mathcal{C}^N)$ for any i . Moreover, $\mathcal{C}^N \subseteq \mathcal{K}_{n-1}$. Indeed, suppose that there exists $H \in \mathcal{C}^N$ that admits K_{n-1} as a minor. Therefore there exists $G \in \mathcal{C}$ such that $H = G[N(v)]$ for some $v \in G$. Taking v together with its neighborhood would give K_n as a minor, that contradicts the fact that $\mathcal{C} \subseteq \mathcal{K}_n$. Hence, any $G_{u,i} \in \mathcal{K}_{n-1}$.

We construct a lid-coloring of G using $4 \cdot \chi_{lid}(\mathcal{C}^N) \cdot \chi(\mathcal{C})^{n-3}$ colors. This coloring is constructed with three different colorings of the vertices of G : c_1 which uses 4 colors, c_2 which uses $\chi_{lid}(\mathcal{C}^N)$ colors and c_3 which is itself composed of $n-3$ colorings with $\chi(\mathcal{C})$ colors. The final color $c(v)$ of a vertex v will be the triplet $(c_1(v), c_2(v), c_3(v))$. Hence the coloring c uses at most $4\chi_{lid}(\mathcal{C}^N)\chi(\mathcal{C})^{n-3}$ colors. The coloring c_1 is used to separate the pairs of vertices that lie in distinct sets $V_{u,i}$. The coloring c_2 separates the pairs of vertices that lie in the same set $V_{u,i}$ and are not twins in $G_{u,i}$. Finally, the coloring c_3 separates the pairs of vertices that lie in the same set $V_{u,i}$, that are twins in $G_{u,i}$ but that are not twins in G .

The coloring c_1 is simply defined by $c_1(v) \equiv i \pmod{4}$ if $v \in V_{u,i}$.

To define c_2 , we define for each i , $0 \leq i \leq s$, a lid-coloring c_2^i of $G_{u,i}$ using colors 1 to $\chi_{lid}(\mathcal{C}^N)$. Then c_2 is defined by $c_2(v) = c_2^i(v)$ if $v \in V_{u,i}$.

We now define the coloring c_3 . Let $V_{u,i}^{id}$ be the set of vertices of $V_{u,i}$ that have a twin in $G_{u,i}$:

$$V_{u,i}^{id} = \{v \in V_{u,i} \mid \exists w \in V_{u,i}, N_{G_{u,i}}[v] = N_{G_{u,i}}[w]\}.$$

Let $G_{u,i}^{id} = G_{u,i}[V_{u,i}^{id}]$. Since the relation “be twin” is transitive (i.e. if u and v are twins, and v and w are twins, then u and w are twins), then $G_{u,i}^{id}$ is clearly a union of cliques. In addition, since $G_{u,i} \in \mathcal{K}_{n-1}$, the connected components of $G_{u,i}^{id}$ are cliques of size at most $n-2$.

Let C be a clique of $G_{u,i}^{id}$. By Corollary 2, there exists a subset $S(C) \subseteq V(G)$ of at most $n-3$ vertices that distinguishes all the pairs of non-twin vertices of C . Note that by definition of C , $S(C) \cap V_{u,i} = \emptyset$, and thus $S(C) \subseteq V_{u,i-1} \cup V_{u,i+1}$.

Let $\mathcal{S} = \{(v, C) \mid v \in S(C) \text{ and } C \text{ is a clique in a graph } G_{u,i}^{id}\}$. We partition \mathcal{S} in $s \times (n-3)$ sets S_i^k , $1 \leq i \leq s$, $1 \leq k \leq n-3$, such that:

- if $(v, C) \in S_i^k$ for some k , then $v \in V_{u,i}$;
- if (v, C) and (w, C') are two elements of S_i^k , then $C \neq C'$.

This partition can be done because each set $S(C)$ has size at most $n-3$.

For each $S_i^k = \{(x_1, C_1), (x_2, C_2), \dots, (x_t, C_t)\}$, we define a graph H_i^k as follows. We start from the graph induced by $V_{u,i} \cup V(C_1) \cup V(C_2) \cup \dots \cup V(C_t)$. Then, for each (x_j, C_j) in S_i^k , we contract C_j in a single vertex y_j and finally, we contract the edge $x_j y_j$ on the vertex x_j . Note that $V_{u,i}$ is the vertex set of H_i^k . Note also that $H_i^k \in \mathcal{C}$ since it is obtained from a subgraph of G by successive edge-contractations. Therefore, $\chi(H_i^k) \leq \chi(\mathcal{C})$.

We now define a proper coloring $c_3^{i,k}$ of H_i^k with colors 1 to $\chi(\mathcal{C})$. Let c_3^k be the coloring of vertices of G defined by $c_3^k(v) = c_3^{i,k}(v)$ if $v \in V_{u,i}$. Finally, c_3 is defined by $c_3(v) = (c_3^1(v), \dots, c_3^{n-3}(v))$, and the final color of v is $c(v) = (c_1(v), c_2(v), c_3(v))$.

We now prove that c is a lid-coloring of G . First, c is a proper coloring. Indeed, two adjacent vertices that are not in the same set $V_{u,i}$ lie in consecutive sets $V_{u,i}$ and $V_{u,i+1}$ and thus have different colors in c_1 , and two adjacent vertices in the same set $V_{u,i}$ have different colors in c_2 (which induces a proper coloring on $V_{u,i}$).

Let now x and y be two adjacent vertices with $N[x] \neq N[y]$. We will prove that $c(N[x]) \neq c(N[y])$. We distinguish three cases.

Case 1: $x \in V_{u,i}$ and $y \in V_{u,i+1}$.

If $x = u$, then y has a neighbor v in $V_{u,i+2} = V_{u,2}$. Indeed, u is taken with minimum degree, so y has at least as many neighbors as u and does not have the same neighborhood than u , implying that y has a neighbor in $V_{u,2}$. Then $c_1(v) = 2 \notin c_1(N[u])$ and so $c(N[x]) \neq c(N[y])$.

Otherwise, x has neighbor v in $V_{u,i-1}$ and $c_1(v) \equiv i-1 \pmod{4} \in c_1(N[x])$. On the other hand, all the neighbors of y belong to $V_{u,i} \cup V_{u,i+1} \cup V_{u,i+2}$ and therefore $c_1(N[y]) \subseteq \{i, i+1, i+2 \pmod{4}\}$. Thus, $c(N[x]) \neq c(N[y])$.

Case 2: x and y belong to $V_{u,i}$ and they are not twins in $V_{u,i}$ (i.e. $N_{V_{u,i}}[x] \neq N_{V_{u,i}}[y]$).

By definition of the coloring c_2^i , there exists a color a that separates x and y , i.e. $a \in c_2^i(N_{V_{u,i}}[x]) \Delta c_2^i(N_{V_{u,i}}[y])$. Then we necessarily have $c(N[x]) \neq c(N[y])$.

Case 3: x and y belong to $V_{u,i}$ and they are twins in $V_{u,i}$ (i.e. $N_{V_{u,i}}[x] = N_{V_{u,i}}[y]$).

In this case, vertices x and y are in the set $V_{u,i}^{id}$. Let C be the clique of $G_{u,i}$ containing x and y . Let $v \in S(C)$ that distinguishes x and y ; thus, $v \in V_{u,j}$ for $j = i-1$ or $j = i+1$. Wlog, $v \in N[x]$ but $v \notin N[y]$. Let S_j^k be the part of \mathcal{S} that contains (v, C) . Suppose that there exists a neighbor w of y such that $c(v) = c(w)$. Then w lies in $V_{u,j}$ because of the coloring c_1 . However, in the graph H_j^k , the vertex v is adjacent to all the neighbors of y in $V_{u,j}$, and in particular is adjacent to w ; therefore, $c_3^{j,k}(v) \neq c_3^{j,k}(w)$, a contradiction. Therefore, the vertex y does not have any neighbor that has the same color as v . Hence, $c(v) \notin c(N[y])$, and $c(N[x]) \neq c(N[y])$.

□

References

- [1] P.N. Balister, O.M. Riordan, and R.H. Schelp. Vertex-distinguishing edge-colorings of graphs. *J. Graph Theory*, **42**:95–109, 2003.
- [2] J. A. Bondy. Induced subsets. *J. Combin. Theory Ser. B*, **12**(2):201–202, 1972.
- [3] A.C. Burriss and R.H. Schelp. Vertex-distinguishing proper edge-colorings. *J. Graph Theory*, **26**:73–83, 1997.
- [4] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater and P. Zhang. The locating-chromatic number of a graph. *Bull. Inst. Combin. Appl.*, **36**:89–101, 2002.
- [5] J. Cerný, M. Hornák, and R. Soták. Observability of a graph. *Math. Slovaca*, **46**(1):21–31, 1996.
- [6] L. Esperet, S. Gravier, M. Montassier, P. Ochem and A. Parreau. Locally identifying coloring of graphs. *Electron. J. Combin.*, **19**(2), 2012.
- [7] F. Foucaud, I. Honkala, T. Laihonen, A. Parreau and G. Perarnau. Locally identifying colouring of graphs with given maximum degree. *Discrete Math.*, **312**(10), 2012.
- [8] M. G. Karpovsky, K. Chakrabarty, and L. B. Levitin. On a new class of codes for identifying vertices in graphs. *IEEE Trans. Inform. Theory*, 44:599–611, 1998.
- [9] A. Lobstein. Identifying and locating-dominating codes in graphs, a bibliography. Published electronically at <http://perso.enst.fr/~lobstein/debutBIBidetlocdom.pdf>.
- [10] J. Nešetřil and P. Ossona de Mendez. Grad and classes with bounded expansion I. Decompositions. *European J. Combin.*, **29**:760-776, 2008.