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Abstract

A proper vertex coloring of a graph is said to be *locally identifying* if the sets of colors in the closed neighborhood of any two adjacent non-twin vertices are distinct. The lid-chromatic number of a graph is the minimum number of colors used by a locally identifying vertex-coloring. In this paper, we prove that for any graph class of bounded expansion, the lid-chromatic number is bounded. Classes of bounded expansion include minor closed classes of graphs. For these latter classes, we give an alternative proof to show that the lid-chromatic number is bounded. This leads to an explicit upper bound for the lid-chromatic number of planar graphs. This answers in a positive way a question of Esperet *et al.* [L. Esperet, S. Gravier, M. Montassier, P. Ochem and A. Parreau. Locally identifying coloring of graphs. *Electronic Journal of Combinatorics*, **19(2)**, 2012.].

1 Introduction

A vertex-coloring is said to be *locally identifying* if (i) the vertex-coloring is proper (i.e. no adjacent vertices receive the same color), and (ii) for any adjacent vertices u, v, the set of colors assigned to the closed neighborhood of u differs from the set of colors assigned to the closed neighborhood of v whenever these neighborhoods are distinct. The *locally identifying chromatic number* of the graph G (or lid-chromatic number, for short), denoted by $\chi_{lid}(G)$, is the smallest number of colors required in any locally identifying coloring of G.

Locally identifying colorings of graphs have been recently introduced by Esperet et al. [6] and later studied by Foucaud et al. [7]. They are related to identifying codes [8, 9], distinguishing colorings [1, 3, 5] and locating-colorings [4]. For

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example, upper bounds on lid-chromatic number have been obtained for bipartite graphs, k-trees, outerplanar graphs and bounded degree graphs. An open question asked by Esperet et al. [6] was to know whether χ_{lid} is bounded for the class of planar graphs. In this paper, we answer positively to this question proving more generally that χ_{lid} is bounded for any class of bounded expansion.

In Section 3, we first give a tight bound of χ_{lid} in term of the tree-depth. Then we use the fact that any class of bounded expansion admits a low tree-depth coloring (that is a k-coloring such that each triplet of colors induces a graph of tree-depth 3, for some constant k) to prove that it has bounded lid-chromatic number.

In Section 4, we focus on minor closed classes of graphs which have bounded expansion and give an alternative bound on the lid-chromatic number, which gives an explicit bound for planar graphs.

The next section is devoted to introduce notation and preliminary results.

2 Notation and preliminary results

Let $G = (V, E)$ be a graph. For any vertex u, we denote by $N_G(u)$ its *neighborhood* in G and by $N_G[u]$ its *closed neighborhood* in G (u together with its adjacent vertices). The notion of neighborhood can be extended to sets as follows: for $X \subseteq V$, $N_G[X] = \{w \in V(G) \mid \exists v \in X, w \in N[v]\}$ and $N_G(X) = N_G[X] \setminus X$. When the considered graph is clearly identified, the subscript is dropped.

The *degree* of vertex u is the size of its neighborhood. The *distance* between two vertices u and v is the number of edges in a shortest path between u and v. For $X \subseteq V$, we denote by $G[X]$ the subgraph of G *induced by* X.

We say that two vertices u and v are *twins* if $N[u] = N[v]$ (although they are often called *true twins* in the literature, we call them *twins* for convenience). In particular, u and v are adjacent vertices. Note that if u and v are adjacent but not twins, there exists a vertex w which is adjacent to exactly one vertex among $\{u, v\}$, i.e. $w \in N[u] \,\Delta\, N[v]$ (where Δ is the symmetric difference between sets). We say that w *distinguishes* u and v, or simply w *distinguishes* the edge uv. For a subset $X \subseteq V$, we say that a subset $Y \subseteq V$ *distinguishes* X if for every pair u, v of non-twin vertices of X, there exists a vertex $w \in Y$ that distinguishes the edge uv.

Let $c : V \to \mathbb{N}$ be a vertex-coloring of G. The coloring c is *proper* if adjacent vertices have distinct colors. We denote by $\chi(G)$ the *chromatic number of* G, i.e. the minimum number of colors in a proper coloring of G. For any $X \subseteq V$, let $c(X)$ be the set of colors that appear on the vertices of X. A *locally identifying coloring* (lid-coloring for short) of G is a proper vertex-coloring c of G such that for any two adjacent vertices u and v that are not twins (i.e. $N[u] \neq N[v]$), we have $c(N[u]) \neq c(N[v])$. A graph G is *k-lid-colorable* if it admits a locally identifying coloring using at most k colors and the minimum number of colors needed for any locally identifying coloring of G is the *locally identifying chromatic number* (lid-chromatic number for short) denoted by $\chi_{lid}(G)$. For a vertex u, we say that u *sees* color a if $a \in c(N[u])$. For two adjacent vertices

u and v, a color that is in the set $c(N[u]) \,\Delta\, c(N[v])$ *separates* u and v, or simply *separates* the edge uv. The notion of chromatic number (resp. lid-chromatic number) can be extended to a class of graphs $\mathscr{C}$ as follows: $\chi(\mathscr{C}) = \sup\{\chi(G), G \in \mathscr{C}\}$ (resp. $\chi_{lid}(\mathscr{C}) = \sup\{\chi_{lid}(G), G \in \mathscr{C}\}$).

The following theorem is due to Bondy [2]:

Theorem 1 (Bondy's theorem [2])**.** *Let $\mathcal{A} = \{A_1, ..., A_n\}$ be a collection of n distinct subsets of a finite set X. There exists a subset X' of X of size at most $n-1$ such that the sets $A_i \cap X'$ are all distinct.*

Corollary 2. *Let C be a n-clique subgraph of G. There exists a vertex subset $S(C) \subseteq V(G)$ of size at most $n-1$ that distinguishes all the pair of non-twin vertices of C.*

Proof. Let C be a n-clique subgraph of G induced by the vertex set $V(C) = \{v_1, v_2, \ldots, v_n\}$. Let $\mathcal{A} = \{N[v_i] \mid v_i \in V(C)\}$ be a collection of distinct subsets of the finite set $X = \bigcup_{1 \le i \le n} N[v_i]$. Note that some v_i's might be twins in G (i.e. $N[v_i] = N[v_j]$ for some $v_i, v_j \in V(C)$) and therefore $|\mathcal{A}|$ could be smaller than n. By Bondy Theorem, there exists $S(C) \subseteq X$ of size at most $|\mathcal{A}| - 1 \le n-1$ such that for any distinct elements A_1, A_2 of $\mathcal{A}$, we have $A_1 \cap S(C) \neq A_2 \cap S(C)$.

Let us prove that $S(C)$ is a set of vertices that distinguish all the pairs of non-twin vertices of C. For a pair of non-twin vertices v_i, v_j of C, we have $N[v_i] \neq N[v_j]$. By definition of $S(C)$, we have $N[v_i] \cap S(C) \neq N[v_j] \cap S(C)$, then there exists $w \in S(C)$ that belongs to $N[v_i] \,\Delta\, N[v_j]$. Therefore, w distinguishes the edge $v_i v_j$. $\square$

3 Bounded expansion classes of graphs

A rooted tree is a tree with a special vertex, called the *root*. The *height* of a vertex x in a rooted tree is the number of vertices on a path from the root to x (hence, the height of the root is 1). The *height* of a rooted tree T is the maximum height of the vertices of T. If x and y are two vertices of T, x is an *ancestor* of y in T if x belongs to the path between y and the root. The *closure* $\mathrm{clos}(T)$ of a rooted tree T is the graph with vertex set $V(T)$ and edge set $\{xy \mid x \text{ is an ancestor of } y \text{ in } T, x \neq y\}$. The *tree-depth* $\mathrm{td}(G)$ of a connected graph G is the minimum height of a rooted tree T such that G is a subgraph of $\mathrm{clos}(T)$. If G is not connected, the tree-depth of G is the maximum tree-depth of its connected components.

Let p be a fixed integer. A *low tree-depth coloring* of a graph G (relatively to p) is a coloring of the vertices of G such that the union of any $i \le p$ color classes induces a graph of tree-depth at most i. Let $\chi_p^{\mathrm{td}}(G)$ be the minimum number of colors required in such a coloring. Note that as tree-depth one graphs and tree-depth two graphs are respectively the stables and star forests, χ_1^{td} and χ_2^{td} respectively correspond to the usual chromatic number and the star chromatic number.

In the following of this section, we first give a tight bound on the lid-chromatic number in terms of tree-depth.

Proposition 3. *For any graph G, $\chi_{lid}(G) \leq 2td(G) - 1$ and this is tight.*

Using this bound, we then bound the lid-chromatic number in terms of χ_3^{td}.

Theorem 4. *For any graph G,*

$$\chi_{lid}(G) \leq 6^{\binom{\chi_3^{td}(G)}{3}}.$$

Classes of graphs of bounded expansion have been introduced by Nešetřil and Ossona de Mendez [10]. These classes contain minor closed classes of graphs and any class of graphs defined by an excluded topological minor. Actually, these classes of graphs are closely related to low tree-depth colorings:

Theorem 5 (Theorem 7.1 [10])**.** *A class of graphs $\mathscr{C}$ has bounded expansion if and only if $\chi_p^{td}(\mathscr{C})$ is bounded for any p.*

We therefore deduce the following corollary from Theorems 4 and 5:

Corollary 6. *For any class $\mathscr{C}$ of bounded expansion, $\chi_{lid}(\mathscr{C})$ is bounded.*

It is in particular true for a class of bounded tree-width. A consequence is that χ_{lid} is bounded for chordal graphs by a function of the clique number (which is equals to the tree-width plus 1 for a chordal graph). It is conjectured by Esperet *et al.* [6] that $\chi_{lid}(G) \leq 2\omega(G)$ if G is chordal.

We now prove Proposition 3.

Proof of Proposition 3. Let us first prove that the bound is tight. Consider the graph H_n obtained from a complete graph, with vertex set $\{a_1, \dots, a_n\}$, by adding a pendant vertex b_i to every a_i but one, say for $1 \leq i < n$. The tree-depth of this graph is at least n as it contains a n-clique. Indeed, given a rooted tree T, two vertices at the same height are non-adjacent in $clos(T)$, we thus need at least n levels. Actually the tree-depth of this graph is at most n since the tree T rooted at a_1, and such that a_i has two sons a_{i+1} and b_i, for $1 \leq i < n$, has height n and is such that $clos(T)$ contains H_n as a subgraph.

Let us show that in any lid-coloring of H_n all the vertices must have distinct colors, and thus use $2n - 1 = 2\mathrm{td}(H_n) - 1$ colors. Indeed, two vertices a_i must have different colors as the coloring is proper. A vertex b_j cannot use the same color as a vertex a_i, as otherwise the vertex a_j would only see the n colors used in the clique, just as a_n. Similarly if two vertices b_i and b_j would use the same color, the vertices a_i and a_j would see the same set of colors.

Let us now focus on the upper bound. We prove the result for a connected graph and by induction on the tree-depth of G, denoted by k. The result is clear for $k = 1$ (the graph is a single vertex).

Let G be a graph of tree-depth $k > 1$ and let T be a rooted tree of height k such that G is a subgraph of $\mathrm{clos}(T)$. If T is a path, the result is clear since there are only k vertices. So assume that T is not a path, and let r be the root of T. Let s be the smallest height such that there are at least two vertices of height $s+1$. We name r_i, for $i \in \{1, ..., s\}$, the unique vertex of height i. Let $R = \{r_1, ..., r_s\}$. Note that each of the vertices of R is adjacent to all the vertices of clos(T). Therefore, we can choose the way we label the s vertices in R (i.e. we can choose the height of each of them in T) without changing clos(T).

Necessarily, $G \setminus R$ has at least two connected components. Let $G_1, \ldots, G_\ell$ be its connected components and thus $\ell \geq 2$. We choose T such that s is minimal. It implies that for each $i \in \{1, ..., s\}$, r_i has neighbors in all the components G_1,...,G_ℓ. Indeed, if it is not the case, by permuting the elements of R (this is possible by the above remark), we can assume without loss of generality that r_s does not have a neighbor in G_ℓ. Therefore, the set of edges $e(r_s, G_\ell) = \{r_s x : x \in V(G_\ell)\}$ of clos(T) are not used by G. Then let T' be the tree obtained from T by moving the whole component G_ℓ one level up in such a way that the root of the subtree corresponding to G_ℓ is now the son of r_{s-1} (instead of r_s previously). Note that clos(T') is isomorphic to clos(T) $\setminus e(r_s, G_\ell)$ and thus G is a subgraph of clos(T'). This new tree T' has two vertices at height s, contradicting the minimality of s.

Any connected component G_j has tree-depth at most $k' = k - s < k$.By induction, for each $j \in \{1, ..., \ell\}$, there exists a lid-coloring c_j of G_j using colors in $\{1, \ldots, 2k'-1\}$. For each c_j, there is a minimum value s_j such that every vertex r_i sees a color in $\{1, \ldots, s_j\}$ in G_j. We choose a $(2k'-1)$-lid-coloring c_j of G_j such that s_j is minimized. Note that for each color $a \leq s_j$, there exists $r_i \in R$ such that r_i sees color a in G_j but no other color of $\{1, \ldots, s_j\}$. Otherwise, after permuting colors a and s_j, every vertex $r_i \in R$ would see a color in $\{1, \ldots, s_j - 1\}$, contradicting the minimality of s_j. Assume without loss of generality that $s_1 \geq s_2 \geq \ldots \geq s_\ell$.

We replace in c_1 the colors $1, 2, \ldots, s_1$ by $1', 2', \ldots, s_1'$. Note that now each vertex r_i sees a color in $\{1', \ldots, s_1'\}$ (in G_1) and a color in $\{1, \ldots, s_2\}$ (in G_2). Furthermore, the other vertices of G (that is the vertices in $G_1, \ldots, G_\ell$) do not have this property since $s_1 \geq s_2$. Thus at this step every edge xr_i with x in some G_j is separated.

Now we color each vertex r_i with color i^*. Let $c : V(G) \to \{1^*, \ldots, s^*\} \cup \{1', \ldots, s_1'\} \cup \{1, \ldots, 2k'-1\}$ be the current coloring of G.

Note that now every distinguishable edge xy in some G_j is separated. Indeed, either xy was distinguished in G_j and it has been separated by c_j, or xy is distinguished by some r_i and it is separated by the color i^*. Note also that c is a proper coloring.

It remains to deal with the edges $r_i r_j$. For that purpose we will refine some color classes. In the following lemma we show that such refinements do not damage what we have done so far.

Claim. *Consider a graph G and a coloring $\varphi : V(G) \longrightarrow \{1, \ldots, k\}$. Consider any refinement φ' of φ, obtained from φ by recoloring with color $k+1$ some vertices colored i, for some i. Any edge xy of G properly colored (resp. separated) by φ is properly colored (resp. separated) by φ'.*

Indeed if $\varphi(x) \neq \varphi(y)$ then $\varphi'(x) \neq \varphi'(y)$, and if $i \in \varphi(N[x]) \,\Delta\, \varphi(N[y])$ then

i or $k+1 \in \varphi'(N[x]) \,\Delta\, \varphi'(N[y])$.

Let us define a relation $\mathcal{R}$ among vertices in R by $r_i \mathcal{R} r_j$ if and only if $c(N[r_i]) = c(N[r_j])$. Let R_1,...,$R_{\bar{s}}$ be the equivalence classes of the relation $\mathcal{R}$ (note that each R_i forms a clique since every r_i has distinct colors). We have $\bar{s} \geq s_1$. Indeed, by definition of s_1 and the coloring c_1, for each color $a \in \{1', \ldots, s_1'\}$, there exists $r_i \in R$ that sees a in G_1 but no other color of $\{1', \ldots, s_1'\}$. This vertex r_i belongs to some equivalence class R_j and thus all the vertices of R_j sees color a in G_1 but no other color of $\{1', \ldots, s_1'\}$.

By Corollary 2, there is a vertex set $S(R_i)$ of size at most $|R_i| - 1$ which distinguishes all pairs of non-twin vertices in R_i. We give to the vertices of $S(R_i)$ new distinct colors. By the previous claim, this last operation does not damage the coloring, and now all the distinguishable edges are separated.

Since for this last operation we need $s - \bar{s}$ new colors, since we used $2k' - 1$ colors $\{1, \ldots, 2k' - 1\}$, s_1 colors $\{1', \ldots, s_1'\}$ and s colors $\{1^*, \ldots, s^*\}$, the total number of colors is $(s - \bar{s}) + (2k' - 1) + s_1 + s = 2k - 1 + s_1 - \bar{s} \leq 2k - 1$. This concludes the proof of the theorem. $\square$

We are now ready to prove Theorem 4:

Proof of Theorem 4. Let α be a low tree-depth coloring of G with parameter $p = 3$ and using $\chi_3^{\mathrm{td}}(G)$ colors. Let $A = \{\alpha_1, \alpha_2, \alpha_3\}$ be a triplet of three distinct colors and let H_A be the subgraph of G induced by the vertices colored by a color of A. Since H_A has tree-depth at most 3, by Proposition 3, H_A admits a lid-coloring c_A with five colors (says colors 1 to 5). We extend c_A to the whole graph by giving color 0 to the vertices in $V(G) \setminus V(H_A)$.

Let $A_1, A_2, \ldots, A_k$ be the $k = \binom{\chi_3^{\mathrm{td}}(G)}{3}$ distinct triplets of colors. We now construct a coloring c of G giving to each vertex x of G the k-uplet

$$(c_{A_1}(x), c_{A_2}(x), \ldots, c_{A_k}(x)).$$

The coloring c is using 6^k colors. Clearly it is a proper coloring: each pair of adjacent vertices will be in some common graph H_A and will receive distinct colors in this graph. Let x and y be two adjacent vertices with $N[x] \neq N[y]$. Let w be a vertex adjacent to only one vertex among x and y. Let $A = \{\alpha(x), \alpha(y), \alpha(w)\}$. Vertices x and y are not twins in the graph H_A. Hence $c_A(N[x]) \neq c_A(N[y])$ and therefore, $c(N[x]) \neq c(N[y])$. $\square$

4 Minor closed classes of graphs

Let G and H be two graphs. H is a *minor* of G if H can be obtained from G with successive edge deletions, vertex deletions and edge contractions. A class $\mathscr{C}$ is *minor closed* if for any graph G of $\mathscr{C}$, for any minor H of G, we have $H \in \mathscr{C}$. The class $\mathscr{C}$ is *proper* if it is not the class of all graphs. Let H be a graph. A *H-minor free graph* is a graph that does not have H as a minor. We denote by $\mathscr{K}_n$ the K_n-minor-free class of graphs. It is clear that any proper minor closed class of graphs is included in the class $\mathscr{K}_n$ for some n. It is folklore that any proper minor closed class of graphs $\mathscr{C}$ has a bounded chromatic number $\chi(\mathscr{C})$.

The class of graphs of bounded expansion includes all the proper minor closed classes of graphs. Thus, by Corollary 6, proper minor closed classes have bounded lid-chromatic number. In this section, we focus on these latter classes and give an alternative upper bound on the lid-chromatic number. This gives us an explicit upper bound for the lid-chromatic number of planar graphs.

Consider any proper minor closed class of graphs $\mathscr{C}$. Since $\mathscr{C}$ is proper, there exists n such that $\mathscr{C}$ does not contain K_n, that is $\mathscr{C} \subseteq \mathscr{K}_n$. Let $\mathscr{C}^N$ be the class of graphs defined by $H \in \mathscr{C}^N$ if and only if there exists $G \in \mathscr{C}$ and $v \in G$ such that $H = G[N(v)]$. Note that $\mathscr{C}^N$ is a minor-closed class of graphs. Indeed, given any $H \in \mathscr{C}^N$, let $G \in \mathscr{C}$ and $v \in V(G)$ such that $H = G[N(v)]$. Let H' be any minor of H. Since $\mathscr{C}$ is minor-closed and H is a subgraph of G, there exists a minor G' of G such that $H' = G'[N(v)]$. Therefore, H' belongs to $\mathscr{C}^N$.

We prove the following result on minor-closed classes of graphs:

Theorem 7. *Let $\mathscr{C}$ be a proper minor closed class of graphs and let $n \geq 3$ be such that $\mathscr{C} \subseteq \mathscr{K}_n$. Then*

$$\chi_{lid}(\mathscr{C}) \leq 4 \cdot \chi_{lid}(\mathscr{C}^N) \cdot \chi(\mathscr{C})^{n-3}$$

The class of trees is exactly the class $\mathscr{K}_3$. Esperet et al. [6] proved the following result.

Proposition 8 ([6])**.** $\chi_{lid}(\mathscr{K}_3) \leq 4$.

It is clear that $\mathscr{K}_3^N$ is the class of stable graphs and therefore, $\chi_{lid}(\mathscr{K}_3^N) = 1$. Note that Theorem 7 implies Proposition 8.

Assume that $\chi_{lid}(\mathscr{K}_{n-1})$ is bounded for some $n \geq 4$. It is clear that $\mathscr{K}_n^N = \mathscr{K}_{n-1}$. Then, by Theorem 7, we have $\chi_{lid}(\mathscr{K}_n) \leq 4 \cdot \chi_{lid}(\mathscr{K}_{n-1}) \cdot \chi(\mathscr{K}_n)^{n-3}$. Since $\chi_{lid}(\mathscr{K}_{n-1})$ and $\chi(\mathscr{K}_n)$ are bounded, $\chi_{lid}(\mathscr{K}_n)$ is bounded.

Esperet et al. [6] also proved the following result.

Proposition 9 ([6])**.** *If G is an outerplanar graph, $\chi_{lid}(G) \leq 20$.*

We can then deduce from Theorem 7 and Proposition 9 the following corollary:

Corollary 10. *Let $\mathscr{P}$ be the class of planar graphs. Then $\chi_{lid}(\mathscr{P}) \leq 1280$.*

Proof. Any graph $G \in \mathscr{P}$ is $\{K_{3,3}, K_5\}$-minor free and thus $\mathscr{P}$ is a proper minor closed class of graphs. Moreover, the neighborhood of any vertex of $G \in \mathscr{P}$ is an outerplanar graph. By Proposition 9, we have $\chi_{lid}(\mathscr{P}^N) \leq 20$. Furthermore, the Four-Color-Theorem gives $\chi(\mathscr{P}) = 4$. By Theorem 7, $\chi_{lid}(\mathscr{P}) \leq 4 \times 20 \times 4^2 = 1280$. □

We finally give the proof of Theorem 7.

Proof of Theorem 7. Let $G \in \mathscr{C}$ and let u be a vertex of minimum degree. For any i, define $V_{u,i}$ as the set of vertices of G at distance exactly i from u and let $G_{u,i} = G[V_{u,i}]$. Let s be the largest distance from a vertex of V to u. In other words, there are $s+1$ nonempty sets $V_{u,i}$ (note that $V_{\{u,0\}} = \{u\}$).

For any i, contracting in G the subgraph $G[V_{u,0} \cup V_{u,1} \cup \ldots \cup V_{u,i-1}]$ in a single vertex x gives a graph $G' \in \mathscr{C}$ such that x is exactly adjacent to every vertex of $G_{u,i}$. Therefore, for any i, $G_{u,i} \in \mathscr{C}^N$. Hence, $\chi_{lid}(G_{u,i}) \leq \chi_{lid}(\mathscr{C}^N)$ for any i. Moreover, $\mathscr{C}^N \subseteq \mathscr{K}_{n-1}$. Indeed, suppose that there exists $H \in \mathscr{C}^N$ that admits K_{n-1} as a minor. Therefore there exists $G \in \mathscr{C}$ such that $H = G[N(v)]$ for some $v \in G$. Taking v together with its neighborhood would give K_n as a minor, that contradicts the fact that $\mathscr{C} \subseteq \mathscr{K}_n$. Hence, any $G_{u,i} \in \mathscr{K}_{n-1}$.

We construct a lid-coloring of G using $4 \cdot \chi_{lid}(\mathscr{C}^N) \cdot \chi(\mathscr{C})^{n-3}$ colors. This coloring is constructed with three different colorings of the vertices of G: c_1 which uses 4 colors, c_2 which uses $\chi_{lid}(\mathscr{C}^N)$ colors and c_3 which is itself composed of $n-3$ colorings with $\chi(\mathscr{C})$ colors. The final color $c(v)$ of a vertex v will be the triplet $(c_1(v), c_2(v), c_3(v))$. Hence the coloring c uses at most $4\chi_{lid}(\mathscr{C}^N)\chi(\mathscr{C})^{n-3}$ colors. The coloring c_1 is used to separate the pairs of vertices that lie in distinct sets $V_{u,i}$. The coloring c_2 separates the pairs of vertices that lie in the same set $V_{u,i}$ and are not twins in $G_{u,i}$. Finally, the coloring c_3 separates the pairs of vertices that lie in the same set $V_{u,i}$, that are twins in $G_{u,i}$ but that are not twins in G.

The coloring c_1 is simply defined by $c_1(v) \equiv i \bmod 4$ if $v \in V_{u,i}$.

To define c_2, we define for each i, $0 \leq i \leq s$, a lid-coloring c_2^i of $G_{u,i}$ using colors 1 to $\chi_{lid}(\mathscr{C}^N)$. Then c_2 is defined by $c_2(v) = c_2^i(v)$ if $v \in V_{u,i}$.

We now define the coloring c_3. Let $V_{u,i}^{id}$ be the set of vertices of $V_{u,i}$ that have a twin in $G_{u,i}$:

$$V_{u,i}^{id} = \{v \in V_{u,i} \mid \exists w \in V_{u,i}, N_{G_{u,i}}[v] = N_{G_{u,i}}[w]\}.$$

Let $G_{u,i}^{id} = G_{u,i}[V_{u,i}^{id}]$. Since the relation "be twin" is transitive (i.e. if u and v are twins, and v and w are twins, then u and w are twins), then $G_{u,i}^{id}$ is clearly a union of cliques. In addition, since $G_{u,i} \in \mathscr{K}_{n-1}$, the connected components of $G_{u,i}^{id}$ are cliques of size at most $n-2$.

Let C be a clique of $G_{u,i}^{id}$. By Corollary 2, there exists a subset $S(C) \subseteq V(G)$ of at most $n-3$ vertices that distinguishes all the pairs of non-twin vertices of C. Note that by definition of C, $S(C) \cap V_{u,i} = \emptyset$, and thus $S(C) \subseteq V_{u,i-1} \cup V_{u,i+1}$.

Let $\mathcal{S} = \{(v, C) \mid v \in S(C)$ and C is a clique in a graph $G_{u,i}^{id}\}$. We partition $\mathcal{S}$ in $s \times (n-3)$ sets S_i^k, $1 \leq i \leq s$, $1 \leq k \leq n-3$, such that:

- if $(v, C) \in S_i^k$ for some k, then $v \in V_{u,i}$;
- if (v, C) and (w, C') are two elements of S_i^k, then $C \neq C'$.

This partition can be done because each set $S(C)$ has size at most $n-3$.

For each $S_i^k = \{(x_1, C_1), (x_2, C_2), \ldots, (x_t, C_t)\}$, we define a graph H_i^k as follows. We start from the graph induced by $V_{u,i} \cup V(C_1) \cup V(C_2) \cup \ldots \cup V(C_t)$. Then, for each (x_j, C_j) in S_i^k, we contract C_j in a single vertex y_j and finally, we contract the edge $x_j y_j$ on the vertex x_j. Note that $V_{u,i}$ is the vertex set of H_i^k. Note also that $H_i^k \in \mathscr{C}$ since it is obtained from a subgraph of G by successive edge-contractions. Therefore, $\chi(H_k^i) \leq \chi(\mathscr{C})$.

We now define a proper coloring $c_3^{i,k}$ of H_i^k with colors 1 to $\chi(\mathscr{C})$. Let c_3^k be the coloring of vertices of G defined by $c_3^k(v) = c_3^{i,k}(v)$ if $v \in V_{u,i}$. Finally, c_3 is defined by $c_3(v) = (c_3^1(v), \dots, c_3^{n-3}(v))$, and the final color of v is $c(v) = (c_1(v), c_2(v), c_3(v))$.

We now prove that c is a lid-coloring of G. First, c is a proper coloring. Indeed, two adjacent vertices that are not in the same set $V_{u,i}$ lie in consecutives set $V_{u,i}$ and $V_{u,i+1}$ and thus have different colors in c_1, and two adjacent vertices in the same set $V_{u,i}$ have different colors in c_2 (which induces a proper coloring on $V_{u,i}$).

Let now x and y be two adjacent vertices with $N[x] \neq N[y]$. We will prove that $c(N[x]) \neq c(N[y])$. We distinguish three cases.

Case 1: $x \in V_{u,i}$ and $y \in V_{u,i+1}$.

If $x = u$, then y has a neighbor v in $V_{u,i+2} = V_{u,2}$. Indeed, u is taken with minimum degree, so y has at least as many neighbors as u and does not have the same neighborhood than u, implying that y has a neighbor in $V_{u,2}$. Then $c_1(v) = 2 \notin c_1(N[u])$ and so $c(N[x]) \neq c(N[y])$.

Otherwise, x has neighbor v in $V_{u,i-1}$ and $c_1(v) \equiv i - 1 \pmod 4 \in c_1(N[x])$. On the other hand, all the neighbors of y belong to $V_{u,i} \cup V_{u,i+1} \cup V_{u,i+2}$ and therefore $c_1(N[y]) \subseteq \{i, i+1, i+2 \pmod 4\}$. Thus, $c(N[x]) \neq c(N[y])$.

Case 2: x and y belong to $V_{u,i}$ and they are not twins in $V_{u,i}$ (i.e. $N_{V_{u,i}}[x] \neq N_{V_{u,i}}[y]$).

By definition of the coloring c_2^i, there exists a color a that separates x and y, i.e. $a \in c_2^i(N_{V_{u,i}}[x]) \,\Delta\, c_2^i(N_{V_{u,i}}[y])$. Then we necessarily have $c(N[x]) \neq c(N[y])$.

Case 3: x and y belong to $V_{u,i}$ and they are twins in $V_{u,i}$ (i.e. $N_{V_{u,i}}[x] = N_{V_{u,i}}[y]$).

In this case, vertices x and y are in the set $V_{u,i}^{id}$. Let C be the clique of $G_{u,i}$ containing x and y. Let $v \in S(C)$ that distinguishes x and y; thus, $v \in V_{u,j}$ for $j = i - 1$ or $j = i + 1$. Wlog, $v \in N[x]$ but $v \notin N[y]$. Let S_j^k be the part of $\mathcal{S}$ that contains (v, C). Suppose that there exists a neighbor w of y such that $c(v) = c(w)$. Then w lies in $V_{u,j}$ because of the coloring c_1. However, in the graph H_j^k, the vertex v is adjacent to all the neighbors of y in $V_{u,j}$, and in particular is adjacent to w; therefore, $c_3^{j,k}(v) \neq c_3^{j,k}(w)$, a contradiction. Therefore, the vertex y does not have any neighbor that has the same color as v. Hence, $c(v) \notin c(N[y])$, and $c(N[x]) \neq c(N[y])$.

$\square$

References

[1] P.N. Balister, O.M. Riordan, and R.H. Schelp. Vertex-distinguishing edge-colorings of graphs. *J. Graph Theory*, **42**:95–109, 2003.

[2] J. A. Bondy. Induced subsets. *J. Combin. Theory Ser. B*, **12**(2):201–202, 1972.

[3] A.C. Burris and R.H. Schelp. Vertex-distinguishing proper edge-colorings. *J. Graph Theory*, **26**:73–83, 1997.

[4] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater and P. Zhang. The locating-chromatic number of a graph. *Bull. Inst. Combin. Appl.*, **36**:89-101, 2002.

[5] J. Cerný, M. Horňák, and R. Soták. Observability of a graph. *Math. Slovaca*, **46**(1):21–31, 1996.

[6] L. Esperet, S. Gravier, M. Montassier, P. Ochem and A. Parreau. Locally identifying coloring of graphs. *Electron. J. Combin.*, **19(2)**, 2012.

[7] F. Foucaud, I. Honkala, T. Laihonen, A. Parreau and G. Perarnau. Locally identifying colouring of graphs with given maximum degree. *Discrete Math.*, **312(10)**, 2012.

[8] M. G. Karpovsky, K. Chakrabarty, and L. B. Levitin. On a new class of codes for identifying vertices in graphs. *IEEE Trans. Inform. Theory*, 44:599–611, 1998.

[9] A. Lobstein. Identifying and locating-dominating codes in graphs, a bibliography. Published electronically at `http://perso.enst.fr/~lobstein/debutBIBidetlocdom.pdf`.

[10] J. Nešestřil and P. Ossona de Mendez. Grad and classes with bounded expansion I. Decompositions. *European J. Combin.*, **29**:760-776, 2008.